Stochastic Methods in Asset Pricing

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Crash Course in Continuous Time Finance – II

(sections 13.3, 13.4)

Andrew Lyasoff

Summary

Equivalent Local Martingale Measures

Tame Trading Strategies

Completeness and Arbitrage: The Fundamental Theorem of Asset Pricing

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for some $\xi \in \mathscr{L}^1(\Omega, \mathscr{G}_T, Q)$, $\quad K_t^{0,\pi} \geq \mathsf{E}^Q\left[(S_T^\circ)^{-1}\,\xi \,|\, \mathscr{G}_t\right] = (S_t^\circ)^{-1}\mathsf{E}^Q\left[e^{-\int_t^T r_s\,\mathrm{d}s}\,\xi \,|\, \mathscr{G}_t\right].$

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Corollary: The initial endowment, x, with which one can initiate a tame trading strategy π that generates the payoff V_T and funds the consumption stream c, must be at least as large as the time 0 market value of the payoffs that the trading strategy is meant to generate. There is no tame trading strategy that "beats the market."

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N.B. The most efficient trading strategies are the ones that are "as good as the market," i.e., generate the payoffs with the smallest possible initial endowment: $x^* = \mathsf{E}^Q\Big[(S_T^\circ)^{-1}V_T + (S^\circ)^{-1}c \bullet \iota_T\Big] < +\infty$.

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Remark: For a self-financing investment-payout strategy everything is in the reverse: if the trading strategy is Q-admissible, the amount, x, which is borrowed at time 0, cannot exceed the expected (under Q) discounted cumulative payments on the loan, i.e.,

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N.B. Transforming the excess returns $X = M + \langle M, \Theta\rangle$ into a local martingale is the same as transforming the normalized excess returns $\beta = W + \langle W, \Theta\rangle$ into a local martingale, i.e., into a Brownian motion (recall that Θ is the local martingale that gives the "pricing rule for marketable risk").

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Due to Girsanov's theorem, we can take $Q \overset{\text{def}}{=} e^{-\Theta_T - \frac{1}{2}\langle\Theta,\Theta\rangle_T} \odot P$, provided that this exponent has expectation $= 1$ (e.g., if Kazamaki's or Novikov's conditions are satisfied).

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First Fundamental Theorem of Asset Pricing:

If no ELMM exists then there is a universally tame trading strategy which represents a *guaranteed* arbitrage. If at least one ELMM exists, then for any such measure Q one can claim that no arbitrage with a Q-tame trading strategy is possible.

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$$P\text{-}\lim_{n\to\infty}\left(x_n + (S^\circ)^{-1}(\pi^n)^\intercal \cdot X_T\right) = \xi\,.$$

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$$\mathsf{P}\text{-}\lim_{n\to\infty}\Big(x_n + (S^\circ)^{-1}(\pi^n)^\intercal \bullet X_T\Big) = \xi\,.$$

Second Fundamental Theorem of Asset Pricing:

A financial market that admits at least one ELMM is complete if and only if there is exactly one ELMM.

An Alternative Characterization:

A financial market with excess-returns process X, interest-rate process r, and information process Y that admits an ELMM is complete if and only if the law of the normalized excess-returns β uniquely defines the law of the process (X, r, Y), within the class of laws that are equivalent to the law of (X, r, Y).

Completeness and Arbitrage: The Fundamental Theorem of Asset Pricing

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In particular, the market would be complete if the normalized excess returns process β generates the market filtration $\mathscr{G}$.