08 Relations.

08_01_Introduction to Relations

Definition. The Cartesian product of two sets A and B, denoted $A \times B$, is the set of $\{(a, b): a \in A \text{ and } b \in B\}$, where (a, b) with $a \in A$ and $b \in B$ is called an ordered pair based on A and B. Note that $(a, b) \neq (b, a)$, in general.

Example. Suppose $X = \{x, y\}$ and $Y = \{1, 2, 3\}$. Then

$$X \times Y = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}.$$
$$Y \times X = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

We also have

$$X \times X = \{(x, x), (x, y), (y, x), (y, y)\}.$$
$$Y \times Y = \{(1, 1), (1, 2), (1, 3),$$
$$(2, 1), (2, 2), (2, 3),$$
$$(3, 1), (3, 2), (3, 3)\}.$$

Note that $X \times Y \neq Y \times X$, but $|X \times Y| = |Y \times X| = |X|*|Y| = 6$.
Also $|X \times X| = |X|*|X| = 4$ and $|Y \times Y| = |Y|*|Y| = 9$.

Suppose A and B are finite sets. In general, $A \times B \neq B \times A$, but $|A \times B| = |A|*|B| = |B|*|A| = |B \times A|$.

Definition. A subset S of the Cartesian product $A \times B$ is called a relation from set A to set B. A subset T of the Cartesian product $A \times A$ is called a relation on set A.

Example. {} and {(x, 1), (x, 3), (y, 2), (y, 3)} are relations from X to Y in the above example. {} and {(1, 1), (2, 3), (3, 1)} are relations on Y in the above example.

Recall that the number of subsets of a finite set S is $2^{|S|}$. Suppose A and B are finite sets. Then the number of relations from A to B is $2^{|A||B|}$. The number of relations on A is $2^{|A||A|}$.

The Domain of a relation R from A to B is a set, denoted Domain(R), consisting of all the first components in the ordered pairs of R. Clearly, Domain(R) $\subseteq$ A.

The Range of a relation R from A to B is a set, denoted Range(R), consisting of all the second components in the ordered pairs of R. Clearly, Range(R) $\subseteq$ B.

Example. Suppose A = {1, 2, 3} and B = {5, 6, 8, 9}. Find a relation R from A to B such that R = {(x, y): x $\in$ A, y $\in$ B, and $3x + 1 < y$}.

[Solution] First, we find A × B as follows.

$$A \times B = \{(1, 5), (1, 6), (1, 8), (1, 9), (2, 5), (2, 6), (2, 8), (2, 9), (3, 5), (3, 6), (3, 8), (3, 9)\}.$$

Check each ordered pair in A × B. If the $3x + 1 < y$ is true, we will put that ordered pair in R. Otherwise we will not put that ordered pair in R. Therefore

$$R = \{(1, 5), (1, 6), (1, 8), (1, 9), (2, 8), (2, 9)\}.$$

The domain of R is $\{1, 2\}$. The range of R is $\{5, 6, 8, 9\}$.

Example. Suppose $A = \{1, 2, 3, 4, 5, 6\}$. Find a relation R on A such that $R = \{(x, y): x \in A, y \in B, x|y\}$.

[Solution] First, we find $A \times A$ as follows.

$$\begin{aligned} A \times A = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ &(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\ &(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\ &(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\ &(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\ &(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} \end{aligned}$$

Check each ordered pair in $A \times A$. If the $x|y$ is true, we will put that ordered pair in R. Otherwise we will not put that ordered pair in R. Therefore

$$\begin{aligned} R = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ &(2, 2), (2, 4), (2, 6), \\ &(3, 3), (3, 6), \\ &(4, 4), \\ &(5, 5), \\ &(6, 6)\} \end{aligned}$$

The domain of R is $\{1, 2, 3, 4, 5, 6\}$. The range of R is also $\{1, 2, 3, 4, 5, 6\}$.