$$u(0)=\begin{bmatrix}1\\2\end{bmatrix}\quad \frac{du_1}{dt} = -u_1+2u_2 \qquad A=\begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix} \qquad \frac{du}{dt}=Au$$

$$\frac{du_2}{dt} = u_1-2u_2$$

$$|A-\lambda I| = \begin{vmatrix}-1-\lambda & 2\\ 1 & -2-\lambda\end{vmatrix} = (\lambda+1)(\lambda+2)-2 = \lambda^2+3\lambda=0$$

$\lambda_1=0 \quad \lambda_2=-3$

$\lambda_1=0 \quad [A-\lambda_1 I] = \begin{bmatrix}-1 & 2\\ 1 & -2\end{bmatrix} \qquad x_1=\begin{bmatrix}2\\1\end{bmatrix}$

$\lambda_2=-3 \quad A-\lambda_2 I = \begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix} \qquad x_2=\begin{bmatrix}-1\\1\end{bmatrix}$

solution: $u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2$

$$c_1\cdot 1\cdot\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}$$

$Sc=u(0) \qquad u(0)=\begin{bmatrix}1\\2\end{bmatrix}$ we can see c_1 and c_2.

$c_1 = c_2 = \frac{1}{3}$

$\begin{cases} ① \text{确定 eigenvalues, eigenvectors} \\ ② \text{find the coefficients} \end{cases}$

$\frac{du}{dt} = \underbrace{Au}$ 耦合.

Set $u = Sv$

$S\frac{dv}{dt} = ASv \qquad \frac{dv}{dt} = S^{-1}ASv = \Lambda v$

$$\begin{cases} \frac{dv_1}{dt} = \lambda_1 v_1 \\ \vdots \end{cases} \qquad \begin{aligned} v(t) &= e^{\Lambda t} v(0) \\ u(t) &= S e^{\Lambda t} S^{-1} u(0) \end{aligned}$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!}$$

$$\underbrace{e^x = \sum_0^n \frac{x^n}{n!}}$$ 泰勒公式

$$\underbrace{(I - At)^{-1} = I + At + \cdots + (At)^n \qquad \frac{1}{1-x} = \sum_0^n x^n}$$

$$e^{At} = SS^{-1} + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1} t^2}{2} + \cdots + \frac{S\Lambda^n S^{-1} t^n}{n!}$$

$$= S\, \underbrace{e^{\Lambda t}}\, S^{-1}$$

So: $e^{At} = S e^{\Lambda t} \cdot S^{-1}$ (A can be diagonalized)

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

two order equation:

example: $y'' + by' + ky = 0$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$u' = Au$

① tranfor to 1-order equation and get A

② get the eigenvalues - - - .