MF702 PROBLEM SET 2: SOLUTIONS

1. Consider a one-period binomial model in which the stock price at time 1 is either $S_1^u = uS_0$ or $S_1^d = dS_0$ with two constants u, d. Show that

$$0 < d < 1 + r < u$$

precludes arbitrage. Here r is one-period interest rate, i.e., \$1 deposit in the bank becomes $1 + r$ after one period. Let X be the value of a portfolio of cash and stocks. For this problem, we need to show that if $X_0 = 0$ and

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0),$$

we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, and this is the case regardless of the choice of the number Δ_0.

Solution: Set $X_0 = 0$. Then

$$X_1 = \Delta_0(S_1 - (1 + r)S_0) = \begin{cases} \Delta_0(u - (1 + r))S_0 & S_1 = uS_0 \\ \Delta_0(d - (1 + r))S_0 & S_1 = dS_0 \end{cases}$$

When $0 < d < 1 + r < u$, X_1 has the same sign as Δ_0 when $S_1 = uS_0$, or the opposite sign as Δ_0 when $S_1 = dS_0$. No matter whether Δ_0 is positive or negative, X_1 cannot be always positive or negative when both events $\{S_1 = uS_0\}$ and $\{S_1 = dS_0\}$ had positive probabilities.

2. Consider a one-period binomial model as in Problem 1 with $S_0 = 4$, $u = 2, d = \frac{1}{2}$, and $r = \frac{1}{4}$. Consider a European call option with strike price $K = 5$ and maturity at time 1.
   (a) Show that the time zero arbitrage-free price of this European call option is 1.20.
   (b) Consider an agent who begins with wealth $X_0 = 0$ and at time zero buys Δ_0 shares of stock and Γ_0 options. The numbers Δ_0 and Γ_0 can be either positive or negative or zero. This leaves the agent with a cash position $-4\Delta_0 - 1.20\Gamma_0$. If this is positive, it is invested in the money market; if it is negative, it represents money borrowed from the money market. At time one, the value of the agent's portfolio of stock, option, and money market assets is

$$X_1 = \Delta_0 S_1 + \Gamma_0(S_1 - 5)_+ - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0).$$

Assume that both S_1^u and S_1^d have positive probability of occurring. Show that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. In other words, one cannot find an arbitrage when the time-zero price of the option is 1.20.

Solution: (a) The call option payoff is 3 when $S_1 = 8$ or 0 when $S_1 = 2$. Consider a replicating portfolio with Δ shares of stock and W cash at time 0. Then

$$\Delta 8 + (1 + \frac{1}{4})W = 3$$
$$\Delta 2 + (1 + \frac{1}{4})W = 0.$$

The solution is $\Delta = \frac{1}{2}$ and $W = -\frac{4}{5}$. The arbitrage-free price of the call option is

$$V_0 = 4 \times \frac{1}{2} - \frac{4}{5} = 1.2.$$

(b) The value of the portfolio at time 1 is

$$X_1 = \begin{cases} 8\Delta_0 + 3\Gamma_0 - \frac{5}{4}(4\Delta_0 + 1.2\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0 & S_1 = 8 \\ 2\Delta_0 - \frac{5}{4}(4\Delta_0 + 1.2\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0 & S_1 = 2 \end{cases}$$

X_1 has opposite sign when $S_1 = 8$ and $S_1 = 2$. Therefore, X_1 cannot be always positive or negative when events $\{S_1 = 8\}$ and $\{S_1 = 2\}$ happen will positive probabilities.

3. Consider a one-period binomial model in which the stock has an initial price of \$100 and one period log return can be either 15% or −5% with equal probabilities. The price of the European call option on this stock with strike price \$115 and maturity after one period is \$0.42.
   (a) What should be the price of a zero-coupon bond that pays \$1 after one period?
   (b) What are the risk-neutral probabilities in this model?
   (c) Now consider a *defaultable* bond that pay off \$1 if the stock price goes up, but only pays off \$0.20 if the stock price goes down. What should be the arbitrage-free price of this security?

Solution: (a) We know the price of the option is given by

$$c = e^{-rT}\Big[\tilde{p}(S_1^u - K)_+ + (1-\tilde{p})(S_1^d - K)_+\Big].$$

We know that $c = 0.42$, $S_1^u = S_0 e^{0.15} = 116.18$, $S_1^d = S_0 e^{-0.05} = 95.12$, and

$$\tilde{p} = \frac{e^r - e^{-0.05}}{e^{0.15} - e^{0.05}} = \frac{e^r - 0.95}{0.21}.$$

We obtain

$$0.42 = e^{-r}\Big[\frac{e^r - 0.95}{0.21}(116.18 - 115)_+ + \frac{1.16 - e^r}{0.21}(95.12 - 115)_+\Big].$$

Solving this equation, we obtain $r \approx 0.0264$. Therefore the price of zero-coupon bond with face value \$1 is $e^{-r} = 0.97$.

(b) The risk-neutral probability is

$$\tilde{p} = \frac{e^r - e^{-0.05}}{e^{1.15} - e^{-0.05}} = 0.3586.$$

(c) From the risk-neutral pricing formula, we obtain

$$\text{Price} = \tilde{\mathbb{E}}[e^{-r}\text{Payoff}] = e^{-0.0264}(1 \times 0.3586 + 0.2 \times (1 - 03586)) = 0.4742.$$

4. Consider a European put option on a non-dividend paying stock where the stock price is \$40, the strike price is \$40, the continuous compounding interest rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months.
   (a) Suppose that the log returns of stock in the first 6 months and the last 6 months of a year are i.i.d. Find the volatility of stock return in 6 months.
   (b) Using the result in (a), calculate u and d according to the Cox-Ross-Rubinstein rule for an one-period (6 months) binomial model.
   (c) Calculate the risk-neutral probability $\tilde{p}$ for a jump up in the stock price in the stock price over a period of 6 months.
   (d) write a computer code that computes the put option price using an N-period model. Plot your option price as a function of N for $N \in [1, 500]$. (Hint: For the N-period model, you need to recalculate the volatility in a period of length $1/N$, then determine the associated u_N and d_N, and recompute $\tilde{p}_N$ using the new u_N and d_N.)
   (e) What value do your computed prices converge to?

Solution: The annual volatility is 0.3. The independence and identical distribution of the returns from $t = 0$ to $t = 1/2$ and from $t = 1/2$ to $t = 1$ implies that

$$\begin{aligned} 0.3 =& \sqrt{\text{Var} \ln \left(\frac{S_1}{S_0}\right)} = \sqrt{\text{Var} \ln \left(\frac{S_1}{S_{1/2}} \frac{S_{1/2}}{S_0}\right)} = \sqrt{\text{Var}\left[\ln \frac{S_1}{S_{1/2}} + \ln \frac{S_{1/2}}{S_0}\right]} \\ =& \sqrt{\text{Var} \ln \frac{S_1}{S_{1/2}} + \text{Var} \ln \frac{S_{1/2}}{S_0}} = \sqrt{2\text{Var} \ln \frac{S_{1/2}}{S_0}}. \end{aligned}$$

Therefore, the volatility of the log return from time $t = 0$ to time $t = 1/2$ is $\frac{0.3}{\sqrt{2}} = 0.21$.

(b) Using the Cox, Ross, Rubinstein rule, we obtain $u = 0.21$ and $d = -0.21$.

(c) The risk-neutral probability of an up-movement is

$$\tilde{p} = \frac{e^{rT} - e^{-0.21}}{e^{0.21} - e^{-0.21}} = \frac{e^{0.04 \times 0.5} - e^{-0.21}}{e^{0.21} - e^{-0.21}} = 0.495.$$

(d) The risk-neutral pricing formula tells us that

$$\begin{aligned} p =& e^{-rT}\Big[\tilde{p}(K-S_T^u)_+ + (1-\tilde{p})(K-S_T^u)_+\Big] \\ =& e^{-0.04\times 0.5}\Big[0.495(40-40e^{021})_+ + 0.505(40-40e^{-0.21})_+\Big] \\ =& 3.75. \end{aligned}$$

(e) For 5 periods we obtain a price of \$3.13, for 50 periods we obtain \$2.95, for 100 periods \$2.96, and for 500 periods also \$2.96.

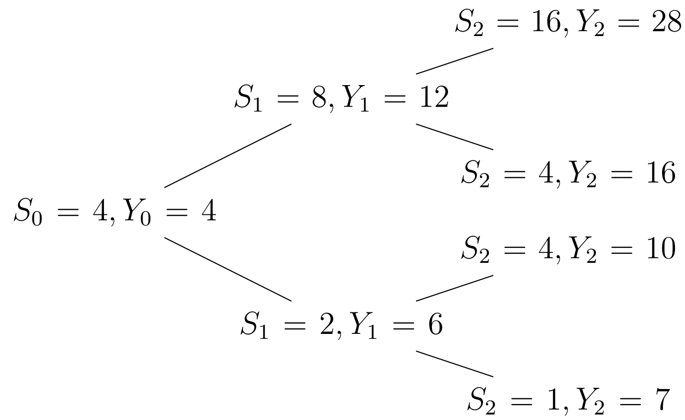
5. Consider a two-period binomial tree model with $S_0 = 4$, $u = 2, d = \frac{1}{2}$, (i.e., $S_1^u = uS_0$ and $S_1^d = dS_0$) and take the one-period simple interest rate $r = \frac{1}{4}$, so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. For $n = 0, 1, 2$, define $Y_n = \sum_{k=0}^n S_k$ to be the sum of the stock prices between times zero and n. Consider an *Asian call option* that expires at time two and has strike $K = 4$ (i.e., whose payoff at time two is $(\frac{1}{3}Y_2 - 4)_+$.) This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_2(s, y) = (\frac{1}{3}y - 4)_+$.
   (a) Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.
   (b) Apply the recursive formula developed in (a) to compute $v_0(4, 4)$, the price of the Asian option at time zero.
   (c) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.

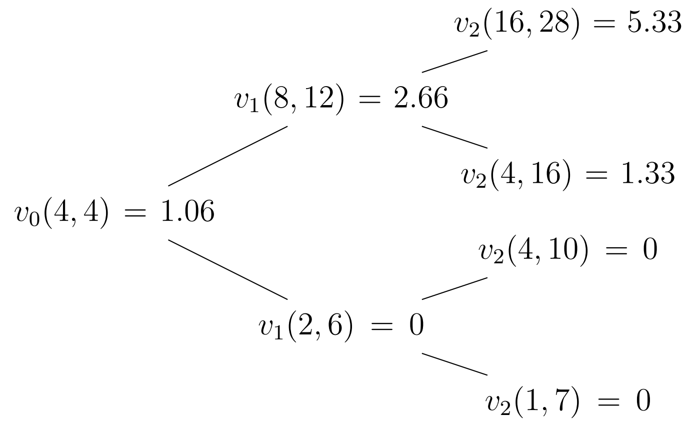
Solution: (a) Starting from a note (s, y) on the binomial tree, it either goes to $(2s, y+2s)$ when the stock prices goes up, or $(\frac{1}{2}s, y + \frac{1}{2}s)$. Therefore

$$v_n(s,y) = (1+\frac{1}{4})^{-1}\Big(\tilde{p}v_{n+1}(2s, y+2s) + (1-\tilde{p})v_{n+1}(\frac{1}{2}s, y+\frac{1}{2}s)\Big), \quad n = 0, 1,$$

and $v_2(s, y) = (\frac{1}{3}y - 4)_+$.

(b) The binomial tree for (s, y) is





(c) At time 0 Invest $\delta_0(4,4)$ shares in stock, and $v_0(4,4) - \delta_0(4,4) \times 4$ in cash. At time 1, if the stock price goes up, we need

$$8\delta_0(4,4) + \frac{5}{4}(1.06 - 4\delta_0(4,4)) = 2.66.$$

The solution is $\delta_0(4,4) = 0.445$. When the stock price goes down, we can verify

$$2 \times 0.445 + \frac{5}{4}(1.06 - 4 \times 0.445) = 0.$$

At time 1 note $(8,12)$, invest $\delta_1(8,12)$ shares in stock and $v_1(8,12) - \delta_1(8,12) \times 8$ in cash. At time 2, if the stock price goes up, we need

$$16\delta_1(8,12) + \frac{5}{4}(2.66 - 8\delta_1(8,12)) = 5.33.$$

The solution is $\delta_1(8,12) = 0.334$. When the stock price goes down, we can check

$$4 \times 0.334 + \frac{5}{4}(2.66 - 8 \times 0.334) = 1.33.$$

At time 1 note $(2,6)$, $\delta_1(2,6) = 0$.

6. Suppose that a stock is currently valued at $S_0 = 100$ and has annual volatility of σ_0. Going forward, you predict that volatility in one year from today will be $\sigma_1 = \sigma^u > \sigma_0$

with probability p, or $\sigma_1 = \sigma^d < \sigma_0$ with probability $1 - p$ for some $p \in (0, 1)$. Figure 1. shows a binomial tree model.

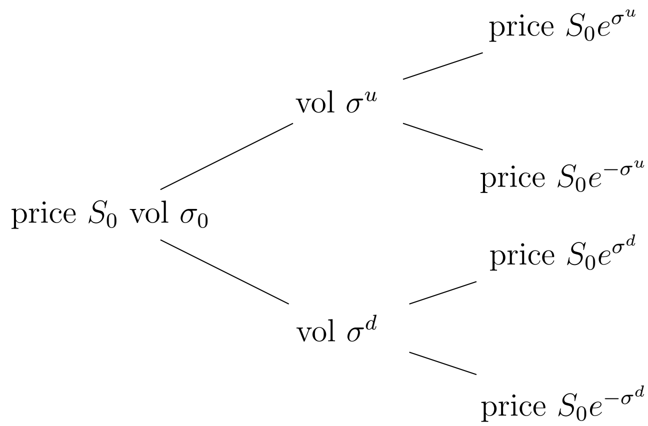


Figure 1. Binomial tree with variable volatility

(a) Consider a probability measure $\tilde{\mathbb{P}}$ with

$$\tilde{\mathbb{P}}(S_1 > S_0) = q \quad \text{and} \quad \tilde{\mathbb{P}}(S_1 \leq S_0) = 1 - q.$$

Show that there exist multiple such measures $\tilde{\mathbb{P}}$ such that the discounted expected period-1 stock price under $\tilde{\mathbb{P}}$ is equal to the initial price S_0. That is, there are multiple $\tilde{\mathbb{P}}$ such that $\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-r} S_1] = S_0$.
Hint: Define $q_\sigma = \tilde{\mathbb{P}}(\sigma_1 = \sigma^u)$, $q^u = \tilde{\mathbb{P}}(S_1 > S_0 | \sigma_1 = \sigma^u)$ and $q^d = \tilde{\mathbb{P}}(S_1 > S_0 | \sigma_1 = \sigma^d)$. Write down the equation that S_1 and S_0 should satisfy under the risk neutral measure $\tilde{\mathbb{P}}$. Does this equation has a unique solution?

(b) Show that this market is not complete.
Hint: Count the number of different states at time 1. Does the number of different states match the degree of freedom of a replicating portfolio at time 1? Think about how many degree of freedom (unknown variables) in a replicating portfolio.

Solution: (a) The risk-neutral probabilities are computed such that the discounted expectation of the stock price is equal to the initial stock price. Therefore,

$$\begin{aligned} S_0 =& \tilde{\mathbb{E}}[e^{-r} S_1] = e^{-r} \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}[S_1 | \sigma_1]\Big] \\ =& e^{-r}\Big[q_\sigma (S_0 e^{\sigma^u} q^u + S_0 e^{-\sigma^u}(1 - q^u)) + (1 - q_\sigma)(S_0 e^{\sigma^d} q^d + S_0 e^{-\sigma^d(1-q^d)})\Big]. \end{aligned}$$

This equation has three unknown (q_σ, q^u, q^d), so it cannot be solved uniquely.

(b) At maturity there are four possible states of the world: $(\sigma_1 = u, S_1 = S_0 e^{\sigma^u})$, $(\sigma_1 = u, S_1 = S_0 e^{-\sigma^u})$, $(\sigma_1 = d, S_1 = S_0 e^{\sigma^d})$, and $(\sigma_1 = d, S_1 = S_0 e^{-\sigma^d})$. However, when constructing a replicating portfolio at time 0, we can only choose the amount of cash and the amount of stocks held in the replicating portfolio. Therefore, we have two degrees of freedom and 4 equations to match. Unless $\sigma^u = \sigma^d$, it is not possible to solve this system, so that a replicating portfolio cannot be constructed.