

Optimal Estimation of the Risk Premium for the Long Run and Asset Allocation: A Case of Compounded Estimation Risk

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ABSTRACT

It is well known that an unbiased forecast of the terminal value of a portfolio requires compounding at the arithmetic mean return over the investment horizon. However, the maximum-likelihood practice, common with academics, of compounding at the *estimator* of mean return results in upward biased and highly inefficient estimates of long-term expected returns. We derive analytically both an unbiased and a small-sample efficient estimator of long-term expected returns for a given sample size and horizon. Both estimators entail penalties that reduce the annual compounding rate as the investment horizon increases. The unbiased estimator, which is far lower than the compounded arithmetic average, is still very inefficient, often more so than a simple geometric estimator known to practitioners. Our small-sample efficient estimator is even lower. These results compound the sobering evidence in recent work that the equity risk premium is lower than suggested by post-1926 data. Our methodology and results are robust to extensions such as predictable returns. We also confirm analytically that parameter uncertainty, properly incorporated, produces optimal asset allocations, in stark contrast to conventional wisdom. Longer investment horizons require *lower*, not higher, allocations to risky assets.

KEYWORDS: arithmetic mean, asset allocation, estimation risk, geometric mean, long-term returns, maximum likelihood, mean-squared error, risk premium, small sample

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Estimates of expected long-term returns, or risk premiums, are crucial inputs in empirical asset pricing and especially portfolio theory. First, it is important to know what wealth a portfolio is expected to generate over the long term, for example, in the context of pension funds and retirement policy. Second, this forecast is an implicit input to the asset allocation decision, the optimal mix of risky and risk-free assets. Portfolio theory is almost always derived under the assumption that the agent knows the parameters of the return distribution. While sophisticated models of time-varying opportunity sets are well developed, the effects of parameter uncertainty on forecasts and optimal investment decisions are not as often discussed.

Recent reconsiderations of the equity risk premium have put the issue of parameter estimation back on the front burner of academic research. Fama and French (2002) and Jagannathan, McGrattan, and Scherbina (2000) make the case that the risk premium, that is, the mean return in excess of the risk-free rate, is less than implied by post-1950 average returns. In addition, more inclusive databases also result in lower historical risk premiums. For example, Dimson, Marsh, and Staunton (2002) and Jacquier, Kane, and Marcus (2003) show that including returns from pre-1926 periods reduces historical average returns. Our results compound this sobering evidence. We show that a downward penalty for estimation risk should be applied to the per-period estimate of the risk premium when it is used for long-term forecasts. This argument is absent in the current literature on the declining equity premium and its implications for long-term investment.

At the core of the discussion is the magnitude of the estimation error in mean returns even with long samples. Of course, given the true parameters of the asset-return distribution, an unbiased forecast of the terminal value of a portfolio obtains by compounding the initial value at the arithmetic mean rate of return over the investment horizon. Substituting a maximum-likelihood, arithmetic average for this true mean is of course asymptotically efficient and often advocated [e.g., Campbell (2001)]. Yet practitioners often prefer geometric averages to simulate future portfolio values, arguing that the arithmetic average produces upward-biased long-term forecasts.

This article studies the effect of estimation error when this unbiased and efficient one-period maximum-likelihood estimator (MLE) is used for multiperiod forecasting. We show that the usual asymptotic arguments for the consistency and efficiency of this common practice, although maximum likelihood, do not apply. This is because the forecasting horizon is often sizable relative to the sample size. Hence the small-sample bias and inefficiency due to Jensen's inequality and the estimation error of the mean do not vanish. Asymptotics here require the ratio of the length of the estimation period to that of the forecasting horizon to be large.

More specifically, we first review the analytic derivation of an unbiased estimator, initially discussed by Blume (1974) and derived analytically in Jacquier, Kane, and Marcus (2003).¹ Then we derive analytically a small-sample efficient

¹ See also Roll (1983), who discusses how, due to Jensen's inequality, different estimates of portfolio mean returns have different biases, for example, first compound and then cross-sectionally averaged or vice versa.

estimator, based on the minimization of mean-squared errors (MSEs). This is an appropriate replacement for the maximum-likelihood procedure, here the arithmetic estimator, when asymptotic conditions are not met. Both these new estimators of expected long-term returns compound initial value at a *weighted average* of the arithmetic and geometric average returns. The weights depend on the ratio of the estimation period to the investment horizon. The small-sample efficient estimator is always lower than the unbiased estimator, which itself is always smaller than the arithmetic estimator. Using the small-sample efficient estimator results in a considerable efficiency gain over the unbiased estimator. In fact, even the practitioner-favored geometric estimator, while biased, is more efficient than the unbiased estimator for large investment horizons.

It is important to recognize that simulation methods of future returns are not immune to these bias and inefficiency problems. Most simulation methods that use sample estimates as inputs to a scenario analysis only reproduce the problems which we describe analytically.

Our initial analysis focuses on the simple case of independently and identically distributed (i.i.d.) log-normal returns, so we verify the robustness of our methodology and results to generalizations of this simple data generating process. Namely, extending our optimal estimator to allow for predictable returns requires only a simple modification of the initial formula. We show that predictability has little effect on the compounding of estimation risk. This is because predictability modifies the data generating process of both past and future data in ways that partially cancel in the computation of the estimator. We also discuss robustness to heteroskedasticity and the estimation of variance.

To illustrate the practical import of these results, we consider an application to a classic problem of optimal asset allocation. We show that, even with i.i.d. returns, the optimal allocation to the risky asset must be reduced compared to the known-parameter case to properly incorporate estimation risk; the longer the horizon, the greater the reduction. This is in stark contrast to conventional wisdom, which recommends increasing the allocation for longer horizons. For realistic values of the inputs, we demonstrate that these effects are large and should be taken into account. Again, we provide easy to implement analytical results.

The next section reviews the biases of the geometric and arithmetic estimators of expected long-term returns and the derivation of an unbiased estimator. Section 2 derives a small-sample efficient estimator which should be used instead of the MLE. We show that the efficiency gain compared to the maximum likelihood and the unbiased estimators is considerable. Section 3 relaxes the i.i.d. assumption and discusses the robustness of the estimators proposed, providing specific results for autocorrelated returns. Section 4 discusses the implications of these results for long-term asset allocation. Section 5 concludes.

1 UNBIASED ESTIMATION OF LONG-TERM EXPECTED RETURNS

1.1 Biases of the Arithmetic and Geometric Estimators

We now show that the two most common competing estimators of expected future long-term returns are both biased. Assume that the one-period return, R_t , of a stock portfolio is lognormally distributed. That is, the log-return $r_t = \ln(1 + R_t)$ is i.i.d. normal with mean μ and standard deviation σ .² Therefore the multiperiod log-return, a sum of one-period log-returns over a future investment horizon of H periods, is normal with mean $H\mu$ and variance $H\sigma^2$. For an investment of \$1, the future portfolio value in H periods, V_H , can be written as

$$V_H = \$1 \times \exp(\mu H + \sigma \sum_{i=1}^H \varepsilon_{t+i}), \quad (1)$$

where $\varepsilon_{t+i} \sim \text{i.i.d. } N(0,1)$.

By the properties of the log-normal distribution, the expected return over H periods is

$$E(V_H) = e^{(\mu + \frac{1}{2}\sigma^2)H} = [1 + E(R)]^H. \quad (2)$$

Equation (2) is the basis for the standard practice of forecasting portfolio value by compounding at the expected rate of return. For example, the Ibbotson Associates publication *Stocks, Bonds, Bills and Inflation Annual Yearbook* simulates future portfolio values using arithmetic averages of past returns. Campbell (2001), who addresses the impact of autocorrelation on long-term forecasts, recommends the substitution of the sample average \bar{R} for $E(R)$ when returns are i.i.d.

However, the sample average holding period return, \bar{R} , is an unbiased but noisy estimator of $E(R)$. Jensen's inequality implies therefore that

$$E([1 + \bar{R}]^H) > [1 + E(\bar{R})]^H = [1 + E(R)]^H = E(V_H). \quad (3)$$

Because of the estimation error in \bar{R} , the compounded sample-average return gives an upward-biased estimate of the expected future portfolio value. That the bias should vanish in a large sample is probably at the root of the typical practice of substituting the maximum-likelihood estimate of $E(R)$ into Equation (2). We will see that this asymptotic intuition is misleading when H is sizeable compared to the sample size T . Blume (1974) first discusses this bias. However, because he assumes that returns are normal rather than lognormal, he does not obtain exact formulas for expected values or bias. Cooper (1996) analyzes the bias in the context of discount factors for capital budgeting purposes. He concludes that the arithmetic mean is usually nearly appropriate, even accounting for estimation

² Autocorrelation in returns modifies the variance of the long-term returns but not the spirit of the following discussion. Section 3 discusses the effect of autocorrelation and heteroskedasticity.

error. However, because discount factors involve powers of the *reciprocal* of the rate of return, the biases he finds differ drastically from those considered here. In our case, Jacquier, Kane, and Marcus (2003) provide an analytical expression for the bias and an unbiased estimator. We review them briefly.

Denote by $\hat{\mu}$, the sample average computed over T past periods of log-returns, that is, the MLE of μ . It can be written as

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^T \ln(1 + R_{-i}) = \frac{1}{T} \left(\mu T + \sigma \sum_{i=1}^T \varepsilon_{-i} \right).$$

As is well known, $\hat{\mu}$ is unbiased with standard deviation σ/\sqrt{T} . It is conveniently written as

$$\hat{\mu} = \mu + \omega \sigma / \sqrt{T}, \quad (4)$$

where $\omega \sim N(0,1)$.

It is well known [e.g., Merton (1980)] that for i.i.d. returns, the precision of the estimator of μ depends only on the calendar span of the historic sample period. That is, $\hat{\mu}$ cannot be made more precise by sampling the data more often, only by sampling for a longer time period. This is at the core of the imprecision of standard estimates of mean returns and market risk premiums. See Fama and French (2002) and Jagannathan, McGrattan, and Scherbina (2000) for recent discussions. In contrast, the estimate of variance can be made arbitrarily precise by sampling more frequently within a given sample period. Therefore, since high-frequency returns data have been available for at least 40 years, the estimation error in σ is a second-order effect. For our purposes we will ignore the estimation error in σ [see Jacquier (2005) for a discussion].

A first estimator for $E(V_H)$, based on the arithmetic average return, replaces $E(R)$ in Equation (2) by \bar{R} . Alternatively, one can insert $\hat{\mu}$ in Equation (2), estimating $1+E(R)$ by $e^{(\hat{\mu} + \frac{1}{2}\sigma^2)}$. $(1 + \bar{R})^H$ and $e^{(\hat{\mu} + \frac{1}{2}\sigma^2)H}$, while not exactly equal in the small sample, have the same probability limit. They are both MLEs of $E(V_H)$. Asymptotically the MLE is invariant to transformation, so substituting $\hat{\mu}$ or \bar{R} appropriately results in a MLE of $E(V_H)$. Because the sampling distribution of $(1 + \bar{R})^H$ is only asymptotically lognormal and requires simulations, we would prefer to use $e^{(\hat{\mu} + \frac{1}{2}\sigma^2)H}$ which has an analytical distribution. We must first convince ourselves, however, that the two estimators have similar properties for realistic values of T and H .

Figure 1 plots the ratios of the sample means and standard deviations of the two estimators for investment horizons, H , from 1 to 40 years, historical sample size $T = 75$ years, $\mu = 0.1$, $\sigma = 0.2$. These values are typical of a scenario where μ is estimated from a long sample of equity returns. For example, the mean and standard deviation of the log-return on the Standard & Poor's 500 (S&P 500) from 1926 to 2001 are 0.099 and 0.196. Figure 1 shows clearly that the two estimators are very similar even for long horizons. The sample means of the

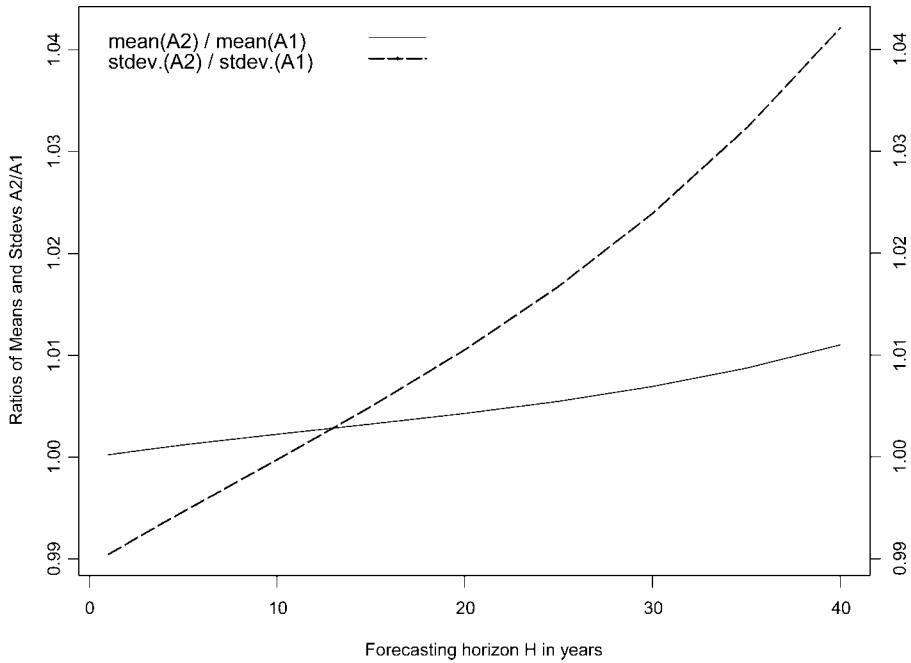


Figure 1 Ratios of sampling means and standard deviations of estimators $A_1 = (1 + \bar{R})^H$ and $A_2 = e^{(\hat{\mu} + \frac{1}{2}\sigma^2)H}$, $\mu = 0.1$, $\sigma = 0.2$, $T = 75$. The mean and standard deviation of A_1 are computed from 2 million draws of the asymptotic distribution of \bar{R} .

two estimators are within 0.5% of each other, even for horizons up to 40 years. Their standard deviations are within 4% of each other. For the rest of the article we will refer to $A = e^{(\hat{\mu} + \frac{1}{2}\sigma^2)H}$ as the arithmetic average estimator.

The bias of the arithmetic average (MLE) estimator A follows from Equation (4). Rewrite A as

$$A = e^{(\hat{\mu} + \frac{1}{2}\sigma^2)H} = e^{(\mu + \omega\sigma/\sqrt{T} + \frac{1}{2}\sigma^2)H} = e^{(\mu + \frac{1}{2}\sigma^2)H} e^{(\omega\sigma/\sqrt{T})H}.$$

Substituting from Equation (2), we obtain

$$E(A) = E(V_H)E\left[e^{\omega\sigma H/\sqrt{T}}\right] = E(V_H)e^{\frac{1}{2}\sigma^2 H^2/T}. \quad (5)$$

Hence the arithmetic average estimator is always biased upward by a factor of $e^{\frac{1}{2}\sigma^2 H^2/T}$.

Note that $\hat{\mu}$ is the logarithm of the geometric mean return. The “geometric average” estimator, denoted G , compounds at the exponential of $\hat{\mu}$ rather than $\hat{\mu} + \frac{1}{2}\sigma^2$. Practitioners often advocate it as an alternative to the arithmetic average estimator A . The argument is often made as follows: “As $G = (P_1/P_{-T})^{1/T}$, compounding at G into the future is the only way to generate forecasts that match the

growth rate observed in the past, for example, from T periods ago until now.” We can write the expectation of G as

$$E(G) = E(e^{\hat{\mu}H}) = E\left[e^{(\mu + \omega\sigma/\sqrt{T})H}\right] = e^{\mu H + \frac{1}{2}\sigma^2 H^2/T} = E(V_H)e^{\frac{1}{2}\sigma^2 (H/T-1)H}. \quad (6)$$

Therefore the intuition in the above argument is correct only in the razor’s edge case for which $H = T$. Then the geometric average estimator results in an unbiased forecast of expected cumulative returns. In all other cases, G is biased. The investment horizon H is of course exogenously set. There is no reason to restrict the estimation period T to be equal to H , even if this were feasible. This would amount to giving up precision in $\hat{\mu}$, just for the sake of removing bias. One clearly wants to use the largest T available, no matter the investment horizon considered.

1.2 Unbiased Forecasts of Long-Term Expected Returns

For any T and H , one can design an unbiased estimator of $E(V_H)$. In fact, direct inspection of Equations (5) and (6) shows that compounding at a rate $\hat{\mu} + \frac{1}{2}\sigma^2(1 - H/T)$ removes the bias.

Formally, we consider estimators in the class as

$$C = e^{(\hat{\mu} + \frac{1}{2}k\sigma^2)H}. \quad (7)$$

These estimators nest the geometric average estimator G ($k = 0$) and the arithmetic average estimator A ($k = 1$). They can be interpreted as using a compounding rate that is a linear combination of A and G with weight on A equal to k ³. The unbiased estimator, U , obtains by solving for the value k_U that sets $E(C)$ in Equation (7) equal to $E(V_H)$. The result is

$$k_U = 1 - H/T. \quad (8)$$

The unbiased estimator, U , is always smaller than the arithmetic estimator, A , since k_U is less than one. Only as the forecast horizon, H , becomes small relative to the estimation period, T , does the MLE become unbiased. The geometric average estimator, G , is biased downward when k_U is positive, that is, when $H < T$. For investment horizons longer than the estimation period, that is, $H > T$, G is biased upward, like A . G is unbiased only for $T = H$.

The bias in A can persist even as $\hat{\mu}$ converges to μ when the sample size T increases, as long as the ratio H/T remains sizeable. For the estimator of expected terminal wealth to converge to its true value, both T and T/H must be large.

³ Considering the seemingly larger class, $e^{(k_1\hat{\mu} + \frac{1}{2}k_2\sigma^2)H}$ adds no generality. One can show that any value $k_1 \neq 1$ leads to *infeasible* estimators, that is, functions of the true parameter. We restrict our analysis to feasible estimators, that is, functions solely of sample statistics. It is also easily verified that feasible estimators in the class, $w_1A + w_2G$, require $w_2 = 0$ and map one to one with those in Equation (7).

2 SMALL-SAMPLE EFFICIENT ESTIMATION

Asymptotics in our case is in T/H , not in T . The previous section showed that for relevant values of T and H , the MLE is biased; large-sample conditions are not met. One can then strongly suspect that it is also not efficient in the relevant small-sample conditions. Rather than counting on asymptotic efficiency, that is, asymptotic minimum MSE, one needs to construct a small-sample efficient estimator for the sample size and horizon considered.

Further, note that unbiasedness is not in itself an estimation goal. Rather, it is typically used if needed to reduce a universe of possible estimators to a manageable set. This can lead to inadmissible estimators, as noted, for example, by Stein (1956). Instead, estimators should be set to minimize a loss function, a measure of average distance to the true parameter. When the problem at hand does not supply a specific loss function, a natural candidate is the MSE, the expectation of the squared deviation of the estimator from the true parameter. MSE can be written as the sum of the variance and squared bias. If tractability is not an issue, there is no reason to impose unbiasedness when searching for a minimum MSE estimator.

A minimum MSE estimator of expected future wealth, again within the class of estimators C in Equation (7), is easily derived. Recall that the class nests A , G , and U , for $k = 1, 0$, and $1 - H/T$. The MSE of an estimator in class C is

$$\begin{aligned} \text{MSE}(C) &= E[C - E(V_H)]^2 = E\left(e^{\hat{\mu}H + \frac{1}{2}\sigma^2 H} - e^{\mu H + \frac{1}{2}\sigma^2 H}\right)^2 \\ &= E\left(e^{2\hat{\mu}H + k\sigma^2 H} - e^{2\mu H + \sigma^2 H} - 2e^{\hat{\mu}H + \frac{1}{2}k\sigma^2 H + \mu H + \frac{1}{2}\sigma^2 H}\right). \end{aligned} \quad (9)$$

Substituting $\hat{\mu} = \mu + \omega\sigma/\sqrt{T}$ from Equation (4) and evaluating the expectation yields

$$\text{MSE}(C) = e^{2\mu H + 2\sigma^2 H^2/T + k\sigma^2 H} + e^{2\mu H + \sigma^2 H} - 2e^{2\mu H + \frac{1}{2}\sigma^2 H^2/T + \frac{1}{2}k\sigma^2 H + \frac{1}{2}\sigma^2 H}.$$

Minimizing this expression over k results in the minimum MSE estimator, denoted M , of $E(V_H)$:

$$k_M = 1 - 3H/T. \quad (10)$$

Hence, for realistic values of T and H , the two popular estimators, arithmetic and geometric, and the unbiased estimator sometimes proposed in the literature are all suboptimal in terms of MSE, the most commonly used risk function. Equation (10) also shows that the best estimator of expected cumulative returns is even lower than the unbiased estimator. Table 1 compares the estimators for relevant ranges of T and H .

Do these different values of k lead to very different estimates of $E(V_H)$? To illustrate the long-term effect of these differences in compounding rates, panel A of Figure 2 plots the various estimates of $E(V_H)$ versus the investment horizon H . The estimates of final wealth diverge dramatically across estimators for longer horizons. For $T = 75$, the arithmetic average A predicts that \$1 will compound to \$120 in 40 years, the unbiased estimator forecasts \$80, and the minimum MSE forecasts only

Table 1 Properties of alternative estimators of cumulative portfolio return. All estimators are of the general form $e^{(\hat{\mu} + \frac{1}{2}k\sigma^2)H}$. $\hat{\mu}$ is the MLE of the mean log-return from a sample of length T . The investment horizon is H years. The values for k are as follows:

Arithmetic estimator (A): $k = 1$
Geometric estimator (G): $k = 0$
Unbiased estimator (U): $k_U = 1 - H/T$
Minimum MSE estimator (M): $k_M = 1 - 3H/T$

Condition		Ordering	Bias < 0	Bias > 0
$T \gg H$	$k_M \approx k_U \approx 1$	$M \approx U \approx A$	G	
$T > 3H$	$0 < k_M < k_U < 1$	$G < M < U < A$	G, M	A
$3H > T > H$	$k_M < 0 < k_U < 1$	$M < G < U < A$	M, G	A
$H > T$	$k_M < k_U < 0$	$M < U < G < A$	M	G, A

\$25. When T is 30 years, U , and especially M , dramatically penalize predictions for investment horizons longer than T . The optimal estimator M penalizes long-horizon forecasts far more than U , as soon as H is sizeable relative to T .

Panel B of Figure 2 plots the effective annual compounding rates of A , G , U , and M versus H , for $T = 75$ years [the length of the Center for Research in Security Prices (CRSP) monthly database] and 30 years (a shorter sample size more relevant for an emerging market). The two horizontal lines in panel B, $A(e^{1 + \frac{1}{2} \times .2^2} = 12.7\%)$ and $G(e^1 = 10.2\%)$, are unaffected by H , while U and M “penalize” the increasing H by linearly decreasing the compounding rate. The effects are large. Even for a lengthy sample period of $T = 75$ years, a 30-year investment horizon, broadly appropriate for a retirement fund, calls for the compounding rate to decline from the arithmetic 12.7%, to about 10% for M . Necessary corrections become dramatic with shorter sample periods. For $T = 30$, the appropriate compounding rate is about 8%, even for a 20-year investment horizon. These results compound the sobering message in Fama and French (2002) and Jagannathan, McGrattan, and Scherbina (2000), that the per-period estimate of the equity premium is lower than once thought.

One may argue that the results for M are specific to the risk function chosen. While this is of course correct, the loss function chosen here has some wide appeal. Mainly it is the small-sample implementation of the efficiency criterion invoked to justify the maximum-likelihood principle. Also, the same estimator is obtained whether one minimizes sampling loss in terms of dollars or returns. Estimators optimal with respect to alternative loss functions, for example, asymmetric, can be derived in a similar fashion [see Jacquier (2005)].

Because the predictions of the various estimators of final wealth differ so considerably, so must their predictive accuracies. Panel A of Figure 3 plots the root MSE of each estimator as a multiple of the value of expected final wealth for $\mu = 0.1$, $\sigma = 0.2$, $T = 60$, as a function of H . By construction, M is most precise at all horizons, and therefore is the lowest curve. We first note that the precision of the

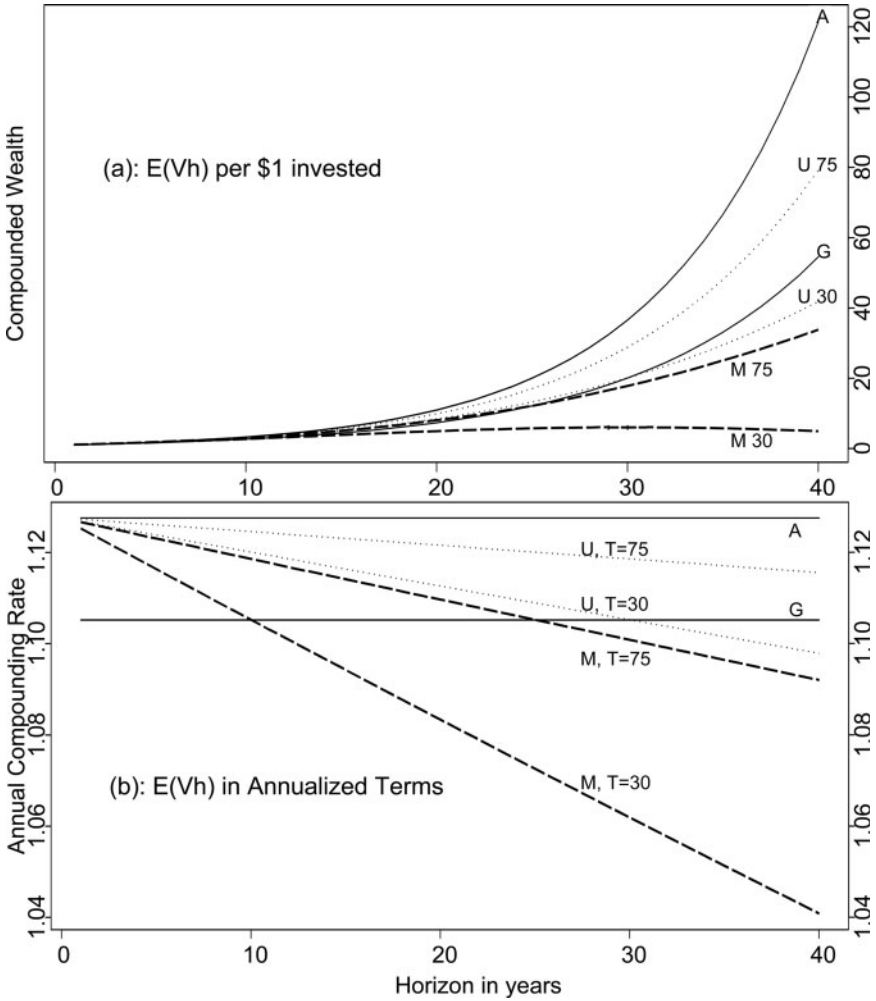


Figure 2 Estimators of long-term expected returns: A, G, U, M. $\hat{\mu} = 0.1$, $\sigma = 0.2$.

MLE, A, largely favored in the academic community, is astonishingly poor. At an horizon of 40 years, its root mean-squared error (RMSE) is nearly 2.5 times the true expected value of final wealth. Such a magnitude renders estimates nearly useless. The second poorest estimator in terms of precision is clearly U, the unbiased estimator. In fact, surprisingly enough, it is the geometric estimator G that is second best. The RMSE of G doesn't diverge much from that of the efficient estimator for horizons shorter than 25 years. Even at an horizon of 40 years, the RMSE of G is only about 30% higher than that of M.

Panel B of Figure 3 gives us further information on the relative precisions of these estimators. There we plot the percentage improvement in RMSE of G, U,

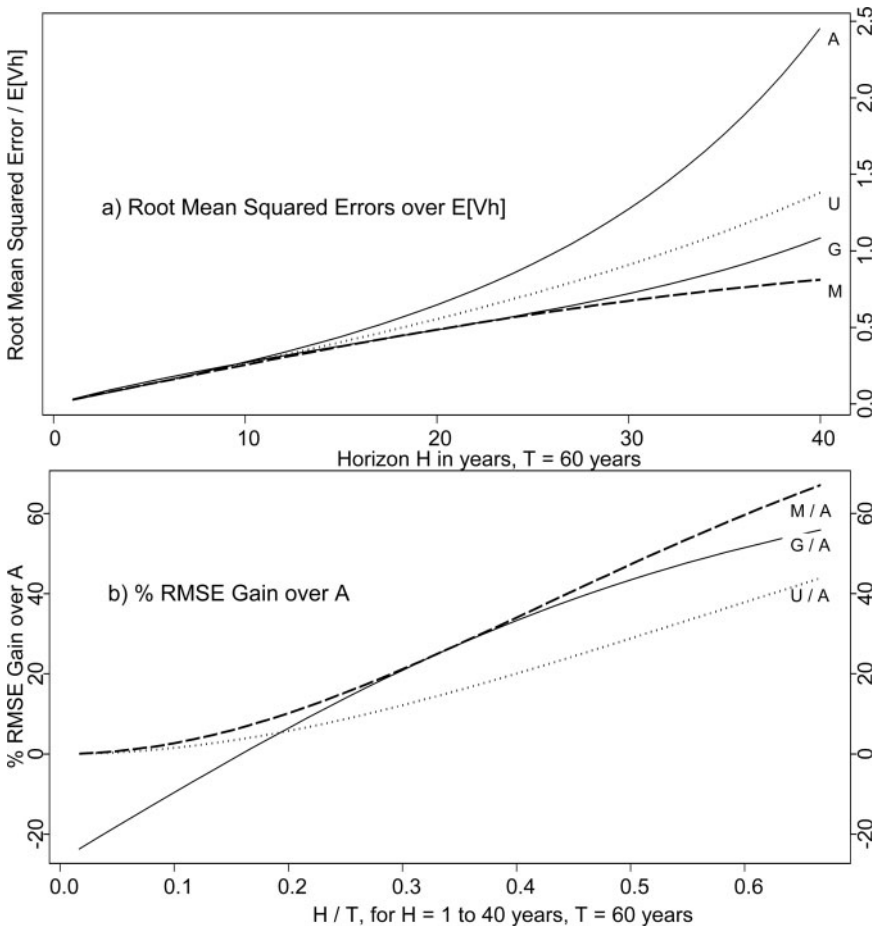


Figure 3 Root mean-squared errors of estimators A , G , U , M . $\hat{\mu} = 0.1$, $\sigma = 0.2$, $T = 60$ years.

and M over A as a function of H/T . Again, by construction, the efficient estimator M establishes an upper bound on the improvement in RMSE. At very short horizons, A , M , and U are all virtually identical with values of $k \approx 1$ [see Equations (8) and (10)], so the improvement over A is negligible. As the horizon extends, the improvement of M becomes dramatic, surpassing 60% at a 40-year horizon. The geometric estimator is best at midrange horizons. Its curve achieves tangency with that of M at $H/T = 1/3$. This follows from Equation (10), which shows that at this point, $k_M = 0$, making G achieve efficiency. Close to this point, on either side of it, G is close to efficient. It should be noted that as soon as H/T is greater than about 0.2, the unbiased estimator U has a substantially higher RMSE than G . In fact, the geometric estimator is less precise than A and U , with a negative “improvement” in RMSE, only at shorter horizons. This is because,

while the estimators M , A , and U converge for very low values of H/T , G remains downward biased. Note that the crossing of the curves for U and G (around $H/T = 0.2$) occurs where G compensates for its squared bias (U has no bias) with an equal reduction in variance. For larger H/T 's, the variance of U is just too large to make its unbiasedness an interesting feature.

To summarize, the catastrophic lack of precision of A , the relatively disappointing imprecision of U , and strong performance of the geometric estimator in the middle range of investment horizons are the striking features of Figure 3.

3 ROBUSTNESS

To produce clear analytical results, so far we have made a number of simplifying assumptions. This section shows that the results are robust to these assumptions.

A large amount of literature beginning with Summers (1986), Fama and French (1988), and Poterba and Summers (1988) has addressed the autocorrelation of long-term returns. Estimates point to negative autocorrelations at lags in the business cycle range. While the strength of the evidence is somewhat disputed, our analysis easily adjusts to autocorrelated returns. Autocorrelation enters the analysis through the variance of two sums of returns, namely H future returns for the forecast and T past returns for the estimate of μ . Given an autocorrelation structure, we introduce the correlation matrix \mathbf{C} and vectors of ones, \mathbf{i} , of dimensions T or H as appropriate. The variance of a sum of log-returns is then $\sigma^2 \mathbf{i}' \mathbf{C} \mathbf{i}$, instead of $T\sigma^2$ or $H\sigma^2$. This affects $E(V_H)$ in Equation (2), where the exponential term becomes $H(\mu + \frac{1}{2} \mathbf{i}' \mathbf{C}_H \mathbf{i} \sigma^2 / H)$. Similarly $\hat{\mu}$ in Equation (4) becomes $\mu + \omega \sigma \sqrt{\mathbf{i}' \mathbf{C}_T \mathbf{i} / T}$. The analysis then follows. Defining $F_T = \mathbf{i}' \mathbf{C}_T \mathbf{i} / T$ and $F_H = \mathbf{i}' \mathbf{C}_H \mathbf{i} / H$, the unbiased and minimum MSE estimators require, respectively,

$$k_U = 1 - \frac{H}{T} \times \frac{F_T}{F_H} \quad (11)$$

$$k_M = 1 - \frac{3H}{T} \times \frac{F_T}{F_H} \quad (12)$$

The correction is similar for other non-i.i.d. specifications. The ratio F_T/F_H more generally can be replaced with the ratio of the variance of the sum of the T returns from the estimation period divided by T to the variance of the sum of the H returns from the forecast period divided by H . It would be straightforward to generalize this approach for other ARMA processes. These corrections require little modification of our basic formulas. One suspects that, for most stable forms of predictability, the new ratios in Equations (11) and (12) may not be far from one.

Figure 4 plots the unbiased and efficient estimators with and without the corrections for autocorrelation. We estimated an MA(4) process on the S&P 500 annual log-returns from 1926 to 2001 and computed k_U and k_M as per Equations (11) and (12). Figure 4 shows that the correction is really a second-order effect,

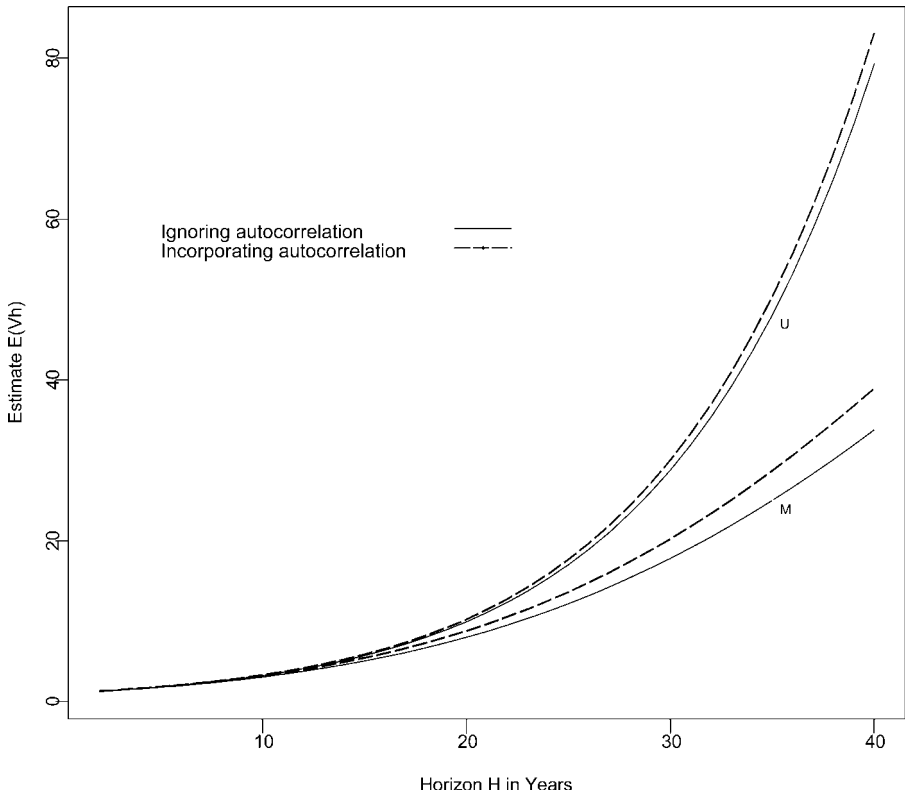


Figure 4 Effect of autocorrelation on estimators U and M . $\hat{\mu} = 0.1$, $\sigma = 0.2$, $T = 75$, MA(4) on annual S&P returns: $\theta = (-0.16, -0.02, -0.16, -0.08)$ estimated on 1926–2001.

given the autocorrelation in the data. Forecasts are barely affected by the correction. Other realistic long-term autocorrelations produced the same results.

For similar reasons, heteroskedasticity poses few problems for these estimators. Heteroskedasticity enters our computation through the variance of the sum of T or H returns. For long investment horizons, this is essentially captured by the unconditional variance, provided that variance is stationary. Empirical estimates of heteroskedasticity, such as stochastic volatility models, imply that conditionality in variance essentially vanishes for forecasting horizons beyond a few years. Note however, that a given form of heteroskedasticity in annual returns would warrant a modification of the standard estimator of μ , as well as the variance of the sum of the next H returns if H is small. However, evidence of heteroskedasticity in annual returns is weak.

Our results can also be used under alternative estimation procedures. For example, Fama and French (2002) note that use of the dividend discount model may provide more precise estimates of the market risk premium than historical averages. This efficiency gain is easily incorporated in our model. Simply convert

the variance reduction in the estimation into an equivalent increase in estimation period, T . Namely, for a more efficient estimator, a reduction in sampling error variance by a factor of E is equivalent to an increase in notional sample size by a factor of E . This is also an intuition for the correction for autocorrelation in Equations (21) and (22), which effectively adjusts H/T for the ratio of “average variance” of the returns in the estimation period to that of the forecasting horizon.

One may also worry that σ is in fact unknown and also estimated. Estimation error in σ introduces nonnormality in the predictive distribution of log-returns [see, e.g., Bawa, Brown, and Klein (1979)]. The variance of the predictive distribution is inflated by a factor of $v/(v-2)$, where v is the sample size used to estimate the variance. Again, as pointed out before, the sampling distribution of $\hat{\sigma}$ converges to σ when the sampling frequency increases. One will therefore benefit from the use of higher-frequency returns for the purpose of estimating σ . The inflation factor will then be extremely close to 1, and the induced nonnormality negligible.

4 OPTIMAL FORECAST VERSUS OPTIMAL ALLOCATION

How important are these issues to economic decisions? We consider in this section an application to a classic problem in finance: the allocation of the portfolio between a risk-free asset and a risky portfolio. Merton (1969) and Samuelson (1969) show that for investors with power utility functions, the optimal allocation to the risky portfolio is $w^* = \frac{\alpha - r_0}{\gamma \sigma^2}$, where $\alpha \equiv \mu + \frac{1}{2}\sigma^2$ is the expected rate of return on the risky asset, r_0 denotes the risk-free return, and γ is the investor’s measure of relative risk aversion. The optimal allocation in these models is independent of the time horizon, and the portfolio is rebalanced continuously to maintain the optimal weights.

To revisit this problem under parameter uncertainty, consider an investor with a power utility function who maximizes the expectation of utility of final wealth given by

$$U(V_H) = \frac{V_H^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \exp[(1-\gamma)\ln(V_H)] \quad (13)$$

Given a capital allocation of w to the risky portfolio and $(1-w)$ to the risk-free asset, where the weight w is maintained constant through continuous rebalancing, the portfolio value at the horizon date is log-normal with parameters⁴

$$\ln(V_H) \sim N(\mu_H, \sigma_H^2) \equiv N[(r_0 + w(\alpha - r_0) - \frac{1}{2}w^2\sigma^2 H, Hw^2\sigma^2)]. \quad (14)$$

Log-normality with these moments of $\ln(V_H)$ implies that expected utility is

$$E[U(V_H)] = \frac{1}{1-\gamma} \exp\{(1-\gamma)H[r_0 + w(\alpha - r_0) - \frac{1}{2}w^2\sigma^2 + \frac{1}{2}(1-\gamma)w^2\sigma^2]\}. \quad (15)$$

⁴ This can be shown formally using, for example, Ito’s lemma. However, it is easier to note that under continuous rebalancing, that is, with fixed portfolio weights, the instantaneous portfolio return is lognormally distributed with constant mean and variance, which implies that the full-period return remains lognormal with the same parameters per unit time.

Maximizing Equation (15) with respect to w yields the well-known optimal allocation $w^* = \frac{\alpha - r_0}{\gamma \sigma^2}$. For i.i.d. returns, the result is independent of the horizon, an implication extensively discussed in the literature.

In contrast, conventional advice is that the allocation should increase with the horizon. This advice is largely motivated in the literature by allowing predictability in expected returns (e.g., Campbell and Viceira (1999), Wachter (2002), Garcia, Detemple, and Rindisbacher (2003), and others) or by appealing to nonportfolio sources of income such as labor [as in Bodie, Merton, and Samuelson (1992)].

The Samuelson-Merton result assumes knowledge of the parameters of the return distribution. Bawa, Brown, and Klein (1979) discuss a “variance inflation” effect due to estimation error on asset allocation in a one-period framework. The effect is not very dramatic for a single period. We now show that it is far greater when the ratio H/T is nontrivial. Barberis (2000) discusses it, but focuses on the interaction of asset allocation with learning, whereas we are more concerned with implications of estimation uncertainty and alternatives to unbiased estimators of expected return, even if no learning or predictability occurs. We now show analytically that the interaction between estimation error and (long) forecasting horizon is very important.

From a decision theoretic perspective, the mere substitution of a point estimate in the optimal allocation in place of the unknown α is incorrect. Rather, the investor, after estimating μ (hence α), has a distribution that represents its uncertainty. This may be a sampling distribution or, for a Bayesian econometrician, the posterior distribution. Therefore $E[U(V_H)]$ in Equation (15) is random, as a nonlinear function of a random variable α . Following decision theory, the proper expected utility to maximize is that resulting from first integrating α out of Equation (15); see Bawa, Brown, and Klein (1979) and Berger (1985) for Bayesian analysis and decision theory, and also Chamberlain (2000) for a recent study. This integration produces the expected utility of wealth given the data, $E[U(V_H | D)]$, to be optimized by the investor. Specifically,

$$E[U(V_H) | D] = \int E[U(V_H) / \alpha] p(\alpha | D) d\alpha. \quad (16)$$

Given a sample size of length T , and no prior information, the posterior distribution of α is simply $N(\hat{\alpha}, \sigma^2/T)$. The result of the integration in Equation (16) is

$$E[U(V_H | D)] = \frac{1}{1 - \gamma} \exp \left\{ (1 - \gamma) H \left[r_0 + w(\hat{\alpha} - r_0) - \frac{1}{2} w^2 \sigma^2 + \frac{1}{2} (1 - \gamma) w^2 \sigma^2 \left(1 + \frac{H}{T} \right) \right] \right\}. \quad (17)$$

While α in Equation (13) is replaced with $\hat{\alpha}$, the last term in Equation (17) is new. It reflects the variance inflation due to the estimation of α . The maximization of

Equation (17) yields the optimal asset allocation:

$$w^* = \frac{\hat{\alpha} - r_0}{\sigma^2 \left[\gamma + \frac{H}{T}(\gamma - 1) \right]}. \quad (18)$$

For $H \ll T$, Equation (18) collapses to the standard optimal allocation. Otherwise, for $\gamma > 1$, the allocation to the risky asset is decreased (relative to the known-parameter case) in favor of a higher allocation to the risk-free asset. Recognition of the uncertainty in the estimate of α leads investors to shy away from risky assets, the more so the greater the H/T ratio. First, this result is precisely contrary to the common advice to invest more in stocks for longer horizons. Second, it happens even if returns are unpredictable. The only exception is log-utility investors, for whom $\gamma = 1$. Because log-utility is linear in α , estimation uncertainty and the horizon H do not affect the location of the optimum.

Note that the problem can easily be written where the investor has a proper prior. The optimal allocation in Equation (18) will then involve the posterior mean rather than $\hat{\alpha}$, and a modified sample size accounting for the prior precision rather than T . One could then deduce from that optimal allocation what type of prior could be consistent with ignoring parameter uncertainty. It will have the feature that the prior mean increases, above the sample mean, with the forecasting horizon.

As in Brennan (1998) and Barberis (2000), our rationale on long-term asset allocation turns on an often overlooked wrinkle in the argument concerning the proposition that stocks are less risky as long-run investments. The result here is driven by the fact that the impact of estimation uncertainty compounds as the investment horizon becomes more distant. Rebalancing does not eliminate this effect. This consideration might well dominate mild negative serial correlation as one evaluates the risk of stocks at different investment horizons. And, while the strength of serial correlation in market returns is still contested, there is no doubt about the considerable estimation error surrounding estimates of mean return.

Figure 5 plots w^* as a function of H for the parameters used in Figure 2, namely $\hat{\mu} = 0.1$, $\sigma = 0.2$. We also use $r_0 = 0.04$, and $\gamma = 2$ and 4. The lines labeled "conventional" show the asset allocation in the known-parameter case. The downward-sloping lines depict the optimal allocation to the risky portfolio when parameter uncertainty is properly incorporated. Figure 5 shows that estimation uncertainty has a dramatic interaction with the investment horizon. The optimal asset allocation may differ considerably for different investment horizons. For a long estimation period of 75 years, the optimal weight on the risky asset falls from 87% under the "conventional" approach to 70% as the investment horizon expands from 1 to 40 years. The effect is far more pronounced when the estimation period is shorter, as it would be for an emerging market or if one believed that the United States was subject to structural breaks. For $T = 30$ years, the allocation falls to 53%.

More risk-averse investors have lower "conventional" allocation to the risky asset. Even for them, the effect of horizon is very strong as a fraction of the conventional allocation. For an investor with $\gamma = 4$, the 40-year conventional allocation is 43%, while the optimal allocation for $T = 30$ is only 20%.

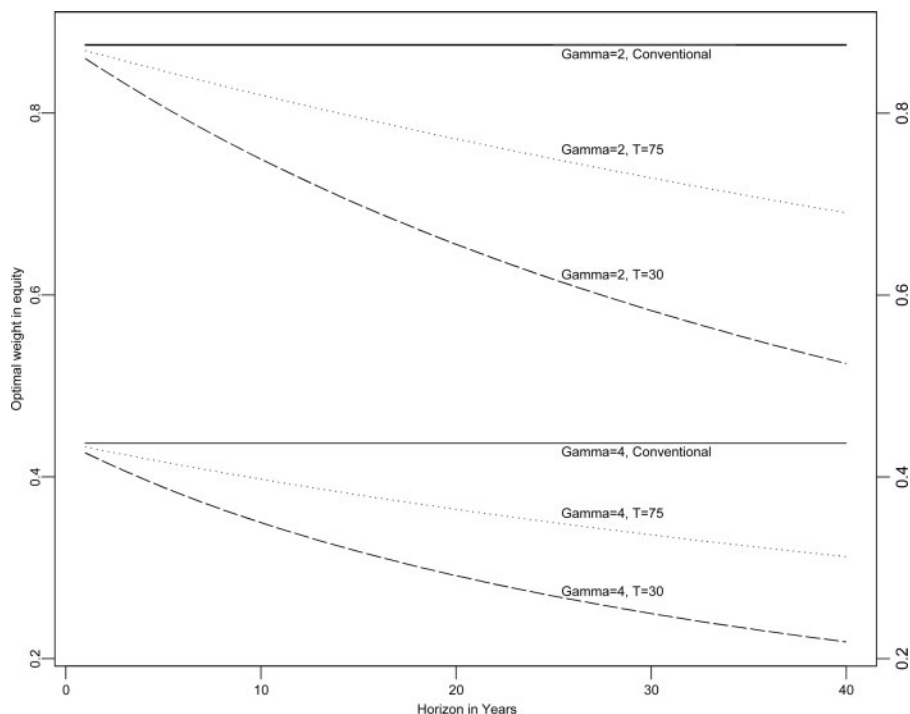


Figure 5 Combined effects of horizon and estimation error on optimal allocation. $\hat{\mu} = 0.1$, $\sigma = 0.2$.

This example is admittedly only indicative in that we assume the choice of w^* is a once-and-for-all decision. A full-blown asset allocation problem [see Brennan (1998) and Barberis (2000)] would allow for dynamic updating of the estimate of μ and the allocation w^* as agents learn about the true distribution of returns from new data, and would give rise to intertemporal hedging demands against changes in the perceived opportunity set along the lines of Merton (1973). These issues would take us far afield from the forecasting problem, however.⁵ Further, the evidence on the predictability of asset returns is far from uncontroversial. While we do not explicitly account for time-varying hedging demands, Equation (18) does provide an analytical guideline to the impact of estimation uncertainty, even in their presence. One may take Equation

⁵ Barberis (2000) presents optimal asset allocations both for buy-and-hold investors who never rebalance and for investors who update parameter estimates as more data become available and periodically rebalance optimally. Our case is somewhere between these extremes in that our investors rebalance but do not update. The greater simplicity of this setting allows us to maintain the focus on forecasting issues as well as to derive a closed-form solution for asset allocation that analytically demonstrates the importance of the key ratio H/T . Symmetrically, Barberis (2000) presents detailed analysis of optimal asset allocation using numerical simulations, but, in contrast to our focus, devotes little attention to properties of alternative estimators of expected return.

(18) as indicative of the importance of estimation risk for simple asset allocation, even in the presence of rebalancing.

5 CONCLUSION

We have considered the problem of forecasting portfolio values over long horizons when the return distribution is estimated from historical data. While recent articles address the estimation of the one-period expected return, few have focused on the formulation of long-term expected returns. Moreover, the literature focuses most exclusively on the arithmetic (maximum-likelihood) and geometric estimators, rarely on unbiased estimators. It has so far ignored naturally efficient estimators such as those minimizing MSE, which can be seen as the small-sample generalization of the principles justifying the maximum likelihood. We derive an analytic small-sample efficient estimator of long-term expected returns. It is far lower than the MLE, the unbiased, and even the geometric average estimator when H is larger than $T/3$. The resulting efficiency gains are spectacular, not only on the MLE, but also on the unbiased estimator for long horizons, and on the geometric estimator for shorter horizons.

We show how the results are easily adjusted for serial correlation. However, realistic values of autocorrelation appear unimportant for the long-horizon forecasts that are the concern of this article. Strong cases are made in recent studies that the estimate of the market risk premium should be revised downward. Our result compounds this by stating that even these lower estimates of mean return should be adjusted further downward when used to predict long-term returns. Our results also show that alternative methods of estimation of the risk premium that can be shown to be more precise [e.g., Fama and French (2002)] are especially valuable if the premium is to be used for long-term forecasts.

We also analytically derive a striking implication for long-term asset allocation. Contrary to conventional wisdom, longer investment horizons imply *lower* allocations to risky assets in order to account for the fact that the estimation error gets compounded at the investor's horizon.

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