09 Principle of Mathematical Induction.

09_03_Additional Examples

Example. Let $P(n)$ be the statement that every set with n elements has $n(n - 1)/2$ subsets containing exactly two elements, where n is an integers and $n \geq 2$. Prove that $P(n)$ is true by using mathematical induction.

[Proof]

Step 1. $P(2)$ is true since any every set with 2 elements has one 2-element subset and $n(n - 1)/2 = 1$ when $n = 2$.

Step 2. Assume $P(n)$ is true when $n = k \geq 2$. Namely, every set with k elements has $k(k - 1)/2$ subsets containing exactly two elements, where k is an integer and $k \geq 2$. Suppose $A := \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be any set with $(k + 1)$ elements. Set $B := A - \{a_1\}$. Thus the 2-element subsets of A are 2-element subsets of B and 2-element sets of $\{a_1, a_2\}, \{a_1, a_3\}, \ldots, \{a_1, a_k\}$, and $\{a_1, a_{k+}1\}$. Since $|B| = k$, The number of 2-element subsets of B is $k(k - 1)/2$. Hence, the number of 2-element subsets of A is

$$k(k - 1)/2 + k = k(k + 1)/2 = (k + 1)(k + 1 - 1)/2.$$

So, $P(k + 1)$ is true.

By the principle of mathematical induction, we have that $P(n)$ is true.

Example. Let X and Y be two square matrices such that $XY = YX$. Prove that $XY^n = YX^n$ is true for any integer n with $n \geq 1$.

[Proof]

Let P(n) be the statement that $XY^n = YX^n$, where n is an integer with $n \geq 1$.

Step 1. P(1) is true because of the given condition of

$$XY = YX.$$

Step 2. Assume that P(n) is true for $n = k \geq 1$, i.e.,

$$XY^k = Y^kX.$$

Now

$$XY^{k+1} = XY^k Y = Y^kXY = Y^kYX = Y^{k+1}X .$$

Hence P(k + 1) is true.

By the principle of mathematical induction, we have that P(n) is true.

Example. Let P(n) be the statement that

$$1 + 1/4 + 1/9 + \ldots + 1/n^2 < 2 - 1/n,$$

where n is an integer at least 2. Prove P(n) is true for all integers at least 2.

[Proof]

Step 1. P(2) is true since $1 + 1/4 = 4/5 < 3/2 = 2 - 1/2$.

Step 2. Assume that P(n) is true for $n = k \geq 2$, i.e.,

$$1 + 1/4 + 1/9 + \ldots + 1/k^2 < 2 - 1/k.$$

Notice that

$$1/(k + 1)^2 < 1/(k(k + 1)).$$

Thus

$$1/(k + 1)^2 < 1/k - 1/(k + 1),$$

$$-1/k + 1/(k + 1)^2 < -1/(k + 1),$$

$$2 - 1/k + 1/(k + 1)^2 < 2 - 1/(k + 1).$$

Therefore

$$1 + 1/4 + 1/9 + \ldots + 1/k^2 + 1/(k + 1)^2$$

$$< 2 - 1/k + 1/(k + 1)^2$$

$$< 2 - 1/(k + 1).$$

Hence P(k + 1) is true.

By the principle of mathematical induction,

we have that P(n) is true.