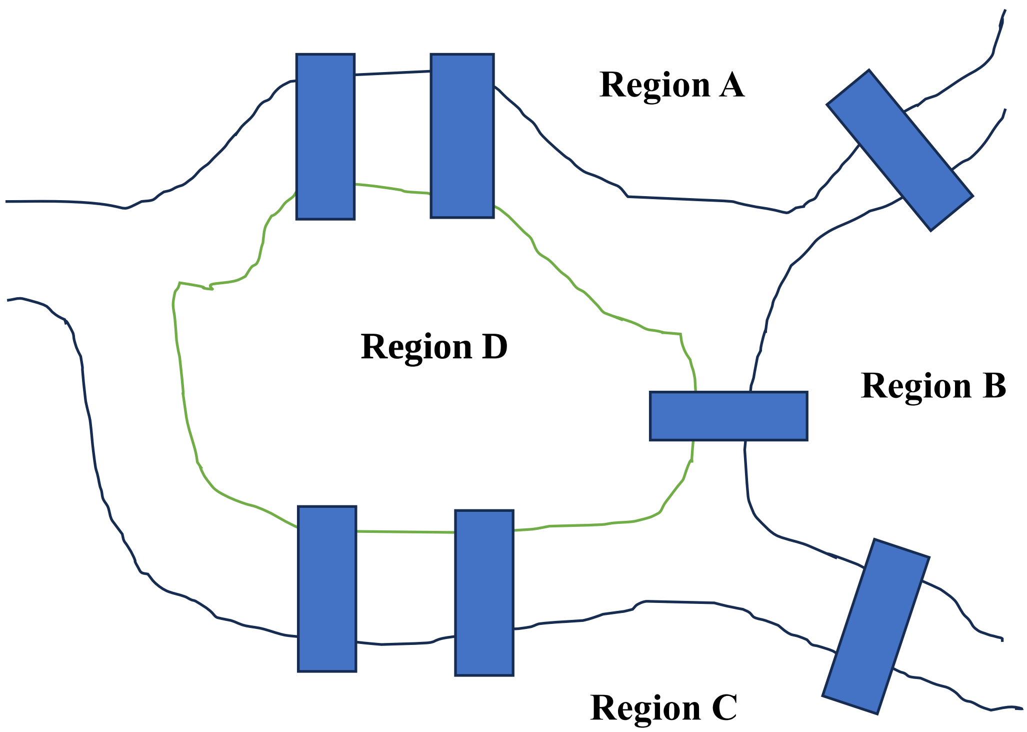
11 Graphs and Tress.

11_01_Introduction to Graph Theory

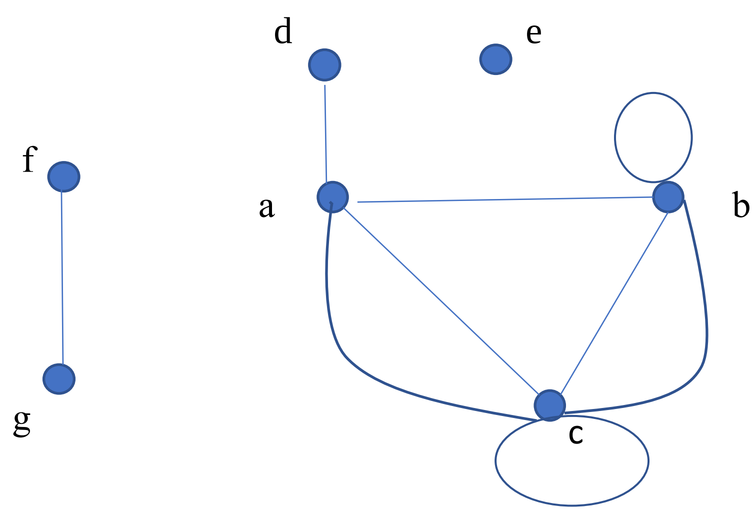
The Königsberg Bridge Problem (or the Seven-Bridge Problem) is a well-known problem in Mathematics. The town of Königsberg, Prussia (now Kalingrad, Russia) was divided into four regions by the branches of the Pregel river. In the 18th century, there were seven bridges that connected the four regions.



People wondered if it was possible to have a traversal plan starting with one region, crossing each bridge exactly once, returning to the starting region. Leonard Euler, one of the greatest mathematicians in history, solved the problem by showing no such traversal plan existed. Euler's solution for the Seven-Bridge

Problem is commonly believed as the first result in Graph Theory, a branch of Mathematics. A good resource for the Seven-Bridge Problem is "https://www.maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-problem".

Definition. A graph is a mathematical structure that consists of a nonempty set of vertices and a set of edges. Each element in the edge set is associated with either one or two vertices in vertex set. If the vertex set is V and the edge set is E, we use G = (V, E) to represent the graph. In the following graph, we have V = {a, b, c, d, e, f, g}, E = {ab, $(ac)_1$, $(ac)_2$, ad, bb, $(bc)_1$, $(bc)_2$, cc, fg}, and G = (V, E).



The graphs in Graph Theory are different from the graphs of functions.

A graph with an infinite vertex set is called an infinite graph.

A graph with a finite vertex set is called a finite graph.

The order of a graph is the number of vertices.

The size of a graph is the number of edges.

When an edge is associated with either one or two vertices in the vertex set, we say the edge is incident with the vertices.

An undirected graph is a graph in which the edges which are associated with vertices have no direction.

A vertex is an isolated vertex if no edges are incident to it.

Two vertices are adjacent if there is an edge joining them.

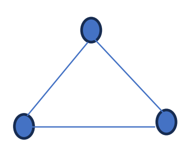
A simple graph is a graph in which each edge joins two different vertices and no two edges join the same pair of vertices.

We study only finite undirected simple graphs in this class.

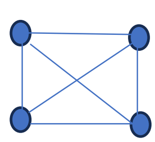
A complete graph on n vertices, denoted K_n, is a simple graph such that each pair of vertices in the graph is joined by one edge. Note that the size of the complete graph K_n is $n(n-1)/2$. Below are some examples of the complete graphs.

K_1

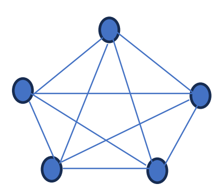
K_2



K_3

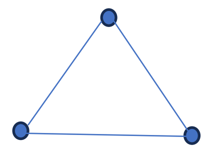


K_4

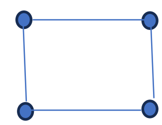


K_5

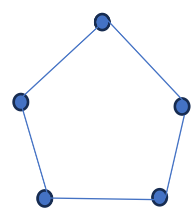
A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. Below are C_3, C_4, C_5, and C_6.



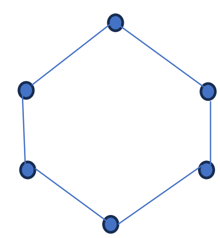
C_3



C_4



C_5



C_6

A path P_n for $n \geq 2$ consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots$, and $\{v_{n-1}, v_n\}$. The Path P_1 is an isolated vertex.

A graph G = (V, E) is a bipartite graph if V can be partitioned into two disjoint subsets A and B such that each edge joins a vertex in A and a vertex in B. In other words, there are no edges join two vertices in A and no edges join two vertices in B.

A complete bipartite graph, denoted $K_{m, n}$, is a graph such that its vertex set can be partitioned into subset A of size m and subset B of size n and there is an edge between each vertex in A and each vertex in B. Note that the size of $K_{m, n}$ is m*n. Below is the bipartite graph $K_{2, 3}$.