08 Relations.

08_02_Equivalence Relations

Definition. Suppose R is relation on a set A. R is reflexive if $(a, a) \in R$ for every element $a \in A$.

Example. The following relations on the set of integers are reflexive.

$R_1 = \{(a, b)$: a and b are integers, $a = b\}$ since $a = a$ is true for any integer a.

$R_2 = \{(a, b)$: a and b are integers, $a \geq b\}$ since $a \geq a$ is true for any integer a.

$R_3 = \{(a, b)$: a and b are integers, $a = b$ or $a = -b\}$ since $a = a$ is true for any integer a and therefore (a, a) is in R_3.

The following relations on the set of integers are not reflexive.

$R_4 = \{(a, b)$: a and b are integers, $a < b\}$ since $7 < 7$ is not true.

$R_5 = \{(a, b)$: a and b are integers, $b = a + 3\}$ since $7 = 7 + 3$ is not true.

$R_6 = \{(a, b)$: a and b are integers, $a + b \leq 17\}$ since $9 + 9 \leq 17$ is not true.

Definition. Suppose R is relation on a set A. R is symmetric if $(a, b) \in R$ if and only if $(b, a) \in R$ for any elements a and b in A.

Example. Decide if the following relations are symmetric.

$R_1 = \{(a, b)$: a and b are integers, $a = b\}$ is symmetric since

$(a, b) \in R_1$ if and only if $(b, a) \in R_1$ for any two integers a and b.

$R_2 = \{(a, b)$: a and b are integers, $a \geq b\}$ is not symmetric since $3 \geq 2$ is true, but $2 \geq 3$ is false.

$R_3 = \{(a, b)$: a and b are integers, $a = b$ or $a = -b\}$ is symmetric since $(a, b) \in R_3$ if and only if $(b, a) \in R_3$ for any two integers a and b.

[Proof] $(a, b) \in R_3$ is equivalent to $(a = b \text{ or } a = -b) \Leftrightarrow$

$(b = a \text{ or } b = -a) \Leftrightarrow (b, a) \in R_3$.

$R_4 = \{(a, b)$: a and b are integers, $a < b\}$ is not symmetric since $5 < 7$ is true, but $7 < 5$ is false.

$R_5 = \{(a, b)$: a and b are integers, $b = a + 3\}$ is not symmetric since $5 = 2 + 3$ is true, but $2 = 5 + 3$ is false.

$R_6 = \{(a, b)$: a and b are integers, $a + b \leq 17\}$ is symmetric since $(a, b) \in R_6$ if and only if $(b, a) \in R_6$ for any two integers a and b.

<u>Definition</u>. Suppose R is relation on a set A. R is <u>transitive</u> if whenever $(a, b) \in R$ and $(b, c) \in R$, we have $(a, c) \in R$ for any elements a, b, and c in A.

<u>Example.</u> Decide if the following relations are transitive.

$R_1 = \{(a, b)$: a and b are integers, $a = b\}$ is transitive since whenever $(a, b) \in R_1$ and $(b, c) \in R_1$, we have $(a, c) \in R_1$ for any integers a, b, and c.

$R_2 = \{(a, b)$: a and b are integers, $a \geq b\}$ is transitive since whenever $(a, b) \in R_2$ and $(b, c) \in R_2$, we have $(a, c) \in R_2$

for any integers a, b, and c.

$R_3 = \{(a, b)$: a and b are integers, $a = b$ or $a = -b\}$ is transitive

since $(a, b) \in R_3$ and $(b, c) \in R_3$, we have $(a, c) \in R_3$ for any integers a, b, and c.

[Proof] Since $(a, b) \in R_3$, we have $a = b$ or $a = -b$.

Since $(b, c) \in R_3$, we have $b = c$ or $b = -c$.

Hence, we have the following four cases.

Case 1. $a = b$, $b = c$. Thus $a = c$.

Case 2. $a = b$, $b = -c$. Thus $a = -c$.

Case 1. $a = -b$, $b = c$. Thus $a = -c$.

Case 1. $a = -b$, $b = -c$. Thus $a = c$.

Therefore $a = c$ or $a = -c$. So $(a, c) \in R_3$.

$R_4 = \{(a, b)$: a and b are integers, $a < b\}$ is transitive

since whenever $(a, b) \in R_4$ and $(b, c) \in R_4$, we have $(a, c) \in R_4$ for any integers a, b, and c.

$R_5 = \{(a, b)$: a and b are integers, $b = a + 3\}$ is not transitive

since $(2, 5) \in R_6$ and $(5, 8) \in R_5$ are true, but $(2, 8) \in R_5$ is false.

$R_6 = \{(a, b)$: a and b are integers, $a + b \leq 17\}$ is not transitive

since $(10, 6) \in R_6$ and $(6, 9) \in R_6$ are true, but $(10, 9) \in R_6$ is false.

<u>Definition</u>. Suppose R is relation on a set A. R is an <u>equivalent relation</u> if R is reflexive, symmetric, and transitive.

<u>Example.</u> Decide if the following relations are equivalent relations.

$R_1 = \{(a, b)$: a and b are integers, $a = b\}$ is an equivalent relation since it is reflexive, symmetric, and transitive.

$R_2 = \{(a, b)$: a and b are integers, $a \geq b\}$ is not an equivalent relation since it is not symmetric although it is reflexive and transitive.

$R_3 = \{(a, b)$: a and b are integers, $a = b$ or $a = -b\}$ is an equivalent relation since it is reflexive, symmetric, and transitive.

$R_4 = \{(a, b)$: a and b are integers, $a < b\}$ is not an equivalent relation. Note that R_4 is not reflexive, R_4 it is not symmetric, R_4 is transitive.

$R_5 = \{(a, b)$: a and b are integers, $b = a + 3\}$ is not an equivalent relation. Note that R_5 is not reflexive, R_5 is not symmetric, R_5 is not transitive.

$R_6 = \{(a, b)$: a and b are integers, $a + b \leq 17\}$ is not an equivalent relation. Note that R_5 is not reflexive, R_5 it is symmetric, R_5 is not transitive.