10 Recursive Sequences.

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Theorem. Let

$$a_1, a_2, \ldots, a_n, \ldots$$

be a recursive sequence such that

$$a_r = c_1*a_{r-1} + c_2*a_{r-2} + \cdots + c_t*a_{r-t}.$$

Then the following equation

$$x^t - c_1*x^{t-1} - c_2*x^{t-2} - \cdots - c_t = 0$$

is the characteristic equation of the sequence.

Suppose the characteristic equation has s distinct roots

$x_1, x_2, \ldots, x_s$ with multiplicities k1, k2, …, ks, respectively,

where ki ≥ 1 for each i with 1 ≤ i ≤ s and k1 + k2 + ⋯ + ks = t.

Then

$$a_i = (p_{1,0}*i^0 + p_{1,1}*i^1 + \cdots + p_{1,k1-1}*i^{k1-1})*(x_1)^i +$$

$$(p_{2,0}*i^0 + p_{2,1}*i^1 + \cdots + p_{2,k2-1}*i^{k2-1})*(x_2)^i +$$

$$\cdots\ \cdots\ \cdots\cdots\cdots\cdots$$

$$(p_{s,0}*i^0 + p_{s,1}*i^1 + \cdots + p_{s,ks-1}*i^{ks-1})*(x_s)^i,$$

where $p_{ui,\,vi}$ can be decided by initial conditions, where

1 ≤ ui ≤ s, 0 ≤ vi ≤ ki, and i = 1, 2, …, s.

Example. Consider the sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

where

$$a_1 = 1,$$

$a_2 = 2,$

$a_3 = 3,$

$a_i = 3*a_{i-1} - 3*a_{i-2} + a_{i-3}$ if $i \geq 4$.

The characteristic equation of the sequence is

$x^3 - 3x^2 + 3x - 1 = 0.$

$(x - 1)^3 = 0.$

Thus,

$x_1 = x_2 = x_3 = 1.$

This equation has a root 1 with multiplicity of 3.

Therefore

$$a_n = c_0 1^n + c_1 1^n n + c_2 1^n n^2$$

Solving the simultaneous equations

$1= a_1 = c_0 + c_1 + c_2$

$2= a_2 = c_0 + 2c_1 + 4c_2$

$3 = a_3 = c_0 + 3c_1 + 9c_2$

we have that

$c_0 = 0$, $c_1 = 1$, and $c_2 = 0$.

Hence $c_n = n$.

Example. Consider the sequence

$a_1, a_2, \ldots, a_n, \ldots$

where

$a_1 = 1,$

$a_2 = 2,$

$a_3 = 3,$

$a_i = -6*a_{i-1} - 12*a_{i-2} - 8*a_{i-3}$ if $i \geq 4$.

The characteristic equation of the sequence is

$$x^3 + 6x^2 + 12x + 8 = 0.$$

$$(x + 2)^3 = 0.$$

Thus,

$$x_1 = x_2 = x_3 = -2.$$

This equation has a root -2 with multiplicity of 3.

Therefore

$$a_n = c_0 (-2)^n + c_1 (-2)^n n + c_2 (-2)^n n^2$$

Solving the simultaneous equations

$$1 = a_1 = -2c_0 - 2c_1 - 2c_2$$

$$2 = a_2 = 4c_0 + 8c_1 + 16c_2$$

$$3 = a_3 = -8c_0 - 24c_1 - 72c_2$$

we have that

$$c_0 = -54/16,\ c_1 = 61/16, \text{ and } c_2 = -15/16.$$

Hence $c_n = (-54 + 61n – 15n^2) (-2)^{n - 4}$.

<u>Example.</u> Consider the sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

where

$$a_i = 18*a_{i-2} - 81*a_{i-4} \text{ if } i \geq 5.$$

The characteristic equation of the sequence is

$$x^4 - 18x^2 + 81 = 0.$$

$$(x^2 - 9)^2 = 0.$$

Thus,

$$x_1 = x_2 = -3;\ x_3 = x_4 = 3.$$

This equation has a root -2 with multiplicity of 4.

Therefore

$$a_n = c_0 (-3)^n + c_1 n (-3)^n + c_2 3^n + c_3 n 3^n.$$