12 Probability

12_01_Sample Spaces, Events, and Probability

An **experiment** is an activity of observation that results in a single outcome which cannot be predicted in advance with certainty. For example, when we roll a die and observe the number on the face of the die. Before the experiment, we know the number on the face of the die is from {1, 2, 3, 4, 5, 6}. But we cannot predicate in advance with certainty which number will be on the face of the die.

A **sample point** is the basic outcome of an experiment which cannot be decomposed into smaller ones. For example, in the experiment of rolling dice above, the sample points are the numbers on the faces of the dice which are 1, 2, 3, 4, 5, and 6.

In the experiment of tossing three coins, we observe the faces of the coins. Then the sample points of this experiment are HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT, where H and T represent HEAD and TAIL of a coin, respectively.

A **sample space** of an experiment is the set of all the possible sample points.

For example, in the experiment of rolling a die, the sample space is {1, 2, 3, 4, 5, 6}.

In the experiment of tossing three coins, the sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

An **event** in an experiment is a subset of the **sample space** of the experiment. An event occurs if an element in the event occurs in the experiment.

For example, in the experiment of rolling a die, the sets of {1, 2, 3, 4, 5, 6}, {1, 3, 5}, {5}, and Ø are subsets of the sample space of {1, 2, 3, 4, 5, 6}, so each of them is an event in the experiment. In an experiment, if the number appearing on the face of a die is 1, then we say events {1, 2, 3, 4, 5, 6} and {1, 3, 5} occur.

In the experiment of tossing three coins, the subsets of {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, {HHT, HTH, THH}, and Ø are subsets of the sample space, so each of them is an event in the experiment. If THH occurs in the experiment, then we say events of {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and {HHT, HTH, THH} occur.

Suppose A and B are two events. Then **A ⊆ B** is defined as if each outcome (or element) in A is also an outcome (or element) in B (or if A occurs, then B occurs). In this case, we say A is contained in B or B contains A.

Suppose A and B are two events. Then **A = B** is defined as both A ⊆ B and B ⊆ A.

Suppose A and B are two events. The **union** of A and B, denoted **A ∪ B**, is a set consisting of the outcomes in A or the outcomes in B. Namely, if A occurs or B occurs, then A ∪ B ocurs.

Suppose A and B are two events. The **intersection** of A and B, denoted **A ∩ B**, is a set consisting of the outcomes in both A and B. Namely, if A occurs and B occurs, then A ∩ B ocurs.

Suppose A and B are two events. The **difference** between A and B, denoted **A - B**, is a set consisting of the outcomes in A and none of the outcomes in B. Namely, if A occurs and B does not occur, then A - B ocurs.

Suppose A is an event. The **complement** of A, denoted $\mathbf{A^C}$, is a set consisting of the outcomes in the sample space but not A. Namely, if A occurs, then A^C does not occur and if A does not occur, then A^C occurs.

Let Ω be sample space of an experiment. Define a function P from a collection of events (this collection of events is a σ-field) to the set of real numbers which satisfies

[1] Nennegativity: $P(E) \geq 0$ for each event E,

[2] Certainty: $P(\Omega) = 1$,

[3] Countable Additivity: if $A_1, A_2, \ldots, A_n, \ldots$ are events such that $A_i \cap A_j = \varnothing$, where $i \neq j$, then

$$P(A_1 \cup A_2 \cup \ldots\ldots) = P(A_1) + P(A_2) + \ldots\ldots$$

Then P(E) is called the probability of the event E.

-$P(\emptyset) = 0$.

[Proof] Note that

$$\Omega = \Omega \cup \emptyset \cup \emptyset \cup \ldots\ldots \text{ and } \Omega, \emptyset, \emptyset, \ldots \text{ are pairwise disjoint.}$$

Thus $P(\Omega) = P(\Omega) + P(\emptyset) + P(\emptyset) + \ldots\ldots$

Hence $1 = 1 + P(\emptyset) + P(\emptyset) + \ldots\ldots$ and therefore $P(\emptyset) = 0$.

-If $A_1, A_2, \ldots, A_n$ are events such that $A_i \cap A_j = \emptyset$, where $i \neq j$, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

[Proof] Note that $A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \emptyset \cup \emptyset \ldots\ldots$ and

$$A_1, A_2, \ldots, A_n, \emptyset, \emptyset, \ldots\ldots \text{ are pairwise disjoint.}$$

Thus $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) + P(\emptyset) + P(\emptyset) + \ldots\ldots$

Hence $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.

-If A is an event, then $P(A^c) = 1 - P(A)$.

[Proof] Note that $\Omega = A \cup A^c$ and $A \cap A^c = \emptyset$. Then

$$1 = P(\Omega) = P(A) + P(A^c). \text{ Thus } P(A^c) = 1 - P(A).$$

-Suppose A and B are events. Then $P(A - B) = P(A) - P(A \cap B)$.

[Proof] Note that $A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$ and

$(A \cap B) \cap (A \cap B^c) = \emptyset$. Thus

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A \cap B) + P(A - B) \text{ and}$$

$$P(A - B) = P(A) - P(A \cap B).$$

-Suppose A and B are events. If $B \subseteq A$, then $P(A - B) = P(A) - P(B)$ and

$P(A) \geq P(B)$.

[Proof] Note that $A \cap B = B$ if $B \subseteq A$. Thus

$$P(A - B) = P(A) - P(A \cap B) = P(A) - P(B).$$

Since $P(A) - P(B) = P(A - B) \geq 0$, we have $P(A) \geq P(B)$.

-Suppose A is an event. Then $P(A) \leq 1$.

[Proof] Since $A \subseteq \Omega$ and $P(\Omega) = 1$, we have $P(A) \leq 1$.

-Suppose A and B are events. Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

[Proof] Note that $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$ and $(A - B)$, $(A \cap B)$, and $(B - A)$ are pairwise disjoint. Thus

$$
\begin{aligned}
P(A \cup B) &= P(A - B) + P(A \cap B) + P(B - A) \\
&= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(B \cap A) \\
&= P(A) + P(B) - P(A \cap B).
\end{aligned}
$$

Example. In the experiment of tossing three coins, we define A as the event of getting at least two heads and B as the event of getting one or two tails. Find the probabilities of the following events.

- $P(A)$,
- $P(B)$,
- $P(A^c)$,
- $P(B^c)$,
- $P(A \cap B)$,
- $P(A \cup B)$,
- $P(A - B)$,
- $P(B - A)$,
- $P(A^c \cap B^c)$,
- $P(A^c \cup B^c)$.

[Solution] The sample space Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, the event A = {HHH, HHT, HTH, THH}, and the event B = {HHT, HTH, HTT, THH, THT, TTH}. Therefore $A \cap B$ = {HHT, HTH, THH} and $A \cup B$ = {HHH, HHT, HTH, HTT, THH, THT, TTH}. Hence

- $P(A) = 4/8 = 1/2$,
- $P(B) = 6/8 = 3/4$,
- $P(A^c) = 1 – 1/2 = 1/2$,
- $P(B^c) = 1 – 3/4 = 1/4$,
- $P(A \cap B) = 3/8$,
- $P(A \cup B) = 7/8$,
- $P(A – B) = P(A) – P(A \cap B) = 1/2 - 3/8 = 1/8$,
- $P(B – A) = P(B) – P(A \cap B) = 3/4 - 3/8 = 3/8$,
- $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 7/8 = 1/8$,
- $P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 3/8 = 5/8$.