Zihan Ma
MF 795
Assignment 6

13.9

$$< \mathcal{M}, \Theta > = < \sigma \cdot W, \Theta^T \cdot W >$$
$$= \begin{pmatrix} < \sigma_1 \cdot W, \Theta^T \cdot W > \\ < \sigma_2 \cdot W, \Theta^T \cdot W > \\ \cdots \end{pmatrix}$$

Note that,

$$< \sigma_1 \cdot W, \Theta^T \cdot W > = \sigma_1 (\theta^T)^T \cdot i$$
$$= \sigma_1 \sigma^{-1} \left(b - r\vec{1}\right) \cdot i$$
$$= (1,0,0, \dots) \begin{pmatrix} b_1 - r \\ b_2 - r \\ \cdots \end{pmatrix} \cdot i$$
$$= (b_1 - r) \cdot i$$

Therefore,

$$< \mathcal{M}, \Theta > = \begin{pmatrix} b_1 - r \\ b_2 - r \\ \cdots \end{pmatrix} \cdot i$$
$$= \left(b - r\vec{1}\right) \cdot i$$

13.12

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} dV_t + V_t d\left(e^{-\int_0^t r_u du}\right) + \frac{1}{2} x \cdot 0$$
$$= e^{-\int_0^t r_u du} dV_t + V_t e^{-\int_0^t r_u du} \cdot (r_t) dt$$

As $dV_t = V_t r_t dt + \pi_t{}^T dX_t - c_t dt$, then

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} (V_t r_t dt + \pi_t{}^T dX_t - c_t dt) - V_t e^{-\int_0^t r_u du} (r_t) dt$$
$$= e^{-\int_0^t r_u du} \pi_t{}^T dX_t - e^{-\int_0^t r_u du} c_t dt$$
$$V_t e^{-\int_0^t r_u du} = V_0 + \int_0^t e^{-\int_0^s r_u du} \pi_s{}^T \, dX_s - \int_0^t e^{-\int_0^s r_u du} c_s \, ds$$
$$\Longrightarrow V_t = e^{\int_0^t r_u du} \left(x + \int_0^t e^{-\int_0^s r_u du} \pi_s{}^T \, dX_s - \int_0^t e^{-\int_0^s r_u du} c_s \, ds \right)$$

13.13

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} dV_t + V_t e^{-\int_0^t r_u du} \cdot (r_t) dt$$

As $dV_t = V_t r_t dt - \pi_t{}^T dX_t - c_t dt$, then

$$d\left(V_t e^{-\int_0^t r_u du}\right) = e^{-\int_0^t r_u du} (V_t r_t dt - \pi_t{}^T dX_t - c_t dt) + V_t e^{-\int_0^t r_u du} (r_t) dt$$
$$= -e^{-\int_0^t r_u du} \pi_t{}^T dX_t - e^{-\int_0^t r_u du} c_t dt$$
$$V_t e^{-\int_0^t r_u du} = V_0 - \int_0^t e^{-\int_0^s r_u du} \pi_s{}^T \, dX_s - \int_0^t e^{-\int_0^s r_u du} c_s \, ds$$
$$\Longrightarrow V_t = e^{\int_0^t r_u du} \left(V_0 - \int_0^t e^{-\int_0^s r_u du} \pi_s{}^T \, dX_s - \int_0^t e^{-\int_0^s r_u du} c_s \, ds \right)$$

13.18

As π is Q-tame, then $K^{0,\pi} - M \geq 0$. Thus, $X = \sigma \cdot (W + \theta \cdot i)$, then X is $(\mathcal{G}, P)$-semimartingale and Q is an ELMM for P, which implies X is $(\mathcal{G}, Q)$ local martingale. Then, we can get $K^{0,\pi} = (S^0)^{-1}\pi^T \cdot X$ is a local martingale. As M is also a $(\mathcal{G}, Q)$ local martingale, then $K^{0,\pi} - M$ is a positive $(\mathcal{G}, Q)$ local martingale, which means $K^{0,\pi} - M$ is a supermartingale.

$$\begin{aligned} E\left[{K_t}^{0,\pi}\middle|\mathcal{F}_s\right] &= E\left[{K_t}^{0,\pi} - M_t + M_t\middle|\mathcal{F}_s\right] \\ &= E\left[{K_t}^{0,\pi} - M_t\middle|\mathcal{F}_s\right] + E[M_t|\mathcal{F}_s] \\ &\leq {K_s}^{0,\pi} - M_s + M_s = {K_s}^{0,\pi} \end{aligned}$$

Therefore, $K^{0,\pi}$ is a $(\mathcal{G}, Q)$ supermartingale.

13.19

As $(S^0)^{-1}\pi^T \cdot X$ is a supermartingale with respect to Q, and $x^* = E^Q[(S_T^0)^{-1}V_T + ((S^0)^{-1}c \cdot i)_T]$, then, $x^* + (S^0)^{-1}\pi^T \cdot X$ is a supermartingale. Therefore, $(S^0)^{-1}V + ((S^0)^{-1}c \cdot i)$ is also a $(\mathcal{G}, Q)$ supermartingale.

$$E^Q[(S_0^0)^{-1}V_0 + ((S^0)^{-1}c \cdot i)_0] = E^Q[V_0] = x^*$$

Therefore, $(S^0)^{-1}V + ((S^0)^{-1}c \cdot i)$ is a $(\mathcal{G}, Q)$ martingale, and $x^* + (S^0)^{-1}\pi^T \cdot X$ is a $(\mathcal{G}, Q)$ martingale.

13.20

$$\beta = \sigma^{-1} \cdot X$$

As X is a $(\mathcal{G}, Q)$-martingale, then β is also a $(\mathcal{G}, Q)$-martingale.

$$\beta = W + \theta \cdot < W, W > = W + \theta \cdot i$$

Then, β is continuous, which also means β is a continuous $(\mathcal{G}, Q)$ local martingale.

$$\beta_0 = W_0 + (\theta \cdot i)_0 = 0$$

$$< \beta, \beta > = < W + \theta \cdot i, W + \theta \cdot i > = < W, W > = i$$

Therefore, we can say that β is a BM w.r.t Q.

14.9

Suppose we have (x, π^+, c) is a Q-tame self-financing strategy and is an upper hedging strategy; (x', π^-, c') is a Q-tame self-financing strategy and is a lower hedging strategy. Then, we have:

$$(S^0)^{-1}V + (S^0)^{-1}c \cdot i = x + (S^0)^{-1}(\pi^+)^T \cdot X$$

$$(S^0)^{-1}V' + (S^0)^{-1}c' \cdot i = x' - (S^0)^{-1}(\pi^-)^T \cdot X$$

As proved in (13.18), $K^{0,\pi^+} = (S^0)^{-1}(\pi^+)^T \cdot X$ is a supermartingale relative to Q; thus,), $K^{0,\pi^-} = -(S^0)^{-1}(\pi^-)^T \cdot X$ is a submartingale.

$$\begin{aligned} &\therefore x \geq E^Q[(S^0)^{-1}V + (S^0)^{-1}c \cdot i]_t \\ &x' \leq E^Q[(S^0)^{-1}V' + (S^0)^{-1}c' \cdot i]_t \\ &\Longrightarrow x - x' \geq E^Q[(S^0)^{-1}(V - V') + (S^0)^{-1}(c - c') \cdot i]_t \end{aligned}$$

As $c_t \geq \varphi_t \geq c_t'$, and $V_T \geq \Phi \geq V_T'$, then $V_T - V_T' \geq 0$, $c - c' \geq 0$. Therefore,

$$(x - x')_T = x - x' \geq E^Q[(S^0)^{-1}(V - V') + (S^0)^{-1}(c - c') \cdot i]_t \geq 0$$

Then, we can claim that $x - x' \geq 0$ for all initial endowment and $\pi^+ \geq \pi^-$.

14.11

(a)

From (14.10), we can have:

$$M_0 = E^Q[(S_T^0)^{-1}\Phi + \int_0^T (S_u^0)^{-1}\,\varphi_u du]$$

Also, from (14.9), we know:

$$V_0^+ = x \geq \pi^+ \geq E^Q[(S^0)^{-1}V + (S^0)^{-1}c \cdot i]_t \geq E^Q[(S_T^0)^{-1}\Phi + ((S_T^0)^{-1}\varphi \cdot i)_T] = M_0$$
$$V_0^- = x' \leq \pi^- \leq E^Q[(S^0)^{-1}V' + (S^0)^{-1}c \cdot i]_t \leq E^Q[(S_T^0)^{-1}\Phi + ((S_T^0)^{-1}\varphi \cdot i)_T] = M_0$$
$$\therefore V_0^+ \geq M_0 \geq V_0^-$$

And, we can further get

$$\pi^+ \geq M_0 \geq \pi^-$$

(b)

$$\pi_t^{\pm} = \pm S_t^0(\sigma_t^T)^{-1}h_t$$
$$K^{0,\pi^{\pm}} = (\pm h^T(\sigma)^{-1} \cdot X)_t = \left(\pm h^T(\sigma)^{-1} \cdot (\sigma \cdot \beta)\right)_t$$
$$= \pm(h^T \cdot \beta)_t = \pm(M_t - M_0)$$

As M_t by definition is a $(\mathcal{G}, Q)$-martingale, and as $K^{0,\pi^{\pm}} = \pm(M_t - M_0)$, then $K^{0,\pi^{\pm}}$ is a martingale. Therefore, $K^{0,\pi^{\pm}} \geq= \pm(M_t - M_0)$, which implies that $\pi_t^{\pm} = \pm S_t^0(\sigma_t^T)^{-1}h_t$ are both Q-tame.

(c)

The investment consumption strategy is (M_0, π^+, φ), then:

$$V^{M_0,\pi^+,\varphi} = (S^0)(M_0 + (S^0)^{-1}(\pi^+)^T \cdot X - (S^0)^{-1}\varphi \cdot i$$
$$V_T = (S_T^0)(M_0 + M_T - M_0 - ((S^0)^{-1}\varphi \cdot i)_T) = (S_T^0)M_T - (S_T^0)((S^0)^{-1}\varphi \cdot i)_T$$
$$M_T = [(S_T^0)^{-1}\Phi + \int_0^T (S_u^0)^{-1}\,\varphi_u du] = [(S_T^0)^{-1}\Phi + ((S_T^0)^{-1}\varphi \cdot i)_T]$$
$$\Longrightarrow V_T = \Phi + S_T^0(((S^0)^{-1}\varphi \cdot i)_T) - S_T^0(((S^0)^{-1}\varphi \cdot i)_T) = \Phi$$
$$\therefore \mathrm{V_T} \geq \Phi$$

As $c = \varphi$, then $c \geq \varphi$, therefore, (M_0, π^+, φ) is an upper hedging strategy, which implies $M_0 \geq \pi^+$.

(d)

The investment consumption strategy is (M_0, π^-, φ), then:

$$V^{\widetilde{M_0,\pi^-},\varphi} = (S^0)(M_0 - (S^0)^{-1}(\pi^+)^T \cdot X - (S^0)^{-1}\varphi \cdot i$$
$$\widetilde{V_T} = (S_T^0)(M_0 + M_T - M_0 - ((S^0)^{-1}\varphi \cdot i)_T) = (S_T^0)M_T - (S_T^0)((S^0)^{-1}\varphi \cdot i)_T$$
$$M_T = [(S_T^0)^{-1}\Phi + \int_0^T (S_u^0)^{-1}\,\varphi_u du] = [(S_T^0)^{-1}\Phi + ((S_T^0)^{-1}\varphi \cdot i)_T]$$
$$\Longrightarrow \tilde{V}_T = \Phi + S_T^0(((S^0)^{-1}\varphi \cdot i)_T) - S_T^0(((S^0)^{-1}\varphi \cdot i)_T) = \Phi$$
$$\therefore \widetilde{\mathrm{V}}_\mathrm{T} \leq \Phi$$

As $c = \varphi$, then $c \leq \varphi$, therefore, (M_0, π^-, φ) is a lower hedging strategy, which implies $M_0 \leq \pi^-$.

(e)

$$\pi^+ \leq M_0 \leq \pi^-$$

And,

$$\pi^+ \geq \pi^-$$

Therefore,

$$
\begin{aligned}
\pi^+ &= \pi^- = M_0 \\
&= \pi \\
&= E^Q[(S_T^0)^{-1}\Phi + \int_0^T (S_u^0)^{-1}\,\varphi_u du|\mathcal{G}_0] \\
&= E^Q[(S_T^0)^{-1}\Phi + \int_0^T (S_u^0)^{-1}\,\varphi_u du]
\end{aligned}
$$

14.13

According to (14.11), we know that π will be an upper hedging strategy if $\pi = M_0$ and $\pi = S_t^0(\sigma_t^T)^{-1}h_t$. Likewise, we can have $-\pi = -S_t^0(\sigma_t^T)^{-1}h_t$ is a lower hedging strategy. Thus, the investment-consumption strategy in (14.11b) is a replicating strategy.

As $V_T = \Phi$, $c = \varphi$, and $(S^0)^{-1}V + (S^0)^{-1}\varphi \cdot i$ is $\mathcal{G}$-martingale, then M_t is a $\mathcal{G}$-martingale. As the market is complete, then the ELMM is unique. Assume $\exists h'$ that also satisfies $M_t - M_0 = h' \cdot \beta$ and $M_t - M_0 = h \cdot \beta$ where β is a BM. Then, obviously

$$
\begin{gathered}
(h-h')\cdot\beta = 0 \\
\Longrightarrow <(h-h')\cdot\beta, (h-h')\cdot\beta> = 0 \\
\Longrightarrow (h-h')^2\cdot t = 0 \\
\therefore h = h'
\end{gathered}
$$

Therefore, we can claim that h is unique, which implies that the replicating strategy is unique.

14.16

In the context of BSM, $\sigma_t = \sigma =$constant, $r_t = r =$ constant. Then the contingent claim $\mathcal{K}_0(T, S_T^4)$:

$$
\begin{aligned}
M_t &= E^Q\left[(S_T^0)^{-1}\Phi + \int_0^T (S_T^0)^{-1}\varphi_u du\,|\mathcal{G}_t\right] \\
&= E^Q[(S_T^0)^{-1}S_T^4|\mathcal{G}_t] \\
&= e^{-rT}E^Q\left[S_0^4 \times e^{4\sigma\beta_T+4rT-2\sigma^2T}\middle|\mathcal{G}_t\right] \\
&= e^{-rT+4rT-2\sigma^2T}E^Q\left[S_0^4 \times e^{4\sigma\beta_T}\middle|\mathcal{G}_t\right] \\
&= S_0^4e^{3rT-2\sigma^2T}E^Q\left[e^{4\sigma\beta_T}|\mathcal{G}_t\right] \\
&= S_0^4e^{3rT-2\sigma^2T}E^Q\left[e^{4\sigma(\beta_T-\beta_t)}e^{4\sigma\beta_t}|\mathcal{G}_t\right] \\
&= S_0^4e^{3rT+8\sigma^2(T-t)-2\sigma^2T+4\sigma\beta_t} \\
D\Phi_s &= 4\sigma S_0^4e^{3rT-2\sigma^2T+8\sigma^2(T-t)+4\sigma\beta_t} \\
E\left[D\Phi_s\middle|\mathcal{G}_s^\beta\right] &= 4\sigma S_0^4e^{3rT+8\sigma^2(T-s)+4\sigma\beta_s-2\sigma^2T} \\
\pi_t &= 4S_0^4e^{3rT-8\sigma^2t+4\sigma\beta_t+6\sigma^2T+r_t} \\
x &= S_0^4e^{3rT+6\sigma^2T}
\end{aligned}
$$

14.17

For $K > 0$, $\Phi = (S_T - K)^+ = (S_T - K)1_{[k,\infty[}(S_T)$

$$
\begin{aligned}
M_t &= E^Q\left[(S_T^0)^{-1}(S_T-K)1_{[k,\infty[}(S_T)\middle|\mathcal{G}_t\right] \\
&= e^{-rT}E^Q\left[\left(S_0e^{\sigma\beta_T+rT-\frac{1}{2}\sigma^2T} - K\right)1_{[a,\infty[}(\beta_T)\middle|\mathcal{G}_t\right]
\end{aligned}
$$

$$\mathrm{a} = \frac{1}{\sigma}\left(\ln\frac{K}{S_0} + \frac{1}{2}\sigma^2\mathrm{T} - \mathrm{rT}\right)$$

$$\Longrightarrow M_t = e^{-rT}(-K)P(\beta_T \geq c) + S_0 e^{-\frac{1}{2}\sigma^2 T}\mathrm{E}^{\mathrm{Q}}\left[\mathrm{e}^{\sigma\beta_T}1\middle|\mathcal{G}_\mathrm{t}\right]$$
$$\times\, \mathrm{E}^{\mathrm{Q}}\left[\mathrm{e}^{\sigma\beta_T}1_{[c,\infty[}(\beta_\mathrm{T})\middle|\mathcal{G}_\mathrm{t}\right]$$

$$h_t = E[DM_t|\mathscr{F}_t^W]$$
$$= \mathrm{E}[\mathrm{DE}[\mathrm{e}^{-\mathrm{rT}}\Phi|\mathcal{G}_\mathrm{t}]|\mathscr{F}_\mathrm{t}^W]$$
$$= \mathrm{e}^{-\mathrm{rT}}\mathrm{E}[\mathrm{D}\Phi|\mathcal{G}_\mathrm{t}]$$
$$= \mathrm{e}^{-rT}\,\mathrm{E_A}\left[\mathrm{D}\left(\mathrm{S_0}\mathrm{e}^{\sigma\beta_T + rT - \frac{1}{2}\sigma^2 T} - \mathrm{K}\right)\middle|\mathcal{G}_\mathrm{t}\right]$$
$$\textit{where } A: B_T \geq a: \frac{1}{\sigma}\left(\ln\frac{K}{S_0} + \frac{1}{2}\sigma^2\mathrm{T} - \mathrm{rT}\right)$$
$$= e^{-rT}\sigma \mathrm{e}^{rT - \frac{1}{2}\sigma^2 T}\mathrm{S_0}\mathrm{E_A}\left[\mathrm{e}^{\sigma\beta_T}\middle|\mathcal{G}_\mathrm{t}\right]$$
$$= S_0\sigma e^{-\frac{1}{2}\sigma^2 T + \sigma\beta_T}\int_{a-\beta_T}^{\infty}\frac{e^{\sigma x}e^{-\frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}\mathrm{dx}$$
$$= \mathrm{S_0}\sigma e^{-\frac{1}{2}\sigma^2 T + \sigma\beta_T + \frac{1}{2}(T-t)\sigma^2}\mathrm{P}\left(\beta_\mathrm{T} \geq \mathrm{a} - (\mathrm{T}-\mathrm{t})\sigma\right)$$

$$S_0\Pi_t = e^{rt}\frac{1}{\sigma}\mathrm{h_t}$$
$$= S_0 e^{\sigma\beta_T + rt - \frac{1}{2}\sigma^2 t}\mathrm{P}\left(\beta_\mathrm{T} \geq \mathrm{a} - (\mathrm{T}-\mathrm{t})\sigma\right)$$
$$\mathrm{P}\left(\beta_\mathrm{T} \geq \mathrm{a} - (\mathrm{T}-\mathrm{t})\sigma\right) = \frac{1}{2}\left(1 - \mathrm{erf}\frac{a-(T-t)\sigma}{\sqrt{2T}}\right)$$

$$\Longrightarrow \Pi_t = \frac{1}{2}S_0 e^{\sigma\beta_T + rt - \frac{1}{2}\sigma^2 t}\left(1 - \mathrm{erf}\frac{a-(T-t)\sigma}{\sqrt{2T}}\right)$$
$$a = \frac{1}{\sigma}\left(\ln\frac{K}{S_0} + \frac{1}{2}\sigma^2\mathrm{T} - \mathrm{rT}\right)$$

14.23

(a)

The wealth process of lower hedging strategy is:

$$(S^0)^{-1}\widetilde{V^-} + (S^0)^{-1}c \cdot t = x^-(S^0)^{-1}(\pi^-)^T \cdot X$$

As π^- is Q-admissible, then the RHS is a sub-martingale. Thus, $\tilde{V}^- \leq \Phi$, $c \leq \varphi$. Then,

$$x^- \leq E^Q[(S_t^0)^{-1}\tilde{V}^- + ((S^0)^{-1}c \cdot t)_t|\mathcal{G}_0]$$
$$\leq\ E^Q[(S_t^0)^{-1}\Phi + ((S^0)^{-1}\varphi \cdot t)_t|\mathcal{G}_0]$$
$$= E^Q[H_t|\mathcal{G}_0]$$
$$= E^Q[H_t]$$

Therefore, we can claim that the initial loan in a lower hedging strategy for the closing time t cannot exeed $E^Q[H_t]$.

(b)
Define the following $(\mathcal{G}^t, Q)$-martingale,

$$M_t \stackrel{\text{def}}{=} E^Q[(S_t^0)^{-1}\Phi_t + ((S^0)^{-1}\varphi \cdot t)_t | \mathcal{G}_{t\wedge t}]$$

By (9.45) and (9.78), we can get M_t is also a $(\mathcal{G}, Q)$-martingale. According to the PRP, M_t can be expressed as $M_t = M_0 + h^T \cdot \beta$ for some predictable h, and we can set $\pi_t^- \stackrel{\text{def}}{=} -(\sigma_t^T)^{-1} S_t^0 h_t$.
Then,

$$\begin{aligned}
(S_{t\wedge t}^0)^{-1}\tilde{V}^- + ((S^0)^{-1}c \cdot t)_{t\wedge t} &= M_0 - ((S^0)^{-1}(\pi^-)^T)_{t\wedge t} = M_t \\
(S_{t\wedge t}^0)^{-1}\tilde{V}^- &= E^Q[(S_t^0)^{-1}\Phi_t + ((S^0)^{-1}\varphi \cdot t)_{t\wedge t} | \mathcal{G}_{t\wedge t}] - ((S^0)^{-1}\varphi \cdot t)_{t\wedge t} \\
&= E^Q\left[(S_t^0)^{-1}\Phi_t + \int_{t\wedge t}^t e^{-\int_{t\wedge t}^s r_s ds} \varphi_u du \middle| \mathcal{G}_{t\wedge t}\right] \\
\Rightarrow \tilde{V}^- = E^Q&\left[e^{-\int_{t\wedge t}^t r_u du}\Phi_t + \int_{t\wedge t}^t e^{-\int_{t\wedge t}^s r_s ds} \varphi_u du \middle| \mathcal{G}_{t\wedge t}\right]
\end{aligned}$$

By definition, we can conclude that $\pi^- = \sup_{t\in\mathcal{T}_{[0,T]}} E^Q[H_t]$.

14.24
π^+ is a Q-tame upper hedging strategy, then.

$$(S^0)^{-1}V^+ + (S^0)^{-1}c \cdot t = x^+ + (S^0)^{-1}(\pi^+)^T \cdot X$$

The RHS is a super-martingale with $V^+ \geq \Phi$, $c \geq \varphi$. Therefore,

$$\begin{aligned}
x^+ &\geq E^Q[(S_t^0)^{-1}V^+ + ((S^0)^{-1}c \cdot t)_t | \mathcal{G}_0] \\
&\geq E^Q[(S_t^0)^{-1}\Phi + ((S^0)^{-1}\varphi \cdot t)_t | \mathcal{G}_0] \\
&= E^Q[H_t | \mathcal{G}_0] \\
&= E^Q[H_t]
\end{aligned}$$

By (14.23), we know that $x^- \leq E^Q[H_t]$, then:

$$x^+ \geq E^Q[H_t] \geq x^-$$
$$\therefore \Pi^+ \geq \Pi^-$$

14.28
Define a uniformly integrable martingale:

$$M_t \stackrel{\text{def}}{=} E\left[\sup_{s\in\mathbb{R}_+} H_s \middle| \mathcal{F}_t\right]$$

Then, by the definition of essential supremum, we can get

$$M_t = E\left[\sup_{s\in\mathbb{R}_+} H_s \middle| \mathcal{F}_t\right] \geq E\left[\sup_{s\in\mathcal{T}[0,+\infty]} H_s | \mathcal{F}_t\right]$$
$$\Rightarrow M_t \geq U_t$$

Particular, $M_t \geq U_t$. Given $E\left[\sup_{t\in\mathbb{R}_+} H_t\right] < \infty$, then,

$$E[|M_t|] = E[M_t] \leq E\left[E\left[\sup_{s\in\mathbb{R}_+} H_s | \mathcal{F}_t\right]\right] = E\left[\sup_{s\in\mathbb{R}_+} H_s\right] < \infty$$

Therefore, M_t is integrable, and according to (4.25), U_t is uniformly integrable for all stopping time. Thus, we can claim that U is of class D.

14.29
Suppose that $\exists$ another right-continuous super-martingale $\widetilde{U}$ such that $\widetilde{U} \geq H$. Then, for any $s \in \mathcal{T}_{[0,\infty]}$, $s \geq t$,

$$E[H_{\mathcal{s}}|\mathcal{F}_t] \le E\big[\widetilde{U}_{\mathcal{s}}\big|\mathcal{F}_t\big] \le \widetilde{U}_t$$
$$\Longrightarrow U_t = ess \sup_{s\in\mathcal{T}_{[0,\infty]}} E[H_{\mathcal{s}}|\mathcal{F}_t] \le \widetilde{U}_t$$

Thus, we can claim that Snell envelope U is the smallest satisfies the conditions.

14.30

If (a) and (b) hold, then

$$U_{t\wedge t^*} = E[U_{t^*}|\mathcal{F}_{t\wedge t^*}]$$
$$\Longrightarrow U_0 = E[U_{t^*}|\mathcal{F}_0] = E[U_{t^*}]$$

Therefore, t^* is optimal.

If t^* is an optimal stopping time, then

$$U_0 = E[H_{t^*}] = E\big[E[H_{t^*}|\mathcal{F}_{\mathcal{s}}]\big] \le E[U_{\mathcal{s}}] \le U_0$$

Therefore,

$$E\big[E[H_{t^*}|\mathcal{F}_{\mathcal{s}}]\big] = E[U_{\mathcal{s}}]$$
$$\Longrightarrow U_{\mathcal{s}} = E[H_{t^*}|\mathcal{F}_{\mathcal{s}}]$$
$$E[U_{\mathcal{s}}] = U_0$$

Then, for any bounded stopping time $\mathcal{s}'$:

$$E\left[U_{\mathcal{s}'}^{t^*}\right] = E[U_{t^*\wedge\mathcal{s}'}] = E[U_0] = U_0$$
$$E[|U_{\mathcal{s}'}|] = E[U_{\mathcal{s}'}] \le E\left[\sup_{t\in\mathbb{R}_+}|H_t|\right] < \infty$$

Therefore, $U^{t^*}$is a martingale, and

$$U_{t^*} = E[H_{t^*}|\mathcal{F}_{t^*}] = H_{t^*}$$

14.33

$$\begin{aligned}(S^0)^{-1}\pi\cdot x &= (S^0)^{-1}(S^0(\sigma^T)^{-1})^T\cdot x\\ &= ((S^0)^{-1}S^0\sigma^{-1}h^T)\cdot x\\ &= (\sigma^{-1}h^T)\cdot x\\ &= h^T\cdot(\sigma^{-1}x)\\ &= h^T\cdot\beta\end{aligned}$$

which is a martingale under Q. Then $(S^0)^{-1}\pi\cdot x$ can be the lower bound by itself as it is a martingale. Therefore, $U_0 = M_0$ and $dC_t \ge \varphi_t dt$, (U_0, π, dC) is Q-admissible.

14.35

Given $\delta = 0$, $e^{-r\cdot i}S$ is a martingale and $e^{-r\cdot i}K$ is a super-martingale under Q. Then $e^{-r\cdot i}(S-K)$ is a sub-martingale. As $(\cdot)^+$ is a convex function, $e^{-r\cdot i}(S-K)^+$ is also a sub-martingale. By (9.37), we can get

$$\begin{aligned}U_t &= ess \sup_{t\in\mathcal{T}_{[t,T]}} E^Q[e^{-rt}(S_t-K)^+|\mathcal{G}_t]\\ &= E^Q[e^{-rT}(S_T-K)^+|\mathcal{G}_t]\end{aligned}$$

as $E^Q\left[e^{-rt_1}\left(S_{t_1}-K\right)^+\middle|\mathcal{G}_t\right] \le E^Q\left[E^Q\left[e^{-rt_2}\left(S_{t_2}-K\right)^+\middle|\mathcal{G}_t\right]\middle|\mathcal{G}_t\right] = E^Q\left[e^{-rt}\left(S_{t_2}-K\right)^+\middle|\mathcal{G}_t\right]$