$$
\begin{cases}
x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 \\
2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2 \\
3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3
\end{cases}
$$

$$
\left[\begin{array}{cccc|c}
1 & 2 & 2 & 2 & b_1 \\
2 & 4 & 6 & 8 & b_2 \\
3 & 6 & 8 & 10 & b_3
\end{array}\right]
$$

Augmented Matrix

增广矩阵

$[A, b]$

$\downarrow$ ①

$$
\begin{bmatrix}
1 & 2 & 2 & 2 & b_1 \\
0 & 0 & 2 & 4 & b_2 - 2b_1 \\
0 & 0 & 2 & 4 & b_3 - 3b_1
\end{bmatrix}
$$

$\downarrow$

pivot columns

$$
\begin{bmatrix}
1 & 2 & 2 & 2 & b_1 \\
0 & 0 & 2 & 4 & b_2 - 2b_1 \\
0 & 0 & 0 & 0 & b_3 - b_2 - b_1
\end{bmatrix}
$$

we can get $b_3 - b_2 - b_1 = 0$

eg: $b_1, b_2, b_3 = 1, 5, 6$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\text{Solvability Condition}}$

b in column space of A

$C(A)$

step 1: find particular solution

set all free varibles to 0. then solve $Ax=b$

$$x_{p_1} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

step 2: X_{null} space

$$x = x_p + x_{null}$$

$$Ax_p = b \Rightarrow A(x_p + x_{null}) = Ax_p + Ax_{null} = b$$

so $x = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} - \\ 1 \end{bmatrix}$$

↓ any linear combination

$\boxed{\text{Full Rank}}$ 满秩.

$m \times n$. martix. rank r. ($r \le m$, $r \le n$)

① Full rank for $r = n$, every pivot columns.

0 free varibles

has 0 or 1 solutions. $x = x_p$ or no solution

② Full rank for $r = m$. every b, has a solution

free varibles : $n - r = n - m$, has ∞ solutions or 0 solution

③ Full rank for $r = m = n$

muse have a unique solution