Zihan Ma
MF 795
Assignment 5

10.52

$$\xi(N) = e^{N-\frac{1}{2}<N,N>}, \xi(N) > 0$$

From (10.34), we know that $\xi(N)$ is a $(\mathcal{F}, P)$ local martingale, therefore, we can say it is a positive local martingale, in other words, it is a super martingale. Thus,

$$\xi(N)_0 = e^{N_0-\frac{1}{2}<N_0,N_0>} = e^0 = 1$$

Therefore, $E^P[\xi(N)_0] = 1 = E^P[\xi(N)_T] \Rightarrow \xi(N)$ is a martingale, by (9.6). Then,

$$E^P[\xi(N)_t|\mathcal{F}_0] = \xi(N)_0$$
$$\Rightarrow E^P[E^P[\xi(N)_t|\mathcal{F}_0]] = E^P[1] = 1$$
$$\Rightarrow E^P[\xi(N)_t|\mathcal{F}_0] = 1$$

10.53

$$\xi(N) = 1 + \xi(N) \cdot N$$
$$(M-<M,N>)(1 + \xi(N) \cdot N) = M_0 + M\xi(N) \cdot N -<M,N$$
$$> \xi(N) \cdot N + \xi(N) \cdot (M-<M,N>)+<M-<M,N>, \xi(N) \cdot N >$$
$$= M_0 + M\xi(N) \cdot N -<M,N> \xi(N) \cdot N + \xi(N) \cdot M - \xi(N) \cdot$$
$$<M,N> +\xi(N) \cdot <M,N> +\xi(N) \cdot <N,<M,N>>$$
$$= M_0 + M\xi(N) \cdot N -<M,N> \xi(N) \cdot N + \xi(N) \cdot M$$

As M and N are continuous local martingales, then $(M-<M,N>)\xi(N)$ is a continuous local martingale. Then, by (9.78), we can say that $M-<M,N>$ is a continuous $(\mathcal{F}, P)$ local martingale.

10.59

$$X = W+<W, \theta^T \cdot W >= W + \theta^T \cdot <W,W >= W + \theta^T \cdot i$$

As W is a BM, then it is a continuous martingale, which implies it is also a continuous $(\mathcal{F}, P)$ local martingale. Then, $\theta^T \cdot W$ is a continuous $(\mathcal{F}, P)$ local martingale. $\Rightarrow W-<W, -N >= X$ is a continuous local martingale w.r.t. $\left(\mathcal{F}, e^{-N_T-\frac{1}{2}<-N,-N>_T} \odot P\right)$, which is the same as $(\mathcal{F}, Q)$. Therefore, X is a $(\mathcal{F}, Q)$ continuous local martingale, and $X_0 = W_0 + \theta^T \cdot 0 = 0$.

$$< X^i, X^j >=< W_i + \theta_i^T \cdot i, W_j + \theta_j^T \cdot i >$$
$$=< W_i, W_j > +< W_i, \theta_j^T \cdot i > +< W_j, \theta_i^T \cdot i > +\theta_i^T \theta_j \cdot <i, i >$$

As i is continuous, increasing and adapted, then it is an increasing process and has finite variation on any finite interval. Then, $< X^i, X^j >=< W_i, W_j >= 0$ when $(i \neq j)$, and $< X^i, X^j >= i$ when $(i = j)$. According to (10.32), X is a d-dimensional BM.

10.66

$$f(W) = |W|$$

Then, the canonical decomposition of it will be:

$$|W_0| + \partial f(W) \cdot W + \partial^2 f(W) \cdot i = W$$

Then,

$$L^0(|W|) = \lim_{\epsilon \searrow 0} \int^t \frac{1}{\epsilon} 1_{[0,\epsilon[}(|W|)d < sign(W) \cdot |W|, sign(W) \cdot |W| >$$

$$L^0(W) = \lim_{\epsilon \searrow 0} \int^t \frac{1}{2\epsilon} 1_{]-\epsilon,\epsilon[}(W) d < W, W >$$

Notice that $1_{]-\epsilon,\epsilon[}(W) = 1_{[0,\epsilon[}(|W|)$, then

$$L^0(W) = \lim_{\epsilon \searrow 0} \int^t \frac{1}{2\epsilon} 1_{]-\epsilon,\epsilon[}(W) d < W, W >$$
$$= \lim_{\epsilon \searrow 0} \int^t \frac{1}{2\epsilon} 1_{[0,\epsilon[}(|W|) d(sign(W)^2 \cdot < |W|, |W| >)$$
$$= \frac{1}{2} \lim_{\epsilon \searrow 0} \int^t \frac{1}{\epsilon} 1_{[0,\epsilon[}(|W|) d < sign(W) \cdot |W|, sign(W) \cdot |W| >$$
$$= \frac{1}{2} L^0(|W|)$$

Therefore, $L^0(|W|) = 2L^0(W)$.

10.67

By the third formula in e_2, with $a = 0, X = |W|$, we can have:

$$|W|^- = -1_{]-\infty,a]}(|W|) \cdot |W| + \frac{1}{2} L^0(|W|)$$

As LHS= 0, and RHS can be rewritten as $-1_{\{0\}}(W) \cdot |W| + L^0(W)$, then we can get

$$L^0(W) = 1_{\{0\}}(W) \cdot |W|$$
$$= 1_{\{0\}}(W) \cdot \left(sign(W) \cdot W + L^0(W)\right)$$
$$= 1_{\{0\}}(W) \cdot L^0(W)$$

Therefore,

$$|W| \cdot L^0(W) = |W| \cdot \left(1_{\{0\}}(W) \cdot L^0(W)\right)$$
$$= |W| 1_{\{0\}}(W) \cdot L^0(W)$$
$$= 0$$

Furthermore, as $dL^0(W) = 1_{\{0\}}(W) dL^0(W) \neq 0$, if $t \in \mathcal{L}_0(W)$, and $dL^0(W) = 1_{\{0\}}(W) dL^0(W) = 0$, if $t \notin \mathcal{L}_0(W)$, then we can conclude that $dL^0(W)$ is supported by $\mathcal{L}_0(W)$.

10.69

a) $f(X) = (X - K)^+$
first derivative:

$$\int \varphi(x) \partial f(x) dx = \int_{-\infty}^{k} \varphi(x)(X - K)^+ dx + \int_{k}^{\infty} \varphi(x)(X - K)^+$$
$$= \int_{k}^{\infty} \varphi(x) dx$$
$$= \int_{\mathbb{R}} \varphi(x) 1_{]k,\infty]} dx$$

Then, $\partial f(x) = 1_{]k,\infty]}$
Second derivative:

$$\int_{\mathbb{R}} \varphi(x) \partial 1_{]k,\infty]} dx = - \int_{\mathbb{R}} \partial \varphi(x) 1_{]k,\infty]} dx$$
$$= - \int_{]k,\infty[} \partial \varphi dx$$

$$= \varphi(k)$$
$$= \int_{\mathbb{R}} \varphi \epsilon_k(dx)$$
$$= \int_{\mathbb{R}} \varphi(x) \delta_k(x) dx$$

Therefore, $\partial 1_{]k,\infty]} = \delta_k(x)$, and the positive measure would be $\epsilon_k(dx)$. And, $\int_k L^0(X)\epsilon_k(da) = L^k(X)$.

Therefore,

$$(X - K)^+ = (X_0 - K)^+ + 1_{]k,\infty]}(X) \cdot X + \frac{1}{2} L^k(X)$$

b) $f(X) = (X - K)^-$
first derivative:

$$\int \varphi(x) \, \partial f(x) dx = \int_{-\infty}^{k} \varphi(x)\partial(-x + k) \, dx + \int_{k}^{\infty} \varphi(x)\partial 0 dx$$
$$= -\int_{-\infty}^{k} \varphi(x) \, dx$$
$$= -\int_{\mathbb{R}} \varphi(x) \, 1_{]-\infty,k]} dx$$

Then, $\partial f(x) = -1_{]-\infty,k]}$
Second derivative:

$$-\int_{\mathbb{R}} \varphi(x) \, \partial 1_{]-\infty,k]} dx = \int_{\mathbb{R}} \partial \varphi(x) 1_{]-\infty,k]} \, dx$$
$$= \int_{]k,\infty[} \partial \varphi dx$$
$$= \varphi(k)$$
$$= \int_{\mathbb{R}} \varphi \epsilon_k(dx)$$
$$= \int_{\mathbb{R}} \varphi(x) \delta_k(x) dx$$

Therefore, $\partial 1_{]-\infty,k]} = \delta_k(x)$, and the positive measure would be $\epsilon_k(dx)$. And, $\int_k L^0(X)\epsilon_k(da) = L^k(X)$.

Therefore,

$$(X - K)^- = (X_0 - K)^- - 1_{]-\infty,k]}(X) \cdot X + \frac{1}{2} L^k(X)$$

c)
$$(X - K)^+ - (X - K)^- = (X_0 - K)^+ - (X_0 - K)^- + \left(1_{]k,\infty]}(X) + 1_{]-\infty,k]}(X)\right) \cdot X$$
$$= X_0 - K + X - X_0$$
$$= X - K$$

$$(X - K)^+ + (X - K)^- = (X_0 - K)^+ - (X_0 - K)^- + \left(1_{]k,\infty]}(X) - 1_{]-\infty,k]}(X)\right) \cdot X + L^k(X)$$
$$= |X_0 - K| + sign(X - K) \cdot X + L^k(X)$$
$$= |X - K|$$

11.37

$$X = X_0 + X \cdot A + Z, A_0 = Z_0 = 0$$
$$e^{-A}X = e^{-A_0}X_0 + e^{-A} \cdot X + X \cdot e^{-A} + < X, e^{-A} >$$
$$= X_0 + e^{-A} \cdot X + X \cdot e^{-A} + < X, e^{-A} >$$
$$= X_0 + e^{-A} \cdot (X_0 + X \cdot A + Z) + X \cdot e^{-A}$$
$$= X_0 + (e^{-A}X) \cdot A + X \cdot e^{-A}$$

As A is continuous with finite variation, then

$$X \cdot e^{-A} = -e^{-A}X \cdot A$$
$$\Rightarrow e^{-A}X = X_0 + (e^{-A}X) \cdot A + e^{-A} \cdot Z - (e^{-A}X) \cdot A$$
$$= X_0 + e^{-A} \cdot Z$$
$$X = e^{A}(X_0 + e^{-A} \cdot Z)$$

11.38

Let $X = X_0 f(s)$, where $s = Y - \frac{1}{2} < Y, Y >$, and $f(s) = e^s$.

$$X = X_0 f(s)$$
$$= X_0 f(0) + X_0 \partial f \cdot s + X_0 \partial^2 f \cdot < s, s >$$
$$= X_0 + X_0 f(s) \cdot Y - X_0 f(s) \cdot < Y, Y > + X_0 f(s) \cdot < Y, Y >$$
$$= X_0 + X \cdot Y$$

Therefore, $X = X_0 e^{Y - \frac{1}{2} < Y, Y >}$ is a solution to equation e_2.

To find if $X = X_0 e^{Y - \frac{1}{2} < Y, Y >}$ is a unique solution, we assume that $\exists$ another X' solution such that $X' = X_0 + X' \cdot Y$.

$$\frac{X'}{X} = X'X^{-1}$$
$$= X_0'X_0^{-1} + X' \cdot X^{-1} + X^{-1} \cdot X' + < X', X^{-1} >$$
$$X' \cdot X^{-1} = X' \cdot \frac{1}{X_0} f^{-1}(s)$$
$$= X' \cdot \frac{1}{X_0}\left(1 + f^{-1}(s) \cdot s + \frac{1}{2} f^{-1}(s) \cdot < s, s > \right)$$
$$= \frac{X'}{X} \cdot \left(s + \frac{1}{2} < s, s > \right)$$
$$= \frac{X'}{X} \cdot (< Y, Y > -Y)$$
$$X^{-1} \cdot X' = \frac{X'}{X} \cdot Y$$
$$< X', X^{-1} > = X' \cdot < Y, X^{-1} >$$
$$= X' \cdot < Y, \frac{1}{X}(< Y, Y > -Y) >$$
$$= \frac{X'}{X} \cdot < Y, < Y, Y > -Y >$$
$$= -\frac{X'}{X} \cdot < Y, Y >$$

Therefore, $\frac{X'}{X} = \frac{X_0'}{X_0} = 1$. Then we can say that $X = X_0 e^{Y - \frac{1}{2} < Y, Y >}$ is a unique solution.

11.39
$$X\xi^{-1}(Y) = X_0 + X \cdot \xi^{-1}(Y) + \xi^{-1}(Y) \cdot X + <X, \xi^{-1}(Y)>$$
As we know that $X = X_0 + X \cdot Y + Z$, then:
$$X\xi^{-1}(Y) = X_0 + X\xi^{-1}(Y) \cdot (<Y,Y> -Y) + X\xi^{-1}(Y) \cdot Y + \xi^{-1}(Y) \cdot Z + X \cdot <Y, \xi^{-1}(Y)> +$$
$$<Z, \xi^{-1}(Y)>$$
$$= X_0 + X\xi^{-1}(Y) \cdot <Y,Y> + \xi^{-1}(Y) \cdot Z - X\xi^{-1}(Y) \cdot <Y,Y> - \xi^{-1}(Y) <Z,Y>$$
$$= X_0 + \xi^{-1}(Y) \cdot Z - \xi^{-1}(Y) <Z,Y>$$
$$\Rightarrow X = \xi(Y)(X_0 + \xi^{-1}(Y) \cdot Z - \xi^{-1}(Y) <Z,Y>)$$

11.42
1)
$$Xe^{bt} = X_0 + X \cdot e^{bt} + e^{bt} \cdot X + <X, e^{bt}>$$
$$= X_0 + be^{bt}X \cdot t + e^{bt} \cdot (X_0 + \sigma W + b(m-X) \cdot t)$$
$$= X_0 + \sigma e^{bt} \cdot W + b(m-X)e^{bt} \cdot t$$
$$= X_0 + \sigma e^{bt} \cdot W + bme^{bt} \cdot t$$
$$\Rightarrow X = e^{-bt}(X_0 + \sigma e^{bt} \cdot W + bme^{bt} \cdot t)$$
$$\Rightarrow X = e^{-bi}\left(X_0 + e^{bi} \cdot (\sigma W + bmi)\right)$$

2) Rewrite X, we can have:
$$X = e^{-bi}X_0 + e^{-bi}\left(e^{bi} \cdot \sigma W + e^{bi} \cdot bmi\right)$$
As X_0 is a Gaussian r.v., then $e^{-bi}X_0$ is a Gaussian process, $e^{bi} \cdot \sigma W$ is a Gaussian process intuitively and $e^{bi} \cdot bmi$ is a scalar process depends on t. Therefore, we can claim that $X = e^{-bi}X_0 + e^{-bi}\left(e^{bi} \cdot \sigma W + e^{bi} \cdot bmi\right)$ is a Gaussian process.
3)
$$E[X_t] = e^{-bt}E[X_0] + \sigma e^{-bt}E[e^{-bt} \cdot W] + bme^{-bt}\left(\frac{te^{bt}-1}{b}\right)$$
$$= e^{-bt}E[X_0] + m - me^{-bt}$$
$$E[X_t^2] = E\left[e^{-2bt}\left(X_0^2 + 2X_0e^{bi} \cdot (\sigma W + bmi) + \left(e^{bi} \cdot (\sigma W + bmi)^2\right)\right)\right]$$
$$= e^{-2bt}E[X_0^2] + 2mE[X_0]e^{-2bt}(e^{bt}-1) + e^{-2bt}E\left[\left(\int e^{bs}d(\sigma W + bmi)^2\right)\right]$$
$$= e^{-2bt}E[X_0^2] + 2mE[X_0]e^{-2bt}(e^{bt}-1) + e^{-2bt}\sigma^2E\left[\left(e^{bt} \cdot \left(W + \frac{bm}{\sigma}i\right)\right)^2\right]$$
$$= e^{-2bt}E[X_0^2] + 2mE[X_0]e^{-2bt}(e^{bt}-1) + e^{-2bt}\sigma^2E\left[\left(f \cdot W + f \cdot \frac{bm}{\sigma}i\right)^2\right]$$
$$= e^{-2bt}E[X_0^2] + 2mE[X_0]e^{-2bt}(e^{bt}-1) + e^{-2bt}\sigma^2E[(f \cdot W)^2]$$
$$+ 2e^{-2bt}\sigma^2E\left[f \cdot W \times f \cdot \frac{bm}{\sigma}i\right] + e^{-2bt}\sigma^2E\left[\left(f \cdot \frac{bm}{\sigma}i\right)^2\right]$$
$$= e^{-2bt}E[X_0^2] + 2mE[X_0]e^{-2bt}(e^{bt}-1) + e^{-2bt}\sigma^2\frac{1}{2b}(e^{2bt}-1)$$
$$+ e^{-2bt}\sigma^2E\left[\frac{(bm)^2}{\sigma^2}\frac{1}{b^2}(e^{bt}-1)^2\right]$$

$$= e^{-2bt}\left(E[X_0{}^2]\right) + 2mE[X_0]e^{-2bt}(e^{bt} - 1) + e^{-2bt}\sigma^2 \frac{1}{2b}(e^{2bt} - 1)$$
$$+ e^{-2bt}m^2(e^{bt} - 1)^2$$
$$Var(X_t) = E[X_t{}^2] - E[X_t]^2$$
$$= e^{-2bt}\left(E[X_0{}^2]\right) + 2mE[X_0]e^{-2bt}(e^{bt} - 1) + e^{-2bt}\sigma^2 \frac{1}{2b}(e^{2bt} - 1)$$
$$+ e^{-2bt}m^2(e^{bt} - 1)^2 - e^{-2bt}E[X_0]^2 - m^2(1 - e^{-bt})^2$$
$$- 2me^{-bt}(1 - e^{-bt})E[X_0]$$
$$= e^{-2bt}Var(X_0) + e^{-2bt}\frac{\sigma^2}{2b}(e^{2bt} - 1)$$

4)
$$E[X_t] = e^{-bt}(E[X_0] - m) + m$$
$$\therefore E[X_0] = m$$
$$Var(X_t) = e^{-2bt}Var(X_0) + e^{-2bt}\frac{\sigma^2}{2b}(e^{2bt} - 1)$$
$$\therefore Var(X_0) = \frac{\sigma^2}{2b}$$

5) When $X < m$, $b(m - X) \cdot i > 0$, then the process will move up. When $X > m$, $b(m - X) \cdot i < 0$, then the process will move down. $dX = \sigma dW + b(m - X)di$, so X sarts from X_0 and will move to m and then float around m and thus fits the mean-reverting pattern.

11.43
$$Xe^{-f\cdot i} = X_0 e^{-f\cdot 0} + X \cdot e^{-f\cdot i} + e^{-f\cdot i} \cdot X + <X, e^{-f\cdot i}>$$
$$= X_0 + X \cdot e^{-f\cdot i} + e^{-f\cdot i} \cdot X + <X, e^{-f\cdot i}>$$
$$= X_0 - Xe^{-f\cdot i}f \cdot i + e^{-f\cdot i} \cdot (X_0 + W + (fX + g) \cdot i)$$
$$= X_0 - Xe^{-f\cdot i}f \cdot i + e^{-f\cdot i} \cdot W + e^{-f\cdot i}(fX + g) \cdot i$$
$$= X_0 + e^{-f\cdot i} \cdot W + e^{-f\cdot i}g \cdot i$$
$$\Rightarrow X = e^{f\cdot i}\left(X_0 + e^{-f\cdot i} \cdot W + e^{-f\cdot i}g \cdot i\right)$$

11.44
As it is a special case of (11.43), we can set:
$$X_0 = x, f = -\frac{1}{T - i}, g = \frac{\xi}{T - i}$$
Then,
$$f \cdot i = -\int_0^t \frac{1}{T - s}ds = \ln(T - s)\,|_0^t = \ln\frac{T - t}{T}$$
$$\Rightarrow e^{-f\cdot i} = \frac{T}{T - t}, e^{-f\cdot i}g = \frac{\xi T}{(T - t)^2}$$
$$e^{-f\cdot i}g \cdot i = \int_0^t \frac{\xi T}{(T - s)^2}ds = \frac{\xi T}{T - s}\,|_0^t = \xi T\left(\frac{1}{T - t} - \frac{1}{T}\right)$$
Therefore,
$$X_t = \frac{T - t}{T}\left(x + \frac{T}{T - t} \cdot W + \xi T\left(\frac{1}{T - t} - \frac{1}{T}\right)\right)$$

$$= \frac{T-t}{T}x + (T-t)\left(\frac{1}{T-t}\cdot W\right) + (T-t)\xi\frac{t}{T(T-t)}$$

$$= \frac{T-t}{T}x + (T-t)\left(\frac{1}{T-t}\cdot W\right) + \frac{t}{T}\xi$$

$$\Rightarrow X = \frac{T-i}{T}x + (T-i)\left(\frac{1}{T-i}\cdot W\right) + \frac{i}{T}\xi$$

11.45

1)

$$E\left[\frac{1}{T-s}\cdot W \times \frac{1}{T-t}\cdot W\right] = \int_0^{s\wedge t} f(u)^2\, du$$

$$= \frac{1}{T-u}\Big|_0^s$$

$$= \frac{s}{T(T-s)}$$

$$E[f\cdot W] = 0$$

Therefore, $f\cdot W$ is a centered Gaussian process.

$$E\left[W_{\frac{t}{T(T-t)}} W_{\frac{s}{T(T-s)}}\right] = \frac{s}{T(T-s)}$$

$$E\left[W_{\frac{t}{T(T-t)}}\right] = 0$$

Therefore, $W_{\frac{t}{T(T-t)}}$ is a Gaussian process. Since $E\left[W_{\frac{t}{T(T-t)}}^2\right] = \frac{t}{T(T-t)}$, then the process converges in L^2-space.

As Gaussian process's law of distribution is determined by the mean and covariance function, and as we can see that both of the processes have the same mean and covariance expression, then they do have the same law.

2)

$$E\left[W_{\frac{1}{t}}^2\right] = \frac{1}{t}$$

$$E\left[\frac{1}{t^2}W_t^2\right] = \frac{1}{t}$$

Therefore, both processes converge in L^2-space, which implies they are also Gaussian processes.

$$E\left[W_{\frac{1}{t}}\right] = E\left[\frac{1}{t}W_t\right] = 0$$

$$E\left[W_{\frac{1}{t}}W_{\frac{1}{s}}\right] = \frac{1}{t}$$

$$E\left[\frac{1}{ts}W_tW_s\right] = \frac{1}{ts}s = \frac{1}{t}$$

Therefore, both processes have same covariance function and same mean, then they have the same distribution law.

3)

Let $t = \frac{T(T-t')}{t'}$, where $t' \in [0, T[$, then $\frac{1}{t} \in \mathbb{R}^+$. Then, $\frac{1}{t} = \frac{t'}{T(T-t')}$, and we can see that $W_{\frac{t'}{T(T-t')}}$

has the same distribution law of $\frac{1}{t} W_t$, where $\frac{1}{t} W_t = \frac{t'}{T(T-t')} W_{\frac{T(T-t')}{t'}}$. Therefore, we can claim

that $W_{\frac{t}{T(T-t)}}$ and $\frac{t}{T(T-t)} W_{\frac{T(T-t)}{t}}$ have the same distribution law.

4)
The Brownian bridge process is $(T - t) \int_0^t \frac{1}{T-s} dW_s$, then clearly it has the same distribution law

as $(T - t) \frac{t}{T(T-t)} W_{\frac{T(T-t)}{t}} \Rightarrow$ it has the same distribution law as $\frac{t}{T} W_{\frac{T(T-t)}{t}}$.

$$\lim_{t \to T} \frac{t}{T} W_{\frac{T(T-t)}{t}} = \lim_{t \to T} W_{\frac{T(T-t)}{t}} = \lim_{t \to 0} W_t = 0 \ \ P - a.s.$$

$$\Rightarrow P\left(\left(\lim_{t \to T} \frac{t}{T} W_{\frac{T(T-t)}{t}}\right) = 0\right) = 1$$

As X_t and $\frac{t}{T} W_{\frac{T(T-t)}{t}}$ have the same distribution law, then $P\left(\left(\lim_{t \to T} X_t\right) = 0\right) = 1$.

11.46

$$B_t = (T - t)f \cdot W + \frac{t}{T} \xi$$

$(T - t)f \cdot W$ is intuitively a Gaussian r.v. as $f \cdot W$ is a Gaussian process.

$$E[B_t^2] = \frac{t^2}{T^2} E[\xi^2] + (T - t)^2 \int_0^t \frac{1}{(T-s)^2} ds$$

$$= \frac{t^2}{T} + (T - t)^2 \frac{1}{T-s}\Big|_0^t$$

$$= \frac{t^2}{T} + \frac{(T-t)^2 t}{T(T-t)}$$

$$= t$$

As B_t converges in L^2-space, it implies it is also a Gaussian process.

$$E[B_t B_s] = (T - t)(T - s) \int^{t \wedge s} \frac{1}{(T-u)^2} du + \frac{ts}{T^2} \times T + (T - t) \frac{s}{T} E[\xi f \cdot W_s]$$

$$+ (T - s) \frac{t}{T} E[\xi f \cdot W_t]$$

$$= (T - t)(T - s) \frac{1}{T-u}\Big|_0^s + \frac{ts}{T}$$

$$= (T - t)(T - s) \frac{s}{T(T-s)} + \frac{ts}{T}$$

$$= \frac{sT - st}{T} + \frac{st}{T}$$

$$= s$$

$$E[B_t] = (T - t)E[f \cdot W_t] + \frac{t}{T} E[\xi] = 0$$

$$\lim_{t \to 0} B_t = Tf \cdot W_0 + \frac{0}{T}$$

Therefore, B_t is a Brownian motion.

$$E[(B_t - B_s)B_s] = E[B_t B_s - B_s{}^2] = s - s = 0$$

As B_t is a Gaussian process, then $B_t - B_s$ are independent from B_s for any $s, t \in [0, T]$ and $s < t$.

11.49

$$U = U_0 + \frac{\sigma}{2}W - \frac{k}{2}U \cdot i$$

$$U^2 = U_0{}^2 + 2U \cdot U + \frac{1}{2} \times 2 \cdot < U, U >$$

$$= U_0{}^2 + 2U \cdot \left(\frac{\sigma}{2}W - \frac{k}{2}U \cdot i\right) + < \frac{\sigma}{2}W - \frac{k}{2}U \cdot i, \frac{\sigma}{2}W - \frac{k}{2}U \cdot i >$$

$$= U_0{}^2 + U\sigma \cdot W - kU^2 \cdot i + \frac{\sigma^2}{4} < W, W > - \frac{\sigma k U}{2} < W, i > + \frac{k^2 U}{4} < i, i >$$

$$= U_0{}^2 + U\sigma \cdot W - kU^2 \cdot i + \frac{\sigma^2}{4}i$$

$$= U_0{}^2 + U\sigma \cdot W - kU^2 \cdot i + \frac{\sigma^2}{4} \cdot i$$

$$= U_0{}^2 + U\sigma \cdot W + \left(\frac{\sigma^2}{4} - kU^2\right) \cdot i$$

$$|U|^2 = \sum_{i=1}^{d} U_i{}^2$$

$$= \sum_{i=1}^{d} U_{io}{}^2 + U_{id}\sigma \cdot W_i + \left(\frac{\sigma^2}{4} - kU_i{}^2\right) \cdot i$$

$$= |U_0|^2 + \sum_{i=1}^{d} U_{id}\sigma \cdot W_i + \frac{d\sigma^2}{4} \cdot i - k|U|^2 \cdot i$$

$$= |U_0|^2 + \sigma U^T dW + \left(\frac{d\sigma^2}{4} - k|U|^2\right) \cdot i$$

$$= |U_0|^2 + \sigma U^T \cdot W + \left(\frac{d\sigma^2}{4} - k|U|^2\right) \cdot i$$

11.52

$$|U|^2 = |U_0|^2 + \sigma U^T \cdot W + \left(\frac{d\sigma^2}{4} - k|U|^2\right) \cdot i$$

$$X = X_0 + \sigma\sqrt{|X|} \cdot W + \left(d\frac{\sigma^2}{4} - kX\right) \cdot i$$

Thus, $|U_t|^{-1}U_t^T \cdot W$ is a Gaussian process.

$$E[|U_t|^{-1}U_t^T \cdot W_t] = E[f \cdot W_t] = 0$$

$$E[f \cdot W_t \times f \cdot W_s] = \int_0^s f^2 \, du$$

$$= \int_0^s |U_u|^{-2} U_u^T U_u \, du$$

$$= \int_0^s |U_u|^{-2} |U_u|^2 du$$

$$= s$$

Therefore, $|U_t|^{-1} U_t^T \cdot W$ is a Brownian Motion. Then,

$$\sigma U^T \cdot W = \sigma |U| |U|^{-1} U^T \cdot W = \sigma |U| \cdot (|U|^{-1} U^T \cdot W)$$

Let $B = |U|^{-1} U^T \cdot W$, then $\sigma |U| \cdot (|U|^{-1} U^T \cdot W) = \sigma |U| \cdot B$

Therefore,

$$|U|^2 = |U_0|^2 + \sigma |U| \cdot B + \left(\frac{d\sigma^2}{4} - k|U|^2\right) \cdot i$$

$$= X_0 + \sigma |U| \cdot B + \left(\frac{d\sigma^2}{4} - k|U|^2\right) \cdot i$$

And,

$$X = X_0 + \sigma \sqrt{|X|} \cdot W + \left(d\frac{\sigma^2}{4} - kX\right) \cdot i$$

Then,

$$X = |U|^2$$
$$\Longrightarrow \sqrt{|X|} = |U|$$

Therefore, X and $|U|^2$ have the same distribution law.

11.57

$$X = X_0 + \sigma \sqrt{|X|} \cdot W + \left(d\frac{\sigma^2}{4} - kX\right) \cdot i$$

$$\frac{Xe^{kt}}{c(t)} = \frac{4kXe^{kt}}{\sigma^2(e^{kt} - 1)}$$

From (11.52), we know that

$$X = |U|^2$$
$$= U_1{}^2 + U_2{}^2 + \cdots + U_d{}^2$$

$$\Longrightarrow \frac{Xe^{kt}}{c(t)} = \sum_{i=1}^d \left(\frac{U_i}{\sqrt{\frac{\sigma^2}{4}(1 - e^{-kt})}}\right)^2$$

Set $U_i = \left(\frac{X}{d}\right)^{\frac{1}{2}} + \frac{\sigma}{2} W - \frac{k}{2} U \cdot i$, then according to (11.42), $U_i = e^{-\frac{k}{2}i}\left(\left(\frac{X}{d}\right)^{\frac{1}{2}} + e^{\frac{k}{2}i} \cdot \left(\frac{\sigma}{2} W + 0\right)\right)$.

As $\frac{X}{d}$ is a constant number, then U_i is a Gaussian process.

$$E[U_{it}] = e^{-\frac{k}{2}t} E[U_0] = e^{-\frac{k}{2}t} \sqrt{\frac{X}{d}}$$

$$Var(U_{it}) = e^{-kt} \times 0 + e^{-kt} \frac{\sigma^2}{4k}(e^{kt} - 1) = \frac{\sigma^2}{4k} - \frac{\sigma^2}{4k}e^{-kt}$$

Clearly, $\sum_{i=1}^{d} \left(\dfrac{U_i}{\sqrt{\frac{\sigma^2}{4}(1-e^{-kt})}} \right)^2$ is a non-central chi-square distribution.

$$\lambda = \sum_{i=1}^{d} \left(\frac{E[U_{it}]}{\sqrt{\frac{\sigma^2}{4k}(1 - e^{-kt})}} \right)^2$$

$$= \sum_{i=1}^{d} \frac{e^{-kt}\frac{X}{d}}{\frac{\sigma^2}{4k}(1 - e^{-kt})}$$

$$= \frac{X}{d\frac{\sigma^2}{4k}(1 - e^{-kt})} \times d$$

$$= \frac{X}{c(t)}$$

Therefore, $\frac{Xe^{kt}}{c(t)} = Y_t \Longrightarrow X_t = e^{-kt}c(t)Y_t$.

11.61

$$Y = \log X$$

$$Y = \log X_0 + \frac{1}{X} \cdot X - \frac{1}{2X^2} \cdot < X, X >$$

$$= \log X_0 + \frac{1}{X} \cdot (X_0 + \sigma X \cdot W + k(\alpha - \log X)X \cdot i) - \frac{1}{2X^2} \cdot ((\sigma X)^2 \cdot i)$$

$$= \log X_0 + \sigma \cdot W + k(\alpha - Y) \cdot i - \frac{\sigma^2}{2} \cdot i$$

$$Y = \log X_0 + \sigma W + k\left(\alpha - \frac{\sigma^2}{2k} - Y\right) \cdot i$$

Let $m = \alpha - \frac{\sigma^2}{2k}$, $Y_0 = \log X_0$, $b = k$, then, clearly, Y is an OU process and $X = e^Y$.

13.9

$$< \mathcal{M}, \Theta > = < \sigma \cdot W, \Theta^T \cdot W >$$

$$= \begin{pmatrix} < \sigma_1 \cdot W, \Theta^T \cdot W > \\ < \sigma_2 \cdot W, \Theta^T \cdot W > \\ \cdots \end{pmatrix}$$

Note that,

$$< \sigma_1 \cdot W, \Theta^T \cdot W > = \sigma_1(\Theta^T)^T \cdot i$$

$$= \sigma_1 \sigma^{-1}(b - r\vec{1}) \cdot i$$

$$= (1,0,0,\dots) \begin{pmatrix} b_1 - r \\ b_2 - r \\ \cdots \end{pmatrix} \cdot i$$

$$= (b_1 - r) \cdot i$$

Therefore,

$$< \mathcal{M}, \Theta > = \begin{pmatrix} b_1 - r \\ b_2 - r \\ \cdots \end{pmatrix} \cdot i$$
$$= (b - r\vec{1}) \cdot i$$

13.12

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} dV_t + V_t d\left(e^{-\int_0^t r_u du}\right) + \frac{1}{2} x \cdot 0$$
$$= e^{-\int_0^t r_u du} dV_t + V_t e^{-\int_0^t r_u du} \cdot (r_t) dt$$

As $dV_t = V_t r_t dt + \pi_t^T dX_t - c_t dt$, then

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du}(V_t r_t dt + \pi_t^T dX_t - c_t dt) - V_t e^{-\int_0^t r_u du}(r_t) dt$$
$$= e^{-\int_0^t r_u du} \pi_t^T dX_t - e^{-\int_0^t r_u du} c_t dt$$
$$V_t e^{-\int_0^t r_u du} = V_0 + \int_0^t e^{-\int_0^s r_u du} \pi_s^T dX_s - \int_0^t e^{-\int_0^s r_u du} c_s ds$$
$$\Rightarrow V_t = e^{\int_0^t r_u du}\left(x + \int_0^t e^{-\int_0^s r_u du} \pi_s^T dX_s - \int_0^t e^{-\int_0^s r_u du} c_s ds\right)$$

13.13

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} dV_t + V_t e^{-\int_0^t r_u du} \cdot (r_t) dt$$

As $dV_t = V_t r_t dt - \pi_t^T dX_t - c_t dt$, then

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du}(V_t r_t dt - \pi_t^T dX_t - c_t dt) + V_t e^{-\int_0^t r_u du}(r_t) dt$$
$$= -e^{-\int_0^t r_u du} \pi_t^T dX_t - e^{-\int_0^t r_u du} c_t dt$$
$$V_t e^{-\int_0^t r_u du} = V_0 - \int_0^t e^{-\int_0^s r_u du} \pi_s^T dX_s - \int_0^t e^{-\int_0^s r_u du} c_s ds$$
$$\Rightarrow V_t = e^{\int_0^t r_u du}\left(V_0 - \int_0^t e^{-\int_0^s r_u du} \pi_s^T dX_s - \int_0^t e^{-\int_0^s r_u du} c_s ds\right)$$

13.18
As π is Q-tame, then $K^{0,\pi} - M \geq 0$. Thus, $X = \sigma \cdot (W + \theta \cdot i)$, then X is $(\mathcal{G}, P)$-semimartingale and Q is an ELMM for P, which implies X is $(\mathcal{G}, Q)$ local martingale. Then, we can get $K^{0,\pi} = (S^0)^{-1} \pi^T \cdot X$ is a local martingale. As M is also a $(\mathcal{G}, Q)$ local martingale, then $K^{0,\pi} - M$ is a positive $(\mathcal{G}, Q)$ local martingale, which means $K^{0,\pi} - M$ is a supermartingale.

$$E\left[K_t^{0,\pi}\middle|\mathcal{F}_s\right] = E\left[K_t^{0,\pi} - M_t + M_t\middle|\mathcal{F}_s\right]$$
$$= E\left[K_t^{0,\pi} - M_t\middle|\mathcal{F}_s\right] + E[M_t|\mathcal{F}_s]$$
$$\leq K_s^{0,\pi} - M_s + M_s = K_s^{0,\pi}$$

Therefore, $K^{0,\pi}$ is a $(\mathcal{G}, Q)$ supermartingale.

13.19
As $(S^0)^{-1} \pi^T \cdot X$ is a supermartingale with respect to Q, and $x^* = E^Q[(S_T^0)^{-1} V_T + ((S^0)^{-1} c \cdot i)_T]$, then, $x^* + (S^0)^{-1} \pi^T \cdot X$ is a supermartingale. Therefore, $(S^0)^{-1} V + ((S^0)^{-1} c \cdot i)$ is also a $(\mathcal{G}, Q)$ supermartingale.

$$E^Q[(S_0^0)^{-1} V_0 + ((S^0)^{-1} c \cdot i)_0] = E^Q[V_0] = x^*$$

Therefore, $(S^0)^{-1}V + ((S^0)^{-1}c \cdot i)$ is a $(\mathcal{G}, Q)$ martingale, and $x^* + (S^0)^{-1}\pi^T \cdot X$ is a $(\mathcal{G}, Q)$ martingale.

13.20

$$\beta = \sigma^{-1} \cdot X$$

As X is a $(\mathcal{G}, Q)$-martingale, then β is also a $(\mathcal{G}, Q)$-martingale.

$$\beta = W + \theta \cdot <W, W> = W + \theta \cdot i$$

Then, β is continuous, which also means β is a continuous $(\mathcal{G}, Q)$ local martingale.

$$\beta_0 = W_0 + (\theta \cdot i)_0 = 0$$
$$<\beta, \beta> = <W + \theta \cdot i, W + \theta \cdot i> = <W, W> = i$$

Therefore, we can say that β is a BM w.r.t Q.

14.5

$$E[\Phi] = E[W_T^4] = 3T^2$$
$$D\Phi = 4W_T^3 1_{[[t_{i-1}, t_i[[}}$$
$$\int_0^T E[4W_T^3 1_{[0,T[}(s)|\mathcal{F}_s^W]dW_s = \int_0^T E[4W_T^3|\mathcal{F}_s^W]dW_s$$
$$= \int_0^T 4E[(W_T - W_s + W_s)^3|\mathcal{F}_s^W]dW_s$$
$$= 4\int_0^T E[(W_T - W_s)^3 + W_s^3 + 3(W_T - W_s)^2 W_s + 3(W_T - W_s)W_s^2|\mathcal{F}_s^W]dW_s$$
$$= \int_0^T 12(T - S)W_s + 4W_s^3 dW_s$$
$$W_T^4 = 3T^2 + \int_0^T (12(T - S)W_s + 4W_s^3)dW_s$$

$$W^4 = W_0^4 + 4W^3 \cdot W + \frac{1}{2} \times 12W^2 \cdot <W, W>$$
$$W^4 = 4W^3 \cdot W + 6W^2 \cdot i$$
$$6W^2 \cdot i = 6(iW^2 - i \cdot W^2)$$
$$= 6[i(2W \cdot W + i) - i \cdot (2W \cdot W + i)]$$
$$= 6\left[2i(W \cdot W) + i^2 - 2Wi \cdot W - \frac{1}{2}i^2\right]$$
$$= 6\left[2i(W \cdot W) - 2Wi \cdot W + \frac{1}{2}i^2\right]$$
$$= 3i^2 + 12i(W \cdot W) - 12Wi \cdot W$$
$$W_T^4 = 3T^2 + \int_0^T 4W^3 dW_s + 12T\int_0^T W_s dW_s - 12\int_0^T sW_s dW_s$$
$$= 3T^2 + \int_0^T (12(T - S)W_s + 4W_s^3)dW_s$$

14.6

$$\Phi = (ae^{W_T} - b)^+ = 1_{[0,+\infty[}(ae^{W_T} - b)$$

$$D\Phi = 1_{[0,+\infty[}(ae^{W_T} - b)ae^{W_T}1_{[0,T[}(s)$$

$$E[D\Phi_s|\mathcal{F}_s^W] = E\left[1_{[0,+\infty[}(ae^{W_T+W_s-W_s} - b)ae^{W_T+W_s-W_s}|\mathcal{F}_s^W\right]$$

Let $y = W_T - W_s$, then

$$E[D\Phi_s|\mathcal{F}_s^W] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-s)}} 1_{[0,+\infty[}(ae^{y+W_s} - b)ae^{y+W_s}e^{\frac{-y^2}{2(T-s)}}$$

$$= \int_{\ln(\frac{b}{a})-W_s}^{\infty} (ae^{y+W_s} - b)ae^{y+W_s}e^{\frac{-y^2}{2(T-s)}}1_{[0,T[}(s)dy$$

$$= A$$

Then,

$$\Phi = E\left[1_{[0,+\infty[}(ae^{W_T} - b)\right] + \int_0^T A\, dW_s$$