07 Matrices

06_04_The Inverse of a Matrix

-Let A be an n x n matrix. A is said to be invertible if there exists an n x n matrix B such that

$$A\,B = B\,A = I,$$

where I is the identity matrix of order n. In this case, B is called the inverse of A

-If an n x n matrix A is invertible, then A has a unique inverse.

[Proof] Suppose B and C are the inverses of A. Then

$$A B = B A = I \quad \text{and} \quad C A = A C = I,$$

where A and B are two n x n matrices and I is the identity matrix of order n. Thus

$$B = B I = B (A C) = (B A) C = I C = C.$$

This completes the proof of the above claim.

-If an n x n matrix A is invertible, then we use A^{-1} to denote its inverse.

-If an n x n matrix A is invertible, then A^{-1} is also invertible and its inverse is A.

[Proof] Since A is invertible, then

$$A A^{-1} = A^{-1} A = I,$$

where I is the identity matrix of order n. Namely,

$$A^{-1} A = A A^{-1} = I.$$

Thus, A^{-1} is invertible and its inverse, denoted $(A^{-1})^{-1}$, is A. This completes the proof of the above claim.

-If n x n matrices A and B are invertible, then AB is also invertible and its inverse is $B^{-1} A^{-1}$.

[Proof] Since $B^{-1} A^{-1} A B = B^{-1} I B = B^{-1} B = I$ and $A B B^{-1} A^{-1} = A I A^{-1} = A A^{-1} = I$, where I is the identity matrix of order n.

This completes the proof of the above claim.

-If an n x n matrices A is invertible, then its transpose, denoted A^T, is also invertible and its inverse is $(A^{-1})^T$.

[Proof] Since A is invertible, then

$$A A^{-1} = A^{-1} A = I,$$

where I is the identity matrix of order n. Thus,

$$(A A^{-1})^T = (A^{-1} A)^T = I^T.$$

Namely,

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I.$$

This completes the proof of the above claim.

-Let

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 8/14 & 2/14 \\ -3/14 & 1/14 \end{bmatrix} = \begin{bmatrix} 8/14 & 2/14 \\ -3/14 & 1/14 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

-Thus

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix}$$

has an inverse of

$$\begin{bmatrix} 8/14 & 2/14 \\ -3/14 & 1/14 \end{bmatrix}$$

-Slove the following system of equations.

$$x - 2y = -3$$
$$3x + 8y = 7$$

[Solution] Let A be

$$\begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix}.$$

Let W be $(x, y)^T$. Then the above equation can be written as

$$AW = b,$$

where $b = (-3, 7)^T$.

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Since A is invertible, A^{-1} exists. Thus,

$$A^{-1}AW = A^{-1}b,$$

Therefore,

$$W = I\,W = A^{-1}b,$$

where I is the identity matrix of order 2.

Hence,

$$W = \begin{bmatrix} 8/14 & 2/14 \\ -3/14 & 1/14 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -5/7 \\ 8/7 \end{bmatrix}$$

So, $x = -5/7$ and $y = 8/7$.

-Find a 2 x 2 matrix M such that

$$\begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix} M = \begin{bmatrix} 1 & 2 \\ -5 & 7 \end{bmatrix}.$$

[Solution] Let A be

$$\begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix}$$

-Since A is invertible, A^{-1} exists. Thus,

$$A^{-1}\begin{bmatrix} 1 & -2 \\ 3 & 8 \end{bmatrix} M = A^{-1}\begin{bmatrix} 1 & 2 \\ -5 & 7 \end{bmatrix}$$

Therefore

$$M = I\,M = \begin{bmatrix} 8/14 & 2/14 \\ -3/14 & 1/14 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} -1/7 & 15/7 \\ -4/7, & 1/14 \end{bmatrix},$$

where I is the identity matrix of order 2.

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-Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $D = ad - bc \neq 0$, then A is invertible and A^{-1} is

$$B = \begin{bmatrix} d/D & -b/D \\ -c/D & a/D \end{bmatrix}.$$

[Proof] A simple computation shows that

$$A\,B = B\,A = I,$$

where I is the identity matrix of order 2.
This completes the proof of the above claim.

Additional topics on the inverse of a matrix will be taught in a course of Linear Algebra.