01 LOGIC

01_03 Tautology Equivalence

A tautology is a proposition which is always true.

Example: $p \vee \sim p$. Recall the following truth table.

p	~p	$p \vee \sim p$
T	F	T
F	T	T

A contradiction is a proposition which is always false.

Example: $p \wedge \sim p$. Recall the following truth table.

p	~p	$p \wedge \sim p$
T	F	F
F	T	F

Note that the negation of a tautology is a contradiction and the negation of a contradiction is a tautology.

A contingency is a proposition which is neither a tautology nor a contradiction. Example: $p \vee \sim q$.

If a biconditional $r \leftrightarrow t$ is a tautology, then the proposition r and proposition t are logically equivalent, denoted by $r \Leftrightarrow t$.

Because of the truth table below, $p \leftrightarrow \sim(\sim p)$ is a tautology, i.e., p and $\sim(\sim p)$ are logically equivalent and $p \Leftrightarrow \sim(\sim p)$.

p	~p	~(~p)	$p \leftrightarrow \sim(\sim p)$
T	F	T	T
F	T	T	T

Because of the truth table below, $(p \wedge q) \leftrightarrow (q \wedge p)$ is a tautology, i.e., $p \wedge q$ and $q \wedge p$ are logically equivalent and $(p \wedge q) \Leftrightarrow (q \wedge p)$.

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

Because of the truth table below, $(p \vee q) \leftrightarrow (q \vee p)$ is a tautology, i.e., $p \vee q$ and $q \vee p$ are logically equivalent and $(p \vee q) \Leftrightarrow (q \vee p)$.

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Because of the truth table below, $(p \oplus q) \leftrightarrow (q \oplus p)$ is a tautology, i.e., $p \oplus q$ and $q \oplus p$ are logically equivalent and $(p \oplus q) \Leftrightarrow (q \oplus p)$.

p	q	$p \oplus q$	$q \oplus p$	$(p \oplus q) \leftrightarrow (q \oplus p)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Because of the truth table below, $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$ is a tautology, i.e., $p \leftrightarrow q$ and $q \leftrightarrow p$ are logically equivalent and $(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$.

p	q	$p \leftrightarrow q$	$q \leftrightarrow p$	$(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

Because of the truth table below, $(p \leftrightarrow q) \leftrightarrow \sim(p \oplus q)$ is a tautology, i.e., $(p \leftrightarrow q)$ and $\sim(p \oplus q)$ are logically equivalent, $\sim(p \leftrightarrow q)$ and $(p \oplus q)$ are logically equivalent), $(p \leftrightarrow q) \Leftrightarrow \sim(p \oplus q)$, and $\sim(p \leftrightarrow q) \Leftrightarrow (p \oplus q)$.

p	q	$p \leftrightarrow q$	$p \oplus q$	$\sim(p \oplus q)$	$(p \leftrightarrow q) \leftrightarrow \sim(p \oplus q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	T	F	T	T

Because of the truth table below, $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ is a tautology, i.e., $(p \rightarrow q)$ and $(\sim p \vee q)$ are logically equivalent and $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Recall the truth table of $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ below.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$ are logically equivalent and

$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$.

Because of the truth table below, $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ is a contingency, i.e., $(p \rightarrow q)$ and $(q \rightarrow p)$ are not logically equivalent.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Because of the truth table below, $(p \wedge q) \leftrightarrow \sim(p \vee q)$ is a contingency, i.e., $(p \wedge q)$ and $\sim(p \vee q)$ are not logically equivalent, and $\sim(p \wedge q)$ and $(p \vee q)$ are not logically equivalent).

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \leftrightarrow \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	F	T	F

Common logical equivalences.

Suppose p is a proposition. Then

$p \vee T \Leftrightarrow T$	$p \wedge F \Leftrightarrow F$
$p \wedge T \Leftrightarrow p$	$p \vee F \Leftrightarrow p$
$p \wedge p \Leftrightarrow p$	$p \vee p \Leftrightarrow p$
$p \Leftrightarrow \sim(\sim p)$	$(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
$p \vee \sim p \Leftrightarrow T$	$p \wedge \sim p \Leftrightarrow F$
$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
$(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$	$(p \oplus q) \Leftrightarrow (q \oplus p)$
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
$p \wedge (q \vee r) \Leftrightarrow$ $(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow$ $(p \vee q) \wedge (p \vee r)$