06 Pigeonhole Principle

06_04 The Applications of Pigeonhole Principle

Example. Show that among any group of n persons with $n \geq 2$, there are always two persons who have the same number of friends.

[Proof] For any person p in the group, the numbers of the friends that p has are 0, 1, 2, …, (n – 1).

If one person in the group has 0 friends, then the numbers of the friends that any p has are 0, 1, 2, …, (n – 2). Since there are n persons, by Pigeonhole Principle, there are always two persons who have the same number of friends.

If one person in the group has (n – 1) friends, then the numbers of the friends that any p has are 1, 2, …, (n – 1). Since there are n persons, by Pigeonhole Principle, there are always two persons who have the same number of friends.

The last case is that any person in the group does not have 0 or (n – 1) friends. then the numbers of the friends that any p has are 1, 2, …, (n – 2). Since there are n persons, by Pigeonhole Principle, there are always two persons who have the same number of friends.

So, the proof is complete.

Example. During one 60-day time period, one table tennis player plays at least one game on each day, but no more than 90 games. Prove that there must be consecutive days in which the player will play 29 games.

[Proof] Let g_i be the total number of games that table tennis player plays from the first day to the ith day. Then, by the given conditions, we have

$$1 \leq g_1 < g_2 < \ldots < g_{60} \leq 90 \text{ and}$$

$$30 \leq g_1 + 29 < g_2 + 29 < \ldots < g_{60} + 29 \leq 119.$$

By Pigeonhole Principle, there exists two distinct integers i and j such that $g_i = g_j + 29$. Namely, $g_i - g_j = 29$. By the definitions of g_i and g_j, that player must play 29 games from Day (j + 1) to Day i.

So, the proof is complete.

Example. Show that in any sequence of m*n + 1 distinct integers there exists a strictly increasing subsequence of integers of length m + 1 or there exists a strictly decreasing subsequence of integers of length n + 1.

[Proof] Suppose, to the contrary, that the claim is not true. Let $i_1, i_2, \ldots, i_{m*n+1}$ be the distinct integers. For each i_r, where $1 \leq r \leq m*n + 1$, we define an ordered pair (a_r, b_r), where a_r is the length of the longest strictly increasing subsequence of integers beginning with i_r and b_r is the length of the longest strictly decreasing subsequence of integers beginning with i_r. Then $1 \leq a_r \leq m$, $1 \leq b_r \leq m$ for each r with $1 \leq r \leq m*n + 1$.

Thus the number of ordered pairs we can formed is less than or equal to m*n. Notice that we have m*n + 1 integers in the given sequence. By the Pigeonhole Principle, we have integers u and v such that $u < v$ and $(a_u, b_u) = (a_v, b_v)$. Thus $a_u = a_v$ and $b_u = b_v$. Now we have the following two possible cases.

<u>Case 1. $i_u < i_v$.</u>

Combining i_u with the longest strictly increasing subsequence of integers beginning with i_v, we obtain a strictly increasing subsequence of integers beginning with i_u. Thus $a_u \geq a_v + 1$, a contradiction.

<u>Case 1. $i_u > i_v$.</u>

Combinin i_u with the longest strictly decreasing subsequence of integers beginning with i_v, we obtain a strictly decreasing subsequence of integers beginning with i_u. Thus $b_u \geq b_v + 1$, a contradiction.

So, the proof is complete.