1\. (a) $\max_{\Delta} \frac{\Delta^T \vec{r}}{\sqrt{\Delta^T \Sigma \Delta}}$, s.t. $\Delta^T \vec{1} = 1$

let $\phi : \Delta \mapsto \begin{bmatrix} \Delta^T r \\ \Delta^T \Sigma \Delta \end{bmatrix}$, $h(x,y) := \frac{x}{\sqrt{y}}$

$f(\Delta) = h \circ \phi(\Delta)$

$$D_\Delta f(\Delta) = (Dh)(\phi(\Delta))(D\phi)(\Delta)$$

$$= \begin{bmatrix} \frac{1}{(\Delta^T\Sigma\Delta)^{\frac{1}{2}}} & -\frac{\Delta^T \vec{r}}{2(\Delta^T\Sigma\Delta)^{\frac{3}{2}}} \end{bmatrix} \begin{bmatrix} \vec{r}^T \\ 2\Delta^T\Sigma \end{bmatrix}$$

$$= \frac{1}{(\Delta^T\Sigma\Delta)^{\frac{3}{2}}} \left((\Delta^T\Sigma\Delta)\vec{r}^T - (\Delta^T\vec{r})\Delta^T\Sigma \right)$$

$D_\Delta f(\Delta) = 0$ $\quad$ $\vec{r}(\Delta^T\Sigma\Delta) - \Sigma\Delta(r^T\Delta) = 0$

$$\Sigma\Delta = \vec{r} \cdot \frac{\Delta^T\Sigma\Delta}{r^T\Delta} = \vec{r} \cdot \lambda \qquad \lambda := \frac{\Delta^T\Sigma\Delta}{\vec{r}^T\Delta}$$

$\Delta = \lambda(\Sigma^{-1}\vec{r})$

need $\beta(1^T\Sigma^{-1}\vec{r}) = 1$ to satisfy $1^T\Delta = 1$

$$\beta = (1^T\Sigma^{-1}\vec{r})^{-1}$$

$$\Delta = \frac{\Sigma^{-1}\vec{r}}{1^T\Sigma^{-1}\vec{r}}$$

(b) $\min_{\Delta} \Delta^T\Sigma\Delta$ s.t. $\Delta^T\vec{1} = 1$

$$L(w) = \frac{1}{2}\Delta^T\Sigma\Delta + \lambda(1 - \Delta^T\vec{1})$$

$$\Sigma\Delta - \lambda 1 = 0$$

$$\Delta = \lambda^{-1}\Sigma^{-1}1 \qquad \lambda = (1'\Sigma^{-1}1)^{-1}$$

$$\Delta_{GMV} = \frac{\Sigma^{-1}\vec{1}}{\vec{1}^T\Sigma^{-1}\vec{1}}$$

2.

(a) $$\Delta_{MSR} = \frac{\Sigma^{-1}(\vec{r}-\vec{r}_f)}{\vec{1}^T\Sigma^{-1}(\vec{r}-\vec{r}_f)} = \begin{bmatrix} 0.56 \\ 0.69 \\ -0.25 \end{bmatrix}$$

(b) $$W = \bar{r}_e \times \frac{\vec{1}^T\Sigma^{-1}\vec{r}_e}{\vec{r}_e^T\Sigma^{-1}\vec{r}_e} \times \frac{\Sigma^{-1}\vec{r}_e}{\vec{1}^T\Sigma^{-1}\vec{r}_e} = \begin{bmatrix} 1.93 \\ 2.37 \\ -0.86 \end{bmatrix}$$

$$ZCB = 1-(1.93+2.37-0.86) = -2.43$$

```python
import numpy as np
sigma=np.matrix([[0.09,-0.03,0.084],[-0.03,0.04,0.012],[0.084,0.01
r=np.matrix([[0.1],[0.09],[0.16]])
ones=np.transpose(np.asmatrix(np.ones(3)))
sigma_inverse=np.linalg.inv(sigma)
rf=np.matrix([[0.02],[0.02],[0.02]])
re=r-rf
p=0.2
msr=(sigma_inverse*re)/(ones.transpose()*sigma_inverse*re)
a=ones.transpose()*sigma_inverse*re
b=re.transpose()*sigma_inverse*re
c=sigma_inverse*re
d=a
alpha=(p*a/b).tolist()[0][0]
print("msr",msr)
print("\nalpha",alpha)
print("delta:",alpha*((c)/(d)))
```

IPython console

Console 2/A

```
703/project/MomentumAnaly
msr [[ 0.5602878 ]
 [ 0.68863874]
 [-0.24892654]]

alpha 3.4377244075640316
delta: [[ 1.92611506]
 [ 2.3673502 ]
 [-0.85574084]]

In [199]:
```

3. (a) $\max_{\Delta} \mathrm{Corr}(r(\Delta), r_I)$, s.t $\Delta^T \vec{1} = 1$

$$\max \frac{\mathrm{Cov}(r(\Delta), r_I)}{\sigma_{r(\Delta)}\sigma_{r_I}}$$

$$= \frac{\Delta^T \gamma}{\Delta^T \Sigma \Delta \sigma_I}$$

let $\phi : \Delta \mapsto \begin{bmatrix} \Delta^T \gamma \\ \Delta^T \Sigma \Delta \sigma_I \end{bmatrix}$ $h(x,y) := \frac{x}{y}$

$f(\Delta) = h \circ \phi(\Delta)$

$$D_\Delta f(\Delta) = (Dh)(\phi(\Delta))(D\phi)(\Delta)$$

$$= \begin{bmatrix} \frac{1}{\Delta^T \Sigma \Delta \sigma_I} & -\frac{\Delta^T \gamma}{(\Delta^T \Sigma \Delta \sigma_I)^2} \end{bmatrix} \begin{bmatrix} \gamma^T \\ 2\sigma_I \Delta^T \Sigma \end{bmatrix}$$

$$= \frac{\sigma_I}{(\sigma_I \Delta^T \Sigma \Delta)^2} \left((\Delta^T \Sigma \Delta)\gamma^T - 2(\Delta^T \gamma)\Delta^T \Sigma \right)$$

$D_\Delta f(\Delta) = 0$ $\qquad \gamma(\Delta^T \Sigma \Delta) - 2\Sigma \Delta(\gamma^T \Delta) = 0$

$$\Sigma \Delta = \gamma \cdot \frac{2\Delta^T \Sigma \Delta}{\gamma^T \Delta} = \vec{r} \cdot \lambda \qquad \lambda := \frac{2\Delta^T \Sigma \Delta}{\gamma^T \Delta}$$

$\Delta = \lambda(\Sigma^{-1} \gamma)$

need $\beta(1^T \Sigma^{-1} \gamma) = 1$ to satisfy $1^T \Delta = 1$

$$\beta = (1^T \Sigma^{-1} \gamma)^{-1}$$

$$\Delta_{corr, I} = \frac{\Sigma^{-1} \gamma}{1^T \Sigma^{-1} \gamma}$$

b $\min Var(r(\Delta)-r_I)$ s.t. $\bar{r}(\Delta)=\bar{r}$ $\Delta^T\vec{1}=1$

$$\min \overline{\left[(\Delta^T r - r_I) - (\Delta^T\vec{r} - \bar{r}_I)\right]^2}$$
$$= [\Delta^T r - r_I - \Delta^T\vec{r} + \bar{r}_I]^2$$
$$= [\Delta^T(r-\bar{r}) - (r_I - \bar{r}_I)]^2$$
$$= (\Delta^T(r-\bar{r}))^2 + (r_I - \bar{r}_I)^2 - 2\Delta^T(r-\bar{r})(r_I-\bar{r}_I)$$
$$= \Delta^T\Sigma\Delta + \sigma_I^2 - 2\Delta^T\gamma$$
$$= 2\times\frac{1}{2}(\Delta^T\Sigma\Delta + \sigma_I^2 - 2\Delta^T\gamma)$$
$$\frac{1}{2}\Delta^T\Sigma\Delta + \frac{1}{2}\sigma_I^2 - \Delta^T\gamma - \lambda_1(\bar{r}(\Delta)-\bar{r}) - \lambda_2(\Delta^T\vec{1}-1)=0$$
$$\frac{1}{2}\Delta^T\Sigma\Delta + \frac{1}{2}\sigma_I^2 - \Delta^T\gamma - \lambda_1(\Delta^T\vec{r}-\bar{r}) - \lambda_2(\Delta^T\vec{1}-1)=0$$

$$\frac{\partial L}{\partial \Delta} = \Sigma\Delta - \gamma - \lambda_1\vec{r} - \lambda_2\vec{1} = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda_1} = \Delta^T\vec{r} - \bar{r} = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda_2} = \Delta^T\vec{1} - 1 = 0 \qquad (3)$$

$$\Delta = \Sigma^{-1}(\gamma + \lambda_1\vec{r} + \lambda_2\vec{1}) \quad \text{by (1)}$$

$$1 = 1^T\Sigma^{-1}\Sigma\Delta = 1^T\Sigma^{-1}\gamma + \lambda_1 1^T\Sigma^{-1}\vec{r} + \lambda_2 1^T\Sigma^{-1}\vec{1}$$

$$\bar{r} = \vec{r}\Sigma^{-1}\Sigma\Delta = \vec{r}^T\Sigma^{-1}\gamma + \lambda_1\vec{r}^T\Sigma^{-1}\vec{r} + \lambda_2\vec{r}^T\Sigma^{-1}\vec{1}$$

Let $a = 1^T\Sigma^{-1}1$ $\quad b = 1^T\Sigma^{-1}\vec{r}$ $\quad c = \vec{r}^T\Sigma^{-1}\vec{r}$

$$1 = 1^T\Sigma^{-1}\gamma + \lambda_1 b + \lambda_2 a$$
$$\bar{r} = \vec{r}^T\Sigma^{-1}\gamma + \lambda_1 c + \lambda_2 b$$

$$\lambda_1 = \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{b}{c}\bar{r} + \frac{b}{c}\vec{r}^T \Sigma^{-1} \gamma}{a - \frac{b^2}{c}}$$

$$\lambda_2 = \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{a}{b}\vec{r}^T \Sigma^{-1} \gamma}{b - \frac{ac}{b}}$$

$$\Delta = \Sigma^{-1}\left(\gamma + \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{b}{c}\bar{r} + \frac{b}{c}\vec{r}^T \Sigma^{-1} \gamma}{a - \frac{b^2}{c}} \gamma + \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{a}{b}\vec{r}^T \Sigma^{-1} \gamma}{b - \frac{ac}{b}} \vec{1} \right)$$

$$\Delta_{MSR} = \frac{\Sigma^{-1}\vec{r}}{1^T \Sigma^{-1} \vec{r}} = \frac{\Sigma^{-1}\vec{r}}{b} \qquad \Delta_{GMV} = \frac{\Sigma^{-1}\vec{1}}{1^T \Sigma^{-1} \vec{1}} = \frac{\Sigma^{-1}\vec{1}}{a} \qquad \Delta_{corr,I} = \frac{\Sigma^{-1}\gamma}{1^T \Sigma^{-1} \gamma}$$

$$\Delta = \Sigma^{-1}\left(\gamma + \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{b}{c}\bar{r} + \frac{b}{c}\vec{r}^T \Sigma^{-1} \gamma}{\frac{ac - b^2}{c}} \gamma + \frac{1 - 1^T \Sigma^{-1} \gamma - \frac{a}{b}\vec{r}^T \Sigma^{-1} \gamma}{\frac{b^2 - ac}{b}} \vec{1} \right)$$