01 LOGIC

01_04 De Morgan's Laws and Their Applications

Augustus De Morgan

De Morgan's Laws

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q,$$

$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q.$$

They can be proved respectively by the truth tables below.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

One application of De Morgan's Laws

Find the negation of the following propositions.

(1) Today is Sunday and $2 + 3 = 5$.

(2) It is raining or $6 > 8$.

[Solution of (1)]

p: Today is Sunday,

q: $2 + 3 = 5$.

By De Morgan's laws,

~(Today is Sunday and $2 + 3 = 5$) = $\sim(p \wedge q) \Leftrightarrow$

$\sim p \vee \sim q$ = Today is not Sunday or $2 + 3 \mathrel{!=} 5$.

[Solution of (2)]

p: It is raining,

q: $6 > 8$.

By De Morgan's laws,

~(It is raining or $6 > 8$) = $\sim(p \vee q) \Leftrightarrow$

$\sim p \wedge \sim q$ = It is not raining and $6 \leq 8$.

One application of De Morgan's Laws

Show that $\sim(p \rightarrow q)$ and $p \wedge \sim q$ are logically equivalent.

[Proof] Recall $p \rightarrow q \Leftrightarrow \sim p \vee q$. Thus, by De Morgan's Laws, $\sim(p \rightarrow q) \Leftrightarrow \sim(\sim p \vee q) \Leftrightarrow \sim(\sim p) \wedge \sim q \Leftrightarrow p \wedge \sim q$.

One application of De Morgan's Laws

Show that $(p \wedge q) \rightarrow p$ is a tautology.

[Proof] Recall $r \rightarrow s \Leftrightarrow \sim r \vee s$. Thus, by De Morgan's Laws, $(p \wedge q) \rightarrow p \Leftrightarrow \sim(p \wedge q) \vee p \Leftrightarrow (\sim p \vee \sim q) \vee p$

$\Leftrightarrow \sim p \vee (\sim q \vee p) \Leftrightarrow \sim p \vee (p \vee \sim q) \Leftrightarrow (\sim p \vee p) \vee \sim q$

$\Leftrightarrow T \vee \sim q \Leftrightarrow T.$

Similarly, we can show that $(p \wedge q) \to q$ is a tautology.

<u>One application of De Morgan's Laws</u>

Show that $p \to p \vee q$ is a tautology.

[Proof] Recall $r \to s \Leftrightarrow \sim r \vee s$. Thus, by De Morgan's

Laws, $p \to p \vee q \Leftrightarrow \sim p \vee (p \vee q) \Leftrightarrow (\sim p \vee p) \vee q$

$\Leftrightarrow T \vee q \Leftrightarrow T.$

Similarly, we can show that $q \to p \vee q$ is a tautology.

More information on Augustus De Morgan can be found at

https://en.wikipedia.org/wiki/Augustus_De_Morgan