09 Principle of Mathematical Induction.

09_02_Additional Examples

Example. Let P(n) be the statement

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

where n is an integer and n ≥ 1. Prove that P(n) is true for all integers at least 1.

[Proof]

Step 1. P(1) is true since both LHS of P(1) and RHS of P(1) are equal to

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Step 2. Assume that P(n) is true for n = k ≥ 1, i.e.,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Now

LHS of P(k + 1) =

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ k+1 & 1 \end{pmatrix} = \text{RHS of } P(k+1).$$

Thus LHS of P(k + 1) = RHS of P(k + 1). Hence P(k + 1) is true.

By the principle of mathematical induction, we have that P(n) is true for all integers at least 1.

Example. Let P(n) be the statement that 3 divides $n^3 + 2n$, where n is a nonnegative integer. Prove that P(n) is true for all integers at least 0.

[Proof]

Step 1. P(0) is true since 3 divides $0 = 0^3 + 2*0$.

Step 2. Assume that P(n) is true for $n = k \geq 1$, i.e.,

$$3 \text{ divides } k^3 + 2k.$$

Now

$$(k + 1)^3 + 2(k + 1)$$
$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= k^3 + 2k + 3(k^2 + k + 1).$$

By assumption, we have that 3 divides $k^3 + 2k$. Obviously,

3 divides $3(k^2 + k + 1)$. Thus 3 divides $k^3 + 2k + 3(k^2 + k + 1)$.

Hence 3 divides $(k + 1)^3 + 2(k + 1)$. So $P(k + 1)$ is true.

By the principle of mathematical induction, we have that $P(n)$ is true for all the integers at least 0.

Example. Let $P(n)$ be the statement $3^n < n!$, where n is an integer and $n \geq 7$. Prove that $P(n)$ is true for all integers at least 7.

[Proof]

Step 1. $P(7)$ is true since $3^7 = 2187 < 5040 = 7!$.

Step 2. Assume that $P(n)$ is true for $n = k \geq 7$, i.e.,

$$3^k < k!, \text{ where } k \geq 7.$$

Then

$$3^{k+1} = 3^k * 3 < k! * (k + 1) = (k + 1)!$$

Hence $P(k + 1)$ is true.

By the principle of mathematical induction, we have that $P(n)$ is true for all the integers $n \geq 7$.