MF702 PROBLEM SET 1: SOLUTIONS

1. Shell will need to buy $100,000$ barrels of oil in 10 days and it is worried about fuel costs. Suppose Shell goes long 100 oil futures contracts, each for $1,000$ barrels of oil, at the current futures price of \$60 per barrel. Suppose futures prices change each day as indicated in the following table

Table 1. Futures prices

Day	0	1	2	3	4	5	6	7	8	9	10
Prices	60	59.50	57.50	57.75	58.00	59.50	60.50	60.75	69.75	61.75	62.50

   (a) What is the mark-to-market profit or loss (in dollars) that Shell will have on each date?
   (b) What is Shell's total profit or loss after 10 days?

Solution: The total cumulative P& L is \$$250,000$.

Day	Price	mark to market P& L	Cumulative P& L
0	60.00		
1	59.50	(50,000)	(50,000)
2	57.50	(200,000)	(250,000)
3	57.75	25,000	(225,000)
4	58.00	25,000	(200,000)
5	59.50	150,000	(50,000)
6	60.50	100,000	50,000
7	60.75	25,000	75,000
8	59.75	(100,000)	(25,000)
9	61.75	200,000	175,000
10	62.50	75,000	250,000

2. Options on futures are exchange-traded instruments that are often written on a variety of indices, such as the S& P 500 index. An option on a futures gives you the right to enter a futures contract at maturity for a predetermined future price. This works as follows: The maturity of the option is T', the strike is K, and the call (put) option gives the holder of option the right to enter a long (short) position on a futures at time T' with maturity T with $T \geq T'$. Options on futures are typically American style. Suppose that $C_t(F)$ is the

price of the call option at time t if the futures price is F, while $P_t(F)$ is the price of the put option. Assume also that $(F_{t,T})_{0\leq t\leq T}$ is the process that describes the futures prices with maturity T.

For this problem, assume that the options are European instead of American, assume, moreover, we don't take into account the mark to margin practice for the future (i.e. treat the future as a forward.) Later we will revisit the pricing of futures options after we learn the risk neutral pricing.

(a) Assume that $T' \leq T$, derive the put-call parity for options on a futures.
(b) Assume that $T' < T$, is your put-call parity the same as (a)?

Hint: When the holder of the European call on a futures exercises at time T', the holder get a futures which allows the holder to buy an asset with price K at time T. The holder will only exercise this European call when $F_{T',T}$ is larger than K, because the option allows the holder to buy an assert with price K at time T, which is cheaper than entering a long position of a futures at time T' with the futures price $F_{T',T}$. Calculate the payoff of this European call at time T. For part (b), think about a portfolio A which has a long position of call option, a portfolio B which has a long position of a put option, a long position of a futures entered at time 0 and a short position of a futures entered at T'.

Solution: We will only solve (b), (a) is a special case. Consider two portfolios

- Portfolio A: a European call futures option plus an amount of cash equal to Ke^{-rT}
- Portfolio B: a European put futures option, plus a long future contract at time 0, a short future contract at time T', plus an amount of cash equal to $F_{0,T}e^{-rT}$, where $F_{0,T}$ is the futures price. (Remember that this is the delivery price at time T. It is costless to enter a futures.)

In portfolio A, the cash can be invested at the risk-free rate, r, and grows to K at time T. Let $F_{T',T}$ be the futures price at maturity of the option. If $F_{T',T} \geq K$, the holder can exercise the European call option to get a futures with the futures price K. At the same time, the holder can enter a short position of a futures with the futures price $F_{T',T}$ at time T'. Then, at time T, the agent uses the long position of the futures buy an asset with price K and sells it with the price $F_{T',T}$. This generate a payoff $F_{T',T} - K$ from the call option at time T. If $F_{T',T} < K$, the holder will not exercise the option. Therefore the value of portfolio A at time T is

$$\max\{F_{T',T}, K\}.$$

In portfolio B, the cash can be invested at the risk-free rate to grow to $F_{0,T}$ at time T. The put option provides a payoff of $\max\{K - F_{T',T}, 0\}$ at time T, similar to the call option argument. Entering a long futures at time 0 allows the holder to buy an asset at time T with price $F_{0,T}$. Entering a short futures at time T', allows the holder to sell an asset at time T with price $F_{T',T}$. Therefore the P& L for the futures position is $F_{T',T} - F_{0,T}$ at time T. The value of portfolio B at time T is therefore

$$F_{0,T} + (F_{T',T} - F_{0,T}) + \max\{K - F_{T',T}, 0\} = \max\{F_{T',T}, K\}.$$

Payoffs of these two portfolios are the same at time T'. Therefore their prices at time 0 must be the same, i.e.,

$$C_0(F_{0,T}) + Ke^{-rT} = P_0(F_{0,T}) + F_{0,T}e^{-rT}.$$

Remember that entering futures are costless.

3. The current price of a stock is \$94 and a 3-months European call with a strike price of \$95 currently sell for \$4.70. An investor who feels that the price of the stock will increase is trying to decide between buying 100 shares and buying $2,000$ options ($=$ 20 contracts). Both strategies involve an investment of \$$9,400$. What advice would you give? How high does the stock price need to rise for the option strategy to be more profitable? Assume that the interest rate is zero.

   Solution: If the investor believes that the stock price will rise by a large amount, then it will be valuable to buy the option instead of buing the stock directly. In order for the option to be more profitable, the price will have to rise to

   $$2000 \times (S_T - 95) - 2000 \times 4.70 \geq 100 \times (S_T - 94).$$

   This means that $S_T \geq 100$.

4. Suppose a corporate treasurer asks you to construct the following derivative: "I will have 1 million GBP to sell in 6 months. If the exchange rate less than 1.25, I want you to give me 1.25. If it is greater than 1.35, I will accept 1.35. If the exchange rate is between 1.25 and 1.35, I will sell the British Pound for the exchange rate." Plot the payoff of this derivative as a function of USD/GBP exchange rate in 6 months. How could you use options to construct the derivative?

   Solution: The derivative can be constructed by entering a European put with strike 1.25 and a short position of European call with strike 1.35, both with 6 months maturity.

5. On September 8, 2020, an investor owns 100 Alphabet shares. The share price on that day was about \$$1,537$ and a December European put option with a strike price of \$$1,500$ costs \$27.30. The investor is comparing two alternatives to limit downside risk. The first involves buying one December European put option contract with a strike price of \$$1,500$. The second involves instructing a broker to see the 100 shares as soon as Alphabet's price reaches $1,500$. Discuss the advantages and disadvantages of the two strategies.

Solution: Buying the put option gives the investor the right to sell the asset at a price of $\$1,500$ in December. Because the option is European, the investor has to wait until December to realize his profits from the option if the price drops. As a result, the investor may be sitting on the losses if the stock loses value before December. Also the investor needs to be able to pay upfront the fee of $100 \times \$27.30 = \$2,730$ to buy put options.

Instructing a broker to sell the stock as soon as the price falls below $\$1,500$ may help the investor to cut the downside risk of a falling stock price before December. However, depending on how fast the broker fills the oder, the price sold may be significantly lower than $\$1,500$ (think about a market crash and everyone wants to sell).