I. Theoretical Analysis

There are k committees, and μ is the maximum transaction processing rate of each committee.

We assume λ is the maximum effective throughput. On average, there are n accounts per transaction. α_m is the probability that a transaction with n accounts involves m shards.

The overall transaction processing rate of the system should satisfy:

$$\lambda \sum_{m=1}^{n} m\alpha_m \leq k\mu. \tag{1}$$

Let χ_i be the random variable where

$$\chi_i = \begin{cases} 1, & \text{if the transaction involves shard } i \\ 0, & \text{if the transaction does not involve shard } i \end{cases} \tag{2}$$

The probability that any account is assigned to shard i is $\frac{1}{k}$. Correspondingly, the probability that an account is not assigned to i is $1 - \frac{1}{k}$.

Thus, the probability that a transaction does not involve shard i is:

$$\mathrm{P}(\chi_i = 0) = \left(1 - \frac{1}{k}\right)^n = \left(\frac{k-1}{k}\right)^n. \tag{3}$$

Correspondingly, the probability that a transaction involves shard i is:

$$\mathrm{P}(\chi_i = 1) = 1 - \mathrm{P}(\chi_i = 0) = 1 - \left(\frac{k-1}{k}\right)^n. \tag{4}$$

The expectation is

$$\begin{aligned} \mathrm{E}(\chi_i) &= 0 \cdot \mathrm{P}(\chi_i = 0) + 1 \cdot \mathrm{P}(\chi_i = 1) \\ &= 1 - \left(\frac{k-1}{k}\right)^n. \end{aligned} \tag{5}$$

Let ξ denote the number of shards processing a transaction with n accounts.

$$\xi = \chi_1 + \chi_2 + \cdots + \chi_k \tag{6}$$

According to the linearity of expectation,

$$\begin{aligned} \mathrm{E}(\xi) &= \mathrm{E}(\chi_1) + \mathrm{E}(\chi_2) + \cdots + \mathrm{E}(\chi_k) \\ &= k \cdot \left[1 - \left(\frac{k-1}{k}\right)^n\right] \end{aligned} \tag{7}$$

According to the definition of expectation,

$$\mathrm{E}(\xi) = \sum_{m=1}^{n} m\alpha_m. \tag{8}$$

Thus,

$$k \cdot \left[1 - \left(\frac{k-1}{k}\right)^n\right] = \sum_{m=1}^{n} m\alpha_m = \mathrm{E}(\xi) \tag{9}$$

Combined with Ineq.(1), we can get

$$\begin{aligned} \lambda \leq \frac{k\mu}{\sum_{m=1}^{n} \alpha_{n,m} m} &= \frac{k\mu}{k \cdot \left[1 - \left(\frac{k-1}{k}\right)^n\right]} \\ &= \frac{\mu}{1 - \left(\frac{k-1}{k}\right)^n} \\ &= \frac{\mu k^n}{k^n - (k-1)^n} \end{aligned} \tag{10}$$

Or

$$\frac{n\lambda}{\mu k} \leq \frac{nk^{n-1}}{k^n - (k-1)^n}. \tag{11}$$

Consider the right side of Ineq.(11)

$$\begin{aligned} &\lim_{k\to\infty} \frac{nk^{n-1}}{k^n - (k-1)^n} \\ &= \lim_{k\to\infty} \frac{nk^{n-1}}{k^n - k^n + nk^{n-1} + o(k^{n-1})} \\ &= 1 \end{aligned} \tag{12}$$

Therefore, as $k \to \infty$, we can have $\lambda \lesssim \dfrac{\mu}{n} k$.