Companion points and partially de Rham cycles.

1. Notation

- p : prime number.
- $n \geq 2$, $n \in \mathbb{Z}$
- $F/\mathbb{Q}$ quadratic imaginary. p splits in F. choose $v|p$, $F_v \simeq \mathbb{Q}_p$

  [Rather, F/F^+ CM, F^+ totally real + technical assumptions]
- G definite unitary group $/\mathbb{Q}$, splits over F, i.e. $G \times_{\mathbb{Q}} F = GL_{n/F}$. (fix the isomorphism)

  and $G(\mathbb{R})$ compact.
- Tame level : $U^p = \prod_{\ell \neq p} U_\ell \in G(\mathbb{A}_f^p) = \prod' G(\mathbb{Q}_\ell)$ open compact subgroup, small enough
- $G = G(\mathbb{Q}_p) \simeq GL_n(F_v) = GL_n(\mathbb{Q}_p)$.
- $L/\mathbb{Q}_p$ $[L:\mathbb{Q}_p] < +\infty$, Large enough coefficient field.

  $\mathcal{O}_L \leftarrow$ ring of integers $\quad k_L = \mathcal{O}_L/\varpi_L \leftarrow$ uniformizer
- p-adic automorphic forms:

  $\hat{S}(U^p, L) := \{ f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/U^p \to L, \text{ continuous} \} \circlearrowleft GL_n(\mathbb{Q}_p) = G$.

  Looks like $\mathcal{C}^{cont}(GL_n(\mathbb{Z}_p), L)^{\oplus s}$. (actually is)
- $\mathbb{T}^{sph}$: some algebra of Hecke operators $/\mathcal{O}_L$

  $\circlearrowleft \hat{S}(U^p, L)$

  Hecke eigenvalues in $\hat{S}(U^p, L)$ $\overset{\text{match Satake parameters and Frobenius eigenvalues}}{\rightsquigarrow}$ $\rho : \mathrm{Gal}(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_p)$ (semisimple)

  $\rightsquigarrow$ $m_\rho \in \mathrm{Spec}\, \mathbb{T}^{sph}$

  Hecke eigenvalues in $\hat{S}(U^p, k_L)$ $\rightsquigarrow$ $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \to GL_n(k_L)$.
- Fix $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \to GL_n(k_L)$ absolutely irreducible, modular.

  $\Pi\ (= \Pi(U^p)) := \hat{S}(U^p, L)_{m_{\bar{\rho}}} \neq 0$.

Expectation: comes from global.

$$\Pi[\rho] := \Pi(U^p)[m_\rho] \xleftrightarrow[\text{p-adic Local Langlands.}]{} \rho_p := \rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

$\circlearrowleft GL_n(\mathbb{Q}_p)$

- $\Pi[\rho]^{an}$ $\circlearrowleft G$ locally analytic representation (Schneider–Teitelbaum)

  $\cap$ $\Pi[\rho]$ subspace of locally analytic vectors (w.r.t. G)

Example : $\mathcal{C}^{cont}(GL_n(\mathbb{Z}_p), L)^{an} = \mathcal{C}^{\ell\text{-}an}(GL_n(\mathbb{Z}_p), L)$

2. Companion points.

$T = \begin{pmatrix} \mathbb{Q}_p^\times & & 0 \\ & \ddots & \\ 0 & & \mathbb{Q}_p^\times \end{pmatrix} = (\mathbb{Q}_p^\times)^n \subset G$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset G$.

- Emerton's Jacquet module

$\pi^{an} \circlearrowleft G$ Locally analytic representation. $\rightsquigarrow$ $J_B(\pi^{an}) \circlearrowleft T$ Locally analytic

- Eigenvariety

$$Y(U^p, \bar\rho) = \left\{ (\rho, \delta) \;\middle|\; \begin{array}{l} \rho: \text{ deformation of } \bar\rho\text{, n-dim'l p-adic cts rep of } \mathrm{Gal}(\bar F/F) \\ \delta: T \to \bar{\mathbb{Q}}_p^\times \text{ continous character} \\ \text{s.t. } \mathrm{Hom}_T(\delta, J_B(\pi[\rho]^{an})) \neq 0 \end{array} \right\}$$

$$\begin{array}{c} Y(U^p,\bar\rho) \\ \Big\downarrow \text{Locally finite} \\ W \end{array} \qquad \begin{array}{c} (\rho, \delta) \\ \big\downarrow \\ \delta|_{(\mathbb{Z}_p^\times)^n} \end{array}$$

$W = \{ \delta_0 : (\mathbb{Z}_p^\times)^n \longrightarrow \bar{\mathbb{Q}}_p^\times \text{, continuous} \}$

$\delta(U_p) = \delta\left(\begin{pmatrix} p^{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right)$

- Companion forms

Fix $(\rho, \delta) \in Y(U^p, \bar\rho)$, in general, $\exists$ $\delta' \neq \delta$ s.t $(\rho, \delta') \in Y(U^p, \bar\rho)$

Conj (Breuil, Hansen) The set $W_H(\rho) := \{\delta' \mid (\rho, \delta') \in Y(U^p, \bar\rho)\}$ is explicitly determined by $\rho_p = \rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$

- Locally analytic socle conjecture.

Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$, $z^\lambda : (z_1, \cdots, z_n) \in (\mathbb{Q}_p^\times)^n \longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}$

Assume $\delta = \underset{\text{locally algebraic characters}}{z^\lambda}\, \delta_{sm}$ where δ_{sm} is a smooth character.

Breuil: $\mathrm{Hom}_T(\delta, J_B(\pi[\rho]^{an})) = \mathrm{Hom}_G(\mathcal{F}(M(\lambda), \delta_{sm}), \pi[\rho]^{an})$

Here $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$: Verma module.

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$, $U(-)$: universal enveloping algebra.

Orlik-Strauch: $\mathcal{F}(M(\lambda), \delta_{sm}) \circlearrowleft G$. Locally analytic.

Jordan Holder factors.

$JH(\mathcal{F}(M(\lambda), \delta_{sm})) \overset{\text{generically}}{=\!=} \{ \mathcal{F}(L(\lambda'), \delta_{sm}) \mid L(\lambda') \in JH(M(\lambda)) \}$

irreducible $U(\mathfrak{g})$-module of highest weight λ'.

Example: $n=3$ $\lambda = (1,1,2)$

$M\begin{pmatrix}1\\1\\2\end{pmatrix} = \left[L\begin{pmatrix}0\\1\\3\end{pmatrix} - L\begin{pmatrix}0\\2\\2\end{pmatrix} - L\begin{pmatrix}1\\1\\2\end{pmatrix} \right]$

Conj (Breuil) Assume $(\rho, \delta) \in Y(U^p, \bar{\rho})$, $\delta = z^\lambda \delta_{sm}$, ρ_p de Rham, "δ_{sm} generic."
then the set $W_B(\rho) = \{(\lambda', \delta_{sm}') \mid \mathcal{F}(L(\lambda'), \delta_{sm}') \hookrightarrow \Pi[\rho]^{an}\}$ is explicitly
determined by ρ_p "companion constituent"

Remark: Conjectures on companion forms / constituents should be viewed as "Locally analytic version of the weight part of Serre's modularity conjecture".

Thm (Regular case: Breuil-Hellmann-Schraen, non-regular: W.)
Assume the Taylor-Wiles assumptions (on $\bar{\rho}$, F/F^+, G, U^p, $p > 2$)
Assume $(\rho, \delta = z^\lambda \delta_{sm}) \in Y(U^p, \bar{\rho})$ and ρ_p is generic crystalline
Then $(\rho, z^{\lambda'}\delta_{sm}) \in Y(U^p, \delta_{sm})$ iff $\mathrm{Hom}_G(\mathcal{F}(L(\lambda'), \delta_{sm}), \Pi[\rho]^{an}) \neq 0$
iff $(\lambda', \delta_{sm}) \in W_B(\rho)$ iff $z^{\lambda'}\delta_{sm} \in W_H(\rho)$.

Rmk: ① ρ_p generic: $\varphi \curvearrowright D_{cris}(\rho_p)$ $(\varphi_1, \dots, \varphi_n)$ eigenvalues,
then $\varphi_i \varphi_j^{-1} \notin \{1, p\}$ $i \neq j$
② regular: Hodge-Tate weights of ρ_p are pairwise distinct
③ regular: same result for δ_{sm}' (BHS)
non-regular: in progress (W.)

3. Partially classical constituents.
$n = 3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$.
① "Classical $\Rightarrow$ de Rham" If $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $L(\lambda)$ finite-dim $\circlearrowleft GL_3(\mathbb{Q}_p)$.
$\mathcal{F}(L(\lambda), \delta_{sm}) = L(\lambda) \otimes \pi_{sm}$ (smooth) "Locally algebraic representation".
Then $\mathrm{Hom}_G(\mathcal{F}(L(\lambda), \delta_{sm}), \Pi[\rho]^{an}) \neq 0 \Rightarrow \rho_p$ de Rham with regular HT weights.

② "finite slope $\Rightarrow$ trianguline" If $\mathrm{Hom}_T(\delta, J_B(\Pi[\rho]^{an})) \neq 0$, (étale in the sense of Kedlaya)
then ρ_p trianguline, i.e. $D_{rig}(\rho_p)$ ((φ, Γ)-module / Robba ring) admits a
filtration $0 \subset D_1 \subset D_2 \subset D_3 = D_{rig}(\rho_p)$ s.t. $D_1, D_2/D_1, D_3/D_2$ rank one (φ, Γ)-modules (up to enlarge L)
(Kisin, Colmez, Kedlaya-Pottharst-Xiao, R.Liu)
(Galois representation attached to Hida's ordinary family are reducible at p)

Thm (W. based on Y. Ding) "Partially classical ⇒ Partially de Rham"

Assume $\mathrm{Hom}_G(\mathcal{F}(L(\lambda), \delta_{sm}), \Pi[\rho]^{an}) \neq 0$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$

$(\Rightarrow)$ $(\rho, z^\lambda \delta_{sm}) \in Y(U^p, \bar\rho)$, $0 \subset D_1 \subset D_2 \subset D_3 = D_{rig}(\rho_p)$ associated with δ_{sm}

① If $\lambda_1 \geq \lambda_2$, then D_2 de Rham

② If $\lambda_2 \geq \lambda_3$, then D_3/D_1 de Rham.

Rmk: $\rho_p : V \circlearrowleft \mathrm{Gal}_{\bar{\mathbb{Q}}_p/\mathbb{Q}_p} \rightsquigarrow D_{rig}(\rho_p) \overset{\text{Berger}}{\rightsquigarrow} W_{dR}(D_{rig}(\rho_p)) = B_{dR} \otimes_{\mathbb{Q}_p} V \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

$D \overset{\text{Berger}}{\rightsquigarrow} W_{dR}(D)$

D de Rham $\Leftrightarrow$ $W_{dR}(D) = W_{dR}(D)^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes_{\mathbb{Q}_p} B_{dR}$.

• Reason: $\lambda = (1, 1, 2)$ (or $\lambda_1 \geq \lambda_2$)

$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset G$. $P = MN$, $M = GL_2(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$

$\mathfrak{p} = \mathrm{Lie}\, P$ $\quad \mathfrak{m} = \mathrm{Lie}\, M$

$L(\lambda)$ quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_M(\lambda)$ parabolic Verma module. (equal in this case)

• $L_M(\lambda)$ f. dim rep of $U(\mathfrak{m})$ of highest weight λ $\circlearrowleft M$

$\Rightarrow$ $\mathcal{O}^P \ni L(\lambda) \circlearrowleft M, U(\mathfrak{g})$

Then $\mathrm{Hom}_G(\mathcal{F}(L(\lambda), \delta_{sm}), \Pi[\rho]^{an}) \neq 0 \xRightarrow{\text{Breuil, Ding}}$

$\mathrm{Hom}_{\mathfrak{m}}(L_M(\lambda), J_P(\Pi[\rho]^{an}))[\delta_{sm}] \neq 0$.

$J_P(-)$: parabolic version Emerton's Jacquet module.

$J_P(\Pi[\rho]^{an}) \circlearrowleft M \rightsquigarrow 0 \subset D_2 \subset D_3 = D_{rig}(\rho_p)$ "paraboline filtration"

rank 2. Bergdall

$\cup$

non-zero locally algebraic rep of M $\qquad$ D_2, D_3/D_2 de Rham

Pf: Ding constructed a partial eigenvariety $(\subset Y(U^p, \bar\rho))$ using $J_P(\Pi^{an})$ (previously Hill-Loeffler) + Global triangulation + density of classical points

4. Local models for the trianguline variety.

$X_{tri}(\bar\rho_p) := \overline{\left\{ (\rho_p, \delta) \;\middle|\; \begin{array}{l} \rho_p : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_n(\bar{\mathbb{Q}}_p^\times), \text{ lifting of } \bar\rho_p, \delta : (\mathbb{Q}_p^\times)^n \to \bar{\mathbb{Q}}_p^\times \\ \exists \{0\} = D_0 \subset D_1 \subset \cdots \subset D_n = D_{rig}(\rho_p) \text{ s.t. } D_i/D_{i-1} = D_{rig}(\delta_i), \delta \text{ regular} \end{array} \right\}}$ Zariski closure.

$x = (\rho_p, \delta) \in X_{tri}(\bar\rho_p)$

$\subset \{(\rho_p, \delta)\}$

BHS, Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin:

$$\begin{array}{ccccc} Y(U^p,\bar\rho) & \hookrightarrow & X_p(\bar\rho) & \hookrightarrow & X_{tri}(\bar\rho_p) \times \mathbb{D}(0,1)^g \times \cdots \\ (\rho,\delta) & & \longmapsto & & ((\rho_p, \text{``}\delta\text{''}), \ z) \end{array}$$

Patched eigenvariety: $X_p(\bar\rho)$.

Pf of the main theorem:

Compare cycles $\mathcal{C}(L(\lambda)) :=$ "$\mathrm{Hom}_G(\mathcal{F}(L(\lambda), \delta_{sm}), \Pi[\rho]^{an})$ with ρ_p, δ_{sm} varying" $\subset X_p(\bar\rho)$

and cycles from the local models of $X_{tri}(\bar\rho_p)$: "partially de Rham property"

- Local models

Assume $\delta = z^h \delta_{sm}$, $h = (h_1, \dots, h_n) \in \mathbb{Z}^n$, δ_{sm} "generic"

$x = (\rho_p, \delta) \in X_{tri}(\bar\rho_p)(L)$

ρ_p is almost de Rham with HT weights h, $\{0\} \subset D_1 \subset \cdots \subset D_n = D_{rig}(\rho_p)$

$(\rho_p, \delta) \overset{\text{Fontaine}}{\rightsquigarrow}$ ① $D_{pdR}(\rho_p)$: n-dim L-space. $\simeq L^n$

② $\mathcal{F}_\bullet$: $\{0\} \subset D_{pdR}(D_1) \subset \cdots \subset D_{pdR}(D_n) = D_{pdR}(\rho_p)$ stabilizer $B \subset GL_n$.

③ $\mathrm{Fil}^\bullet$: HT filtration stabilizer $P \subset GL_n$

④ ν: nilpotent operator $\circlearrowright D_{pdR}(\rho_p)$, $\mathcal{F}_\bullet$, $\mathrm{Fil}^\bullet$.

$\Rightarrow$

$x_{pdR} \in X_p := \{(\nu, g_1B, g_2P) \in \mathfrak{gl}_n \times G/B \times G/P \mid \underset{\text{adjoint action}}{\mathrm{ad}(g_1^{-1})}\nu \in \mathrm{Lie}\, B, \ \mathrm{ad}(g_2^{-1})\nu \in \mathrm{Lie}\, P\}$

Fact $X_P = \bigcup_{w \in S_n/W_P} X_{P,w}$ ← irreducible components

W_P: Weyl gp of Levi of P.

Take $w \in S_n/W_P$, $h_1' < \cdots < h_n'$ s.t. $w(h_1', \dots, h_n') = (h_1, \dots, h_n)$

Thm (BHS, when h regular, W.)

Recall δ_{sm} generic. Then up to formally smooth maps, $\exists$ isomorphism

$$\widehat{X_{tri}(\bar\rho_p)}_x \simeq \widehat{X_{P,w}}_{,x_{pdR}}$$

And $X_{tri}(\bar\rho_p)$ is irreducible at x.

- Generalized Steinberg variety.

$Z_P := \{(\nu, g_1B, g_2P) \in X_P \mid \nu \text{ nilpotent}\} \subset X_P$

$Z_P = \bigcup_{w \in S_n/W_P} Z_{P,w}$ equidimensional.

$Z_{P,w}$: irred components of Z_P.

· Let $\lambda = (h_n', \cdots, h_1') + (0, 1, \cdots, n-1)$

BHS: $\quad \mathcal{C}(L(ww_0 \cdot \lambda)) \longleftrightarrow$ "$Z_{P,w}$"

Let $Q = \begin{pmatrix} \boxed{*}_{i} & * \\ & \boxed{*}_{n-i} \end{pmatrix}$ $\quad M_Q = GL_i \times GL_{n-i}$, $\; N_Q = \mathrm{Mat}_{i,n-i}$

Note that $\quad D_n = D_{\mathrm{rig}}(\rho_p)$ is de Rham $\Leftrightarrow$ v is zero

$\qquad D_i,\ D_n/D_{n-i}$ is de Rham $\Leftrightarrow$ v is zero on $\mathfrak{F}_i$, $\;\mathfrak{F}_n/\mathfrak{F}_{n-i}$

$\qquad\qquad \Leftrightarrow \mathrm{ad}(g_1^{-1}) v \in \mathrm{Lie}\, N_Q$

Thm ("Partially classical $\Rightarrow$ partially de Rham")

For $w \in W/W_P$, $Z_{P,w} \subset \{(v, g_1B, g_2P) \in Z_P \mid \mathrm{ad}(g_1^{-1}) v \in \mathrm{Lie}\, N_Q\}$ if and only if $L(ww_0 \cdot \lambda) \in \mathcal{O}^{\mathrm{Lie}\, Q}$ ($\Leftrightarrow$ $ww_0 \cdot \lambda$ is a dominant weight for M_Q)

Example: $n = 3$ $\qquad P = \begin{pmatrix} \boxed{} & \\ & \square \end{pmatrix}$ $\qquad g_1B = B$ $\qquad s_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$ $s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$

$\lambda = (1, 1, 2)$.

$$S_3/W_P = \{ e, \; s_2, \; s_1s_2 \}$$

$$v = \begin{pmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{pmatrix} \qquad g_2P = \begin{pmatrix} 1 & & \\ & 1 & \\ x & y & 1 \end{pmatrix} P \subset G/P$$

$$Z_P \text{ "=" } \mathrm{Spec}(L[x, y, a, b, c] / (\mathrm{ad}\, g_2^{-1} v \in \mathrm{Lie}\, P))$$

$$\| $$

$$\bigcup_{w \in S_3/W_P} \mathrm{Spec}(L[x, y, a, b, c] / I_w)$$

				$L(ww_0 \cdot \lambda)$
$I_e = (x, y)$	$\longleftrightarrow$	$Z_{P,e}$	$\longleftrightarrow$	$L\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$
$I_{s_2} = (x, c)$	$\longleftrightarrow$	Z_{P,s_2}	$\longleftrightarrow$	$L\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$
$I_{s_1s_2} = (a, xb + yc)$	$\longleftrightarrow$	Z_{P,s_1s_2}	$\longleftrightarrow$	$L\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Rmk: When $P = B$, same if we replace $Z_{P,w}$ by $\overset{Z_{P,w}\,?}{C_{P,w}}$: "characteristic cycle associated with the G-equivariant D-module $\mathcal{L}(ww_0 \cdot \lambda)$ on $G/B \times G/B$ localized from $L(ww_0 \cdot \lambda)$"