No-arbitrage Pricing of Options

Goals:

- Understand the no-arbitrage pricing argument
- Study the binomial model for option pricing
- Introduce risk neutral pricing and complete marekts

Relevant literature:

- Shreve Ch. 1, Hull Ch. 12

Arbitrage

If you invest in the financial markets, you will try to exploit every opportunity to make money without costs: If there is a *free lunch*, you will take it!

An arbitrage is the possibility to make a profit on an investment without having to pay anything upfront for it.

Formally, it is defined as a portfolio with value process X and maturity T such that $X_0 = 0$ and

1. $\mathbb{P}(X_T \geq 0) = 1$,
2. $\mathbb{P}(X_T > 0) > 0$.

In other words, the portfolio X costs nothing at time 0, does not loss value at time T, but will pay off at time T with positive probability.

No-arbitrage assumption

- Arbitrages are very rare in financial markets
- Whenever there is an arbitrage, large investors jump in and exploit the arbitrage opportunity
- This increases the demand for the arbitrage portfolio, increasing its price
- Supply and demand in the markets balance each other out eventually and the arbitrage opportunity disappears

Throughout the course, we will assume that there exist no arbitrage opportunities. Later in the course, we will derive conditions under which the no-arbitrage assumption is guaranteed.

Replication

- The no-arbitrage assumption facilitates the pricing of financial assets (including derivatives)

- If two assets with the same cash flows at all times, then they have have the same prices. If not, we can long the under-valued asset and short the over-valued asset to generate arbitrage

- We will repeatedly use this replication argument to compute prices of derivatives

A simple model

- Two dates: $t = 0, t = 1$ (one period)
- Two states at $t = 1$: 'u' and 'd'

$$\begin{tikzcd} & S_u \\ S_0 \arrow[ur, dash] \arrow[dr, dash] & \\ & S_d \end{tikzcd}$$

(a) State Variable

$$\begin{tikzcd} & V_1^u \\ V_1^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & V_1^d \end{tikzcd}$$

(b) Security 1

$$\begin{tikzcd} & V_2^u \\ V_2^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & V_2^d \end{tikzcd}$$

(c) Security 2

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(b) Security 1

$$\begin{tikzcd} & V_2^u \\ V_2^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & V_2^d \end{tikzcd}$$

(c) Security 2

$$\begin{tikzcd} & 4 \\ 2 \arrow[ur, dash] \arrow[dr, dash] & \\ & 1 \end{tikzcd}$$

(a) Security 1

$$\begin{tikzcd} & 3 \\ 2.25 \arrow[ur, dash] \arrow[dr, dash] & \\ & 2 \end{tikzcd}$$

(b) Security 2

$$\begin{tikzcd} & 2 \\ V_3^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & 3 \end{tikzcd}$$

(c) Security 3

Question: What is the fair (no-arbitrage) price for security 3, V_3^0?

Replication

- Replicate the payoffs, V_3^u and V_3^d, by combining security 1 and security 2 in a portfolio
- Solve for x_1 and x_2:

$$V_1^u x_1 + V_2^u x_2 = V_3^u$$
$$V_1^d x_1 + V_2^d x_2 = V_3^d$$

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$$V_1^u x_1 + V_2^u x_2 = V_3^u$$
$$V_1^d x_1 + V_2^d x_2 = V_3^d$$

- Solution: $x_1 = -1$ and $x_2 = 2$
- Fair value for security 3:

$$\begin{aligned} V_3^0 &= x_1 V_1^0 + x_2 V_2^0 \\ &= -1(2) + 2(2.25) = 2.5 \end{aligned}$$

Replication

- Replicate the payoffs, V_3^u and V_3^d, by combining security 1 and security 2 in a portfolio
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$$
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V_1^d x_1 + V_2^d x_2 &= V_3^d
\end{aligned}
$$

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- Fair value for security 3:

$$
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V_3^0 &= x_1 V_1^0 + x_2 V_2^0 \\
&= -1(2) + 2(2.25) = 2.5
\end{aligned}
$$

- If $V_3^0 < 2.5$, we can short 2 units of security 2, long a unit of security 1 and 3.
  - At time 0: cash flow $2(2.25) - 2 - V_3^0 > 0$;
  - At time 1:
    state d: $1 + 3 - 2(2) +$ time 0 cash flow with interest;
    state u: $4 + 2 - 2(3) +$ time 0 cash flow with interest.
    Both are larger than 0. Arbitrage!

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    state d: $1 + 3 - 2(2) +$ time 0 cash flow with interest;
    state u: $4 + 2 - 2(3) +$ time 0 cash flow with interest.
    Both are larger than 0. Arbitrage!
- How can you realize arbitrage if $V_3^0 > 2.5$?

European options

- An European call option gives you the right (not the obligation) to buy a stock at the maturity T for the strike price K
- Payoff of an European call option at maturity is

$$(S_T - K)_+ = \max\{S_T - K, 0\}$$

- An European put option gives you the right (not the obligation) to sell a stock at the maturity T for the strike price K
- Payoff of an European call option at maturity is

$$(K - S_T)_+ = \max\{K - S_T, 0\}$$

Put-call parity

The payoffs of European call and put satisfy

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

Let C_0 and P_0 be prices of call and put options at time 0

r is the continuous compounding interest rate.

No-arbitrage implies that

$$C_0 - P_0 = S_0 - Ke^{-rT}.$$

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$$C_0 - P_0 = S_0 - Ke^{-rT}.$$

If $C_0 - P_0 > S_0 - Ke^{-rT}$,

- Time 0: short a unit of call and Ke^{-rT} of zero-coupon bond, long a unit of put and long a share of stock.
- Time T:
    - If $S_T > K$, the call option is in the money, the put option is out of money. Sell the stock at price S_T, cover the zero-coupon bond with payoff K and the call option with payoff $S_T - K$
    - If $S_T \leq K$, the call option is in the money, the put option is in the money. Sell the stock at price S_T, exercise the put option to get payoff $K - S_T$, and cover the zero-coupon bond with payoff K
    - Net liability zero at time T, but positive profit at time 0, arbitrage!

Pricing an European call

Consider an European call option with T and $K = 21$.

$$
\begin{array}{ccc}
 & & S_T^u = 22 \\
 & \nearrow & \\
S_0 = 20 & & \\
 & \searrow & \\
 & & S_T^d = 18
\end{array}
$$

(a) Stock prices

$$
\begin{array}{ccc}
 & & V_T^u = (S_T^u - K)_+ = 1 \\
 & \nearrow & \\
V^0 = ? & & \\
 & \searrow & \\
 & & V_T^d = (S_T^d - K)_+ = 0
\end{array}
$$

(b) Option payoffs

Pricing an European call

Consider an European call option with T and $K = 21$.

$$S_T^u = 22$$
$$S_0 = 20$$
$$S_T^d = 18$$

(a) Stock prices

$$V_T^u = (S_T^u - K)_+ = 1$$
$$V^0 = ?$$
$$V_T^d = (S_T^d - K)_+ = 0$$

(b) Option payoffs

Replicating portfolio: Δ shares in stock and W dollar in bond at time 0. At time T:

$$\begin{aligned} 22\Delta + We^{rT} &= 1, \quad S_T = S_T^u, \\ 18\Delta + We^{rT} &= 0, \quad S_T = S_T^d. \end{aligned}$$

Assume that $T = 3$ months and $r = 0.5\%$. Then the solution is

$$\Delta = \frac{1}{4} \quad W = -18\frac{1}{4}e^{-0.005\frac{3}{12}} \approx -4.49.$$

Borrow 4.49 dollar and buy $1/4$ shares of stocks at time 0.
The call option price at time 0 is

$$V^0 = \frac{1}{4}S_0 + W = 0.51.$$

Pricing a European put option

Now that we know the price of an European call, we can compute the price of an European put with the same strike and the same maturity on the same stock.

The put-call parity tells us that

$$C_0 - P_0 = S_0 - Ke^{-rT}.$$

We obtain

$$P_0 = C_0 + Ke^{-rT} - S_0 = 0.51 + 21e^{-0.005\frac{3}{12}} - 20 = 1.48.$$

Assumptions

We were able to price the option because of several assumptions:

1. *There exists no arbitrage.* This assumption is mild. Arbitrage opportunities exist, but it's expensive to exploit them. Only very large investors, such as banks and hedge funds that have close to unlimited access to cash, can actually exploit arbitrage opportunities - and they do! Most investors, however, cannot exploit arbitrage.

2. *Any amount of the share can be traded.* This is not very realistic. In general, you can only buy entire shares. The above arbitrage argument still goes through if, e.g., you scale up the argument to price 4 call options instead of 1. Then, you would need to buy 1 share of the stock to replicate the payoff of 4 calls.

Assumptions

3. *Borrowing and lending interest rates are the same.* This is not the case in reality. The arbitrage arguments above, nevertheless, can be generalized to include different rates for borrowing and lending. In this case, the price for a long call and a short call would be different.

4. *The stock price at maturity can only take on two values.* This is also not realistic. We will relax this assumption later on. However, the binomial model provides a very basic framework to obtain important results on option pricing that hold even in more complex models.

The binomial pricing model

Suppose the strike price is K, maturity is T, and that the stock price S_T can take on the following values:

- $S_T^u = S_{\underset{0}{\cancel{t}}} e^{uT}$ with probability p
- $S_T^d = S_{\underset{0}{\cancel{t}}} e^{dT}$ with probability $q = 1 - p$

Assume that $d < u$ so that S_T^d is a *down*-price and S_T^u an *up*-price.

In order for any investor to actually hold this stock, there has to be some positive probability that the stock will return more than a cash investment (otherwise nobody would buy the stock):

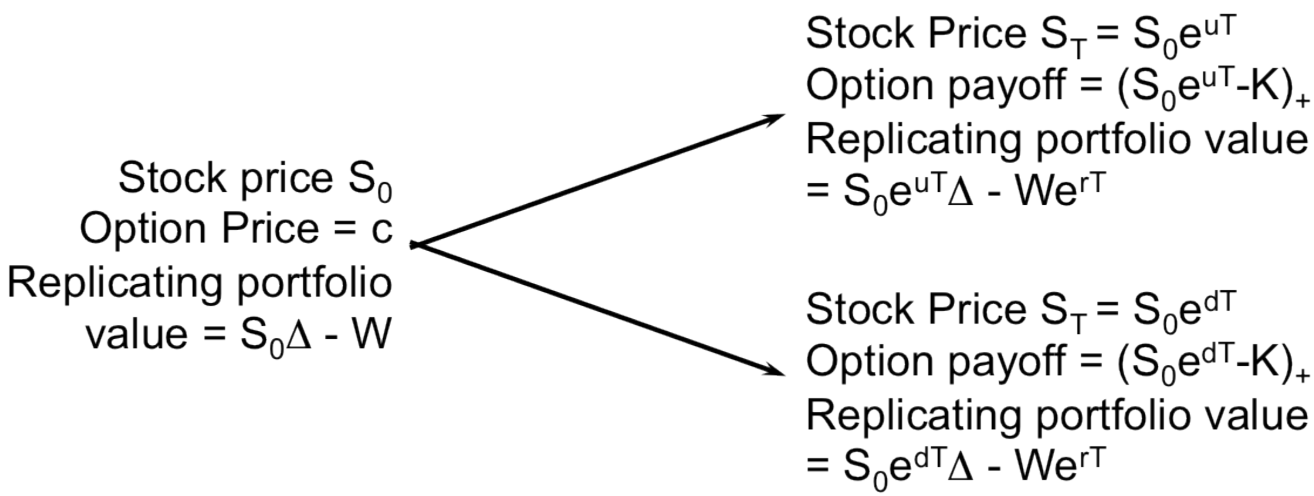
$$S_T^u = S_{\underset{0}{\cancel{t}}} e^{uT} > S_0 e^{rT} \Leftrightarrow u > r$$

Similarly, in order for anybody to sell the stock, there has to be some risk that the stock will return less than a loan:

$$S_T^d = S_{\underset{0}{\cancel{t}}} e^{dT} < S_0 e^{rT} \Leftrightarrow d < r$$

The binomial pricing model

The replicating portfolio holds a loan of $\$W$ and Δ shares of the stock



We assume that $d < r < u$

The binomial pricing model

In order for the portfolio to replicate the option, we have to set

$$S_T^u \Delta - W e^{rT} = (S_T^u - K)_+$$
$$S_T^d \Delta - W e^{rT} = (S_T^d - K)_+$$

We assume that $S_T^d < K < S_T^u$. This guarantees that the option pays off with positive probability but not with certainty (there is risk). Then:

$$S_T^u \Delta - W e^{rT} = S_T^u - K$$
$$S_T^d \Delta - W e^{rT} = 0$$

The binomial pricing model

We obtain:

$$\Delta = \frac{S_T^u - K}{S_T^u - S_T^d}$$

$$W = S_T^d e^{-rT} \frac{S_T^u - K}{S_T^u - S_T^d}$$

The replication argument then implies that $c = S_0\Delta - W$:

$$c = S_0 \frac{S_T^u - K}{S_T^u - S_T^d} - S_T^d e^{-rT} \frac{S_T^u - K}{S_T^u - S_T^d} = \frac{S_T^u - K}{S_T^u - S_T^d} \left(S_0 - S_T^d e^{-rT}\right).$$

In other words,

$$c = S_0 \left(1 - e^{(d-r)T}\right) \frac{e^{uT} - \frac{K}{S_0}}{e^{uT} - e^{dT}}.$$

Since $d < r < u$ and $S_T^u > K$, we have that $c > 0$.

Delta hedge

The replicating portfolio matches the payoff of the call option. Thus, by holding a loan of $\$W$ and Δ shares of the stock, the portfolio will always pay off as much as the call option, *no matter what stock price* S_T *we get at time* T.

If you were to hold a long position of the call option and a short position of the replicating portfolio, you would never lose money if the stock price moves (you would not make any money either...). The replicating portfolio hedges the risk of a moving stock price.

Because of this, the price

$$c = S_0 \left(1 - e^{(d-r)T}\right) \frac{e^{uT} - \frac{K}{S_0}}{e^{uT} - e^{dT}}$$

is also known as the **delta hedge** price of a European call option.

No-arbitrage put price

The put-call parity tells us that

$$p = Ke^{-rt} - S_0 + c.$$

Thus:

$$p = Ke^{-rT} - S_0 + S_0 \left(1 - e^{(d-r)T}\right) \frac{e^{uT} - \frac{K}{S_0}}{e^{uT} - e^{dT}}$$

Important observation

We assumed that $S_T = S_T^u$ with probability p and $S_T = S_T^d$ with probability $q = 1 - p$.

Where do these probabilities show up?

- If you paid attention closely, you will have noticed that these probabilities **never** show up
- We matched the payoff of the option for every possible state at time T
- The likelihood of each state is not important
- This is **always** the case and is due to the fact that investors require compensation in order to hold risky assets in every state of the world

Risk-neutral price of call

Let's re-write the call price in a different way. Define

$$\tilde{p} = \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}}.$$

Our assumption that $d < r < u$ implies that $0 < \tilde{p} < 1$. Thus, we can view $(\tilde{p}, 1 - \tilde{p})$ as a probability distribution on $\{S_T^u, S_T^d\}$. Let $\tilde{\mathbb{P}}$ be a probability measure such that

- $S_T = S_T^u$ with probability $\tilde{p}$
- $S_T = S_T^d$ with probability $1 - \tilde{p}$

Risk-neutral price of a European call

Then we can rewrite c as:

$$
\begin{aligned}
c &= S_0\left(1-e^{(d-r)T}\right)\frac{e^{uT}-\frac{K}{S_0}}{e^{uT}-e^{dT}} \\
&= e^{-rT}\left(S_0e^{uT}-K\right)\frac{e^{rT}-e^{dT}}{e^{uT}-e^{dT}} \\
&= e^{-rT}\left[\tilde{p}\left(S_T^u-K\right)\right] = e^{-rT}\left[\tilde{p}\left(S_T^u-K\right)+(1-\tilde{p})0\right] \\
&= e^{-rT}\left[\tilde{p}\left(S_T^u-K\right)_+ + (1-\tilde{p})\left(S_T^d-K\right)_+\right]
\end{aligned}
$$

Thus, if we write $\tilde{\mathbb{E}}$ for the expectation computed relative to the probability measure $\tilde{\mathbb{P}}$, we obtain:

$$c = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T-K\right)_+\right]$$

Risk-neutral price of a European call

This is the **risk-neutral price** of a European call option:

$$c = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)_+\right]$$

It is the expected payoff of the option, discounted by the interest rate, and computed with respect to the **risk-neutral** probabilities

$$\tilde{\mathbb{P}}\left[S_T = S_T^u\right] = \tilde{p} = \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}},$$

$$\tilde{\mathbb{P}}\left[S_T = S_T^d\right] = 1 - \tilde{p} = \frac{e^{uT} - e^{rT}}{e^{uT} - e^{dT}}.$$

Comments on risk-neutral valuation

$$c = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)_+\right]$$

The fact that the price of a call is some expectation of its payoff is not surprising: Any rational buyer would not pay more for an asset than the expected profit she can make, and any rational seller would not sell for less than that expected profit.

Nevertheless, it is surprising that this expectation is computed relative to the risk-neutral distribution of the future stock price and *not* relative to its actual (*physical*) distribution.

Why is this?

Risk-neutral pricing

Let's compute the expected stock price under $\tilde{\mathbb{P}}$:

$$\begin{aligned}
\tilde{\mathbb{E}}[S_T] &= S_T^u \tilde{p} + S_T^d (1 - \tilde{p}) \\
&= S_0 e^{uT} \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}} + S_0 e^{dT} \frac{e^{uT} - e^{rT}}{e^{uT} - e^{dT}} \\
&= S_0 \frac{e^{(u+r)}T - e^{(u+d)T} + e^{(u+d)T} - e^{(r+d)T}}{e^{uT} - e^{dT}} \\
&= S_0 e^{rT} \frac{e^{uT} - e^{dT}}{e^{uT} - e^{dT}} = S_0 e^{rT}
\end{aligned}$$

Under the risk-neutral probability, the investor expects that the stock will grow with the risk-free interest rate. That is, the stock is as good as a risk-free investment – there are no significant risks and the price can match its expected profit!

Risk-neutral pricing

Let's compute the expected stock price under $\tilde{\mathbb{P}}$:

$$
\begin{aligned}
\tilde{\mathbb{E}}[S_T] &= S_T^u \tilde{p} + S_T^d (1 - \tilde{p}) \\
&= S_0 e^{uT} \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}} + S_0 e^{dT} \frac{e^{uT} - e^{rT}}{e^{uT} - e^{dT}} \\
&= S_0 \frac{e^{(u+r)}T - e^{(u+d)T} + e^{(u+d)T} - e^{(r+d)T}}{e^{uT} - e^{dT}} \\
&= S_0 e^{rT} \frac{e^{uT} - e^{dT}}{e^{uT} - e^{dT}} = S_0 e^{rT}
\end{aligned}
$$

Under the risk-neutral probability measure, investors are indifferent between a risky and a risk-free investment. ~~Supply and demand can therefore agree to transact. This is **why** the investor prices the option as the expected profit under $\tilde{\mathbb{P}}$ and not under $\mathbb{P}$.~~

Comments on risk-neutral valuation

In order for an investor to invest in a call option, the investor has to believe that the option will provide enough benefit to her even in the worst case.

Think of it this way:

- You want to play the lottery and know that the jackpot this week is \$1 million. Suppose that you can win the lottery with 1% probability. How much would you pay for a lottery ticket?
- You would certainly not pay $\$1$ million $\times\ 0.01 = \$10,000$, right? The probability that you will win is too small in your eyes to justify this price.

Comments on risk-neutral valuation

The same holds for the call option!

- The no-arbitrage assumption implies that $\mathbb{P}[S_T \geq S_0 e^{rT}] > 0$ (otherwise nobody would hold the stock)
- Nevertheless, the probability that the option will pay off $(S_T = S_T^u)$ is too small in the eyes of the investors in order to justify an investment in the option
- When pricing the option, the investor will *demand* that she is *compensated* for carrying the risk that the option may not pay off
- Because of this, she adjusts the probability distribution of S_T in order to reflect her preferences on risks
  - This means that the option price will be lower than implied by the expected discounted payoff under the physical distribution

Risk preferences

An investor that requires compensation for holding risky positions is known as **risk averse**. She dislikes losses and will only take on risks if she believes that the reward for carrying those risks is justified.

An investor that is indifferent towards risks is called **risk neutral**.

When pricing the option, a risk averse investor modifies her beliefs on the distribution of the stock return to make her *risk neutral* towards holding the option.

Original example

Suppose we want to price an option on a stock with maturity $T = 3$ months and strike $K = \$21$. The stock price today is $S_0 = \$20$, the spot rate is $r = 0.5\%$, and

- $S_T = S_T^u = \$22$ with probability $p = 0.6$
- $S_T = S_T^d = \$18$ with probability $q = 1 - p = 0.4$

We then have:

$$u = \frac{1}{T} \ln\left(\frac{22}{20}\right) \approx 0.38, \quad d = \frac{1}{T} \ln\left(\frac{18}{20}\right) \approx -0.42,$$

$$\tilde{p} = \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}} \approx 0.51 < p, \quad 1 - \tilde{p} \approx 0.49 > q,$$

$$\begin{aligned} c &= e^{-rT}\left[\tilde{p}(S_T^u - K)_+ + (1-\tilde{p})(S_T^d - K)_+\right] \\ &= e^{-rT}\tilde{p}(S_T^u - K) \approx 0.51 \end{aligned}$$

This is the same price as before!

No-arbitrage and risk neutral price

We have seen that there are two forms to write the call price:

1. *No-arbitrage* or *delta-hedge price*. This price is based only on a replication argument:

$$c = S_0 \frac{e^{uT} - \frac{K}{S_0}}{e^{uT} - e^{dT}} \left(1 - e^{(d-r)T}\right)$$

2. *Risk-neutral price*. This price is based on the risk preferences of the investor:

$$c = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)_+\right]$$

Both prices are the same if *(i)* there exists no arbitrage, *(ii)* any amount of a stock can be traded, and *(iii)* borrowing and lending costs the same, and *(iv)* the stock price can only jump up or down from time zero to maturity.

State price

Consider any two assets i and j

$$\begin{tikzcd}[row sep=small] & V_i^u \\ V_i^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & V_i^d \end{tikzcd}$$

(a) Asset i

$$\begin{tikzcd}[row sep=small] & V_j^u \\ V_j^0 \arrow[ur, dash] \arrow[dr, dash] & \\ & V_j^d \end{tikzcd}$$

(b) Asset j

From the risk-neutral evaluation

$$V_i^0 = e^{-rT}[\tilde{p}V_i^u + (1-\tilde{p})V_i^d], \quad V_j^0 = e^{-rT}[\tilde{p}V_j^u + (1-\tilde{p})V_j^d].$$

From the previous equations, we obtain

$$e^{-rT}\tilde{p} = \frac{V_i^0 - e^{-rT}V_i^d}{V_i^u - V_i^d} = \frac{V_j^0 - e^{-rT}V_j^d}{V_j^u - V_j^d}$$

$$e^{-rT}(1-\tilde{p}) = \frac{e^{-rT}V_i^u - V_i^0}{V_i^u - V_i^d} = \frac{e^{-rT}V_j^u - V_j^0}{V_j^u - V_j^d}$$

$e^{-rT}\tilde{p}$ is the no-arbitrage price of a security which pays 1 in the S_u state and nothing in the S_d states.
$e^{-rT}(1-\tilde{p})$ is the no-arbitrage price of a security which pays 1 in the S_d state and nothing in the S_u states.

Market price of risk

Consider the objective ('true') probabilistics $\mathbb{P}[S_u] = p$ and $\mathbb{P}[S_d] = 1 - p$.

From the equation in the previous slides,

$$
\begin{aligned}
& -e^{-rT}\tilde{p}(1-p) + e^{-rT}(1-\tilde{p})p \\
=& (1-p)\frac{e^{-rT}V_i^d - V_i^0}{V_i^u - V_i^d} + p\frac{e^{-rT}V_i^u - V_i^0}{V_i^u - V_i^d} \\
=& (1-p)\frac{e^{-rT}V_j^d - V_j^0}{V_j^u - V_j^d} + p\frac{e^{-rT}V_j^u - V_j^0}{V_j^u - V_j^d}
\end{aligned}
$$

Rearranging and using that $pV_i^u + (1-p)V_i^d = \mathbb{E}^{\mathbb{P}}[V_i]$

$$\frac{e^{-rT}\mathbb{E}^{\mathbb{P}}[V_i] - V_i^0}{V_i^u - V_i^d} = \frac{e^{-rT}\mathbb{E}^{\mathbb{P}}[V_j] - V_j^0}{V_j^u - V_j^d}$$

Therefore, for any security i

$$\lambda = \frac{e^{-rT}\mathbb{E}^{\mathbb{P}}[V_i] - V_i^0}{V_i^u - V_i^d}.$$

λ is called the Market price of risk
Price of security i is

$$V_i^0 = e^{-rT}\mathbb{E}^{\mathbb{P}}[V_i] - \lambda(V_i^u - V_i^d).$$

Market price of risk

For given security prices, the objective (true) state probabilities determine λ

For $V_0 = 2,\ V^u = 4, V^d = 1, e^{-rT} = 0.95$

p	$e^{-rT}\mathbb{E}^{\mathbb{P}}[V]$	λ	$\lambda[V_u - V_d]$	V_0
0.1	1.24	-0.26	-0.77	2.00
0.2	1.52	-0.16	-0.48	2.00
0.3	1.81	-0.07	-0.20	2.00
0.3684	2.00	0.00	0.00	2.00
0.4	2.09	0.03	0.09	2.00
0.5	2.38	0.13	0.38	2.00

0.3864 is the risk neutral probability $\tilde{p}$, whose associate market price of risk is 0.

Multiperiod binomial model

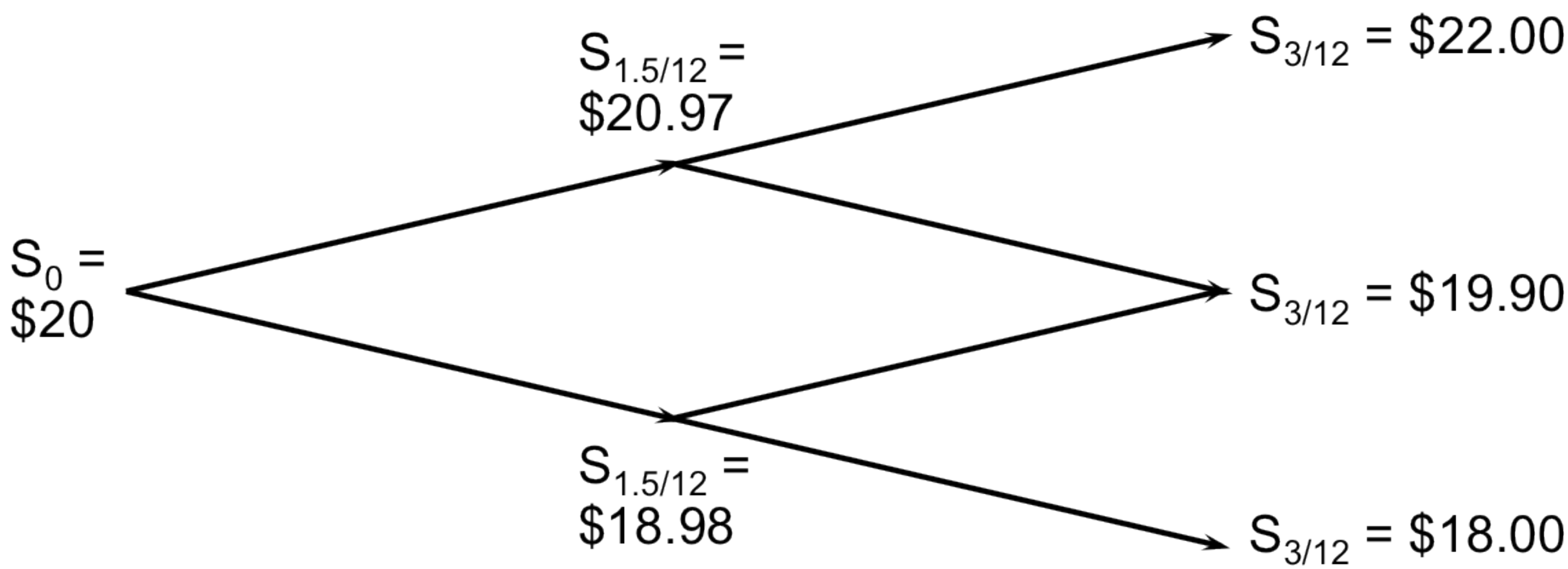
The binomial model we introduced earlier is not very realistic:

- It assumes that the stock price can only take on two values at maturity
- It does not consider what happens to the stock price between time 0 and maturity

A more meaningful model is the multiperiod binomial model

Example of multiperiod binomial model

Suppose that at times $t = 1.5$ and 3 months, the stock price can only move up or down from its previous value:



Example of multiperiod binomial model

In particular, assume the following multiperiod binomial model:

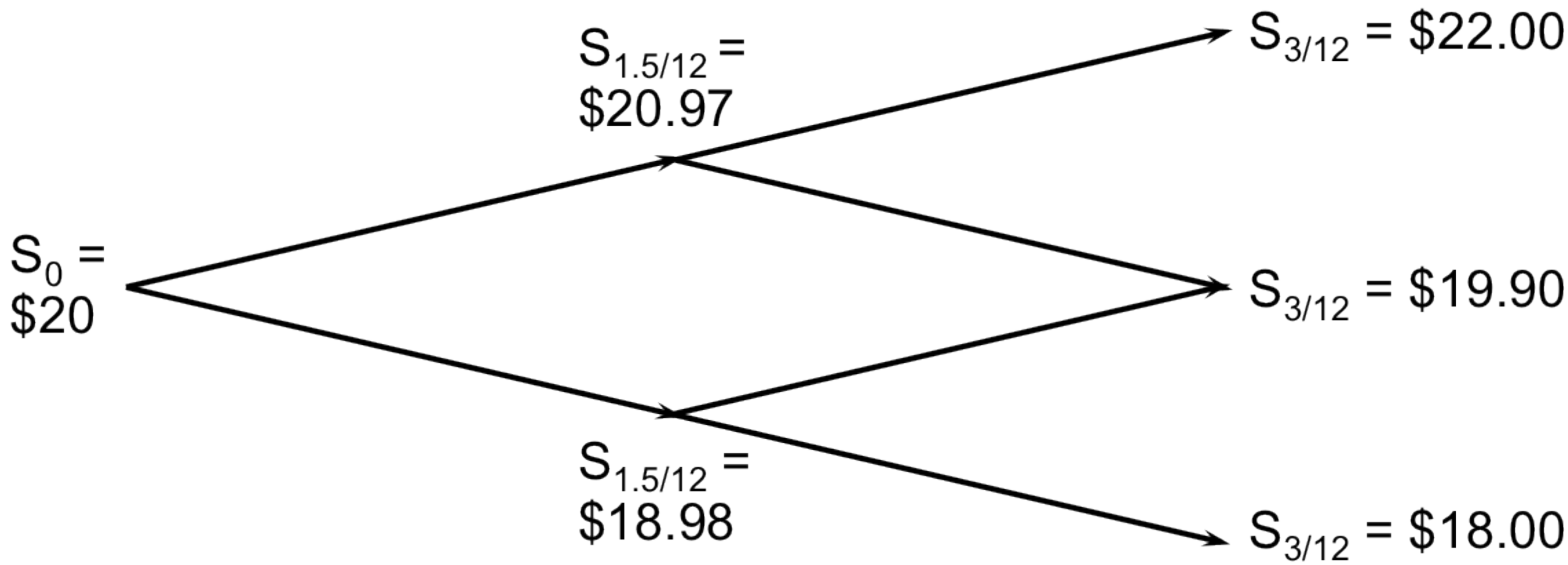
$$S_{1.5/12} = \begin{cases} S^u = S_0 e^{u\frac{1.5}{12}} & \text{with probability } p = 0.6 \\ S^d = S_0 e^{d\frac{1.5}{12}} & \text{with probability } p = 0.4 \end{cases}$$

$$S_{3/12} = \begin{cases} S^{uu} = S_0 e^{u\frac{3}{12}} & \text{with probability } 0.36 \\ S^{ud} = S_0 e^{(u+d)\frac{1.5}{12}} & \text{with probability } 0.48 \\ S^{dd} = S_0 e^{d\frac{3}{12}} & \text{with probability } 0.16 \end{cases}$$

In this case, the number of price jumps up until time T is binomially distributed.

Assume $S_0 = \$20$ and $r = 0.5\%$. Take $u = 0.38$ and $d = -0.42$.

Example of multiperiod binomial model

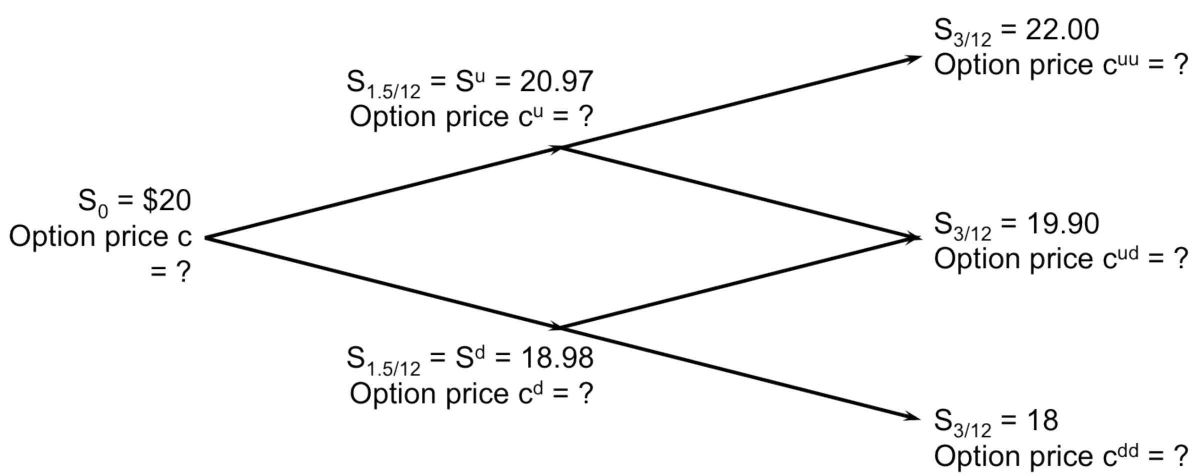


What is the price of a European call option with strike $K = \$21$ and maturity $T = 3$ months?

Backward induction

The standard approach to compute the price of an option in the multiperiod binomial model is to start at maturity and go backwards in time. This is known as **backward induction**.

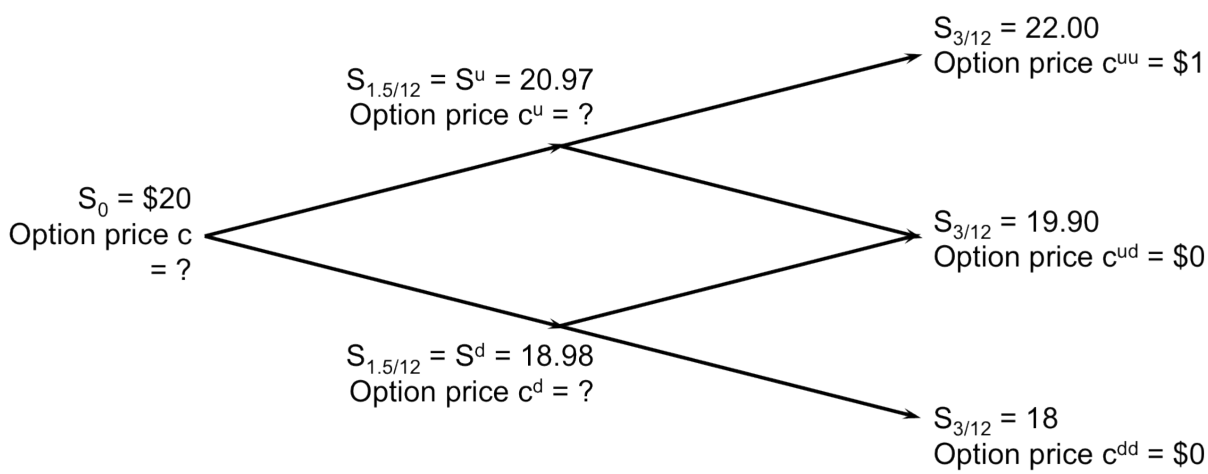
We will compute the price of the option at each node of the multiperiod binomial tree:



Backward induction

At maturity T, we have:

- If $S_T = S^{uu}$, then $c^{uu} = (S_T - K)_+ = (22 - 21)_+ = 1$
- If $S_T = S^{ud}$, then $c^{ud} = (S_T - K)_+ = (19.90 - 21)_+ = 0$
- If $S_T = S^{dd}$, then $c^{dd} = (S_T - K)_+ = (18 - 21)_+ = 0$



Backward induction

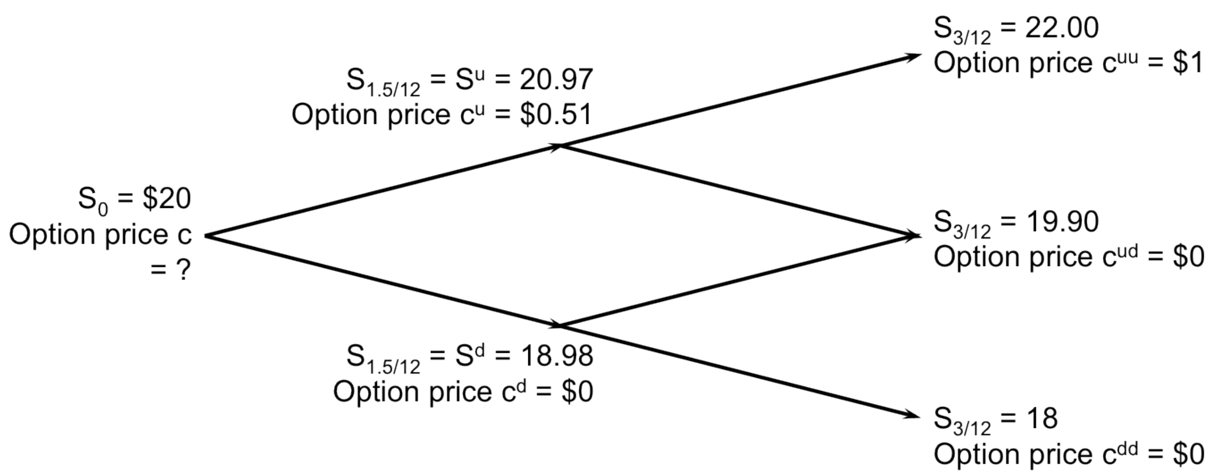
At time $t = 1.5$ months, we can use the risk-neutral pricing formula from before for each possible value of $S_{1.5/12}$. With $\tilde{p} = 0.51$, we have:

- If $S_{1.5/12} = S^u = \$20.97$, then

$$c^u = e^{-r\frac{1.5}{12}} \left[\tilde{p}c^{uu} + (1-\tilde{p})c^{ud}\right] = \$0.51$$

- If $S_{1.5/12} = S^d = \$18.98$, then

$$c^d = e^{-r\frac{1.5}{12}} \left[\tilde{p}c^{ud} + (1-\tilde{p})c^{dd}\right] = \$0$$

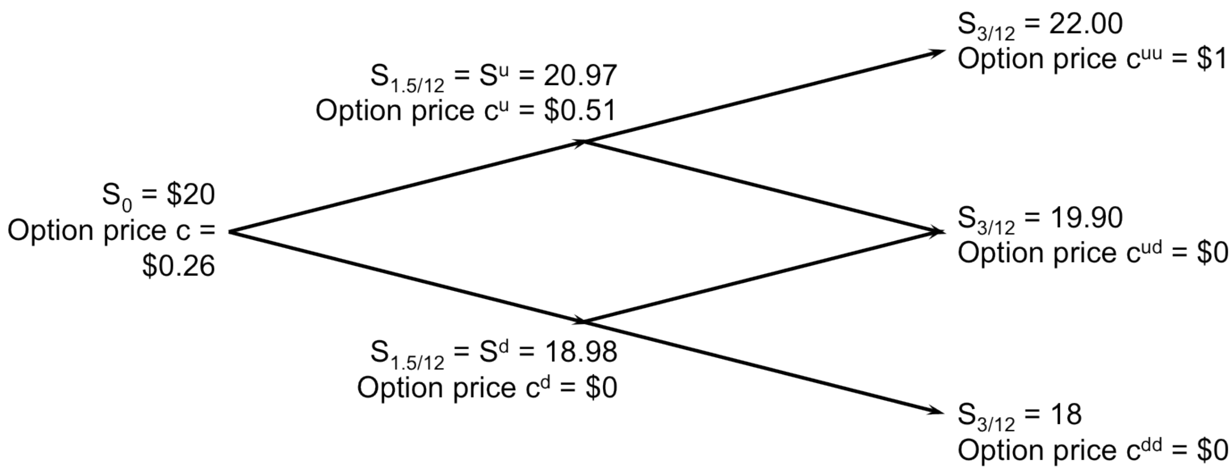


Backward induction

At time $t = 0$ months, we again use the risk-neutral pricing formula:

$$c = e^{-r\frac{1.5}{12}}\left[\tilde{p}c^u + (1-\tilde{p})c^d\right] = \$0.26$$

We can do this because we can always sell the option at time $t = 1.5$ months for the risk-neutral price. Thus, we can view the call option at time $t = 0$ as a European option with maturity of 1.5 months and payoff equal to the price of the option at $t = 1.5$ months



Multiperiod delta hedge

Alternatively, we can construct a portfolio that replicates the option at all times.

- You want to choose the cash and the shares you hold at each point of time as to replicate the value of the option in each node of the tree
- Thus, you have two decisions to make:
  1. What portfolio to build at time $t = 0$
  2. How to readjust your portfolio at time $t = 1.5$ months after the stock price has changed

Let's take a deeper look

Multiperiod delta hedge

- Let v denote the value of the portfolio. We want to compute v_0, the initial value of the portfolio, which is equal to the option price c for a replicating portfolio
- At time $t = 0$, the investor buys Δ_0 shares of the asset and invest the rest $(v_0 - \Delta_0 S_0)$ in savings
- At time $t = 1.5/12$, the investor faces two potential scenarios:
  1. If $S_{1.5/12} = S^u$, then the value of the portfolio is $v_{1.5/12} = v^u = \Delta_0 S^u + e^{r1.5/12}(v_0 - \Delta_0 S_0)$. Now, she may want to hold Δ^u shares to reflect her views on the future development of the stock
  2. If $S_{1.5/12} = S^d$, then the value of the portfolio is $v_{1.5/12} = v^d = \Delta_0 S^d + e^{r1.5/12}(v_0 - \Delta_0 S_0)$. Now, she may want to hold Δ^d shares to reflect her views on the future development of the stock

Multiperiod delta hedge

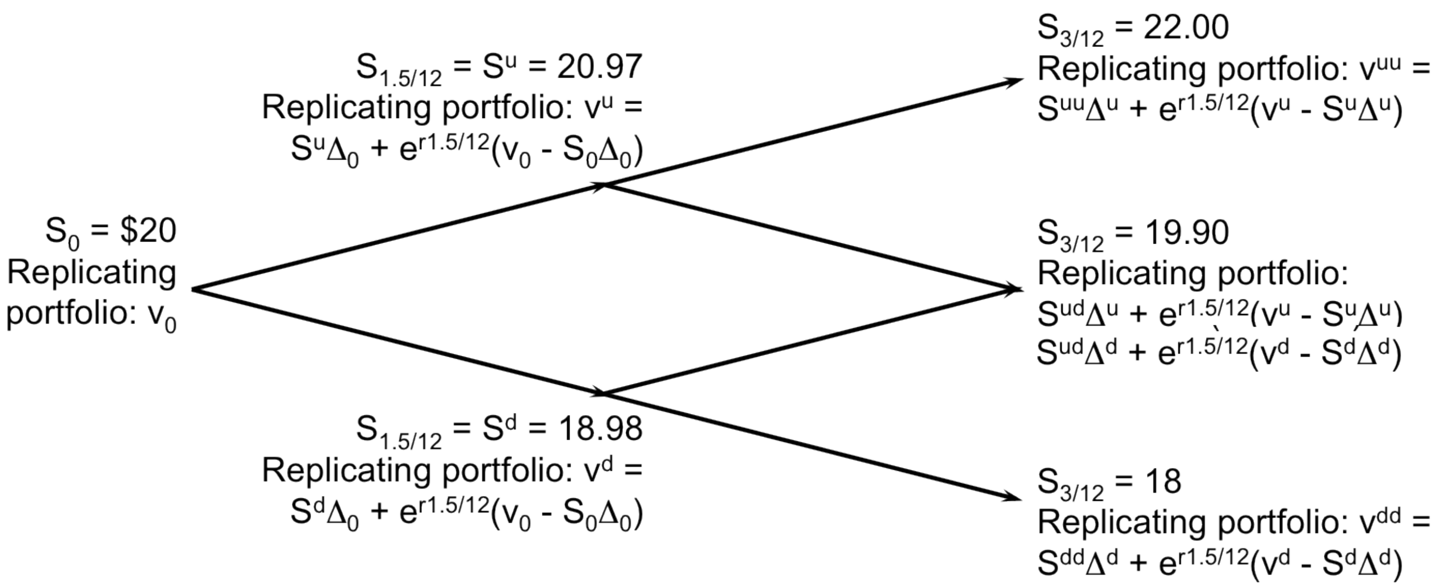
- At maturity $T = 3/12$, the investor faces three potential scenarios:
  1. If $S_{3/12} = S^{uu}$, then the value of the portfolio is
     $v_{3/12} = v^{uu} = \Delta^u S^{uu} + e^{r1.5/12}(v^u - \Delta^u S^u)$
  2. If $S_{3/12} = S^{dd}$, then the value of the portfolio is
     $v_{3/12} = v^{dd} = \Delta^d S^{dd} + e^{r1.5/12}(v^d - \Delta^d S^d)$
  3. If $S_{3/12} = S^{ud}$, then the value of the portfolio is

$$v_{3/12} = \begin{cases} v^{ud} = \Delta^u S^{ud} + e^{r1.5/12}(v^u - \Delta^u S^u) & \text{if } S_{1.5/12} = S^u \\ v^{du} = \Delta^d S^{ud} + e^{r1.5/12}(v^d - \Delta^d S^d) & \text{if } S_{1.5/12} = S^d \end{cases}$$

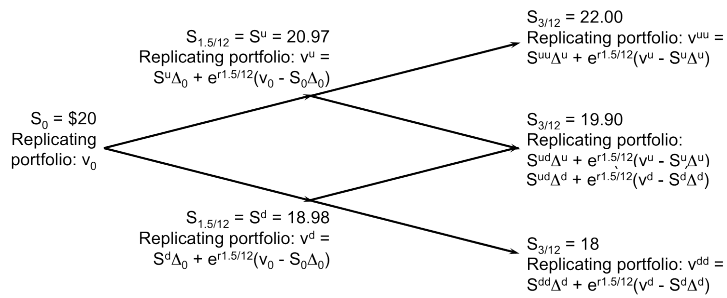
- She liquidates the portfolio at maturity:

$$v_t = 0 \quad \text{for} \quad t > T$$

Multiperiod delta hedge



Multiperiod delta hedge



We have 6 unknown and 6 equations:

1. Replication at maturity if $S_{3/12} = S^{uu}$:

$$\Delta^{\overset{u}{\cancel{d}}} S^{uu} + e^{r1.5/12}(v^u - \Delta^u S^u) = c^{uu}$$

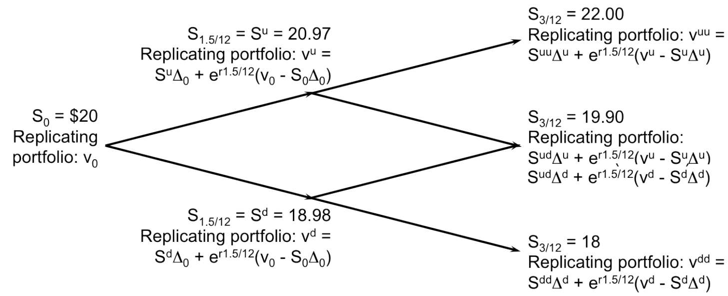
2. Replication at maturity if $S_{3/12} = S^{dd}$:

$$\Delta^d S^{dd} + e^{r1.5/12}(v^d - \Delta^d S^d) = c^{dd}$$

3. Replication at maturity if $S_{1.5/12} = S^u$ and $S_{3/12} = S^{ud}$:

$$\Delta^{\overset{u}{\cancel{d}}} S^{ud} + e^{r1.5/12}(v^{\overset{u}{\cancel{d}}} - \Delta^{\overset{u}{\cancel{d}}} S^{\overset{u}{\cancel{d}}}) = c^{ud}$$

Multiperiod delta hedge



4. Replication at maturity if $S_{1.5/12} = S^d$ and $S_{3/12} = S^{ud}$:

$$\Delta^d S^{ud} + e^{r1.5/12}(v^d - \Delta^d S^d) = c^{ud}$$

5. Replication at time $t = 1.5$ months when $S_{1.5/12} = S^u$:

$$\Delta_0 S^u + e^{r1.5/12}(v_0 - \Delta_0 S_0) = c^u$$

6. Replication at time $t = 1.5$ months when $S_{1.5/12} = S^d$:

$$\Delta_0 S^d + e^{r1.5/12}(v_0 - \Delta_0 S_0) = c^d$$

Multiperiod delta hedge

Thus, a replicating porfolio is constructed by solving the linear system:

$$
\begin{aligned}
\Delta^u S^{uu} + e^{r1.5/12}(v^u - \Delta^u S^u) &= c^{uu} \\
\Delta^d S^{dd} + e^{r1.5/12}(v^d - \Delta^d S^d) &= c^{dd} \\
\Delta^d S^{ud} + e^{r1.5/12}(v^d - \Delta^d S^d) &= c^{ud} \\
\Delta^d S^{ud} + e^{r1.5/12}(v^d - \Delta^d S^d) &= \Delta^u S^{ud} + e^{r1.5/12}(v^u - \Delta^u S^u) \\
\Delta_0 S^u + e^{r1.5/12}(v_0 - \Delta_0 S_0) &= c^u \\
\Delta_0 S^d + e^{r1.5/12}(v_0 - \Delta_0 S_0) &= c^d
\end{aligned}
$$

As we will see next, there exists a solution to the above system for any multiperiod binomial model. As a result, the risk-neutral price of the option is also a no-arbitrage price.

Notation

Before we proceed, let fix some notation:

- We consider a multiperiod binomial model with N periods with Δt time between periods; i.e., $N = \frac{T}{\Delta t}$
- For $1 \leq n \leq N$, let $\omega_n \in \Omega = \{u, d\}$ denote if the stock price goes up (u) or down (d) in the n-th period
- Let $S_n(\omega_1 \ldots \omega_n)$ denote the stock price in the n-th period if the stock price goes up and down according to $\omega_1 \ldots \omega_n$
- Let $c_n(\omega_1 \ldots \omega_n)$ denote the value of a European call option in the n-th period if the stock price moved as in $\omega_1 \ldots \omega_n$
- Let $\Delta_n(\omega_1 \ldots \omega_n)$ denote the number of shares of a replicating portfolio in the n-th period if the stock price moved as in $\omega_1 \ldots \omega_n$
- Let $v_n(\omega_1 \ldots \omega_n)$ denote the value of a replicating portfolio in the n-th period if the stock price moved as in $\omega_1 \ldots \omega_n$

Valuation in multiperiod binomial model

Theorem (Valuation). *Assume that* $d < r < u$ *and let* $\tilde{p} = \frac{e^{r\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$. *Define the risk-neutral price of the option recursively through*

$$c_N(\omega_1 \dots \omega_N) = (S_N(\omega_1 \dots \omega_N) - K)_+ ,$$
$$c_n(\omega_1 \dots \omega_n) = e^{-r\Delta t} \left[\tilde{p} c_{n+1}(\omega_1 \dots \omega_n u) + (1 - \tilde{p}) c_{n+1}(\omega_1 \dots \omega_n d)\right]$$

for $0 \leq n < N$. *Set* $v_0 = c_0$ *and*

$$\Delta_n(\omega_1 \dots \omega_n) = \frac{c_{n+1}(\omega_1 \dots \omega_n u) - c_{n+1}(\omega_1 \dots \omega_n d)}{S_{n+1}(\omega_1 \dots \omega_n u) - S_{n+1}(\omega_1 \dots \omega_n d)}$$
$$v_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) = \Delta_n(\omega_1 \dots \omega_n) S_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) + e^{r\Delta t} \left(v_n(\omega_1 \dots \omega_n) - S_n(\omega_1 \dots \omega_n) \Delta_n(\omega_1 \dots \omega_n)\right)$$

Then, $v_n(\omega_1 \dots \omega_N) = c_n(\omega_1 \dots \omega_N)$ *for any* $n \leq N$ *and* $\omega_1 \dots \omega_N$. *The portfolio replicates the option.*

Replication in multiperiod binomial model

Proof. We prove the claim by forward induction. Fix $N \in \mathbb{N}$. The definitions imply the claim for $n = 0$.

Now suppose the claim holds for $n < N$. We want to show that it also holds for $n + 1$. Take any $\omega_1 \dots \omega_n$ and assume first that $\omega_{n+1} = u$. The induction assumption and the definitions of Δ_n, $\tilde{p}$, and c_n imply that (ignoring the dependence on $\omega_1 \dots \omega_n$ for simplicity)

$$\begin{aligned}
v_{n+1}(u) &= \Delta_n S_{n+1}(u) + e^{r\Delta t}\left(c_n - S_n \Delta_n\right) \\
&= \Delta_n S_n \left(e^{u\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t} c_n \\
&= \left(c_{n+1}(u) - c_{n+1}(d)\right) \frac{e^{u\Delta t} - e^{r\Delta t}}{e^{u\Delta t} - e^{d\Delta t}} + e^{r\Delta t} c_n \\
&= \left(c_{n+1}(u) - c_{n+1}(d)\right)(1 - \tilde{p}) + e^{r\Delta t} c_n \\
&= \left(c_{n+1}(u) - c_{n+1}(d)\right)(1 - \tilde{p}) + \tilde{p} c_{n+1}(u) + (1 - \tilde{p}) c_{n+1}(d)
\end{aligned}$$

Thus,

$$v_{n+1}(u) = c_{n+1}(u).$$

A similar argument leads to

$$v_{n+1}(d) = c_{n+1}(d).$$

The claim follows. $\square$

Completeness

The theorem above shows us how to construct a replicating portfolio and compute a no-arbitrage price for a European call option based on the multiperiod binomial model

- The arguments can be generalized for any European derivative with payoff $c_N = f(S_N)$ at maturity $T = N\Delta t$
- Here, f is some non-negative function of the stock price
- For a European call option, $f(S_T) = (S_T - K)_+$

The multiperiod binomial model is said to be **complete** because every (European) derivative on the stock can be replicated by a portfolio consisting of savings and the stock. A market is **complete** when every derivative can be replicated with the existing assets

Completeness

- If a market is complete, then every European derivative on a stock can be replicated using the available assets
- Therefore, there cannot exist any arbitrage in a market that consists of savings, stocks, and European derivatives on the stocks
- In complete markets, every European security can be priced by backwards induction using the risk-neutral pricing formula
- The replication theorem tells us that the price implied by the risk-neutral pricing formula is the same as the price implied by a replicating portfolio
- As a result, every European derivative in the binomial model has a unique price, and these prices admit no arbitrage

Choosing u and d

So far, we have fixed u and d arbitrarily. What is a meaningful choice for u and d?

- If you are an investor hoping to buy an option, you are worried about the risks you will face if when you own the option (e.g., the stock price falling when you buy a call)
- Because of this, u and d should be chosen in a way to reflect the riskiness of the option

Choosing u and d

Cox, Ross, and Rubinstein propose to choose u and d as to match the volatility of the stock:

- Suppose the stock has volatility of σ over the time period Δt
- Then, set $p = 1 - q = \frac{1}{2}$ and

$$u = \sigma,$$

$$d = -\sigma.$$

This ensures that

$$\mathrm{Var}\left(\ln\left(\frac{S_{\Delta t}}{S_0}\right)\right) = \sigma^2$$

Dividend paying stocks

So far, we have only considered stocks that do not pay any dividends. What is the price of call option on a dividend paying stock?

- If a stock guarantees a dividend payment, then this payment is risk-free in the eye of the investor
- Thus, the investor will remove any dividend payments from her calculations of the risk-neutral return of the stock

Dividend paying stocks

Suppose the stock pays a continuous dividend yield of δ. The risk-neutral price of a European call on this stock is the same as the risk-neutral price of a non-dividend paying stock after updating the risk-neutral probabilities:

- If the stock pays a dividend, then the correct risk neutral probabilities are:

$$\tilde{p} = \frac{e^{(r-\delta)\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$

$$\tilde{q} = 1 - \tilde{p} = \frac{e^{u\Delta t} - e^{(r-\delta)\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$

- For no-arbitrage, we have to assume that $d < r - \delta < u$

Radon-Nikodym derivative

Consider a finite sample space Ω

Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$. Both give positive probability to every element of Ω

Define

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$$

Z is a random variable. It is called the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$

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Z is a random variable. It is called the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$

Proposition

(i) $\mathbb{P}(Z > 0) = 1$;

(ii) $\mathbb{E}^{\mathbb{P}}[Z] = 1$;

(iii) *For any random variable* Y, $\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY]$.

Proof

(i) follows from the assumption that $\tilde{\mathbb{P}}(\omega) > 0$ for every $\omega \in \Omega$.

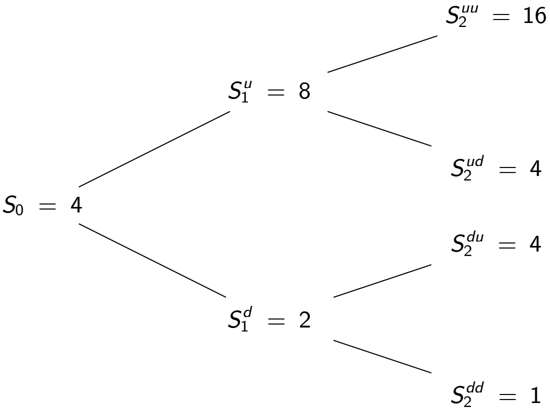
(ii) Can be verified by

$$\mathbb{E}^{\mathbb{P}}[Z] = \sum_{\omega} Z(\omega)\mathbb{P}(\omega) = \sum_{\omega} \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) = \sum_{\omega} \tilde{\mathbb{P}}(\omega) = 1.$$

(iii)

$$\begin{aligned}\mathbb{E}^{\tilde{\mathbb{P}}}[Y] &= \sum_{\omega} Y(\omega)\tilde{\mathbb{P}}(\omega) = \sum_{\omega} Y(\omega)\frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) \\ &= \sum_{\omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[ZY].\end{aligned}$$

Example



The sample space is

$$\Omega = \{uu, ud, du, dd\}.$$

Example

We take $p = \frac{2}{3}$. Then the physical probability measure is

$$\mathbb{P}(uu) = \frac{4}{9}, \quad \mathbb{P}(ud) = \mathbb{P}(du) = \frac{2}{9}, \quad \mathbb{P}(dd) = \frac{1}{9}.$$

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Take one period simple interest rate to be $r = \frac{1}{4}$. To find the risk neutral probability $\tilde{p}$, we need

$$S_0 = (1+r)^{-1}\Big(\tilde{p}S_1^u + (1-\tilde{p})S_1^d\Big).$$

We can solve for $\tilde{p} = \frac{1}{2}$. Similarly, the second period risk neutral problem is also $\tilde{p} = \frac{1}{2}$.

The risk neutral probability is

$$\tilde{\mathbb{P}}(uu) = \tilde{\mathbb{P}}(ud) = \tilde{\mathbb{P}}(du) = \tilde{\mathbb{P}}(dd) = \frac{1}{4}.$$

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The risk neutral probability is

$$\tilde{\mathbb{P}}(uu) = \tilde{\mathbb{P}}(ud) = \tilde{\mathbb{P}}(du) = \tilde{\mathbb{P}}(dd) = \frac{1}{4}.$$

Therefore, the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is

$$Z(uu) = \frac{9}{16}, \quad Z(ud) = Z(du) = \frac{9}{8}, \quad Z(dd) = \frac{9}{4}.$$

Z is the re-weighting factor between $\tilde{\mathbb{P}}$ and $\mathbb{P}$.
We can check that $\mathbb{E}^{\mathbb{P}}[Z] = 1$.

Example

Consider a lookback option with payoffs

$$V_2 = \max_{0 \leq n \leq 2} S_n - S_2.$$

The payoffs at $n = 2$ are

$$V_2(uu) = 0, \quad V_2(ud) = 4, \quad V_2(du) = 0, \quad V_2(dd) = 3.$$

Example

Consider a lookback option with payoffs

$$V_2 = \max_{0 \le n \le 2} S_n - S_2.$$

The payoffs at $n = 2$ are

$$V_2(uu) = 0, \quad V_2(ud) = 4, \quad V_2(du) = 0, \quad V_2(dd) = 3.$$

Then the option price at time 0 is

$$V_0 = (1+r)^{-2}\mathbb{E}^{\tilde{P}}[V_2] = \left(\frac{5}{4}\right)^{-2}\left(\frac{1}{4}V_2(ud) + \frac{1}{4}V_2(dd)\right) = 1.12.$$

It can also be obtained via

$$V_0 = (1+r)^{-2}\mathbb{E}^{\mathbb{P}}[ZV_2] = \left(\frac{5}{4}\right)^{-2}\left(\frac{2}{9}\frac{9}{8}V_2(ud) + \frac{1}{9}\frac{9}{4}V_2(dd)\right) = 1.12.$$

Radon-Nikodym derivative process

Consider a N-period binomial model

Z depends on N coin tosses in the model. To get related random variables that depend on fewer coin tossed, we can estimate Z based on the information at time $n < N$ via

$$Z_n = \mathbb{E}_n^{\mathbb{P}} Z, \quad n = 0, 1, \ldots, N.$$

Z_n is measurable with respect to $\mathcal{F}_n$.

Proposition

Z_n, $n = 0, 1, \ldots, N$, is a martingale under $\mathbb{P}$.

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Proposition

Z_n, $n = 0, 1, \ldots, N$, is a martingale under $\mathbb{P}$.

In the previous example,

$$\begin{aligned} Z_1(u) =& \mathbb{E}_1^{\mathbb{P}}[Z] = \frac{2}{3} Z(uu) + \frac{1}{3} Z(ud) = \frac{3}{4}, \\ Z_1(d) =& \mathbb{E}_1^{\mathbb{P}}[Z] = \frac{2}{3} Z(du) + \frac{1}{3} Z(dd) = \frac{3}{2}. \end{aligned}$$

Radon-Nikodym derivative process

Proposition

Let Y be a random variable measurable with respect to $\mathcal{F}_n$. Then

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[Z_n Y].$$

Radon-Nikodym derivative process

Proposition

Let Y be a random variable measurable with respect to $\mathcal{F}_n$. Then

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[Z_n Y].$$

Proof:

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}_n[ZY]] = \mathbb{E}^{\mathbb{P}}[Y\mathbb{E}^{\mathbb{P}}_n[Z]] = \mathbb{E}^{\mathbb{P}}[YZ_n].$$

Radon-Nikodym derivative process

Proposition

Let Y be a random variable measurable with respect to $\mathcal{F}_n$. Then

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[Z_n Y].$$

Proof:

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}_n^{\mathbb{P}}[ZY]] = \mathbb{E}^{\mathbb{P}}[Y\mathbb{E}_n^{\mathbb{P}}[Z]] = \mathbb{E}^{\mathbb{P}}[YZ_n].$$

Proposition

Let $n \leq m$ and Y be a random variable measurable with respect to $\mathcal{F}_m$. Then

$$\mathbb{E}_n^{\tilde{\mathbb{P}}}[Y] = \frac{1}{Z_n}\mathbb{E}_n^{\mathbb{P}}[Z_m Y].$$

Radon-Nikodym derivative process

Proposition

Let Y be a random variable measurable with respect to $\mathcal{F}_n$. Then

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[Z_n Y].$$

Proof:

$$\mathbb{E}^{\tilde{\mathbb{P}}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}_n^{\mathbb{P}}[ZY]] = \mathbb{E}^{\mathbb{P}}[Y\mathbb{E}_n^{\mathbb{P}}[Z]] = \mathbb{E}^{\mathbb{P}}[YZ_n].$$

Proposition

Let $n \leq m$ and Y be a random variable measurable with respect to $\mathcal{F}_m$. Then

$$\mathbb{E}_n^{\tilde{\mathbb{P}}}[Y] = \frac{1}{Z_n}\mathbb{E}_n^{\mathbb{P}}[Z_m Y].$$

Proof: Let $\omega_1, \ldots, \omega_n$ be given,

$$\mathbb{E}_n^{\tilde{\mathbb{P}}}[Y](\omega_1, \ldots, \omega_n) = \sum_{\omega_{n+1}, \ldots, \omega_m} Y(\omega_1, \ldots, \omega_m)\tilde{p}^{\#u(\omega_{n+1}, \ldots, \omega_m)}\tilde{q}^{\#d(\omega_{n+1}, \omega_m)}$$

$$= \left(\frac{p}{\tilde{p}}\right)^{\#u(\omega_1\ldots\omega_n)}\left(\frac{q}{\tilde{q}}\right)^{\#d(\omega_1\ldots\omega_n)} \sum_{\omega_{n+1}\cdots\omega_m}\Bigg[Y(\omega_1\ldots\omega_m)\left(\frac{\tilde{p}}{p}\right)^{\#u(\omega_1\ldots\omega_m)}\left(\frac{\tilde{q}}{q}\right)^{\#d(\omega_1\ldots\omega_m)}$$

$$\cdot\, p^{\#u(\omega_{n+1}\cdots\omega_m)}q^{\#d(\omega_{n+1}\cdots\omega_m)}\Bigg] = \frac{1}{Z_n(\omega_1\ldots\omega_n)}\mathbb{E}_n^{\mathbb{P}}[YZ_m](\omega_1\ldots\omega_n).$$

Summary

- The risk-neutral price of a European derivative with maturity T and payoff function $g(S_T)$ is

$$p = \tilde{\mathbb{E}}\left[e^{-rT} g(S_T)\right],$$

where $\tilde{\mathbb{E}}$ indicates that the expectation is computed with respect to the risk-neutral probability distribution of the stock price:

$$\mathbb{P}[S_T = e^{uT} S_0] = \tilde{p} = \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}}$$

$$\mathbb{P}[S_T = e^{dT} S_0] = 1 - \tilde{p} = \frac{e^{uT} - e^{rT}}{e^{uT} - e^{dT}}$$

- The risk-neutral price is also a no-arbitrage price

Summary

- In the multiperiod binomial model with time step Δt, a replicating portfolio for a European derivative with payoff $g(S_T)$ has initial value $v_0 = \pi$ and is recursively defined through:

$$\begin{aligned}
\pi_N &= g(S_N) \\
\pi_n &= e^{-r\Delta t}\left[\tilde{p}\pi_{n+1}(u) + (1-\tilde{p})\pi_{n+1}(d)\right] \quad (0 \le n < N) \\
v_{n+1} &= \Delta_n S_{n+1} + e^{r\Delta t}\left(v_n - S_n \Delta_n\right) \quad (0 < n < N) \\
\Delta_n &= \frac{\pi_{n+1}(u) - \pi_{n+1}(d)}{S_{n+1}(u) - S_{n+1}(d)} \quad (0 \le n < N)
\end{aligned}$$

- The replicating portfolio satisfies $v_n = \pi_n$ for all $0 \le n \le N$
- Δ_n can be interpreted as the sensitivity of the derivative price to changes in the stock price

Extension: Path-dependent options

The no-arbitrage pricing theorem also holds for European options whose payoff at maturity T does not only depend on the stock price S_T at time T, but also on all previous stock prices before time T:

$$c_N = f(S_n : 0 \leq n \leq N)$$

→ E.g.: Consider an Asian call option that pays off at maturity the positive part of the average difference between the stock price and the strike from time 0 to maturity:

$$c_N(\omega_1 \dots \omega_N) = \left(\frac{1}{N} \sum_{n=1}^{N} S_n(\omega_1 \dots \omega_n) - K \right)_+$$