Stochastic Methods in Asset Pricing

The MIT Press (2017)

Crash Course in Continuous Time Finance – I

(sections 13.1, 13.2)

Andrew Lyasoff

Summary

Security Prices, Returns, and Excess Returns

Self-Financing Trading Strategies

Investment-Consumption Strategies

Investment-Payout Strategies

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σ, β, and r are predictable (for $\mathscr{F}$) and $\int_0^t |\sigma_s|^2\,\mathrm{d}s < \infty$, $\int_0^t |b_s|\,\mathrm{d}s < \infty$, $\int_0^t |r_s|\,\mathrm{d}s < \infty$ (P-a.s.)

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There are no-redundant securities: $\sigma_t(\omega)$ is of full-rank $\Leftrightarrow$ $\sigma_t(\omega)\sigma_t(\omega)^\intercal$ is full-rank $\mathrm{d}P(\omega) \times \mathrm{d}t$-a.e.

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Naturally, r is the interest-rate process. N.B. $\mathscr{A}$ may include dividend income.

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$$X_t = Z_t - Z_0 - \int_0^t r_s\vec{1}\,\mathrm{d}s = \int_0^t \sigma_s\,\mathrm{d}W_s + \int_0^t (b_s - r_s\vec{1})\,\mathrm{d}s\,, \quad t \geq 0\,.$$

N.B. We work exclusively with excess returns.

INFORMATION PROCESS: some continuous and $\mathscr{F}$-adapted process Y

INFORMATION FILTRATION: $(\mathscr{G}_t \overset{\text{def}}{=} \mathscr{F}_t^{X,r,Y}\,,\, t \in \mathbb{R}_+)$, assuming $\mathscr{G}_0 = \{\Omega, \emptyset\}$

REALIZED VOLATILITY: $\langle X, X\rangle$ = the matrix process with elements $\langle X^i, X^j\rangle$, $1 \leq i, j \leq n$

$$\langle X, X\rangle_t = \int_0^t \sigma_s\sigma_s^{\intercal}\,\mathrm{d}s\,, \quad t \geq 0\,, \quad \text{or} \quad \langle X, X\rangle = \sigma\sigma^{\intercal}\bullet\iota$$

N.B. $\langle X, X\rangle$ is adapted to $\mathscr{F}^X \subset \mathscr{G}$ $\quad\Rightarrow\quad$ $(\sigma\sigma^{\intercal})$ is observable but σ may not be!!

However: there is an $\mathscr{F}^X$-adapted (square) $n \times n$ full-rank matrix $\alpha_t(\omega)$ with $\alpha_t\alpha_t^{\intercal} = \sigma_t\sigma_t^{\intercal}$

$$X = \alpha\bullet\left(\alpha^{-1}\sigma\bullet W + \alpha^{-1}(b - r\vec{1})\bullet\iota\right) = \alpha\bullet\left(B + \theta\bullet\iota\right)\,, \qquad \theta_t = \alpha_t^{-1}(b_t - r_t\vec{1})$$

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$x \in \mathbb{R}$ is the initial investment; π is an $\mathbb{R}^n$-valued and predictable for $\mathscr{G}$ process that gives the investor's trading strategy (portfolio process); c is an $\mathbb{R}_+$-valued and predictable for $\mathscr{G}$ process that gives the investor's consumption rate (consumption strategy). These objects are assumed to satisfy (a.s.):

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We now turn to the dynamics of the wealth process, $V = V^{x,\pi,c}$, which (x, π, c) generates.

$$\begin{aligned} V_{t+\mathrm{d}t} &= \big(V_t - \pi_t{}^\intercal\,\vec{1}\big)(1 + r_t\,\mathrm{d}t) + \pi_t{}^\intercal(\vec{1} + \mathrm{d}Z_t) - c_t\,\mathrm{d}t \\ &= V_t + V_t\,r_t\,\mathrm{d}t + \pi_t{}^\intercal(\mathrm{d}Z_t - r_t\vec{1}\,\mathrm{d}t) - c_t\,\mathrm{d}t \\ &= V_t + V_t\,r_t\,\mathrm{d}t + \pi_t{}^\intercal\,\mathrm{d}X_t - c_t\,\mathrm{d}t\,. \end{aligned}$$

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N.B. The economic interpretation of this identity is very important.

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A self-financing investment-payout strategy

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debt dynamics

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N.B. A self-financing investment-payout strategy= self-financing investment-consumption strategy with *negative initial endowment* and *negative consumption*.