HOMEWORK OF HARMONIC ANALYSIS

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1. Hardy-Littlewood maximal operator

1.1. approximation to the identity.

1.1.1. Suppose $g \in \mathcal{S}$ is a tempered function, ϕ_t is an approximation to the identity, prove $\lim_{t\to 0}\phi_t = \delta_0$.

Proof. $\because$ $g \in \mathcal{S}$ is a tempered function, $\therefore$ $\|g\|_1 < \infty$, There exists a infinity M, $M = \sup_{x\in\mathbb{R}^n} |g(x)| < \infty$. $\because$ $\phi_t(g) = \int_{\mathbb{R}^n} g(y)\phi_t(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} g(y)\frac{1}{t^n}\phi(\frac{y}{t})\,\mathrm{d}y = \int_{\mathbb{R}^n} g(ty)\phi(y)\,\mathrm{d}y$, $|g(ty)\phi(y)| \le M|\phi(y)|$, $\phi(y)$ is integrable, by Lebesgue control convergence theorem, $g(ty)\phi(ty)$ is integrable and
$\lim_{t\to 0}\phi_t(g) = \int_{\mathbb{R}^n}(\lim_{t\to 0} g(ty))\phi(y)\,\mathrm{d}y = g(0) = \delta_0(g)$. Hence $\lim_{t\to 0}\phi_t = \delta_0$ $\square$

1.2. weakly bounded operator.

1.2.1. Suppose $T_t(t > 0)$ is a family of sublinear operators. T^* is the maximal operator of them. proof that $\{f \in L^p(\mathbb{R}^n) : \lim_{t\to 0} T_tf(x)$ exists for a.e. $x \in \mathbb{R}^n\}$ is closed subset of $\mathbb{R}^n$.

Proof. Suppose $f_n \in \Gamma \triangleq \{f \in L^p(\mathbb{R}^n) : \lim_{t\to 0} T_tf(x)$ exists for a.e. $x \in \mathbb{R}^n\}$, and $f_n \to f$.Let $A_s \triangleq \{x \in \mathbb{R}^n : |\overline{\lim}_{t\to 0}T_tf(x) - \underline{\lim}_{t\to 0}T_tf(x)| > s\}$, for $\forall s$, we need to proof that $|A_s = 0|$. $\because$ $|\overline{\lim}_{t\to 0}T_tf(x) - \underline{\lim}_{t\to 0}T_tf| = |\overline{\lim}_{t\to 0}T_tf(x) - \underline{\lim}_{t\to 0}T_tf_n + \underline{\lim}_{t\to 0}T_tf_n - \overline{\lim}_{t\to 0}T_tf_n + \overline{\lim}_{t\to 0}T_tf_n\underline{\lim}_{t\to 0}T_tf| \le |\overline{\lim}_{t\to 0}T_tf(x) - \underline{\lim}_{t\to 0}T_tf_n| + |\underline{\lim}_{t\to 0}T_tf_n - \overline{\lim}_{t\to 0}T_tf_n| + |\overline{\lim}_{t\to 0}T_tf_n\underline{\lim}_{t\to 0}T_tf| = |\overline{\lim}_{t\to 0}T_tf(x) - \underline{\lim}_{t\to 0}T_tf_n| + |\overline{\lim}_{t\to 0}T_tf_n\underline{\lim}_{t\to 0}T_tf| \le |\overline{\lim}_{t\to 0}T_t(f - f_n)| + |\overline{\lim}_{t\to 0}T_t(f_n - f)| \le 2T^*(f - f_n)$

$\therefore A_s \subset \{x \in \mathbb{R}^n : T^*(f - f_n)(x) > \frac{s}{2}\}$, because T^* is a weak-(p, q) operator, we have $|A_s| \le (\frac{C\|f-f_n\|_p}{\frac{s}{2}})^q \to 0$, as $n \to \infty$. $\therefore |A_s| = 0$. $\square$

1.3. Marcinkiewicz theorem.

1.3.1. Prove $\int_X |f(x)|^p\mathrm{d}\mu(x) = p\int_0^\infty s^{p-1}\lambda_f(s)\,\mathrm{d}s$.

Proof. $\because \int_X |f(x)|^p\mathrm{d}\mu(x) = \int_X\int_0^{|f(x)|} ps^{p-1}\mathrm{d}s\,\mathrm{d}\mu(x) = \int_X\int_o^\infty ps^{p-1}\chi_{\{s<|f(x)|\}}\mathrm{d}s\,\mathrm{d}\mu(x)$, by Fubini theorem, $= \int_0^\infty ps^{p-1}\int_X \chi_{\{|f(x)|>s\}}\mathrm{d}\mu(x)\,\mathrm{d}s = \int_0^\infty ps^{p-1}\lambda_f(s)\mathrm{d}s$. $\square$

1.3.2. Suppose sublinear operator T is a weak-(p_0, p_0) and weak-(p_1, p_1), prove $\|T\|_(p, p) \le C(p)\|T\|^{1-\theta}_{(p_0,p_0)}\|T\|^{\theta}_{(p_1,p_1)}$, when $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. $\square$

1.4. Riesz-Thorin interpolation theorem.

1.4.1. The assume f is finity in three-lines theorem can be reduced to $|f(z)| \leq e^{c|y|}, \forall z = x + iy$.

Proof. It suffices to prove that $\limsup\limits_{|y|\to\infty, x\in[a,b]} |f_\epsilon(x+iy)| = 0$.

$\because |f_\epsilon(x+iy)| = \left|\dfrac{e^{\epsilon z^2} f(z)}{M_a^{\frac{b-z}{b-a}} M_b^{\frac{z-a}{b-a}}}\right| = \left|\dfrac{e^{\epsilon(x^2-y^2)} f(z)}{M_a^{\frac{b-z}{b-a}} M_b^{\frac{z-a}{b-a}}}\right| \leq \left|\dfrac{e^{\epsilon(x^2-y^2)-c|y|}}{M_a^{\frac{b-z}{b-a}} M_b^{\frac{z-a}{b-a}}}\right| \leq Ce^{\epsilon(x^2-y^2)-c|y|}$

$\therefore |f_\epsilon(x+iy)| \to 0$, as $|y| \to \infty$. □

1.5. **Hardy-Littlewood maximal function.**

1.5.1. Show that for $\forall x \in U$, $\exists B(x,r)$ s.t. $U \subset B(x,r)$, and $|U| \geq C|B(x,r)|$.

Proof. □

1.5.2. Show that $Mf(x)$ is lower-semicontinuity.

Proof. Let $A \triangleq \{x \in \mathbb{R}^n : Mf(x) > s\}$, It suffices to prove A is open.
For $\forall x \in A$, $\exists B$ is an open rectangle, s.t $x \in B$, and $\frac{1}{|B|}\int_B f(x)\,\mathrm{d}x > s$.
$\therefore \forall y \in B$, $\frac{1}{|B|}\int_B f(x)\,\mathrm{d}x > s$, therefore $y \in A$, $B \subset A$, then A is open. □

1.5.3. Suppose $0 \leq \phi \in L^1(\mathbb{R}^n)$ is increasely, prove that there exists $\{\phi_k\}$ s.t. $\{\phi_k\}$ is a sequence of simple functions, and $\phi_k \to \phi$ increasely.

Proof. □

1.6. **Binary maximal function.**

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