10 Recursive Sequences.

10_02_Recursive_Sequences 2

Theorem. Let

$$a_1, a_2, \ldots, a_n, \ldots$$

be a recursive sequence such that

$$a_1 = c_1,$$

$$a_2 = c_2,$$

$$a_i = A*a_{i-1} + B*a_{i-2} \text{ if } i \geq 3.$$

Then the following equation

$$x^2 - A*x - B = 0$$

is called the characteristic equation of the sequence.

[1] Suppose $x^2 - A*x - B = 0$ has two distinct roots u and v. Then

$$a_i = C*u^i + D*v^i,$$

where C and D can be decided by $a_1 = c_1$ and $a_2 = c_2$.

[2] Suppose $x^2 - A*x - B = 0$ has two identical roots u and u. Then

$$a_i = C*u^i + D*i*u^i,$$

where C and D can be decided by $a_1 = c_1$ and $a_2 = c_2$.

Example. Consider the Fibonacci sequence

$$f_1, f_2, \ldots, f_n, \ldots$$

where

$$f_1 = 1,$$

$$f_2 = 1,$$

$$f_i = f_{i-1} + f_{i-2} \text{ if } i \geq 3.$$

the characteristic equation of the sequence is

$$x^2 - x - 1 = 0.$$

This equation has two distinct roots

$$u = (1 + \sqrt{5})/2 \text{ and}$$

$$v = (1 - \sqrt{5})/2.$$

Solving the simultaneous equations

$$1 = a_1 = C*u^1 + D*v^1,$$

$$1 = a_2 = C*u^2 + D*v^2,$$

we have that

$$C = 1/\sqrt{5} \text{ and}$$

$$D = -1/\sqrt{5}.$$

Thus

$$f_i = C*u^i + D*v^i$$

$$= [((1 + \sqrt{5})/2)^i - ((1 - \sqrt{5})/2)^i]/\sqrt{5}$$

<u>Example.</u> Consider the sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

where

$$a_1 = 1,$$

$$a_2 = 2,$$

$$a_i = 3*a_{i-1} + 4*a_{i-2} \text{ if } i \geq 3.$$

the characteristic equation of the sequence is

$$x^2 - 3x - 4 = 0.$$

This equation has two distinct roots

$$u = 4 \text{ and}$$

$$v = -1.$$

Solving the simultaneous equations

$$1 = a_1 = C*u^1 + D*v^1,$$

$$2 = a_2 = C*u^2 + D*v^2,$$

we have that

$$C = 3/20 \text{ and}$$

$$D = -2/5.$$

Thus

$$f_i = C*u^i + D*v^i$$

$$= 3*4^i/20 - 2*(-1)^i/5.$$

<u>Example.</u> Consider the sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

where

$$a_1 = 1,$$

$$a_2 = 2,$$

$$a_i = 6*a_{i-1} - 9*a_{i-2} \text{ if } i \geq 3.$$

the characteristic equation of the sequence is

$$x^2 - 6x + 9 = 0.$$

This equation has two identical roots

$$u = 3 \text{ and}$$

$$v = 3.$$

Solving the simultaneous equations

$$1 = a_1 = C*u^1 + D*1*u^1,$$

$$2 = a_2 = C*u^2 + D*2*u^2,$$

we have that

$$C = 4/9 \text{ and}$$

$$D = -1/9.$$

Thus

$$f_i = C*u^i + D*i*u^i$$

$$= 4*3^i/9 - i*3^i/9.$$