07 Matrices

07_02_Matrix Operations

-Two matrices A and B are **equal** if r(A) = r(B), c(A) =c(B), and all the corresponding entries are equal.

-Suppose two matrices A and B are equal. Find x, y, and z

$$A = \begin{bmatrix} x - y & 3x \\ 50 & x + y \\ 40 & 2021 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & y + z \\ 50 & 30 \\ 40 & 2021 \end{bmatrix}$$

Two Matrices Are Equal

-Suppose two matrices A and B are equal. Find x, y, and z

$$A = \begin{bmatrix} x - y & 3x \\ 50 & x + y \\ 40 & 2021 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & y + z \\ 50 & 30 \\ 40 & 2021 \end{bmatrix}$$

-[Solution] Since A and B are equal, we have

$x - y = 10$, $x + y = 30$, and $3x = y + z$. Thus

$x - y + x + y = 10 + 30$. Namely, $2x = 40$, and $x = 20$.

Thus $y = 30 - x = 30 - 20 = 10$.

Hence $z = 3x - y = 3*20 - 10 = 50$.

Two Matrices Are Equal

-You can work a similar one below after today's lecture.

-Suppose two matrices A and B are equal. Find x, y, and z

$$A = \begin{bmatrix} 2x - y & 70 \\ 5y & x + 2y \\ 60 & 2022 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 70 \\ 2x + z & 30 \\ 60 & 2022 \end{bmatrix}$$

The Sum or Addition of Two Matrices

Definition. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two m x n matrices. The sum of **A** and **B**, denoted **A** + **B**, is the m x n matrix such that $a_{ij} + b_{ij}$ is its (i, j)-element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

Example.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

The requirements for computing the sum of A and B are $r(A) = r(B)$ and $c(A) = c(B)$, otherwise $A + B$ is not defined.

The Sum or Addition of Two Matrices

-For any two matrices A and B, if $r(A) = r(B)$ and $c(A) = c(B)$, then $A + B = B + A$.
Namely, the matrix addition is commutative.

-For any three matrices A, B and D, if $r(A) = r(B) = r(D)$ and $c(A) = c(B) = c(D)$, then $(A + B) + D = A + (B + D)$.
Namely, the matrix addition is associative.

The Difference between Two Matrices

Definition. Let **A** = [a_{ij}] and **B** = [b_{ij}] be two m x n matrices. The difference of **A** and **B**, denoted **A** - **B**, is the m x n matrix such that a_{ij} - b_{ij} is its (i, j)-element. In other words, **A** - **B** = [a_{ij} - b_{ij}].

Example.

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

The requirements for computing the difference between A and B are r(A) = r(B) and c(A) = c(B), otherwise A - B is not defined.

The Multiplication between a Number and a Matrix

A matrix can be multiplied by a number. Let k be a number and

A = (a_{ij}) is a matrix. Then

kA = **Ak** is defined as **(ka_{ij})** = **($a_{ij}k$)**

Example, If k = 4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

The Multiplication between a Number and a Matrix

$$4\times\begin{bmatrix}3 & -1\\2 & 1\\2 & -3\\4 & 1\end{bmatrix}=\begin{bmatrix}3 & -1\\2 & 1\\2 & -3\\4 & 1\end{bmatrix}\times 4=\begin{bmatrix}12 & -4\\8 & 4\\8 & -12\\16 & 4\end{bmatrix}$$

An Example for Mixed Operations on Matrices

Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 3 \\ -2 & -1 \\ 2 & 4 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 0 & 4 \\ 5 & 6 \\ 7 & 4 \end{bmatrix}$$

Then $3A + 2B - 4D =$

$$\begin{bmatrix} 3*1 & 3*2 \\ 3*4 & 3*5 \\ 3*0 & 3*(-1) \end{bmatrix} + \begin{bmatrix} 2*(-3) & 2*3 \\ 2*(-2) & 2*(-1) \\ 2*2 & 2*4 \end{bmatrix} - \begin{bmatrix} 4*0 & 4*4 \\ 4*5 & 4*6 \\ 4*7 & 4*4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -12 & -11 \\ -24 & -11 \end{bmatrix}$$

You can work on a similar one after today's lecture

Suppose

$$A = \begin{bmatrix} -1 & 4 \\ 3 & -2 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 5 \\ 8 & -6 \\ -3 & 2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & 0 \\ 3 & 9 \\ 8 & 2 \end{bmatrix}$$

Find $2A + 3B - 5D$.

The Multiplication of Two Matrices

-The multiplication of a row matrix and a column matrix.

Suppose X is a row matrix and Y is a column matrix such that $c(X) = r(Y)$. For instance, $c(X) = r(Y) = 5$.

$$X = [x_1, x_2, x_3, x_4, x_5],\ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}. \text{ Then } XY = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5.$$

An Example for the Multiplication of Two Matrices

-When $c(A) = r(B)$, we can multiply A and B, otherwise AB is not defined.

-Suppose A is a 2x3 matrix and B is a 3x4 matrix.

$$A_{2x3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B_{3x4} = \begin{bmatrix} 7 & 8 & 9 & 0 \\ -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{bmatrix}$$

Define $M = A_{2x3} B_{3x4}$. Then M has 2 rows and 4 columns.

Namely, $M = M_{2x4} = (m_{ij})_{2x4}$.

An Example for the Multiplication of Two Matrices

-Now A is a 2x3 matrix and B is a 3x4 matrix.

$$A_{2x3}\,B_{3x4} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 & 9 & 0 \\ -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1*7 + 2*(-1) + 3*(-5) & 1*8 + 2*(-2) + 3*(-6) & 1*9 + 2*(-3) + 3*(-7) & 1*0 + 2*(-4) + 3*(-8) \\ 4*7 + 5*(-1) + 6*(-5) & 4*8 + 5*(-2) + 6*(-6) & 4*9 + 5*(-3) + 6*(-7) & 4*0 + 5*(-4) + 6*(-8) \end{bmatrix}_{2x4}$$

$$= \begin{bmatrix} -10 & -14 & -18 & -32 \\ -7 & -14 & -21 & -68 \end{bmatrix}_{2x4}$$

Use the ith row in the left matrix to multiply the jth column in the right matrix to obtain (i, j)-elment m_{ij} in M = AB, where $1 \le i \le r(A)$ and $1 \le j \le c(B)$.

An Example for the Multiplication of Two Matrices

-Another scheme for computing $A_{2x3}\ B_{3x4}$.

$$A_{2x3}= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{bmatrix}$$

$$B_{3x4} = \begin{bmatrix} 7 & 8 & 9 & 0 \\ -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{bmatrix}$$

m_{11}=1*7 + 2*(-1) + 3*(-5), m_{12} =1*8 + 2*(-2) + 3*(-6), m_{13} =1*9 + 2*(-3) + 3*(-7), m_{14} =1*0 + 2*(-4) + 3*(-8)

m_{21}= 4*7 + 5*(-1) + 6*(-5), m_{21}=4*8 + 5*(-2) + 6*(-6), m_{23}=4*9 + 5*(-3) + 6*(-7), m_{24} =4*0 + 5*(-4) + 6*(-8)

$$= \begin{bmatrix} -10 & -14 & -18 & -32 \\ -7 & -14 & -21 & -68 \end{bmatrix}_{2x4}$$

Use the ith row in the left matrix to multiply the jth column in the right matrix to obtain (i, j)-elemnt m_{ij} in M = AB, where $1 \le i \le r(A)$ and $1 \le j \le c(B)$.

Multiplication of Two Matrices

Two matrices can be multiplied together only when the number of columns in the left matrix equals the number of rows in the right matrix.

$$A_{mxk}B_{kxn} = C_{mxn}$$

Illustrations of Matrix Multiplication

The Multiplication of $\mathbf{A}_{mxk} = [a_{ij}]_{mxk}$ and $\mathbf{B}_{kxn} = [b_{ij}]_{kxn}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix Multiplication Is NOT Commutative

-A_{3x2} is a 3x2 matrix and B_{2x4} is a 2x4 matrix.
We can multiply A_{3x2} and B_{2x4}. Namely, $A_{3x2}\,B_{2x4}$ is well-defined. But we cannot multiply B_{2x4} and $A_{3x2.}$

-A_{3x2} is a 3x2 matrix and B_{2x3} is a 2x3 matrix.
We can multiply A_{3x2} and B_{2x3}. We can also multiply B_{2x3} and $A_{3x2.}$ But $A_{3x2}\,B_{2x3}$ is a 3x3 matrix, $B_{2x3}A_{3x2}$ is a 2x2 matrix. So $A_{3x2}\,B_{2x3} \neq B_{2x3}A_{3x2.}$

Matrix Multiplication Is NOT Commutative

$$A = \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$$

$$\text{then } AB = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$$

Matrix Multiplication Is Not Commutative

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$AB \neq BA$$

Transposes of Matrices

Definition. Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The transpose of $\mathbf{A}$, denoted by $\mathbf{A}^t$ or $\mathbf{A}^T$, the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and

$j = 1, 2, \ldots, m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of Matrices

A square matrix $\mathbf{A} = [a_{ij}]$ is called symmetric

if $a_{ij} = a_{ji}$ for i and j with $1 \le i \le n$ and $1 \le j \le n$.

Suppose A is a square matrix. Then $A = A^t$ if and only

if A is symmetric.

$$\text{The matrix } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is square.}$$

This matrix is equal to its transpose. It is symmetric.

Summary

-Multiplication of two matrices

$$A_{mxk}\, B_{kxn} = C_{mxn} = (c_{ij})_{mxn.}$$

c_{ij} = the multiplication of
the i^{th} row of A and
the j^{th} column of B.

-Matrix multiplication is not commutative.