04 Counting Methods

04_04_Identities

Pascal's Identity. Suppose n and k are integers such that $1 \le k \le n$. Then

$$C(n + 1, k) = C(n, k) + C(n, k - 1).$$

[Proof1] We prove it algebraically. Notice that

$$\begin{aligned} & C(n, k) + C(n, k - 1) \\ &= n!/(k!*(n - k)!) + n!/((k - 1)!*(n - k + 1)!) \\ &= [n!*(n - k + 1) + n!*k]/[k!*(n - k + 1)!] \\ &= [n!*(n + 1)]/[k!*(n + 1 - k)!] \\ &= (n + 1)!/[k!*(n + 1 - k)!] \\ &= C(n + 1, k). \end{aligned}$$

[Proof2] We prove it combinatorically. Let A be a set of size (n + 1). Then C(n + 1, k) is the number of k-element subsets of A. Let x be any element in A. Then the k-element subsets of A can be categorized into two sets Y and Z.

$$Y = \{S: S \subseteq A, |S| = k, \text{ and } x \in S\},$$

$$Z = \{S: S \subseteq A, |S| = k, \text{ and } x \notin S\}.$$

Note that Z is a set contains all the k-element subsets of $A - \{x\}$. Thus

$$|Z| = C(n, k).$$

Define

$$W = \{S - \{x\}: S \subseteq A, |S| = k, \text{ and } x \in S\}.$$

Then, there is a bijection between set Y and W. Thus,

$|Y| = |W|$. Note that W is a set contains all the $(k – 1)$-element subsets of $A – \{x\}$. Thus

$$|W| = C(n, k – 1).$$

Hence,

$C(n + 1, k)$ = the number of k-element subsets of A.

$$= |Y| + |Z|$$

$$= |W| + |Z|$$

$$= C(n, k) + C(n, k – 1).$$

[Proof3] Note that $C(n + 1, k)$ is the number of paths from $(0, 0)$ to $(n + 1 - k, k)$ in which only horizontal moves to east and the vertical moves to north are allowed.

It is observed that each path from $(0, 0)$ to $(n + 1 - k, k)$ in which only horizontal moves to east and the vertical moves to north are allowed can be formed by the following two ways.

[1] Appending one horizontal grid edge to a path from $(0, 0)$ to $(n + 1 - k - 1, k)$ in which only horizontal moves to east and the vertical moves to north are allowed.

[2] Appending one vertical grid edge to a path from $(0, 0)$ to $(n + 1 - k, k - 1)$ in which only horizontal moves to east and the vertical moves to north are allowed.

The number of paths can be formed in [1] is

$$C(n + 1 - k - 1 + k, k) = C(n, k).$$

The number of paths can be formed in [1] is

$$C(n + 1 - k + k - 1, k) = C(n, k - 1).$$

Thus

$$C(n + 1, k) = c(n, k) + c(n, k - 1).$$

<u>Identity.</u> Suppose m, n, and k are integers such that $0 \leq k \leq m$ and $0 < k \leq n$. Then

$$C(m + n, k) = C(m, 0)*C(n, k) + C(m, 1)*C(n, k - 1) + \ldots + C(m, r)*C(n, k - r) + \ldots + C(m, k)*C(n, 0).$$

[Proof] Suppose there are m red balls and n blue balls in a bag. Then the number of choosing k balls in the ball is C(m + n, k). Choosing k balls in the bag can also be accomplished in the following ways.

[0] Choose 0 red balls and k blue balls in the bag, or

[1] Choose 1 red ball and (k – 1) blue balls in the bag, or

[2] Choose 2 red balls and (k – 2) blue balls in the bag, or

………...

[r] Choose r red balls and (k – r) blue balls in the bag, or

………...

[k] Choose k red balls and 0 blue balls in the bag.

Note that the number of ways of choosing 0 red balls and k blue balls in the bag is C(m, 0)*C(n, k);

the number of ways of choosing 1 red ball and (k – 1)blue balls in the bag is C(m, 1)*C(n, k - 1);

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the number of ways of choosing k red balls and 0 blue balls in the bag is C(m, k)*C(n, 0).

Therefore

$$C(m + n, k) = C(m, 0)*C(n, k) + C(m, 1)*C(n, k - 1) + \ldots + C(m, r)*C(n, k - r) + \ldots + C(m, k)*C(n, 0).$$

<u>Corollary.</u> Let m and n be positive integers. Show that

$$C(m + n, 2) = C(m, 2) + C(n, 2) + m*n.$$

[Proof] From the above identity, we have that

$$C(m + n, 2) = C(m, 0)*C(n, 2) + C(m, 1)*C(n, 1) + C(m, 2)*C(r, 0)$$

$$= C(m, 2) + C(n, 2) + m*n$$