12 Probability

12_03_Additional_Examples

Example 1. Assign 3 persons in 5 rooms. Find the probabilities of the following events.

[1] Three persons are assigned in the first three rooms and each person is assigned exactly in one room.

[2] Three persons are assigned in any three rooms and each person is assigned exactly in one room.

[Solution]

[1] Each person can be assigned in any of the 5 rooms. So, the sample space size is 5^3. Since 3 persons can be arranged in any order in the first three rooms. So, the size of the event is 3!. Thus, the probability is

$$3!/5^3 = 0.048.$$

[2] Since there are C(5, 3) ways of choosing any three rooms, the probability is

$$C(5, 3) * 3!/5^3 = 0.48.$$

Example 1 can be extended to the following form.

Example 2. Assign m persons in n rooms, where $m \leq n$. Then

[1] The probability of the m persons are assigned in a fixed set of m rooms and each person is assigned exactly in one

room is

$$m!/n^m.$$

[2] The probability of m persons are assigned in any m rooms and each person is assigned exactly in one room is

$$C(m, n) * n!/n^m.$$

<u>Example 3.</u> There are a_1 red balls, a_2 green balls, and a_3 blue balls in a bag. Randomly select x balls from the bag such that x_1 of them are red, x_2 of them is green, and x_3 of them are blue. Find the probability of this event.

[Solution]

There are

$$C(a_1 + a_2 + a_3, x) = C(a_1 + a_2 + a_3, x_1 + x_2 + x_3)$$

ways of selecting the x balls.

Clearly, the size of the event is

$$C(a_1, x_1)*C(a_2, x_2)*C(a_3, x_3).$$

Thus, the probability is

$$C(a_1, x_1)*C(a_2, x_2)*C(a_3, x_3)/C(a_1 + a_2 + a_3, x_1 + x_2 + x_3).$$

<u>Example 4.</u> There are a_1 red balls, a_2 green balls, and a_3 blue balls in the first bag. There are b_1 red balls, b_2 green balls, and b_3 blue balls in the second bag. Randomly and independently select one ball from the first bag and one ball from the second bag. Find the probability of that the two balls have the same color.

[Solution]

Let A be the event of two balls have the same color.

Let B_1 be the event of the ball selected from the first bag is a red ball.

Let C_1 be the event of the ball selected from the second bag is a red ball.

Let B_2 be the event of the ball selected from the first bag is a green ball.

Let C_2 be the event of the ball selected from the second bag is a green ball.

Let B_3 be the event of the ball selected from the first bag is a blue ball.

Let C_3 be the event of the ball selected from the second bag is a blue ball.

Set $D_1 := B_1 \cap C_1$, $D_2 := B_2 \cap C_2$, and $D_3 := B_3 \cap C_3$.

Then A is the disjoint union of D_1, D_2, and D_3. Thus,

$$P(A) = P(D_1) + P(D_2) + P(D_3)$$
$$= P(B_1 \cap C_1) + P(B_2 \cap C_2) + P(B_3 \cap C_3)$$

Since B_1 and C_1, B_2 and C_2, B_3 and C_3 are independent, respectively, we have

$$P(B_1 \cap C_1) = P(B_1)*P(C_1),$$
$$P(B_2 \cap C_2) = P(B_2)*P(C_2),$$
$$P(B_3 \cap C_3) = P(B_3)*P(C_3).$$

Hence

$$P(A) = [a_1/(a_1 + a_2 + a_3)]*[b_1/(b_1 + b_2 + b_3)] + [a_2/(a_1 + a_2 + a_3)]*[b_2/(b_1 + b_2 + b_3)] + [a_3/(a_1 + a_2 + a_3)]*[b_3/(b_1 + b_2 + b_3)]$$
$$= (a_1b_1 + a_2b_2 + a_3b_3)/[(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)].$$

Example 5. Let A and B be two events. Suppose $P(A) = a$, $P(B) = b$, and $P(A \mid B^c) = c$. Find

[1] $P(B^c \cap A)$.

[3] $P(A \cap B)$.

[4] $P(A \cup B)$.

[Solution]

[1] $P(B^c \cap A) = P(A \mid B^c) * P(B^c) = c*(1 - P(B)) = c*(1 - b)$.

[2] $A \cap B = A - B^c \cap A$ and $B^c \cap A \subseteq A$. Thus,

$$P(A \cap B) = P(A - B^c \cap A) = P(A) - P(B^c \cap A)$$
$$= a - c*(1 - b).$$

[3] $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= a + b - (a - c*(1 - b)) = b + c*(1 - b).$$

Example 6. Let A and B be two independent events. Prove that

[1] A^c and B are independent.

[2] A and B^c are independent.

[3] A^c and B^c are independent.

[Proof]

[1] Since $A^c \cap B = B - A \cap B$ and $A \cap B \subseteq B$, we have that $P(A^c \cap B) = P(B) - P(A \cap B)$

$$= P(B) - P(A)*P(B)$$

//since A and B are independent

$$= (1 - P(A))*P(B)$$
$$= P(A^c) * P(B).$$

Thus, A^c and B are independent.

[2] Since A and B are independent, B and A are independent.

Applying [1], we have B^c and A are independent.

Namely, A and B^c are independent.

[3] By [1], we have A^c and B are independent.

Applying [2], we have that A^c and B^c are independent.