12 Probability

12_04_Law of Total Probability and Theorem of Bayes

Example. In one year, an insurance company classified its customers into four categories of C_1(very safe), C_2(safe), C_3(neutral), and C_4(dangerous) and found that 10%, 20%, 60%, and 10% of the customers are in C_1(very safe), C_2(safe), C_3(neutral), and C_4(dangerous), respectively. It is also found that the probabilities of having an accident for the customers in C_1(very safe), C_2(safe), C_3(neutral), and C_4(dangerous) are 0.02, 0.05, 0.10, and 0.20, respectively.

[1] Find the probability of a random customer who had an accident in that year.

[2] If one customer had an accident in that year, find the probabilities of that customer who are in C_1 (very safe), C_2(safe), C_3(neutral), and C_4(dangerous), respectively.

[Solution] [1]

Let A be the event that a customer who had an accident in that year.

Let B_1 be the event that a customers is from C_1(very safe).

Let B_2 be the event that a customer is from C_2(safe).

Let B_3 be the event that a customer is from C_3(neutral).

Let B_4 be the event that a customer is from C_4(dangerous).

Thus $P(B_1) = 0.10$, $P(B_2) = 0.20$, $P(B_3) = 0.60$, and $P(B_4) = 0.10$.

Then $A = A \cap \Omega = A \cap (B_1 \cup B_2 \cup B_3 \cup B_4)$

$$= (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4).$$

Note that $(A \cap B_i)$ and $(A \cap B_j)$ are pairwise disjoint. We have

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + P(A \cap B_4)$$

$$= P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2) + P(A \mid B_3) P(B_3) + P(A \mid B_4) P(B_4)$$

$$= 0.02*0.10 + 0.05*0.20 + 0.10*0.60 + 0.20*0.10$$

$$= 0.092.$$

[Solution] [2]

If one customer had an accident in that year, the probability of that customer are in C_1 (very safe) is

$P(B_1 \mid A) = P(B_1 \cap A)/P(A) =$

$P(A \mid B_1)\, P(B_1)/(P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3) + P(A \mid B_4)\, P(B_4))$

$= 0.02*0.10/(0.02*0.10 + 0.05*0.20 + 0.10*0.60 + 0.20*0.10)$

$= 0.002/0.092 \approx 0.0217.$

If one customer had an accident in that year, the probability of that customer are in C_2 (safe) is

$P(B_2 \mid A) = P(B_2 \cap A)/P(A) =$

$P(A \mid B_2)\, P(B_2)/(P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3) + P(A \mid B_4)\, P(B_4))$

$= 0.05*0.20\ /(0.02*0.10 + 0.05*0.20 + 0.10*0.60 + 0.20*0.10)$

$= 0.010/0.092 \approx 0.1087.$

If one customer had an accident in that year, the probability of that customer are in C_3 (neutral) is

$P(B_3 | A) = P(B_3 \cap A)/ P(A) =$

$P(A | B_3)\ P(B_3)/(P(A | B_1)\ P(B_1) + P(A | B_2)\ P(B_2) + P(A | B_3)\ P(B_3) + P(A | B_4)\ P(B_4))$

$= 0.10*0.60/(0.02*0.10 + 0.05*0.20 + 0.10*0.60 + 0.20*0.10)$

$= 0.06/0.092 \approx 0.6522.$

If one customer had an accident in that year, the probability of that customer are in C_4 (dangerous) is

$P(B_4 | A) = P(B_4 \cap A)/P(A) =$

$P(A | B_4)\ P(B_4)/(P(A | B_1)\ P(B_1) + P(A | B_2)\ P(B_2) + P(A | B_3)\ P(B_3) + P(A | B_4)\ P(B_4))$

$= 0.20*0.10/(0.02*0.10 + 0.05*0.20 + 0.10*0.60 + 0.20*0.10)$

$= 0.020/0.092 \approx 0.2174.$

The Law of Total Probability. If $B_1, B_2, \ldots, B_n$ form a partition of the sample space Ω**.** Then for any event A we have

$$P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + \ldots + P(A \mid B_n)\, P(B_n).$$

Bayes' Theorem. If $B_1, B_2, \ldots, B_n$ form a partition of the sample space Ω**.** Then for any event A we have

$$P(B_i \mid A) =$$

$$P(A \mid B_i)\, P(B_i)/((P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + \ldots + P(A \mid B_n)\, P(B_n)),$$

for each i with $1 \le i \le n$.