04 Counting Methods

04_01_Permutations

Permutation. Let S be a set of $\{x_1, x_2, \ldots, x_n\}$. A r-permutation based on S is a string P of length r, where $0 \le r \le n$, such that

[1] All the elements in P are from S.

[2] The elements in P are pairwise distinct.

The total number of r-permutations based S is denoted by P(n, r).

Example. Let S be {1, 2, 3, 4}. Then the 0-permutation based S is the empty string. All the 1-permutations based on S are 1, 2, 3, 4. All the 2-permutations based on S are 12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43. All the 3-permutations based on S are 123, 132, 213, 231, 312, 321, 124, 142, 214, 241, 412, 421, 134, 143, 314, 341, 413, 431, 234, 243, 324, 342, 423, 432. All the 4-permutations based on S are 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321.

Thus P(4, 0) = 1, P(4, 1) = 4, P(4, 2) = 12, P(4, 3) = 24, and P(4, 4) = 24.

Using Product Rule, we can prove that

$$P(n, r) = n*(n - 1)*\ldots*(n - r + 1).$$

We define $P(n, n) := n! := n*(n - 1)*\ldots*3*2*1.$

n! is read as n factorial.

Note that P(n, 0) = 1 and 0! = 1.

Then we have

$$P(n, r) = n!/(n - r)!.$$

Example. List all the 2-permutations and 3-permutations based on {x, y, z}.

[Solution]. All the 2-permutations based on {x, y, z} are xy, yx, xz, zx, yz, zy. All the 3-permutations based on {x, y, z} are xyz, xzy, yxz, yzx, zxy, zyx.

Example. Find P(17, 5).

[Solution] $P(17, 5) = 17*16*15*14*13 = 742{,}560$.

Example. How many ways are there to select a first scholarship winner, a second scholarship winner, a third scholarship winner from 17 applicants who have applied for the scholarship?

[Solution]. $P(17, 3) = 17*16*15 = 4080$.

Example. Suppose that a salesperson is assigned to visit ten different cities. The salesperson can start with any one of the ten cities. How many possible visitation plans are there?

[Solution]. $P(10, 10) = 10! = 10*9*8*7*6*5*4*3*2*1 =$

$= 3{,}628{,}800.$

Example. How many permutations of {a, b, c, d, e, f} begin with e.

[Solution]. $P(5, 5) = 5! = 120$.