Stochastic Methods in Asset Pricing

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The Coin Toss Space and the Random Walk

(section 1.4)

Andrew Lyasoff

Summary

Event Trees and Partitions of the Sample Space

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The Natural Probability Measure on the Coin Toss Space

Purely Atomic Measures and Completion of the σ-field

First Borel-Cantelli Lemma

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N.B. A function $P : \mathscr{P} \mapsto [0, 1]$ such that $\sum_{A \in \mathscr{P}} P(A) = 1$ defines a probability measure on $\sigma(\mathscr{P})$.

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The random walk is the family $\mathcal{Z}_t : \Omega_\infty \mapsto \mathbb{Z}$, $t \in \mathbb{N}$, defined by $\mathcal{Z}_t(\omega) = \mathcal{X}_0(\omega) + \mathcal{X}_1(\omega) + \cdots + \mathcal{X}_t(\omega)$.

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Unless we improve this picture, the following events will not have a well defined probability:

$$\left\{\omega \in \Omega:\ \text{the random walk } \left(\mathcal{Z}_t(\omega) \in \mathbb{Z}\right)_{t\in\mathbb{N}} \text{ crosses } 0 \in \mathbb{Z} \text{ infinitely often}\right\};$$

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Due to Caratheodory's extension theorem and Ionescu Tulcea's theorem, P_* has unique extension to a probability measure, P_∞, on the σ-field $\mathscr{F}_\infty \stackrel{\text{def}}{=} \sigma(\mathscr{A})$, to which the above events belong.

The Natural Probability Measure on the Coin Toss Space

Intuitively, if the coin is symmetric, and the coin tosses are random and are not influencing one another, all configurations $\omega \in \Omega_\infty$ should be equally likely to occur. But if we were to assign the same probability $P(\omega) = p$ to every configuration, as the configurations are infinitely many, the only choice would be $p = 0$.

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N.B. $\mathscr{A}$ is not a σ-field and P_* is not a probability measure!

Unless we improve this picture, the following events will not have a well defined probability:

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