MF795: STOCHASTIC METHODS IN ASSET PRICING I
Fall Semester 2020

Assignment № 1
Due on: 9/22/2020

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MF 795 Homework 1

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1 (0.4)Exercise: An experiment consists of tossing a coin countably infinitely many times. An *elementary outcome* from this experiment is an infinite sequence of the symbols H and T, which we can write as $(\varepsilon_i \in \{\text{H, T}\})_{i\in N}$. Prove that the collection, Ω_∞, of all such sequences is uncountable.

ANSWER:
Assume that the sequence $x = (x_k \in \mathit{\Omega})_{k\in\mathbb{N}}$ is countable, from the definition, $x = \{x_1, x_2, x_3, ...\}$.
x_1 : $\mathbf{T}TTTTTTT...$
x_2 : $H\mathbf{H}HHHHHH...$
x_3 : $HT\mathbf{T}HTHTH...$
Then there must be one element of Ω_∞ that is not contained in the sequence x, which is HTH.... Since the first toss is different from the first toss in x_1, the second toss is different from the second toss in x_2, and the third toss is different from the third toss in x_3, etc, this is a new sequence which does not in Ω_∞, and this is a contradiction with Ω_∞ is countable.

2 (0.5)Exercise: Prove that the open interval]0,1[has the same cardinality as the entire real line $\mathbb{R}$.

ANSWER:
Firstly, we need to prove that the open interval]0,1[has the same cardinality as the real line.
Define f:(0,π)$\rightarrow$ $\mathbb{R}$ by

$$f(x) = \tan(x + \frac{\pi}{2}).$$

We need to show that the function is bijective. Since $f^{-1}(x) = \arctan(x + \frac{\pi}{2})$ is the inverse of f, we can say that the function f is bijective.
So, $]0, \pi[$ and $\mathbb{R}$ have the same cardinality.
Then we need to show that]0,1[and $]0, \pi[$have the the same cardinality.
Define g:]0,1[$\rightarrow$]$0, \pi$[by

$$g(x) = \pi x.$$

Then, the inverse is

$$g^{-1}(x) = \frac{x}{\pi}.$$

It is easy to show that $g(x)$ is the inverse of $g^{-1}(x)$. Therefore, g is a bijection, so]0,1[and]0,π[have the same cardinality. By transitivity,]0,1[and $\mathbb{R}$ have the same cardinality.

3 (0.6)Exercise: Give an example of a one-to-one mapping from the interval [0,1] onto to the interval [0,1[. Then construct a one-to-one mapping from the closed interval [0,1] onto the open interval]0,1[.

ANSWER:
(1)From the hint given by Dr. Andrew Lyasoff, we need to define a function $f : [0,1] \to [0,1)$

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{for } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

which $n \in \mathbb{N}_{++}$
This is injective: Let $f(x) = f(y)$ for some $x, y \in [0,1]$. This means they must be in one of the two categories and hence must be either both $\frac{1}{n}$ or both otherwise. In both cases, $x = y$ showing the function is injective.
This is surjective: Let $a \in [0,1)$, then we must have one of the following:
1. That $a = \frac{1}{n}$ for some $n \in \mathbb{N}$, in which case $f(\frac{1}{n}) = a$.
2. That $a \neq \frac{1}{n}$ and so $f(a) = a$ since $a \in [0,1) \subset [0,1]$
Since f is injective and surjective, it is one-to-one from [0,1] to [0,1)

(2) Let $f : [0,1] \to (0,1)$ be given by

$$f(x) = \begin{cases} \frac{1}{2} & \text{for x=0} \\ \frac{1}{n+2} & \text{for } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

This is injective: Let $f(x) = f(y)$ for some $x, y \in [0,1]$. This means they must be in one of the three categories and hence must be either both 0, both $\frac{1}{n}$ or both otherwise. In all cases, $x = y$ showing the function is injective.
This is surjective: Let $a \in (0,1)$, then we must have one of the following:
1. That $a = \frac{1}{2}$, in which case $f(0) = a$
2. That $a = \frac{1}{n+2}$ for some $n \in \mathbb{N}$, in which case $f(\frac{1}{n}) = a$.
3. That $a \neq \frac{1}{n}$ and so $f(a) = a$ since $a \in (0,1) \subset [0,1]$
Since f is injective and surjective, it is one-to-one from [0,1] to (0,1)

4 (0.7)Exercise: Obviously, the set $\mathbb{N}$ is countable by definition, but is the set $\mathbb{N} \times \mathbb{N}$ countable? What about the set $\mathbb{N} \times ... \times \mathbb{N}$(n-times)? Prove that the set of all rational numbers is countable. Prove that the union of any countable collection of countable sets is a countable set. Prove that the set of all points in $\mathbb{R}^2$ with rational coordinates is countable.

ANSWER:
(1). Let $f : N \times N \to N$ be defined by

$$f(n, m) = 2^n 3^m,$$

where $n, m \in N$. This is an injection, since if $f(n, m) = f(n', m')$, then $2^n 3^m = 2^{n'} 3^{m'}$, which implies $(n, m) = (n', m')$.

(2). Let $f : N \times ... \times N \to N$ be defined by

$$f(n, m, o, ..., p) = 2^n 3^m 5^o ... (n^{th}\ prime\ number)^p,$$

where $n, m, o, ..., p \in N$, all base numbers are prime number. By Euclid's theorem, there are infinite prime number. This is an injection, since if $f(n, m, o, ..., p) = f(n', m', o', ..., p')$, then $2^n 3^m 5^o ... 7^p = 2^{n'} 3^{m'} 5^{o'} ... 7^{p'}$, which implies $(n, m, o, ..., p) = (n', m', o', ..., p')$.

(3). Let $Q_+ \to N \times N$ be defined by

$$f(q) = (m, n),$$

where $n, m \in N, q = \frac{m}{n}$. This is injective, since if $f(q) = f(q')$, then (m,n)=(m',n'), which implies $q = q'$. Since $N \times N$ is countable by first part, Q_+ is countable.
Let $Q_+ \to Q_-$ be defined by

$$f(a) = -a,$$

$where a \in Q_+$. This is injective, since if $f(a) = f(b)$, then $-a = -b$, which implies $a = b$. So $Q_-$is countable. Hence set of all rational numbers is countable by part 4, the union of any countable collection of countable sets is a countable set.

(4). Let $f : A_1 \cup A_2 ... \cup A_N \to \mathbb{N} \times \mathbb{N}$ be defined by

$$f(a_{n,m}) = (n, m),$$

where $A_1 = \{a_{1,1}, a_{1,2}, a_{1,3}, ...\}, A_2 = \{a_{2,1}, a_{2,2}, a_{2,3}, ...\}, A_3 = \{a_{n,1}, a_{n,2}, a_{n,3}, ...\}...$
$n, m \in \mathbb{N}$
This is an injection, since if $f(a_{n,m}) = f(a_{n',m'})$, then $(n, m) = (n', m')$, which implies $a_{n,m} = a_{n',m'}$. By part 1, $\mathbb{N} \times \mathbb{N}$ is countable,so $A_1 \cup A_2 ... \cup A_N$ is countable.
(5) Let $f : R \times R \to N \times N$ be defined by

$$f(r_a, r_b) = (a, b)$$

where $a, b \in N$. Since R is countable, there is a natural number pair with a real number. This is injective, since if $f(r_a, r_b) = f(r_{a'}, r_{b'})$, then $(a, b) = (a', b')$, which implies $r_a = r_{a'}, r_b = r_{b'}$. Since $N \times N$ is countable, $R \times R$ is countable.

5 (1.1)Exercise: Find the values for x and y that give:
(a)$V_1(H) = 1$ and $V_1(T) = 0$;
(b)$V_1(H) = 0$ and $V_1(T) = 1$.

ANSWER:
(a)

$$V_1(H) = 1$$
$$V_1(T) = 0$$

$$xS_1(H) + yB_1(H) = 1$$
$$xS_1(T) + yB_1(T) = 0$$

$$xuS_0 + y(1+r)B_0 = 1 \qquad (1)$$
$$xdS_0 + y(1+r)B_0 = 0 \qquad (2)$$

$$x = \frac{1 - y(1+r)B_0}{uS_0} \qquad by\ (1)$$

Put x into (2):

$$\frac{1 - y(1+r)B_0}{uS_0} dS_0 + y(1+r)B_0 = 0$$

Though simplify,

$$y = \frac{d}{(d-u)(1+r)B_0}$$

Put y into previous x:

$$x = \frac{1}{(u-d)S_0}$$

(b)

$$V_1(H) = 0$$
$$V_1(T) = 1$$

$$xS_1(H) + yB_1(H) = 0$$
$$xS_1(T) + yB_1(T) = 1$$

$$xuS_0 + y(1+r)B_0 = 0 \qquad (1)$$
$$xdS_0 + y(1+r)B_0 = 1 \qquad (2)$$

$$x = \frac{-y(1+r)B_0}{uS_0} \qquad by\ (1)$$

Put x into (2):

$$\frac{-y(1+r)B_0}{uS_0} dS_0 + y(1+r)B_0 = 0$$

Though simplify,

$$y = \frac{u}{(u-d)(1+r)B_0}$$

Put y into previous x:

$$x = \frac{1}{(d-u)S_0}$$

6 (1.2)Exercise: Calculate Φ_0^H and Φ_0^T as functions of S_0, B_0, d, u and r.

ANSWER: By 1.1,

$$x = x_H^* = \frac{1}{(u-d)S_0}$$

$$y = y_H^* = \frac{d}{(d-u)(1+r)B_0}$$

By definition,

$$\Phi_0^H = x_H^* S_0 + y_h^* B_0$$

$$\Phi_0^H = \frac{1}{(u-d)S_0} S_0 + \frac{d}{(d-u)(1+r)B_0} B_0$$

After simplification:

$$\Phi_0^H = \frac{1+r-d}{(u-d)(1+r)}$$

By 1.1,

$$x = x_T^* = \frac{1}{(d-u)S_0}$$

$$y = y_T^* = \frac{u}{(u-d)(1+r)B_0}$$

By definition,

$$\Phi_0^T = x_T^* S_0 + y_T^* B_0$$

$$\Phi_0^T = \frac{1}{(d-u)S_0} S_0 + \frac{u}{(u-d)(1+r)B_0} B_0$$

After simplification,

$$\Phi_0^T = \frac{u-r-1}{(u-d)(1+r)}$$

7 (1.7)Exercise: Using simple algebra, prove the relation

$$\Phi_0^H + \Phi_0^T = \frac{1}{1+r}.$$

ANSWER:
$\Phi_0^H + \Phi_0^T = \frac{1+r-d}{(u-d)(1+r)} + \frac{u-r-1}{(u-d)(1+r)} = \frac{1+r-d+u-r-1}{(u-d)(1+r)} = \frac{1}{1+r}$

8 (1.9)Exercise: Using elementary algebra, prove that

$$S_0 = \frac{1}{1+r}(S_1(H)\tilde{p} + S_1(T)\tilde{q}) \qquad \textit{and} \qquad B_0 = \frac{1}{1+r}(B_1(H)\tilde{p} + B_1(T)\tilde{q}).$$

ANSWER:
(1)
$\frac{1}{1+r}(S_1(H)\tilde{p} + S_1(T)\tilde{q})$
$= \frac{1}{1+r}(uS_0\frac{1+r-d}{u-d} + dS_0\frac{u-1-r}{u-d})$
$= S_0\frac{1}{1+r}(u\frac{1+r-d}{u-d} + d\frac{u-1-r}{u-d})$
$= S_0\frac{1}{1+r}(\frac{u+ur-ud}{u-d} + \frac{du-d-dr}{u-d})$
$= S_0\frac{1}{1+r}\frac{(u-d)(1+r)}{u-d}$
$= S_0$
(2)
$\frac{1}{1+r}(B_1(H)\tilde{p} + B_1(T)\tilde{q})$
$= \frac{1}{1+r}((1+r)B_0\frac{1+r-d}{u-d} + (1+r)B_0\frac{u-1-r}{u-d})$
$= \frac{1}{1+r}(1+r)B_0\frac{1+r-d+u-1-r}{u-d}$
$= B_0$

9 (1.12)Exercise: In the setting of the last example, calculate the probability for at least one ace in a hand of 13 cards.

ANSWER:
P(at least one ace in a hand of 13 cards)=1-P(no ace in a hand of 13 cards)=1-$\binom{48}{13}/\binom{52}{13} = 14498/20825 \approx 0.696$

10 (1.14)Exercise: In the context of the last example, calculate $P(A)$, where $A = \{n \in \mathbb{N}$:n is even$\}$.

ANSWER:
$P(A) = \sum_{\omega\in\{n\in\mathbb{N}:n\ is\ even\}} e^{-c}\frac{c_\omega}{\omega!} = e^{-c}\sum_{i\in\mathbb{N}}\frac{c^{2i}}{(2i)!} = e^{-c}\sum_{i=0}^{\infty}\frac{c^{2i}}{(2i)!}$
Let $f(x) = \sum_{i=0}^{\infty}\frac{x^{2i}}{(2i)!}$, and $f'(x) = \sum_{i=1}^{\infty}\frac{x^{2i-1}}{(2i-1)!}$. By (1.13 EXAMPLE), f(x)+f'(x)=e^x

y' + y = e^x
P(x)=1 Q(x)=e^x
By deferential equation fomular, $y = e_{-\int P(x)dx}[\int e^{\int P(x)dx}Q(x)dx + c]$
y=f(x)=$\frac{1}{2}e^x + \frac{1}{2}e^{-x}$
P(A)=$e^{-c}f(x) = e^{-c}(\frac{1}{2}e^c + \frac{1}{2}e^{-c}) = \frac{1}{2} + \frac{1}{2}e^{-2c}$

11 (1.15)Exercise: Let $\Omega = \{1, 2, ...\}$. Suppose that the mass placed at the integer ω is $\frac{C}{\omega^2}$ for every $\omega \in \Omega$. Choose the constant C so that the total mass of Ω equals 1 - just as in (1.13). Would it be possible to make the same choice if the mass placed at the integer ω is $\frac{C}{\omega}$?

ANSWER:
Since $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, $\frac{6}{\pi^2}\sum_{i=1}^{\infty} \frac{1}{i^2} = 1$. So $C = \frac{6}{\pi^2}$.
It is not possible to make the same choice if the mass placed at the integer ω is $\frac{C}{\omega}$ because $\sum_{x=1}^{\infty} \frac{1}{x}$ diverges.

12 (1.20)Exercise: Prove that a nonempty collection of subsets, $\mathscr{A} \subseteq 2^{\Omega}$, is an algebra if and only if it is closed under finite union and complement, that is,A,B $\in \mathscr{A}$ implies $A \cup B \in \mathscr{A}$, and $A \in \mathscr{A}$ implies $A^C \in \mathscr{A}$.

ANSWER:
Prove:
$\mathscr{A}$ is an algebra $\Leftrightarrow$ $\mathscr{A}$ is closed under finite union and complement
$\Rightarrow$
closed under finite union and complement is the definition of algebra
$\Leftarrow$
$A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$ is closed under complement
$\cup_{i \in N_{|n}} A_i \in \mathscr{A}\ for\ (A_i \in \mathscr{A})_{i \in N_{|n}}$ is closed under finite union
To prive $\emptyset \in \mathscr{A}$
If $A \in \mathscr{A}$, then $A \cup A^c = \Omega \in \mathscr{A}$. Since $\Omega \in \mathscr{A}$, $\Omega^c = \emptyset \in \mathscr{A}$

13 (1.22)Exercise: Prove that the intersection of *any* (countable or uncountable) family of σ-fields (over one and the same sample space Ω) is always a σ-field, and explain why the same claim cannot be made about a union of σ-fields - even about the union of just two σ-fields.

ANSWER:
Intersection:

Assume $\mathscr{A}, \mathscr{B}, ...$ are σ field. We need to prove $\mathscr{A} \cap \mathscr{B} \cap ...$ is σ field.
By definition, $\emptyset \in \mathscr{A}$, $\emptyset \in \mathscr{B}$,.... So, $\emptyset \in \mathscr{A} \cap \mathscr{B} \cap ...$
If $A \in \mathscr{A} \cap \mathscr{B} \cap ...$, then $A \in \mathscr{A}$, $A \in \mathscr{B}$,.... So, $A^c \in \mathscr{A}$, $A^c \in \mathscr{B}$,..... Thus, $A^c \in \mathscr{A} \cap \mathscr{B} \cap ....$
For any sequence $(A_i \in \mathscr{A} \cap \mathscr{B} \cap ...)_{I \in N}$, then $\cup_{i \in N} A_i \in \mathscr{A}$, $\cup_{i \in N} A_i \in \mathscr{B}$....
So, $\cup_{i \in N} A_i \in \mathscr{A} \cap \mathscr{B} \cap ...$

Union:
Proof union of σ-fields is not a σ-fields by contradiction.
Assume $\Omega = \{a, b, c, d\}$:
$\mathscr{A} = \{\{a\}, \{b, c, d\}, \Omega, \emptyset\}$
$\mathscr{B} = \{\{a\}, \{b, c, d\}, \Omega, \emptyset\}$
$\mathscr{A} \cup \mathscr{B}$ is a σ-fields.
Then $\mathscr{A} \cup \mathscr{B} = \{\{a\}, \{b\}, \{a, c, d\}], \{b, c, d\}, \Omega, \emptyset\}$. Since $\{a\} \cup \{b\} \notin \mathscr{A} \cup \mathscr{B}$, this is a contradiction. So union of σ-fields is not a σ-fields.

14 (1.28)Exercise: Let$(\mathbb{X}, \mathcal{S}, \mu)$ be any measure space and let $\emptyset \neq A \in \mathcal{S}$ be any nonempty element of $\mathcal{S}$. Prove that the set function

$$A \cap \mathcal{S} \ni B \rightsquigarrow \mu(B)$$

is a measure on the σ-field $A \cap \mathcal{S}$. It is called $the measure induced on A by \mu$ and is nothing but the restriction $\mu \upharpoonright (A \cap \mathcal{S})$. Since there is no risk of ambiguity, this restriction is still denoted by μ, and the measure space $(A, A \cap \mathcal{S}, \mu \upharpoonright (A \cap \mathcal{S}))$ is often written more succinctly in the form $(A, A \cap \mathcal{S}, \mu)$.

Conversely, show that if μ is a measure on $A \cap \mathcal{S}$, for some set $A \in \mathcal{S}$, then μ gives rise to a measure on $\mathcal{S}$ given by $\mu(B) = \mu(B \cap A)$ for any $B \in \mathcal{S}$.

ANSWER:
Since $(\mathbb{X}, \mathcal{S}, \mu)$ is an measure space, we have $\mu(\cup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$. Given $\emptyset \neq A \in \mathcal{S}$, let $\mathcal{S} = A_1, A_2, ...$ $A_1 \cap \mathcal{S}$=$\{A_1 \cap A_1, A_1 \cap A_2, A_1 \cap A_3, ...\}$. Since $\mathcal{S}$ is a $\sigma-$ field, $A_1 \cap A_{i_{i \in \mathbb{N}}} \in \mathcal{S}$. So, μ is a measure on the σ-field $A \cap \mathcal{S}$.
Conversly, if μ is a measure on $A \cap \mathcal{S}$, for some set $A \in \mathcal{S}$, $then$ $\mu(\cup_{i \in \mathbb{N}} A_1 \cap A_i) = \sum_{i \in \mathbb{N}} \mu(A_1 \cap A_i)$. Let $B \in \mathcal{S}$,Since $B \cap A_i \in A_1 \cap A_{i_{i \in \mathbb{N}}}$ and let $\mu(B) = \mu(B \cap A)$, so μ is a measure on $\mathcal{S}$.

15 (1.36)Exercise: Consider the following chain of partitions of Ω :

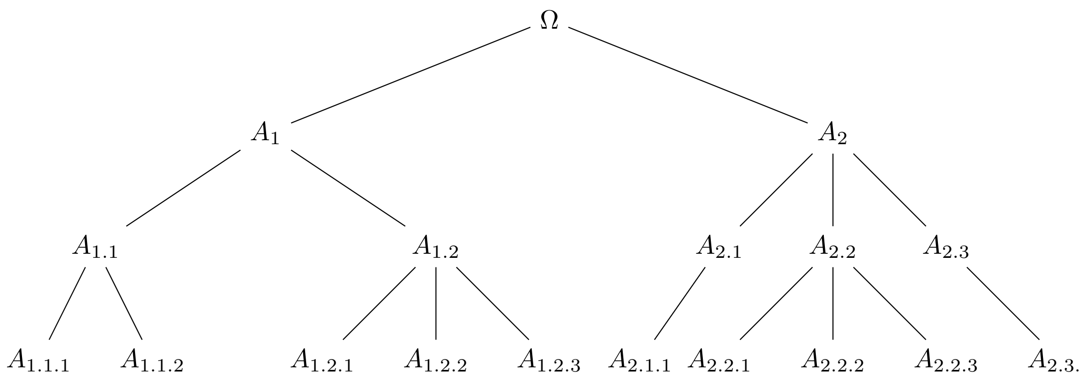
$$\mathscr{P}_0 = \{\Omega\},$$

$$\mathscr{P}_1 = \{A_1, A_2\},$$

$$\mathscr{P}_2 = \{A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}, A_{2,3}\},$$

$$\mathscr{P}_3 = \{A_{1,1,1}, A_{1,1,2}, A_{1,2,1}, A_{1,2,2}, A_{1,2,3}, A_{2,1}, A_{2,2,1}, A_{2,2,2}, A_{2,2,3}, A_{2,3}\},$$

where the addition of a subscript means partitioning of the respective subset; for example, $A_{1,1} \cup A_{1,2} = A_1$, $A_{1,2,1} \cup A_{1,2,2} \cup A_{1,2,3} = A_{1,2}$, etc. Plot the event tree associated with the chain of partitions $\{\mathscr{P}_0 \mathscr{P}_1 \mathscr{P}_2 \mathscr{P}_3\}$.



16 (1.38)Exercise: (follow-up to (1.36))There are 10 level-3 nodes in the event-tree introduced in (1.36), or, to put it another way, the partition $\mathscr{P}_3$ consists 10 non-overlapping subsets of Ω. Suppose that all elements of $\mathscr{P}_3$ have equal probability, so that $P(A) = \frac{1}{10}$ for every $A \in \mathscr{P}_3$. Calculate the probabilities of the following events from the σ-field $sigma(\mathscr{P}_3)$:

$$A_1, \quad A_2, \quad A_{1,2} \cup A_2, \quad A_1 \cup A_{2,1} \cup A_{2,2,2} \cup A_{2,2,3}.$$

ANSWER:
$P(A_1) = P(A_{1.1}) \cap P(A_{1.2}) = P(A_{1.1.1} \cap A_{1.1.2} \cap A_{1.2.1} \cap A_{1.2.2} \cap A_{1.2.3}) = P(A_{1.1.1}) + P(A_{1.1.2}) + P(A_{1.2.1}) + P(A_{1.2.2}) + P(A_{1.2.3}) = \frac{1}{2}$

$P(A_2) = P(A_{2.1}) \cap P(A_{2.2}) \cap P(a_{2.3}) = P(A_{2.1.1} \cap A_{2.2.1} \cap A_{2.2.2} \cap A_{2.2.3} \cap A_{2.3.1}) = P(A_{2.1.1}) + P(A_{2.2.1}) + P(A_{2.2.2}) + P(A_{2.2.3}) + P(A_{2.3.1}) = \frac{1}{2}$

$P(A_{1.2} \cup A_2) = P(A_{1.2.1} \cup A_{1.2.2} \cup A_{1.2.3} \cup A_2) = P(a_{1.2.1}) + P(A_{1.2.2}) + P(A_{1.2.3}) + P(A_2) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{2} = \frac{4}{5}$

$P(A_1 \cup A_{2.1} \cup A_{2.2.2} \cup A_{2.2.3}) = P(A_1 \cup A_{2.1.1} \cup A_{2.2.2} \cup A_{2.2.3}) = P(A_1) + P(A_{2.1}) + P(A_{2.2.2}) + P(A_{2.2.3}) = \frac{1}{2} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{5}$

17 (1.40)Exercise: Prove that all probability measures P_t, $t \in \mathbb{N}$, are compatible, in the sense that for every fixed $u > t, P_u(A) = P_t(A)$ for every $A \in \mathscr{F}_t$.

ANSWER:
Set $A = \cup_{i=1}^{n_t} A_{t,i}$ where $A_{t,i} \in \mathscr{P}_t$. Since $\mathscr{P}_{t+1}$ is a refinement of $\mathscr{P}_t$, we will have $A = \cup_{i=1}^{n_{t+1}} A_{t+1,i}$ where $A_{t_1,i} \in A_{t,j}$.
Given $P_t(A_{t,i}) = \frac{1}{2^t}$, $A_{t,i} = A_{t+1,i_1} \cup A_{t+1,i_2}$, $P_{t+1}(A_{t,i}) = P_{t+1}(A_{t+1,i_1}) + P_{t+1}(A_{t+1,i_2}) = \frac{1}{2^{t+1}} + \frac{1}{2^{t+1}} = \frac{1}{2^t} = P_t(A_{t,i})$. This is true for all t, by recursion, it is true for all u, t, where u,t$\in \mathbb{N}$. So all probability measures $P_t, t \in \mathbb{N}$, are compatible.

18 (1.46)Exercise: Prove that for any $\omega \in \Omega_\infty$, the singleton $\{\omega\}$ is a measurable set (that is $\{\Omega\} \in \mathscr{F}_\infty$) and that $P_\infty(\{\omega\}) = 0$.

ANSWER:
Assume that we have Ω_∞ and $A \in \mathcal{A} = \cup_{t=1}^{\infty} \mathcal{F}_t$, where $\mathcal{F}_t = \sigma(P_t)$. Let $F_\infty = \sigma(\mathcal{A})$. Assume singleton $\{\omega\} = \{+1, +1, -1, -1, ...\}$, $A_1 \in P_1$,$\{\omega\} \in A_1$; $A_2 \in P_2$, $\{\omega\} \in A_2$... So, $\omega \in A_t$ with the consequence. The singleton $\{\omega\} = \cap_{t=1}^{\infty} A_t$ because it cannot contain two element. If there are two elements, there will be another partition. So $\{\omega\} = \cap_{t=1}^{\infty} A_t$. Since $A_t \in \mathcal{F}_t \in \mathcal{F}_\infty$, $\{\omega\} \in \mathcal{F}_\infty$
Since $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$ and $A = \cap_{i \in \mathbb{N}} A_i$, then P(A)=$lim_i P(A_i)$.
So here is $P(\cap_{t=1}^{\infty} A_t) = lim_{t \to \infty} \frac{1}{2^t} = 0$

19 (1.49)Exercise: Explain why the random event in $\mathfrak{E}_1$ can be described as "infinitely many events A_i occur" and can be expressed as $\{A_i$ i.o.$\}$.

ANSWER:
$\cap_{i \in \mathbb{N}} \cup_{j \geq i} A_j$ can be explained with two part. Firstly, $\cup_{j \geq i} A_j$ means that if this union happen, there is at least one A_j happens. Secondly, intersect means and. At this whole event, this means whatever from which A_i, there are alway some A_j happen after the A_i. So there are infinitely many events A_i occur.

20 (1.51)EXERCISE: Prove – yet again, but this time using the first Borel–Cantelli lemma – that the probability to toss a coin infinitely many times and to never observe "heads," after some, however large, finite number of tosses, is 0.

ANSWER:
If we want to know the probability of never observe "heads", we can use the first Borel-Cantelli lemma. Let A_i be the event that the first i^{th} tosses don't have heads. $\sum^{i \in \mathbb{N}} p(A_i) = \sum_{i=1}^{\infty} 2^{-i} = 1 < \infty$, so $p(limsup_i A_i) = 0$

21 (1.58)EXERCISE: Prove that if the set function $\mu_* : \mathscr{R} \mapsto \mathbb{R}_+$, defined through the relation $\mathfrak{C}_2$ for some increasing function $F : \mathbb{R} \mapsto \mathbb{R}$ can be claimed to be countably additive, then the function $F(\cdot)$ must also be right-continuous, that is to say, must be an increasing càdlàg function.

ANSWER:
If μ_* is countably additive, then $\mu_*(]a, b + \frac{1}{n}]) \to \mu_*(]a, b])$.
So, $F(b + \frac{1}{n}) - F(a)$ converge to $F(b) - F(a)$.
So, $F(b + \frac{1}{n})$ converge to $F(b)$ which means F is right continuous.

22 (1.63)Exercise: Prove that the Lebesgue measure Λ (or, more generally, Λ^n) is nonatomic: for any $x \in \mathbb{R}$, one has $\{x\} \in \mathcal{B}(\mathbb{R})$ and $\Lambda(\{x\}) = 0$. Conclude that for every $-\infty < a < b < +\infty$,

$$\Lambda([a, b]) = \Lambda(]a, b]) = \Lambda([a, b[) = \Lambda(]a, b[).$$

Prove that if $C \subset \mathbb{R}$ is any countable set inside $\mathbb{R}$, then $C \in \mathcal{B}(\mathbb{R})$ and $\Lambda(C) = 0$. Conclude that the set $\mathbb{Q}$ of all rational numbers is a Borel set and $\Lambda(\mathbb{Q}) = 0$. Prove that the set of all points in $\mathbb{R}^2$ that have rational coordinates is a Borel set and Lebesgue measure of that set is 0.

ANSWER:
(1): $\Lambda(\{x\}) = \Lambda(]x - \frac{1}{n}, x]) = lim_{n\to\infty} \frac{1}{n} = 0$
(2):
$\Lambda([a, b]) = \Lambda(\{a\} \cup]a, b]) = 0 + (b - a) = b - a$
$\Lambda(]a, b]) = b - a$
$\Lambda([a, b[) = \Lambda(\{a\} \cup]a, b] \setminus \{b\}) = b - a$
$\Lambda(]a, b[) = \Lambda(]a, b] \setminus \{b\})$=b-a
So these four are equal.
(3):
Since $C \subset \mathbb{R}$ is any countable set inside $\mathbb{R}$, every element in C is singleton. Since $\{x\} = \cap_{n\in\mathbb{N}}]x - \frac{1}{n}, x + \frac{1}{n}[$, every singleton is Borel Set. So C is Borel Set which is union of singleton.
Since C is countable, and $\Lambda(\{x\}) = 0$, $\Lambda(C) = 0$.
(4)
$\forall \{x\} = (\{x_1\}, \{x_2\}) \in \mathbb{R}^2$
$(\{x_1\}, \{x_2\}) = (\cap_{n\to\infty}(x_1 - \frac{1}{n}, x_1 + \frac{1}{n}), \cap_{n\to\infty}(x_2 - \frac{1}{n}, x_2 + \frac{1}{n}))$, so it is Borel set.
$\Lambda^2(x) = \Lambda(\cap_{n\to\infty}(x_1 - \frac{1}{n}, x_1 + \frac{1}{n}))\Lambda(\cap_{n\to\infty}(x_2 - \frac{1}{n}, x_2 + \frac{1}{n}))$
$= lim_{x\to\infty}()(x_1 + \frac{1}{n} - x_1 + \frac{1}{n})(x_2 + \frac{1}{n} - x_2 + \frac{1}{n}) = lim_{n\to\infty}(\frac{2}{n})^2 = 0$

23 (1.65)Prove that if μ is any purely atomic measure on $\mathbb{R}$, then $\mu \perp \Lambda$. Explain how one can construct a purely atomic measure Q on $\mathbb{R}$ that is supported on the (countable) set of rational numbers $\mathbb{Q}$, in that Q($\mathbb{R} \setminus \mathbb{Q}$)=0.

ANSWER:
(1)Let $(\mathbb{X}, \mathcal{S}, \mu)$ be a measure. $\mu(A) = \sum_{x\in\mathcal{X}\cap A} \mu(\{x\})$ for all $A \in \mathcal{S}$ (=0 if

$\mathcal{X} \cap A = \emptyset$)
By definition, $\mathcal{X}$is countable so $\mathcal{X} \cap A$is countable. So $\mu(A^C) = 0$. $\Lambda(A) = 0$by previous questions. So $\mu(A^C) = \Lambda(A) = 0$
(2). Let $Q(A) = A \cap 1$, then $Q(\mathbb{R} \setminus \mathbb{Q}) = 0$

24 (1.68)Exercise: Prove that the Cantor set C is a Borel set ($C \in \mathcal{B}([0,1])$) and that $\Lambda(C) = 0$. Construct a nonatomic probability measure P on $\mathcal{B}([0,1])$ which is supported by the Cantor set C, in that $P(C^C) = 0$, and is therefore singular to $\Lambda(P \perp \Lambda)$.

ANSWER:
(1)Since every time the orgin set change to two smaller Borel sets, the intersection of many Borel set is still Borel set. (2) A_i is a sequence of sets, $\Lambda(\cap_{i \in \mathbb{N}} A_i) = lim_{i \to \infty} (\frac{2}{3})^n = 0$
(2) Since $P_\infty(C) = 1$, we need to extend P_∞ to $\mathcal{B}([0,1])$, where $P_\infty([0,1] \setminus C) = 0 = \Lambda(C)$

25 (1.69)Exercise: Prove that the set of real numbers [0,1] is an uncountably infinite. Conclude that the same property must hold also for the set of all reals $\mathbb{R}$.

ANSWER:
Since Cantor set on [0,1] is part of real number on [0,1] and Cantor set on [0,1] is uncountably infinite, set of real numbers [0,1] is also uncountably infinite. Since real number on [0,1] is part of whole real number set, all reals is uncountably infinite.

26 (1.70)Exercise: Consider the graph of the function $y = x^2$ as a subset of $\mathbb{R}^2$:

$$\Gamma = \{(x, x^2) := \infty < x < \infty\} \subset \mathbb{R}^2$$

. Prove that $\Gamma \in \mathcal{B}(\mathbb{R}^2)$ and compute $\Lambda^2(\Gamma)$.

ANSWER:
Suppose we have [a,b]*[a,b], P_i is partion with 2^i elements. Let $A_i$be the union of rectangles in P_i cover the curve. $A_1 \subseteq A_2 \subseteq A_3 \subseteq ..., \Gamma = \cap_{n=1}^{\infty} A_i$,every rectangle is a borel set on $\mathbb{R}^2$ so the Γ is borel set.
Since this is a outer continuity, $\Lambda^2(\Gamma) = lim_{i \to \infty} \sum_{i=1}^{2^n} (t_i - t_{i-1})^2 = 0$