MF702 PROBLEM SET 4

1.

(i) Use Itô's formula to compute dW_t^4, where $\{W_t; t \geq 0\}$ is a Wiener process. Then write W_T^4 as the sum of a time integral and an integral with respect to dW_t.

(ii) Take expectations on both sides of the formula you obtained in (i), use the face that $\mathbb{E}W_t^2 = t$, derive the formula $\mathbb{E}W_T^4 = 3T^2$.

(iii) Use the method of (i) and (ii) to derive a formula for $\mathbb{E}W_T^6$.

2. (Solving the Vasicek equation). The Vasicek interest rate stochastic differential equation is

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t,$$

where α, β and σ are constants. We are going to solve this equation in this exercise.

(i) Use Itô's formula to compute $d(e^{\beta t}R_t)$. Simplify it so that you have a formula for $d(e^{\beta t}R_t)$ that does not involve R_t.

(ii) Integrate the equation you obtained in (i) and solve for R_t.

3. For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x , is

$$c(t,x) = x\Phi(d_+(T-t,x)) - Ke^{-r(T-t)}\Phi(d_-(T-t,x)),$$

where

$$\begin{aligned} d_+(\tau,x) =& \frac{1}{\sigma\sqrt{\tau}}\Big[\log\frac{x}{K} + \Big(r + \frac{\sigma^2}{2}\Big)\tau\Big], \\ d_-(\tau,x) =& d_+(\tau,x) - \sigma\sqrt{\tau}, \end{aligned}$$

and $\Phi(y)$ is the cumulative standard normal distribution

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x), \quad 0 \leq t < T, x > 0, \tag{1}$$

where c_t is the time derivative, c_x and c_{xx} are first and second partial derivatives with respect to x. c also satisfies the terminal condition

$$\lim_{t \uparrow T} c(t,x) = (x-K)_+, \quad x > 0, x \neq K. \tag{2}$$

For this exercise, we abbreviate $c(t, x)$ as c and $d_\pm(T-t, x)$ as $d_\pm$.

(i) Verify the equation

$$Ke^{-r(T-t)}\Phi'(d_-) = x\Phi'(d_+).$$

(ii) Show that $c_x = \Phi(d_+)$. This is the *delta* of the option. (Be careful! Remember that d_+ is a function of x.)

(iii) Show that

$$c_t = -rKe^{-r(T-t)}\Phi(d_-) - \frac{\sigma x}{2\sqrt{T-t}}\Phi'(d_+).$$

This is the *theta* of the option.

(iv) Use the formulas above to show that c satisfies (1).

(v) Show that for $x > K$, $\lim_{t \uparrow T} d_\pm = \infty$, but for $0 < x < K$, $\lim_{t \uparrow T} d_\pm = -\infty$. Use these equalities to derive the terminal condition (2).