Stochastic Methods in Asset Pricing

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Put-Call Symmetry

(section 14.5)

Andrew Lyasoff

Summary

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$$\mathrm{d}X_t \stackrel{\text{def}}{=} \frac{1 + r^e\,\mathrm{d}t}{S_t^{e/u}} S_{t+\mathrm{d}t}^{e/u} - (1 + r^u\,\mathrm{d}t)$$

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If X is a semimartingale with local variance $\mathrm{d}\langle X, X\rangle_t = \sigma^2\,\mathrm{d}t$, then it must be that $X = \sigma\beta$, so that

$$\mathrm{d}S_t^{e/u} = S_t^{e/u}\big(\sigma\,\mathrm{d}\beta_t + (r^u - r^e)\,\mathrm{d}t\big)\,, \qquad \text{where } \beta \text{ is a } (\mathscr{G}, Q^u)\text{-Brownian motion.}$$

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Exercise: Prove that the dynamics of the \$/€ exchange rate are governed by

$$\mathrm{d}S_t^{u/e} = S_t^{u/e}\Big(\sigma\,\mathrm{d}(-\beta_t^*) + (r^e - r^u)\mathrm{d}t\Big)\,,$$

where $\beta_t^* = \beta_t - \sigma t$. Explain why (β_t^*), or equivalently $(-\beta_t^*)$, must be a $(\mathscr{G}_t, Q^e)$-Brownian motion, where (Q^e) is the pricing measure for EU traded assets.

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$C\big(t, S_t^{e/u}, K, \sigma^2, r^u, r^e\big) \stackrel{\text{def}}{=}$ the price in \$ of a the right to purchase €1 at (strike) price \$$K$ (a call).

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This derivative contract is the same as the right to sell \$$K$ for the price of €1, which is no different from the right to sell K identical contracts, every one of which gives its holder the right to sell \$1 for €$(1/K)$.

The payoff from each of these K contracts, at time $\not{t} \geq t$, expressed in € is $\left(\dfrac{1}{K} - S_{\not{t}}^{u/e}\right)^+$.

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The total price of K put contracts in € is $K\,P\big(t, S_t^{u/e}, 1/K, \sigma^2, r^e, r^u\big)\,.$

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The total price of K put contracts in € is $K\,P\big(t, S_t^{u/e}, 1/K, \sigma^2, r^e, r^u\big)$.

Converted into \$ the price is $S_t^{e/u}\,K\,P\big(t, S_t^{u/e}, 1/K, \sigma^2, r^e, r^u\big)$.

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$$S_t^{e/u}\, K\, P\big(t, S_t^{u/e}, 1/K, \sigma^2, r^e, r^u\big) = P\big(t, K, S_t^{e/u}, \sigma^2, r^e, r^u\big)\,.$$

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HINT: Suppose that at time t one has $S_t^{e/u} = x$, so that $S_t^{u/e} = 1/x$. The left side above is the same as the price of $x\,K$ put options (priced in €).

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$$x\,K\left(\frac{1}{K} - S_{\not{t}}^{u/e}\right)^{+} = \big(x - x\,K\,S_{\not{t}}^{u/e}\big)^{+}\,.$$

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Since $S_t^{u/e} = \dfrac{1}{x}$, we have $x\,K\,S_t^{u/e} = K$, so that $x\,K\,S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$ is the same geometric Brownian motion as $S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$, i.e., a geometric Brownian motion (under Q^e) of volatility σ and growth rate $r^e - r^u$, except $x\,K\,S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$, starts at time t from position K instead of $\dfrac{1}{x}$.

Exercise: Whether the put is of European-style or American-style, one must have

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$$x\,K\left(\frac{1}{K} - S_{\mathscr{t}}^{u/e}\right)^+ = \big(x - x\,K\,S_{\mathscr{t}}^{u/e}\big)^+\,.$$

Since $S_t^{u/e} = \dfrac{1}{x}$, we have $x\,K\,S_t^{u/e} = K$, so that $x\,K\,S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$ is the same geometric Brownian motion as $S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$, i.e., a geometric Brownian motion (under Q^e) of volatility σ and growth rate $r^e - r^u$, except $x\,K\,S_{\mathscr{t}}^{u/e}$, $\mathscr{t} \geq t$, starts at time t from position K instead of $\dfrac{1}{x}$. The price (in €) at time t of an option with this payoff can be expressed as $P\big(t, K, x, \sigma^2, r^e, r^u\big)$ (the spot is K and the strike is $x = S_t^{e/u}$). As a result, we have

$$C\big(t, S_t^{e/u}, K, \sigma^2, r^u, r^e\big) = P\big(t, K, S_t^{e/u}, \sigma^2, r^e, r^u\big)\,,$$

Exercise: Whether the put is of European-style or American-style, one must have

$$S_t^{e/u}\,K\,P\big(t, S_t^{u/e}, 1/K, \sigma^2, r^e, r^u\big) = P\big(t, K, S_t^{e/u}, \sigma^2, r^e, r^u\big)\,.$$

HINT: Suppose that at time t one has $S_t^{e/u} = x$, so that $S_t^{u/e} = 1/x$. The left side above is the same as the price of $x\,K$ put options (priced in €). If exercised at time $\tau \geq t$, the aggregate payoff (in €) from these options would be

$$x\,K\left(\frac{1}{K} - S_\tau^{u/e}\right)^+ = \big(x - x\,K\,S_\tau^{u/e}\big)^+\,.$$

Since $S_t^{u/e} = \dfrac{1}{x}$, we have $x\,K\,S_t^{u/e} = K$, so that $x\,K\,S_\tau^{u/e}$, $\tau \geq t$ is the same geometric Brownian motion as $S_\tau^{u/e}$, $\tau \geq t$, i.e., a geometric Brownian motion (under Q^e) of volatility σ and growth rate $r^e - r^u$, except $x\,K\,S_\tau^{u/e}$, $\tau \geq t$, starts at time t from position K instead of $\dfrac{1}{x}$. The price (in €) at time t of an option with this payoff can be expressed as $P\big(t, K, x, \sigma^2, r^e, r^u\big)$ (the spot is K and the strike is $x = S_t^{e/u}$). As a result, we have

$$C\big(t, S_t^{e/u}, K, \sigma^2, r^u, r^e\big) = P\big(t, K, S_t^{e/u}, \sigma^2, r^e, r^u\big)\,,$$

which is to say, in the most general terms:

$$\mathrm{Call}\big(t, S_t, K, \sigma^2, r, \delta\big) = \mathrm{Put}\big(t, K, S_t, \sigma^2, \delta, r\big)\,.$$