Capstone Assignment

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December 10, 2019

Problem 1

A *continuous semimartingale* is any $\mathscr{F}-$adapted process, S, that can be written as the sum, $S = M + A$, of a continuous local martingale M and a continuous and $\mathscr{F}-$adapted process A that starts from 0 and has sample paths that a.s. have finite variation on finite intervals. If a decomposition of this form exists, the local martingale component, M, and the finite variation component, A, are unique, and the identification $S = M + A$ is known as the *the canonical semimartingale decomposition of* S.

Assuming that S is a semimartingale with canonical decomposition $S = M + A$, show that every one of the following processes is also a semimartingale and identify their respective canonical semimartingale decomposition.
(a) $e^S \equiv (e^{S_t})_{t\in\mathbb{R}_+}$; (b) $S^2 \equiv (S_t^2)_{t\in\mathbb{R}_+}$; (c) $S^n \equiv (S_t^n)_{t\in\mathbb{R}_+}$ for some $n \in \mathbb{N}_{++}$;
(d) $S \cdot \imath \equiv (\int_0^t S_u \mathrm{d}x)_{t\in\mathbb{R}_{++}}$; (e) $S \cdot W + S \cdot \imath \equiv (\int_0^t S_u \mathrm{d}W_u + \int_0^t S_u \mathrm{d}u)_{t\in\mathbb{R}_{++}}$; and
(f)$f(S, \imath) \equiv \big(f(S_t, t)\big)_{t\in\mathbb{R}_+}$
for some function $f : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ such that derivatives $\frac{\partial^2}{\partial x^2} f(x, t)$ and $\frac{\partial}{\partial t} f(x, t)$ exist and are continuous.

Proof. (a) for the continuous function $f(x) = e^x$ and has continuous first and second-order derivatives. Then from Ito's formula e^S is a continuous semimartingale because S is a continuous semimartingale.
Again from Ito's formula,
$e^S = e^{S_0} + e^S \cdot (M + A) + \frac{1}{2} e^S \cdot \langle S, S\rangle = e^{M_0} + e^S \cdot M + e^S \cdot (A + \frac{1}{2}\langle M, M\rangle)$
$e^{M_0} + e^S \cdot M$ is a continuous local martingale because a stochastic integral with respect to the continuous local martingale is also a continuous local martingale.
$A + \frac{1}{2}\langle M, M\rangle$ is of finite variation and starts from 0, then $e^S \cdot (A + \frac{1}{2}\langle M, M\rangle)$ is of finite variation and starts from 0 because a stochastic integral starts from 0 and is of finite variation with respect to a finite variation.
Thus its local martingale part is

$$e^{M_0} + e^S \cdot M$$

its finite variation part is

$$e^S \cdot (A + \frac{1}{2}\langle M, M\rangle)$$

(b) for the continuous function $f(x) = x^2$ and has continuous first and second-order derivatives. Then from Ito's formula S^2 is a continuous semimartingale because S is a continuous semimartingale.
Again from Ito's formula,

$$S^2 = S_0^2 + 2S \cdot (M + A) + \langle S, S\rangle = M_0^2 + 2S \cdot M + 2S \cdot A + \langle M, M\rangle$$

It is the same as the (a), its local martingale part is

$$M_0^2 + 2S \cdot M$$

its finite variation part is

$$2S \cdot A + \langle M, M\rangle$$

(c) for the continuous function $f(x) = x^n$ for some $n \in \mathbb{N}_{++}$ and has continuous first and second-order derivatives. Then from Ito's formula S^n is a continuous semimartingale because S is a continuous semimartingale.
If $n = 1, 2$ then from the problem and part(b) we know its canonical decomposition, if $n \geq 3$, then from Ito's formula,

$$S^n = S_0^n + nS^{n-1} \cdot S + \frac{1}{2}n(n-1)S^{n-2} \cdot \langle S, S\rangle = M_0^n + nS^{n-1} \cdot M + nS^{n-1} \cdot A + \frac{1}{2}n(n-1)S^{n-2} \cdot \langle M, M\rangle$$

Thus its local martingale part is

$$M_0^n + nS^{n-1} \cdot M$$

its finite variation part is

$$nS^{n-1} \cdot A + \frac{1}{2}n(n-1)S^{n-2} \cdot \langle M, M\rangle$$

(d) $\imath$ can be seen as a semimartingale with zero local martingale part, then $S \cdot \imath$ is a continuous semimartingale because $\imath$ is a continuous local martingale. In particular, $S \cdot \imath$ is of finite variation and starts from 0, then its local martingale part is 0, finite variation part is $S \cdot \imath$

(e) $S \cdot W$ is a continuous local martingale, of course semimartingale, $S \cdot \imath$ is of finite variation, of course semimartingale. the sum of semimartingale is a semimartingale. In particular, its local martingale part is $S \cdot W$, finite variation part is $S \cdot \imath$.

(f) the first partial derivative $\frac{\partial}{\partial x}f(x,t)$ exists and continuous because the second partial derivative exists and continuous, then from Ito's formula $f(S, \imath)$ is a continuous semimartingale because S and $\imath$ is a continuous semimartingale. Again, from Ito's formula,

$$f(S, \imath) = f(M_0, 0) + \frac{\partial}{\partial t}f(S, \imath) \cdot \imath + \frac{\partial}{\partial x}f(S, \imath) \cdot S + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(S, \imath) \cdot \langle M, M\rangle$$

Then its local martingale part is

$$f(M_0,0)+\frac{\partial}{\partial x}f(S,\imath)\cdot M$$

its finite variation part is

$$\frac{\partial}{\partial t}f(S,\imath)\cdot\imath+\frac{\partial}{\partial x}f(S,\imath)\cdot A+\frac{1}{2}\frac{\partial^2}{\partial x^2}f(S,\imath)\cdot\langle M,M\rangle$$

$\square$

Problem 2

Answer the questions in Problem 1 in the special case where S is an Itô process of the form $S=S_0+\sigma\cdot W+b\cdot\imath$, i.e.,

$$S_t=S_0+\int_0^t\sigma_u \mathrm{d}W_u+\int_0^t b_u \mathrm{d}u$$

and explain what are the conditions for the process $\sigma(w,t)$ and $b(w,t)$ under which the integrals above are well defined.
Solution: First we need

$$\sigma\in\mathscr{L}^2_{\text{loc}}(\mathbb{R}_+),b\in\mathscr{L}_{\text{loc}}(\mathbb{R}_+)$$

(a) $e^S=e^{S_0}+\sigma e^S\cdot W+(\frac{1}{2}\sigma^2+b)e^S\cdot\imath$ is a Itô process.

(b) $S^2=S_0^2+2\sigma S\cdot W+(2bS+\sigma^2)\cdot\imath$ is a Itô process.

(c) $n=1,2$, S^n is a Itô process. If $n\geq 3$

$$S^n=S_0^n+n\sigma S^{n-1}\cdot W+[nbS^{n-1}+\frac{1}{2}n(n-1)\sigma^2S^{n-2}]\cdot\imath$$

is a Itô process.

(d) it is a Itô process with Brownian motion integral part is 0 and starts from 0.

(e) $S\cdot W+S\cdot\imath$ is a Itô process and starts from 0.

(f)

$$f(S,\imath)=f(S_0,0)+\sigma\frac{\partial}{\partial x}f(S,\imath)\cdot W+[\frac{\partial}{\partial t}f(S,\imath)\cdot+b\frac{\partial}{\partial x}f(S,\imath)\cdot+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f(S,\imath)]\cdot\imath$$

is a Itô process

$\square$

Problem 3

Consider the function $\sigma : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $b : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and give the conditions under which the following stochastic differential equation has a unique strong solution S :

$$S = S_0 + \sigma(S, \imath) \cdot W + b(S, \imath) \cdot \imath, \qquad \text{i.e.} \qquad S_t = S_0 + \int_0^t \sigma(S_u, u) \mathrm{d}W_u + \int_0^t b(S_u, u) \mathrm{d}u, \qquad t \in \mathbb{R}_+$$

Explain what it means for a stochastic differential equation to have a strong solution, and what it means for a strong solution to be unique. Answer the questions in Problem 1 in the special case where the semimartingale S is the solution to the above equation and, finally, write a stochastic equation of diffusion type that the process $X = e^S$, i.e., $X_t = e^{S_t}, t \in \mathbb{R}_+$, satisfies($S$ is again the solution to the above equation).

Solution: According to the globally Lipschitz condition, if there exists a universal constant $C > 0$,

$$|\sigma(x, t) - \sigma(y, t)| + |b(x, t) - b(y, t)| \leq C|x - y|$$

According to the linear growth condition,

$$\sigma^2(x, t) + b^2(x, t) \leq C^2(1 + x^2)$$

for every $t \in \mathbb{R}_+, x, y \in \mathbb{R}$. In addition, if S_0 is independent with the Brownian motion and $E[S_0^2] < \infty$.

Then there exists a strong solution.

In addition, if for every $R \in \mathbb{R}_{++}$, there is a constant $C_R \in \mathbb{R}_{++}$ such that

$$|\sigma(x, t) - \sigma(y, t)| + |b(x, t) - b(y, t)| \leq C_R|x - y|$$

for every $t \in \mathbb{R}_+, |x|, |y| \leq R$. Then strong uniqueness holds.

If a stochastic differential equation has a strong solution, it means that give a initial X_0 and Brownian motion W, then $S = G(W, X_0)$ for some deterministic function G.

Uniqueness means that if given two strong solution S, $\tilde{S}$ for initial X_0 and W. If

$$P[S_t = \tilde{S}_t, 0 \leq t < \infty] = 1$$

Then strong uniqueness holds.

(a) $e^S = e^{S_0} + \sigma(S, \imath)e^S \cdot W + [\frac{1}{2}\sigma^2(S, \imath) + b(S, \imath)]e^S \cdot \imath$ is a Itô process.

(b) $S^2 = S_0^2 + 2\sigma(S, \imath)S \cdot W + [2b(S, \imath)S + \sigma^2(S, \imath)] \cdot \imath$ is a Itô process.

(c) $n = 1, 2$, S^n is a Itô process. If $n \geq 3$

$$S^n = S_0^n + n\sigma(S, \imath)S^{n-1} \cdot W + [nb(S, \imath)S^{n-1} + \frac{1}{2}n(n-1)\sigma^2(S, \imath)S^{n-2}] \cdot \imath$$

is a Itô process.

(d) it is a Itô process with Brownian motion integral part is 0 and starts from 0.

(e) $S \cdot W + S \cdot \imath$ is a Itô process and starts from 0.

(f)

$$f(S,\imath) = f(S_0,0) + \sigma(S,\imath)\frac{\partial}{\partial x}f(S,\imath)\cdot W + [\frac{\partial}{\partial t}f(S,\imath)\cdot + b(S,\imath)\frac{\partial}{\partial x}f(S,\imath)\cdot + \frac{1}{2}\sigma^2(S,\imath)\frac{\partial^2}{\partial x^2}f(S,\imath)]\cdot\imath$$

is a Itô process

From (a) we can get that

$$X = X_0 + \sigma(\log X,\imath)X \cdot W + [\frac{1}{2}\sigma^2(\log X,\imath) + b(\log X,\imath)]X \cdot \imath$$

Where diffusion

$$a(X,\imath) = \sigma(\log X,\imath)X$$

drift is

$$c(X,\imath) = [\frac{1}{2}\sigma^2(\log X,\imath) + b(\log X,\imath)]X$$

□

Problem 4

Answer the questions in Problem 3 in the special case where S is a geometric Brownian motion, i.e., $\sigma(S_t,t) = \sigma S_t$ and $b(S_t,t) = bS_t$ for some constants $\sigma, b \in \mathbb{R}$.
Solution: As for the geometric Brownian motion, it must have a strong unique solution because it is linear in coefficient.
Then we have

$$S = S_0 + S \cdot (\sigma W + b\imath) \tag{1}$$

We multiply both sides $e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath}$
From Itô formula we have

$$e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath} = 1 + e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath} \cdot (-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath) + \frac{1}{2}\sigma^2 e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath} \cdot \imath \tag{2}$$

Then from integration by parts we have

$$e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath}S = S_0 + e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath} \cdot S + S \cdot e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath} + \langle S, e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath}\rangle$$

Then plugging in (1) and (2) We can get

$$e^{-\sigma W - b\imath + \frac{1}{2}\sigma^2\imath}S = S_0$$

Thus

$$S = S_0 e^{\sigma W + (b - \frac{1}{2}\sigma^2)\imath}$$

It's the same as in the problem 1 but to change $\sigma(S,\imath)$ to σS and $b(S,\imath)$ to bS.

As for $X = e^S$, the diffusion $a(X, \imath) = \sigma X \log X$, the drift is

$$c(X, \imath) = (\frac{1}{2}\sigma^2 \log^2 X + b \log X)X$$

$\square$

Problem 5

Give the solution, X, to the following linear stochastic differential equations, in which σ, b, m, and k are fixed scalars:
(a) $X = X_0 + b\imath + \sigma X \cdot W$; (b) $X = X_0 + bX \cdot \imath + \sigma X \cdot W$; (c) $X = X_0 + (bX - m) \cdot \imath + \sigma X \cdot W$

$$\text{(d)} X = X_0 + (bX - m) \cdot \imath + (\sigma X + K) \cdot W$$

Solution: As for the SDEs which have the linear form

$$X = X_0 + X \cdot Y + Z \tag{3}$$

We divide X by $\mathscr{E}(Y) = e^{Y - \frac{1}{2}\langle Y, Y\rangle}$ and apply Itô formula, first we have

$$\mathscr{E}^{-1}(Y) = 1 - \mathscr{E}^{-1}(Y) \cdot Y + \mathscr{E}^{-1}(Y) \cdot \langle Y, Y\rangle \tag{4}$$

Then we plugging in (3) and (4) into $\mathscr{E}^{-1}(Y)X$ and use Itô formula :

$$\mathscr{E}^{-1}(Y)X = X_0 + \mathscr{E}^{-1}(Y) \cdot (X_0 + X \cdot Y + Z) + X \cdot (1 - \mathscr{E}^{-1}(Y) \cdot Y + \mathscr{E}^{-1}(Y) \cdot \langle Y, Y\rangle) - \langle \mathscr{E}^{-1}(Y) \cdot Y, X \cdot Y + Z\rangle$$

Simplify the equation we get

$$\mathscr{E}^{-1}(Y)X = X_0 + \mathscr{E}^{-1}(Y) \cdot Z - \mathscr{E}^{-1}(Y) \cdot \langle Y, Z\rangle$$

Then

$$X = \mathscr{E}(Y)(X_0 + \mathscr{E}^{-1}(Y) \cdot Z - \mathscr{E}^{-1}(Y) \cdot \langle Y, Z\rangle)$$

(a) here $Y = \sigma W$, then $\mathscr{E}(Y) = e^{\sigma W - \frac{1}{2}\sigma^2 \imath}$, $Z = b\imath$ then $\langle Y, Z\rangle = 0$
Thus

$$X = e^{\sigma W - \frac{1}{2}\sigma^2 \imath}(X_0 + be^{-\sigma W + \frac{1}{2}\sigma^2 \imath} \cdot \imath)$$

i.e.

$$X_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}(X_0 + b\int_0^t e^{-\sigma W_u + \frac{1}{2}\sigma^2 u} \mathrm{d}u)$$

(b) here $Y = \sigma W + b\imath$, $Z = 0$, then $\mathscr{E}(Y) = e^{\sigma W + (b - \frac{1}{2}\sigma^2)\imath}$
Thus

$$X = e^{\sigma W + (b - \frac{1}{2}\sigma^2)\imath} X_0$$

(c) here $Y = \sigma W + b\imath$, $Z = -m\imath$, then $\mathscr{E}(Y) = e^{\sigma W + (b - \frac{1}{2}\sigma^2)\imath}$, $\langle Y, Z \rangle = 0$
Thus
$$X = e^{\sigma W + (b-\frac{1}{2}\sigma^2)\imath}(X_0 - me^{-\sigma W - (b - \frac{1}{2}\sigma^2)\imath} \cdot \imath)$$
i.e.
$$X_t = e^{\sigma W_t + (b - \frac{1}{2}\sigma^2)t}(X_0 - m \int_0^t e^{-\sigma W_u - (b - \frac{1}{2}\sigma^2)u} \mathrm{d}u)$$

(d) here $Y = \sigma W + b\imath, Z = kW - m\imath$, then $\langle Y, Z \rangle = k\sigma\imath$
Thus
$$X = e^{\sigma W + (b - \frac{1}{2}\sigma^2)\imath}\big(X_0 + ke^{-\sigma W - (b - \frac{1}{2}\sigma^2)\imath} \cdot W - (m + k\sigma)e^{-\sigma W - (b - \frac{1}{2}\sigma^2)\imath} \cdot \imath\big)$$
i.e.
$$X_t = e^{\sigma W_t + (b - \frac{1}{2}\sigma^2)t}\big(X_0 + k \int_0^t e^{-\sigma W_u - (b - \frac{1}{2}\sigma^2)u} \mathrm{d}W_u - (m + k\sigma) \int_0^t e^{-\sigma W_s - (b - \frac{1}{2}\sigma^2)s} \mathrm{d}s\big)$$

$\square$

Problem 6

Give the solution, X, to the following linear stochastic differential equations, in which σ, b, m, and k are predictable (for the filtration $\mathscr{F}$) and stochastic processes with a.s. locally bounded sample paths (i.e. sample paths that are bounded on finite intervals.):
(a) $X = X_0 + b \cdot \imath + \sigma X \cdot W$; (b)$X = X_0 + bX \cdot \imath + \sigma X \cdot W$; (c) $X = X_0 + (bX - m) \cdot \imath + \sigma X \cdot W$

$$\text{(d)}X = X_0 + (bX - m) \cdot \imath + (\sigma X + K) \cdot W$$

Solution:

(a) here $Y = \sigma \cdot W$, then $\mathscr{E}(Y) = e^{\sigma \cdot W - \frac{1}{2}\sigma^2 \cdot \imath}$, $Z = b \cdot \imath$ then $\langle Y, Z \rangle = 0$
Thus
$$X = e^{\sigma \cdot W - \frac{1}{2}\sigma^2 \cdot \imath}(X_0 + be^{-\sigma \cdot W + \frac{1}{2}\sigma^2 \cdot \imath} \cdot \imath)$$
i.e.
$$X_t = e^{\int_0^t \sigma_u \mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma_u^2 \mathrm{d}u}(X_0 + \int_0^t b_s e^{-\int_0^s \sigma_r \mathrm{d}W_r + \frac{1}{2}\int_0^s \sigma_r^2 \mathrm{d}r} \mathrm{d}s)$$

(b) here $Y = \sigma \cdot W + b \cdot \imath$, $Z = 0$, then $\mathscr{E}(Y) = e^{\sigma \cdot W + (b - \frac{1}{2}\sigma^2) \cdot \imath}$
Thus
$$X = e^{\sigma \cdot W + (b - \frac{1}{2}\sigma^2) \cdot \imath} X_0$$
i.e.
$$X_t = e^{\int_0^t \sigma_s \mathrm{d}W_s + \int_0^t b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u} X_0$$

(c) here $Y = \sigma \cdot W + b \cdot \imath$, $Z = -m \cdot \imath$, then $\mathscr{E}(Y) = e^{\sigma \cdot W + (b - \frac{1}{2}\sigma^2)\cdot \imath}$, $\langle Y, Z \rangle = 0$
Thus
$$X = e^{\sigma \cdot W + (b - \frac{1}{2}\sigma^2)\cdot \imath}(X_0 - m e^{-\sigma \cdot W - (b - \frac{1}{2}\sigma^2)\cdot \imath} \cdot \imath)$$
i.e.
$$X_t = e^{\int_0^t \sigma_s \mathrm{d}W_s + \int_0^t b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u}(X_0 - \int_0^t m_v e^{-\int_0^v \sigma_s \mathrm{d}W_s - \int_0^v b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u} \mathrm{d}v)$$

(d) here $Y = \sigma \cdot W + b \cdot \imath, Z = k \cdot W - m \cdot \imath$, then $\langle Y, Z \rangle = k\sigma \cdot \imath$
Thus
$$X = e^{\sigma \cdot W + (b - \frac{1}{2}\sigma^2)\cdot \imath}\big(X_0 + k e^{-\sigma \cdot W - (b - \frac{1}{2}\sigma^2)\cdot \imath} \cdot W - (m + k\sigma) e^{-\sigma \cdot W - (b - \frac{1}{2}\sigma^2)\cdot \imath} \cdot \imath\big)$$
i.e.
$X_t = e^{\int_0^t \sigma_s \mathrm{d}W_s + \int_0^t b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u}$
$\big(X_0 + \int_0^t k_v e^{-\int_0^v \sigma_s \mathrm{d}W_s - \int_0^v b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u} \mathrm{d}W_v - \int_0^t (m_r + k_r \sigma_r) e^{-\int_0^r \sigma_s \mathrm{d}W_s - \int_0^r b_u - \frac{1}{2}\sigma_u^2 \mathrm{d}u} \mathrm{d}r\big)$

$\square$

Problem 7

Let θ be some jointly measurable and adapted $\mathbb{R}^d-$ valued stochastic process, such that the sample paths of the process $|\theta|^2 \cdot \imath$, i.e., the process $\int_0^t |\theta_s|^2 \mathrm{d}s, t \in \mathbb{R}_+$, do not explode (remain finite) in finite time ($|\theta_s|$ stands for the Euclidean norm of $\theta_s \in \mathbb{R}^d$). Assuming that W is a $d-$dimensional Brownian motion, and that r is some jointly measurable and adapated $\mathbb{R}-$valued process with locally integrable sample paths, consider the process

$$X = e^{-\theta^T \cdot W - (r + \frac{1}{2}\theta^T\theta)\cdot \imath}, \quad \text{i.e.,} \quad X_t = e^{-\int_0^t \theta_s^T \mathrm{d}W_s - \int_0^t (r_s + \frac{1}{2}\theta_s^T\theta_s)\mathrm{d}s}, \qquad t \in \mathbb{R}_+$$

and show that this process is a semimartingale by identifying its canonical decomposition into a continuous local martingale and a continuous process of finite variation. Write a stochatsic equation that the process X satisifies.
Solution: First we set the vector

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_d \end{bmatrix}, W = \begin{bmatrix} W_1 \\ \vdots \\ W_d \end{bmatrix}$$

First X is a continuous semimartingale because $-\theta^T \cdot W - (r + \frac{1}{2}\theta^T\theta) \cdot \imath$ is a continuous semimartingale and $f(x) = e^x$ is continuous and its first and second-order derivatives exist and continuous.
Then from Itô formula, we have
$X = e^{-\theta^T \cdot W - (r + \frac{1}{2}\theta^T\theta)\cdot \imath} = e^{-\sum_{i=1}^d \theta_i \cdot W_i - (r + \frac{1}{2}\sum_{j=1}^d \theta_j^2)\cdot \imath}$
$= 1 + e^{-\sum_{i=1}^d \theta_i \cdot W_i - (r + \frac{1}{2}\sum_{j=1}^d \theta_j^2)\cdot \imath} \cdot \big(-\sum_{i=1}^d \theta_i \cdot W_i - (r + \frac{1}{2}\sum_{j=1}^d \theta_j^2) \cdot \imath\big)$

$+ \frac{1}{2}e^{-\sum_{i=1}^{d}\theta_i \cdot W_i - (r+\frac{1}{2}\sum_{j=1}^{d}\theta_j^2)\cdot \imath} \cdot \left(\sum_{i=1}^{d}\theta_i^2 \cdot \imath\right)$
$= 1 - \sum_{i=1}^{d} X\theta_i \cdot W_i - r \cdot \imath$
$1-\sum_{i=1}^{d} X\theta_i \cdot W_i$ is a continuous local martingale because a stochastic integral with respect to a continuous local martingale is a local martingale and the sum of continuous local martingale is a continuous local martingale. (of course the stochastic integral is well-defined because $\int_0^t |\theta_s|^2 \mathrm{d}s$, do not explode (remain finite) in finite time.
$-r \cdot \imath$ is of finite variation because r is a process with locally integrable sample paths.
Then X satisifies the stochastic equation :

$$X = 1 - \theta^T X \cdot W - r \cdot \imath$$

i.e.

$$X_t = 1 - \sum_{i=1}^{d} \int_0^t (\theta_i)_s X_s \mathrm{d}(W_i)_s - \int_0^t r_u \mathrm{d}u$$

$\square$