Stochastic Methods in Asset Pricing

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American-Style Contingent Claims

(section 14.4)

Andrew Lyasoff

Summary

Basic Definitions and Terminology

Upper and Lower Hedging Strategies

The Pricing of American-Style Contingent Claims

The Snell Envelope

The Arbitrage Price of American-Style Contingent Claims

No Early Exercise for American Calls on Non-Dividend Underlying Assets

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IMPORTANT REMARK: The long and the short positions on an American-style contingent claim are no longer mirror images of one another!

Upper and Lower Hedging Strategies

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Upper hedging strategy (a.k.a. *super-replicating strategy*) for $\mathscr{K}(T,\Phi,\varphi)$ is any choice of a Q-admissible self-financing investment-consumption strategy (x,π^+,c), such that (a.s.) $c_t \geq \varphi_t$ and $V_t^{x,\pi^+,c} \geq \Phi_t$ for all $t \in [0,T]$.

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Lower hedging strategy for $\mathscr{K}(T,\Phi,\varphi)$ is any choice of a stopping time $\tau \in \mathscr{T}_{[0,T]}$ and a Q-tame self-financing investment-payout strategy (x,π^-,c), such that (a.s.) $\tilde{V}_\tau^{x,\pi^-,c} \leq \Phi_\tau$ and $c_{t\wedge\tau} \leq \varphi_{t\wedge\tau}$ for all $t \in [0,T]$ (the trading strategy π^- and the payout rate c need to be defined only until time τ).

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The aggregate discounted payoff from $\mathscr{K}(T, \boldsymbol{\Phi}, \varphi)$ is:

$$H_{\mathscr{t}} = (S_{\mathscr{t}}^\circ)^{-1}\boldsymbol{\Phi}_{\mathscr{t}} + \int_0^{\mathscr{t}} (S_u^\circ)^{-1}\varphi_u \mathrm{d}u\,.$$

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We only consider *complete* and *arbitrage-free* markets and require that: $\mathsf{E}^Q\left[\sup_{t\in[0,T]} H_t\right] < \infty$.

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HINT:
$$x - \int_0^{\mathcal{t}} (S_u^\circ)^{-1}(\pi_u^-)^\intercal \,\mathrm{d}X_u = (S_{\mathcal{t}}^\circ)^{-1} V_{\mathcal{t}}^- + \int_0^{\mathcal{t}} (S_u^\circ)^{-1} c_u \,\mathrm{d}u\,.$$

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$$M_t \overset{\text{def}}{=} \mathsf{E}^{Q}\left[(S_{\mathcal{t}}^{\circ})^{-1}\boldsymbol{\Phi}_{\mathcal{t}} + \int_0^{\mathcal{t}} (S_u^{\circ})^{-1}\varphi_u\,\mathrm{d}u \,\middle|\, \mathscr{G}_{t\wedge\mathcal{t}}\right],\ t\in[0,T],\ \text{is a } (\mathscr{G}^{\mathcal{t}}, Q)\text{-martingale} \Rightarrow (\mathscr{G}, Q)\text{-martingale}$$

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HINT:
$$x - \int_0^{\mathscr{t}} (S^\circ_u)^{-1}(\pi^-_u)^\intercal \,\mathrm{d}X_u = (S^\circ_{\mathscr{t}})^{-1}V^-_{\mathscr{t}} + \int_0^{\mathscr{t}} (S^\circ_u)^{-1} c_u \,\mathrm{d}u\,.$$

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PRP $\Rightarrow$ $M_t = M_0 + \int_0^t h_s^\intercal \,\mathrm{d}\beta_s$ $\Rightarrow$ $\pi^-_t \overset{\text{def}}{=} -(\sigma_t^\intercal)^{-1} S^\circ_t\, h_t$ $\Rightarrow$ with (debt process) $\tilde V \overset{\text{def}}{=} \tilde V^{M_0,\pi^-,\varphi}$,

$$(S^\circ_{t\wedge\mathscr{t}})^{-1}\tilde V^-_{t\wedge\mathscr{t}} + \int_0^{t\wedge\mathscr{t}} (S^\circ_u)^{-1}\varphi_u \,\mathrm{d}u = M_0 - \int_0^{t\wedge\mathscr{t}} (S^\circ_u)^{-1}(\pi^-_u)^\intercal \,\mathrm{d}X_u = M_{t\wedge\mathscr{t}} = M_t\,.$$

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HINT: $$x - \int_0^{\mathscr{t}} (S_u^\circ)^{-1}(\pi_u^-)^\intercal \,\mathrm{d}X_u = (S_{\mathscr{t}}^\circ)^{-1} V_{\mathscr{t}}^- + \int_0^{\mathscr{t}} (S_u^\circ)^{-1} c_u \,\mathrm{d}u\,.$$

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Exercise: Let x denote the initial wealth in some (arbitrarily chosen) upper hedging strategy and let the stopping time $t \in \mathscr{T}_{[0,T]}$ be arbitrarily chosen. Prove that $x \geq \mathsf{E}^{Q}[H_{t}]$ and conclude that $\Pi^{+} \geq \Pi^{-}$.

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HINT: If V is the wealth generated by some upper hedging startegy, then

$$(S^\circ_t)^{-1}\Phi_t + \int_0^t (S^\circ_u)^{-1}\varphi_u \, \mathrm{d}u \leq (S^\circ_t)^{-1}V^+_t + \int_0^t (S^\circ_u)^{-1}c_u \, \mathrm{d}u = x + \int_0^t (S^\circ_u)^{-1}(\pi^+_u)^\intercal \, \mathrm{d}X_u \, .$$

What does the optional stopping theorem for supermartingales say?

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The r.v. η with these properties is P-a.s. unique and is called *the essential supremum* (a.k.a. *the essential upper bound*) of the family $(\xi_i)_{i\in\mathbb{I}}$ end is denoted by $\operatorname{ess\,sup}_{i\in\mathbb{I}} \xi_i$.

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Exercise: The Snell envelope U is the smallest positive and right-continuous supermartingale that dominates H. HINT: Positive supermartingales always converge. If $\tilde{U}$ is another positive and right-continuous supermartingale with $\tilde{U}_t \geq H_t$, then, for any $\mathscr{t} \in \mathscr{T}_{[t,\infty]}$, $\mathsf{E}[H_{\mathscr{t}}\,|\,\mathscr{F}_t] \leq \mathsf{E}[\tilde{U}_{\mathscr{t}}\,|\,\mathscr{F}_t] \leq \tilde{U}_t$ $\Rightarrow$ $U_t = \operatorname{ess\,sup}_{\mathscr{t}\in\mathscr{T}_{[t,\infty[}} \mathsf{E}[H_{\mathscr{t}}\,|\,\mathscr{F}_t] \leq \tilde{U}_t$.

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In the case of a finite time horizon $T > 0$ and a continuous and positive reward process H optimal (finite) stopping time always exists provided that $\mathsf{E}\big[\sup_{t\in[0,T]} H_t\big] < \infty$.

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$$(S^\circ)^{-1} S = e^{-r\imath} e^{Z - \frac{1}{2}\langle Z,Z\rangle} = e^{Z_0} e^{-r\imath} e^{(Z-Z_0) - \frac{1}{2}\langle Z,Z\rangle} = S_0 e^{-r\imath} e^{\sigma\beta + (r-\delta)\imath - \frac{1}{2}\sigma^2\imath} = S_0 e^{\sigma\beta - \delta\imath - \frac{1}{2}\sigma^2\imath} .$$

For a US-call we have:

$$U_t = \operatorname{ess\,sup}_{\mathcal{t}\in\mathscr{T}_{[t,T]}} \mathsf{E}^Q\left[e^{-r\mathcal{t}}\left(S_{\mathcal{t}} - K\right)^+ \mid \mathscr{G}_t\right] = \operatorname{ess\,sup}_{\mathcal{t}\in\mathscr{T}_{[t,T]}} \mathsf{E}^Q\left[\left(e^{-r\mathcal{t}} S_{\mathcal{t}} - e^{-r\mathcal{t}} K\right)^+ \mid \mathscr{G}_t\right] .$$

No Early Exercise for American Calls on Non-Dividend Underlying Assets

If dividends are paid continuously at rate $\delta S_t\,\mathrm{d}t$ then the excess-returns process is $X = Z - Z_0 + \delta\iota - r\iota$.
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With $\delta = 0$, $e^{-r\iota}S = S_0e^{\sigma\beta-\frac{1}{2}\sigma^2\iota}$ is a Q-martingale, $e^{-r\iota}K$ is a “supermartingale,”
and $e^{-r\iota}(S-K)^+$ is a Q-submartingale (see (14.35), (7.5) and (9.2b)).

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By the optional stopping theorem for submartingales (why does it apply?),

$$e^{-r\mathscr{t}}\left(S_{\mathscr{t}} - K\right)^+ \le \mathsf{E}^Q\left[e^{-rT}\left(S_T - K\right)^+ \mid \mathscr{G}_{\mathscr{t}}\right] \quad \Rightarrow \quad \mathsf{E}^Q\left[e^{-r\mathscr{t}}\left(S_{\mathscr{t}} - K\right)^+\right] \le \mathsf{E}^Q\left[e^{-rT}\left(S_T - K\right)^+\right]$$

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and therefore if $\delta = 0$, then $\ell^* = T$ and $\Pi = U_0 = \mathsf{E}^Q\left[e^{-rT}\left(S_T - K\right)^+\right]$.