# MF702 PROBLEM SET 6

**1.** Consider $M$ assets whose return is a $M$-dimensional random vector $r$. The expected value of $r$ is a $M$-dimensional column vector $\overrightarrow{r}$ and the covariance matrix is $\Sigma$.

(a) The Maximum Sharpe Ratio portfolio $\Delta_{MSR}$ solves

$$\max_{\Delta} \frac{\Delta^\top \overrightarrow{r}}{\sqrt{\Delta^\top \Sigma \Delta}}, \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Use the Lagrange multiplier method to show that $\Delta_{MSR} = \frac{\Sigma^{-1}\overrightarrow{r}}{\overrightarrow{1}\Sigma^{-1}\overrightarrow{r}}$.

(b) The Global Minimum Variance portfolio $\Delta_{GMV}$ solves

$$\min_{\Delta} \Delta^\top \Sigma \Delta, \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Show that $\Delta_{GMV} = \frac{\Sigma^{-1}\overrightarrow{1}}{\overrightarrow{1}^\top \Sigma^{-1}\overrightarrow{1}}$.

**2.** You can invest in three assets with expected returns and return covariance matrix given by

$$\overrightarrow{r} = \begin{pmatrix} 0.10 \\ 0.09 \\ 0.16 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.09 & -0.03 & 0.084 \\ -0.03 & 0.04 & 0.012 \\ 0.084 & 0.012 & 0.16 \end{pmatrix}.$$

You can also invest in the risk-free asset with return $r_f = 0.02$.

(a) What is the market portfolio ($\Delta_{MSR}$) in this setting? Compute the optimal weights invested in each risky security for this portfolio.

(b) Find the minimum-volatility portfolio which achieves an expected return of $\bar{r} = 0.20$.

**3.** (ETF construction) Often we wish to track an index $I$, but for practical reasons (such as wishing to avoid transactions costs) we cannot invest in all the securities which comprise $I$. In this setting, we wish to construct a portfolio that mimics the index most closely, in the sense of minimizing the tracking error, while still maintaining a target level of return.

More precisely, suppose we may invest in a subset of the index, comprising of $M$ stocks. The stocks have (random) returns $r_1, \ldots, r_M$ with expected values $\overrightarrow{r} = (\bar{r}_1, \ldots, \bar{r}_M)$ and covariance matrix $\Sigma = \{\Sigma_{mk}\}_{m,k=1}^{M}$ where $\Sigma_{mk} = Cov(r_m, r_k)$. On the other hand, the index $I$ has random return $r_I$ with expected value $\bar{r}_I$ and variance $\sigma_I^2$. We are able to estimate the covariance of the index and stock returns, obtaining the vector $\Upsilon = \{\Upsilon_m\}_{m=1}^M$, $\Upsilon_m = Cov(r_m, r_I)$. Lastly, we assume the investor may not allocate money to the ZCB.

We wish to find the portfolio weights $\Delta = (\Delta_1, \ldots, \Delta_M)$ whose random return $r(\Delta)$ minimize

$$Var(r(\Delta) - r_I), \tag{1}$$

still yielding an expected return of $\bar{r}(\Delta) = \bar{r}$, where $\bar{r}$ is the target return.

(a) Consider the (related) problem of trying to find the portfolio which is most closely correlated with the index, but has no opinion about the expected return: i.e.

$$\max_{\Delta} Corr(r(\Delta), r_I), \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Explicitly identify the optimal portfolio $\Delta_{Corr,I}$.

(b) Now, come back to the problem in (1):

$$Var(r(\Delta) - r_I), \quad \text{s.t.} \bar{r}(\Delta) = \bar{r}, \Delta^\top \overrightarrow{1} = 1.$$

Show that the optimal portfolio admits the decomposition

$$\Delta^* = \alpha_{C,I}\Delta_{Corr,I} + \alpha_M \Delta_{MSR} + \alpha_G \Delta_{GMV}.$$

Explicitly identify the constants $\alpha_{C,I}$, $\alpha_M$, and $\alpha_G$.