10 Recursive Sequences.

10_04_The Master_Theorem

Theorem. Suppose T(n) is a function defined on the set of nonnegative integers and satisfies the following recurrence

$$T(n) = a\,T(n/b) + f(n),$$

where a and b are constants such that $a \geq 1$ and $b > 1$ and n/b can be thought as either [n/b], the floor of n/b, or {n/b}, the ceiling of n/b.

[1] If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log_2 n)$.

[2] If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

[3] If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and sufficient large n, then $T(n) = \Theta(f(n))$.

Example. Suppose T(n) is a function defined on the set of nonnegative integers and satisfies the following recurrence

$$T(n) = 2\,T(n/2) + 10\,n.$$

Since $f(n) = 10\,n = \Theta(n^{\log_b a})$, where b = 2 and a = 2, we, by [1] in the master theorem above, have

$$T(n) = \Theta(n^{\log_b a} \log_2 n) = \Theta(n \log_2 n).$$

Example. Suppose T(n) is a function defined on the set of nonnegative integers and satisfies the following recurrence

$$T(n) = 7\,T(n/2) + 5\,n^2.$$

Note that $\log_b a \approx 2.803$ when $a = 7$ and $b = 2$.

Thus $f(n) = 5\,n^2 = O(n^{\log_b a - \varepsilon})$ for a constant $\varepsilon = 0.1 > 0$.

We, by [2] in the master theorem above, have

$$T(n) = \Theta(n^{\log_b a}) \approx \Theta(n^{2.803}) \neq \Theta(n^3).$$

<u>Example.</u> Suppose T(n) is a function defined on the set of nonnegative integers and satisfies the following recurrence

$$T(n) = 7\,T(n/2) + n^3.$$

Note that $\log_b a \approx 2.803$ when $a = 7$ and $b = 2$.

Thus $f(n) = n^3 = \Omega(n^{\log_b a + \varepsilon})$ for a constant $\varepsilon = 0.1 > 0$.

Also $af(n/b) \leq cf(n)$ for $c = 15/16 < 1$ and $n \geq 1$.

we, by [3] in the master theorem above, have

$$T(n) = \Theta(f(n)) = \Theta(n^3).$$