Stochastic Methods in Asset Pricing

The MIT Press (2017)

# European-Style Contingent Claims

(sections 14.1, 14.2)

Andrew Lyasoff

# Summary

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Résumé of Maliavin Calculus

Basic Definitions and Terminology

Hedging

The Price of an European-Style Contingent Claim

Replication of European-Style Contingent Claims

The Model of F. Black, M. Scholes, and R. C. Merton

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Let $\Phi = F(W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$, be a cylindrical function of $W$,

for some choice of the subdivision $0 = t_0 < t_1 < \ldots < t_n = T$ and the function $F \in \mathscr{C}_b^1(\mathbb{R}^n; \mathbb{R})$.

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Exercise: If $f(t) = \sum_{i=1}^{n} a_i\, 1_{]t_{i-1},t_i]}(t)$, $a_i \in \mathbb{R}$, then $\mathsf{E}\Big[\Phi \int_0^T f(t)\,\mathrm{d}W_t\Big] = \mathsf{E}\Big[\int_0^T f(s)\, D\Phi_s\,\mathrm{d}s\Big]$.

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Conclude that if $\psi = \sum_{i=1}^{n} \xi_{i-1} \, 1_{]\!]t_{i-1}, t_i]\!]}$, for $\xi_{i-1} \in L^2(\Omega, \mathscr{F}_{t_{i-1}}^W, \mathsf{P})$, $1 \le i \le n$,

then
$$\mathsf{E}\left[\left(\Phi - \mathsf{E}[\Phi]\right) \int_0^T \psi_s \,\mathrm{d}W_s\right] = \mathsf{E}\left[\int_0^T \psi_s \mathsf{E}\left[D\Phi_s \,|\, \mathscr{F}_s^W\right] \mathrm{d}s\right].$$

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Let $\mathbb{D}^2_1 \stackrel{\text{def}}{=}$ the completion of the set of all cylindrical r.v. $\Phi$ with respect to the norm $\|\cdot\|$.

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Let $\mathbb{D}_1^2 \stackrel{\text{def}}{=}$ the completion of the set of all cylindrical r.v. $\boldsymbol{\Phi}$ with respect to the norm $\|\cdot\|$.

We then have

**Clark-Ocone formula:**

$$\boldsymbol{\Phi} - \mathsf{E}[\boldsymbol{\Phi}] = \int_0^T \mathsf{E}\big[D\boldsymbol{\Phi}_s \,|\, \mathscr{F}_s^W\big]\,\mathrm{d}W_s, \quad \text{for any } \boldsymbol{\Phi} \in \mathbb{D}_1^2.$$

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which follows trivially from Itô’s formula!

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Exercise: In an arbitrage-free market one must have $\Pi^- \leq \Pi \leq \Pi^+$, $\quad \Pi \stackrel{\text{def}}{=}$ price of $\mathscr{K}_0(T, \boldsymbol{\Phi}, \varphi)$.

*What is $\Pi$, then?* May be $\Pi = M_0 \equiv \mathsf{E}^Q\Big[(S^\circ_T)^{-1}\boldsymbol{\Phi} + \int_0^T (S^\circ_u)^{-1}\varphi_u \,\mathrm{d}u\Big]$, $\quad t \in [0,T]$? May be …

Assume that $\mathsf{E}^Q\big[(S^\circ_T)^{-1}\boldsymbol{\Phi}\big] < \infty$ and $\mathsf{E}^Q\Big[\int_0^T (S^\circ_u)^{-1}\varphi_u \,\mathrm{d}u\Big] < \infty$,

and define the martingale: $M_t \stackrel{\text{def}}{=} \mathsf{E}^Q\Big[(S^\circ_T)^{-1}\boldsymbol{\Phi} + \int_0^T (S^\circ_u)^{-1}\varphi_u \,\mathrm{d}u \,\Big|\, \mathscr{G}_t\Big]$, $\quad t \in [0,T]$.

From the PRP: $M_t - M_0 = \int_0^t h_s{}^\intercal \,\mathrm{d}\beta_s$, $\quad t \in [0,T]$, for some predictable $h$.

Define $\pi^\pm_t \stackrel{\text{def}}{=} \pm(\sigma_t{}^\intercal)^{-1} S^\circ_t\, h_t, t \in [0,T]$ $\;\Rightarrow\;$ $\pm\int_0^t (S^\circ_u)^{-1}(\pi^\pm_u)^\intercal \,\mathrm{d}X_u = \int_0^t h_s{}^\intercal \,\mathrm{d}\beta_s$. $\quad$ ($Q$-tame?)

$$M_t = (S^\circ_t)^{-1}\mathsf{E}^Q\Big[e^{-\int_t^T r_u \mathrm{d}u}\boldsymbol{\Phi} + \int_t^T e^{-\int_t^s r_u \mathrm{d}u}\varphi_s \,\mathrm{d}s \,\Big|\, \mathscr{G}_t\Big] + \int_0^t (S^\circ_u)^{-1}\varphi_u \,\mathrm{d}u, \quad t \in [0,T].$$

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Since $\varPi^- \leq \varPi^+$, it must be that $\varPi^- = M_0 = \varPi^+$.

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# Replication of European-Style Contingent Claims

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The upper hedge $(M_0 = \Pi, \pi^+, \varphi)$ *replicates* a long position on $\mathscr{K}_0(T, \boldsymbol{\Phi}, \varphi)$: for a rational investor there is no difference between buying the option for the amount $\Pi = M_0$, or, instead, using this same amount to initiate the investment-consumption strategy $(\Pi, \pi^+, c \equiv \varphi)$.

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The lower hedge $(M_0 = \mathit{\Pi}, \pi^-, \varphi)$ *replicates* the underwriting of $\mathscr{K}_0(T, \boldsymbol{\Phi}, \varphi)$: for a rational investor there is no difference between underwriting the option and collecting the amount $\mathit{\Pi} = M_0$, or, instead, borrowing the same amount and amortizing the loan through the investment-payout strategy $(\mathit{\Pi}, \pi^+, \varphi)$.

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N.B. $(M_0, \pi^+, \varphi)$ and $(M_0, \pi^-, \varphi) = (M_0, -\pi^+, \varphi)$ can be treated as “one strategy.” It is called “the replicating strategy for $\mathscr{K}_0(T, \boldsymbol{\Phi}, \varphi)$.”

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Exercise: If the market is free of arbitrage and complete, there is precisely one replicating strategy. HINT: If $(M_0 = \mathit{\Pi}, \pi^+, \varphi)$ is a replicating strategy, then it must generate wealth process $(V_t, t \in [0, T])$ with $V_T = \boldsymbol{\Phi}$ and dynamics:

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Thus either side represents a martingale (why?) and the process $(S^\circ)^{-1} (\pi^+)^\intercal \sigma$ is unique. (Why?)

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For a contingent claim with $\varphi_t = 0$ and $\Phi = F(S_T)$, for some $F : \mathbb{R}_+ \mapsto \mathbb{R}$ that has at most polynomial growth,

$$\text{Price of } \mathscr{K}_0(T, F(S_T)) = \mathsf{E}^Q\left[e^{-rT} F\left(S_0 e^{\sigma\beta_T + rT - \frac{1}{2}\sigma^2 T}\right)\right] = \frac{e^{-rT}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} F\left(S_0 e^{\sigma x + rT - \frac{1}{2}\sigma^2 T}\right) e^{-\frac{x^2}{2T}}\,\mathrm{d}x\,.$$