MF702 Midterm Exam

October 28, 2020

Please write your answers on separate paper. Write down clearly the question number for each solution. Mark your submission with a page number in each page. Scan or take pictures of your solutions and update them to the assignment session on Questromtools.

1. (35 points)

(a) (7 points) What are three major differences between forward and futures?

(b) (7 points) What are the main assumptions for Black-Scholes-Merton pricing formula?

(c) (7 points) Construct a long position of a forward with the forward price E with two options, each with strike price E.

(d) (7 points) In early March 2020 when COVID-19 started to spread globally, volatility skew for equity indices became more steep (i.e., the slop of implied volatility with respect to moneyness because more negative). Give an explanation of this phenomenon.

(e) (7 points) Consider a geometric Brownian motion

$$dX_t = \alpha X_t dt + \sigma X_t dZ_t,$$

where Z is a 1-dimensional Wiener process. Use Itô's formula to calculate the dynamics of X^p for some constant $p > 0$.

2. (25 points) Consider a 3-month *up-and-in* European call barrier option. Consider a two-period binomial model for this option. The annual log return for the up-side stock price is $u = 0.1$ and the annual log return for the down-side stock price is $d = -0.05$. This means that the stock price either becomes $S_i e^{u \times 1.5/12}$ or $S_i e^{d \times 1.5/12}$ after one period, for $i = 0$ or 1. The annual continuously compounding interest rate is $r = 0.02$.
At the end of the second period, the payoff of this barrier option is

$$V_2 = \begin{cases} (S_2 - K)_+ & \text{if } M_2 \geq B \\ 0 & \text{otherwise} \end{cases},$$

where K is the strike price, B is the *barrier* level, and $M_2 = \max_{0 \le i \le 2} S_i$. In other words, if M_2 is larger than the barrier level B, the option is knocked in and the payoff at maturity is the standard European call; if M_2 does not reach B, the option is worthless.
Assume $B = 101$ and $K = 100$ for this problem.

(a) (4 points) Draw a binomial tree model and write down stock prices on each node of the tree. Calculate the risk-neutral probabilities for this binomial tree model.

(b) (8 points) Use the risk-neutral pricing method, find the risk-neutral price of this European call barrier option at time 0.

(c) (10 points) If the seller of this option construct a replication strategy for this option, how many shares of stock the seller needs to buy/sell at time 0? Verify the replication strategy indeed replicates the barrier option payoff at the end of second period.

(d) (3 points) If the barrier level B increases, will the option price for the European call barrier option increase or decrease? Please explain your reason.

3. (15 points) Let $T_c < T$ and $K > 0$ be given, a *chooser option* is a contract sold at time zero that confers on its owner the right to receive either a European call or a put time time T_c. The owner of the chooser option may wait until time T_c before choosing. The European call or put chosen expires at time T with strike price K.

(a) (3 points) Let c_t and p_t be the European call and put prices at time t. Write down the value of the chooser option at time T_c.

(b) (12 points) Let r be the annual continuously compounding interest rate. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time T and having strike price K, and a call, expiring at time T_c and having strike price $Ke^{-r(T-T_c)}$. (Hint: use put-call parity at time T_c.)

4. (25 points) Consider a financial market in which investors can trade a risky stock with price S_0 at time 0 and borrow or lend at the risk-free rate $r > 0$ (one-period simple interest rate). Consider an one-period market model with three states: $\Omega = \{u, m, d\}$ with $S_1^u > S_1^m > S_1^d > 0$ and probabilities $p_u = \mathbb{P}[S_1 = S_1^u] > 0$, $p_m = \mathbb{P}[S_1 = S_1^m] > 0$, and $p_d = \mathbb{P}[S_1 = S_1^d] = 1 - p_u - p_m > 0$.

(a) (5 points) Is the market complete?

(b) (8 points) Show that the risk-neutral probability measure is not uniquely determined. What is the maximum possible range of the risk-neutral probability $\tilde{p}_u = \widetilde{\mathbb{P}}[S_1 = S_1(u)]$?

(c) (6 points) Consider a European call option with maturity at time 1 and strike price $K = S_1^m$. Use your answers in part (b) to find the range of possible prices for this European call option at time zero.

(d) (6 points) Suppose that the European call option considered in (c) is traded liquidly in the market with the price c, which is in the range of the risk-neutral prices you found in (c). Is the risk-neutral measure uniquely determined?