02 Set Theory

02_04_Set_Identities

Common identities

$$A \cup \emptyset = A, \ A \cap U = A,$$

$$A \cup U = U, \ A \cap \emptyset = \emptyset,$$

$$A \cup A = A, \ A \cap A = A,$$

$$((A \cap B) \cap C) \ = (A \cap (B \cap C)),$$

$$((A \cup B) \cup C) \ = (A \cup (B \cup C)),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

De Morgan's laws.

$$(A \cap B)^c = A^c \cup B^c, \ (A \cup B)^c = A^c \cap B^c.$$

Prove that $(A^c)^c = A$.

[Proof] Let S be a set. First, we have the following observations.

If $x \in S$, then $x \notin S^c$ (1)

If $x \notin S$, then $x \in S^c$ (2)

If $x \in S^c$, then $x \notin S$ (3)

If $x \notin S^c$, then $x \in S$ (4)

We first prove that $A \subseteq (A^c)^c$.

For each $x \in A$, we, by (1), have that $x \notin A^c$. From (2) with $S = A^c$, we have that $x \in (A^c)^c$. Thus $A \subseteq (A^c)^c$.

Next, we prove that $(A^c)^c \subseteq A$.

For each $x \in (A^c)^c$, we, from (3) with $S = A^c$, have that $x \notin A^c$. we, by (4), have that $x \in A$. Thus $(A^c)^c \subseteq A$. Therefore $(A^c)^c = A$.

<u>Prove that $A - B = A \cap B^c$.</u>

[Proof] We first prove that $A - B \subseteq A \cap B^c$.

For each $x \in A - B$, we have $x \in A$ and $x \notin B$. Namely, $x \in A$ and $x \in B^c$. Thus $x \in A \cap B^c$. Therefore $A - B \subseteq A \cap B^c$.

Next, we prove that $A \cap B^c \subseteq A - B$.

For each $x \in A \cap B^c$, we have that $x \in A$ and $x \in B^c$. Namely, $x \in A$ and $x \notin B$. Thus $x \in A - B$. Therefore $A \cap B^c \subseteq A - B$.

Hence $A - B = A \cap B^c$.

<u>Prove that $(A \cap B)^c = A^c \cup B^c$.</u>

[Proof] We first prove that $(A \cap B)^c \subseteq A^c \cup B^c$.

For each $x \in (A \cap B)^c$, $x \notin A \cap B$. Thus $x \notin A$ or $x \notin B$. Therefore $x \in A^c$ or $x \in B^c$. Hence $x \in A^c \cup B^c$. So $(A \cap B)^c \subseteq A^c \cup B^c$.

Next, we prove that $A^c \cup B^c \subseteq (A \cap B)^c$.

For each $x \in A^c \cup B^c$, $x \in A^c$ or $x \in B^c$. Thus $x \notin A$ or $x \notin B$. Therefore $x \notin A \cap B$. Hence $x \in (A \cap B)^c$. So $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore $(A \cap B)^c = A^c \cup B^c$.

Prove that $(A \cup B)^c = A^c \cap B^c$.

[Proof] Applying the above identity by replacing A by A^c and replacing B by B^c, we have $(A^c \cap B^c)^c = (A^c)^c \cup (B^c)^c$. Thus $(A^c \cap B^c)^c = A \cup B$. Therefore $((A^c \cap B^c)^c)^c = (A \cup B)^c$. Hence $A^c \cap B^c = (A \cup B)^c$.

Exercise

Prove that $(A \cap B)^c = A^c \cup B^c$ by using $(A \cup B)^c = A^c \cap B^c$.

Prove that $A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y)$.

[Proof] We first prove that

$$A \cap (X \cup Y) \subseteq (A \cap X) \cup (A \cap Y).$$

For each $x \in A \cap (X \cup Y)$, we have $x \in A$ and $x \in (X \cup Y)$.

Namely, we have $x \in A$ and $x \in X$ or $x \in A$ and $x \in Y$.

If $x \in A$ and $x \in X$, then $x \in A \cap X \subseteq (A \cap X) \cup (A \cap Y)$.

If $x \in A$ and $x \in Y$, then $x \in A \cap Y \subseteq (A \cap X) \cup (A \cap Y)$.

Thus $x \in (A \cap X) \cup (A \cap Y)$. Hence

$$A \cap (X \cup Y) \subseteq (A \cap X) \cup (A \cap Y).$$

Next, we will prove that

$$(A \cap X) \cup (A \cap Y) \subseteq A \cap (X \cup Y).$$

For each $x \in (A \cap X) \cup (A \cap Y)$, we have that $x \in (A \cap X)$ or $x \in (A \cap Y)$.

If $x \in (A \cap X)$, then $x \in (A \cap X) \subseteq A \cap (X \cup Y)$.

If $x \in (A \cap Y)$, then $x \in (A \cap Y) \subseteq A \cap (X \cup Y)$.

Thus $x \in A \cap (X \cup Y)$. Hence

$$(A \cap X) \cup (A \cap Y) \subseteq A \cap (X \cup Y).$$

(Alternatively, since $(A \cap X) \subseteq A \cap (X \cup Y)$ and $(A \cap Y) \subseteq A \cap (X \cup Y)$, we have

$$(A \cap X) \cup (A \cap Y) \subseteq A \cap (X \cup Y).)$$

Therefore

$$A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y).$$

<u>Prove that $A = (A \cap B) \cup (A \cap B^c)$.</u>

[Proof] Notice that $A \cap U = A$, $B \cup B^c = U$, and $A \cap (X \cup Y) = (A \cap X) \cup (A \cap Y)$. We have that $A = A \cap U = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$.

<u>Prove that $A \cup (X \cap Y) = (A \cup X) \cap (A \cup Y)$.</u>

[Proof] $A \cup (X \cap Y)$

$= \{x : x \in A \text{ or } x \in (X \cap Y)\}$

$= \{x: x \in A \text{ or } (x \in X \text{ and } x \in Y)\}$

$= \{x: (x \in A \text{ or } x \in X) \text{ and } (x \in A \text{ or } x \in Y)\}$

// by $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$= \{x : x \in (A \cup X) \cap (A \cup Y)\}$

$= (A \cup X) \cap (A \cup Y).$