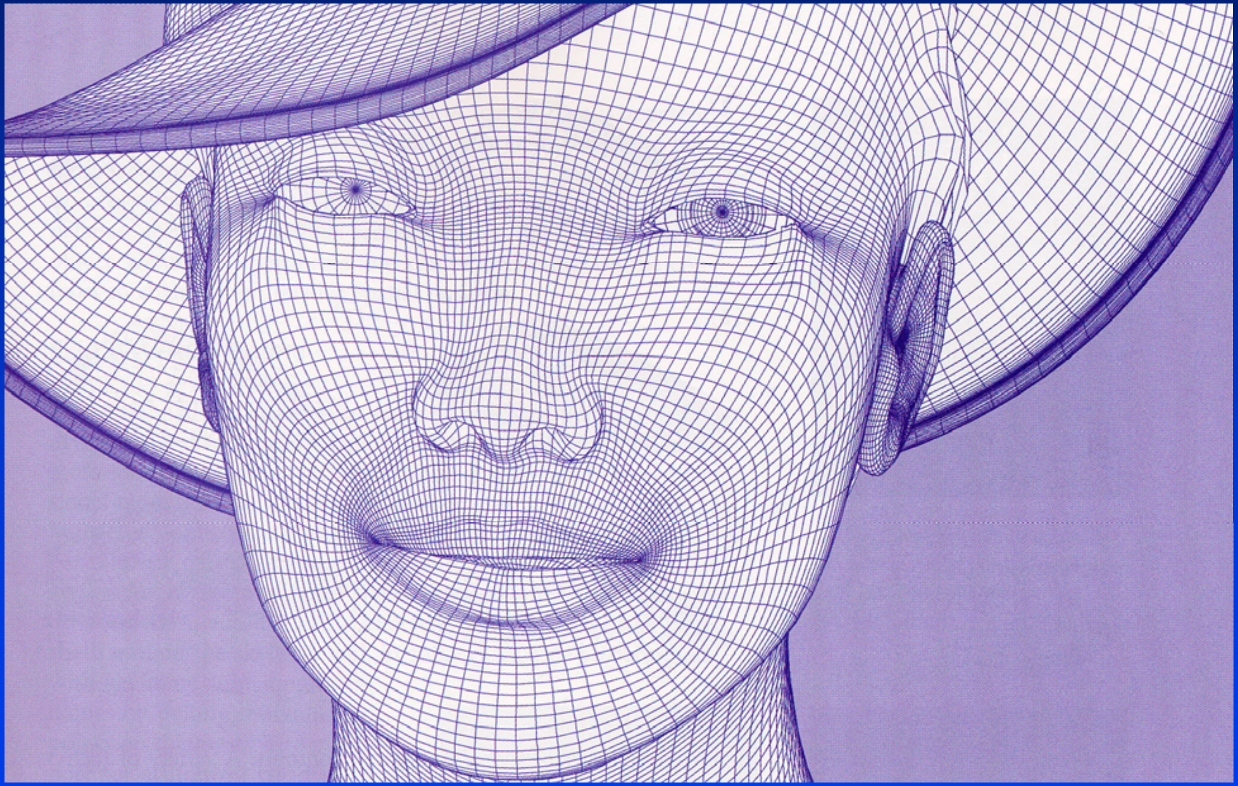
Parametric Curves and Surfaces I



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Higher Degree Curves and Surfaces

- Way to model objects, specify motion curves
- Compact
  - Smooth curve/surface possible with few control points
- Easier to use for modeling
  - User has small set of parameters that control large DOF
- Image rendering more complex
  - Direct or Polygonal approximation
- 3D “rendering” simpler
  - Numerically Controller (NC) milling of mathematically defined surface

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Explicit equation

- $y=f(x)$, $z=g(x)$
- Cannot get multiple y or z for single x
- Not rotationally invariant
- Cannot describe curves with infinite slopes

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Implicit equation

- $f(x, y, z)=0$
- Have to solve non-linear equation for each point
- Difficult to move along the curve
- More solution than we want (need to specify additional constraint)
- Difficult to join two curves (such that tangents are same)
- Derivatives not given in terms of direction of motion

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Parametric equation

- $x=x(t)$, $y=y(t)$, $z=z(t)$
- As t varies from 0 to 1, function sweeps out a curve (or surface in the case of parametric surfaces in u and v)
- t not plotted

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Cubic

- $$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
  $$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
  $$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

$$0 \le t \le 1$$

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- Lower degree poly – too little control (cannot be made to pass through specific point with specified derivatives)
- Higher degree poly – too much complexity (greater the degree, greater the amount of oscillations)
- Need 4 known vector equations to solve for the 4 unknowns
- Depending on the constraints, different curve

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Matrix form

$$Q(t) = [x(t)\ \ y(t)\ \ z(t)] = [\overbrace{t^3\ \ t^2\ \ t\ \ 1}^{T}]\overbrace{\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}}^{C}$$

- Parametric tangent vector

$$\frac{d}{dt}Q(t) = [3a_x t^2 + 2b_x t + c_x \quad 3a_y t^2 + 2b_y t + c_y \quad 3a_z t^2 + 2b_z t + c_z]$$

  – Velocity of a point on the curve w.r.t. t

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Continuity between two curves

- G^0 geometric continuity – 2 curve segments joined
- G^1 geometric continuity – 2 curve segments joined & direction of tangent same
- C^n parametric continuity – 2 curves joined & n^{th} tangent vector, $\frac{d^n}{dt^n}Q(t)$, same

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- In general $C^1 => G^1$

  except when $\dfrac{dQ}{dt} = [\,0\,,\,0\,,\,0\,]$

  can have tangent vector same, direction of tangent different

- Cannot determine tangent vector (magnitude) by looking at resulting curve: t not plotted

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Basis functions (Blending functions)

- $Q = T \bullet C$
- Rewrite $C = M \bullet G$
  - *M*: 4 x 4 “basis” matrix
  - *G*: Geometry vector (of vectors)

$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z} \\ g_{4x} & g_{4y} & g_{4z} \end{bmatrix}$$

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- $$\begin{aligned} x(t) = {} & (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} + \\ & (t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x} + \\ & (t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} + \\ & (t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x} \end{aligned}$$

  $y(t),\ z(t)\ \ similar$

- $$x(t) = \sum_{k=0}^{3} g_{kx} B_k(t)$$

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- Can think of Q as weighted sum of geometry matrix G where weights are the “blending functions” B
- Another way of looking : reconstruction of a function Q by “filtering” 4 samples
  - Basis functions are the filters

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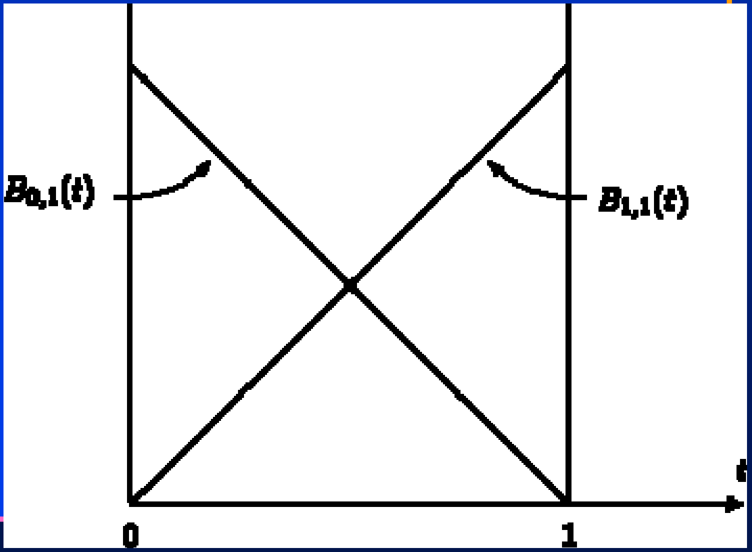
Linear interpolation

- $$Q(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = TMG$$
- *M* is "basis" matrix of coefficients and different for different curves
- *G* is geometric vector
- *TM* are the basis functions
  - *1*-*t* and *t*



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To Transform Curve

- Just x-form *G*
- Not for non-affine transform like perspective transform (ok for rational curves…to come later)

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Bézier Curves

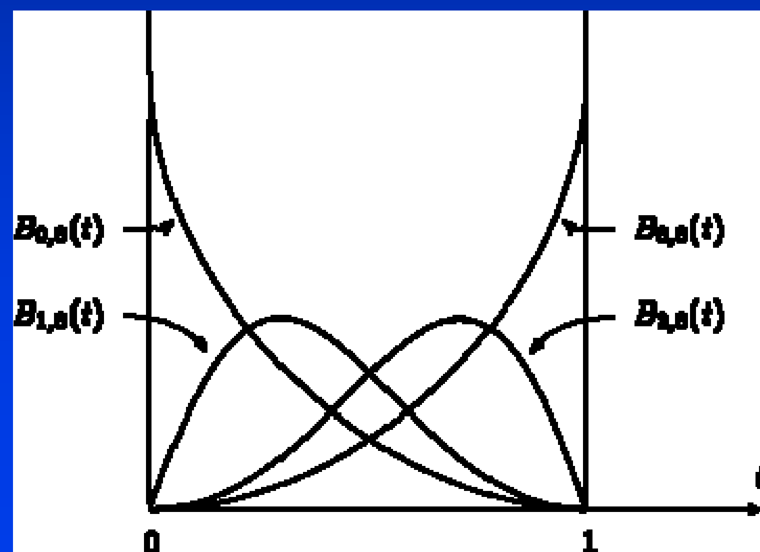
- $$Q(t) = \sum_{k=0}^{3} P_k B_k(t)$$
- Where basis functions are given by the Bernstein cubic polynomials

$$B_0(t) = (1-t)^3$$

$$B_1(t) = 3t(1-t)^2$$

$$B_2(t) = 3t^2(1-t)$$

$$B_3(t) = (t)^3$$



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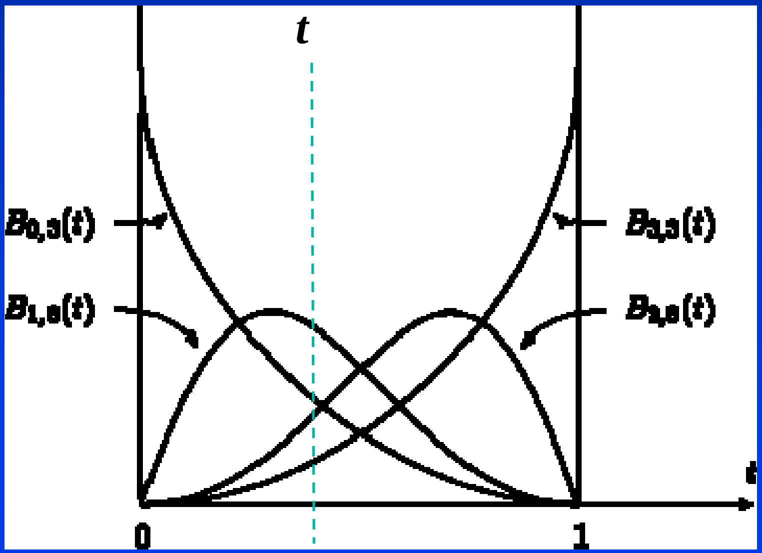
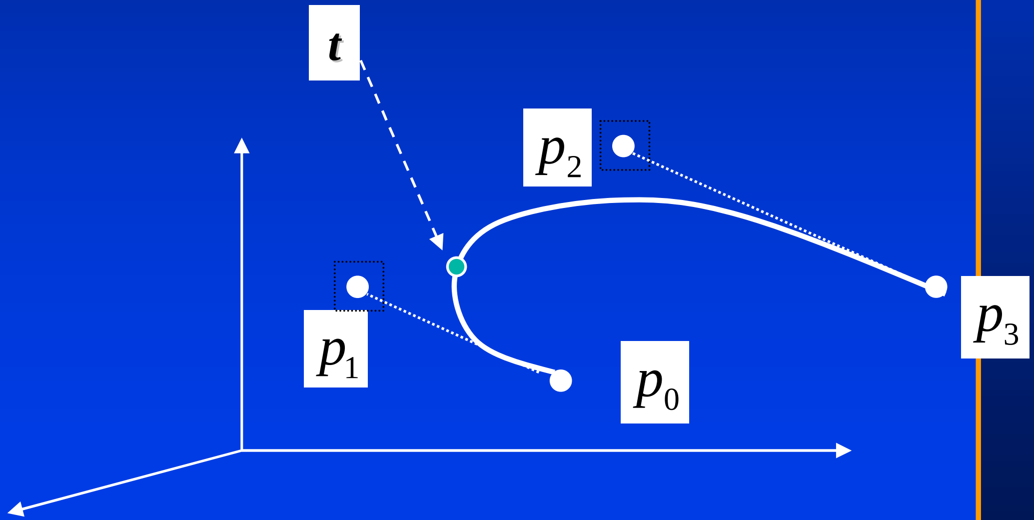
- At $t=0$, $Q(0) = P_0$
- At $t=1$, $Q(1) = P_3$
- At any t, the 4 blending functions give weights for 4 control points





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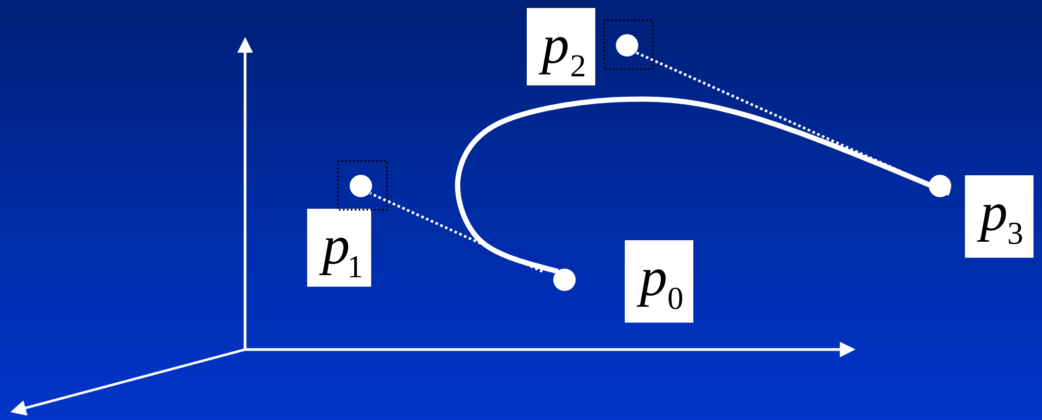

$$Q'(0) = 3(p_1 - p_0)$$

$$Q'(1) = 3(p_2 - p_3)$$

- Basis functions chosen such that p_1-p_0 gives the first point's tangent and p_2-p_3 gives the last point's tangent
- This plus the beginning and end points make up the 4 constraints

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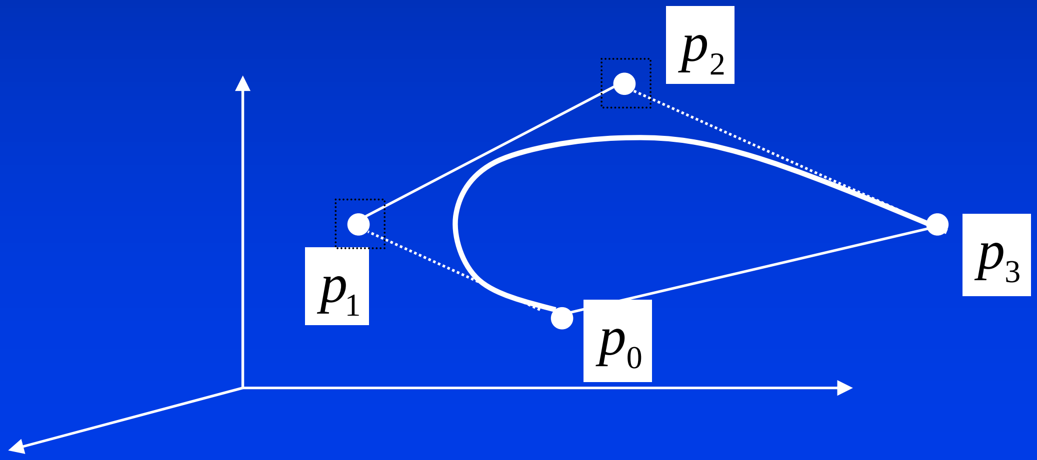
- Curve contained in "convex hull" formed by control points
  - Basis functions sum to unity for all t
  - Useful for defining bounding box



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Matrix form

$$Q(t) = \sum_{k=0}^{3} P_k B_k(t)$$

$$Q(t) = P_0(1-t)^3 + P_1 3t(1-t)^2 + P_2 3t^2(1-t) + P_3(t)^3$$

$$Q(t) = T\,M\,G$$

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

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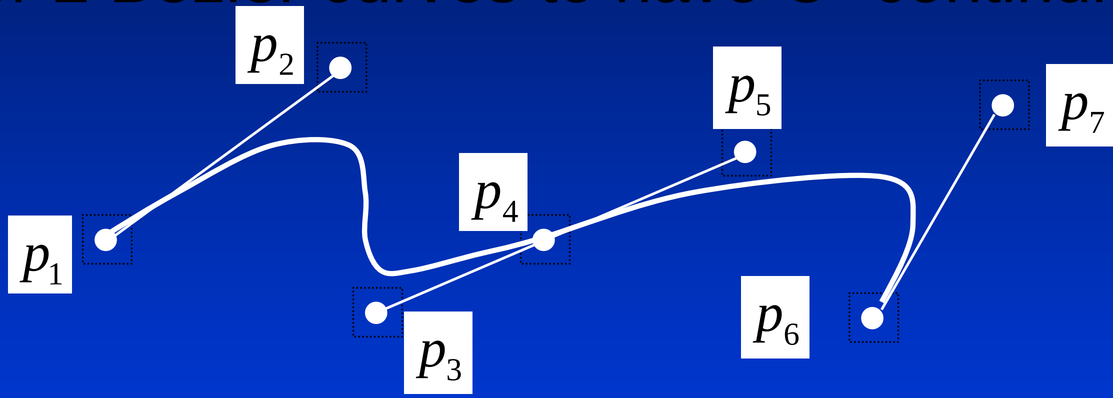
Combine Bezier curves

- For 2 Bézier curves to have C^1 continuity



- $p_3 - p_4 - p_5$ collinear and equally spaced
- If not equally spaced, G^1 continuous

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Problems with Bezier curves

- Global control
  - All weighting function non-zero everywhere except at the ends (for one curve)
  - Means that small changes in one control point changes the whole curve
- Continuity between curves explicitly controlled by the linearity of the control points

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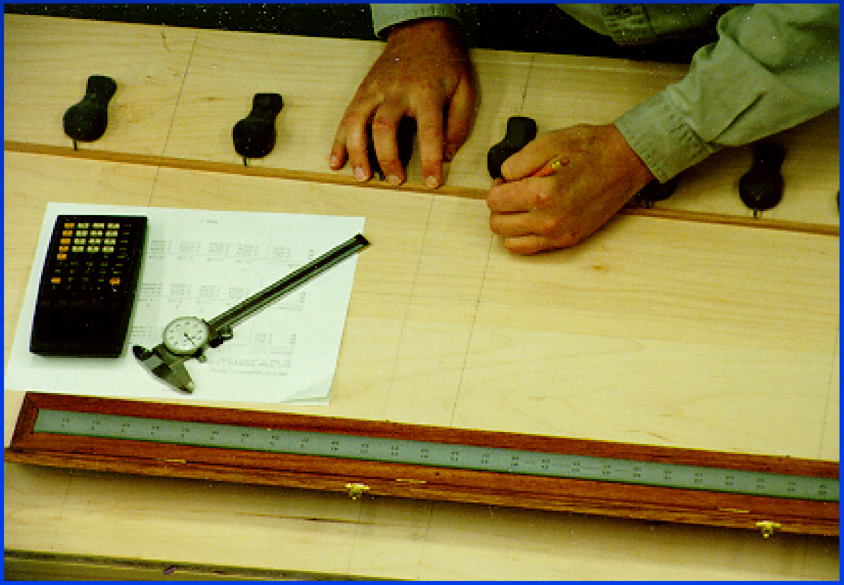
Splines

- Natural spline
  - Global control
  - At least C^2 continuous
  - Flexible strip & “ducks” or “whales”



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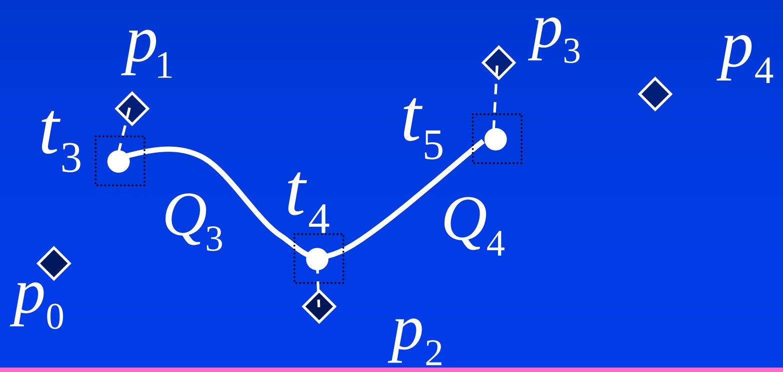
- Knot: where two curves join
- Interpolating
  - Control point = knot point
- Approximating
  - Control point ≠ knot point



$$\text{control pts}: p_0 \ldots p_m$$

$$\text{curves} \quad : Q_3 \ldots Q_m$$

$$\text{Parameter for } Q_i \ :$$

$$t_i \le t \le t_{i+1}$$

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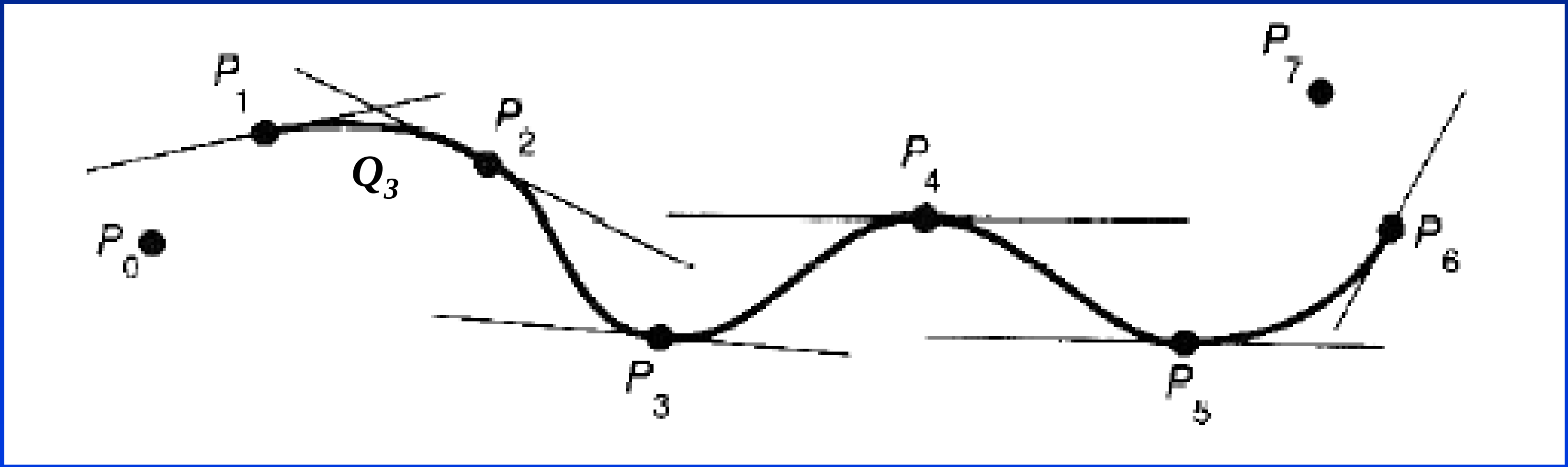
Cardinal splines

- Interpolating
- C^1 continuity



- Assume tangent at p_j given by $\overline{p_{j-1}\, p_{j+1}}$

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$$Q(t) = At^3 + Bt^2 + Ct + D$$

or

$$Q(t) = [t^3 \ \ t^2 \ \ t \ \ 1]\,[A \ \ B \ \ C \ \ D]^T$$

$$Q(0) = p_{i-2} = D$$

$$Q(1) = p_{i-1} = A + B + C + D$$

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- Parametric derivative

$$Q'(t) = [3t^2 \;\; 2t \;\; 1 \;\; 0]\,[A \;\; B \;\; C \;\; D]^T$$

- At t=0, beginning tangent and at t=1, end tangent

$$Q'(0) = a(p_{i-1} - p_{i-3}) \equiv S_0 = C$$
$$Q'(1) = a(p_i - p_{i-2}) \equiv S_1 = 3A + 2B + C$$

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- 4 constraints from previous two slides give

$$M_1$$

$$\begin{bmatrix} p_{i-2} \\ p_{i-1} \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

- Definitions of S_0 and S_1 give

$$\begin{bmatrix} p_{i-2} \\ p_{i-1} \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & a \end{bmatrix} \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$

$$M_2$$

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- Equating the two equations

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = M_1^{-1} M_2 \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$

- Multiply out $M_1^{-1}M_2$
  - Call this M

$$M_1^{-1} M_2 = \begin{bmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \equiv M$$

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Matrix form of Cardinal Spline

$$Q(t) = [t^3 \ \ t^2 \ \ t \ \ 1][A \ \ B \ \ C \ \ D]^T$$

$$Q(t) = [t^3 \ \ t^2 \ \ t \ \ 1]M \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$

$$M = \begin{bmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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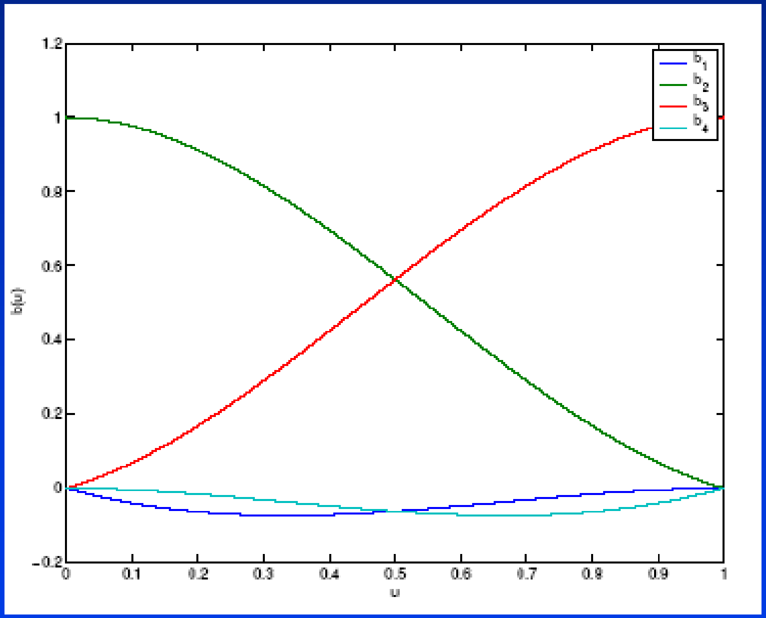
- Blending functions for Catmull-Rom spline



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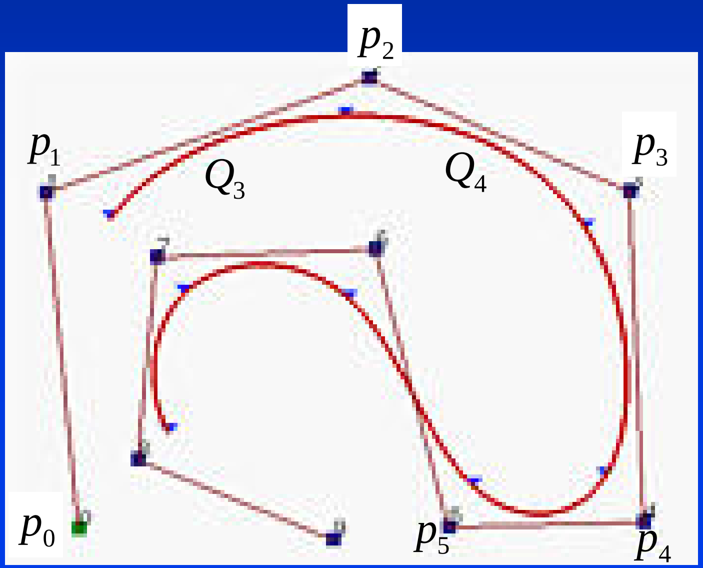
Uniform (Nonrational) B-splines (“basis spline”)

- Approximating
- C^0, C^1, C^2 continuous



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$$Q_i(t) = \sum_{k=0}^{3} p_{i-3+k} B_{i-3+k}(t)$$

- p: control points (geometric vector components)
- B: basis functions (or blending functions)
- i: segment number
- k: local control point index
- t: 0 to 1

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- For parameter range $t_i \le t \le t_{i+1}$

$$B_i = \frac{1}{6} t^3$$

$$B_{i-1} = \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)$$

$$B_{i-2} = \frac{1}{6}(3t^3 - 6t^2 + 4)$$

$$B_{i-3} = \frac{1}{6}(1 - t)^3$$

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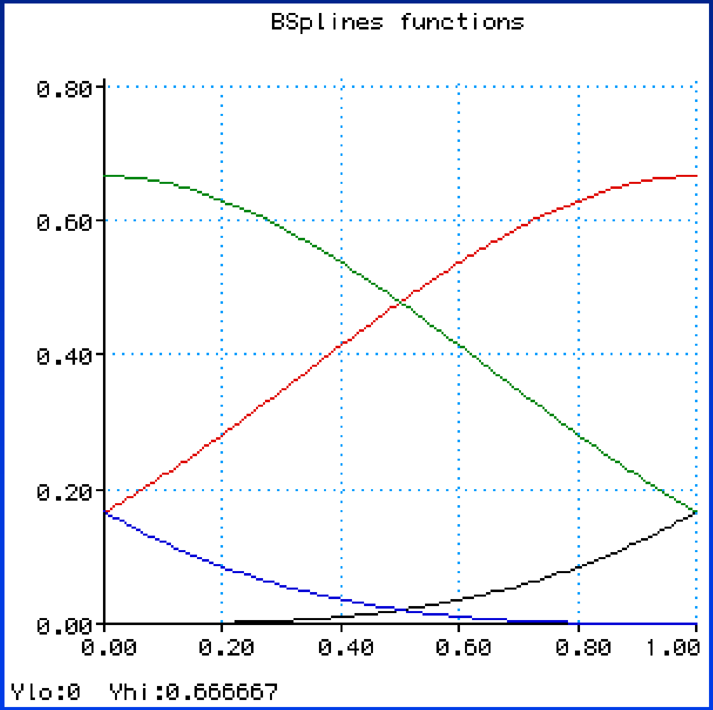
- Basis functions in range $t_i \leq t \leq t_{i+1}$



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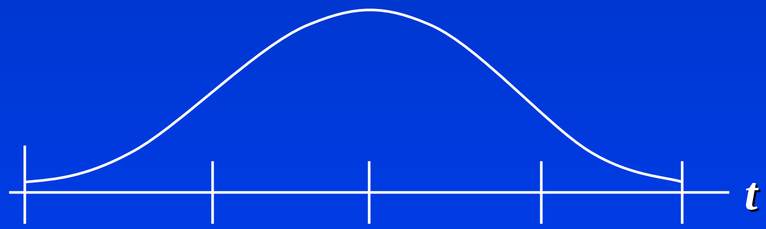
Define as one B-spline curve

- i: non-local control point number
- t: global parameter (not just 0 to 1)
- Basis function in entire range given by shifting right given equations by 0, 1, 2, 3, 4 units to right

$$Q(t) = \sum_{i=0}^{m} p_i B_i(t)$$



$$B_i = \frac{1}{6} t^3$$

$$B_{i-1} = \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)$$

$$B_{i-2} = \frac{1}{6}(3t^3 - 6t^2 + 4)$$

$$B_{i-3} = \frac{1}{6}(1 - t)^3$$

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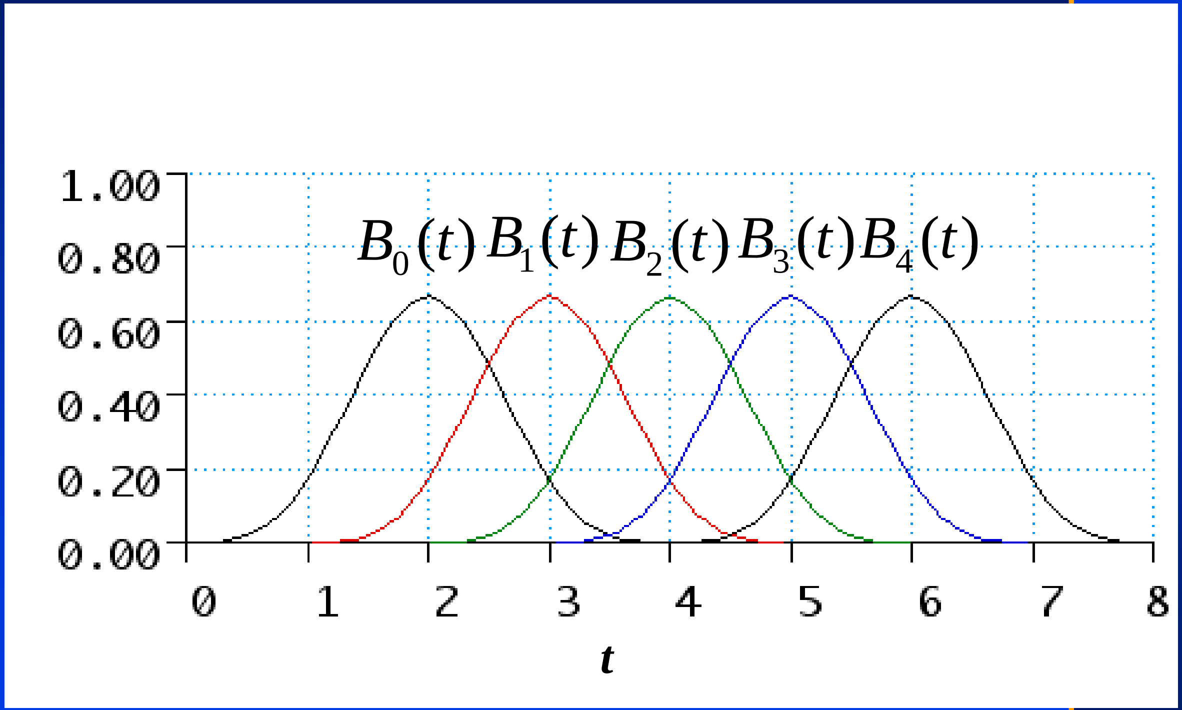
- For the useful range, basis functions sum to 1
  - t ≥ 3 for curve
- For example on right, two curves between t=3 and 5



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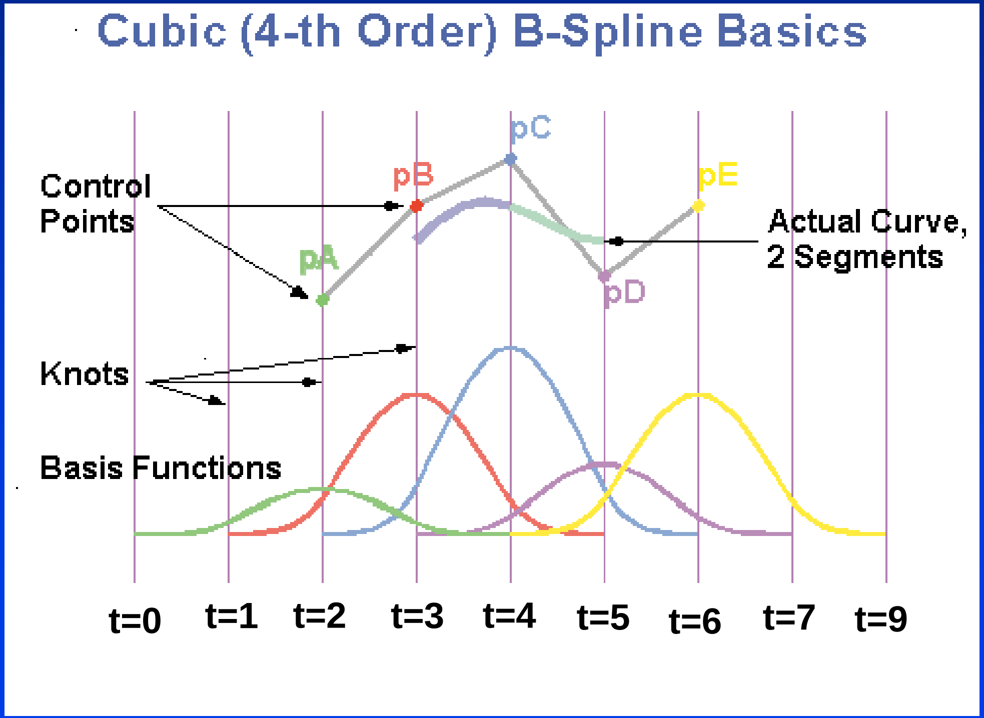
- Curve segments end/start at **equal** intervals with respect to global parameter *t*: “uniform”
  - [0, 1, 2, … m]
  - Called knot vector
- On right, 9 knots
  - Basis functions shown weighted by value of control points



Sequin, C.

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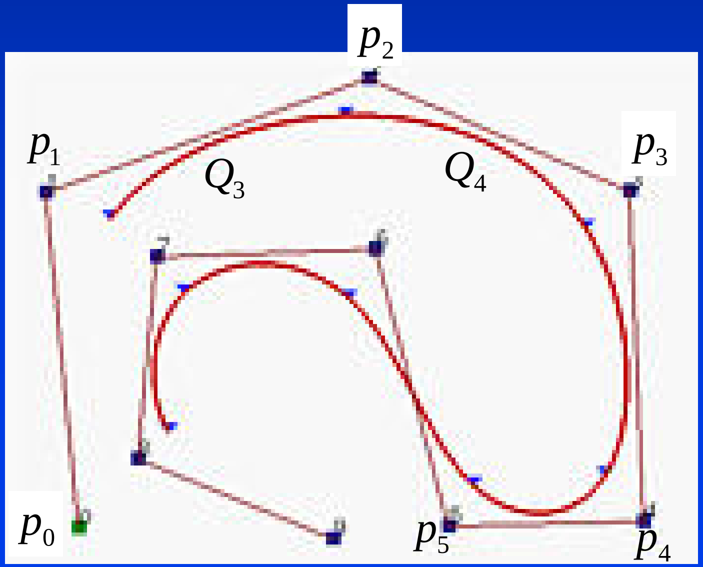
- For the B-spline on the right:
knot vector is [0,1,2,3,4,5,6,7,8,9,10,11,12,13] with useful knot values at [3,4,5,6,7,8,9,10] where basis function sum to 1



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Matrix form

$$Q_i(t) = T_i\, M\, G_i$$

$$M = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

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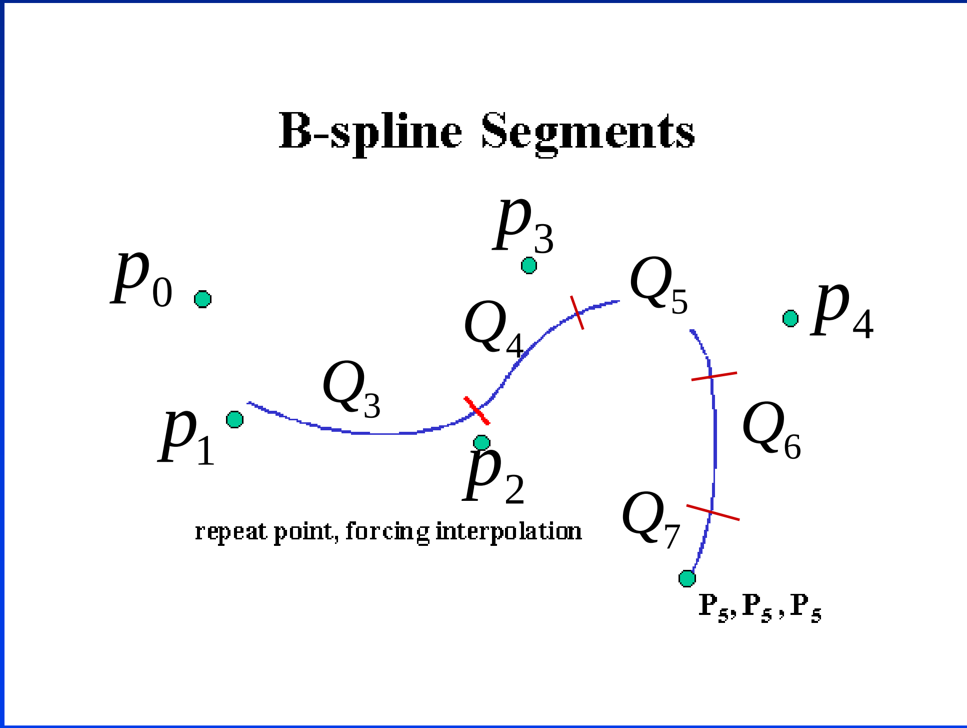
Multiplicity of Control Points

- Example: Last control point used three times
- Q5 uses p5 once
- Q6 uses p5 twice
- Q7 uses p5 three times



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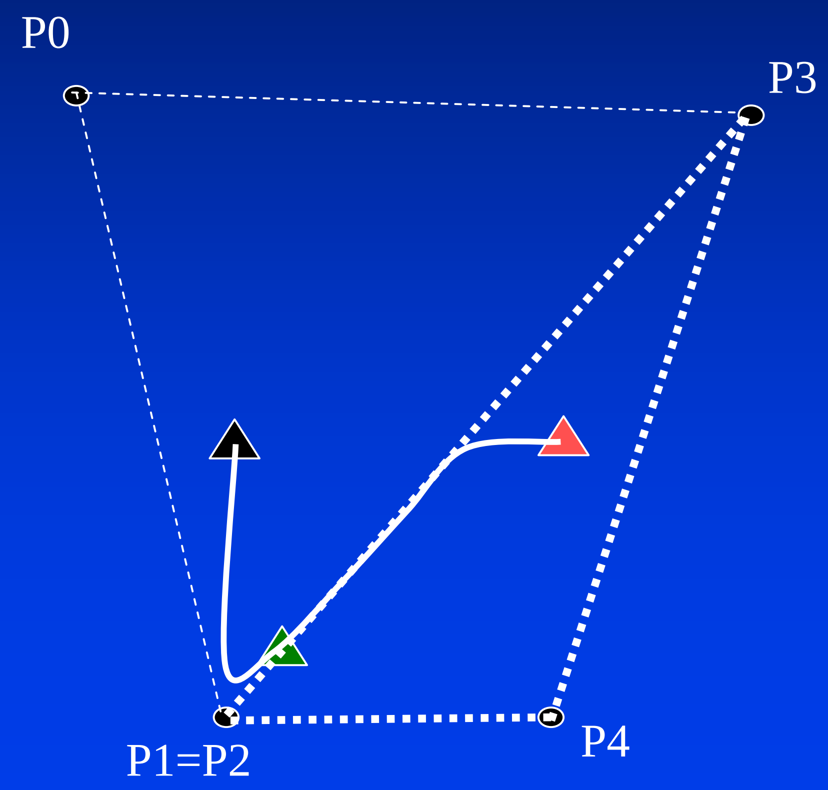
- Example: Interior control point used twice
- Continuity C^2G^1 :
  - Continuity within still C^2
  - Continuity across knot G^1



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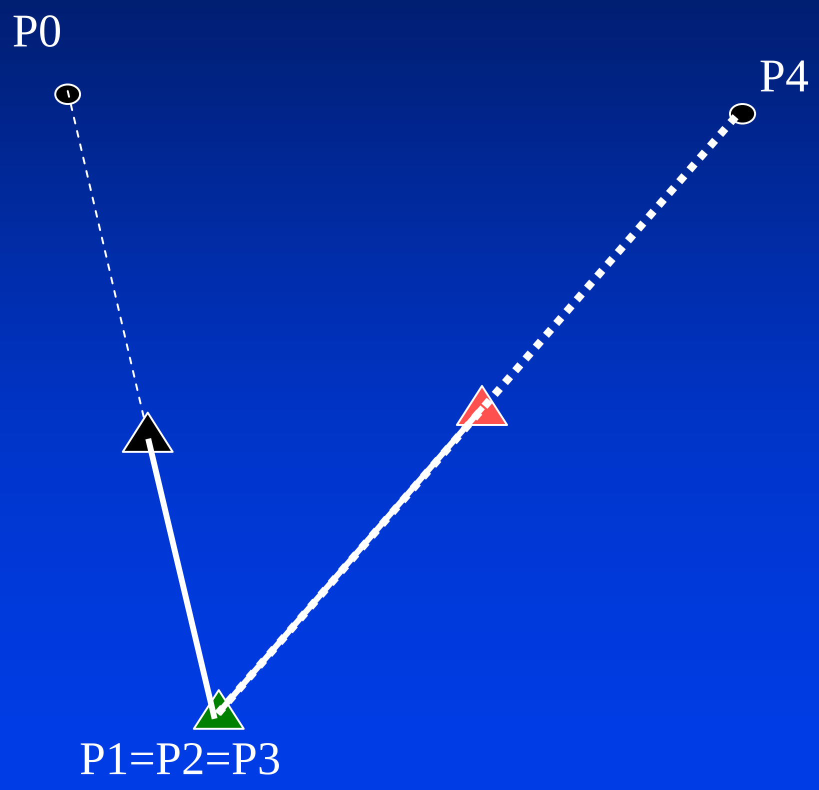
- Example: Interior control point used three times
- Continuity C^2G^0
  - Continuity within still C^2
  - Continuity across knot G^0



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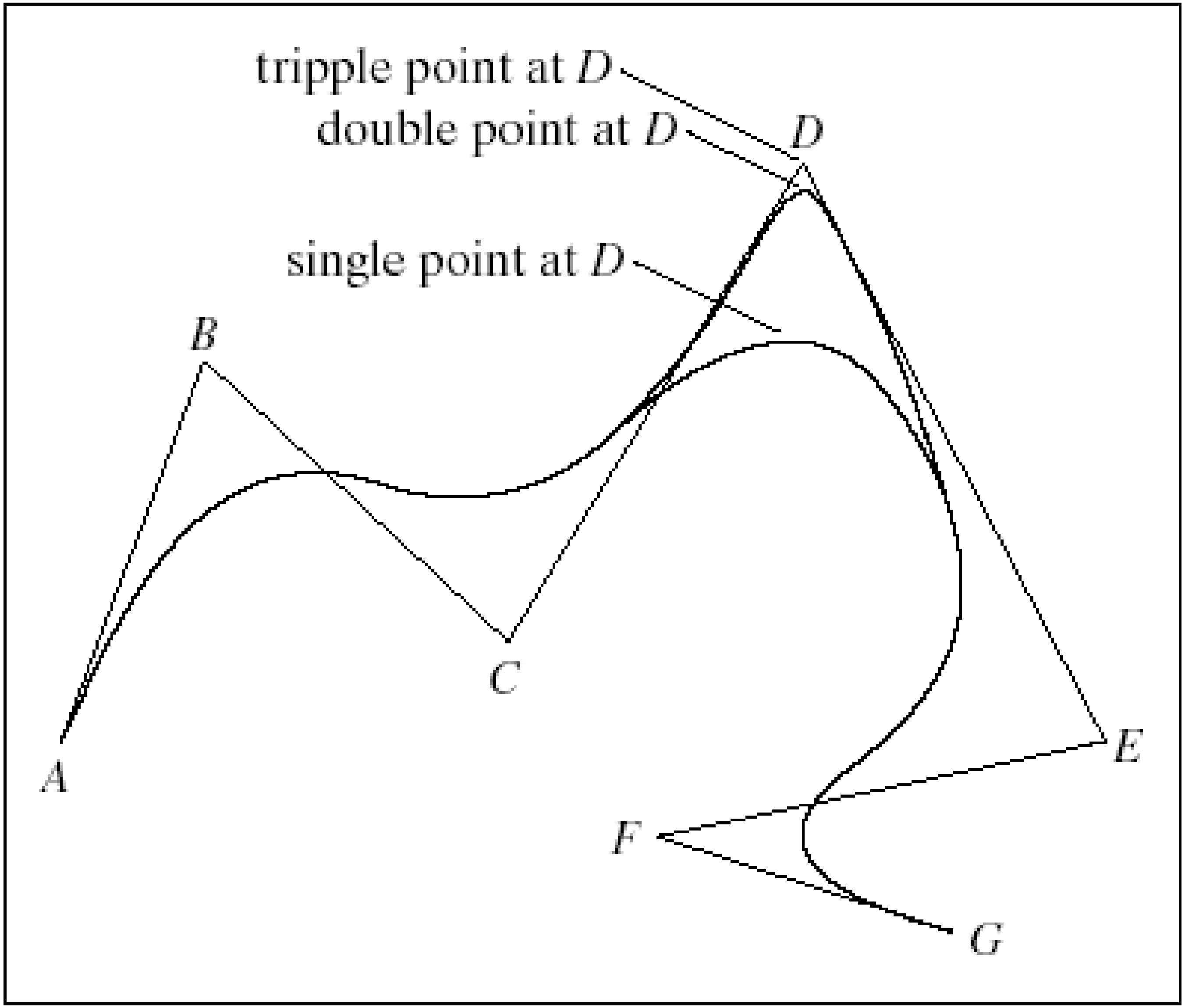

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Nonuniform (Nonrational) B-Spline

- Parameter interval between knot values need not be uniform
  - Most common is reducing intervals between successive knots to 0 (multiplicity of knots)
  - When $t_i = t_{i+1}$ multiple knot and Q_i becomes a point
- Basis functions can vary from curve segment to curve segment
- Continuity can be reduced from C^2 to C^1 to C^0 to none
- C^0: curve ***interpolates*** control point w/o straight line segment as in uniform B-spline w/ multiplicity of control points
- Possible to add additional knots and control points

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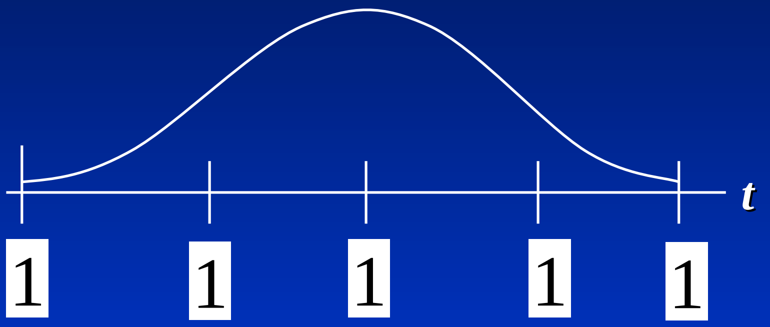
Effect of knot multiplicity
On single basis function

- [0,1,2,3,4]
  - Uniform interval between knot values
  - Right figure shows basis function with multiplicity of one for each knot (usual B-spline as before)



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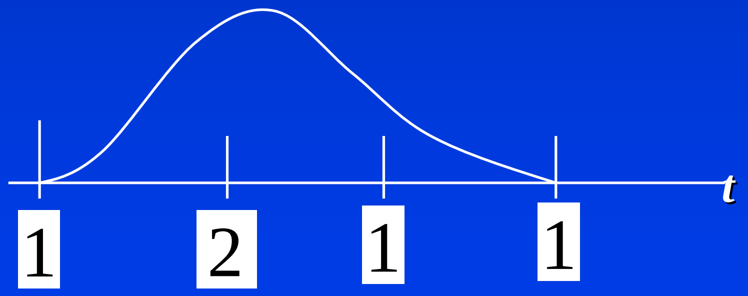
- [0,1,1,2,3]
  - Interval between second and third knot values is 0
  - B_{i-1} (second segment) shrinks to 0
  - Eliminate second derivative continuity, first derivative continuity remain



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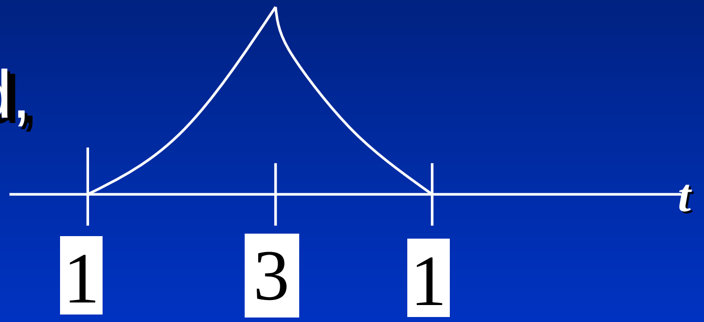
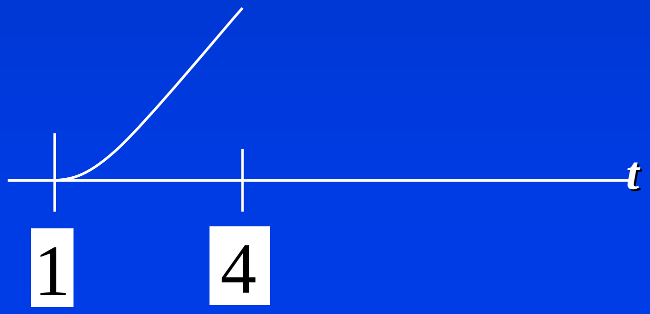
- [0,1,1,1,2]
  - Interval between second, third, fourth knot values is 0
  - Only positional continuity



- [0,1,1,1,1]
  - Interval between second, third, fourth, and fifth knot values is 0
  - No positional continuity



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Knot multiplicity	Continuity Conditions	Continuity
1	Positional tangential curvature	C^2
2	Positional tangential	C^1
3	positional	C^0
4	none	none

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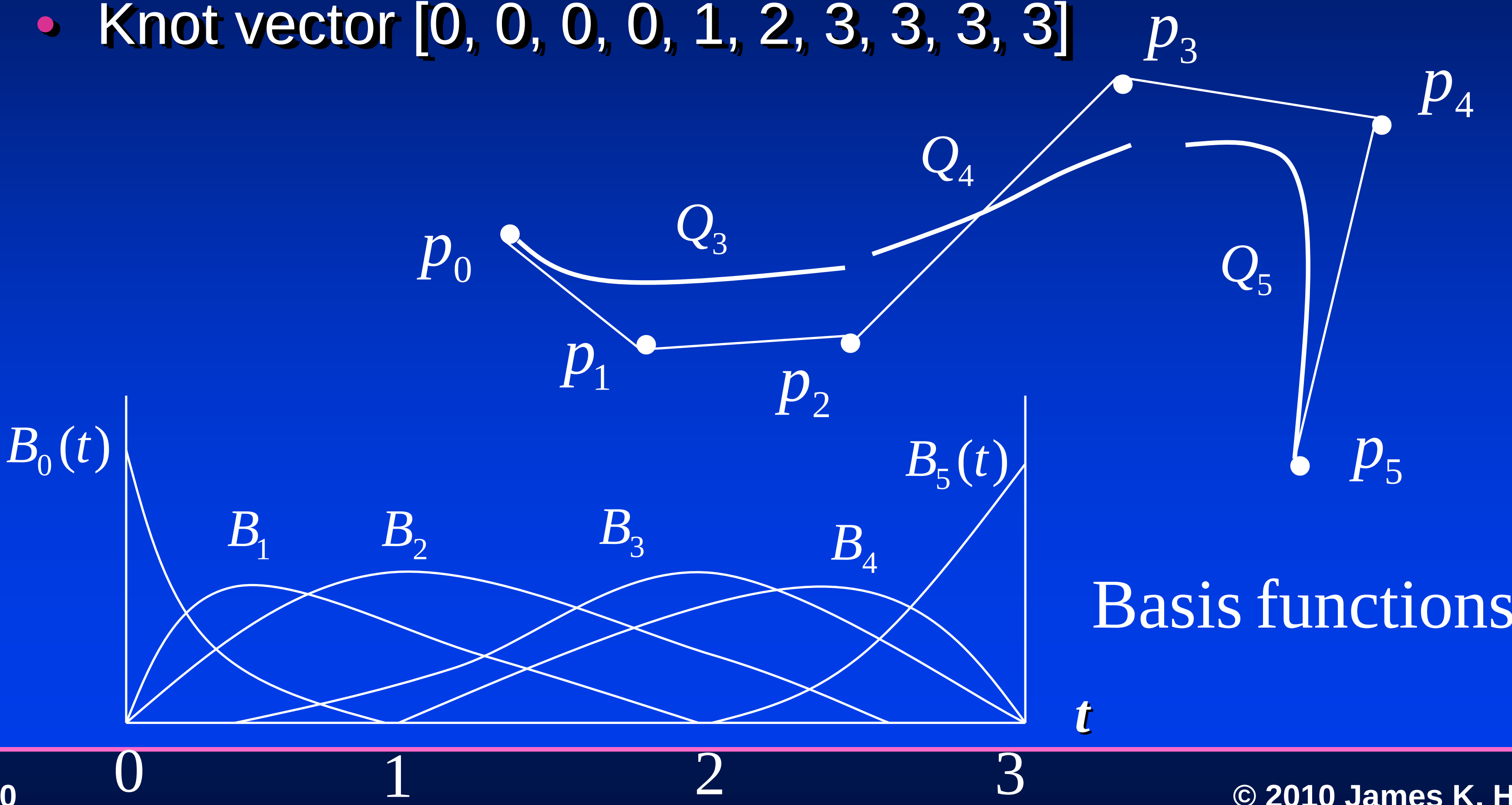
Example of knot multiplicity on ends

- Knot vector [0, 0, 0, 0, 1, 2, 3, 3, 3, 3]



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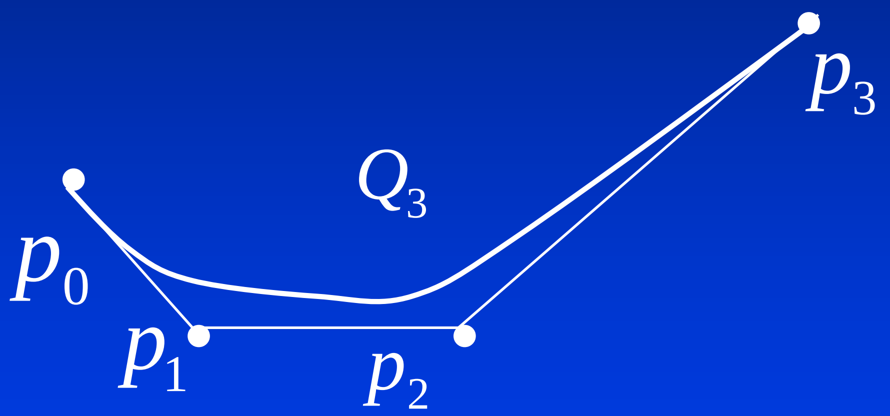
- Knot vector [0, 0, 0, 0, 1, 1, 1, 1]



  – equal to Bézier curve

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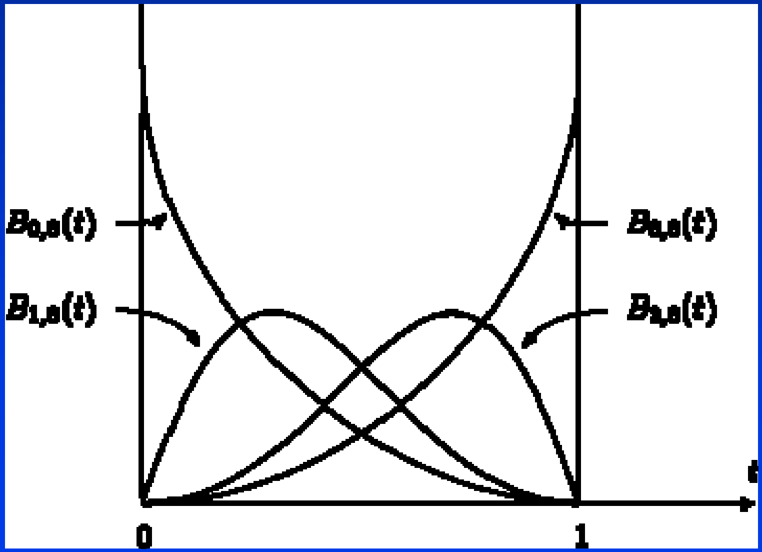
- Basis functions are Bezier basis functions
- Bezier a subset of B-splines



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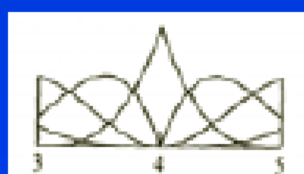
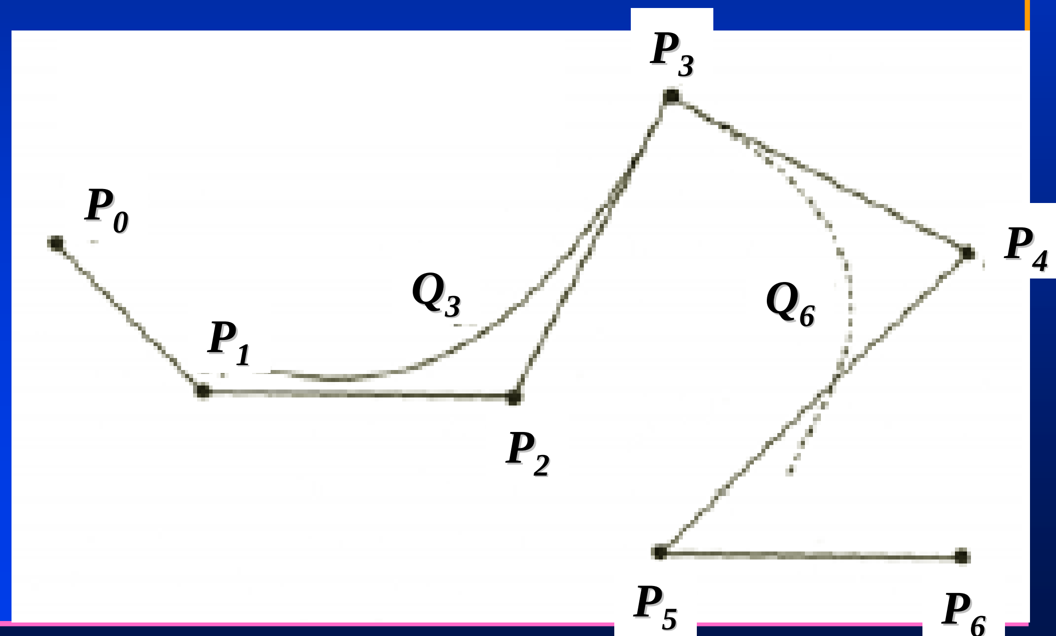
Example of knot multiplicity in interior knot

- Triple knot vector [0, 1, 2, 3, 4, 4, 4, 5, 6, 7, 8]
- Q_4 and Q_5 shrink to zero
- C^0 continuity between Q_3 and Q_6





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Next: Parametric surfaces

Turbo Squid

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