I. Theoretical Analysis

There are k committees, and μ is the maximum transaction processing rate of each committee.

We assume λ is the maximum effective throughput. On average, there are n accounts per transaction. ~~$\alpha_{n,m}$~~ α_m is the probability that a transaction with n accounts involves m shards.

The overall transaction processing rate of the system ~~is:~~ should satisfy:

$$k\mu =\lambda \sum_{m=1}^{n} m\alpha_{n,m m} \le k\mu. \tag{1}$$

Let χ_i be the random variable where

$$\chi_i = \begin{cases} 1, & \text{if the transaction involves shard } i \\ 0, & \text{if the transaction does not involve shard } i \end{cases} \tag{2}$$

The probability that any account is assigned to shard i is $\frac{1}{k}$. Correspondingly, the probability that an account is not assigned to i is $1-\frac{1}{k}$.

Thus, the probability that a transaction does not involve shard i is:

$$P\,\mathrm{P}(\chi_i = 0) = \left(1-\frac{1}{k}\right)^n = \left(\frac{k-1}{k}\right)^n. \tag{3}$$

Correspondingly, the probability that a transaction involves shard i is:

$$P\,\mathrm{P}(\chi_i = 1) = 1-P1-\mathrm{P}(\chi_i = 0) = 1-\left(\frac{k-1}{k}\right)^n. \tag{4}$$

The expectation is

$$\begin{aligned} \mathrm{E}(\chi_i) &= 0\cdot \mathrm{P}(\chi_i = 0) + 1\cdot \mathrm{P}(\chi_i = 1) \\ &= 1-\left(\frac{k-1}{k}\right)^n. \end{aligned} \tag{5}$$

Let ~~ξ_n~~ ξ denote the number of shards processing a transaction with n accounts.

$$\xi_n = \chi_1 + \chi_2 + \cdots + \chi_k \tag{6}$$

According to the linearity of expectation,

$$\begin{aligned} \mathrm{E}(\xi) &= \mathrm{E}(\chi_1) + \mathrm{E}(\chi_2) + \cdots + \mathrm{E}(\chi_k) \\ &= k\cdot\left[1-\left(\frac{k-1}{k}\right)^n\right] \end{aligned} \tag{7}$$

According to the definition of expectation,

$$E\,\mathrm{E}(\xi_n) = \sum_{m=1}^{n} \alpha_{n,m} m\alpha_m. \tag{8}$$

Thus,

$$E(\xi_n) = k\cdot\left[1-\left(\frac{k-1}{k}\right)^n\right] = \sum_{m=1}^{n} \alpha_{n,m} m\alpha_m = \mathrm{E}\,(\xi) \tag{9}$$

Combined with ~~Eq~~Ineq.(1), we can get

$$\begin{aligned} \lambda &\le \frac{k\mu}{\sum_{m=1}^{n}\alpha_{n,m}m} = \frac{k\mu}{k\cdot\left[1-\left(\frac{k-1}{k}\right)^n\right]} \\ &= \frac{\mu}{1-\left(\frac{k-1}{k}\right)^n} \\ &= \frac{\mu k^n}{k^n-(k-1)^n} \end{aligned} \tag{10}$$

Or

$$\frac{\partial\lambda}{\partial k} = \frac{\mu n k^{n-1}(k-1)^{n-1}}{[k^n-(k-1)^n]^2} > 0 \frac{n\lambda}{\mu k} \le \frac{nk^{n-1}}{k^n-(k-1)^n}. \tag{11}$$

~~Eq.(10) can be transformed into:~~ Consider the right side of Ineq.(11)

$$\frac{\mu k}{n\lambda} = \frac{k^n-(k-1)^n}{nk^{n-1}}$$

$$\begin{aligned} &\lim_{k\to\infty} \frac{nk^{n-1}}{k^n-(k-1)^n} \\ &= \lim_{k\to\infty} \frac{nk^{n-1}}{k^n-k^n+nk^{n-1}+o(k^{n-1})} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{k\to\infty}\frac{\mu k}{n\lambda} &= \lim_{k\to\infty}\frac{k^n-(k-1)^n}{nk^{n-1}} \\ &= \lim_{k\to\infty}\frac{k^n-k^n+nk^{n-1}+o(k^{n-1})}{nk^{n-1}} \\ &= 1 \end{aligned}$$

Therefore, as $k\to\infty$, we can have ~~$\lambda \sim \frac{\mu}{n}k$~~ $\lambda \le \frac{\mu}{n}k$.