MF702 PROBLEM SET 2

1. Consider a one-period binomial model in which the stock price at time 1 is either $S_1^u = uS_0$ or $S_1^d = dS_0$ with two constants u, d. Show that

$$0 < d < 1 + r < u$$

precludes arbitrage. Here r is one-period interest rate, i.e., \$1 deposit in the bank becomes $1 + r$ after one period. Let X be the value of a portfolio of cash and stocks. For this problem, we need to show that if $X_0 = 0$ and

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0),$$

we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, and this is the case regardless of the choice of the number Δ_0.

2. Consider a one-period binomial model as in Problem 1 with $S_0 = 4$, $u = 2, d = \frac{1}{2}$, and $r = \frac{1}{4}$. Consider a European call option with strike price $K = 5$ and maturity at time 1.
   (a) Show that the time zero arbitrage-free price of this European call option is 1.20.
   (b) Consider an agent who begins with wealth $X_0 = 0$ and at time zero buys Δ_0 shares of stock and Γ_0 options. The numbers Δ_0 and Γ_0 can be either positive or negative or zero. This leaves the agent with a cash position $-4\Delta_0 - 1.20\Gamma_0$. If this is positive, it is invested in the money market; if it is negative, it represents money borrowed from the money market. At time one, the value of the agent's portfolio of stock, option, and money market assets is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)_+ - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0).$$

   Assume that both S_1^u and S_1^d have positive probability of occurring. Show that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. In other words, one cannot find an arbitrage when the time-zero price of the option is 1.20.

3. Consider a one-period binomial model in which the stock has an initial price of \$100 and can go up 15% or down 5% with equal probabilities. The price of the European call option on this stock with strike price \$115 and maturity after one period is \$0.42.
   (a) What should be the price of a zero-coupon bond that pays \$1 after one period?
   (b) What are the risk-neutral probabilities in this model?

(c) Now consider a *defaultable* bond that pay off \$1 if the stock price goes up, but only pays off \$0.20 if the stock price goes down. What should be the arbitrage-free price of this security?

4. Consider a European put option on a non-dividend paying stock where the stock price is \$40, the strike price is \$40, the continuous compounding interest rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months.
   (a) Suppose that the log returns of stock in the first 6 months and the last 6 months of a year are i.i.d. Find the volatility of stock return in 6 months.
   (b) Using the result in (a), calculate u and d according to the Cox-Ross-Rubinstein rule for an one-period binomial model.
   (c) Calculate the risk-neutral probability $\tilde{p}$ for a jump up in the stock price in the stock price over a period of 6 months.
   (d) write a computer code that computes the put option price using an N-period model. Plot your option price as a function of N for $N \in [1, 500]$.
   (e) What value do your computed prices converge to?

5. Consider a three-period binomial tree model with $S_0 = 4$, $u = 2, d = \frac{1}{2}$, and take the one-period simple interest rate $r = \frac{1}{4}$, so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. For $n = 0, 1, 2, 3$, define $Y_n = \sum_{k=0}^{n} S_k$ to be the sum of the stock prices between times zero and n. Consider an *Asian call option* that expires at time two and has strike $K = 4$ (i.e., whose payoff at time three is $(\frac{1}{4}Y_t - 4)_+$.) This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_3(s, y) = (\frac{1}{4}y - 4)_+$.
   (a) Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.
   (b) Apply the recursive formula developed in (a) to compute $v_0(4, 4)$, the price of the Asian option at time zero.
   (c) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.

6. Suppose that a stock is currently valued at $S_0 = 100$ and has annual volatility of σ_0. Going forward, you predict that volatility in one year from today will be $\sigma_1 = \sigma^u > 0$ with probability p, or $\sigma_1 = \sigma^d < \sigma_0$ with probability $1 - p$ for some $p \in (0, 1)$. Figure 1. shows a binomial tree model.
   (a) Consider a probability measure $\tilde{\mathbb{P}}$ with

$$\tilde{\mathbb{P}}(S_1 > S_0) = q \quad \text{and} \quad \tilde{\mathbb{P}}(S_1 \leq S_0) = 1 - q.$$

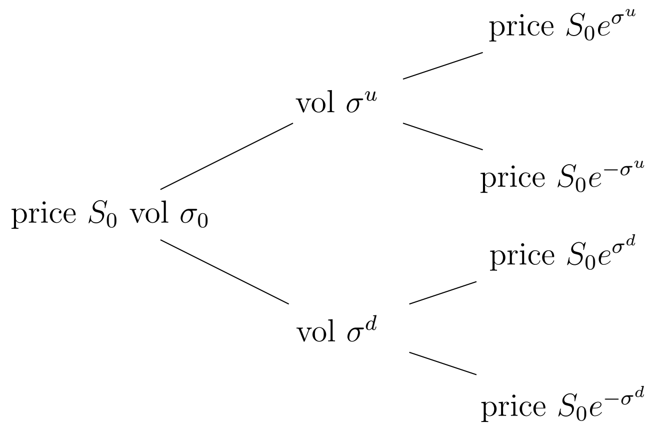

FIGURE 1. Binomial tree with variable volatility

Show that there exist multiple such measures $\tilde{\mathbb{P}}$ such that the discounted expected period-1 stock price under $\tilde{\mathbb{P}}$ is equal to the initial price S_0. That is, there are multiple $\tilde{\mathbb{P}}$ such that $\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-r}S_1] = S_0$.

Hint: Define $q_\sigma = \tilde{\mathbb{P}}(\sigma_1 = \sigma^u)$, $q^u = \tilde{\mathbb{P}}(S_1 > S_0|\sigma_1 = \sigma^u)$ and $q^d = \tilde{\mathbb{P}}(S_1 > S_0|\sigma_1 = \sigma^d)$. Write down the equation that S_1 and S_0 should satisfy under the risk neutral measure $\tilde{\mathbb{P}}$. Does this equation has a unique solution?

(b) Show that this market is not complete.

Hint: Count the number of different states at time 1. Does the number of different states match the number of different values of a replicating portfolio at time 1?