HOMEWORK OF HARMONIC ANALYSIS

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1. LECTURE 1

1.1. **1.** Suppose $g \in \mathcal{S}$ is a tempered function, ϕ_t is an approximation to the identity, prove $\lim_{t\to 0} \phi_t = \delta_0$.

Proof. $\because$ $g \in \mathcal{S}$ is a tempered function, $\therefore$ $\|g\|_1 < \infty$, There exists a infinity M, $M = \sup_{x\in\mathbb{R}^n} |g(x)| < \infty$. $\because$ $\phi_t(g) = \int_{\mathbb{R}^n} g(y)\phi_t(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} g(y)\frac{1}{t^n}\phi(\frac{y}{t})\,\mathrm{d}y = \int_{\mathbb{R}^n} g(ty)\phi(y)\,\mathrm{d}y$, $|g(ty)\phi(y)| \le M|\phi(y)|$, $\phi(y)$ is integrable, by Lebesgue control convergence theorem, $g(ty)\phi(ty)$ is integrable and
$\lim_{t\to 0}\phi_t(g) = \int_{\mathbb{R}^n}(\lim_{t\to 0} g(ty))\phi(y)\,\mathrm{d}y = g(0) = \delta_0(g)$. Hence $\lim_{t\to 0}\phi_t = \delta_0$

1.2. **2.** Suppose $T_t(t>0)$ is a family of sublinear operators. T^* is the maximal operator of them. proof that $\{f \in L^p(\mathbb{R}^n) : \lim_{t\to 0} T_t f(x) \text{ exists for a.e. } x \in \mathbb{R}^n\}$ is closed subset of $\mathbb{R}^n$.

Proof. Suppose $f_n \in \Gamma \triangleq \{f \in L^p(\mathbb{R}^n) : \lim_{t\to 0} T_t f(x) \text{ exists for a.e. } x \in \mathbb{R}^n\}$, and $f_n \to f$.Let $A_s \triangleq \{x \in \mathbb{R}^n : |\overline{\lim}_{t\to 0} T_t f(x) - \underline{\lim}_{t\to 0} T_t f(x)| > s\}$, for $\forall s$, we need to proof that $|A_s = 0|$. $\because$ $|\overline{\lim}_{t\to 0} T_t f(x) - \underline{\lim}_{t\to 0} T_t f| = |\overline{\lim}_{t\to 0} T_t f(x) - \underline{\lim}_{t\to 0} T_t f_n + \underline{\lim}_{t\to 0} T_t f_n - \overline{\lim}_{t\to 0} T_t f_n + \overline{\lim}_{t\to 0} T_t f_n \underline{\lim}_{t\to 0} T_t f| \le |\overline{\lim}_{t\to 0} T_t f(x) - \underline{\lim}_{t\to 0} T_t f_n| + |\underline{\lim}_{t\to 0} T_t f_n - \overline{\lim}_{t\to 0} T_t f_n| + |\overline{\lim}_{t\to 0} T_t f_n \underline{\lim}_{t\to 0} T_t f| = |\overline{\lim}_{t\to 0} T_t f(x) - \underline{\lim}_{t\to 0} T_t f_n| + |\overline{\lim}_{t\to 0} T_t f_n \underline{\lim}_{t\to 0} T_t f| \le |\overline{\lim}_{t\to 0} T_t(f - f_n)| + |\overline{\lim}_{t\to 0} T_t(f_n - f)| \le 2T^*(f - f_n)$

$\therefore$ $A_s \subset \{x \in \mathbb{R}^n : T^*(f - f_n)(x) > \frac{s}{2}\}$, because T^* is a weak-(p,q) operator, we have $|A_s| \le (\frac{C\|f - f_n\|_p}{\frac{s}{2}})^q \to 0$, as $n \to \infty$. $\therefore$ $|A_s| = 0$.

1.3. **3.** Prove $\int_X |f(x)|^p \mathrm{d}\mu(x) = p\int_0^\infty s^{p-1}\lambda_f(s)\,\mathrm{d}s$.

Proof. $\because$ $\int_X |f(x)|^p \mathrm{d}\mu(x) = \int_X \int_0^{|f(x)|} ps^{p-1}\mathrm{d}s\,\mathrm{d}\mu(x) = \int_X \int_o^\infty ps^{p-1}\chi_{\{s<|f(x)|\}}\mathrm{d}s\,\mathrm{d}\mu(x)$, by Fubini theorem, $= \int_0^\infty ps^{p-1}\int_X \chi_{\{|f(x)|>s\}}\mathrm{d}\mu(x)\,\mathrm{d}s = \int_0^\infty ps^{p-1}\lambda_f(s)\mathrm{d}s$. ☐

1.4. **3.** Suppose sublinear operator T is a weak-(p_0,p_0) and weak-(p_1,p_1), prove $\|T\|_(p,p) \le C(p)\|T\|_{(p_0,p_0)}^{1-\theta}\|T\|_{(p_1,p_1)}^{\theta}$, when $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. *E-mail address*: xushijie@mail.ustc.edu.cn