

Gaussian Distribution

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1 Standard Gaussian Distribution

The probability density function of the standard n -dimensional Gaussian distribution is given by:

$$p(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}x^\top x\right) \quad (1)$$

where $x \in \mathbb{R}^n$, the covariance matrix is I , and the mean is 0.

2 Linear Transformation of Gaussian Distribution

If we have a linear transformation $y = Ax + b$ of a Gaussian distribution with mean b and covariance matrix AA^\top , then the probability density function of y is given by:

$$p(y) = \frac{1}{\sqrt{(2\pi)^n |AA^\top|}} \exp\left(-\frac{1}{2}(y - b)^\top (AA^\top)^{-1}(y - b)\right) \quad (2)$$

So the probability density function of the Gaussian distribution is given by:

$$p(y) = \frac{\sqrt{|B|}}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}(y - \mu)^\top B(y - \mu)\right) \quad (3)$$

where B is a symmetric positive definite matrix. If the corresponding random variable is Y , then we have:

$$Y \sim \mathcal{N}(\mu, B^{-1}) \quad (4)$$

$$\mathbb{E}[Y] = \mu \quad (5)$$

$$\text{Cov}[Y] = B^{-1} \quad (6)$$

$$(B^{-1})_{ij} = \mathbb{E}[(Y_i - \mu_i)(Y_j - \mu_j)] \quad (7)$$

2.1 Characteristic Function

The characteristic function of the generalized Gaussian distribution is given by:

$$\phi(t) = \exp\left(-\frac{1}{2}t^\top Bt + i\mu^\top t\right) \quad (8)$$

Proof. Since B is a symmetric positive definite, we can write $B^{-1} = PP^\top$, where P is an invertible matrix. Consider the random variable X as a standard Gaussian distribution, and $Z = PX + \mu$. Then we have:

$$p(z) = \frac{|P|}{2\pi^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(z - \mu)^\top (P^{-1})^\top P^{-1}(z - \mu)\right) \quad (9)$$

$$= \frac{|P|}{2\pi^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(z - \mu)^\top B(z - \mu)\right) \quad (10)$$

$$= \sqrt{\frac{|B|}{(2\pi)^n}} \exp\left(-\frac{1}{2}(z - \mu)^\top B(z - \mu)\right) \quad (11)$$

which means Z is a generalized Gaussian distribution with mean μ and covariance matrix B .

The characteristic function of Z is given by:

$$\phi(t) = E[\exp(it^\top Z)] \quad (12)$$

$$= E[\exp(it^\top (PX + \mu))] \quad (13)$$

$$= E[\exp(i(P^\top t)^\top X)] \exp(i\mu^\top t) \quad (14)$$

$$= \exp\left(-\frac{1}{2}(P^\top t)^\top P^\top t\right) \exp(i\mu^\top t) \quad (15)$$

$$= \exp\left(-\frac{1}{2}t^\top Bt\right) \exp(i\mu^\top t) \quad (16)$$

Equation 15 is because X is a standard Gaussian distribution, so its characteristic function is $\exp\left(-\frac{1}{2}t^\top t\right)$. \square

3 Convergence of Gaussian Distribution

Theorem 1 (Convergence of Gaussian Distribution). *Suppose $X_k : \Omega \rightarrow \mathbb{R}^n$ is normal for all k and that $X_k \rightarrow X$ in $L^2(\Omega)$, i.e. $E[|X_k - X|^2] \rightarrow 0$ as $k \rightarrow \infty$. Then X is normal.*

Proof.

$$E[\exp(it^\top X_k)] - E[\exp(it^\top X)]^2 \leq E[|X_k - X|^2] |t|^2 \rightarrow 0$$

as $k \rightarrow \infty$. So $E[\exp(it^\top X_k)] \rightarrow E[\exp(it^\top X)]$ as $k \rightarrow \infty$. So X is normal. \square

4 Co-variance Matrix

$$\phi(u) = \exp\left(-\frac{1}{2}u^\top Bu + i\mu^\top t\right)$$

$$\mathbb{E}[X_i X_j] = -\frac{\partial^2 \phi}{\partial u_i \partial u_j}(0) \quad (17)$$

$$= -\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \bigg|_{t=0} \exp\left(-\frac{1}{2}t^\top Bt + i\mu^\top t\right) \quad (18)$$

$$= \mu_i \mu_j + \frac{1}{2}B_{ij} + \frac{1}{2}\delta_{ij} \quad (19)$$

$$\text{Cov}[X_i, X_j] = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] \quad (20)$$

$$= \mathbb{E}[X_i X_j] - \mu_i \mathbb{E}[X_j] - \mu_j \mathbb{E}[X_i] + \mu_i \mu_j \quad (21)$$

$$= \frac{1}{2}B_{ij} + \frac{1}{2}\delta_{ij} \quad (22)$$