03 Counting Methods

03_02_the Principle of Inclusion and Exclusion

If x is a real number, the floor of x, denoted [x], is the largest integer which is less than or equal to x. The ceiling of x, denoted {x}, is the smallest integer which is greater than or equal to x. For example, if x = -7, 0, or 7, then [x] = {x} = x. If x = 3.14, then [x] = 3 and {x} = 4. If x = - 2.718, then [x] = -3 and {x} = -2. Note that $x - 1 < [x] \le x \le \{x\} < x + 1$ and $[x] = x = \{x\}$ if x is an integer.

Example.

[1] Find the number of positive integers which are less than 201 and are divisible by 5.

[2] Find the number of positive integers which are less than 201 and are divisible by 7.

[3] Find the number of positive integers which are less than 201 and are divisible by 5 and 7.

[4] Find the number of positive integers which are less than 201 and are divisible by 5 or 7.

[5] Find the number of positive integers which are less than 201 and are divisible by 5 and are not divisible by 7.

[6] Find the number of positive integers which are less than 201 and are not divisible by 5 and are divisible by 7.

[7] Find the number of positive integers which are less than 201 and are not divisible by 5 and are not divisible by 7.

[Solution] Define A = {a: a is an integer, $0 < a < 201$, $5|a$},

B = {b: b is an integer, $0 < b < 201$, $5|b$},

U = {u: u is an integer, $0 < u < 201$}.

[1] The answer is $[200/5] = 40 = |A|$.

[2] The answer is $[200/7] = 28 = |B|$.

[3] Note that 5 and 7 are relatively prime, a positive integer n is divisible by 5 and 7 if and only if n is divisible by 5*7 = 35.

The answer is $[200/35] = 5 = |A \cap B|$.

[4] The answer is $|A \cup B| = |A| + |B| - |A \cap B| = 63$.

[5] The answer is $|A - B| = |A - A \cap B| = |A| - |A \cap B| = 35$.

[6] The answer is $|B - A| = |B - B \cap A| = |B| - |B \cap A| = 23$.

[7] The answer is $|A^c \cap B^c| = |(A \cup B)^c| = |U| - |A \cup B|$

$= 200 - 63 = 137$.

Note that the following example needs the concepts of permutation and combination.

Let U be {1, 2, 3, 4}. A derangement of U is a permutation wxyz of 1, 2, 3, and 4 such that $w \neq 1$, $x \neq 2$, $y \neq 3$, and $z \neq 4$.

<u>Example.</u> Find the number of derangements of U.

[Solution] Let A, B, C, and D be finite sets. We first prove that

$$|A \cup B \cup C \cup D|$$
$$= |A| + |B| + |C| + |D|$$
$$– (|A \cap B| + |A \cap C| + |A \cap D| +$$

$$|B \cap C| + |B \cap D + |C \cap D|)$$

$$+ (|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|)$$

$$- |A \cap B \cap C \cap D|.$$

[Proof] $|A \cup B \cup C \cup D| = |A| + |B \cup C \cup D| - |A \cap (B \cup C \cup D)|$

$= |A| + |B| + |C| + |D| - (|B \cap C| + (|B \cap D + |C \cap D|) +$

$|B \cap C \cap D| - |(A \cap B) \cup (A \cap C) \cup (A \cap D)|$ --------[1]

Now

$$|(A \cap B) \cup (A \cap C) \cup (A \cap D)|$$

$$= |A \cap B| + |A \cap C| + |A \cap D|$$

$$- (|A \cap B \cap A \cap C| + |A \cap B \cap A \cap D| + |A \cap C \cap A \cap D|)$$

$$+ |A \cap B \cap A \cap C \cap A \cap D|$$

$$= |A \cap B| + |A \cap C| + |A \cap D|$$

$$- (|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|)$$

$$+ |A \cap B \cap C \cap D| \text{ --------[2]}$$

Plugging [2] into [1], we obtain the desired result.

So, the proof is complete.

Suppose wxyz is a permutation of the elements in U.

Define $A = \{wxyz: w = 1\}$,

$B = \{wxyz: x = 2\}$,

$C = \{wxyz: y = 3\}$,

$D = \{wxyz: z = 4\}$,

Then $|A| = |B| = |C| = |D| = 3!$,

$|A \cap B| = |A \cap C| = |A \cap D| =$

$|B \cap C| = |B \cap D = |C \cap D| = 2!$,

$|A \cap B \cap C| = |A \cap B \cap D| =$

$|A \cap C \cap D| = |B \cap C \cap D| = 1 = 1!$,

$|A \cap B \cap C \cap D| = 0!$.

Thus $|A \cup B \cup C \cup D| = 4*3! – 6*2! + 4*1! – 1*0!$

$= C(4, 3)*3! – C(4, 2)*2! + C(4, 1)*1! – C(4,0)*0!$

Hence, the answer is $| A^c \cap B^c \cap C^c \cap D^c| =$

$|(A \cup B \cup C \cup D)^c| = |U|! - |A \cup B \cup C \cup D| =$

$4! - C(4, 3)*3! + C(4, 2)*2! - C(4, 1)*1! + C(4,0)*0!$

$= 4!(1 – 1/1! + 1/2! – 1/3! + 1/4!) = 9$.

<u>Exercise</u>. Find the numbers of derangements of

$\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5, 6\}, \ldots\ldots,$

$\{1, 2, 3, 4, 5, \ldots, n\}$.