Stochastic Methods in Asset Pricing

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The Martingale Solution to Merton's Problem

(section 14.3)

Andrew Lyasoff

Summary

Prerequisites from Convex Analysis

Merton's Investment-Consumption Problem

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$$\mathsf{E}\Big[\big(e^{-\beta\iota}U(c)\big)\bullet\iota_T + e^{-\beta T}\bar{U}(V_T)\Big],$$

where $V_t \equiv V_t^{x^*,\pi,c}$, $t \in [0,T]$, is the wealth process generated by the strategy (π, c) and the initial endowment x^*.

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By the estimates from the convex analysis prerequisites:

$$\begin{aligned} L(\pi, c, \phi) &= \phi x^* + \mathsf{E}\Big[e^{-\beta\iota}\Big(U(c) - \phi e^{\beta\iota}(S^\circ)^{-1} R\, c\Big)\boldsymbol{\cdot}\iota_T + e^{-\beta T}\Big(\bar{U}(V_T) - \phi e^{\beta T}(S^\circ_T)^{-1} R_T V_T\Big)\Big] \\ &\le \phi x^* + \mathsf{E}\Big[e^{-\beta\iota}\Big((-U)^*\big(-\phi e^{\beta\iota}(S^\circ)^{-1} R\big)\Big)\boldsymbol{\cdot}\iota_T + e^{-\beta T}(-\bar{U})^*\big(-\phi e^{\beta T}(S^\circ_T)^{-1} R_T\big)\Big]\,. \end{aligned}$$

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N.B. $\sup_{c,\pi}$ can be attained (for fixed ϕ) only with

$$c^\phi_t = (\partial U)^{-1}\big(\phi e^{\beta t}(S^\circ_t)^{-1}R_t\big) \quad\text{and}\quad V^\phi_T = (\partial\bar{U})^{-1}\big(\phi e^{\beta T}(S^\circ_T)^{-1}R_T\big)\,.$$

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$$U(c_t^\phi) \geq U(c_t) + \phi e^{\beta t}(S_t^\circ)^{-1} R_t (c_t^\phi - c_t) \quad \text{for all } \; t \in [0, T]\,,$$

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HINT: $U(c_t^\phi) - \big(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\big) c_t^\phi \geq U(c_t) - \big(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\big) c_t.$

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HINT: $\quad U(c_t^\phi) - \big(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\big)c_t^\phi \geq U(c_t) - \big(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\big)c_t.$

Can the budget constraint be satisfied with $c = c^\phi$ and $V_T = V_T^\phi$, chosen as in the previous slide (as explicit functions of ϕ), for a particular ϕ? We must solve for ϕ from

$$\mathsf{E}\Big[\big((S^\circ)^{-1} R\, c^\phi\big)\bullet\iota_T + (S_T^\circ)^{-1} R_T V_T^\phi\Big] = x^*\,.$$

Martingale solution to Merton's Investment-Consumption Problem

Exercise: If (x^*, π, c) is *any* Q-tame investment-consumption strategy, and if $\mathcal{V}$ is the associated wealth process, then

$$U(c_t^\phi) \geq U(c_t) + \phi e^{\beta t}(S_t^\circ)^{-1} R_t (c_t^\phi - c_t) \quad \text{for all } \ t \in [0, T],$$

and

$$\bar{U}(V_T^\phi) \geq \bar{U}(\mathcal{V}_T) + \phi e^{\beta T}(S_T^\circ)^{-1} R_T \left(V_T^\phi - \mathcal{V}_T\right).$$

HINT: $U(c_t^\phi) - \left(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\right) c_t^\phi \geq U(c_t) - \left(\phi e^{\beta t}(S_t^\circ)^{-1} R_t\right) c_t.$

Can the budget constraint be satisfied with $c = c^\phi$ and $V_T = V_T^\phi$, chosen as in the previous slide (as explicit functions of ϕ), for a particular ϕ? We must solve for ϕ from

$$\mathsf{E}\left[\left((S^\circ)^{-1} R\, c^\phi\right) \bullet \iota_T + (S_T^\circ)^{-1} R_T V_T^\phi\right] = x^*.$$

If this equation has a solution $\phi = \phi_*$, then we must find a Q-tame portfolio process π^*, which, together with the consumption process c^{ϕ_*} and the endowment x^*, generates a wealth process with final value that exactly matches $V_T^{\phi_*}$.

Martingale solution to Merton's Investment-Consumption Problem

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and

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HINT: $U(c_t^{\phi}) - \big(\phi e^{\beta t}(S_t^{\circ})^{-1} R_t\big) c_t^{\phi} \geq U(c_t) - \big(\phi e^{\beta t}(S_t^{\circ})^{-1} R_t\big) c_t.$

Can the budget constraint be satisfied with $c = c^{\phi}$ and $V_T = V_T^{\phi}$, chosen as in the previous slide (as explicit functions of ϕ), for a particular ϕ? We must solve for ϕ from

$$\mathsf{E}\Big[\big((S^{\circ})^{-1} R \, c^{\phi}\big) {\boldsymbol\cdot} \iota_T + (S_T^{\circ})^{-1} R_T V_T^{\phi}\Big] = x^* \, .$$

If this equation has a solution $\phi = \phi_*$, then we must find a Q-tame portfolio process π^*, which, together with the consumption process c^{ϕ_*} and the endowment x^*, generates a wealth process with final value that exactly matches $V_T^{\phi_*}$.

N.B. This step is the same as the replication of an European-style contingent claim with intertemporal payoff rate $\varphi = c^{\phi_*}$ and with final (closing) payoff $\boldsymbol{\Phi} = V_T^{\phi_*}$.

Martingale solution to Merton's Investment-Consumption Problem

Martingale solution to Merton's Investment-Consumption Problem

Martingale solution to Merton's Investment-Consumption Problem

Exercise:

Suppose (x^*, π^*, c^{ϕ_*}) is chosen as in the previous slide, and let (x^*, π, c) be *any* other Q-tame investment consumption strategy that is efficient for the initial investment x^*, that is, (x^*, π, c) generates wealth process $\mathcal{V}$ such that $\mathsf{E}\Big[\big((S^\circ)^{-1} R\, c\big){\bullet}\iota_T + (S^\circ_T)^{-1} R_T \mathcal{V}_T\Big] = x^*$. Prove that $\mathsf{E}\Big[\big(e^{-\beta\iota} U(c^{\phi_*})\big){\bullet}\iota_T + e^{-\beta T}\bar{U}(V_T^{\phi_*})\Big] \geq \mathsf{E}\Big[\big(e^{-\beta\iota} U(c)\big){\bullet}\iota_T + e^{-\beta T}\bar{U}(\mathcal{V}_T)\Big]$, and conclude that the strategy (x^*, π^*, c^{ϕ_*}) is indeed the optimal strategy.

Martingale solution to Merton's Investment-Consumption Problem

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Suppose (x^*,π^*,c^{ϕ_*}) is chosen as in the previous slide, and let (x^*,π,c) be *any* other Q-tame investment consumption strategy that is efficient for the initial investment x^*, that is, (x^*,π,c) generates wealth process $\mathcal{V}$ such that $\mathsf{E}\Big[\big((S^\circ)^{-1}R\,c\big)\bullet\iota_T + (S^\circ_T)^{-1}R_T\mathcal{V}_T\Big] = x^*$. Prove that $\mathsf{E}\Big[\big(e^{-\beta\iota}U(c^{\phi_*})\big)\bullet\iota_T + e^{-\beta T}\bar{U}(V_T^{\phi_*})\Big] \geq \mathsf{E}\Big[\big(e^{-\beta\iota}U(c)\big)\bullet\iota_T + e^{-\beta T}\bar{U}(\mathcal{V}_T)\Big]$, and conclude that the strategy (x^*,π^*,c^{ϕ_*}) is indeed the optimal strategy.

HINT: From the previous exercise

$$\Big(e^{-\beta\iota}\big(U(c^{\phi_*})-U(c)\big)\Big)\bullet\iota_T+e^{-\beta T}\big(\bar{U}(V_T^{\phi_*})-U(\mathcal{V}_T)\big) \geq \phi\Big(\big((S^\circ)^{-1}R(c^{\phi_*}-c)\big)\bullet\iota_T+(S^\circ_T)^{-1}R_T\big(V_T^{\phi_*}-\mathcal{V}_T\big)\Big),$$

where the expectation of the right side of the inequality is 0.

Martingale solution to Merton's Investment-Consumption Problem

Exercise:

Suppose (x^*, π^*, c^{ϕ_*}) is chosen as in the previous slide, and let (x^*, π, c) be *any* other Q-tame investment consumption strategy that is efficient for the initial investment x^*, that is, (x^*, π, c) generates wealth process $\mathcal{V}$ such that $\mathsf{E}\Big[\big((S^\circ)^{-1} R\, c\big)\bullet\iota_T + (S^\circ_T)^{-1} R_T \mathcal{V}_T\Big] = x^*$. Prove that $\mathsf{E}\Big[\big(e^{-\beta\iota} U(c^{\phi_*})\big)\bullet\iota_T + e^{-\beta T}\bar{U}(V^{\phi_*}_T)\Big] \geq \mathsf{E}\Big[\big(e^{-\beta\iota} U(c)\big)\bullet\iota_T + e^{-\beta T}\bar{U}(\mathcal{V}_T)\Big]$, and conclude that the strategy (x^*, π^*, c^{ϕ_*}) is indeed the optimal strategy.

HINT: From the previous exercise

$$\left(e^{-\beta\iota}\left(U(c^{\phi_*})-U(c)\right)\right)\bullet\iota_T+e^{-\beta T}\left(\bar{U}(V^{\phi_*}_T)-U(\mathcal{V}_T)\right) \geq \phi\Big(\big((S^\circ)^{-1}R(c^{\phi_*}-c)\big)\bullet\iota_T+(S^\circ_T)^{-1}R_T\left(V^{\phi_*}_T-\mathcal{V}_T\right)\Big),$$

where the expectation of the right side of the inequality is 0 (why?).

Martingale solution to Merton's Investment-Consumption Problem

Exercise:

Suppose (x^*, π^*, c^{ϕ_*}) is chosen as in the previous slide, and let (x^*, π, c) be *any* other Q-tame investment consumption strategy that is efficient for the initial investment x^*, that is, (x^*, π, c) generates wealth process $\mathcal{V}$ such that $\mathsf{E}\Big[\big((S^\circ)^{-1} R\, c\big) {}^\bullet\iota_T + (S^\circ_T)^{-1} R_T \mathcal{V}_T\Big] = x^*$. Prove that

$\mathsf{E}\Big[\big(e^{-\beta\iota} U(c^{\phi_*})\big) {}^\bullet\iota_T + e^{-\beta T} \bar{U}(V^{\phi_*}_T)\Big] \geq \mathsf{E}\Big[\big(e^{-\beta\iota} U(c)\big) {}^\bullet\iota_T + e^{-\beta T} \bar{U}(\mathcal{V}_T)\Big]$, and conclude that the strategy (x^*, π^*, c^{ϕ_*}) is indeed the optimal strategy.

HINT: From the previous exercise

$$\left(e^{-\beta\iota}\left(U(c^{\phi_*}) - U(c)\right)\right) {}^\bullet\iota_T + e^{-\beta T}\left(\bar{U}(V^{\phi_*}_T) - U(\mathcal{V}_T)\right) \geq \phi\Big(\big((S^\circ)^{-1} R(c^{\phi_*} - c)\big) {}^\bullet\iota_T + (S^\circ_T)^{-1} R_T \left(V^{\phi_*}_T - \mathcal{V}_T\right)\Big),$$

where the expectation of the right side of the inequality is 0.