07_03_Matrix Multiplication

-An Algorithm for Multiplication of Two Matrices

Inputs: Matrix $A_{m \times n} = (a_{ij})$ and Matrix $B_{n \times p} = (b_{ij})$

Output: Matrix $C_{m \times p} = (c_{ij}) = A_{m \times n} B_{n \times p}$.

for i from 0 to (m – 1)

 for j from 0 to (p – 1)

  $c_{ij} = 0$

  for r from 0 to (n – 1)

   $c_{ij} = c_{ij} + a_{ir} * b_{rj}$

The Multiplication of Two Matrices

In the multiplication of Matrix $A_{m \times n}$ and Matrix $B_{n \times p}$,

-the number of matrix entry multiplications is (m*p*n),
-the number of matrix entry addition is (m*p*(n – 1)).

Thus, the cost for the multiplication of Matrix $A_{m \times n}$ and Matrix $B_{n \times p}$ is the sum of the cost of multiplying matrix entries (m*p*n) times and the cost of adding matrix entries (m*p*(n – 1)) times.

An Example

In the multiplication of Matrix $A_{7 \times 11}$ and Matrix $B_{11 \times 17}$,

[1] the number of matrix entry multiplications:

$$7*17*11 = 1309,$$

[2] the number of matrix entry additions:

$$7*17*(11 - 1) = 1190.$$

The Multiplication of Two Square Matrices

In the multiplication of Matrix $A_{n \times n}$ and Matrix $B_{n \times n}$,

-the number of matrix entry multiplications is n^3.
-the number of matrix entry addition is $(n^2(n - 1))$.

The information on designing algorithms to reduce the cost for the multiplication of two square matrices can be found at

https://en.wikipedia.org/wiki/Strassen_algorithm
https://en.wikipedia.org/wiki/Computational_complexity_of_matrix_multiplication

The Matrix-Chain Multiplication

-Let $A_{7 \times 11}$, $B_{11 \times 17}$, and $C_{17 \times 5}$ be three matrices such that all the entries in the three matrices are real numbers.

The multiplication of $A_{7 \times 11}$, $B_{11 \times 17}$, and $C_{17 \times 5}$ can be carried out in the following two ways.

[1] $((A_{7 \times 11}B_{11 \times 17})\ C_{17 \times 5})$,

[2] $(A_{7 \times 11}(\ B_{11 \times 17}\ C_{17 \times 5}))$.

The Matrix-Chain Multiplication

In $((A_{7 \times 11}B_{11 \times 17})\ C_{17 \times 5})$,

Let $X_{7 \times 17} = A_{7 \times 11}B_{11 \times 17.}$ Then $((A_{7 \times 11}B_{11 \times 17})\ C_{17 \times 5}) = (X_{7 \times 17}\ C_{17 \times 5})$.

We need 7*17*11 = 1309 real number multiplications to compute $X_{7 \times 17}$. We need 7*5*17 = 595 real number multiplications to compute $(X_{7 \times 17}\ C_{17 \times 5})$. Thus, the total numbers of real number multiplications to compute $((A_{7 \times 11}B_{11 \times 17})\ C_{17 \times 5})$ is 1904 = 1309 + 595.

The Matrix-Chain Multiplication

In $(A_{7 \times 11} (B_{11 \times 17} C_{17 \times 5}))$,

Let $Y_{11 \times 5} = B_{11 \times 17} C_{17 \times 5}$. Then $(A_{7 \times 11} (B_{11 \times 17} C_{17 \times 5})) = (A_{7 \times 11} Y_{11 \times 5})$.

We need 11*5*17 = 935 real number multiplications to compute $Y_{11 \times 5}$. We need 7*5*11 = 385 real number multiplications to compute $(A_{7 \times 11} Y_{17 \times 5})$. Thus, the total numbers of real number multiplications to compute $(A_{7 \times 11} (B_{11 \times 17} C_{17 \times 5}))$ is 1320 = 935 + 385.

The Matrix-Chain Multiplication

Hence, in terms of counting the number of real number multiplications, $((A_{7 \times 11} B_{11 \times 17})\ C_{17 \times 5})$ is more expensive than $(A_{7 \times 11}(\ B_{11 \times 17}\ C_{17 \times 5}))$.

-The question is how we multiply effectively the matrices A_1, A_2, …, and A_n of dimensions of $b_1 \times b_2$, $b_2 \times b_3$, …, $b_n \times b_{n+1}$, respectively.

More information on this question can found at

https://en.wikipedia.org/wiki/Matrix_chain_multiplication