10 Recursive Sequences.

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An arithmetic sequence is a sequence of numbers

$$a_1, a_2, \ldots, a_n, \ldots$$

such that $a_i = a_1 + (i – 1)*d = a_{i-1} + d$, where i is an integer with $i \geq 2$ and d is a constant.

Note that sum of the first n terms in the sequence is

$$S_n := a_1 + a_2 + \ldots + a_n$$
$$= (a_1 + a_n)*n/2$$
$$= (2*a_1 + (n – 1)*d)*n/2$$

Example. Let $a_i = i$, where i = 1, 2, 3, … …. Then

$$a_1, a_2, \ldots, a_n, \ldots$$

form an arithmetic sequence with d = 1 and

$$S_n = a_1 + a_2 + \ldots + a_n$$
$$= 1 + 2 + 3 + ... + n$$
$$= n(n + 1)/2.$$

Example. Let $a_i = 2*i - 1$, where i = 1, 2, 3, … …. Then

$$a_1, a_2, \ldots, a_n, \ldots$$

form an arithmetic sequence with d = 2 and

$$S_n = a_1 + a_2 + \ldots + a_n$$
$$= 1 + 3 + 5 + ... + (2*n – 1)$$
$$= n(1 + 2*n - 1)/2 = n^2.$$

A geometric sequence is a sequence of numbers

$$a_1, a_2, \ldots, a_n, \ldots$$

such that $a_i = a_1 * r^{i-1} = a_{i-1} * r$, where $i \geq 2$ and r is a constant.

Note that if r = 1 the sum of the first n terms in the sequence is

$$S_n := a_1 + a_2 + \ldots + a_n$$
$$= n * a_1$$

and if $r \neq 1$ the sum of the first n terms in the sequence is

$$S_n := a_1 + a_2 + \ldots + a_n$$
$$= (a_1(r^n - 1))/(r - 1).$$

Example. Let $a_i = 2^i$, where i = 1, 2, 3, … …. Then

$$a_1, a_2, \ldots, a_n, \ldots$$

form a geometric sequence with r = 2 and

$$S_n = a_1 + a_2 + \ldots + a_n$$
$$= 2^1 + 2^2 + \ldots + 2^n$$
$$= 2(2^n - 1)/(2 - 1)$$
$$= 2^{n+1} - 2.$$

Example. Let $a_i = (1/3)^i$, where i = 1, 2, 3, … …. Then

$$a_1, a_2, \ldots, a_n, \ldots$$

form a geometric sequence with r = 1/3 and

$$S_n = a_1 + a_2 + \ldots + a_n$$
$$= (1/3)^1 + (1/3)^2 + \ldots + (1/3)^n$$
$$= (1/3)((1/3)^n - 1)/(1/3 - 1)$$

$$= (1/2)(1 - (1/3)^n).$$

<u>A recursive sequence</u> is a sequence of numbers

$$a_1, a_2, \ldots, a_n, \ldots$$

such that the ith term can be decided by the terms appearing before the ith term, where $i \geq 2$.

An arithmetic sequence is a recursive sequence.

A geometric sequence is also a recursive sequence.

<u>Example.</u> Let

$$a_1, a_2, \ldots, a_n, \ldots$$

be a recursive sequence such that

$$a_1 = 1,$$

$$a_i = 2*a_{i-1} + 3 \text{ if } i \geq 2.$$

Find a_2, a_3, a_4, and a_5.

[Solution]

$$a_2 = 2*a_1 + 3 = 2*1 + 3 = 5,$$

$$a_3 = 2*a_2 + 3 = 2*5 + 3 = 13,$$

$$a_4 = 2*a_3 + 3 = 2*13 + 3 = 29,$$

$$a_5 = 2*a_4 + 3 = 2*29 + 3 = 61.$$

<u>Example.</u> Let

$$a_1, a_2, \ldots, a_n, \ldots$$

be a recursive sequence such that

$$a_1 = 1,$$

$$a_2 = 2,$$

$$a_i = 4*a_{i-1} - 3*a_{i-2} \text{ if } i \geq 3.$$

Find a_3, a_4, and a_5.

[Solution]

$$a_3 = 4*a_2 - 3*a_1 = 4*2 - 3*1 = 5,$$

$$a_4 = 4*a_3 - 3*a_2 = 4*5 - 3*2 = 14,$$

$$a_5 = 4*a_4 - 3*a_3 = 4*14 - 3*5 = 41.$$