# 03 Counting Methods

## 03_01_the Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion for two sets.

Suppose A and B are two finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In addition, if U is a finite set and A, B, and S are subsets, then we have

$$|U| = |S \cup S^c| = |S| + |S^c| - |S \cap S^c| = |S| + |S^c|,$$

$$|A^c \cap B^c| = |(A \cup B)^c| = |U - (A \cup B)| = |U| - |A \cup B|.$$

We also have

$$|A - B| = |A - A \cap B| = |A| - |A \cap B|,$$

$$|B - A| = |B - B \cap A| = |B| - |B \cap A|,$$

$$|A \oplus B| = |(A - B) \cup (B - A)|$$

$$= |A - B| + |B - A| - |(A - B) \cap (B - A)|$$

$$= |A - B| + |B - A|$$

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Example. Suppose $|U| = 37$, $|A| = 17$, $|B| = 11$, and $|A \cap B| = 7$.

Find $|A \cup B| = |A| + |B| - |A \cap B| = 17 + 11 - 7 = 21$,

$$|A - B| = |A - A \cap B| = |A| - |A \cap B| = 17 - 7 = 10.$$

$$\neq |A| - |B|,$$

$$|B - A| = |B - B \cap A| = |B| - |B \cap A| = 11 - 7 = 4$$

$$\neq |B| - |A|,$$

$$|A \oplus B| = |(A - B) \cup (B - A)|$$

$$= |A - B| + |B - A| = 10 + 4 = 14, \text{ and}$$

$$|A^c \cap B^c| = |(A \cup B)^c| = |U - (A \cup B)| = |U| - |A \cup B|$$
$$= 37 - 21 = 16.$$

<u>Example.</u> At a university, there are 2000 seniors. Among them, 1024 are taking a course in business, 777 are taking a course in computer science, and 512 are taking courses in both business and computer science. Find the number of students who are taking a course either in business or in computer science, Find also the number of students who are taking courses neither in business nor in computer science.

[Solution] Set U := the set of the seniors at the university.

Then $|U| = 2000$.

Set A := the set of the seniors who are taking a course in business. Then $|A| = 1024$.

Set B := the set of the seniors who are taking a course in computer science. Then $|B| = 777$.

Thus $A \cap B$ is a set of the seniors who are taking courses in both business and computer science and $|A \cap B| = 512$.

Note also that $A \cup B$ is a set of seniors who are taking courses either in business or in computer science and $A^c \cap B^c = (A \cup B)^c$ is a set of seniors who are taking courses neither in business nor in computer science.

$|A \cup B| = |A| + |B| - |A \cap B| = 1024 + 777 - 512 = 1289$,

$|A^c \cap B^c| = |(A \cup B)^c| = |U| - |A \cup B| = 2000 - 1289 = 711$.

The Principle of Inclusion and Exclusion for three sets.

Suppose A, B, and D are three finite sets. Then

$$|A \cup B \cup D| = |A| + |B| + |D| - |A \cap B| - |B \cap D| - |D \cap A| + |A \cap B \cap D|.$$

In addition, if U is a finite set and A, B, and D are subsets, then we have

$$|A^c \cap B^c \cap D^c| = |(A \cup B \cup D)^c| = |U - (A \cup B \cup D)| = |U| - |A \cup B \cup D|.$$

Example. At a university, there are 2000 seniors. Among them, 1024 are taking a course in business, 777 are taking a course in computer science, 555 are taking a course in mathematics. Furthermore, 512 are taking courses in both business and computer science. 380 are taking courses in both computer science and mathematics, 300 are taking courses in both mathematics and business, and 200 are taking courses in computer science, mathematics, and business. Find the number of seniors who are taking courses in mathematics, business, or computer science. Find also the number of seniors who are not taking any courses in business, computer science, or mathematics.

[Solution] Set U := the set of the seniors at the university.

Then $|U| = 2000$.

Set A := the set of the seniors who are taking a course in business. Then $|A| = 1024$.

Set B := the set of the seniors who are taking a course in computer science. Then $|B| = 777$.

Set D := the set of the seniors who are taking a course in mathematics. Then $|D| = 555$.

Thus $A \cap B$ is a set of the seniors who are taking courses in both business and computer science and $|A \cap B| = 512$;

$B \cap D$ is a set of the seniors who are taking courses in both computer science and mathematics and $|B \cap D| = 380$;

$D \cap A$ is a set of the seniors who are taking courses in both mathematics and business and $|D \cap A| = 300$;

$A \cap B \cap D$ is a set of the seniors who are taking courses in computer science, mathematics, and business and $|A \cap B \cap D| = 200$.

Note also that $A \cup B \cup D$ is a set of seniors who are taking courses in business, computer science, or mathematics and $(A \cup B \cup D)^c$ is a set of seniors who are not taking any courses in business, computer science, or mathematics. Therefore

$|A \cup B \cup D| = |A| + |B| + |D| - |A \cap B| - |B \cap D| - |D \cap A| + |A \cap B \cap D| = 1024 + 777 + 555 – 512 – 380 – 300 + 200 = 1364$.

$|(A \cup B \cup D)^c| = |U – (A \cup B \cup D )|= |U| - |A \cup B \cup D| = 2000 – 1364 = 636$.