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MF 795
Assignment 4

9.31
By the law of iterated logarithm, we can have:

$$\lim_{t\to\infty}\frac{W_t}{\sqrt{2t(\log(\log t))}} \le 1$$

Then,

$$\lim_{t\to\infty} W_t \le \lim_{t\to\infty}\sqrt{2t(\log(\log t))}$$
$$\Longrightarrow \lim_{t\to\infty}(W_t - \frac{t}{2}) \le \lim_{t\to\infty}\left(\sqrt{2t(\log(\log t))} - \frac{t}{2}\right)$$

As $\sqrt{2t(\log(\log t))}$ grows much slower than $\frac{t}{2}$, then when $t\to\infty$, $RHS = -\infty$. Therefore,

$$\lim_{t\to\infty}(W_t - \frac{t}{2}) = -\infty$$

And,

$$\lim_{t\to\infty} e^{W_t - \frac{t}{2}} = 0$$

9.32

$$\begin{aligned} E[X_t^+] &= E[W_t^+] \\ &= \int_0^\infty \frac{x}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_0^\infty e^{-\frac{u}{t}}\,du \\ &= \frac{-t}{\sqrt{2\pi t}} \times \left[e^{-\frac{u}{t}}\right]_0^\infty \\ &= -\frac{\sqrt{t}}{\sqrt{2\pi}} \times (0-1) \\ &= \frac{\sqrt{t}}{\sqrt{2\pi}} \end{aligned}$$

Therefore, when $t\to\infty$, $E[X_t^+]\to\infty$. Then, it does not follow the convergence theorem.

$$\begin{aligned} E[X_t^+] &= E[W^2] \\ &= \int_0^\infty \frac{x^2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}dx \\ &= \frac{t}{2} \end{aligned}$$

Therefore, when $t\to\infty$, $E[X_t^+]\to\infty$. Then, it does not follow the convergence theorem.

9.35
By (9.33), we know that those three conditions are equivalent, then we only need to prove one of them to let the other two also be satisfied. Suppose we let $R_\infty = \frac{dQ}{dP}$, then $E[R_t|\mathcal{F}_s] = E\left[\frac{dQ}{dP} \upharpoonright \mathcal{F}_t \middle| \mathcal{F}_s\right]$. As $Q \precsim P$, then $Q(A) = \int_A R_t dP = E[R_t 1_A] = E[R_\infty 1_A] = E\big[E[R_\infty 1_A|\mathcal{F}_t]\big] =$

$E\left[1_A E[R_\infty|\mathcal{F}_t]\right] \Longrightarrow R_t = E[R_\infty|\mathcal{F}_t]$. Now, if we can prove that R is a martingale, we can automatically get it is also uniformly integrable. Suppose $0 \le s < t < \infty$, then $E[R_t|\mathcal{F}_s] = E[E[R_\infty|\mathcal{F}_t]|\mathcal{F}_s] = E[R_\infty|\mathcal{F}_s] = R_s$. Therefore, R is a uniformly integrable martingale. Then the limit exists, $\lim_{t\to\infty} R_t = R_\infty = \frac{dQ}{dP}$.

9.40
As X is a uniformly integrable martingale, then it is a sub-martingale as well as a super-martingale. Therefore, $X_{\mathscr{s}} = E[X_{\mathscr{t}}|\mathcal{F}_{\mathscr{s}}]$, and when $\mathscr{t} = \infty$, $X_{\mathscr{s}} = E[X_\infty|\mathcal{F}_{\mathscr{s}}]$. As $X_{\mathscr{t}} = E[X_\infty|\mathcal{F}_{\mathscr{t}}]$, for all choices of stopping time $\mathscr{t}$. And $\mathcal{F}_{\mathscr{t}} \subset \mathcal{F}_\infty \subseteq \mathcal{F}$, therefore, all $\mathcal{F}_{\mathscr{t}}$ is a sub-σ-field. Then by (4.24), we can say that the entire collection of $X_{\mathscr{t}}$ is uniformly integrable.

9.44
Note that as $\mathcal{F}_t^{\mathscr{t}} \equiv \mathcal{F}_{t\wedge \mathscr{t}} \subseteq \mathcal{F}_t$. Since X is right continuous and adapted, then it is progressively measurable. Then, by (8.56), we can get $X^{\mathscr{t}} \prec \mathcal{F}^{\mathscr{t}}$ and $X^{\mathscr{t}}$ is cadlag. Suppose, we have a choice of a bounded stopping time for $\mathcal{F}^{\mathscr{t}}$, say $\mathscr{s}$, then $\{\mathscr{s} \le t\} \in \mathcal{F}_t^{\mathscr{t}} = \mathcal{F}_{t\wedge\mathscr{t}} \subseteq \mathcal{F}_t$. Then, $\{\mathscr{s} \le t\} \in \mathcal{F}_t$. Therefore, $\mathscr{s}$ is $\mathcal{F}$-stopping time, as $\mathscr{t}$ is $\mathcal{F}$-stopping time as well, then $\mathscr{s} \wedge \mathscr{t}$ is $\mathcal{F}$-stopping time and $E[|X_{\mathscr{s}\wedge\mathscr{t}}|] < \infty$. Also, because $\mathscr{s} \le T$, then $\mathscr{s} \wedge \mathscr{t} \le T$. Since X is a martingale w.r.t. $\mathcal{F}$, then:

$$E[X_{\mathscr{s}\wedge\mathscr{t}}] = E[X_0]$$

Therefore, $X^{\mathscr{t}}$ is $\mathcal{F}^{\mathscr{t}}$-martingale.

9.45
Suppose X is an $\mathcal{F}^{\mathscr{t}}$ martingale. Note that as $\mathcal{F}_t^{\mathscr{t}} \equiv \mathcal{F}_{t\wedge\mathscr{t}} \subseteq \mathcal{F}_t$, then if $X \prec \mathcal{F}^{\mathscr{t}}$, then $X \prec \mathcal{F}_t$. As X is an $\mathcal{F}^{\mathscr{t}}$ martingale and $t \wedge \mathscr{t}$ is a bounded $\mathcal{F}^{\mathscr{t}}$-stopping time, then $X_{t\wedge\mathscr{t}} = E[X_t|\mathcal{F}_{t\wedge\mathscr{t}}] = X_t \Longrightarrow X_{t\wedge\mathscr{t}} = X_t \Longrightarrow X_{t\wedge\mathscr{t}}$ is the modification of X_t. Furthermore, because X_t and $X_{t\wedge\mathscr{t}}$ are both right-continuous, then we can say X_t and $X_{t\wedge\mathscr{t}}$ are indistinguishable. If $\mathscr{s}$ is any bounded $\mathcal{F}$-stopping time, then $X_{\mathscr{s}\wedge\mathscr{t}} = X_{\mathscr{s}}$ and $\mathscr{s} \wedge \mathscr{t}$ is a bounded $\mathcal{F}^{\mathscr{t}}$-stopping time and it is also a bounded $\mathcal{F}$-stopping time. Therefore, we have:

$$E[X_{\mathscr{s}\wedge\mathscr{t}}] = E[X_{\mathscr{s}}] = E[X_0]$$
$$E[|X_{\mathscr{s}}|] = E[|X_{\mathscr{s}\wedge\mathscr{t}}|] < \infty$$

Therefore, X is an $\mathcal{F}$-martingale.

9.68
(a) Suppose $\mathscr{t}_i = i$. Then conditions (a), $\mathscr{t}_{i+1} \le \mathscr{t}_i$, and (b), $\lim_i \mathscr{t}_i = \infty$, of (9.66) are satisfied. Now, we are left to show that if $\mathscr{t}_i$ reduces M. When $i = 0$, $\mathscr{t}_0 = 0 \Longrightarrow E\left[1_{\{\mathscr{t}_0>0\}}|X_t^0|\right] = 0 \Longrightarrow 1_{\{\mathscr{t}_0>0\}}X_t^0$ is a uniformly integrable martingale. When $i > 0$, $1_{\{\mathscr{t}_i>0\}} = 1 \Longrightarrow 1_{\{\mathscr{t}_i>0\}}X_t^{\mathscr{t}_i}$ is a uniformly integrable martingale.

(b) Suppose M, N are two local martingales, then let $\mathscr{s}$ and $\mathscr{t}$ be the two localizing sequence respectively. Let $\mathscr{u} = \mathscr{s} \wedge \mathscr{t}$. Then conditions (a), $\mathscr{t}_{i+1} \le \mathscr{t}_i$, and (b), $\lim_i \mathscr{t}_i = \infty$, of (9.66) are satisfied. Now, we are left to show that if $\mathscr{u}_i$ reduces M. Since, $\mathscr{s}_i$ reduces M and $\mathscr{t}_i$ reduces N, and they are both stopping times for M and N, respectively, then $\mathscr{s}_i \wedge \mathscr{t}_i$ is also a stopping time for M and N. By (9.65), we can claim that because $\mathscr{s}_i \wedge \mathscr{t}_i \le \mathscr{s}_i$ or $\mathscr{t}_i$, then $\mathscr{s}_i \wedge \mathscr{t}_i$ also reduces M and $N \Longrightarrow 1_{\{\mathscr{t}_i\wedge\mathscr{s}_i>0\}}M_t^{\mathscr{t}_i\wedge\mathscr{s}_i}, 1_{\{\mathscr{t}_i\wedge\mathscr{s}_i>0\}}N_t^{\mathscr{t}_i\wedge\mathscr{s}_i}$ are uniformly integrable martingales, which implies that $1_{\{t_i \wedge s_i > 0\}} M_t^{t_i \wedge s_i} \pm N_t^{t_i \wedge s_i}$ is also a uniformly integrable martingales. Then, $t_i \wedge s_i$ also reduces $M \pm N$. Therefore, $M \pm N$ is also a local martingale.

(c) Suppose t_i is the localizing sequence for X. Then, conditions (a), $t_{i+1} \leq t_i$, and (b), $\lim_i t_i = \infty$, of (9.66) are satisfied and $1_{\{t_i>0\}} X_t^{t_i}$ is a uniformly integrable martingale. Let Y be a stopped local martingale, say $X_{s \wedge t}$. Then, we have $1_{\{t_i>0\}} Y_t^{t_i} = 1_{\{t_i>0\}} X_{s \wedge t}^{t_i} = 1_{\{t_i>0\}} X_{s \wedge t \wedge t_i}$. As t_i reduces X, then $t_i \wedge s$ reduces X too $\Longrightarrow \left(X_t^{t_i \wedge s} - X_0^{t_i \wedge s}\right)$ is a u.i. martingale $\Longrightarrow$ $\left(X_{t \wedge s}^{t_i} - X_{0 \wedge s}^{t_i}\right)$ is also a u.i. martingale $\Longrightarrow t_i$ reduces $X_{s \wedge t}$. Therefore, a stopped local martingale is still a local martingale.

(d) Let t_i be the localizing sequence for local martingale X. Then,

$$
\begin{aligned}
X_s^{t_n} &= E\left[X_t^{t_n} \middle| \mathcal{F}_s\right] \\
X_s &= \lim_{n \to \infty} X_s^{t_n} \\
&= \lim_{n \to \infty} \inf E\left[X_t^{t_n} \middle| \mathcal{F}_s\right] \\
&\geq E\left[\lim_{n \to \infty} \inf X_t^{t_n} \,\middle|\, \mathcal{F}_s\right] \\
&= E[X_t | \mathcal{F}_s]
\end{aligned}
$$

9.78

(a) ($\Longrightarrow$) $M \prec \mathcal{F}$ is an $(\mathcal{F}, Q)$-martingale $\Longrightarrow E^Q[M_t | \mathcal{F}_s] = M_s$. By (3.45), we know that $E^Q[M_t | \mathcal{F}_s] = \frac{1}{E^P[R_t | \mathcal{F}_s]} E^P[M_t R_t | \mathcal{F}_s] = \frac{1}{R_s} E^P[M_t R_t | \mathcal{F}_s] \Longrightarrow M_s R_s = E^P[M_t R_t | \mathcal{F}_s]$. Also, as $M \prec \mathcal{F}$, and $Q, P \prec \mathcal{F}$, then $MR \prec \mathcal{F}$. Thus, we know $E[M_t] < \infty$ and $E[R_t] = 1 \Longrightarrow E[M_t R_t] < \infty$. Therefore, MR is an $(\mathcal{F}, Q)$-martingale.
($\Longleftarrow$) MR is an $(\mathcal{F}, Q)$-martingale, then $E[M_t R_t] < \infty$ and $E[R_t] = 1 \Longrightarrow E[M_t] < \infty$. Also, as $MR \prec \mathcal{F}$, $Q, P \prec \mathcal{F} \Longrightarrow R \prec \mathcal{F}$, then $M \prec \mathcal{F}$. From the above, we have $M_s R_s = E^P[M_t R_t | \mathcal{F}_s] \Longrightarrow$ $M_s = \frac{1}{R_s} E^P[M_t R_t | \mathcal{F}_s] = \frac{1}{E^P[R_t | \mathcal{F}_s]} E^P[M_t R_t | \mathcal{F}_s] = E^Q[M_t | \mathcal{F}_s]$. Therefore, M is an $(\mathcal{F}, Q)$-martingale.

(b) ($\Longrightarrow$) $M \prec \mathcal{F}$ is a local $(\mathcal{F}, Q)$-martingale, then $\exists$ a localizing sequence t_i such that $1_{\{t_i>0\}} M^{t_i}$ is a martingale w.r.t. Q. Then, $E^Q\left[1_{\{t_i>0\}} M^{t_i}\right] < \infty \Longrightarrow E^P\left[1_{\{t_i>0\}} M^{t_i} R^{t_i}\right] < \infty$, we are left to prove that $1_{\{t_i>0\}} M^{t_i} R^{t_i}$ is a martingale. As $M \prec \mathcal{F}$, $R \prec \mathcal{F}$, $M^{t_i} \prec \mathcal{F}^{t_i} \Longrightarrow M^{t_i} R^{t_i} \prec \mathcal{F}^{t_i}$, also because $M^{t_i} \prec \mathcal{F} \Longrightarrow R^{t_i} \prec \mathcal{F}$. Then, we can get $E^Q\left[1_{\{t_i>0\}} M^{t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] = 1_{\{t_i>0\}} M_{s \wedge t_i}$. $\Longrightarrow E^Q\left[1_{\{t_i>0\}} M_{t \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] = \frac{1}{E^P\left[R_{s \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right]} E^P\left[1_{\{t_i>0\}} M_{t \wedge t_i} R_{s \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] \Longrightarrow 1_{\{t_i>0\}} M_{s \wedge t_i} = \frac{1}{R_{s \wedge t_i}} E^P\left[1_{\{t_i>0\}} M_{t \wedge t_i} R_{s \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] \Longrightarrow 1_{\{t_i>0\}} M_{s \wedge t_i} R_{s \wedge t_i} = E^P\left[1_{\{t_i>0\}} M_{t \wedge t_i} R_{s \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] \Longrightarrow$ $1_{\{t_i>0\}} M_{s \wedge t_i} R_{s \wedge t_i}$ is a local $(\mathcal{F}, P)$-martingale.
($\Longleftarrow$) MR is local $(\mathcal{F}, P)$-martingale$\Longrightarrow 1_{\{t_i>0\}} M^{t_i} R^{t_i}$ is a martingale. As $M^{t_i} R^{t_i} \prec \mathcal{F}^{t_i} \Longrightarrow$ $M^{t_i} R^{t_i} \prec \mathcal{F}$, as $R^{t_i} \prec \mathcal{F} \Longrightarrow M^{t_i} \prec \mathcal{F}$. Thus, because $E\left[1_{\{t_i>0\}} M^{t_i} R^{t_i}\right] < \infty \Longrightarrow$ $E\left[1_{\{t_i>0\}} M^{t_i}\right] < \infty$. Now, we are left to show that $1_{\{t_i>0\}} M^{t_i}$ is a martingale. From statement above, we know $1_{\{t_i>0\}} M_{s \wedge t_i} R_{s \wedge t_i} = E^P\left[1_{\{t_i>0\}} M_{t \wedge t_i} R_{s \wedge t_i} \middle| \mathcal{F}_{s \wedge t_i}\right] \Longrightarrow 1_{\{t_i>0\}} M_{s \wedge t_i} =$

$\frac{1}{R_{s\wedge t_i}} E^P\left[1_{\{t_i>0\}} M_{t\wedge t_i} R_{s\wedge t_i} \middle| \mathcal{F}_{s\wedge t_i}\right] \Rightarrow 1_{\{t_i>0\}} M_{s\wedge t_i} =$
$\frac{1}{E^P\left[R_{s\wedge t_i} \middle| \mathcal{F}_{s\wedge t_i}\right]} E^P\left[1_{\{t_i>0\}} M_{t\wedge t_i} R_{s\wedge t_i} \middle| \mathcal{F}_{s\wedge t_i}\right] = E^Q\left[1_{\{t_i>0\}} M_{t\wedge t_i} \middle| \mathcal{F}_{s\wedge t_i}\right] \Rightarrow 1_{\{t_i>0\}} M^{t_i} R^{t_i}$ is a $(\mathcal{F}, Q)$-martingale.

10.4

(a) Note that X can be expressed as a finite sum, $X = \sum_{i\in\mathbb{N}_{|n}} X^i = \xi 1_{[[0]]} + \sum_{i\in\mathbb{N}_{|n}} \eta_i 1_{]]t_i,t_{i+1}]]} \Rightarrow$
$X \cdot W = \sum_{i\in\mathbb{N}_{|n}} X^i \cdot W = 0 + \sum_{i\in\mathbb{N}_{|n}} \eta_i (W_{t_{i+1}\wedge t} - W_{t_i\wedge t}) = \sum_{i\in\mathbb{N}_{|n}} \eta_i (W_{t_{i+1}\wedge t} - W_{t_i\wedge t})$.
Suppose that $s \in]t_k, t_{k+1}]$, and $t \in]t_l, t_{l+1}]$, where $t_l > t_k$. Now we want to prove that $E[X \cdot W_t | \mathcal{F}_s] = X \cdot W_s$. We can rewrite $X \cdot W_t$ as:

$$X \cdot W_t = \sum_{i=0}^{k-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) + \eta_k\left(W_{t_{k+1}} - W_{t_k}\right) + \sum_{i=k+1}^{l-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) + \eta_l\left(W_t - W_{t_l}\right)$$

1st term:

$$E\left[\sum_{i=0}^{k-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) | \mathcal{F}_s\right] = \sum_{i=0}^{k-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right)$$

2nd term:

$$\begin{aligned} E[\eta_k\left(W_{t_{k+1}} - W_{t_k}\right) | \mathcal{F}_s] &= \eta_k E[W_{t_{k+1}} - W_{t_k} | \mathcal{F}_s] \\ &= \eta_k\left[E\left[W_{t_{k+1}} \middle| \mathcal{F}_s\right] - W_{t_k}\right] \\ &= \eta_k\left(W_s - W_{t_k}\right) \end{aligned}$$

3rd term:

$$\begin{aligned} E\left[\sum_{i=k+1}^{l-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) | \mathcal{F}_s\right] &= \sum_{i=k+1}^{l-1} E\left[\eta_i\left(W_{t_{i+1}} - W_{t_i}\right) \middle| \mathcal{F}_s\right] \\ &= \sum_{i=k+1}^{l-1} E\left[E\left[\eta_i\left(W_{t_{i+1}} - W_{t_i}\right) \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_{t_i}\right] \\ &= \sum_{i=k+1}^{l-1} E\left[\eta_i\left(W_{t_{i+1}} - W_{t_i}\right) \middle| \mathcal{F}_{t_i}\right] \\ &= 0 \end{aligned}$$

4th term:

$$\begin{aligned} E\left[\eta_l\left(W_t - W_{t_l}\right) | \mathcal{F}_s\right] &= E\left[E\left[\eta_l\left(W_t - W_{t_l}\right) \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_{t_i}\right] \\ &= E\left[\eta_l\left(W_t - W_{t_l}\right) \middle| \mathcal{F}_{t_i}\right] \\ &= 0 \end{aligned}$$

Therefore, $E[X \cdot W_t | \mathcal{F}_s] = \sum_{i=0}^{k-1} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) + \eta_k\left(W_s - W_{t_k}\right) = \sum_{i=0}^{k} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right) = X \cdot W_s$.

(b) $E[(X \cdot W_t)^2] = E\left[\left(\sum_{i=1}^{n} \eta_i\left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] = E[\sum_{i=1}^{n} {\eta_i}^2\left(W_{t_{i+1}} - W_{t_i}\right)^2] +$
$E\left[\sum_{i=0}^{n} \sum_{i\neq j} \eta_i \eta_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right]$. Note that $E\left[\eta_i \eta_j\left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right] = E\left[\eta_i \eta_j\left(W_{t_{i+1}} - W_{t_i}\right)\right] E\left[\left(W_{t_{j+1}} - W_{t_j}\right)\right] = 0$

$$\Longrightarrow E[(X \cdot W_t)^2] = E[\sum_{i=1}^{n} \eta_i{}^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2]$$
$$= \sum_{i=0}^{n} E[\eta_i{}^2] E\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]$$
$$= \sum_{i=0}^{n} E[\eta_i{}^2](\, t_{i+1} - t_i)$$
$$= \int_0^t E[X_s^2] ds$$

10.5

(a) As $X, Y \in \mathfrak{G}$, suppose $s < t$, then $X \cdot W_s = \sum_{i=0}^{l} \eta_i \left(W_{t_{i+1}} - W_{t_i}\right)$ and $Y \cdot W_t = \sum_{j=0}^{k} \xi_j \left(W_{t_{j+1}} - W_{t_j}\right)$

$$\Longrightarrow X \cdot W_s \times Y \cdot W_t = \sum_{i=1}^{l} \sum_{j=1}^{k} \eta_i \xi_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)$$
$$= \sum_{i=1}^{l} \eta_i \xi_i \left(W_{t_{i+1}} - W_{t_i}\right)^2$$
$$\Longrightarrow E[X \cdot W_s \times Y \cdot W_t] = E\left[\sum_{i=1}^{l} \eta_i \xi_i \left(W_{t_{i+1}} - W_{t_i}\right)^2\right]$$
$$= \sum_{i=1}^{l} E[\eta_i \xi_i](t_{i+1} - t_i)$$
$$= \int_0^s E[X_r Y_r] dr$$

If $s \geq t$, then $E[X \cdot W_s \times Y \cdot W_t] = \int_0^t E[X_r Y_r] dr$, therefore, $E[X \cdot W_s \times Y \cdot W_t] = \int_0^{s \wedge t} E[X_r Y_r] dr$.

(b)

$$E[(X \cdot W_s - Y \cdot W_t)^2] = E[(X \cdot W_s)^2] + E[(Y \cdot W_t)^2] - 2E[X \cdot W_s \times Y \cdot W_t]$$
$$= \int_0^t E[X_s^2] ds + E[Y_s^2] ds - 2E[X_s Y_s] ds$$
$$= \int_0^t E[(X_s - Y_s)^2] ds$$
$$= \int_0^t E[(X_r - Y_r)^2] dr$$

10.25

$$AM = A_0 M_0 + A \cdot M + M \cdot A + < A, M >$$
$$\because A_0 = 0 \Longrightarrow AM - M \cdot A = A \cdot M + < A, M >= A \cdot M$$

If A is only cadlag, then:

$$\begin{aligned}
A_t M_t &= \sum\nolimits_{i\in\mathbb{N}_{|n-1}} \left(A_{t_{i+1}^-} M_{t_{i+1}} - A_{t_i^-} M_{t_i}\right) + A_0 M_0 \\
&= \sum\nolimits_{i\in\mathbb{N}_{|n-1}} \left(A_{t_{i+1}^-} M_{t_{i+1}} - A_{t_i^-} M_{t_{i+1}} + A_{t_i^-} M_{t_{i+1}} - A_{t_i^-} M_{t_i}\right) \\
&= \sum\nolimits_{i\in\mathbb{N}_{|n-1}} (\left(A_{t_{i+1}^-} - A_{t_i^-}\right) M_{t_{i+1}} + A_{t_i^-}(M_{t_{i+1}} - M_{t_i})) \\
&= \sum\nolimits_{i\in\mathbb{N}_{|n-1}} \left(A_{t_{i+1}^-} - A_{t_i^-}\right) M_{t_i} + A_{t_i^-}\left(M_{t_{i+1}} - M_{t_i}\right) + \left(A_{t_{i+1}^-} - A_{t_i^-}\right)\left(M_{t_{i+1}} - M_{t_i}\right) \\
&= M\cdot A + A_-\cdot M + < A_-, M > \\
&= M\cdot A + A_-\cdot M \\
&\Longrightarrow AM - M\cdot A = A_-\cdot M
\end{aligned}$$

Therefore, by (10.16), a stochastic integral w.r.t. a local martingale is also a local martingale, then $AM - M\cdot A$ is also a local martingale.

10.26

$$XYZ = X_0Y_0Z_0 + X\cdot(YZ) + (YZ)\cdot X + < X, YZ >$$

Where $YZ = Y_0Z_0 + Y\cdot Z + Z\cdot Y + < Y, Z >$, then:

$$\begin{aligned}
X\cdot(YZ) &= X\cdot(Y_0Z_0 + Y\cdot Z + Z\cdot Y + < Y, Z >) \\
&= X\cdot(Y\cdot Z) + X\cdot(Z\cdot Y) + X\cdot < Y, Z > \\
&= (XY)\cdot Z + (XZ)\cdot Y + X\cdot < Y, Z >
\end{aligned}$$

And

$$\begin{aligned}
&< X, YZ > = < X, (Y_0Z_0 + Y\cdot Z + Z\cdot Y + < Y, Z >) > \\
&= < X, Y\cdot Z > + < X, Z\cdot Y > \\
&= Y\cdot < X, Z > + Z\cdot < X, Y >
\end{aligned}$$

Therefore,

$$XYZ = X_0Y_0Z_0 + (XY)\cdot Z + (XZ)\cdot Y + X\cdot < Y, Z > + (YZ)\cdot X + Y\cdot < X, Z > + Z\cdot < X, Y >$$

10.27

<u>Base case:</u> n=2. $LHS = X^2 = X_0^2 + X\cdot X + X\cdot X + < X, X > = X_0^2 + 2X\cdot X + < X, X >$; $RHS = X_0^2 + 2X\cdot X + < X, X > = LHS$.

<u>Induction hypothesis:</u> assume the statement holds true for n=k. now we need to prove it also holds when n=k+1.

$$\begin{aligned}
X^{k+1} &= XX^k \\
&= X_0^{k+1} + X\cdot X^k + X^k\cdot X + < X, X^k >
\end{aligned}$$

Note that,

$$\begin{aligned}
X^k &= X_0^k + k(X^{k-1})\cdot X + \frac{1}{2}k(k-1)(X^{k-2})\cdot < X, X > \\
< X, X^k > &= X^{k-1}\cdot < X, X >
\end{aligned}$$

Therefore,

$$\begin{aligned}
X\cdot X^k &= X\cdot\left(X_0^k + k(X^{k-1})\cdot X + \frac{1}{2}k(k-1)(X^{k-2})\cdot < X, X >\right) \\
&= kX^k\cdot X + \frac{1}{2}k(k-1)(X^{k-1})\cdot < X, X > \\
X^{k+1} &= X_0^{k+1} + (k+1)X^k\cdot X + \frac{1}{2}k(k+1)(X^{k-1})\cdot < X, X >
\end{aligned}$$

Therefore, the statement is true for every $n \geq 2$.

10.29

Denote $f(X,Y) = XY$, $X^1 = X$, $X^2 = Y$, then according to Ito's Formula, we can have:

$$\frac{\partial XY}{\partial X} = Y, \frac{\partial XY}{\partial Y} = X, \frac{\partial^2 XY}{\partial X \partial Y} = 1, \frac{\partial^2 XY}{\partial X^2} = 0, \frac{\partial^2 XY}{\partial Y^2} = 0$$

Therefore,

$$XY = X_0 Y_0 + \sum_{i \in \mathbb{N}_{|n}} \partial_i f(X,Y) \cdot X^i + \frac{1}{2} \sum_{i \in \mathbb{N}_{|n}} \sum_{j \in \mathbb{N}_{|n}} \partial_i \partial_j f(X,Y) \cdot < X^i, X^j >$$
$$= X_0 Y_0 + Y \cdot X + X \cdot Y + \frac{1}{2}(0 \cdot < X,X > +1 \cdot < X,Y > +1 \cdot < Y,X > +0 \cdot < Y,Y >)$$
$$= X_0 Y_0 + Y \cdot X + X \cdot Y + < X,Y >$$

10.31

(a) By Ito's Formula, we have

$$f(X,Y) = f(X_0,Y_0) + \partial_x f \cdot X + \partial_y f \cdot Y + \frac{1}{2}(\partial_x \partial_x f \cdot < X,X > +2 \times \partial_x \partial_y f \cdot < X,Y > +\partial_y \partial_y f \cdot < Y,Y >)$$

As X, Y are continuous semimartingales, and f is a sufficiently continuous and smooth function, then $\partial_x f$, $\partial_y f$, $\partial_x \partial_x f$, $\partial_x \partial_y f$ and $\partial_y \partial_y f$ are all continuous. Therefore, $f(X,Y)$ is also a continuous semimartingale. Denote $Y = M' + A'$ and $X = M + A$, then,

$$f(X,Y) = f(X_0,Y_0) + \partial_x f \cdot (M + A) + \partial_y f \cdot (M' + A') + \frac{1}{2} \partial_x \partial_x f \cdot < M^2 > +\partial_x \partial_y f \cdot < M, M' > + \frac{1}{2} \partial_y \partial_y f \cdot < M'^2 >$$

Local martingale part: $f(X_0,Y_0) + \partial_x f \cdot M + \partial_y f \cdot M'$

Finite variation part: $\partial_x f \cdot A + \partial_y f \cdot A' + \frac{1}{2} \partial_x \partial_x f \cdot < M^2 > +\partial_x \partial_y f \cdot < M, M' > + \frac{1}{2} \partial_y \partial_y f \cdot < M'^2 >$

(b) $f(X,Y) = e^{X+Y}$, then by Ito's Formula we can get:

$$f(X,Y) = e^{X_0+Y_0} + e^{X+Y} \cdot X + e^{X+Y} \cdot Y + \frac{1}{2}(e^{X+Y} \cdot < X,X > +2 \times e^{X+Y} \cdot < X,Y > +e^{X+Y} \cdot < Y,Y >)$$

Denote $Y = M' + A'$ and $X = M + A$, then,

$$f(X,Y) = e^{X_0+Y_0} + e^{X+Y} \cdot (M + A) + e^{X+Y} \cdot (M' + A') + \frac{1}{2}(e^{X+Y} \cdot < X,X > +2 \times e^{X+Y} \cdot < X,Y > +e^{X+Y} \cdot < Y,Y >)$$

Local martingale part: $e^{X_0+Y_0} + e^{X+Y} \cdot M + e^{X+Y} \cdot M'$

Finite variation part: $e^{X+Y} \cdot A + e^{X+Y} \cdot A' + \frac{1}{2}(e^{X+Y} \cdot < X,X > +2 \times e^{X+Y} \cdot < X,Y > +e^{X+Y} \cdot < Y,Y >)$.

10.34

$f(Y) = e^{X - \frac{1}{2} < X,X >}$, where $Y = X - \frac{1}{2} < X,X >$. Then by Ito's Formula, we will have:

$$f(Y) = e^{Y_0} + e^{Y} \cdot (Y) + \frac{1}{2} e^{Y} \cdot < Y, Y >$$
$$= e^{Y_0} + e^{Y} \cdot X - \frac{1}{2} e^{Y} \cdot < X, X > + \frac{1}{2} e^{Y} \cdot < Y, Y >$$

Note that $< Y, Y > = < X - \frac{1}{2} < X, X >, X - \frac{1}{2} < X, X \gg = < X, X > - < X, < X, X \gg + \frac{1}{4} < < X, X >, < X, X > > = < X, X >$

$$\Longrightarrow f(Y) = e^{X_0} + e^{X - \frac{1}{2} < X, X >} \cdot X = e^{X_0} + f(Y) \cdot X$$

Then, if X is a local martingale, so is $f(X) \cdot X$. Therefore, $f(X)$ must be a local martingale.

10.35

$$\log M = \log M_0 + \frac{1}{M} \cdot M - \frac{1}{2M^2} \cdot < M, M >$$
$$= L_0 + \frac{1}{M} \cdot M - \frac{1}{2M^2} \cdot < M, M >$$
$$\Longrightarrow M = e^{L_0 + \frac{1}{M} \cdot M - \frac{1}{2M^2} \cdot < M, M >}$$

Let $L = L_0 + \frac{1}{M} \cdot M \Longrightarrow < L, L > = < L_0 + \frac{1}{M} \cdot M, L_0 + \frac{1}{M} \cdot M > = \frac{1}{M^2} < M, M >$. Since M is a continuous and strictly positive local martingale, then $L_0 + \frac{1}{M} \cdot M$ is a continuous martingale, which implies that L is a continuous martingale. Therefore, $M = e^{L - \frac{1}{2} < L, L >}$.

10.38

We know that $< W, B >_t = Q^{\mathcal{P}_n}(W, B)_t = \lim_{\substack{n \to \infty \\ |p_n| \to 0}} \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)\left(B_{t_i} - B_{t_{i-1}}\right)$

$$\Longrightarrow E\left[\lim_{\substack{n \to \infty \\ |p_n| \to 0}} \left(\sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}}\right)\left(B_{t_i} - B_{t_{i-1}}\right)\right)^2\right]$$
$$= \lim_{\substack{n \to \infty \\ |p_n| \to 0}} \sum_{i=1}^{n} (t_i - t_i)^2 \leq \lim_{\substack{n \to \infty \\ |p_n| \to 0}} |p_n| t = 0$$

Therefore, $E[(Q^{\mathcal{P}_n}(W, B)_t)^2] \to 0$ when $|p_n| \to 0$. Thus, we can get that W and Bs' joint quadratic variation is null.

10.39

$$WW = W_0 W_0 + W \cdot W + W \cdot W + < W, W >$$
$$W \cdot W = \frac{W^2 - < W, W >}{2} = \frac{W^2 - i}{2}$$

10.40

(a)

$$Wi = W_0 i_0 + i \cdot W + W \cdot i + < W, i >$$

We know that $i = < W, W >$. Since W is a continuous martingale, then it is automatically a continuous local martingale, which implies that $< W, W >$ is an increasing process. Therefore, $< W, W >$ has finite variation on every finite interval, which means $< W, i > = 0$.

$$\Longrightarrow Wi = W \cdot i + i \cdot W$$
$$\Longrightarrow W \cdot i = Wi - i \cdot W$$

$$\Leftrightarrow \int^t W_s ds = tW_t - \int^t s dW_s$$

(b)

$$\begin{aligned} E[W_t(i \cdot W_t)] &= E[W_t(iW - i \cdot W_t)] \\ &= E[tW_t^2] - E[W_t(W_t \cdot t)] \\ &= tE[W_t^2] - E\left[W_t \int_0^t W_s ds\right] \\ &= t^2 - E\left[W_t \int_0^t W_s ds\right] \end{aligned}$$

$$\begin{aligned} E\left[W_t \int_0^t W_s ds\right] &= \int_0^t E[W_s{}^2] ds \\ &= \int_0^t s\, ds \\ &= \frac{t^2}{2} \end{aligned}$$

$$\Rightarrow E[W_t(i \cdot W_t)] = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

10.41

Let $Y = \sigma W + \left(b - \frac{1}{2}\sigma^2\right) i$ and $f(x) = e^x$. Then, we have:

$$\begin{aligned} f(Y) &= f(Y_0) + \partial_Y f \cdot Y + \frac{1}{2}\partial_Y^2 f \cdot < Y, Y > \\ &= 1 + f(Y) \cdot \left(\sigma W + \left(b - \frac{1}{2}\sigma^2\right) i\right) + \frac{1}{2} f(Y)\sigma^2 \cdot i \\ &= 1 + X \cdot \sigma W + \left(b - \frac{\sigma^2}{2}\right) X \cdot i + \frac{\sigma^2}{2} X \cdot i \\ &= 1 + \sigma X \cdot W + bX \cdot i \end{aligned}$$

10.43

$$A_t \cdot W = \begin{pmatrix} \cos\alpha_t W^1 - \sin\alpha_t W^2 \\ \sin\alpha_t W^1 + \cos\alpha_t W^2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

$$< Z_1, Z_1 > = \int_0^t (sin^2\alpha_s + cos^2\alpha_s) ds = t$$

$$< Z_1, Z_2 > = \int_0^t (\sin\alpha_s \cos\alpha_s - \cos\alpha_s \sin\alpha_s) = 0$$

$$< Z_2, Z_2 > = \int_0^t (sin^2\alpha_s + cos^2\alpha_s) ds = t$$

As W^1 and W^2 are continuous martingales, then they are local continuous martingales. Therefore, both Z_1 and Z_2 are continuous local martingales. Thus, $\int_0^0 \cos\alpha_s\, dW_s = 0$ and $\int_0^0 \sin\alpha_s\, dW_s = 0$, then $Z_1(0) = Z_2(0) = 0$. Therefore, Z_1 and Z_2 are independent BM.

10.44

(a) Like (10.43), we can get:

$$Z_1 = \cos\alpha_t W^1 - \sin\alpha_t W^2$$
$$Z_2 = \sin\beta_t W^1 + \cos\beta_t W^2$$
$$< Z_1, Z_1 > = \int_0^t (sin^2\alpha_s + cos^2\alpha_s)ds = t$$
$$< Z_1, Z_2 >= \int_0^t \cos\alpha_s \sin\beta_s - \sin\alpha_s \cos\beta_s\, ds$$
$$= \int_0^t \sin(\beta_s - \alpha_s)ds$$
$$< Z_2, Z_2 > = \int_0^t (sin^2\alpha_s + cos^2\alpha_s)ds = t$$

As W^1 and W^2 are continuous martingales, then they are local continuous martingales. Therefore, both Z_1 and Z_2 are continuous local martingales. Thus, $\int_0^0 \cos\alpha_s\, dW_s = 0$ and $\int_0^0 \sin\alpha_s\, dW_s = 0$, then $Z_1(0) = Z_2(0) = 0$. Therefore, Z_1 and Z_2 are BM.

(b) $X = e^{Z_1+Z_2+i}$, then by Ito's Formula, we can get:

$$X = X_0 + e^{Z_1+Z_2+i}\cdot(Z_1+Z_2+i) + \frac{1}{2}e^{Z_1+Z_2+i}\cdot < Z_1+Z_2+i, Z_1+Z_2+i >$$
$$= 1 + e^{Z_1+Z_2+i}\cdot(Z_1+Z_2) + e^{Z_1+Z_2+i}\cdot i + \frac{1}{2}e^{Z_1+Z_2+i}\cdot < Z_1+Z_2, Z_1+Z_2 >$$

As Z_1 and Z_2 are continuous local martingales, then $Z_1 + Z_2$ is also a continuous local martingale, therefore:

Local martingale part: $1 + e^{Z_1+Z_2+i}\cdot(Z_1+Z_2)$

Finite variation part: $e^{Z_1+Z_2+i}\cdot i + \frac{1}{2}e^{Z_1+Z_2+i}\cdot < Z_1+Z_2, Z_1+Z_2 >$

10.45

We have $X = X_0 + \sigma\cdot W + b\cdot i$, and $\sigma = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}$, then we can have:

$$X^1 = X^1(0) + W_1 + 2W_2 + i\cdot i$$
$$X^2 = X^2(0) + 3W_1 + W_3 - i\cdot i$$
$$X^3 = X^3(0) + W_2 + 4W_3 + i$$

As W_1, W_2, W_3 are continuous martingales, then they are automatically local martingales.
By Ito's Formula, we can have:

$$f(a,b,c,d) = f(a_0,b_0,c_0,d_0) + \sum_i \partial_i f\cdot X^i + \frac{1}{2}\sum_i\sum_j \partial_i\partial_j f\cdot < X^i, X^j >$$
$$\sum_i \partial_i f\cdot X^i = \partial_a f\cdot a + \partial_b f\cdot b + \partial_c f\cdot c + \partial_d f\cdot d$$
$$\frac{1}{2}\sum_i\sum_j \partial_i\partial_j f\cdot < X^i, X^j >$$
$$= \frac{1}{2}(\partial_a^2 f\cdot < a,a > + \partial_b^2 f\cdot < b,b > + \partial_c^2 f\cdot < c,c > + \partial_d^2 f\cdot < d,d >) + \partial_a\partial_b f\cdot < a,b > + \partial_a\partial_c f\cdot < a,c > + \partial_a\partial_d f\cdot < a,d > + \partial_b\partial_c f\cdot < b,c > + \partial_b\partial_d f\cdot < b,d > + \partial_c\partial_d f\cdot < c,d >$$

Denote $a = M_a + A_a$, $b = M_b + A_b$, $c = M_c + A_c$, $d = M_d + A_d$

$$f(a,b,c,d) = f(a_0,b_0,c_0,d_0) + \partial_a f \cdot (M_a + A_a) + \partial_b f \cdot (M_b + A_b) + \partial_c f \cdot (M_c + A_c) + \partial_d f \cdot (M_d + A_d) + \frac{1}{2}(\partial_a^2 f \cdot < a,a > + \partial_b^2 f \cdot < b,b > + \partial_c^2 f \cdot < c,c > + \partial_d^2 f \cdot < d,d >) + \partial_a \partial_b f \cdot < a,b > + \partial_a \partial_c f \cdot < a,c > + \partial_a \partial_d f \cdot < a,d > + \partial_b \partial_c f \cdot < b,c > + \partial_b \partial_d f \cdot < b,d > + \partial_c \partial_d f \cdot < c,d >$$

Local continuous martingale: $f(a_0,b_0,c_0,d_0) + \partial_a f \cdot M_a + \partial_b f \cdot M_b + \partial_c f \cdot M_c + \partial_d f \cdot M_d$

Finite variation: $\partial_a f \cdot A_a + \partial_b f \cdot A_b + \partial_c f \cdot A_c + \partial_d f \cdot A_d + \frac{1}{2}(\partial_a^2 f \cdot < a,a > + \partial_b^2 f \cdot < b,b > + \partial_c^2 f \cdot < c,c > + \partial_d^2 f \cdot < d,d >) + \partial_a \partial_b f \cdot < a,b > + \partial_a \partial_c f \cdot < a,c > + \partial_a \partial_d f \cdot < a,d > + \partial_b \partial_c f \cdot < b,c > + \partial_b \partial_d f \cdot < b,d > + \partial_c \partial_d f \cdot < c,d >$

As X^1, X^2, X^3, i are semimartingales and continuous, then $f(X^1, X^2, X^3, i)$ is also a continuous semimartingale.

Local continuous martingale: $\partial_{X^1} f \cdot (W_1 + 2W_2) + \partial_{X^2} f \cdot (3W_1 + W_3) + \partial_{X^3} f \cdot (W_2 + 4W_3)$

Finite variation: $\partial_{X^1} f \cdot (i \cdot i) - \partial_{X^2} f \cdot (i \cdot i) + \partial_{X^3} f \cdot i + \partial_i f \cdot i + \frac{1}{2}(\partial_a^2 f \cdot < W_1 + 2W_2, W_1 + 2W_2 > + \partial_b^2 f \cdot < 3W_1 + W_3, 3W_1 + W_3 > + \partial_c^2 f \cdot < W_2 + 4W_3, W_2 + 4W_3 >) + \partial_{X^1} \partial_{X^2} f \cdot < W_1 + 2W_2, 3W_1 + W_3 > + \partial_{X^1} \partial_{X^3} f \cdot < W_1 + 2W_2, W_2 + 4W_3 > + \partial_{X^2} \partial_{X^3} f < 3W_1 + W_3, W_2 + 4W_3 > = \partial_{X^1} f \cdot (i \cdot i) - \partial_{X^2} f \cdot (i \cdot i) + \partial_{X^3} f \cdot i + \partial_i f \cdot i + \frac{1}{2}\partial_a^2 f \cdot 5i + \frac{1}{2}\partial_b^2 f \cdot 10i + \frac{1}{2}\partial_c^2 f \cdot 17i + 3\partial_{X^1} \partial_{X^2} f \cdot i + 2\partial_{X^1} \partial_{X^3} f \cdot i + 4\partial_{X^2} \partial_{X^3} f \cdot i.$