08 Relations.

08_03_Equivalence Relations and Partitions

Definition. Let R be an equivalent relation on a set A. For each $a \in A$, $R[a] = \{x: x \in A, (a, x) \in R\}$ is called the equivalent class represented by a. Note that $a \in R[a]$ for each $a \in A$ ($\neq \emptyset$).

Example. Suppose $A = \{-2, -1, 0, 1, 2\}$. Define a relation $R = \{(a, b) : a \in A, b \in A, \text{ and } (a = b \text{ or } a = -b)\}$ on A. Then R is an equivalent relation on A. We further have

$$R[-2] = \{-2, 2\} = R[2],$$
$$R[-1] = \{-1, 1\} = R[1],$$
$$R[0] = \{0\}.$$

Theorem 1. Let R be an equivalent class on a set A. Then the following statements are equivalent.

[1] $(a, b) \in R$.

[2] $R[a] = R[b]$.

[3] $R[a] \cap R[b] \neq \emptyset$.

[Proof]. [1] => [2] For each $x \in R[a]$, we have $(a, x) \in R$. Since R is symmetric, we have $(x, a) \in R$. we have $(x, b) \in R$ since R is transitive, $(x, a) \in R$, and $(a, b) \in R$, Since R is symmetric, $(b, x) \in R$. Thus $x \in R[b]$. Hence $R[a] \subseteq R[b]$.

Similarly, we can prove that $R[b] \subseteq R[a]$.

Therefore $R[a] = R[b]$.

[2] => [3]. Since $a \in R[a] = R[b]$, $a \in R[a] \cap R[b] \neq \emptyset$.

[3] => [1]. Choose $x \in R[a] \cap R[b] \neq \emptyset$. Then $x \in R[a]$ and $x \in R[b]$. Thus $(a, x) \in R$ and $(b, x) \in R$. Since R is symmetric, we have $(a, x) \in R$ and $(x, b) \in R$. Since R is transitive, we have $(a, b) \in R$. ■

From Theorem 1, we have that the following statements are equivalent.

[1] $(a, b) \notin R$.

[2] $R[a] \neq R[b]$.

[3] $R[a] \cap R[b] = \emptyset$.

Definition. A collection of disjoint nonempty subsets S_i, where $i \in I$ and I is an index set, of a set S is a partition of S if $S = \bigcup_{i \in I} S_i$. When $I = \{1, 2, 3, \ldots, n\}$, we write $S = S_1 \cup S_2 \cup \ldots \cup S_n$.

Example. Suppose $A = \{-2, -1, 0, 1, 2\}$. Then $\{-2, 2\}$, $\{-1, 1\}$, and $\{0\}$ form a partition of A.

Example. Suppose $A = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Namely, A is the set of integers. Then

$A_0 = \{0\}$,

$A_1 = \{-1, 1\}$,

$A_2 = \{-2, 2\}$,

……

$A_n = \{-n, n\}$,

……

form a partition of A.

Example. Suppose A = {0, 1, 2, 3, …}. Namely, A is the set of nonnegative integers. Then

$$A_0 = \{0,\ 7,\ 14,\ 21, \dots\},$$
$$A_1 = \{1,\ 8,\ 15,\ 22, \dots\},$$
$$A_2 = \{2,\ 9,\ 16,\ 23, \dots\},$$
$$A_3 = \{3,\ 10,\ 17,\ 24, \dots\},$$
$$A_4 = \{4,\ 11,\ 18,\ 25, \dots\},$$
$$A_5 = \{5,\ 12,\ 19,\ 26, \dots\},$$
$$A_6 = \{6,\ 13,\ 20,\ 27, \dots\},$$

form a partition of A.

Theorem 2. Let R be an equivalent class on a set A. Then the distinct equivalent classes based on R form a partition of A.

[Proof] Note first that each equivalent class is nonempty.

By Theorem 1, $\bigcup_{a \in A} R[a]$ = the union of distinct equivalent classes based on R. It is obvious that $A \subseteq \bigcup_{a \in A} R[a]$ and $\bigcup_{a \in A} R[a] \subseteq A$. Note that the distinct equivalent classes are also disjoint. Thus A = the disjoint union of distinct equivalent classes based on R. Hence the distinct equivalent classes based on R form a partition of A. ■

Example. Suppose A = {-2, -1, 0, 1, 2}. Define a relation R ={(a, b) : a ∈ A, b ∈ A, and (a = b or a = -b)} on A. Then R is an equivalent relation on A. The partition of A based on R can be found as follows.

$R[-1] = \{x: x \in A, (-1, x) \in R\} = \{-1, 1\} = R[1]$,

$R[0] = \{x: x \in A, (0, x) \in R\} = \{0\}$,

$R[-2] = \{x: x \in A, (-2, x) \in R\} = \{-2, 2\} = R[2]$.

Clearly, $R[-1] = \{-1, 1\}$, $R[0] = \{0\}$, and $R[-2] = \{-2, 2\}$ form a partition of A.

<u>Example.</u> Suppose A be the set of nonnegative integers. Define a relation $R = \{(a, b) : a \in A, b \in A, \text{ and } a \bmod 7 = b \bmod 7\}$ on A. Prove that R is an equivalent relation on A and find the partition of A based on R.

[Proof] Note that $a \bmod 7 = a \bmod 7$. So $(a, a) \in R$ and R is reflexive.

Note that $(a, b) \in R$ if and only if $a \bmod 7 = b \bmod 7$ if and only if $b \bmod 7 = a \bmod 7$ if and only if $(b, a) \in R$. So R is symmetric.

If $(a, b) \in R$ and $(b, c) \in R$, then $a \bmod 7 = b \bmod 7$ and $b \bmod 7 = c \bmod 7$. Thus $a \bmod 7 = c \bmod 7$ and $(a, c) \in R$. So R is transitive.

Hence R is an equivalent relation on A.

The partition of A based on R can be found as follows.

$$\begin{aligned} R[0] &= \{x : x \in A \text{ and } (0, x) \in R\} \\ &= \{x : x \in A \text{ and } 0 \bmod 7 = x \bmod 7\} \\ &= \{x : x \in A, 0 = x \bmod 7\} \\ &= \{x : x \in A, x \bmod 7 = 0\} \end{aligned}$$

$= \{0, 7, 14, 21, \dots\},$

$R[1] = \{x : x \in A \text{ and } (1, x) \in R\}$

$= \{x : x \in A \text{ and } 1 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 1 = x \bmod 7\}$

$= \{x : x \in A, x \bmod 7 = 1\}$

$= \{1, 8, 15, 22, \dots\},$

$R[2] = \{x : x \in A \text{ and } (2, x) \in R\}$

$= \{x : x \in A \text{ and } 2 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 2 = x \bmod 7\}$

$= \{x : x \in A, x \bmod 7 = 2\}$

$= \{2, 9, 16, 23, \dots\},$

$R[3] = \{x : x \in A \text{ and } (3, x) \in R\}$

$= \{x : x \in A \text{ and } 3 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 3 = x \bmod 7\}$

$= \{x : x \in A, x \bmod 7 = 3\}$

$= \{3, 10, 17, 24, \dots\},$

$R[4] = \{x : x \in A \text{ and } (4, x) \in R\}$

$= \{x : x \in A \text{ and } 4 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 4 = x \bmod 7\}$

$= \{x : x \in A, x \bmod 7 = 4\}$

$= \{4, 11, 18, 25, \dots\},$

$R[5] = \{x : x \in A \text{ and } (5, x) \in R\}$

$= \{x : x \in A \text{ and } 5 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 5 = x \bmod 7\}$

$= \{x : x \in A, x \bmod 7 = 5\}$

$= \{5, 12, 19, 26, \dots\},$

$R[6] = \{x : x \in A \text{ and } (6, x) \in R\}$

$= \{x : x \in A \text{ and } 6 \bmod 7 = x \bmod 7\}$

$= \{x : x \in A, 6 = x \bmod 7\}$

$= \{x : x \in A, \ x \bmod 7 = 6\}$

$= \{6, 13, \ 20, \ 27, \dots \}$,

Clearly, R[0], R[1], R[2], R[3], R[4], R[5], and R[6] form a partition of A.

<u>Theorem 3.</u> Suppose $\{P_i, i \in I\}$ form a partition of A. Then $R = \cup_{i \in I} (P_i \times P_i)$ is an equivalent relation on A.

[Proof] Since $\{P_i, i \in I\}$ form a partition of A, for each $a \in A$, $a \in P_k$ for some $k \in I$. Thus $(a, a) \in P_k \times P_k \subseteq \cup_{i \in I} (P_i \times P_i) = R$. Therefore R is reflexive.

If $(a, b) \in R$, then there exists an index $k \in I$ such that $(a, b) \in P_k \times P_k$. Thus $a \in P_k$ and $b \in P_k$. Hence $(b, a) \in P_k \times P_k \subseteq \cup_{i \in I} (P_i \times P_i) = R$.

If $(b, a) \in R$, then there exists an index $k \in I$ such that $(b, a) \in P_k \times P_k$. Thus $b \in P_k$ and $a \in P_k$. Hence $(a, b) \in P_k \times P_k \subseteq \cup_{i \in I} (P_i \times P_i) = R$.

Therefore R is symmetric.

If $(a, b) \in R$ and $(b, c) \in R$ then there exists an index $k \in I$ and an index $j \in I$ such that $(a, b) \in P_k \times P_k$ and $(b, c) \in P_j \times P_j$. Thus $a \in P_k$, $b \in P_k$ and $b \in P_j$, $c \in P_j$. Thus $j = k$. Hence $(a, c) \in P_k \times P_k \subseteq \cup_{i \in I} (P_i \times P_i) = R$.

Therefore R is transitive.

So R is an equivalent relation on A. ■

When I = {1, 2, 3, …, n} in Theorem 3, we have that $(P_1 \times P_1) \cup (P_2 \times P_2) \cup \ldots \cup (P_n \times P_n)$ is an equivalent relation on A if $P_1, P_2, \ldots, P_n$ form a partition of A.

Note that $P_1 = \{-1, 1\}$, $P_2 = \{0\}$, and $P_3 = \{-2, 2\}$ form a partition of A = {-2, -1, 0, 1, 2}. Then, by Theorem 3, we have that $(P_1 \times P_1) \cup (P_2 \times P_2) \cup (P_3 \times P_3) =$

{(-1, -1), (-1, 1)
(1, -1), (1, 1),
(0, 0),
(-2, -2), (-2, 2)
(2, -2), (2, 2)}

is an equivalent relation on A.

Note that we can use Theorem 3 to prove a given relation is an equivalent relation.

<u>Example.</u> Suppose A = {1, 2, 3, 4, 5} and R =

{(1, 1), (1, 4),
(4, 1), (4, 4),
(3, 3),
(4, 4),
(5, 5)}

Prove that R is an equivalent relation on A.

[Proof] Let $P_1 = \{1, 4\}$, $P_2 = \{3\}$, $P_3 = \{4\}$, and $P_4 = \{5\}$. Then P_1, P_2, P_3, and P_4 form a partition of A and $R = (P_1 \times P_1) \cup (P_2 \times P_2) \cup (P_3 \times P_3) \cup (P_4 \times P_4)$. Hence, by Theorem 3, R is an equivalent relation on A. ■

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