Application of eigenvalues:

①: Markov Matrix

②: Fourier series. 傅里叶级数 (projection problem)

一、Markov Matrix

two properties: $\begin{cases} ① \text{ all entries} \geq 0 \\ ② \text{ all columns add to } \underline{1} \end{cases}$

ensures one eigenvalue is 1

all other eigenvalues $|\lambda_i| \leq 1$

eigenvector $x_1 \geq 0$

eg: $A = \begin{bmatrix} 1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$

保证 $\underbrace{|A-I|=0}$, all columns of $A-I$ add to 0

$\longrightarrow$ $A-I$ is singular, so $|A-I|=0$

eigenvalues of A = eigenvalues of A^T

eg: $u_{k+1} = A u_k$, A is Markov Matrix (可认为是概率问题)

$$\begin{bmatrix} u_a \\ u_b \end{bmatrix}_{k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix}_k \qquad \text{assume} \begin{bmatrix} u_a \\ u_b \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

how to get $\begin{bmatrix} u_a \\ u_b \end{bmatrix}_{100}$?

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \qquad \lambda_1 = 1 \qquad \lambda_2 = 0.7$$

$$\lambda_1 = 1 \qquad A - \lambda_1 I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.7 \qquad A - \lambda_2 I = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot 0.7^k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} \qquad \begin{cases} c_1 = \frac{1000}{3} \\ c_2 = \frac{2000}{3} \end{cases}$$

so $u_k = \frac{1000}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} \cdot 0.7^k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

二. Fourier series.

projections with orthonormal basis. : $q_1\ q_2 \cdots q_n$

Any $v = x_1 q_1 + \cdots + x_n q_n = [q_1 \cdots q_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

so $q_1^T \cdot v = x_1$ $\qquad Qx = v$

$\vdots$

$q_n^T \cdot v = x_n$ $\qquad x = Q^{-1} v = Q^T v$

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

$f(x) = f(x + 2\pi)$

内积定义: $f^T g = \int_0^{2\pi} f(x) g(x)\, dx$

左右对 $\cos x$ 取内积

$$\int_0^{2\pi} f(x)\cos x\,dx = \int_0^{2\pi} a_1 \cos^2 x\,dx = a_1 \cdot \pi$$

so $\quad a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$