11 Graphs and Tress.

11_04_Eulerian_Hamiltonian_Graphs

Definition. A trail in a graph G is an alternating sequence of vertices and edges $u_1\ e_1\ u_2\ e_2\ u_3\ e_3\ \ldots\ldots\ u_{n-1}\ e_{n-1}\ u_n$ such that for each i with $1 \leq i \leq (n-1)$, $e_i = u_i u_{i+1}$ and all the edges $e_1, e_2, \ldots e_{n-1}$ are distinct.

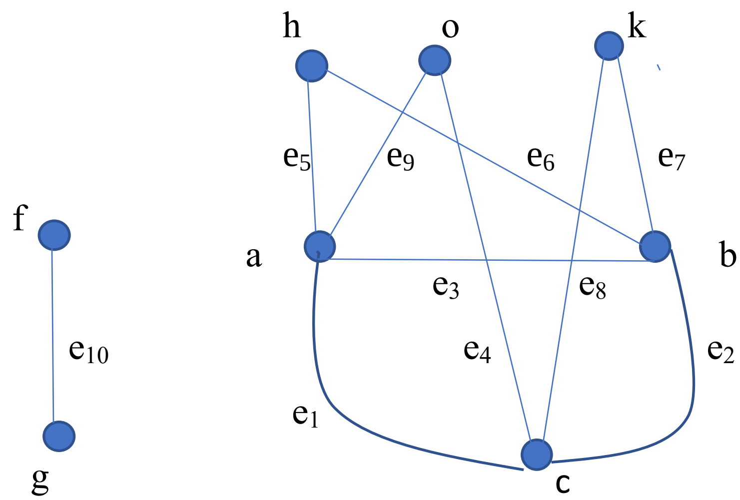
If G is a simple graph, then a trail in G can be uniquely determined by the sequence of the vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$.

Definition. A closed trail is a trail $u_1\ e_1\ u_2\ e_2\ u_3\ e_3\ \ldots u_{n-1}\ e_{n-1}\ u_n\ e_n\ u_1$ such that the initial and terminal vertices of the trail coincide.

If G is a simple graph, then a closed trail in G can be uniquely determined by the sequence of the vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n, u_1$.

Definition. An Euler tour in a graph G is a closed trail that traverses each edge in G exactly once.

Definition. A graph G is Eulerian if G has an Euler tour.



For example, T: = a e_1 c e_2 b e_7 k e_8 c e_4 o is a trail in the graph above. Since the graph above is a simple graph, we write T as a sequence of vertices. Namely, T: = a c b k c o.

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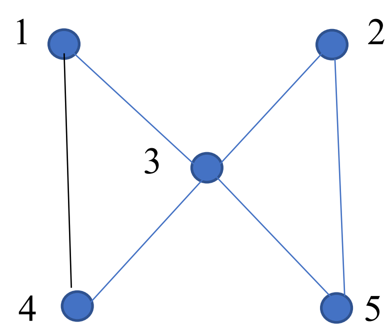

The graph above is Eulerian since it has an Euler tour of

1 3 5 2 3 4 1

<u>Theorem 1.</u> Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if the degrees of all the vertices in G are even.

[Proof] Suppose the degrees of all the vertices in a connected graph $G = (V, E)$ are even and G is not Eulerian. Clearly, $|V| \geq 2$ and $|E| \geq 1$. Let G be a graph that satisfies the above conditions and has the minimum number of edges. Next, we prove that G has a closed trail. Choose a longest path

$$P: = u_1\, u_2 \ldots u_k$$

in G. Since P is a longest path, u_1 cannot be adjacent to any vertices outside P otherwise we would have a path which is longer than P. Since the degree of u_1 in G is at least two, u_1 must adjacent to a vertex u_s, where $3 \leq s \leq k$. Thus $u_1\, u_2 \ldots u_s\, u_1$ is a closed trail. Choose a closed trail C which has the largest number of edges in G. Since C is not an Euler tour, there must be a component $H = (V(H), E(H))$ in $G - E(C)$ such that $|E(H)| \geq 1$. Notice that each vertex in C has an even degree associated with the closed trail C and each vertex in G has an even degree. Thus each vertex in H has an even degree associated with H. Clearly, $|E(H)| < |E(G)|$. Then the choice of G implies that H must have an Euler tour $C(H)$. Since G is connected, there must a vertex u is in both $V(C)$ and $V(C(H))$. Without loss of generality, we can assume that u is the initial and terminal vertices of both C and $C(H)$. Thus the combination of C and $C(H)$

yields a closed trail such that the number of edges in the newly created closed trail is greater the number of edges in the closed trail C, a contradiction.

Suppose G is Eulerian. Then there is an Eluer tour C in G which has the same initial and terminal vertex, say v. For any vertex u in C which is not the same as u, Each appearance of u in C will count exactly two edges incident with u. Notice that C traverses all edges in G. Then the degree of u must be even. Each appearance of v in C as an internal vertex will count exactly two edges incident with v. Since v is both initial and terminal vertex of C, the degree of v must be even. ■

Definition. A Hamiltonian cycle in a graph G is a cycle contains all the vertices of G.

Definition. A graph G is Hamiltonian if G has a Hamiltonian cycle.

Example. K_n with $n \geq 3$, C_n with $n \geq 3$, and $K_{n, n}$ with $n \geq 2$ are Hamiltonian. P_n with $n \geq 3$ and $K_{m, n}$ with $m \neq n$, $m \geq 2$, and $n \geq 2$ are not Hamiltonian.

The following is the famous Dirac's theorem in Graph Theory.

Theorem 2. Let $G = (V, E)$ be a graph with $|V| \geq 3$. If the degree of each vertex in G is at least $|V|/2$, then G is Hamiltonian.

[Proof] Suppose a graph G satisfies the conditions in Theorem 2 and G is not Hamiltonian.
We first prove that G is connected. Suppose, to the contrary, that G is disconnected. Then there is a connected component, say H, in G such that $|V(H)| \leq |V|/2$. This is a contradiction since under this circumstance the degree of each vertex in H cannot be at least $|V|/2$.
Choose a longest path

$$P := u_1\, u_2 \ldots u_k$$

in G. Since P is a longest path, u_1 and u_k cannot be adjacent to any vertices outside P otherwise we would have paths which are longer than P. Define

$$S := \{u_i : u_1u_i \in E\}, \quad T := \{u_j : u_{j-1}u_k \in E\}$$

Then we claim that $S \cap T = \emptyset$.
Suppose, to the contrary, that $S \cap T \neq \emptyset$. Then there exist a vertex u_s such that $u_s \in S \cap T$. Thus $u_1u_s \in E$ and $u_{s-1}u_k \in E$. Thus G has a cycle

$$C := u_1\, u_s\, u_{s+1}\, u_{s+2} \ldots u_k\, u_{s-1}\, u_{s-2} \ldots u_1.$$

Notice that G is not Hamiltonian, Thus $V - V(C) \neq \emptyset$.
Since G is connected, there exists a vertex $w \in V - V(C)$ and a vertex u_t on P such that $wu_t \in E$. Hence G has a path which is longer than P, a contradiction.

Now

$$
\begin{aligned}
|V| &= |V|/2 + |V|/2 \le d(u_1) + d(u_k) = |S| + |T| \\
&= |S \cup T| + |S \cap T| = |S \cup T| \\
&\le |V(P) - \{u_1\}| = |V(P)| - 1 \le |V| - 1,
\end{aligned}
$$

a contradiction. ■