(1) (a)

$S_0 = 50$

$S_1^u = 55 \to S_1^{u'} = 53$

$S_1^d = 47$

$S_2^{uu} = 57$

$S_2^{ud} = 48$

$S_2^{dd} = 41$

Period 1. $50 = \check{p} \cdot (1+r)^{-1} \cdot 55 + (1-\check{p}) \cdot (1+r)^{-1} \cdot 47$

$50 = \check{p} \cdot 55/1.02 + (1-\check{p}) \cdot 47/1.02$

$\check{p} = 0.5 \qquad 1-\check{p} = 0.5$

Period 2. $53 = \check{p} \cdot (1+r)^{-1} \cdot 57 + (1-\check{p}) \cdot (1+r)^{-1} \cdot 48$

$S_1^u \to S_2^{uu} \quad 53 = \check{p} \cdot 57/1.02 + (1-\check{p}) \cdot 48/1.02$

$\check{p} = 0.67 \qquad 1-\check{p} = 0.33$

$47 = \check{p} \cdot (1+r)^{-1} \cdot 48 + (1-\check{p}) \cdot (1+r)^{-1} \cdot 41$

$S_1^d \to S_2^{ud} \quad 47 = \check{p} \cdot 48/1.02 + (1-\check{p}) \cdot 41/1.02$

$\check{p} = 0.99 \qquad 1-\check{p} = 0.01$

(b), (C)

$g(S_2^{uu}) = 57-45 = 12$

$\check{E}_1[\frac{1}{1.02}\Pi_2] = 0.67 \times \frac{1}{1.02} \times 12 + 0.33 \times \frac{1}{1.02} \times 3 = 8.85$

$g(S_1^u) = 55-45+2 = 12$

$\Pi_1 = \max(8.85, 12) = 12$ exercise.

$g(S_0) = 50-45 = 5$

$g(S_2^{ud}) = 48-45 = 3$ exercise

$E[\frac{1}{1.02}\Pi_1] = 0.5 \times \frac{1}{1.02} \times (12+2.91) = 7.31$

$\Pi = \max(7.31, 5) = 7.31$

Don't exercise.

$g(S_1^d) = 47-45 = 2$

$g(S_2^{dd}) = (41-45)_+ = 0$ Don't exercise

$\check{E}_1[\frac{1}{1.02}\Pi_2] = 0.99 \times \frac{1}{1.02} \times 3 = 2.91$

$\Pi_1 = \max(2, 2.91) = 2.91$ Don't exercise.

(2)

(a)

$S_0 = 100$

$S_1^u = 100e^{\frac{0.15}{2}} = 107.79$

$S_1^d = 100e^{\frac{-0.10}{2}} = 95.12$

$S_2^{uu} = 100e^{0.15} = 116.18$

$S_2^{ud} = S_2^{du} = 100e^{\frac{0.15-0.1}{2}} = 102.53$

$S_2^{dd} = 100e^{-0.1} = 90.48$

means can be exercised at that price.

(b)

$$\tilde{p} = \frac{e^{\frac{0.02}{2}} - e^{-\frac{0.1}{2}}}{e^{\frac{0.15}{2}} - e^{-\frac{0.1}{2}}} = 0.46 \qquad 1-\tilde{p} = 0.54$$

$g(S_0) = (95-100)_+ = 0$

$E[e^{-rst}\Pi_1] = 2.41 \times 0.54 e^{-\frac{0.02}{2}}$

$= 1.29$

$\Pi = \max(0, 1.29) = 1.29$

$\Pi_1 = 0$

$\Pi_1 = 4.52(1-\tilde{p})e^{-\frac{0.02}{2}}$

$= 4.52 \times 0.54 e^{-\frac{0.02}{2}}$

$= 2.41$

$\Pi_2 = (95-116.18)_+ = 0$

$\Pi_2 = (95-102.53)_+ = 0$

$\Pi_2 = (95-90.48)_+ = 4.52$

(c) We should only exercise when $S^{uu}$.

(3). Need to prove $(S_n - K)_+ \le \tilde{E}[e^{-r(m-n)}(S_m - K)_+]$

By Jensen's inequality, if $f$ is convex, $f(E[X]) \le E[f(x)]$

Let $f: x \to (x-K)_+$, $f$ is convex.

Since $\{S_t e^{-rt}\}_{t\ge0}$ is a martingale, $E[S_T e^{-rT} | \mathcal{F}_t] = S_t e^{-rt}$

$E[f(S_t e^{-rt})] = E[f(E[S_T e^{-rT}|\mathcal{F}_t])] \le E[E[f(S_T e^{-rT})|\mathcal{F}_t]] = E[f(S_T e^{-rT})]$

Payoff at $t = (S_t - K)_+$

$\tilde{E}[e^{-rt}(S_t - K)_+] \le \tilde{E}(e^{-rt}S_t - e^{-rT}K)_+$

$E[(S_t e^{-rt} - K)_+] \le E[(S_T e^{-rT} - K)_+]$

$E[(S_t e^{-rt} - Ke^{-rt})_+] \le E[(S_T e^{-rT} - Ke^{-rT})_+]$

$(S_t - K)e^{-rn} \le E[(S_m - K)_+ e^{-rm}]$ $\quad 0 \le n \le m \le T$

$(S_t - K) \le E[(S_m - K)_+ e^{-r(m-n)}]$

(4).

(a)

$$
\begin{tikzcd}
 & & C_2^{uu}=100e^{2u} \\
 & C_1^u = 100e^u \arrow[ur] \arrow[dr] & \\
C_0=100 \arrow[ur] \arrow[dr] & & C_2^{ud}=C_2^{du}=100\,e^{u+d} \\
 & C_1^d=100e^d \arrow[ur] \arrow[dr] & \\
 & & C_2^{dd}=100e^{2d}
\end{tikzcd}
$$

(b)

Payoff

$$
\begin{tikzcd}[column sep=large]
 & & \text{Debt} = 100 \\
 & \begin{array}{l} g_1(C_1^u)=100e^u \\ E_1=(\breve{p}100+(1-\breve{p})100)e^{-r} \\ \quad =100e^{-r} \\ \Pi_1=\max(100e^u,100e^{-r})=100e^u \end{array} \arrow[ur, "\breve{p}"] \arrow[dr, "1-\breve{p}"] & \\
\Pi_0=e^{-r}\left(100e^u\breve{p}+\max(g_1(C_1^d),E_1)(1-\breve{p})\right) \arrow[ur, "\breve{p}"] \arrow[dr, "1-\breve{p}"'] & & \text{Debt} = 100 \\
 & \begin{array}{l} g_1(C_1^d)=100e^d \\ E_1=(\breve{p}100+(1-\breve{p})100e^{2d})e^{-r} \\ \Pi_1=\max(g_1(C_1^d),E_1) \end{array} \arrow[ur, "\breve{p}"] \arrow[dr, "1-\breve{p}"] & \\
 & & \text{Debt} = 100e^{2d}
\end{tikzcd}
$$