03 Counting Methods

03_04_the Product Rule

Example.

[1] Let A be the set of bit strings of length 8 starting with 01.

$$|A| = 2^6 = 64.$$

[2] Let B be the set of bit strings of length 8 ending with 110.

$$|A| = 2^5 = 32.$$

[3] Let C be the set of bit strings of length 8 starting with 01 and ending with 110.

$$|A \cap B| = 2^3 = 8.$$

[4] Let D be the set of bit strings of length 8 staring with 01 or ending with 110.

$$|A \cup B| = |A| + |B| - |A \cap B| = 2^6 + 2^5 - 2^3 = 88.$$

Example. A string based on an alphabet of a finite size n is a palindrome if its reversal is identical to itself.

Find the number of palindromes of length k based on an alphabet of a finite size n.

[Solution]

If k is even, a palindrome must be in the following form

$$x_1x_2 \ldots x_{k/2}\, x_{k/2} \ldots x_2x_1.$$

Note that x_i, where $1 \le i \le k/2$, can be any element in the alphabet of size n.

Thus, the number of palindromes of length k based on an alphabet of a finite size n is $n^{k/2} = n^{\{k/2\}}$.

If k is odd, a palindrome must be in the following form

$$x_1x_2 \ldots x_{[k/2]}\, y\, x_{[k/2]} \ldots x_2x_1.$$

Note that y and x_i, where $1 \leq i \leq [k/2]$, can be any element in the alphabet of size n. Thus, the number of palindromes of length k based on an alphabet of a finite size n is $n^{[k/2]+1} = n^{\{k/2\}}$.

Combining the two cases above, we have that the number of palindromes of length k based on an alphabet of a finite size n is $n^{\{k/2\}}$.

<u>Example.</u> Find the number of factors of 360.

[Solution] $360 = 2^3*3^2*5^1$. Then each factor of 360 is of form $2^i*3^j*5^k$, where i, j and k are integers with $0 \leq i \leq 3$, $0 \leq j \leq 2$, and $0 \leq k \leq 1$. Thus, the number of factors, including 1 and 360 itself, of 360 is 4*3*2 = 24.

<u>The Fundamental Theorem of Arithmetic</u> ensures that every positive integer n that is at least 2 can be written uniquely as a prime or as the product of a collection of primes, where the primes are arranged in order of nondecreasing size. Namely,

$$n = (p_1)^{k1} *(p_2)^{k2}*\ldots* (p_s)^{ks},$$

where $p_1, p_2, \ldots, p_s$ are primes such that $p_1 < p_2 < \ldots < p_s$. Then the number of factors of n is $(k_1 + 1)*(k_2 + 1)*\ldots*(k_s + 1)$.

Example. How many functions are there from a set with 4 elements to a set with 3 elements.
[Solution] Each element in the first set can be mapped to any elements in the second set. Thus, the number of functions from a set with 4 elements to a set with 3 elements is $3^4 = 81$.

In general, the number of functions from a finite set A to a finite set B is $|B|^{|A|}$.

Example. How many one-to-one functions are there from a set with 3 elements to a set with 4 elements.
[Solution] Let a_1, a_2, and a_3 be the elements in the first set. The element a_1 can be mapped to any 4 elements in the second set. Since the functions are one-to-one, the element a_2 can be mapped to any 3 elements that don't possess preimages in the second set. The element a_3 can be mapped to any 2 elements that don't have preimages in the second set. Thus, the number of the one-to-one functions from a set with 3 elements to a set with 4 elements is $4*3*2 = 24$.

In general, the number of one-to-one functions from a finite set A to a finite set B is

$$|B|*(|B| - 1)*(|B| - 2)*\ldots*(|B| - |A| + 1) =$$

$$|B|!/(|B| - |A|)!,$$

where $|A| \leq |B|$.