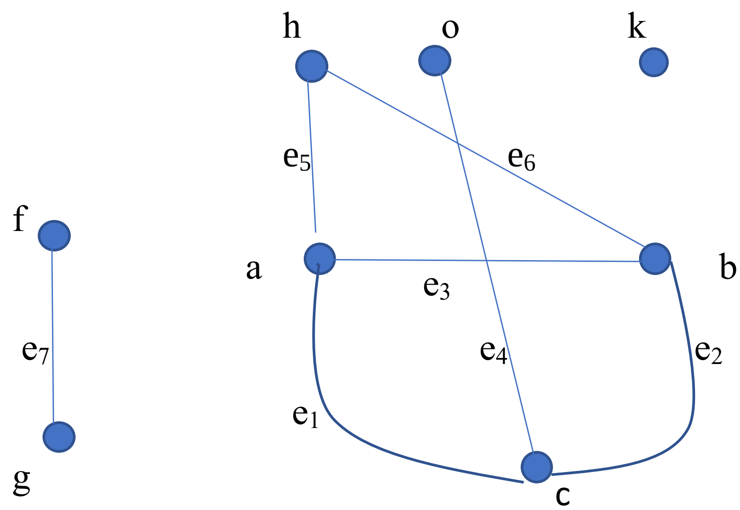
11 Graphs and Tress.

11_03_Trees

Definition. A path is an alternating sequence of vertices and edges $u_1\ e_1\ u_2\ e_2\ u_3\ e_3 \ldots\ldots u_{n-1}\ e_{n-1}\ u_n$ such that for each i with $1 \le i \le (n-1)$, $e_i = u_i u_{i+1}$ and all the vertices $u_1, u_2, \ldots u_{n-1}, u_n$ are distinct.



For example, a e_1 c e_2 b e_6 h and f e_7 g are paths in the graph above.

Definition. A cycle is a path such that the initial and terminal vertices of the path coincide.

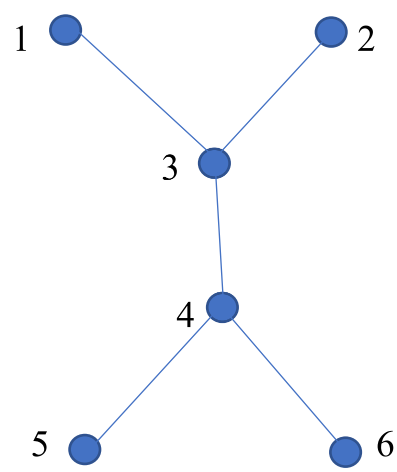
For example, a e_1 c e_2 b e_6 h e_5 a and c e_2 b e_3 a e_1 c are cycles in the graph above.

Definition. A graph is connected if every pair of vertices is connected by a path in the graph. If a graph is not connected, then we say the graph is disconnected.

For example, the first graph in this section is disconnected. The graphs such as C_{17}, P_{11}, K_7, and $K_{6,7}$ are connected.

<u>Definition.</u> A tree is a connected graph that does not have any cycle.

For example, the first graph in this section is not a tree. P_6 and the following graph are trees.



<u>Theorem 1.</u> Let G = (V, E) be a tree of order n. Then the number of edges in G is (n – 1).

[Proof] We will prove Theorem 1 by mathematical induction on the number of vertices.

P(n): the number of edges in a tree of order n is (n – 1), where $n \geq 1$.

$P(n)$ is true when $n = 1$ since now the tree is P_1 and and the number of edges in P_1 is $0 = (n - 1)$.

Suppose $P(n)$ is true for $n = k$, Namely, the number of edges in a tree of order k is $(k - 1)$, where $k \geq 1$.

We need to prove $P(n)$ is true for $n = (k + 1)$, Namely, we need to prove that the number of edges in a tree of order $(k + 1)$ is k, where $k \geq 1$.

Since the tree of order 2 is P_2 and the number of edges in P_2 is $1 = (2 - 1)$. Thus $P(2)$ is true. We now assume that $k \geq 2$. Let $T = (V, E)$ be a tree of order $(k + 1)$. Choose a longest path $P[u, v]$ in T, where u and v are the two end vertices of the path $P[u, v]$.

We claim that $d(u) = 1$. Suppose, to the contrary, $d(u) \geq 2$. Since $P[u, v]$ is a longest path in T, u is not adjacent to any vertex outside the path $P[u,v]$ otherwise we have another path in T which is longer than $P[u, v]$. Thus u is adjacent to at least two vertices on $P[u, v]$. This implies T has a cycle, a contradiction.

Now construct a graph $T_1 = (V_1, E_1)$ that is obtained from T by deleting vertex u and the edge incident with u. Clearly, T_1 is a tree of order k. Thus $|E_1| = |V_1| - 1$. Note that $|E| = |E_1| + 1$ and $|V| = |V_1| + 1$. Hence $|E| = |V| - 1 = k$.

By the principle of mathematical induction, $P(n)$ is true. So the proof of Theorem 1 is complete. ∎

Example. Suppose $T = (V, E)$ is a tree and the sum of degrees of vertices in T is 10.

[1] Find the number of edges of T.

[2] Find the number of vertices of T.

[3] Construct a tree satisfying all the conditions above.

[4] Find the adjacency matrix of the tree constructed in [3].

[Solution]

[1] From the Handshaking Theorem, we have

$2|E|$ = the sum of degrees of vertices in $T = 10$.

Thus $|E| = 5$.

[2] Since T is a tree, we, by Theorem 1 in this section, have $|V| - 1 = |E|$. Hence $|V| = 6$.

[3] The second graph in this section is a tree satisfying all the conditions. Other trees such as P_6 are also correct answers for this question.

[4] The adjacency matrix of the tree constructed in [3] is as follows.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$