06 Pigeonhole Principle

06_03 The Applications of Pigeonhole Principle

Example. Show that among any group of 8 integers, we can always find two of them such that their difference is a multiple of 7.

[Proof] For any integer n, when it is divided by 7, the possible remainders are 0, 1, 2, 3, 4, 5, and 6. From Pigeonhole Principle, we can always find two integers, say a and b, among the eight integers such that we get the same remainder, say r, when a and b are divided by 7. Thus we have two integers s and t such that $a = 7s + r$ and $b = 7t + r$. Hence $a - b = 7s + r - 7t - r = 7(s - t)$. So, $7|(a - b)$. The proof is complete.

The above example can be further generalized as follows.

Example. Let k be a positive integer. Show that among any group of $(k + 1)$ integers, we can always find two of them such that their difference is a multiple of k.

[Proof] For any integer n, when it is divided by k, the possible remainders are 0, 1, 2, 3, … , and $(k - 1)$. From Pigeonhole Principle, we can always find two integers, say a and b, among the $(k + 1)$ integers such that we get the same remainder, say r, when a and b are divided by k. Thus we have two integers s and t such that $a = ks + r$

and $b = kt + r$. Hence $a - b = ks + r - kt - r = k(s - t)$.
So, $k|(a - b)$. The proof is complete.

Example. Show that choose any 5 points in a square with edge length 2, we can always find 2 of them such that their distance is less than or equal to $\sqrt{2}$.
[Proof] Cut the square with edge length 2 into four smaller square such that the edge length of each of them is 1. This is possible by using the middle horizontal line and middle vertical line for the square. From Pigeonhole Principle, we can always find two points such that they are in the same square with unit length 1. Note that the distance between any pair of points in a square with edge length 1 is less than or equal to $\sqrt{2}$. Thus we can always find 2 points such that their distance is less than or equal to $\sqrt{2}$. Therefore, the proof is complete.

Example. Suppose that the relationship between each pair of persons is either friend or enemy. Show that among any group of 6 persons, we can always find 3 of them such that the three persons are either mutual friends or mutual enemies.
[Proof] We use 1, 2, 3, 4, 5, and 6 to denote the 6 persons. The relationships between Person 1 and Persons 2, 3, 4, 5, and 6 are either friend or enemy. From Pigeonhole Principle, Person 1 has three friends among Persons 2, 3, 4, 5, and 6 or Person 1 has three enemies among Persons 2, 3,4, 5, and 6.

-In the first case, without loss of generality, we assume that Person 1 has three friends Persons 2, 3, and 4.

- -If Persons 2 and 3 are friends, then Persons 1, 2, and 3 are mutual friends.
- -If Persons 3 and 4 are friends, then Persons 1, 3, and 4 are mutual friends.
- -If Persons 4 and 2 are friends, then Persons 1, 4, and 2 are mutual friends.
- -If any subcase above does not happen, Then Persons 2, 3, and 4 are mutual enemies.

-In the second case, without loss of generality, we assume that Person 1 has three enemies Persons 4, 5, and 6.

- -If Persons 4 and 5 are enemies, then Persons 1, 4, and 5 are mutual enemies.
- -If Persons 5 and 6 are enemies, then Persons 1, 5, and 6 are mutual enemies.
- -If Persons 6 and 4 are enemies, then Persons 1, 6, and 4 are mutual friends.
- -If any subcase above does not happen, Then Persons 4, 5, and 6 are mutual friends.

Therefore, the proof is complete.