06 Pigeonhole Principle

06_01 Pigeonhole Principle and Examples

Pigeonhole Principle. Suppose n is a positive integer. If n + 1 or more pigeons (in general, objects) are placed into n pigeonholes (in general, n boxes), then at least one pigeon hole (in general, box) contains two or more pigeons (in general, objects.)

[Proof]. Suppose the number of pigeons (in general, objects) in each of the pigeonholes (in general, boxes) is less than 2. Then the number of pigeons (in general, objects) in each of the pigeonholes (in general, boxes) is less than or equal to 1. Then the total number of pigeons (in general, objects) would be less than or equal to n. This contradicts the given condition that we have n + 1 pigeons (in general, objects.)

Example. Among any collection of 13 or more people, there must be at least two of them who were born in the same month.

[Proof] The number of pigeons or objects is 13 or more. Each month plays the role of a pigeonhole or box. Namely, the number of pigeonholes or boxes is 12. Place a person into the month she or he was born. The conclusion follows from the Pigeonhole Principle.

Example. Among any collection of 27 or more English words, there must be at least two of them ends with the same letter.

[Proof] The number of pigeons or objects is 27 or more. Each letter in English alphabet plays the role of a pigeonhole or box.

If a word ends with A, it will be placed in the pigeonhole or box represented by the letter A.

If a word ends with B, it will be placed in the pigeonhole or box represented by the letter B.

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If a word ends with Z, it will be placed in the pigeonhole or box represented by the letter Z.

The conclusion follows from the Pigeonhole Principle.

<u>Example.</u> Show that if any six integers are selected from $S = \{1, 2, 3, \ldots, 9, 10\}$, then we can always find two of them such that their sum is 11.

[Proof1]. One can prove it by listing all the 6-element subsets of S and checking that each of those 6-element subsets has two elements such that their sum is 11. For instance, {1, 2, 3, 4, 5, 6} has two elements 5 and 6 and their sum is 11. Note that the number of 6-elemnt sunsets of S is C(10, 6) = 210.

[Proof2] The six selected integers play the roles of the pigeons. Construct pigeonholes: {1, 10}, {2, 9}, {3, 8}, {4, 7}, {5, 6}. The assignment rule is that the selected integer k will be placed into the set containing it. By Pigeonhole Principle, two of them will be in the same set and their sum is 11.

Recall if x is a real number, the floor of x, denoted [x], the largest integer which is less than or equal to x. The ceiling, of x, denoted $\{x\}$, is the smallest integer which is greater than or equal to x. For example, if $x = -7, 0$, or 7, then $[x] = \{x\} = x$. If $x = 3.14$, then $[x] = 3$ and $\{x\} = 4$. If $x = -2.718$, then $[x] = -3$ and $\{x\} = -2$. Note that $x - 1 < [x] \leq x \leq \{x\} < x + 1$ and $[x] = x = \{x\}$ if x is an integer.

<u>The Generalized Pigeonhole Principle</u> If n pigeons (or objects) are placed into k pigeonholes (or boxes), then there exists at least one pigeonhole (or box) containing at least $\{n/k\}$ pigeons (or objects.)

[Proof]. Suppose the number of pigeons (or, objects) in each of the pigeonholes (or, boxes) is less than $\{n/k\}$. Then the number of pigeons (or, objects) in each of the pigeonholes (or, boxes) is less than or equal to $\{n/k\} - 1$. Then the total number of pigeons (or, objects) would be less than or equal to $k(\{n/k\} - 1) < k(n/k + 1 - 1) = n$. This contradicts the given condition that we have n pigeons (or, objects.)

<u>Example.</u> Among 1024 people there are at least $\{1024/12\} = 86$ of them who were born in the same month. Among 1024 people there are at least $\{1024/365\} = 3$ of them who were born in the same day if the year is not a leap year. Among 1024 people there are at least $\{1024/366\} = 3$ of them who were born in the same day if the year is a leap year.