

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right] \leftarrow \begin{array}{l} \text{Augmented matrix} \\ \text{增广矩阵} \\ [A, b] \end{array}$$

↓ ①

$$\left[ \begin{array}{cccc|c} \textcircled{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \textcircled{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array} \right]$$

↓

$$\left[ \begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right] \begin{array}{l} \text{pivot columns} \\ \rightarrow \text{we can get } b_3 - b_2 - b_1 = 0 \end{array}$$

eg:  $b_1, b_2, b_3 = 1, 5, 6$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solvability Condition

$b$  in column space of  $A$   
 $C(A)$

step 1: find particular solution

set all free variables to 0. then solve  $Ax=b$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

step 2:  $x_{null}$  space

$$x = x_p + x_{null}$$

$$Ax_p = b \Rightarrow A(x_p + x_{null}) = Ax_p + Ax_{null} = b$$

$$\therefore x = \begin{bmatrix} -2 \\ 0 \\ 3 \\ \frac{1}{2} \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

↓ any linear combination

**Full Rank** 满秩.

$m \times n$  matrix rank  $r$ . ( $r \leq m, r \leq n$ )

① Full rank for  $r = n$ , every pivot columns,  
 0 free variables  
 has 0 or 1 solutions.  $A = A_p$  or no solution

② Full rank for  $r = m$ . every  $b$ , has a solution  
 free variables:  $n - r = n - m$ , has  $\infty$  solutions  
 or 0 solution

③ Full rank for  $r = m = n$   
 must have a unique solution