Critical Points Transferring between two Diffeomorphic Manifolds

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1 Introduction

1.1 We we have

M, N are manifolds, $F : M \to N$ is a diffeomorphism. $h : N \to \mathbb{R}$ denotes the depth function of Cell surface N.

$$M \xrightarrow{F} N \xrightarrow{h} \mathbb{R} \tag{1}$$

1.2 We want

We want to find the representative points for protrusions on N, namely the critical points of h. Specifically, we want to find the set of points $p \in N$ such as $\mathrm{d}h_p = 0$. [1, Exercise 11.24]

Depth function h is a smooth function on N, so it is difficult to calculate properties of h directly. One can define the induced depth function $H = h \circ F$ to manipulate h on M. Also, it is natural to consider the $F^*(\mathrm{d}h) : TM \to \mathbb{R}$ on M as a corresponding of $\mathrm{d}h$ on N:

$$F_p^*(\mathrm{d}h)(v) = \mathrm{d}h_{F(p)}\left(dF_p(v)\right), \quad v \in T_pM \tag{2}$$

We have the following lemma to calculate $F^*(\mathrm{d}h)$. [1, Proposition 11.25]

Lemma 1. *Let M, N be smooth manifolds, $F : M \to N$ be a diffeomorphism. Let $h : N \to \mathbb{R}$ be a smooth function. Denote $H = h \circ F$. We have*

$$F^*(\mathrm{d}h) = \mathrm{d}H \tag{3}$$

Proof of Lemma 1. $\forall p \in M, v \in T_pM$, we have

$$\begin{aligned}
\left(F^*\mathrm{d}h\right)_p(v) &= \left(\mathrm{d}F_p^*\left(\mathrm{d}h_{F(p)}\right)\right)(v) \\
&= \mathrm{d}h_{F(p)}\left(\mathrm{d}F_p(v)\right) \\
&= \mathrm{d}F_p(v)h \\
&= v(h \circ F) \\
&= \mathrm{d}(h \circ F)_p(v)
\end{aligned}$$

$\square$

Theorem 2. *Let M, N be smooth manifolds, $F : M \to N$ be a diffeomorphism. Let $h : N \to \mathbb{R}$ be a smooth function. If $P \subseteq N$ is the critical point set of h and $Q \subseteq M$ is the critical point set of $H = h \circ F$, then $F^{-1}(P) = Q$.*

Proof of Theorem 2. It suffices to show that $F^{-1}(P) \subseteq Q$ and $Q \subseteq F^{-1}(P)$.

1) $Q \subseteq F^{-1}(P)$:
Consider $q \in Q$ and $v \in T_{F(q)}N$. We have

$$\mathrm{d}H_q = 0.$$

Since F is a diffeomorphism, we have $\mathrm{d}F_q^{-1}(v) \in T_qM$. By definition, we have

$$\mathrm{d}h_{F(q)}(v) = [F_q^*(\mathrm{d}h)](\mathrm{d}F_q^{-1}(v)) \tag{4}$$

And from Lemma 1, noticing $\mathrm{d}H_q = 0$, we have

$$[F_q^*(\mathrm{d}h)](\mathrm{d}F_q^{-1}(v)) = \mathrm{d}H_q(\mathrm{d}F_q^{-1}(v)) = 0 \tag{5}$$

Thus

$$\mathrm{d}h_{F(q)}(v) = 0, \quad \forall v \in T_{F(q)}N \tag{6}$$

which is exactly

$$F(q) \in P \tag{7}$$
$$\Rightarrow \quad q \in F^{-1}(P) \tag{8}$$
$$\Rightarrow \quad Q \subseteq F^{-1}(P) \tag{9}$$

2) $F^{-1}(P) \subseteq Q$:
For all $p \in P$,

$$\mathrm{d}h_p = 0 \tag{10}$$

Similar to the proof of 1), $\forall v \in T_{F^{-1}(p)}M$, we have

$$\mathrm{d}H_{F^{-1}(p)}(v) = \mathrm{d}h_p(\mathrm{d}F_{F^{-1}(p)}(v)) = 0. \tag{11}$$

Thus

$$F^{-1}(p) \in Q, \tag{12}$$

which gives

$$F^{-1}(P) \subseteq Q \tag{13}$$

$\square$

From Theorem 2, we can calculate the critical point set of H to get the critical point set of h.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*, ser. Graduate Texts in Mathematics. Springer, 2013.