Math147 Notes - Kathryn Hare

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1 Real Numbers

1.1 Completeness Axiom of Real Numbers

Every non-empty subset of the real numbers that is bounded above has a least upper bound. (Similarly every non-empty subset of reals bounded below has a least upper bound)

1.2 Characterization of the Least Upper Bound (Greatest Lower Bound)

A is an upper bound if:

1. A is an upper bound
2. For all $z \in \mathbb{R}$, if $z < A$, then

1.3 Archimedean Property

Given any real number x, there is some $N \in \mathbb{N}$ such that $N > x$

1.4 Squeeze Theorem

Theorem 1 *Suppose $(x_n) \leq (y_n) \leq (z_n)$ for all $n \in \mathbb{N}^*$. Assume $(x_n) \longrightarrow L$ and $(z_n) \longrightarrow L$. Then $(y_n) \longrightarrow L$.*

For functions: If $f(x) \leq g(x) \leq h(x)$ for all x "near" a, and if $lim_{x \to a} f(x) = L = lim_{x \to a} h(x)$, then also $lim_{x \to a} g(x) = L$.

1.5 Bounded Sequences

Say (x_n) is bounded if there is some C with $|x_n| \leq C$ for all $n \in \mathbb{N}$.

1.6 Monotonic Convergence Theorem (MCT)

Theorem 2 *If (x_n) is a monotonic sequence that is bounded, then (x_n) converges.*

Proof of MCT Suppose (x_n) is increasing and bounded above. Let $A = \{x_n : n = 1, 2, 3...\}$. Note that this set is non-empty and bounded above.
By the Completeness Axiom of $\mathbb{R}$, A has a LUB, call it L.
Let $\epsilon > 0$. Since L is an upper bound for A, $x_n \leq L$ for all $n \in \mathbb{N}$, hence $x_n < L + \epsilon$ (for

all n). Since L is LUB(A), and $L - \epsilon < L$, there is some $x_N \in A$ with $x_N > L - \epsilon$. Since (x_n) is increasing, $x_n \geq x_N$ if $n \geq N$. $x_n > L - \epsilon$. Hence for all $n \geq N, L - \epsilon < x_n < L + \epsilon$ and therefore $(x_n) \longrightarrow L$.

1.7 Nested Interval Property

Given any collection of nested intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq ...$ with $(b_n - a_n) \longrightarrow 0$, there is a unique $x_n \in [a_n, b_n]$ for every n.

Proposition 1 *Every sequence has a monotonic subsequence.*

1.8 Bolzano-Weierstrass Theorem

Theorem 3 *Every bounded sequence has a convergent subsequence.*

Proof of BWT By the previous proposition, every sequence has a monotonic subsequence. Since the original sequence is bounded, this subsequence is also bounded, thus by MCT, it converges.

1.9 Cauchy Sequences

A sequence (x_n) is called Cauchy if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that $|x_n - x_m| < \epsilon$ if $n, m \geq N$

1. Any convergent sequence is Cauchy.

2. Any cauchy sequence is bounded.

Theorem 4 *Every Cauchy sequence converges.*

Proof. Since the sequence (x_n) is Cauchy it is bounded. By the BW theorem, it has a convergent subsequence $(x_n)_{k=1}^{\infty}$ with limit L. Let $\epsilon > 0$. Since (x_n) is Cauchy, there is some N such that $|x_n - x_m| < \frac{\epsilon}{2}$ if $n, m \geq N$. Furthermore, since $(x_{n_k}) \longrightarrow L$, there is some index $n_k > N$ where $|x_{n_k} - L| < \frac{\epsilon}{2}$. Let $n \geq N$. Then $|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

2 Limits of Functions

Say f has limit $L \in \mathbb{R}$ at point p if for every $\epsilon > 0$ there is some $\delta > 0$ such that whenever $0 < |x - p| < \delta$, then $|f(x) - L| < \epsilon$.

2.1 Continuous Functions

For $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, f is continuous at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$ (one-sided limit if a is an endpoint of A) For every $\epsilon > 0$, there is some $\delta > 0$, so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

2.2 Sequential Characterization

$f : A \longrightarrow \mathbb{R}$ is continuous at $a \in A$ if and only if whenever (x_n) is a sequence from A (meaning every $(x_n) \in A$) with $(x_n) \to a$, then $(f(x_n)) \to f(a)$.

2.3 Composition of Functions

$f : A \longrightarrow \mathbb{R} g : B \longrightarrow \mathbb{R}, B \supseteq$ Range of f

Theorem 5 *If $f : A \longrightarrow \mathbb{R}$ is continuous at $a \in A$ and g is continuous at $f(a)$, then $g \circ f(x)$ is continuous at a.*

Proof. Show if $(x_n) \to a$ $(x_n \in A)$, then $(g \circ f(x_n)) \to g \circ f(a)$.
Let $(x_n) \to a$, $x_n \in A$. f is continuous at a. Thus $(f(x_n)) \to f(a)$.
Let $f(x_n) = y_n$. $y_n \in$ Range $f \subseteq B$. $(y_n) \to f(a)$ and g is continuous at $f(a)$. $(g(y_n)) \to g(f(a)) = g \circ f(a)$. Thus $g(f(x_n)) \to g \circ f(a)$. Therefore $g \circ f$ is continuous at a.

2.4 Intermediate Value Theorem

Theorem 6 *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Assume $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in [a, b]$ with $f(c) = 0$.*

Proof. Let $A = \{x \in [a, b] : f(x) < 0\}$. A is not empty as $a \in A$. Also, A is bounded. By the Completeness Axiom of $\mathbb{R}$, A has a LUB, call it L. $a \leq L \leq b$ (since b is an UB for A). Hence f is continuous at L. Notice

2.5 Extreme Value Theorem

2.6 Increasing / Decreasing Function

f is (strictly) increasing if whenever $x_2 > x_1$, then $f(x_2) \geq f(x_1)$. If $f : [a, b] \to \mathbb{R}$ is continuous and 1 - 1, then it is either strictly increasing or strictly decreasing.

2.7 Inverse Trig Functions

- $sine \to y = arcsin(x) \Leftrightarrow siny = x$ and $y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- $cosine \to y = arccos(x) \Leftrightarrow cosy = x$ and $y = [0, \pi]$.
- $tan \to y = arctan(x) \Leftrightarrow tany = x$ and $y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2.8 Logarithm Function

3 Differentiation

Say f is differentiable at a if $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists and then we denote this limit by $f'(a)$. If f, g are differentiable at a,

1. $f \pm g$ is differentiable at a, and $(f \pm g)'(a) = f'(a) \pm g'(a)$.
2. $f \cdot g$ is differentiable at a, and $(fg)'(a) = f'(a)g(a) + g'(a)f'(a)$.

3.1 Chain Rule

$f : A \to B \subseteq \mathbb{R}$, $g : B \to \mathbb{R}$ If f is differentiable at a, and g is differentiable at $f(a)$ then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

3.2 Caratheodory Theorem

Theorem 7 *If F is differentiable at a, then there is a function Φ which is continuous at a and $F(x) - F(a) = \Phi(x)(x - a)$ for all x, and $\Phi(a) = F'(a)$.*

3.3 Derivatives of Inverses

Let $f : (c, d) - > \mathbb{R}$ be a continuous, $1 - 1$ function on (c, d). Suppose f is differentiable at $a \in (c, d)$ and $f'(a) \neq 0$. Then f^{-1} is differentiable at $f(a)$ and $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$. (or, write $b = f(a) \leftrightarrow f^{-1}(b) = a$ and see $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$.

3.4 Implicit Differentiation

A point x is a local maximum for f if there is some $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$ and $y \in$ Domain f, $f(y) \leq f(x)$. x is a (global) maximum if $f(y) \leq f(x)$ for all $y \in$ Domain f.

3.5 Critical Points Theorem (CPT)

Theorem 8 *If f has a local max or min at $x \in (a, b) \subseteq$ Domain of f and if f is differentiable at x, then $f'(x) = 0$.*

Proof. f is differentiable at x so $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists.
Get $\delta > 0$ from definition of local min/max if $z \in (x - \delta, x + \delta)$ then $f(x) \geq f(z)$. If $|h| < \delta$, then $x + h \epsilon (x - \delta, x + \delta)$.
Therefore, $f(x+h) - f(x) \leq 0$. If $0 < h < \delta$, then $\frac{f(x+h)-f(x)}{h} \leq 0$, so $\lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} \geq$

0. If $-\delta < h < 0$, then $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0 \Rightarrow$ since limit exists, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$, so $f'(x) = 0$.

3.6 Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

3.7 Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$

Proof. Since f is continuous on $[a, b]$ it has a global maximum and minimum by EVT. If one of these occurs at some point $c \in (a, b)$, then since f is differentiable there, $f'(c) = 0$ by CPT. Otherwise, the global max and min occur at a and b, and $f(a) = f(b)$, and that implies that f is constant, and so $f'(c) = 0$ for every $c \in (a, b)$.

3.8 Increasing Function Theorem

If $f'(x) \geq 0$ for every $x \in (a, b)$ and f is continuous on $[a, b]$, then f is increasing on $[a, b]$.

3.9 First Derivative Test

Assume f is continuous on $[a, b]$ and $c \in [a, b]$.

1. If $f' \geq 0$ on (a, c) and $f' \leq 0$ on (c, b) then c in local max.
2. If $f' \leq 0$ on (a, c) and $f' \geq 0$ on (c, b) then c in local min.
3. If f' is some sign on both sides (i.e. > 0 or < 0 on both sides) then c is neither local max/min.

3.10 Second Derivative

Say f is concave up on interval I if $f'(x)$ is strictly increasing on I. Say f is concave down on I if $f'(x)$ is strictly decreasing on I. Call c an inflection point if $f'(c)$ exists at c and the concavity of f changes at c. $\exists\delta > 0$ such that f is concave up on $(c - \delta, c)$ and down on $(c, c + \delta)$ (or vice versa).

3.11 Second Derivative Test

Suppose $f'(c) = 0$.

1. If $f''(c) > 0$ (< 0) then f has a local min (max) atc.
2. If $f''(c) = 0$, then anything is possible.

3.12 Cauchy Mean Value Theorem

If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ such that $(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$.

3.13 L'Hopital's Rule

Assume f, g are differentiable on interval $I = [a - \delta, a + \delta]$ except possibly at a. Assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ and either $\lim_{x\to a} f = 0$ or ∞. Suppose g, g' are non-zero at I, except perhaps at a. If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Proof. $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$. Recall f, g were defined and differentiable on $I = [a - \delta_0, a + \delta_0]$ for some $\delta_0 > 0$ except possibly at a. (Re)define f and g at a by setting $f(a) = g(a) = 0$. This does not change f' or g' at points $x \neq a$. Our new f, g are still differentiable on I except possibly at a. Since $\lim_{x\to a} f = f(a) = 0$. To see that $lim_{x\to a} \frac{f}{g} = L$, we have to take any $\epsilon > 0$ and find $\delta > 0$ such that $0 < |x - a| < \delta$, then $|\frac{f}{g}(x) - L| < \epsilon$. Know that $\lim_{x\to a} \frac{f'}{g'} = L$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \implies |\frac{f'}{g'}(x) - L| < \epsilon$. Let $\delta = min(\delta_0, \delta_1) > 0$. Let $0 < |x - a| < \delta$.

Suppose $x > a$. We have f, g continuous on $[a, x]$ and differentiable on (a, x). So CMVT applies. Hence there exists a $c \in (a, x)$ such that, $(f(x) - f(a))g'(c) = f'(c)(g'(x) - g'(a))$. By assumption, $g'(c) \neq 0$ since $c \neq a$. Also, $g(x) - g(a) \neq 0$ since $g(a) = 0$ and $g(z) \neq 0$ for any $z \in I$, except $z = a$. Divide to get:

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Thus,

$$|\frac{f(x)}{g(x)} - L| = |\frac{f(x) - f(a)}{g(x) - g(a)} - L| = |\frac{f'(c)}{g'(c)} - L|$$

for some $c \in (a, x)$. Hence $0 < |c - a| < \delta \leq delta_1$, so $|\frac{f(x)}{g(x)} - L| = |\frac{f'(c)}{g'(c)} - L| < \epsilon$.

3.14 Taylor Polynomials

3.15 Taylor's Theorem