09 Principle of Mathematical Induction.

09_01_Principle of Mathematical Induction

The Principle of Mathematical Induction. To prove P(n), a statement involving an integer n and having its initial value n_0, is true for all integers n such that $n \geq n_0$, we need to complete the following two steps.

Step 1 (or Basic Step): We prove $P(n_0)$ is true.

Step 2 (or Inductive Step): We prove that P(k + 1) is true under the assumption that P(k) is true, where $k \geq n_0$.

Next, we use LHS and RHS to represent respectively Left-Hand Side and Right-Hand Side.

Example. Let P(n) be the statement $1 + 2 + 3 + ... + n = n(n + 1)/2$, where n is an integer and $n \geq 1$. Prove that P(n) is true for all integers at least 1.

[Proof]

Step 1. P(1) is true since LHS of P(1) = 1 = 1*(1 + 1)/2 = RHS of P(1).

Step 2. Assume that P(n) is true for $n = k \geq 1$, i.e.,

$$1 + 2 + 3 + ... + k = k(k + 1)/2, \text{ where } k \geq 1.$$

Now

$$\text{LHS of } P(k + 1) – \text{RHS of } P(k + 1)$$

$$= 1 + 2 + 3 + ... + k + (k + 1) – (k + 1)(k + 1 + 1)/2$$

$$= k(k + 1)/2 + (k + 1) – (k + 1) (k + 2)/2$$

$$= (k + 1)[k/2 + 1 – (k + 2)/2]$$

$$= (k + 1)[k/2 + 2/2 – (k + 2)/2]$$

$$= (k + 1) [(k + 2)/2 – (k + 2)/2]$$

$$= (k + 1) * 0/2 = 0.$$

Thus LHS of P(k + 1) = RHS of P(k + 1). Hence P(k + 1) is true.

By the principle of mathematical induction, we have that P(n):

$1 + 2 + 3 + ... + n = n(n + 1)/2$, where $n \geq 1$, is true.

Example. Let P(n) be the statement $1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$, where n is an integer and $n \geq 1$. Prove that P(n) is true for all integers at least 1.

[Proof]

Step 1. P(1) is true since LHS of P(1) = $1^2 = 1*(1 + 1)(2*1 + 1)/6$

= RHS of P(1).

Step 2. Assume that P(n) is true for $n = k \geq 1$, i.e.,

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Now

$$\text{LHS of } P(k + 1) = 1^2 + 2^2 + \ldots + k^2 + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{k+1}{6}(k(2k+1)+6(k+1)) = \frac{k+1}{6}(2k^2+7k+6)$$

$$= \frac{k+1}{6}(k+2)(2k+3) = \frac{(k+1)(k+2)(2k+3)}{6}$$

= RHS of P(k + 1).

By the principle of mathematical induction, we have that P(n):

$1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$, where $n \geq 1$, is true.