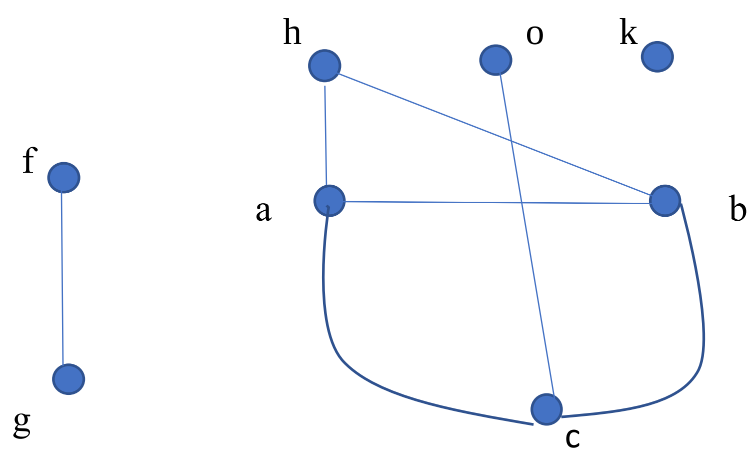
11 Graphs and Tress.

11_02_The Handshaking Theorem

Definition. The degree of a vertex u in an undirected graph G is the number of edges incident with it. We use $d_G(u)$ (or d(u) if G is clear from the context) to denote it.

Consider the following G = (V, E), where V = {a, b, c, o, f, g, h, k}, E = {ab, ac, ah, bc, bh, co, fg}.



We have d(a) = 3, d(b) = 3, d(c) = 3, d(o) = 1, d(f) = 1, d(g) = 1, d(h) = 2, d(k) = 0.

A simple calculation shows that the sum of vertex degrees is 14 which is twice the number of edges. In fact, this result is true for any graph.

Theorem 1 (the Handshaking Theorem). Let G = (V, E) be a graph. Then the sum of degrees of vertices in G is equal to twice the number of edges in G.

[Proof] In the sum of degrees of vertices in G, each edge is counted exactly two times. Thus

$$\Sigma_{v \in V} d(v) = 2|E|. \quad \blacksquare$$

A graph is k-regular if the degrees of all the vertices in the graph are equal to k.

Example. Find the number of edges in a 7-regular graph of order 18.

[Solution] By the Handshaking Theorem, we have 2|E| = 7 *18. Thus |E| = 63.

Example. Can we have a 17-regular graph of order 37?

[Solution] The answer is NO since the sum of degrees of vertices in a graph must be even, but 17*37 = 629, which is odd.

For a graph G = (V, E), we define Odd(G) = {u: u ∈ V, d(u) is odd}.

Corollary 1. Let G = (V, E) be a graph. Then |Odd(G)| is even.

[Proof] Define EVEN(G) = {u: u ∈ V, d(u) is even}. Set S:= Odd(G) and T:= EVEN(G). Then S ∪ T = V and S ∩ T = ∅. By the Handshaking Theorem, we have

$$2|E| = \Sigma_{v \in V} d(v) = \Sigma_{v \in S} d(v) + \Sigma_{v \in T} d(v).$$

Note that 2|E| and $\Sigma_{v \in T} d(v)$ are even. Thus $\Sigma_{v \in S} d(v)$ is even. Therefore |S| must even otherwise $\Sigma_{v \in S} d(v)$ cannot be even. ■

We often use matrices to represent graphs. The adjacency matrix of a graph is one of the representations.

Suppose that G = (V, E) is a graph with V = $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix, denoted $A(G) = [a_{ij}]$, of G, with respect to the above ordering of vertices, is an $n \times n$ zero-one matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ if v_i and v_j are not adjacent,

The adjacency matrix of the graph at the beginning of this section is as follows.

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} a & b & c & f & g & h & k & o \end{array} \\
\begin{array}{c} a \\ b \\ c \\ f \\ g \\ h \\ k \\ o \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Note that an adjacency matrix of a graph is symmetric zero-one matrix such that all the diagonal entries are equal to 0.

The summation of the entries in one row or column is equal to the degree of the corresponding vertex.

Each edge in a graph is corresponding to two 1's in the the adjacency matrix of the graph.

The above observations can yield another proof of the Handshaking Theorem as follows.

[Proof] Let G = (V, E) be a graph of order n. Then

2|E| = the sum of all the entries in A(G)

= the sum of all the entries in the first row in A(G) +

the sum of all the entries in the second row in A(G) +

… … … … …

the sum of all the entries in the nth row in A(G)

= the degree of the first vertex +

the degree of the second vertex +

… … … … …

the degree of the nth vertex

= the sum of degrees of the vertices in the graph G.