Gaussian Distribution

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1 Standard Gaussian Distribution

The probability density function of the standard n-dimensional Gaussian distribution is given by:

$$p(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}x^\top x\right) \tag{1}$$

where $x \in \mathbb{R}^n$, the covariance matrix is I, and the mean is 0.

2 Linear Transformation of Gaussian Distribution

If we have a linear transformation $y = Ax + b$ of a Gaussian distribution with mean b and covariance matrix $AA^\top$, then the probability density function of y is given by:

$$p(y) = \frac{1}{\sqrt{(2\pi)^n |AA^\top|}} \exp\left(-\frac{1}{2}(y-b)^\top (AA^\top)^{-1}(y-b)\right) \tag{2}$$

So the probability density function of the Gaussian distribution is given by:

$$p(y) = \frac{\sqrt{|B|}}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}(y-\mu)^\top B(y-\mu)\right) \tag{3}$$

where B is a symmetric positive definite matrix. If the corresponding random variable is Y, then we have:

$$Y \sim \mathcal{N}(\mu, B^{-1}) \tag{4}$$
$$\mathrm{E}[Y] = \mu \tag{5}$$
$$\mathrm{Cov}[Y] = B^{-1} \tag{6}$$
$$(B^{-1})_{ij} = \mathrm{E}[(Y_i - \mu_i)(Y_j - \mu_j)] \tag{7}$$

2.1 Characteristic Function

The characteristic function of the generalized Gaussian distribution is given by:

$$\phi(t) = \exp\left(-\frac{1}{2}t^\top Bt + i\mu^\top t\right) \tag{8}$$

Proof. Since B is a symmetric positive definite, we can write $B^{-1} = PP^\top$, where P is an invertible matrix. Consider the random variable X as a standard Gaussian distribution, and $Z = PX + \mu$. Then we have:

$$\begin{aligned}
p(z) &= \frac{|P|}{2\pi^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(z-\mu)^\top (P^{-1})^\top P^{-1}(z-\mu)\right) && (9)\\
&= \frac{|P|}{2\pi^{\frac{n}{2}}} \exp\left(-\frac{1}{2}(z-\mu)^\top B(z-\mu)\right) && (10)\\
&= \sqrt{\frac{|B|}{(2\pi)^n}} \exp\left(-\frac{1}{2}(z-\mu)^\top B(z-\mu)\right) && (11)
\end{aligned}$$

which means Z is a generalized Gaussian distribution with mean μ and covariance matrix B.

The characteristic function of Z is given by:

$$\begin{aligned}
\phi(t) &= \mathrm{E}[\exp(it^\top Z)] && (12)\\
&= \mathrm{E}[\exp(it^\top (PX+\mu))] && (13)\\
&= \mathrm{E}[\exp(i(P^\top t)^\top X)]\exp(i\mu^\top t) && (14)\\
&= \exp\left(-\frac{1}{2}(P^\top t)^\top P^\top t\right)\exp(i\mu^\top t) && (15)\\
&= \exp\left(-\frac{1}{2}t^\top Bt\right)\exp(i\mu^\top t) && (16)
\end{aligned}$$

Equation 15 is because X is a standard Gaussian distribution, so its characteristic function is $\exp\left(-\frac{1}{2}t^\top t\right)$. □

3 Convergence of Gaussian Distribution

Theorem 1 (Convergence of Gaussian Distribution)**.** *Suppose $X_k : \Omega \to \mathbb{R}^n$ is normal for all k and that $X_k \to X$ in $L^2(\Omega)$, i.e. $\mathrm{E}[|X_k - X|^2] \to 0$ as $k \to \infty$. Then X is normal.*

Proof.

$$\mathrm{E}[\exp(it^\top X_k)] - \mathrm{E}[\exp(it^\top X)]^2 \leq \mathrm{E}[|X_k - X|^2]|t|^2 \to 0$$

as $k \to \infty$. So $\mathrm{E}[\exp(it^\top X_k)] \to \mathrm{E}[\exp(it^\top X)]$ as $k \to \infty$. So X is normal. □

4 Co-variance Matrix

$$\phi(u) = \exp\left(-\frac{1}{2}u^\top Bu + i\mu^\top t\right)$$

$$\begin{aligned}
\mathrm{E}[X_iX_j] &= -\frac{\partial^2\phi}{\partial u_i \partial u_j}(0) && (17)\\
&= -\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\bigg|_{t=0} \exp\left(-\frac{1}{2}t^\top Bt + i\mu^\top t\right) && (18)\\
&= \mu_i\mu_j + \frac{1}{2}B_{ij} + \frac{1}{2}\delta_{ij} && (19)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}[X_i, X_j] &= \mathrm{E}[(X_i - \mu_i)(X_j - \mu_j)] && (20)\\
&= \mathrm{E}[X_iX_j] - \mu_i \mathrm{E}[X_j] - \mu_j \mathrm{E}[X_i] + \mu_i\mu_j && (21)\\
&= \frac{1}{2}B_{ij} + \frac{1}{2}\delta_{ij} && (22)
\end{aligned}$$

5 Co-variance Matrix for Brownian Motion (2-stage)

$$\begin{aligned}
P^0(B_{t_1} \in F_1, B_{t_2} \in F_2) &= \int_{F_1\times F_2} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2\\
\mathrm{E}[B_{t_1}B_{t_2}] &= \int_{\mathbb{R}^2} x_1x_2 p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2\\
&= \int_{\mathbb{R}^2} x_1x_2 \frac{\exp\left(-\frac{x_1^2}{2t_1}\right)}{(\sqrt{2\pi t_1})^n}\frac{\exp\left(-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right)}{(\sqrt{2\pi(t_2-t_1)})^n}\mathrm{d}x_1\mathrm{d}x_2\\
&= (2\pi)^{-n}[t_1(t_2 - t_1)]^{-n/2}\int_{\mathbb{R}^2} x_1x_2\exp\left(-\frac{x_1^2}{2t_1} - \frac{(x_2 - x_1)^2}{2(t_2 - t_1)}\right)\mathrm{d}x_1\mathrm{d}x_2\\
&\triangleq (2\pi)^{-n}[t_1(t_2 - t_1)]^{-n/2}I
\end{aligned}$$

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} x_1x_2\exp\left(-\frac{x_1^2}{2t_1} - \frac{(x_2 - x_1)^2}{2(t_2 - t_1)}\right)\mathrm{d}x_1\mathrm{d}x_2\\
&= \int_{-\infty}^{\infty} x_1\exp\left(-\frac{x_1^2}{2t_1}\right)\mathrm{d}x_1 \int_{-\infty}^{\infty} x_2\exp\left(-\frac{(x_2 - x_1)^2}{2(t_2 - t_1)}\right)\mathrm{d}x_2\\
&= \int_{-\infty}^{\infty} x_1^2\exp\left(-\frac{x_1^2}{2t_1}\right)\mathrm{d}x_1 C_1\\
&= t_1^2 C_1 C_2
\end{aligned}$$

Thus we have:

$$\mathrm{E}[B_{t_1} B_{t_2}] = t_1^2 \tag{23}$$