06 Pigeonhole Principle

06_02 Pigeonhole Principle and Examples

Another Generalization of Pigeonhole Principle Suppose $m_1, m_2, \ldots, m_n$ are positive integers. If we place $m_1 + m_2 + \ldots + m_n - n + 1$ pigeons (in general, objects) into k pigeonholes (in general, boxes), then there exists an i, where $i \in \{1, 2, \ldots, n\}$, such that Pigeonhole i (in general, Box i) has at least m_i pigeons.

[Proof]. Suppose, to the contrary, that for each i, $i \in \{1, 2, \ldots, n\}$, the number of pigeons in Pigeonhole i is less than or equal to $m_i - 1$. Then

$m_1 + m_2 + \ldots + m_n - n + 1$

= the total number of pigeons

= the number of pigeons in Pigeonhole 1 +

the number of pigeons in Pigeonhole 2 +

…………

the number of pigeons in Pigeonhole n

$\leq (m_1 - 1) + (m_2 - 1) + \ldots + (m_n - 1)$

$= m_1 + m_2 + \ldots + m_k - n,$

a contradiction.

Note that if $m_1 \geq 2$, $m_2 \geq 2$, …, $m_n \geq 2$, then

$$m_1 + m_2 + \ldots + m_n - n + 1 \geq n + 1.$$

The above generalized Pigeonhole Principle implies the first version of the Pigeonhole Principle.

Example. Suppose that each student in one Computer Science class of 30 students is a sophomore, a junior, or a senior. Show that

[1] Show that there are at least 16 sophomores, at least 10 juniors, or at least 6 seniors.

[2] Show that there are at least 18 sophomores, at least 11 juniors, or at least 3 seniors.

[Proof].

[1] Let $n = 3$, $m_1 = 16$, $m_2 = 10$, and $m_3 = 6$.

Then $30 = 16 + 10 + 6 - 3 + 1$

$= m_1 + m_2 + m_3 - n + 1.$

Apply the above generalized Pigeonhole Principle, we have the desired result.

[2] Let $n = 3$, $m_1 = 18$, $m_2 = 11$, and $m_3 = 3$.

Then $30 = 18 + 11 + 3 - 3 + 1$

$= m_1 + m_2 + m_3 - n + 1.$

Apply the above generalized Pigeonhole Principle, we have the desired result.

Example. Let $a_1, a_2, \ldots, a_n$ be a sequence of integers. Show that there exist integers i and j, where $i < j$, such that $(a_{i+1} + a_{i+2} + \ldots + a_j)$ is divisible by n, where $i \in \{0, 1, 2, \ldots, (n-1)\}$ and $j \in \{1, 2, \ldots, n\}$.

[Proof]. Define $s_1 = a_1,$

$s_2 = a_1 + a_2,$

$\ldots \ldots$

$s_k = a_1 + a_2 + \ldots + a_k,$

$$\dots \dots$$

$$s_n = a_1 + a_2 + \dots \dots \dots + a_n.$$

If there exists an integer r such that s_r is divisible by n, then we have the desired result with $i = 0$ and $j = r$.

Now we consider the case that s_r is not divisible by n for each $r \in \{1, 2, \dots, n\}$. Then when s_r is divided by n, where $r \in \{1, 2, \dots, n\}$, the possible remainders are 1, 2, …, (n – 1). By Pigeonhole Principle, there exist integers i and j, where $i < j$, $i \in \{0, 1, 2, \dots, (n-1)\}$, and $j \in \{1, 2, \dots, n\}$, such that $s_i \bmod n = s_j \bmod n$. Thus $(s_j - s_i) \bmod n = 0$. Hence $(a_{i+1} + a_{i+2} + \dots + a_j) \bmod n = 0$. So $(a_{i+1} + a_{i+2} + \dots + a_j)$ is divisible by n.