BERNSTEIN-ZELEVINSKY DUALITY FOR LOCALLY ANALYTIC PRINCIPAL SERIES REPRESENTATIONS

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Abstract. We consider certain dual of the Kohlhaase-Schraen resolutions for locally analytic principal series representations of p-adic Lie groups in the case of integral weights. The dual complexes calculate the expected Bernstein-Zelevinsky dual of the locally analytic representations and lead to the Grothendieck-Serre duality of coherent sheaves on patched eigenvarieties.

Contents

1. INTRODUCTION

Let p be a prime number. In this paper, we determine the Bernstein-Zelevinsky dual of a locally analytic principal series representation of a split reductive p-adic Lie group, induced from a locally algebraic character of a maximal torus. Take $d \geq 2$ and $G = \mathrm{GL}_d(\mathbb{Q}_p)$ in this introduction.

1.1. **Motivation.** Let π be a smooth representation of G over $\mathbb{C}$, the Bernstein-Zelevinsky duality (also called cohomological duality) is given by ([Ber92, §IV.5], see also [Far06])

$$\pi \mapsto \mathbb{D}_{\mathrm{BZ}}(\pi) := R\mathrm{Hom}_G(\pi, \mathcal{C}_c^{\mathrm{sm}}(G, \mathbb{C}))$$

where $\mathcal{C}_c^{\mathrm{sm}}(G, \mathbb{C})$ denotes the space of compactly supported locally constant functions on G, which is a bimodule over G via the left and right translations. An interesting property is that the duality should intertwine with the Grothendieck-Serre duality for coherent sheaves on the stack of Langlands parameters under the categorical Langlands correspondences (e.g. [Zhu20, Conj. 4.5.1 (1)]).

For p-adic representations of G, notably in the categorical p-adic local Langlands program of Emerton-Gee-Hellmann presented in [EGH23], similar dualities are proposed for smooth representations in natural characteristics [EGH23, Conj. 6.1.14]. It is also expected that such duality exists for locally analytic representations [EGH23, Rem. 6.2.22]. Recent work of Hellmann-Hernandez-Schraen [HHS24] gives strong evidence in this direction by establishing Serre duality for some sheaves on (patched) eigenvarieties. The property of these coherent sheaves is vital for *loc. cit.* to produce multiplicities of p-adic automorphic eigenforms.

In this paper, for a locally analytic representation π over a p-adic coefficient field E, we naively generalize the duality by

$$(1.1) \qquad\qquad \pi \mapsto \mathbb{D}_{\mathrm{BZ}}(\pi) := R\mathrm{Hom}_G(\pi, \mathcal{C}_c^{\mathrm{la}}(G, E))$$

where $\mathcal{C}_c^{\mathrm{la}}(G, E)$ is the space of compactly supported locally analytic functions on G.

Remark 1.1. Since (1.1) involves the derived category of locally analytic representations (that may not be admissible in the sense of Schneider-Teitelbaum [ST03]), we can and will use the solid formalism of locally analytic representations by Rodrigues Jacinto-Rodríguez Camargo [RJRC22, RJRC23] to make the definition rigorous.

We will not consider the general theory in this paper. Rather, we would like to calculate explicitly $\mathbb{D}_{\mathrm{BZ}}(\pi)$ when π is a locally analytic principal series representation. This will be enough to answer partially the expectation in [EGH23]. Whatever the definition of $\mathbb{D}_{\mathrm{BZ}}$, we expect that

$$\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_I^G \mathcal{W}) \simeq \mathrm{Hom}_G(\text{c-Ind}_I^G \mathcal{W}, \mathcal{C}_c^{\mathrm{la}}(G, E)) = \text{c-Ind}_I^G \mathcal{W}^\vee$$

where c-Ind$_I^G$ denotes the compact induction when I is a compact open subgroup of G, $\mathcal{W}$ is certain locally analytic representation of I (Theorem 5.3) and $\mathcal{W}^\vee$ is the continuous E-linear dual of $\mathcal{W}$. This would allow us to calculate $\mathbb{D}_{\mathrm{BZ}}(\pi)$ using a resolution of π by compactly induced representations as done for smooth representations by Schneider-Stuhler in [SS97].

1.2. **Duality for Kohlhaase-Schraen resolutions.** The resolution we consider is that of Kohlhaase-Schraen in [KS12]. Let B be the Borel subgroup of G of upper-triangular matrices with the maximal diagonal torus T. Let $\chi : T \to E^{\times}$ be a continuous character. Let $\pi = \operatorname{Ind}_B^G \chi$ be the locally analytic parabolic induction. Kohlhaase-Schraen found a presentation of π as a (derived) quotient of certain compactly induced representation:

$$\operatorname{Ind}_B^G \chi \simeq \text{c-Ind}_I^G \mathcal{W}_\sharp \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}.$$

Here I is the Iwahori subgroup of G, $\mathcal{W}_\sharp$ is a locally analytic representation of I (see (1.4) for the precise definition) and $\mathcal{H} = E[U_1, \cdots, U_d]$, where $U_1, \cdots, U_d \in \operatorname{End}_G(\text{c-Ind}_I^G \mathcal{W}_\sharp)$, is a polynomial (Iwahori-Hecke) algebra acting on c-Ind$_I^G \mathcal{W}_\sharp$ with the maximal ideal $\mathfrak{m} = (U_1 - 1, \cdots, U_d - 1)$ [1]. Such presentation arises from a Koszul resolution:
(1.2)
$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp := [\text{c-Ind}_I^G \mathcal{W}_\sharp \to \cdots \to \wedge^i \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp \to \cdots \to \text{c-Ind}_I^G \mathcal{W}_\sharp] \xrightarrow{\sim} \operatorname{Ind}_B^G \chi$$

in the derived category. Taking account of the expectation that $\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_I^G \mathcal{W}_\sharp) = \text{c-Ind}_I^G \mathcal{W}_\sharp^\vee$, and the self-duality of the Koszul complex $\wedge^\bullet \mathcal{H}^d$, $\mathbb{D}_{\mathrm{BZ}}(\operatorname{Ind}_B^G \chi)$ should be given by the following dual complex

$$\operatorname{Hom}_G(\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp, \mathcal{C}_c^{\mathrm{la}}(G, E)) \simeq \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp^\vee[-d]$$

with a shift of degree d. Let $\overline{B}$ be the opposite Borel subgroup. Let $\mathfrak{g}, \overline{\mathfrak{b}}$ be the Lie algebras of G and $\overline{B}$ with the universal enveloping algebras $U(\mathfrak{g}), U(\overline{\mathfrak{b}})$. The theorem below determines this dual.

Theorem 1.2. *Suppose that χ is a locally algebraic character with weight $\lambda \in \mathbb{Z}^n$ and let $\chi_{\mathrm{sm}} : T \to E^{\times}$ be the smooth part of χ. Then there is a quasi-isomorphism*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp^\vee[-d] \simeq \mathcal{F}_{\overline{B}}^G(\overline{M}(\lambda)^\vee, \mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}}))$$

where $\overline{M}(\lambda)^\vee = (U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \lambda)^\vee$ is the dual Verma module in the BGG category $\mathcal{O}^{\overline{\mathfrak{b}}}$ for the highest weight λ.

Here $\mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}}) = \chi_{\mathrm{sm}}^{-1}[-d]$ and $\mathcal{F}_{\overline{B}}^G(\overline{M}(\lambda)^\vee, \chi_{\mathrm{sm}}^{-1})$ is a locally analytic representation constructed using the functor $\mathcal{F}_{\overline{B}}^G(-,-)$ in [OS15] from certain $\mathfrak{g}$-modules and smooth representations of T. It admits the same Jordan-Hölder factors as $\operatorname{Ind}_{\overline{B}}^G \chi^{-1} = \mathcal{F}_{\overline{B}}^G(\overline{M}(\lambda), \chi_{\mathrm{sm}}^{-1})$. Similarly, $\operatorname{Ind}_B^G \chi = \mathcal{F}_B^G(M(-\lambda), \chi_{\mathrm{sm}})$ where $M(-\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (-\lambda) \in \mathcal{O}^{\mathfrak{b}}$ is the Verma module. We have a general result for representations that are in the image of the functor $\mathcal{F}_B^G$.

Theorem 1.3 (Theorem 3.15 and Theorem 3.21). *Let M be a $\mathfrak{g}$-module in the BGG category $\mathcal{O}^{\mathfrak{b}}$ with algebraic weights and χ_{sm} be a smooth character of T, then there exists a Koszul resolution of $\mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp \simeq \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$$

for some Banach I-representation $\mathcal{W}_\sharp$ such that there exists a quasi-isomorphism

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_\sharp^\vee[-d] \simeq \mathcal{F}_{\overline{B}}^G(\operatorname{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty}, \mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}}))$$

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[1] The ideal $\mathfrak{m}$ differs from that in [KS12] because of our normalizations of the U_i-operators (Remark 2.5).

where $\mathrm{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty} \in \mathcal{O}^{\overline{\mathfrak{b}}}$ is certain BGG dual of $M \in \mathcal{O}^{\mathfrak{b}}$ appeared in Breuil's adjunction formula [Bre15, Prop. 4.2] (see (1.3)).

The calculation leads to the following conjecture.

Conjecture 1.4. *Let P be a parabolic subgroup of G (containing B) with Lie algebra $\mathfrak{p}$ and the opposite $\overline{P}$. Let M be a $\mathfrak{g}$-module in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ with algebraic weights and V be a finitely presented admissible smooth representation of the Levi factor of P, then we have*

$$\mathbb{D}_{\mathrm{BZ}}(\mathcal{F}_P^G(M, V)) = \mathcal{F}_{\overline{P}}^G(\mathrm{Hom}_E(M, E)^{\mathfrak{n}_{\overline{P}}^\infty}, \mathbb{D}_{\mathrm{BZ}}(V))$$

where $\mathfrak{n}_{\overline{P}}$ is the nilradical of the Lie algebra of $\overline{P}$.

Theorem 1.5 (Theorem 5.5)**.** *Conjecture 1.4 is true if $P = B$, V is a smooth character of T and $\mathbb{D}_{\mathrm{BZ}}$ is defined by (1.1) (Definition 5.4).*

Remark 1.6. If $M = E$ is the trivial $\mathfrak{g}$-module, then $\mathcal{F}_P^G(M, V) = (\mathrm{Ind}_P^G V)^{\mathrm{sm}}$ is the smooth parabolic induction. In this case the conjecture says that $\mathbb{D}_{\mathrm{BZ}}((\mathrm{Ind}_P^G V)^{\mathrm{sm}}) = (\mathrm{Ind}_{\overline{P}}^G \mathbb{D}_{\mathrm{BZ}}(V))^{\mathrm{sm}}$, which matches with the duality for smooth representations [Ber92, Thm. IV.31]. As in *loc. cit.*, one possible approach to proving the conjecture would be establishing the second adjointness theorem for the functor $\mathcal{F}_P^G$, in some way generalizing Breuil's adjunction formula [Bre15, Prop. 4.2] (which is based on Emerton's adjunction [Eme07], see also [BHS19, Lem. 5.2.1]):

$$(1.3) \qquad \mathrm{Hom}_G(\mathcal{F}_P^G(M, V), \Pi) = \mathrm{Hom}_{(\mathfrak{g}, \overline{P})}(\mathrm{Hom}_E(M, E)^{\mathfrak{n}_{\overline{P}}^\infty} \otimes_E \mathcal{C}_c^{\mathrm{sm}}(N_{\overline{P}}, V), \Pi)$$

to the derived setting.

Remark 1.7. Let $\tau : g \mapsto (g^t)^{-1}$ be the Chevalley involution (inverse transpose) of $G = \mathrm{GL}_d(\mathbb{Q}_p)$ which switches B and $\overline{B}$. Let $\mathbb{D} = \tau \circ \mathbb{D}_{\mathrm{BZ}}$ be the twist by the Chevalley involution of our duality functor. Then the above theorem says that $\mathbb{D}(\mathcal{F}_B^G(M, \chi_{\mathrm{sm}})) = \mathcal{F}_B^G(M^\vee, \chi_{\mathrm{sm}})$ up to a degree shift, where $(-)^\vee$ denotes the dual in the BGG category $\mathcal{O}^{\mathfrak{b}}$. This is exactly the duality expected in [EGH23, Rem. 6.2.22], and discussed in [HHS24, Thm. 1.4].

1.3. **Duality for coherent sheaves.** Using the Koszul resolutions, we can verify the categorical expectation for the Bernstein-Zelevinsky duality in some global setting. The I-representations as $\mathcal{W}_\sharp$ and $\mathcal{W}_\sharp^\vee$ define coefficient systems on locally symmetric spaces with the Iwahori level at p. The cohomologies or homologies of these coefficient systems, which define overconvergent p-adic automorphic forms (e.g., [Urb11, AS08, Loe11, HN17]), should be related by Poincaré duality and induce certain Serre duality of coherent sheaves on the (patched) eigenvariety after taking finite slope parts. In this paper, we will work in simple, and more local, settings of abstract patched eigenvarieties.

In §5.3, using patched completed homologies in [CEG$^+$16], from a locally analytic representation π appeared in Theorem 1.3, we can attach two coherent sheaves $\mathfrak{A}_\infty^{\mathrm{rig}}(\pi), \mathfrak{A}_\infty'^{\mathrm{rig}}(\pi)$ supported respectively on two rigid analytic spaces $\mathfrak{X}$ and $\mathfrak{X}'$. Here $\mathfrak{X}$ (resp. $\mathfrak{X}'$) is roughly the space of deformations of a mod-p Galois representation $\overline{\rho}$ (resp. the dual $\overline{\rho}^\vee = \tau \circ \overline{\rho}$ up to a twist where τ is given by the inverse transpose). The Chevalley involution induces an isomorphism $\eta : \mathfrak{X} \simeq \mathfrak{X}'$. The following theorem is a formal consequence of Theorem 1.3 and the Poincaré duality of completed cohomologies.

Theorem 1.8 (Theorem 5.12). *Let $\pi = \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ be as in Theorem 1.3. There exists an isomorphism*

$$\mathbb{D}_{\mathrm{GS}}(\mathfrak{A}_\infty^{\mathrm{rig}}(\pi)) \simeq \eta^* \mathfrak{A}_\infty^{',\mathrm{rig}}(\mathbb{D}_{\mathrm{BZ}}(\pi)).$$

of coherent sheaves on $\mathfrak{X}$, where $\mathbb{D}_{\mathrm{GS}}(-)$ denotes the Grothendieck-Serre duality for coherent sheaves on $\mathfrak{X}$.

1.4. **Construction and proof.** Now we go into more (technical) details of Theorem 1.3, by explaining the two types of resolutions $\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W}_\sharp$ and $\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W}_\flat$ of a principal series $\mathrm{Ind}_B^G \chi$ where χ is locally algebraic of weight λ.

Take $n \geq 1$. Let $\mathcal{C}^{I_n-\mathrm{an}}(I, E)$ be functions on I that is rigid analytic on all cosets for certain open normal subgroup $I_n \subset I$. Let $\mathcal{D}^{I_n-\mathrm{an}}(I, E) := \mathcal{C}^{I_n-\mathrm{an}}(I, E)'$ be the continuous dual of $\mathcal{C}^{I_n-\mathrm{an}}(I, E)$, the I_n-analytic distribution algebra. Consider the I-subspace

$$\mathrm{Ind}_B^G \chi(BI) = \{f \in \mathcal{C}^{\mathrm{la}}(G, E) \mid f(bg) = \chi(b)f(g), \forall b \in B, \mathrm{supp}(f) \subset BI\} \subset \mathrm{Ind}_B^G \chi$$

of functions in $\mathrm{Ind}_B^G \chi$ supported on $BI \subset G$. Then the representation $\mathcal{W}_\sharp$ of I in (1.2) is defined to be

$$(1.4) \qquad \mathcal{W}_\sharp := \mathrm{Ind}_B^G \chi(BI) \cap \mathcal{C}^{I_n-\mathrm{an}}(I, E)$$

of functions that are "I_n-rigid analytic" which is naturally a module over $\mathcal{D}^{I_n-\mathrm{an}}(I, E)$.

Let $\overline{N}$ be the unipotent radical of $\overline{B}$. Using that $B \backslash BI = I \cap \overline{N}$, we may identify the space $\mathrm{Ind}_B^G \chi(BI) = \mathrm{Ind}_{I \cap B}^I \chi$ with $\mathcal{C}^{\mathrm{la}}(I \cap \overline{N}, E)$, the space of locally analytic functions on $I \cap \overline{N}$. There are inclusions of subspaces

$$(1.5) \qquad \mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E) \subset \mathcal{W}_\sharp \subset \mathcal{C}^{\mathrm{la}}(I \cap \overline{N}, E) = \mathrm{Ind}_B^G \chi(BI) \subset \mathrm{Ind}_B^G \chi$$

where $\mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$ is the space of polynomial (algebraic) functions on $I \cap \overline{N}$. The maps in (1.5) are equivariant for a subalgebra $\mathcal{D}(\overline{B} \cap I, \mathfrak{g}) \subset \mathcal{D}^{I_n-\mathrm{an}}(I, E)$ generated by $U(\mathfrak{g})$ and the distribution algebra of $\overline{B} \cap I$. Moreover, as a $\mathfrak{g}$-module, $\mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$ can be identified to the dual Verma module $\overline{M}(\lambda)^\vee$ in $\mathcal{O}^{\overline{\mathfrak{b}}}$ whose appearance is also the first step for (1.3).

The first inclusion of (1.5) induces a natural map

$$\mathcal{W}_\flat := \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\overline{B} \cap I, \mathfrak{g})} \mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E) \to \mathcal{W}_\sharp.$$

Together with the Kohlhaase-Schraen resolution (1.2), we arrive at least a map of complexes

$$(1.6) \qquad \wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W}_\flat \to \wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W}_\sharp \to \mathrm{Ind}_B^G \chi.$$

Note that in contrast to the locally analytic representation $\mathcal{W}_\sharp$, the I-representation $\mathcal{W}_\flat$ is finite over the distribution algebra $\mathcal{D}^{I_n-\mathrm{an}}(I, E)$ and is usually considered as a (continuous linear) dual of a locally analytic representation as $\mathcal{W}_\flat^\vee$ (even though $\mathcal{W}_\flat, \mathcal{W}_\sharp^\vee$ are still locally analytic representations). The left-hand side of (1.6) will be the complex that is "dual" to the original Kohlhaase-Schraen resolution in Theorem 1.3 for $\mathbb{D}_{\mathrm{BZ}}(\mathrm{Ind}_B^G \chi) = \mathcal{F}_{\overline{B}}^G(\overline{M}(\lambda)^\vee, \mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}}))$.

We prove that (1.6) is a quasi-isomorphism, thus obtaining the second type resolution of $\mathrm{Ind}_B^G \chi$ in Theorem 1.3. The major difficulty lies in the surjectivity of the map

$$(1.7) \qquad \text{c-Ind}_I^G \mathcal{W}_\flat \to \mathrm{Ind}_B^G \chi.$$

The image of $\mathcal{W}_\flat$ is a union of I-translations of some Banach completion of $\mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$. Our proof of the surjectivity boils down to some statement similar to Proposition 1.9 below

on different completions of the $U(\mathfrak{g})$-module $\overline{M}(\lambda)^\vee \simeq \mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E) \simeq \mathcal{C}^{\mathrm{pol}}(\overline{N}, E)$. We let $\widehat{U}(\mathfrak{g})$ be the Arens-Michael envelope of the universal enveloping algebra $U(\mathfrak{g})$, which is a Fréchet completion of $U(\mathfrak{g})$. We write $\mathbf{G} = \mathrm{GL}_n \supset \overline{\mathbf{B}} \supset \overline{\mathbf{N}}$ for the algebraic groups such that $\overline{B} = \overline{\mathbf{B}}(\mathbb{Q}_p), \overline{N} = \overline{\mathbf{N}}(\mathbb{Q}_p)$. The space $\mathcal{C}^{\mathrm{an}}(\overline{N}, E)$ of rigid analytic functions on the analytification $\overline{\mathbf{N}}^{\mathrm{an}}$ of $\overline{\mathbf{N}}_E$ is a Fréchet completion of the space $\mathcal{C}^{\mathrm{pol}}(\overline{N}, E)$ of all algebraic functions.

Proposition 1.9. *The isomorphism* $\overline{M}(\lambda)^\vee \simeq \mathcal{C}^{\mathrm{pol}}(\overline{N}, E)$ *extends to a surjection (even an isomorphism)* $\widehat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M}(\lambda)^\vee \to \mathcal{C}^{\mathrm{an}}(\overline{N}, E)$.

Except for the case $\mathbf{G} = \mathrm{SL}_2$ where we calculate explicitly (Example 3.14), the proof of the above statement uses essentially the theory of rigid analytic $\widehat{\mathcal{D}}$-modules and the Beilinson-Bernstein localization for $\widehat{U}(\mathfrak{g})$-modules of Ardakov-Wadsley (see for example [Ard14], even though the actual proof will only use the theory developed in [AW13] for Banach completions). The reason is that it's easier to compare the two types of completions of $U(\mathfrak{g})$-modules locally after the localization. In the case that $\lambda = 0$, the dual Verma module $\overline{M}(\lambda)^\vee$ is localized to the $\mathcal{D}$-module $j_* \mathcal{O}_{\overline{\mathbf{N}}}$ where $j : \overline{\mathbf{N}} \simeq \overline{\mathbf{N}}w_0 \hookrightarrow \mathbf{G}/\overline{\mathbf{B}}$. Proposition 1.9 is roughly equivalent to that the localization of $\widehat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M}(\lambda)^\vee$ is the $\widehat{\mathcal{D}}$-module $j_*^{\mathrm{an}} \mathcal{O}_{\overline{\mathbf{N}}^{\mathrm{an}}}$. We emphasize the non-trivial fact that the direct image $j_*^{\mathrm{an}} \mathcal{O}_{\overline{\mathbf{N}}^{\mathrm{an}}}$ is coadmissible over $\widehat{\mathcal{D}}$ which implies that $\mathcal{C}^{\mathrm{an}}(\overline{N}, E)$ is coadmissible over $\widehat{U}(\mathfrak{g})$ by taking the global section. Such direct images of $\widehat{\mathcal{D}}$-modules were studied by Bitoun-Bode in [BB21] and our proof is largely inspired by their methods.

Remark 1.10. As shown by [BB21], the direct image of a coadmissible $\widehat{\mathcal{D}}$-module may not be coadmissible. This is not the case for us with the integral weights but remains a serious issue for more general weights. We don't know whether the resolution (1.6) holds if the weights of the character χ are not "p-adically non-Liouville" in some way, even when $\mathbf{G} = \mathrm{SL}_2$. This is the reason that we restrict to locally algebraic characters for parabolic inductions. The duality of representations of general weights or families of representations is still mysterious to the authors.

1.5. **Overview.** We review the construction of Kohlhaase-Schraen in §2. In §3, we establish the resolutions for representations and study dualities between complexes, thus proving the main Theorem 1.3. The proof of the surjectivity of (1.7) (and Proposition 1.9) is postponed to §4. In the last section §5, we embrace the solid formalism to discuss the duality between representations and coherent sheaves (Theorem 1.8).

1.7. **Notation and convention.** Let L be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O} = \mathcal{O}_L$ and a uniformizer $\varpi = \varpi_L$. We take a finite large enough extension E of $\mathbb{Q}_p$ as the coefficient field.

Fix $\mathbf{G}$ to be a connected split reductive group over L with a Borel subgroup $\mathbf{B} = \mathbf{TN}$ where $\mathbf{T}$ is a split maximal torus. We fix also split models $\mathbf{G}_{\mathcal{O}_L} \supset \mathbf{B}_{\mathcal{O}_L} \supset \mathbf{T}_{\mathcal{O}_L}$ over $\mathcal{O}_L$ of $\mathbf{G}$. Let d be the rank of $\mathbf{T}$. Write $G = \mathbf{G}(L) \supset B = \mathbf{B}(L) \supset T = \mathbf{T}(L)$ and similarly $N = \mathbf{N}(L)$. Denote by $\overline{B} = \overline{N}T$ the opposite Borel subgroup. Let $\Phi \supset \Phi_+ \supset \Delta$ be the set of roots, positive roots and simple roots of $\mathbf{G}$, corresponding to $\mathbf{B}$. Let $X_*(\mathbf{T})$ (resp. $X^*(\mathbf{T})$) be the lattice of coweights (resp. weights) of $\mathbf{T}$.

Write $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$, etc. for the base change to E of the $\mathbb{Q}_p$-Lie algebras of G, B, T, N, etc. Write $U(\mathfrak{g}), U(\mathfrak{b})$, etc. for the universal enveloping algebras.

Set $K = \mathbf{G}_{\mathcal{O}_L}(\mathcal{O}_L)$, a hyperspecial maximal compact open subgroup of G. Write $B_0 = K \cap B, T_0 = T \cap K$, etc. Set $T^- = \{t \in T \mid t(K \cap \overline{N})t^{-1} \subset K \cap \overline{N}\}$ and $T^+ = (T^-)^{-1}$.

For a p-adic Lie group G, we write $\mathcal{C}^{\mathrm{la}}(G, E)$ for the space of locally $\mathbb{Q}_p$-analytic functions on G with values in E. Write $\mathcal{D}(G, E) = \mathcal{C}^{\mathrm{la}}(G, E)'$ for the distribution algebra.

Let I be the preimage of $\mathbf{B}_{\mathcal{O}_L}(\mathcal{O}_L/\varpi_L)$ under the reduction map $K \to \mathbf{G}_{\mathcal{O}_L}(\mathcal{O}_L/\varpi_L)$. The Iwahori subgroup I admits the Iwahori decomposition $I = (I \cap \overline{N})(I \cap T)(I \cap N)$. There exists and we fix a sequence of good open normal subgroups $\cdots \subset I_{n+1} \subset I_n \subset \cdots \subset I$ where $n \geq 1$ forming a neighborhood basis of the neutral element of G and admitting Iwahori decompositions $I_n = (I_n \cap \overline{B})(I_n \cap T)(I_n \cap B)$, coming from decompositions of rigid analytic groups $\mathbf{I}_n$ as in [Eme06, Prop. 4.1.3].

We let $\mathcal{C}^{\mathrm{an}}(I_n, E)$ be the rigid analytic functions on the rigid space $(\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{I}_n) \otimes_L E$. Set $\mathcal{C}^{I_n-\mathrm{an}}(I, E) = \oplus_{g \in I/I_n} g.\mathcal{C}^{\mathrm{an}}(I_n, E) \subset \mathcal{C}^{\mathrm{la}}(I, E)$. Since I_n is normal in I and any element $g \in I$ induces a rigid analytic automorphism of $\mathbf{I}_n$ by conjugation, we have $g.\mathcal{C}^{\mathrm{an}}(I_n, E) = \mathcal{C}^{\mathrm{an}}(I_n g^{-1}, E) = \mathcal{C}^{\mathrm{an}}(g^{-1}I_n, E)$ for all $g \in I$. The space $\mathcal{C}^{I_n-\mathrm{an}}(I, E)$ consists of I_n-rigid analytic functions on I. Let $\mathcal{D}^{I_n-\mathrm{an}}(I, E)$ be the continuous linear dual of $\mathcal{C}^{I_n-\mathrm{an}}(I, E)$.

We use τ to denote the Chevalley involution (inverse transpose) of $\mathbf{G}, G$ or $\mathfrak{g}$. For $M \in \mathcal{O}^{\mathfrak{b}}$, we will write $\tau(M)^\vee = \mathrm{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty}$. See Remark 3.4.

2. Kohlhaase-Schraen resolutions

In this section, we review and extend (a little bit of) the work of Kohlhaase-Schraen in [KS12, §2] for resolutions of parabolic inductions.

2.1. **The Koszul complexes.** We start with some general considerations of compact inductions (from the Iwahori subgroup I) and Hecke operators. Then we turn to Kohlhaase-Schraen resolutions for parabolic inductions (from the Borel subgroup B)

Suppose that $\mathcal{W}$ is a representation of the Iwahori subgroup I of the split p-adic reductive group G over E. Consider the abstract compactly induced representation

$$\text{c-Ind}_I^G \mathcal{W} = \{f : G \to \mathcal{W} \text{ compactly supported } \mid f(xg) = x.f(g), \forall x \in I\}$$

on which G acts by right translations. If $\mathcal{W}$ is a locally analytic representation of I, c-Ind$_I^G \mathcal{W}$ can be equipped with the locally convex direct sum topology so that c-Ind$_I^G \mathcal{W}$ is a locally analytic representation of G.

For $g \in G, w \in \mathcal{W}$, we use the notation $[g, w] = g[1, w]$ to denote the function in c-Ind$_I^G \mathcal{W}$ supported on Ig^{-1} sending $ig^{-1} \in Ig^{-1}$ to $i.w$. Then $[g, w] = [gi, i^{-1}w]$ for $i \in I$. We also use

the notation $[g, \mathcal{W}] = [gI, \mathcal{W}]$ to denote the space of all functions in c-Ind$_I^G \mathcal{W}$ supported in Ig^{-1}. For a subset $S \subset G$, write $[S, \mathcal{W}] = \sum_{s \in S}[s, \mathcal{W}] \subset$ c-Ind$_I^G \mathcal{W}$. Then c-Ind$_I^G \mathcal{W} = \oplus_{g \in G/I}[g, \mathcal{W}] = \oplus_{g \in I \backslash G/I}[IgI, \mathcal{W}]$.

The Hecke (endomorphism) algebra $\text{End}_G(\text{c-Ind}_I^G \mathcal{W}) = \text{Hom}_I(W, (\text{c-Ind}_I^G \mathcal{W})|_I)$ acts on c-Ind$_I^G \mathcal{W}$, where the equality follows from the usual Frobenius reciprocity. For $t \in T$,

$$\text{Hom}_I(\mathcal{W}, (\text{Ind}_I^{ItI} \mathcal{W})|_I) = \text{Hom}_I(\mathcal{W}, \text{Ind}_{I \cap tIt^{-1}}^I [t, \mathcal{W}]) = \text{Hom}_{I \cap tIt^{-1}}(W, [t, W])$$

gives rise to a subspace of $\text{End}_G(\text{c-Ind}_I^G \mathcal{W})$ via $\text{Ind}_I^{ItI} \mathcal{W} = [ItI, \mathcal{W}] \subset$ c-Ind$_I^G \mathcal{W}$.

Assume that for $t \in T^-$, there is an element $U_t \in \text{Hom}_{I \cap tIt^{-1}}(\mathcal{W}, [t, \mathcal{W}]) \subset \text{End}_G(\text{c-Ind}_I^G \mathcal{W})$. Equivalently, we make t^{-1} act on $\mathcal{W}$ by an operator $\psi_t : \mathcal{W} \to \mathcal{W}$ in the sense that the map $w \mapsto [t, \psi_t.w]$ is in $\text{Hom}_{I \cap tIt^{-1}}(\mathcal{W}, [t, \mathcal{W}])$. More explicitly, ψ_t satisfies the condition

$$(t^{-1}xt)\psi_t(w) = \psi_t(xw)$$

for $x \in I \cap tIt^{-1}, w \in \mathcal{W}$. The corresponding action of U_t on c-Ind$_I^G \mathcal{W}$ is given by

$$(2.1) \qquad\qquad U_t[g, w] = \sum_{xtI \in ItI/I} [gxt, \psi_t(x^{-1}w)], \forall g \in G, w \in \mathcal{W}$$

by [KS12, (2.2)] (here $xtI \in ItI/I$ is equivalent to $x \in I/(I \cap tIt^{-1})$).

Lemma 2.1. *Suppose that $\psi_{t_1} \circ \psi_{t_2} : \mathcal{W} \to \mathcal{W}$ is equal to $\psi_{t_1 t_2}$ for all $t_1, t_2 \in T^-$ (in other words, $\mathcal{W}$ becomes a module over the semi-group IT^+I), then $U_{t_1} U_{t_2} = U_{t_1 t_2}$.*

Proof. By the Iwahori decomposition, $ItI/I = (I \cap \overline{N})/t(I \cap \overline{N})t^{-1} \cdot tI/I$ for any $t \in T^-$. We calculate that

$$U_{t_1} U_{t_2}[g, w]$$
$$= \sum_{xt_1 I \in It_1 I/I} \sum_{yt_2 I \in It_2 I/I} [gyt_2 xt_1, \psi_{t_1}(x^{-1}\psi_{t_2}(y^{-1}w))]$$
$$= \sum_{t_2 xt_2^{-1} \in t_2(I \cap \overline{N})t_2^{-1}/t_1 t_2(I \cap \overline{N})(t_1 t_2)^{-1}, y \in (I \cap \overline{N})/t_2(I \cap \overline{N})t_2^{-1}} [gy(t_2 xt_2^{-1})t_2 t_1, \psi_{t_1}\psi_{t_2}((yt_2 xt_2^{-1})^{-1}w)]$$
$$= \sum_{x \in (I \cap \overline{N})/t_1 t_2(I \cap \overline{N})(t_1 t_2)^{-1}} [gxt_2 t_1, \psi_{t_1}\psi_{t_2}(x^{-1}w)],$$

where for the second equality, we used that $x^{-1}\psi_{t_2}(-) = \psi_{t_2}(t_2 x^{-1} t_2^{-1}-)$ since $t_2 x^{-1} t_2^{-1} \in t_2(I \cap \overline{N})t_2^{-1} \subset I \cap t_2 It_2^{-1}$. The result follows. $\square$

There is an isomorphism $T/T_0 \simeq X_*(\mathbf{T})$ of $\mathbb{Z}$-modules characterized by $\langle \chi, t \rangle = \text{val}_\varpi(\chi(t))$ for $\chi \in X^*(\mathbf{T}), t \in T$, where val_ϖ denotes the ϖ-adic valuation. Under the identification $T^-/T_0 = X_*(\mathbf{T})^- = \{\mu \in X_*(\mathbf{T}) \mid \langle \alpha, \mu \rangle \leq 0, \forall \alpha \in \Phi_+\}$ corresponds to the set of antidominant coweights. It contains $X_*(\mathbf{T})_0 = \{\mu \in X_*(\mathbf{T}) \mid \langle \alpha, \mu \rangle = 0, \forall \alpha \in \Phi\} \simeq \mathbb{Z}^{d-|\Delta|}$. Write $\mathbf{Z}$ for the center of $\mathbf{G}$ and $Z = \mathbf{Z}(L)$. Then $\mathbf{Z} \subset \mathbf{T}$ ([Mil17, Prop. 21.7]) and $Z/(Z \cap K) \simeq X_*(\mathbf{T})_0$. Choose generators $z_1, \cdots, z_{d-|\Delta|}$ of $Z/(Z \cap K)$. And we choose a splitting $X_*(\mathbf{T}) \simeq X_*(\mathbf{T})' \oplus X_*(\mathbf{T})_0$ where $X_*(\mathbf{T})' \simeq \mathbb{Z}^{|\Delta|}$ is dual to $X^*(\mathbf{T}) \cap \mathbb{Q}\Phi$. It's easy to see that the splitting induces an isomorphism $X_*(\mathbf{T})^- \simeq \mathbb{N}^{|\Delta|} \oplus \mathbb{Z}^{d-|\Delta|}$ as monoids and we can choose generators $t_\alpha \in T^-/T_0, \alpha \in \Delta$ for the

free monoid $X_*(\mathbf{T})^- \cap X_*(\mathbf{T})' \simeq \mathbb{N}^{|\Delta|}$ (we use the same notation as in [KS12, §1] but we don't require $\langle \alpha, t_\alpha \rangle = -1$!).

Let $\mathcal{H} = E[T^-/T_0] = E[X_*(\mathbf{T})^-]$ be the monoid ring. Then $\mathcal{H} = E[z_i^\pm, t_\alpha, i = 1, \cdots, d - |\Delta|, \alpha \in \Delta]$.

Choose an ordering of Δ and we rewrite $(t_1, \cdots, t_d)$ for the sequence $(z_1, \cdots, z_{d-|\Delta|}, t_\alpha, \alpha \in \Delta)$ of elements in $\mathcal{H}$. We write $\mathfrak{m} \subset \mathcal{H}$ for the ideal $(t_1 - 1, \cdots, t_d - 1)$.

From now on we will always work in the situation that given $\mathcal{W}$, there are operators ψ_t for $t \in T^-$ making $\mathcal{W}$ an IT^+I-module as in Lemma 2.1. This implies that c-$\mathrm{Ind}_I^G \mathcal{W}$ is an $\mathcal{H}$-module given by (2.1) (note that U_t is the identity map if $t \in T_0$.). And we will write the corresponding Hecke operators $U_t \in \mathcal{H}$ for $t \in T^-/T_0$. If $f : \mathcal{W}_1 \to \mathcal{W}_2$ is an IT^+I-equivariant map, then the induced G-map c-$\mathrm{Ind}_I^G \mathcal{W}_1 \to$ c-$\mathrm{Ind}_I^G \mathcal{W}_2$ is $\mathcal{H}$-equivariant.

Recall for the sequence of elements $(U_{t_1} - 1, \cdots, U_{t_d} - 1)$, there is a homological complex $\wedge^\bullet \mathcal{H}^d$ of $\mathcal{H}$-modules with differentials

$$\wedge^i \mathcal{H}^d \to \wedge^{i-1} \mathcal{H}^d : f_1 \wedge \cdots \wedge f_i \mapsto \sum_{j=1}^i (-1)^{j+1} \varphi(f_j) f_1 \wedge \cdots \wedge \widehat{f_j} \wedge \cdots \wedge f_i$$

where $\varphi : \mathcal{H}^d \to \mathcal{H}, (h_1, \cdots, h_d) \mapsto \sum_{i=1}^d h_i (U_{t_i} - 1)$ [Sta24, Tag 0621]. Note that

$$H_i(\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W}) = \mathrm{Tor}_i^\mathcal{H}(\mathcal{H}/\mathfrak{m}, \text{c-Ind}_I^G \mathcal{W}).$$

Proposition 2.2. *The sequence $(U_{t_1} - 1, \cdots, U_{t_d} - 1)$ is Koszul regular for the $\mathcal{H}$-module* c-$\mathrm{Ind}_I^G \mathcal{W}$*: the natural map*

$$\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W} \to \text{c-Ind}_I^G \mathcal{W} \otimes_\mathcal{H} \mathcal{H}/\mathfrak{m}$$

is a quasi-isomorphism of $E[G]$-modules

Proof. This is [KS12, Thm. 2.5] with the same proof sketched below. We first reduce the statement to the adjoint case G/Z. The operators $U_{z_i}, i = 1, \cdots, d - |\Delta|$ acts on each $[gIZ, \mathcal{W}]$ for $g \in G/IZ$. Since $gIZ/gI = gZ/g(Z \cap I) = g(Z/(Z \cap I)) = \prod_i z_i^\mathbb{Z}$, it's easy to see that $U_{z_1} - 1, \cdots, U_{z_{d-|\Delta|}} - 1$ form regular sequences on c-$\mathrm{Ind}_I^G \mathcal{W}$ and c-$\mathrm{Ind}_I^G \mathcal{W}/(U_{z_i} - 1, i = 1, \cdots, d - |\Delta|) \simeq$ c-$\mathrm{Ind}_{IZ}^G \mathcal{W}$.

We now consider the action of $\mathcal{H}' = E[U_{t_\alpha}, \alpha \in \Delta]$ on c-$\mathrm{Ind}_{IZ}^G \mathcal{W}$. Let $\mathfrak{m}' = U_{t_\alpha} - 1, \alpha \in \Delta$. The statement is equivalent to that $\mathrm{Tor}_q^{\mathcal{H}'}(\mathcal{H}'/\mathfrak{m}', \text{c-Ind}_{IZ}^G \mathcal{W}) = 0$ for $q \geq 1$. Recall [KS12, Lem. 1.4], see also [Her11, Lem. 2.20]: $G = \coprod_{t \in T/T_0} KtI$ and for any $g \in G$ there is $t \in T^-$ such that $gItI \subset KT^- I$. Using the lemma, as in the proof of [KS12, Thm. 2.5], we have $\mathrm{Tor}_q^{\mathcal{H}'}(\mathcal{H}'/\mathfrak{m}', \text{c-Ind}_{IZ}^G \mathcal{W}) \simeq \mathrm{Tor}_q^{\mathcal{H}'}(\mathcal{H}'/\mathfrak{m}', \text{c-Ind}_{IZ}^G \mathcal{W} \otimes_{\mathcal{H}'} \mathcal{H}'[\frac{1}{U_{t_\alpha} - 1}, \alpha \in \Delta]) \simeq \mathrm{Tor}_q^{\mathcal{H}'}(\mathcal{H}'/\mathfrak{m}', \text{c-Ind}_{IZ}^G \mathcal{W}(G^-))$ where $G^- = KT^- IZ = \coprod_{\underline{m} \in \mathbb{N}^\Delta} K \prod_\alpha t_\alpha^{m_\alpha} IZ$ and

$$\text{c-Ind}_{IZ}^G \mathcal{W}(G^-) := [G^-, \mathcal{W}] \subset \text{c-Ind}_{IZ}^G \mathcal{W}.$$

We remain to show that $U_{t_\alpha} - 1, \alpha \in \Delta$ form a regular sequence on c-$\mathrm{Ind}_{IZ}^G \mathcal{W}(G^-)$ as in [KS12, Thm. 2.6] for an arbitrary ordering $\alpha_1, \cdots, \alpha_{|\Delta|}$ of Δ. We prove by induction on $1 \leq j \leq |\Delta|$. Suppose that $(U_{t_{\alpha_j}} - 1)f = \sum_{i=1}^{j-1}(U_{t_{\alpha_i}} - 1)f^i$ for some $f, f^1, \cdots, f^{i-1} \in$ c-$\mathrm{Ind}_{IZ}^G \mathcal{W}(G^-)$. Then formally in $\prod_{g \in G^-/IZ}[g, \mathcal{W}]$, $f = \sum_{i=1}^{j-1}(U_{t_{\alpha_i}} - 1)(U_{t_{\alpha_j}} - 1)^{-1} f^i =$

$\sum_{i=1}^{j-1}(U_{t_{\alpha_i}} - 1)\sum_{k\geq 0}(-1)U_{t_{\alpha_j}}^k f^i$. Consider

$$V_r = \coprod_{\underline{m}\in\mathbb{N}^\Delta, m_{\alpha_j}=r} K t_{\alpha_1}^{m_{\alpha_1}}\cdots t_{\alpha_{|\Delta|}}^{m_{\alpha_{|\Delta|}}} IZ.$$

Then $U_{t_{\alpha_j}} : [V_r, \mathcal{W}] \to [V_{r+1}, \mathcal{W}]$ and $U_{t_{\alpha_i}} - 1 : [V_r, \mathcal{W}] \to [V_r, \mathcal{W}]$ for $i \neq j$ by [KS12, Lem. 1.3]. Let $g^i = \sum_{k\geq 0}(-1)U_{t_{\alpha_j}}^k f^i$. Write $f = \sum_{r\geq 0} f^{(r)}$ and $g^i = \sum_{r\geq 0} g^{i,(r)}$ where $f^{(r)}, g^{i,(r)} \in [V_r, \mathcal{W}] \subset$ c-Ind$_{IZ}^G \mathcal{W}(G)$ (this is possible since f, f^i are compactly supported modulo Z). There exists s such that $f^{(r)} = 0$ for all $r \geq s$. Then $f = \sum_{i=1}^{j-1}(U_{t_{\alpha_i}} - 1)\sum_{r=0}^{s} g^{i,(r)}$, which implies that f lies in the subspace of c-Ind$_{IZ}^G \mathcal{W}(G^-)$ generated by images of $U_{t_{\alpha_i}} - 1, i = 1, \cdots, j-1$.   □

Corollary 2.3. *Let $\mathcal{W}_1 \hookrightarrow \mathcal{W}_2$ be an injection of IT^+I-representations equipped with the corresponding ψ_t-actions. Then the induced $\mathcal{H}$-modules map c-Ind$_I^G \mathcal{W}_1 \to$ c-Ind$_I^G \mathcal{W}_2$ induces an injection*

$$\text{c-Ind}_I^G \mathcal{W}_1 \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \text{c-Ind}_I^G \mathcal{W}_2 \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m}.$$

Proof. By Proposition 2.2, $\text{Tor}_1^{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, \text{c-Ind}_I^G(\mathcal{W}_2/\mathcal{W}_1)) = 0$.   □

Now we consider parabolic inductions. Let U be a locally analytic representation of the torus T inflated to a representation of B with the same notation. Recall the locally analytic parabolic induction $\text{Ind}_B^G U$ as in [Eme07, §2.1]:

$$\text{Ind}_B^G U = \{f \in \mathcal{C}^{\text{la}}(G, U) \mid f(bg) = b.f(g), \forall b \in B\}$$

with the action of G given by right translations. For an open subset $V \in B\backslash G$, e.g., $V = BI$, let $\text{Ind}_B^G U(V) \subset \text{Ind}_B^G U$ be the subspace consisting of functions supported in V. Let $\text{Res}_V : \text{Ind}_B^G U \to \text{Ind}_B^G U(V), f \mapsto f \cdot \mathbf{1}_V$ be the restriction map where $\mathbf{1}_V$ denotes the characteristic function of V.

We take an I-invariant subspace $\mathcal{W} \subset \text{Ind}_B^G U(BI)$ such that for any $t \in T^-$, $\text{Res}_{BI}(t^{-1}.\mathcal{W}) \subset \mathcal{W}$ as subspaces of $\text{Ind}_B^G U(BI)$.

Definition 2.4. In the above situation, we set

$$\psi_t : \mathcal{W} \to \mathcal{W}, f(-) \mapsto \text{Res}_{BI}(t^{-1}.f) = \text{Res}_{BI}(f(-t^{-1})) = f(-t^{-1})\mathbf{1}_{BI}$$

for $t \in T^-$. And let $U_t \in T^-$ be the corresponding Hecke operator.

Remark 2.5. Our definition of ψ_t is *slightly different* from that of [KS12], where $\psi_t.f(-) = \text{Res}_{BI}f(t - t^{-1}) = \chi(t)\text{Res}_{BI}(f(-t^{-1}))$ when U is a character χ. Our price is that the representation c-Ind$_I^G \mathcal{W} \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m}$ depends implicitly on $\chi(t_i), 1 \leq i \leq d$.

The defined actions satisfy the condition of Lemma 2.1.

Lemma 2.6. *In the above definition, we have $\psi_t(xw) = (t^{-1}xt)\psi_t(w)$ for $x \in tIt^{-1} \cap I$. And for $t_1, t_2 \in T^-$, $\psi_{t_1}\psi_{t_2} = \psi_{t_1 t_2}$.*

Proof. The map Res_{BI} is I-equivariant. Hence $\psi_t(xw) = \text{Res}_{BI}(t^{-1}xw) = \text{Res}_{BI}(t^{-1}xtt^{-1}w) = (t^{-1}xt)\text{Res}_{BI}(t^{-1}w) = (t^{-1}xt)\psi_t(w)$. And $\psi_{t_1}\psi_{t_2}f = \psi_{t_2}(f(-t_1^{-1})\mathbf{1}_{BI}) = f(-t_2^{-1}t_1^{-1})\mathbf{1}_{BIt_2}\mathbf{1}_{BI} = f(-t_2^{-1}t_1^{-1})\mathbf{1}_{BI}$ since $BIt_2 \supset BI$.   □

By the Frobenius reciprocity, there is G-map

$$(2.2) \qquad \Phi_{\mathcal{W}} : \text{c-Ind}_I^G \mathcal{W} \to \text{Ind}_B^G U : [g, w] \mapsto g.w.$$

Proposition 2.7. *For $t_1, t_2 \in T^-$, $U_{t_1 t_2} = U_{t_1} U_{t_2}$ making $\text{c-Ind}_I^G \mathcal{W}$ an $\mathcal{H}$-module. Moreover, the map (2.2) factors through an injection*

$$\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \text{Ind}_B^G U.$$

Proof. The first statement is Lemma 2.1 and Lemma 2.6. The proof for the injection is the same as for [KS12, Prop. 2.4]. We first recall the key input [KS12, Prop. 1.2]: suppose that $t \in T^-, k, k' \in K$ such that $ktIB \cap k'tIB \neq \emptyset$, then $ktIB = k'tIB, ktI = k'tI$ and $kI = k'I$ (the proof of *loc. cit.*, which cites [SS91, §4, Prop. 7], works for the general split reductive group G).

We check that for $t \in T^-$

$$\Phi_{\mathcal{W}}(U_t[g, w]) = \sum_{xtI \in ItI/I} gxt.\psi_t(x^{-1}w) = \sum_{xtI \in ItI/I} gxt.(t^{-1}x^{-1}w \cdot \mathbf{1}_{BI})$$

$$= \sum_{xtI \in ItI/I} gxtt^{-1}x^{-1}(w\mathbf{1}_{BIt^{-1}x^{-1}}) = g.w = \Phi_{\mathcal{W}}([g, w])$$

since $BI = \coprod_{It^{-1}x^{-1} \in It^{-1}I/I} BIt^{-1}x^{-1}$ (by [KS12, Prop. 1.2] and $I/(I \cap tIt^{-1}) = (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}$).

By definition, we have $U_z[g, w] = [gz, z^{-1}.w]$ for $z \in Z, w \in \mathcal{W}$. Hence $\Psi_{\mathcal{W}}$ factors through $\text{c-Ind}_I^G \mathcal{W}/(U_{z_1} - 1, \cdots, U_{z_{d-|\Delta|}} - 1) \simeq \text{c-Ind}_{IZ}^G \mathcal{W}$ where the action of the center Z on $\mathcal{W}$ comes from the embedding $\mathcal{W} \subset \text{Ind}_B^G U$. The quotient $\text{c-Ind}_{IZ}^G \mathcal{W}$ inherits the actions of $U_{t_\alpha}, \alpha \in \Delta$. We only need to show that the kernel of $\Phi'_{\mathcal{W}} : \text{c-Ind}_{IZ}^G \mathcal{W} \to \text{Ind}_B^G U$ lies in the subspace $\mathcal{V}$ spanned by $(U_{t_\alpha} - 1)[g, w]$ for $g \in G, w \in \mathcal{W}$ and $\alpha \in \Delta$. The proof goes as for [KS12, Prop. 2.4], and we give a sketch in our notation. By [KS12, Lem. 1.4] (recalled in the proof of Proposition 2.2), $G = \coprod_{t \in T/T_0 Z} KtIZ$ and for any $g \in G$, there exists $t \in T^-$ such that $gItI \subset KT^-I$. Assume that $f \in \ker(\Phi'_{\mathcal{W}})$ and we prove that $f \in \mathcal{V}$. Since $U_t([g, w]) \subset [gItIZ, \mathcal{W}]$ and $\mathcal{V} \subset \ker(\Phi'_{\mathcal{W}})$, we may assume that $f \in [KT^-IZ, \mathcal{W}]$. Since $Kt_1I \cdot It_2I \subset Kt_1t_2I$, up to add elements in $\mathcal{V}$, we may assume that $f \in [Kt^nIZ, \mathcal{W}]$ for some n large enough and $t = \prod_\alpha t_\alpha$. Write $Kt^nIZ = \coprod_{j \in J} k_j t^n IZ$ and $f = \sum_j [k_j t^n, w_j]$. Then $\Phi'_{\mathcal{W}}(f) = \sum_j k_j t^n.w_j \in \text{Ind}_B^G U$. The support of each $k_j t^n.w_j$ as a function on G is contained in $BI(k_j t^n)^{-1} = (k_j t^n IB)^{-1}$ which are pairwisely disjoint for different j by [KS12, Prop. 1.2] recalled above. Hence $\Phi'_{\mathcal{W}}(f) = 0$ implies that $w_j = 0$ for all j. We conclude that $f = 0$. $\qquad \square$

2.2. **Change the levels.** For the actual application to the duality (Theorem 3.21), we need the flexibility to shrink (or change) the Iwahori subgroup I to other open compact subgroups of G for the compact inductions in the Kohlhaase-Schraen resolutions. In this subsection, I denotes an open compact subgroup such that $K \cap T \subset I \subset K$ admitting the Iwahori decomposition $I = (I \cap N)(I \cap T)(I \cap \overline{N})$. For example, we can take I to be the opposite Iwahori subgroup. We still let $\mathcal{H} = E[T^-/(T \cap K)] \supset \mathfrak{m} = (U_{t_i} - 1)_{i=1,\cdots,d}$. The discussions on Hecke operators at the beginning of §2.1, including Lemma 2.1, apply for the compact induction $\text{c-Ind}_I^G \mathcal{W}$ which will be a $\mathcal{H}$-module if $\mathcal{W}$ is an IT^+I-module equipped with the corresponding ψ_t-actions.

Remark 2.8. One can also shrink the torus $T \cap I$: take I' such that $I' \cap T \subsetneq I \cap T$ and replace $\mathcal{H}$ by the algebra $E[T^-/(I' \cap T)]$. See [Oll14] for the example when I' is the pro-p Iwahori subgroup.

We first take a subgroup I' such that $T \cap K \subset I \cap \overline{B} \subset I' \subset I$ with the Iwahori decomposition such that

$$(2.3) \qquad I'/(tI't^{-1} \cap I') = I/(tIt^{-1} \cap I) = (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}, \forall t \in T^-.$$

This assumption implies that $BI' = BI = B(I \cap \overline{N})$.

Example 2.9. We can and will take $I' = (I \cap \overline{N})(I \cap T)(\overline{I} \cap N) = \overline{I} \cap I$ where I is the Iwahori subgroup and $\overline{I}$ is the Iwahori subgroup for the opposite Borel $\overline{B}$.

Suppose that $\mathcal{W}$ is a representation of IT^+I with operators $\psi_t, t \in T^-$ satisfying the condition of Lemma 2.1. We consider $\mathcal{W}' := \mathcal{W}|_{I'}$. Then operators ψ_t still act on $\mathcal{W}'$. We can similarly define Hecke operators U_t acting on c-Ind$_{I'}^G \mathcal{W}'$ as in (2.1) for $t \in T^-$ making c-Ind$_{I'}^G \mathcal{W}'$ an $\mathcal{H}$-module. We get a Koszul complex $\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}}$ c-Ind$_{I'}^G \mathcal{W}'$.

Conversely, suppose that $\mathcal{W}'$ is a representation of $I'T^+I'$ with ψ_t-actions and $\mathcal{W} = $ c-Ind$_{I'}^I \mathcal{W}'$.

Lemma 2.10. *There are ψ_t-actions on c-Ind$_{I'}^I \mathcal{W}'$ satisfying the condition of Lemma 2.1 such that*

$$\psi_t[g, w] = [t^{-1}gt, \psi_t(w)]$$

for $[g, w] \in$ c-Ind$_{I'}^I \mathcal{W}', g \in I \cap N$ and $(t^{-1}xt)\psi_t(-) = \psi_t(x-)$ for $x \in I \cap tIt^{-1}$.

Proof. Since c-Ind$_{I'}^I \mathcal{W}' = \oplus_{g \in I/I'} [g, \mathcal{W}]$ and $I/I' = (I \cap N)/(I' \cap N)$, ψ_t is already defined by the formula and we check it is well-defined. For $n \in B \cap I'$, $\psi_t[gn, n^{-1}w] = [t^{-1}gnt, \psi_t(n^{-1}w)] = [t^{-1}gt, t^{-1}nt\psi_t(n^{-1}w)] = [t^{-1}gt, \psi_t(w)]$ since $n \in I' \cap tI't^{-1}$. For $x \in I \cap tIt^{-1}$ and $g \in I \cap N \subset I \cap tIt^{-1}$, we can write $xg = g'i'$ for $g' \in I \cap N, i' \in I \cap \overline{B} = I' \cap \overline{B}$. Since $g', g'i' = xg \in I \cap tIt^{-1}$, we see $i' \in \overline{B} \cap I \cap tIt^{-1} = t(I \cap \overline{B})t^{-1}$. Then $\psi_t[xg, w] = \psi_t[g', i'.w] = [t^{-1}g't, \psi_t(i'w)] = [t^{-1}g't, t^{-1}i't\psi_t(w)] = [t^{-1}xgt, \psi_t(w)] = t^{-1}xt[t^{-1}gt, \psi_t(w)] = (t^{-1}xt)\psi_t([g, w])$. $\qquad \square$

Proposition 2.11. *Let $T \cap K \subset I' \subset I$ be subgroups with the Iwahori decomposition satisfying (2.3).*

(1) *If $\mathcal{W}' = \mathcal{W}|_{I'}$ with the restricted ψ_t-actions, then there is an $\mathcal{H}$-equivariant morphism c-Ind$_{I'}^G \mathcal{W}' \to$ c-Ind$_I^G \mathcal{W}$ which induces a quasi-isomorphism*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{I'}^G \mathcal{W}' \to \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}.$$

(2) *If $\mathcal{W} = $ c-Ind$_{I'}^I \mathcal{W}'$ with the induced ψ_t-actions given by Lemma 2.10, then there is an $\mathcal{H}$-equivariant isomorphism c-Ind$_{I'}^G \mathcal{W}' \simeq$ c-Ind$_I^G \mathcal{W}$ which induces a quasi-isomorphism*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{I'}^G \mathcal{W}' \to \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}.$$

Proof. (1) The map c-Ind$_{I'}^G \mathcal{W}' \to$ c-Ind$_I^G \mathcal{W}$, denoted by pr, is induced by the Frobenius reciprocity applying to the I'-inclusion $\mathcal{W}' = [I, \mathcal{W}]|_{I'} \subset ($c-Ind$_I^G \mathcal{W})|_{I'}$. It is also induced by c-Ind$_{I'}^I \mathcal{W}' \to \mathcal{W} : \sum_{i \in I/I'} [i, w_i] \mapsto \sum_{i \in I/I'} iw_i$. Then pr$([g, [i, w]]) = [g, iw] = [gi, w]$ for $g \in G, [i, w] \in$ c-Ind$_{I'}^I \mathcal{W}'$. We see pr$([g, w]) = [g, w]$ for $g \in G, w \in \mathcal{W}'$.

Let $t \in T^-$, $w \in \mathcal{W}'$ and $[i, w] \in$ c-Ind$_{I'}^I \mathcal{W}' \subset$ c-Ind$_{I'}^G \mathcal{W}'$. Then by definition $U_t[i, w] = \sum_{a \in (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}} [iat, \psi_t(a^{-1}w)]$. While for $[1, iw] \in$ c-Ind$_I^G \mathcal{W}$ and $i \in I$, $U_t[1, iw] = U_t[i, w] = $

$\sum_{a \in (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}}[iat, \psi_t(a^{-1}w)]$. By the explicit formula, the morphism pr commutes with the operator U_t. The kernel of pr is spanned by G-translations of elements $\sum_{i \in I/I'}[i, w_i]$ such that $\sum_{i \in I/I'} iw_i = 0$. If $i \in I$ and $a \in I \cap \overline{N}$, write (uniquely) $ia = a_i x_{a,i}$ such that $a_i \in I \cap \overline{N}$ and $x_{a,i} \in I \cap B \subset I \cap tIt^{-1}$. Then $t^{-1}x_{a,i}t\psi_t(a^{-1}w_i) = \psi_t(x_{a,i}a^{-1}w_i) = \psi_t(a_i^{-1}iw_i)$. Suppose that t is chosen such that $t^{-1}(I \cap B)t \subset I'$ (this is possible since $I \cap T = I' \cap T$ by our assumption). In this case we have

$$U_t(x) = \sum_{i \in I/I'} \sum_{a \in (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}} [iat, \psi_t(a^{-1}w_i)] = \sum_{i,a}[a_i t, \psi_t(a_i^{-1}iw_i)] = \sum_{a'}\sum_i [a't, \psi_t(a'^{-1}iw_i)] = 0$$

for $x = \sum_{i \in I/I'}[i, w_i]$ in the kernel of pr so that $\sum_i iw_i = 0$. For the third equality we used that when a ranges over $(I \cap \overline{N})/t(I \cap \overline{N})t^{-1} = I/(I \cap tIt^{-1})$, $a_i(I \cap tIt^{-1}) = i^{-1}a(I \cap tIt^{-1}) \in I/(I \cap tIt^{-1})$ ranges over the same set.

Let $\mathcal{H}' = E[U_{t_i}^{\pm}, i = 1, \cdots, d]$. The fact that $U_t(x) = 0$ for $x \in \ker(\mathrm{pr})$ implies that pr induces an $\mathcal{H}'$-isomorphism: $\mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \otimes_{\mathcal{H}} \mathcal{H}' \simeq \mathrm{c\text{-}Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}} \mathcal{H}'$. The map pr also induces a map of the Koszul complexes

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \to \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \mathrm{c\text{-}Ind}_I^G \mathcal{W}.$$

The induced maps of homologies are $\mathrm{Tor}_\bullet^{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}') \to \mathrm{Tor}_\bullet^{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, \mathrm{c\text{-}Ind}_I^G \mathcal{W})$ which are isomorphisms since for any $\mathcal{H}$-module M, $\mathrm{Tor}_\bullet^{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, M) \simeq \mathrm{Tor}_\bullet^{\mathcal{H}'}(\mathcal{H}'/\mathfrak{m}, M \otimes_{\mathcal{H}} \mathcal{H}')$ [Sta24, Tag 00M8]).

(2) Since $\mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' = \mathrm{c\text{-}Ind}_I^G \mathcal{W}$, we only need to check that the Hecke actions coincide. Take $t \in T^-$. Suppose that $[g, w] \in \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}'$ and $g \in I$. For $x \in I \cap \overline{N} = I' \cap \overline{N}$, we can write $gx = x'g'$ where $x' \in I \cap \overline{N}$ and $g' \in I \cap B \subset I \cap tIt^{-1}$. Then $[gxt, \psi_t(x^{-1}w)] = [x'g't, \psi_t(x^{-1}w)] = [x't, [t^{-1}g't, \psi_t(x^{-1}w)]] = [x't, \psi_t([(x')^{-1}g, w])]$. Note that when x ranges over $I \cap \overline{N}/t(I \cap \overline{N})t^{-1} = I/(I \cap tIt^{-1})$, x' ranges over the same set. Hence

$$U_t[g, w] = \sum_{x \in I'/(I' \cap tI't^{-1})} [gxt, \psi_t(x^{-1}w)] = \sum_{x \in I/(I \cap tIt^{-1})} [xt, \psi_t([x^{-1}g, w])].$$

The result follows by definitions. $\qquad \square$

Corollary 2.12. *Let I be the Iwahori subgroup and $\mathcal{W}'$ be an I'-subrepresentation of $\mathrm{Ind}_B^G U(BI)$ (in the notation of Proposition 2.7) that is stable under the operators $\psi_t : f \mapsto f(-t^{-1}) \cdot \mathbf{1}_{BI}$ for all $t \in T^-$. Then the induced map $\mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \to \mathrm{Ind}_B^G U$ factors through an injection*

$$\mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \mathrm{Ind}_B^G U.$$

Proof. Write $\mathcal{W}_0 = \mathrm{Ind}_B^G U(BI)$. Let

$$\mathcal{W}'' := \mathrm{coker}(\mathrm{c\text{-}Ind}_{I'}^I \mathcal{W}' \to \mathrm{c\text{-}Ind}_{I'}^I \mathcal{W}_0|_{I'}) = \mathrm{c\text{-}Ind}_{I'}^I \mathrm{coker}(\mathcal{W}' \to \mathcal{W}_0|_{I'})$$

with the induced ψ_t-actions. We have a short exact sequence of $\mathcal{H}$-modules

$$0 \to \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \to \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}_0|_{I'} \to \mathrm{c\text{-}Ind}_I^G \mathcal{W}'' \to 0.$$

By (the proof of) Proposition 2.2, $\mathrm{Tor}_1^{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, \mathrm{c\text{-}Ind}_I^G \mathcal{W}'') = 0$. Hence by Proposition 2.7 and Proposition 2.11, the composite

$$\mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}' \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \mathrm{c\text{-}Ind}_{I'}^G \mathcal{W}_0|_{I'} \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \simeq \mathrm{c\text{-}Ind}_I^G \mathcal{W}_0 \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \mathrm{Ind}_B^G U$$

is an injection. $\qquad \square$

We will also need the case when $T \cap K \subset I' \subset I$, $I' \cap \overline{N} \subsetneq I \cap \overline{N}$ while $I' \cap N = I \cap N$. Let $\mathcal{W}$ be an IT^+I-representation as before and let $\mathcal{W}' = \mathcal{W}|_{I'T^+I'}$ with the restriction of ψ_t-actions satisfying the condition of Lemma 2.1.

Proposition 2.13. *In the above situation, there is an $\mathcal{H}$-equivariant morphism $\text{c-Ind}_I^G \mathcal{W} \to$ $\text{c-Ind}_{I'}^G \mathcal{W}'$ which induces a quasi-isomorphism*

$$\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G \mathcal{W} \to \wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_{I'}^G \mathcal{W}'.$$

Proof. There is a natural I-injection

$$\mathcal{W} \hookrightarrow \text{Ind}_{I'}^I \mathcal{W}|_{I'} \simeq \text{c-Ind}_{I'}^I \mathcal{W}|_{I'} : w \mapsto \frac{1}{|I/I'|} \sum_{x \in I/I'} [x, x^{-1}w].$$

Write ι for the induced map $\text{c-Ind}_I^G \mathcal{W} \hookrightarrow \text{c-Ind}_{I'}^G \mathcal{W}'$. We see that

$$
\begin{aligned}
|I/I'| \iota(U_t[g, w]) &= \sum_{h \in I/I'} \sum_{x \in I/(I \cap tIt^{-1})} [gxth, h^{-1}\psi_t(x^{-1}w)] \\
&= \sum_{tht^{-1} \in tIt^{-1}/tI't^{-1}} \sum_{x \in I/(I \cap tIt^{-1})} [gx(tht^{-1})t, h^{-1}\psi_t(x^{-1}w)] \\
&= \sum_{x \in (I \cap \overline{N})/t(I' \cap \overline{N})t^{-1}} [gxt, \psi_t(x^{-1}w)].
\end{aligned}
$$

where we used that $I/(I \cap tIt^{-1}) = (I \cap \overline{N})/t(I \cap \overline{N})t^{-1}$, $I/I' = (I \cap \overline{N})/(I' \cap \overline{N})$, and $h^{-1}\psi_t(x^{-1}w) = \psi_t(th^{-1}t^{-1}x^{-1}w)$ if $th^{-1}t^{-1} \in t(I \cap \overline{N})t^{-1} \subset I \cap tIt^{-1}$. While

$$|I/I'| U_t \iota([g, w]) = \sum_{h \in I/I'} \sum_{x \in I'/(I' \cap tI't^{-1})} [ghxt, \psi_t(x^{-1}h^{-1}w)] = \sum_{x \in I/(I' \cap tI't^{-1})} [gxt, \psi_t(x^{-1}w)].$$

Hence ι is $\mathcal{H}$-equivariant. For any $[1, w] \in \text{c-Ind}_{I'}^G \mathcal{W}'$, we have

$$U_t[1, w] = \sum_{x \in (I' \cap \overline{N})/t(I' \cap \overline{N})t^{-1}} [xt, \psi_t(x^{-1}w)] = \sum_{x \in t^{-1}(I' \cap \overline{N})t/(I' \cap N)} [tx, \psi_t(tx^{-1}t^{-1}w)].$$

Take $t \in T^-$ such that $I \cap \overline{N} \subset t^{-1}(I' \cap \overline{N})t$. Then we can write the last term as

$$
\begin{aligned}
&\sum_{x \in t^{-1}(I' \cap \overline{N})t/(I \cap \overline{N})} \sum_{y \in (I \cap \overline{N})/(I' \cap \overline{N})} [txy, \psi_t(ty^{-1}t^{-1}tx^{-1}t^{-1}w)] \\
&= \sum_{x \in t^{-1}(I' \cap \overline{N})t/(I \cap \overline{N})} \sum_{y \in (I \cap \overline{N})/(I' \cap \overline{N})} [txy, y^{-1}\psi_t(tx^{-1}t^{-1}w)]
\end{aligned}
$$

since $ty^{-1}t^{-1} \in I \cap tIt^{-1}$. Now

$$\sum_{y \in (I \cap \overline{N})/(I' \cap \overline{N})} [txy, y^{-1}\psi_t(tx^{-1}t^{-1}w)] = \sum_{y \in I/I'} [txy, y^{-1}\psi_t(tx^{-1}t^{-1}w)] = \iota([tx, \psi_t(tx^{-1}t^{-1}w)])$$

lies in the image of ι. We conclude that U_t annihilates $\text{coker}(\iota)$. Using the argument as for (1) of Proposition 2.11, the quasi-isomorphism follows. $\square$

Remark 2.14. Proposition 2.13 is the representation theoretic analog of [Eme06, Prop. 3.4.11].

2.3. **Examples.** We first recall the examples in [KS12]. Let $\chi : T \to E^\times$ be a continuous character. We take good open normal subgroups $I_n = \mathbf{I}_n(L), n \geq 1$ of I which admit Iwahori decompositions. Fix n large enough such that χ is rigid analytic on $T \cap I_n$. Then Kohlhaase-Schraen takes the I-subspace

$$\mathcal{W}_{\sharp,n} := \mathrm{Ind}_B^G \chi(BI)^{I_n-\mathrm{an}} \subset \mathrm{Ind}_B^G \chi$$

the subspace of I_n-rigid analytic functions in $\mathrm{Ind}_B^G \chi$ that are supported on the open subset BI. Since $\mathcal{C}^{I_n-\mathrm{an}}(I, E) = \mathcal{C}^{(I_n \cap B)-\mathrm{an}}(I \cap B, E) \widehat{\otimes}_E \mathcal{C}^{(I_n \cap \overline{N})-\mathrm{an}}(I \cap \overline{N}, E)$, we see as an $I \cap \overline{N}$-representation, $\mathcal{W}_{\sharp,n}$ can be identified with $\mathcal{C}^{(I_n \cap \overline{N})-\mathrm{an}}(I \cap \overline{N}, E)$ the space of $I_n \cap \overline{N}$-analytic functions on $I \cap \overline{N}$. As $t^{-1}(I \cap \overline{N})t \supset I \cap \overline{N}$ for $t \in T^-$, there is a factorization

$$\mathrm{Res}_{BI} \circ \mathrm{Ad}(t) : \mathcal{C}^{(I_n \cap \overline{N})-\mathrm{an}}(I \cap \overline{N}, E) \to \mathcal{C}^{t^{-1}(I_n \cap \overline{N})t-\mathrm{an}}(t^{-1}(I \cap \overline{N})t, E) \to \mathcal{C}^{(I_n \cap \overline{N})-\mathrm{an}}(I \cap \overline{N}, E).$$

Since $f(-t^{-1}) = \chi(t)^{-1} f(t - t^{-1})$, the map ψ_t in Definition 2.4 stabilizes $\mathcal{W}_{\sharp,n}$.

Proposition 2.15 ([KS12, Prop. 2.4]). *For $\mathcal{W}_{\sharp,n}$ as above, the map $\Phi_{\mathcal{W}_{\sharp,n}} : \mathrm{c\text{-}Ind}_I^G \mathcal{W}_{\sharp,n} \to \mathrm{Ind}_B^G \chi$ is surjective. And $\Phi_{\mathcal{W}_{\sharp,n}}$ induces a quasi-isomorphism (see also Proposition 2.7 and Proposition 2.2)*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \mathrm{c\text{-}Ind}_I^G \mathcal{W}_{\sharp,n} \simeq \mathrm{Ind}_B^G \chi.$$

Corollary 2.16. *For any sub I-representation $\mathcal{W}_\sharp$ such that there exists $m \geq n$ and inclusions $\mathcal{W}_{\sharp,n} \subset \mathcal{W}_\sharp \subset \mathcal{W}_{\sharp,m}$ and $\mathcal{W}_\sharp$ is stable under ψ_t for all $t \in T^-$, the map $\Phi_{\mathcal{W}_\sharp}$ induces a quasi-isomorphism*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \mathrm{c\text{-}Ind}_I^G \mathcal{W}_\sharp \simeq \mathrm{Ind}_B^G \chi.$$

Proof. Use Proposition 2.7, Proposition 2.2 (Corollary 2.3) and the surjectivity of $\Phi_{\mathcal{W}_{\sharp,n}}$. $\square$

We now give examples of $\mathcal{W}_\sharp$ that satisfy the assumption of Corollary 2.16, whose duals will be more convenient for us. Take an open normal subgroup $H \subset I$ with Iwahori decomposition $H = (H \cap \overline{N})(H \cap T)(H \cap N)$ and such that H is uniform pro-p, see [ST03, §4] and the proof of [OS15, Thm. 5.5]. Equip H with a p-valuation ω and ordered basis $h_1, \cdots, h_{d_H}$ for $d_H = \dim_{\mathbb{Q}_p} H$ compatible with the Iwahori decomposition and root groups, see [OS10, §3.3.3]. The ordered basis induces $\mathbb{Z}_p^{d_H} \simeq H$ as p-adic manifolds. Upon shrinking H, we assume that $\omega(h_1) = \cdots = \omega(h_{d_H}) = \omega \in \mathbb{N}$ using the method in [Laz65, Prop. III.3.1.3].

The distribution algebra $\mathcal{D}(H, E)$ is the continuous dual of $\mathcal{C}^{\mathrm{la}}(H, E)$. We consider its Banach completions

$$\mathcal{D}_r(H, E) = \{ \sum_{\alpha \in \mathbb{N}^{d_H}} d_\alpha \mathbf{b}^\alpha \mid d_\alpha \in E, \lim_{\alpha \to \infty} |d_\alpha|_p r^{\sum_i \alpha_i \omega(h_i)} = 0 \}$$

for $r < 1$ where $b_i = h_i - 1$ and $\mathbf{b}^\alpha = (h_1 - 1)^{\alpha_1} \cdots (h_{d_H} - 1)^{\alpha_{d_H}}$. Since I normalizes H, the conjugation of $x \in I$ induces an automorphism of $D_r(H, E)$ (see the discussion in the proof of [ST03, Thm. 5.1]). We let $\mathcal{D}_r(I, E) = E[I] \otimes_{E[H]} \mathcal{D}_r(H)$. Let $\mathcal{C}^r(I, E)$ be the continuous E-dual of $\mathcal{D}_r(I, E)$ with the open compact topology, a Smith space with an open compact lattice over $\mathcal{O}_L$. Then $\mathcal{C}^r(I, E) = \prod_{g \in I/H} g.C^r(H, E)$ where

$$\mathcal{C}^r(H, E) = \{ \sum_{\alpha \in \mathbb{N}^{d_H}} c_\alpha \binom{x_1}{\alpha_1} \cdots \binom{x_{d_H}}{\alpha_{d_H}} \mid \sup_\alpha |c_\alpha|_p r^{-\sum_i \alpha_i \omega(h_i)} < +\infty \}$$

for $x = (x_1, \cdots, x_{d_H}) \in \mathbb{Z}_p^{d_H} \simeq H$ in terms of Mahler's basis. Moreover, similarly as to $\mathcal{C}^{(I_n \cap \overline{N})-\text{an}}(I, E)$, we have (cf. [RJRC22, Thm. 3.40 (2)] and [OS10, Prop. 3.3.4])

$$\mathcal{C}^r(I, E) = (\mathcal{D}_r(I, E))' = (\mathcal{D}_r(B \cap I, E)\widehat{\otimes}_E \mathcal{D}_r(I \cap \overline{N}, E))' = \mathcal{C}^r(B \cap I, E)\widehat{\otimes}_E \mathcal{C}^r(I \cap \overline{N}, E).$$

By [Laz65, III.1.3.8] (and see also [RJRC22, §4.1.3]), $\mathcal{C}^r(I, E) \supset \mathcal{C}^{I_m-\text{an}}(I, E)$ for r sufficiently close to 1. We set

$$\mathcal{W}_{\sharp,r} := \text{Ind}_B^G \chi(BI) \cap \mathcal{C}^r(I, E).$$

Then $\mathcal{W}_{\sharp,m} \subset \mathcal{W}_{\sharp,r}$ for r sufficiently close to 1.

Lemma 2.17. *The operator ψ_t in Definition 2.4 stabilizes $\mathcal{W}_{\sharp,r}$ for $r < 1$ sufficiently close to 1. Hence the conclusion of Corollary 2.16 holds for such $\mathcal{W}_{\sharp,r}$.*

Proof. As in the case for $\mathcal{C}^{I_n-\text{an}}(I, E)$ and $\mathcal{W}_{\sharp,n}$, we need to show that if $f \in \mathcal{C}^r(I \cap \overline{N}, E)$, then $\text{Res}_{I \cap \overline{N}}(f(t - t^{-1}))$ is still in $\mathcal{C}^r(I \cap \overline{N}, E)$. Since $I \cap \overline{N}$ is a product of its root subgroups and the conjugation of t acts on each of them, we reduce to prove the similar result for $\mathcal{O}_L \subset L = \mathbf{G}_{m/L}(L)$ and for the action of ϖ_L on $\mathcal{O}_L$ or $\mathcal{C}^r(\mathcal{O}_L, E)$, namely to show that $f(-) \mapsto f(\varpi_L -)$ sends $\mathcal{C}^r(\mathcal{O}_L, E)$ into $\mathcal{C}^r(\mathcal{O}_L, E) \subset \mathcal{C}(\mathcal{O}_L, E)$. Choose a $\mathbb{Z}_p$-basis $1 = e_1, \cdots, e_{d_L}$ of $\mathcal{O}_L$ with coordinates $x_1, \cdots, x_{d_L} \in \mathbb{Z}_p$. Then $\times \varpi_L$ factors as $\mathcal{O}_L \xrightarrow{\tau} \mathbb{Z}_p^{d_L} \xrightarrow{\eta} \mathcal{O}_L = \mathbb{Z}_p^{d_L}$ where τ is a $\mathbb{Z}_p$-linear automorphism of $\mathcal{O}_L$ and $\eta : (x_1, \cdots, x_{d_L}) \mapsto (p^{a_1} x_1, \cdots, p^{a_{d_L}} x_{d_L})$. Since the definition of $\mathcal{D}_r(\mathcal{O}_L, E)$, as well as $\mathcal{C}^r(\mathcal{O}_L, E)$, is independent of the choice of ordered basis, τ induces an automorphism of $\mathcal{C}^r(\mathcal{O}_L, E)$ and we may assume that $L = \mathbb{Q}_p$ and $\varpi_L = p$. For simplicity, we assume that $\omega(1) = 1$. Take n such that $p^{-\frac{p^{-n}}{p-1}} < r \leq p^{-\frac{p^{-(n+1)}}{p-1}}$. Then $\mathcal{C}^{p^n \mathbb{Z}_p-\text{an}}(\mathbb{Z}_p, E) \subset \mathcal{C}^r(\mathbb{Z}_p, E) \subset \mathcal{C}^{p^{n+1}\mathbb{Z}_p-\text{an}}(\mathbb{Z}_p, E) = \{\sum_{\alpha \geq 0} c_\alpha \binom{x}{\alpha} | \varliminf_{\alpha \to +\infty} |c_\alpha|_p p^{\frac{p^{-(n+1)}\alpha - s(\alpha)}{p-1}} = 0\}$ by [Laz65, III.1.3.8] where $s(\sum \alpha_i p^i) = \sum_i \alpha_i$ if $\alpha_i \in \{0, 1, \cdots, p-1\}$. The result follows. See also [JN19, Cor. 3.3.10] and Lemma 3.17 for the dual version. $\square$

3. Dual complexes

In this section, we establish the main theorem on resolutions of representations constructed from the functor $\mathcal{F}_B^G$ in [OS15] (Theorem 3.15) and study the duality between these complexes (Theorem 3.21).

3.1. **Banach modules over the distribution algebras.** We recall the definitions of various distribution algebras and use them to construct locally analytic I-representations as a continuation of §2.3.

Let $\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)$ be the subalgebra $U(\mathfrak{g})\mathcal{D}(\overline{B} \cap I, E)$ generated by $U(\mathfrak{g})$ and $\mathcal{D}(\overline{B} \cap I, E)$ inside $\mathcal{D}(I, E)$, which was considered in [OS15, §3.4] or [SS16, §4]. Define similarly $\mathcal{D}(\mathfrak{g}, \overline{B})$. Note that $\mathcal{D}(\mathfrak{g}, \overline{B} \cap I) = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \mathcal{D}(\overline{B} \cap I)$ [OS15, Prop. 3.5].

Definition 3.1. If $\overline{M}$ is $\mathcal{D}(\mathfrak{g}, \overline{B})$-module such that $\overline{M}$ is finitely presented over $U(\mathfrak{g})$ and $\overline{B}$ acts locally finitely (i.e., $\overline{M}$ is the union of its finite-dimensional $\overline{B}$-subspaces), we let

$$\mathcal{W}_{\sharp,r}(\overline{M}) := \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M}.$$

Since $\overline{M}$ is assumed to be a finitely presented $\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)$-module, $\mathcal{W}_{\natural,r}(\overline{M})$ is a finitely presented $\mathcal{D}_r(I, E)$-module.

By definition, $\mathcal{D}_r(I, E) = E[I] \otimes_{E[I_n]} \mathcal{D}_r(H, E)$ while

$$\mathcal{D}_r(H, E) = E[H] \otimes_{E[H^m]} D_{r^{p^m}}(H^m) = E[H] \otimes_{E[H^m]} U_r(\mathfrak{g})$$

for m such that $\frac{1}{p} < r^{p^m} < p^{-\frac{1}{p-1}}$ ([OS15, (5.5.6)] for $p \geq 2$) where $U_r(\mathfrak{g})$ is the closure of $U(\mathfrak{g})$ in $\mathcal{D}_r(I, E)$. Moreover, by [OS15, Sublemma 5.6], the closure of $\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)$ in $\mathcal{D}_r(I, E)$ is a subring $\mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I)$ such that $\mathcal{D}_r(I, E) = \oplus_{I/H^m(\overline{B} \cap I)} \delta_g \mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I)$ where δ_g denote Dirac distributions and $H^m(\overline{B} \cap I) = H(\overline{B} \cap I) \cap \mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I)$.

For $\overline{M}$ in Definition 3.1, we write $\overline{M}_r := U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M} = \mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M}$ ([SS16, Lem. 4.6]). Then we get

$$\mathcal{W}_{\natural,r}(\overline{M}) = \oplus_{g \in I/H^m(\overline{B} \cap I)} \delta_r \overline{M}_r.$$

Note that $\widehat{U}(\mathfrak{g}) = \varprojlim_r U_r(\mathfrak{g})$ is a Fréchet-Stein algebra with the Fréchet-Stein structure (cf. [ST03, §3], [SS16, Prop. 4.8]). Let $\widehat{\overline{M}} := \widehat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M} = \varprojlim_r \overline{M}_r$. Similarly, $\mathcal{D}(I, E) = \varprojlim_r \mathcal{D}_r(I, E)$.

Lemma 3.2. *The functors $\overline{M} \mapsto \mathcal{W}_{\natural,r}(\overline{M})$ are exact on $\overline{M}$ that are finitely presented as $U(\mathfrak{g})$-modules and $\mathcal{W}_{\natural,r}(\overline{M}) \neq 0$ for r sufficiently close to 1. Moreover, if $\overline{M}$ is a simple $U(\mathfrak{g})$-module, then $\overline{M}_r$ is a simple $U_r(\mathfrak{g})$-module.*

Proof. The exactness follows from the flatness of $U_r(\mathfrak{g})$ over $\widehat{U}(\mathfrak{g})$ by [ST03, Rem. 3.2] and that $\widehat{U}(\mathfrak{g})$ is flat over $U(\mathfrak{g})$ [Sch13a, Thm. 4.3.3]. The simplicity of $\overline{M}_r$ when $\overline{M}$ is simple is well known [OS15, Thm. 5.7] and we give a sketch. We may suppose that $\overline{M}_r \neq 0$ (cf. [Sch13a, Lem. 4.3.6]), hence $\overline{M} \subset \overline{M}_r$. Let $0 \neq N \subset \overline{M}_r$ be a sub-$U_r(\mathfrak{g})$-module. We may assume that N is finitely generated since $U_r(\mathfrak{g})$ is a Noetherian Banach algebra, N is closed in $\overline{M}_r$ equipped with the induced canonical Banach topology. One then shows that the $\mathfrak{t}$-weight spaces of N are contained in $\overline{M}$, cf. [FdL99, Cor. 1.3.22] or see [Sch13a, Prop. 2.0.1]. If $N \neq 0$, then N contains $\overline{M}$ and $N = \overline{M}_r$. $\square$

Lemma 3.3. *Suppose that $t \in T^-$ and $\overline{M}$ is a $\mathcal{D}(\mathfrak{g}, \overline{B})$-module such that $\overline{M}|_{U(\mathfrak{g})} \in \mathcal{O}_{\mathrm{alg}}^{\overline{\mathfrak{b}}}$ ([OS15, Def. 2.6]) and is locally $\overline{B}$-finite. The action of the Dirac distribution $\delta_{t^{-1}} \in \mathcal{D}(\mathfrak{g}, \overline{B})$ on $\overline{M}$ extends continuously to a map $\overline{M}_r \to \overline{M}_r$ and induces an operator $\psi_t : \mathcal{W}_{\natural,r}(\overline{M}) \to \mathcal{W}_{\natural,r}(\overline{M})$ satisfying $(t^{-1}xt)\psi_t(-) = \psi_t(x-), \forall x \in I \cap tIt^{-1}$.*

Proof. The $\mathcal{D}(\mathfrak{g}, \overline{B})$-module $\overline{M}$ admits a presentation

$$\mathcal{D}(\mathfrak{g}, \overline{B}) \otimes_{\mathcal{D}(\overline{B}, E)} \sigma' \to \mathcal{D}(\mathfrak{g}, \overline{B}) \otimes_{\mathcal{D}(\overline{B}, E)} \sigma \to \overline{M} \to 0$$

where σ, σ' are finite-dimensional $\overline{B}$-representations over E. This is possible since $\overline{B}$ acts on $\overline{M}$ locally finitely and the same statement holds for $\mathcal{D}(\mathfrak{g}, \overline{B}) \otimes_{\mathcal{D}(\overline{B}, E)} \sigma = \mathcal{D}(\mathfrak{g}, \overline{B} \cap I) \otimes_{\mathcal{D}(\overline{B} \cap I, E)} \sigma = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \sigma$. By base change we get the exact sequence

$$\mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I) \otimes_{\mathcal{D}(\overline{B} \cap I, E)} \sigma' \to \mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I) \otimes_{\mathcal{D}(\overline{B} \cap I, E)} \sigma \to \overline{M}_r \to 0.$$

To extend the action of $\delta_{t^{-1}}$ to $\overline{M}_r$, it's enough to verify the existence of the extensions for $\mathcal{D}_r(\mathfrak{g}, \overline{B} \cap I) \otimes_{\mathcal{D}(\overline{B} \cap I, E)} \tau = U_r(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \tau, \tau = \sigma, \sigma'$ (and use the uniqueness of such extensions).

While $U_r(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \tau = U_r(\mathfrak{g}) \otimes_{U_r(\overline{\mathfrak{b}})} \tau = U_r(\mathfrak{n}) \otimes_E \tau$ using $U_r(\mathfrak{g}) = U_r(\mathfrak{n}) \widehat{\otimes}_E U_r(\overline{\mathfrak{b}})$. We can get the $\delta_{t^{-1}}$-action on $U_r(\mathfrak{n}) \otimes_E \tau$ since $\delta_{t^{-1}} U_r(\mathfrak{n}) \subset U_r(\mathfrak{n}) \delta_{t^{-1}}$ for $t \in T^-$ and τ is $\overline{B}$-stable. As an I-representation, $\mathcal{W}_{\natural,r}(\overline{M}) = \text{c-Ind}_{H^m(\overline{B} \cap I)}^I \overline{M}_r$. The ψ_t-operator on $\mathcal{W}_{\natural,r}(\overline{M})$ is defined using Lemma 2.10. $\qquad \square$

3.2. **The resolution for principal series.** We continue with the principal series $\text{Ind}_B^G \chi$ from §2.3 and with the notation there. We will construct I-representations $\mathcal{W}_{\natural,r}$ for $\text{Ind}_B^G \chi$ that do not satisfy the assumption but might satisfy the conclusion of Corollary 2.16. Let $\lambda = \text{wt}(\chi) \in (E^d)^{[L:\mathbb{Q}_p]}$ be the weight of the locally analytic character χ. We assume that $\lambda \in (\mathbb{Z}^d)^{[L:\mathbb{Q}_p]}$, namely χ is locally algebraic.

Consider the subspace $\mathcal{C}^{\text{pol}}(I \cap \overline{N}, E) = \mathcal{C}^{\text{pol}}(\overline{N}, E)\mathbf{1}_{I \cap \overline{N}} \subset \mathcal{W}_{\natural,r} \subset \text{Ind}_B^G \chi$ consisting of polynomial functions on $I \cap \overline{N}$, identified to the coordinate ring of $(\text{Res}_{L/\mathbb{Q}_p} \overline{\mathbf{N}}) \otimes_{\mathbb{Q}_p} E$. By [Eme07, Lem. 2.5.8], $\mathcal{C}^{\text{pol}}(I \cap \overline{N}, E)$ is a $\mathcal{D}(\mathfrak{g}, \overline{B})$-module. Actually, by the explicit description in *loc. cit.* (see also Remark 3.4 below), its restriction to $U(\mathfrak{g})$ is the dual Verma module $\overline{M}(\lambda)^\vee$ in the category $\mathcal{O}^{\overline{\mathfrak{b}}}$ (the BGG dual of the Verma module of the highest weight λ) when restricted to $\mathfrak{g}$. In the following, we write $M(\lambda)^\vee$ for the $\mathcal{D}(\mathfrak{g}, \overline{B})$-module $\mathcal{C}^{\text{pol}}(I \cap \overline{N}, E)$.

More generally, by [Bre15, Prop. 3.6], there exists a natural injection of $(\mathfrak{g}, \overline{B} \cap I)$-modules (where $M \in \mathcal{O}_{\text{alg}}^{\overline{\mathfrak{b}}}$ and χ_{sm} is a smooth character of T, see more in §3.3),

$$(3.1) \qquad \text{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty} \otimes_E \chi_{\text{sm}} \hookrightarrow \text{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty} \otimes \mathcal{C}_c^\infty(\overline{N}, \chi_{\text{sm}}) \subset \mathcal{F}_B^G(M, \chi_{\text{sm}}).$$

Here $\text{Hom}_E(M, E)$ is a $\mathfrak{g}$-module induced by the $\mathfrak{g}$-action on M and the involution of $\mathfrak{g}$ given by multiplying -1. The superscript $\mathfrak{n}_{\overline{B}}^\infty$ denotes the subspace of elements killed by finite powers of $\mathfrak{n}_{\overline{B}} = \overline{\mathfrak{n}}$. The first injection of (3.1) sends χ_{sm} to the constant function $\mathbf{1}_{I \cap \overline{N}}$ on $I \cap \overline{N}$. And the image of the last injection is the space of locally polynomial functions compactly supported on $\overline{N}$ inside $\mathcal{F}_B^G(M, \chi_{\text{sm}})$ [Bre15, (16) & Prop. 3.6]. The image of $\text{Hom}_E(M, E)^{\mathfrak{n}_{\overline{B}}^\infty} \otimes_E \chi_{\text{sm}}$ in $\mathcal{W}_{\natural,r}$ (see §2.3) consists of polynomial functions on $I \cap \overline{N}$.

Remark 3.4. We have $\text{Ind}_B^G \chi = \mathcal{F}_B^G(M(-\lambda), \chi_{\text{sm}})$ in the notation of [OS15]. There are isomorphisms of $U(\mathfrak{g})$-modules (in the last identity the involution of multiplying -1 of $\mathfrak{g}$ switches left and right $U(\mathfrak{b})$-modules)

$$\mathcal{C}^{\text{pol}}(I \cap \overline{N}, E) \simeq \text{Sym}^\bullet(\overline{\mathfrak{n}}^\vee) \simeq \text{Hom}_E(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \chi^{-1}, E)^{\overline{\mathfrak{n}}^\infty} = \text{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}), \chi)^{\overline{\mathfrak{n}}^\infty}.$$

Note that $\text{Hom}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (-\lambda), E)^{\mathfrak{n}_{\overline{B}}^\infty}$ is the dual Verma module in $\mathcal{O}^{\overline{\mathfrak{b}}}$ for the Verma module $U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \lambda \in \mathcal{O}^{\overline{\mathfrak{b}}}$. (According to [Hum08, §3.2], $(U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \lambda)^\vee$ is constructed by taking the left $U(\mathfrak{g})$-module $\text{Hom}_E(U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \lambda, E)^{\mathfrak{n}_{\overline{B}}^\infty}$ and then composing with the Chevalley involution (inverse transpose) of $\mathfrak{g}$.)

Since $\mathcal{W}_{\natural,r}$, as well as its dual, is a $\mathcal{D}_r(I, E)$-module, extending the module structure over $\mathcal{D}(I, E)$, the $\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)$-map $\overline{M}(\lambda)^\vee \hookrightarrow \mathcal{W}_{\natural,r}$ from (3.1), extends to a continuous map

$$i_{\mathcal{W}_{\natural,r}} : \mathcal{W}_{\natural,r} := \mathcal{W}_{\natural,r}(\overline{M}(\lambda)^\vee) = \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M}(\lambda)^\vee \to \mathcal{W}_{\natural,r} \subset \text{Ind}_B^G \chi.$$

Lemma 3.5. *The map* $i_{\mathcal{W}_{\natural,r}} : \mathcal{W}_{\natural,r} \to \mathcal{W}_{\natural,r}$ *is* ψ_t-*equivariant for* $t \in T^-$, *where the* ψ_t-*actions on* $\mathcal{W}_{\natural,r}$ *are given by Lemma 3.3 for the* $\mathcal{D}(\mathfrak{g}, \overline{B})$-*module* $\mathcal{C}^{\text{pol}}(I \cap \overline{N}, E) \simeq \mathcal{C}^{\text{pol}}(\overline{N}, E)$.

Proof. Write $\overline{M}$ for $\overline{M}(\lambda)^\vee$. For $f \in \mathcal{C}^{\mathrm{pol}}(\overline{N}, E) \overset{\mathrm{Res}_{BI}}{\to} \mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$, ψ_t is just the action of t^{-1} on the $\mathcal{D}(\mathfrak{g}, \overline{B})$-module $\mathcal{C}^{\mathrm{pol}}(\overline{N}, E)$ (composed with Res_{BI} on $\mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$). Hence the map $\overline{M} \hookrightarrow \mathrm{Ind}_B^G \chi(BI)$ from (3.1) is ψ_t-equivariant by Definition 2.4. By continuity, the same holds for $\overline{M}_r \subset \mathcal{W}_{\natural,r}(\overline{M}) \to \mathrm{Ind}_B^G \chi(BI)$. For $[g, f] \in \mathcal{W}_{\natural,r}(\overline{M}) = \text{c-Ind}_{H^m(\overline{B} \cap I)}^I \overline{M}_r$, the element $\psi_t(i_{\mathcal{W}_{\natural,r}}([g, f])) \in \mathrm{Ind}_B^G \chi(BI)$, where $g \in N \cap I$, is equal to $f(-t^{-1}g) \cdot \mathbf{1}_{BI}$. While (see Lemma 2.10)

$$i_{\mathcal{W}_{\natural,r}}([t^{-1}gt, \psi_t(f)]) = (t^{-1}gt)(f(-t^{-1})\mathbf{1}_{BI}) = f(-(t^{-1}gt)t^{-1})\mathbf{1}_{BI} = f(-t^{-1}g)|_{BI}$$

since $t^{-1}gt \in I$ if $g \in N \cap I$ and $t \in T^-$. Hence $i_{\mathcal{W}_{\natural,r}}$ is ψ_t-equivariant. $\qquad\square$

Proposition 3.6. *The map* $\Phi_{\mathcal{W}_{\natural,r}} : \text{c-Ind}_I^G \mathcal{W}_{\natural,r} \to \mathrm{Ind}_B^G \chi$ *induced by* $i_{\mathcal{W}_{\natural,r}}$ *factors through an injection (where* $\mathcal{H}, \mathfrak{m}$ *are given in §2.1)*

$$\text{c-Ind}_I^G \mathcal{W}_{\natural,r} \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \mathrm{Ind}_B^G \chi.$$

Proof. By the discussion before Lemma 3.2, $\text{c-Ind}_I^G \mathcal{W}_{\natural,r} = \text{c-Ind}_{I'}^G (\overline{M}(\lambda)^\vee)_r$ where $I' = H^m(\overline{B} \cap I)$ contains $I \cap \overline{B}$ with the Iwahori decomposition. Since the map $\overline{M}(\lambda)^\vee \hookrightarrow \mathcal{W}_{\natural,r}$ is injective, so is the map $(\overline{M}(\lambda)^\vee)_r \hookrightarrow \mathcal{W}_{\natural,r}$ as $(\overline{M}(\lambda)^\vee)_r$ has the same length over $U_r(\mathfrak{g})$ as the length of $\overline{M}(\lambda)^\vee$ over $U(\mathfrak{g})$ by Lemma 3.2. Moreover, the ψ_t-actions on $\mathcal{W}_{\natural,r}$ given by Lemma 3.3 is the one in §2.2 induced from the ψ_t-actions on $(\overline{M}(\lambda)^\vee)_r$. Apply (2) of Proposition 2.11 and Corollary 2.12 for $\mathcal{W}' = (\overline{M}(\lambda)^\vee)_r$, we get the injectivity. $\qquad\square$

Remark 3.7. If $\overline{M}(\lambda)^\vee$ is simple, then the module $\mathcal{W}_{\natural,r}$ is topologically of finite length as a $\mathcal{D}_r(I, E)$-module, with the same length as the $U(\mathfrak{g})$-module $\overline{M}(\lambda)^\vee$ by results in [OS15]. In this case the map $i_{W_{\natural,r}}$ is injective and Proposition 3.6 follows directly from Proposition 2.7. But this is not true in general. It may happen that $\overline{M} \subset \overline{M}(\lambda)^\vee$ is a $\mathcal{D}(\mathfrak{g}, \overline{P} \cap I)$-module for some parabolic subgroup $\overline{P} \supsetneq \overline{B}$ (for example if χ_{sm} is trivial, $\overline{M} = E$ and $\overline{P} = G$). Then the map $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M} \to \mathcal{W}_{\natural,r}$ factors through the quotient $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{P} \cap I)} \overline{M}$.

3.3. **Beyond dual Verma modules.** We extend the Kohlhaase-Schraen resolutions for principal series in the last sections to representations $\mathcal{F}_B^G(M)$ constructed from some $\mathcal{D}(\mathfrak{g}, B)$-modules M in [OS15] extended in [Bre16, Appendice] or [SS16, §4] that we will review below.

Consider a $\mathcal{D}(\mathfrak{g}, B)$-module M such that $M \in \mathcal{O}^{\flat}_{\mathrm{alg}}$ and such that we can choose a presentation

$$0 \to \mathfrak{d} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \sigma \to M \to 0$$

for a finite dimensional B-representation σ. For such a presentation and r sufficiently close to 1 (which we will assume from now on), consider

$$\mathcal{W}_{\natural,r}(\sigma^\vee) := (\mathrm{Ind}_B^G \sigma^\vee)(BI) \cap \mathcal{C}_r(I, \sigma) \subset \mathrm{Ind}_B^G \sigma^\vee$$

where $\sigma^\vee = \mathrm{Hom}_E(\sigma, E)$ with the subspace topology of $\mathcal{C}_r(I, \sigma)$. The continuous E-dual of $\mathcal{W}_{\natural,r}(\sigma^\vee)$ is equal to $\mathcal{D}_r(I, E) \otimes_{D(B,E)} \sigma$. By definition

$$\mathcal{F}_B^G(M) = (\mathrm{Ind}_B^G \sigma^\vee)^{\mathfrak{d}} = ((\mathcal{D}(G, E) \otimes_{D(B,E)} \sigma)/\mathcal{D}(G, E)\mathfrak{d})'$$

where the last superscript denotes the strong dual. Explicitly, the action of $\mathfrak{x} \otimes y \in U(\mathfrak{g}) \otimes \sigma$ on $\mathrm{Ind}_B^G \sigma^\vee$ sends $f : G \to \sigma^\vee$ to $g \mapsto (\mathfrak{x} \cdot f)(g)(y) \in \mathcal{C}^{\mathrm{la}}(G, E)$ where the Lie algebra action $\mathfrak{x}\cdot$ is

induced by left G-translations. We also consider

$$\mathcal{W}_{\natural,r}(\sigma^\vee) := \mathcal{D}_r(I,E) \otimes_{\mathcal{D}(\mathfrak{g},I\cap\overline{B})} \mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee)$$

where $\mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee)$ is the space of $(I\cap\overline{N})$-polynomial functions supported on I and with values in $\sigma^\vee$. Then $\mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee) = \mathrm{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}),\sigma^\vee)^{\overline{\mathfrak{n}}^\infty} = \mathrm{Hom}_E(U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\sigma, E)^{\overline{\mathfrak{n}}^\infty}$ (see [Eme07, (2.5.7)] or discussions around (3.1)). As in Lemma 3.5, there is a natural ψ_t-equivariant $\mathcal{D}_r(I,E)$-map $i_{\mathcal{W}_{\natural,r}(\sigma^\vee)} : \mathcal{W}_{\natural,r}(\sigma^\vee) \to \mathcal{W}_{\sharp,r}(\sigma^\vee)$ where we equip $\mathcal{W}_{\sharp,r}(\sigma^\vee)$ with the ψ_t-actions given by Definition 2.4.

There is a ψ_t-stable sub-$\mathcal{D}_r(I,E)$-module

$$\mathcal{W}_{\natural,r}(\tau(M)^\vee) = \mathcal{D}_r(I,E) \otimes_{\mathcal{D}(\mathfrak{g},\overline{B}\cap I)} \mathrm{Hom}_E(M,E)^{\overline{\mathfrak{n}}^\infty}$$

of $\mathcal{W}_{\natural,r}(\sigma^\vee)$ where for short we write $\tau(M)^\vee := \mathrm{Hom}_E(M,E)^{\overline{\mathfrak{n}}^\infty}$ with the usual dual left actions of $U(\mathfrak{g})$ and B. Consider also the I-subspace

$$\mathcal{W}_{\sharp,r}(\tau(M)^\vee) := \mathcal{W}_{\natural,r}(\sigma^\vee)^{\mathfrak{d}} = \mathcal{W}_{\natural,r}(\sigma^\vee) \cap \mathcal{F}_B^G(M)$$

of $\mathcal{W}_{\sharp,r}(\sigma^\vee)$ which is ψ_t-stable (since the action of $\mathfrak{d}$ commutes with right translations and restrictions on G which are used to define ψ_t) and has the continuous dual given by $\mathcal{D}_r(I,E)\otimes_{\mathcal{D}(\mathfrak{g},B\cap I)}M$ (see the proof of [OS15, Prop. 3.11]).

Lemma 3.8. *The composite map $\mathcal{W}_{\natural,r}(\tau(M)^\vee) \subset \mathcal{W}_{\natural,r}(\sigma^\vee) \overset{i_{\mathcal{W}_{\natural,r}(\sigma^\vee)}}{\to} \mathcal{W}_{\sharp,r}(\sigma^\vee)$ factors through a ψ_t-equivariant I-map $i_{\mathcal{W}_{\natural,r}(\tau(M)^\vee)} : \mathcal{W}_{\natural,r}(\tau(M)^\vee) \to \mathcal{W}_{\sharp,r}(\tau(M)^\vee) \subset \mathcal{W}_{\sharp,r}(\sigma^\vee)$. Moreover, $i_{\mathcal{W}_{\natural,r}(\tau(M)^\vee)}$ is independent of the presentations of M and is induced by (3.1).*

Proof. Since $\mathcal{W}_{\natural,r}(\tau(M)^\vee)$ is generated as a $\mathcal{D}_r(I,E)$-module by $\tau(M)^\vee$ and $\mathcal{W}_{\sharp,r}(\tau(M)^\vee)$ is a $\mathcal{D}_r(I,E)$-submodule, it's enough to show that $\mathrm{Hom}_E(M,E)^{\overline{\mathfrak{n}}^\infty} \subset \mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee)$ is contained in the subspace $\mathcal{W}_{\sharp,r}(\sigma^\vee)^{\mathfrak{d}} \cap \mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee) = \mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee)^{\mathfrak{d}}$. By [Bre15, Lem. 3.3], $\mathcal{C}^{\mathrm{pol}}(I\cap\overline{N},\sigma^\vee)^{\mathfrak{d}} = \mathrm{Hom}_E(U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\sigma, E)^{\overline{\mathfrak{n}}^\infty,\mathfrak{d}} = \mathrm{Hom}_E((U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\sigma)/\mathfrak{d}, E)^{\overline{\mathfrak{n}}^\infty}$. The factorization follows. The independence of the presentations follows from [Bre15, Prop. 3.4 (ii)]. $\qquad\square$

The ψ_t-equivariant map $i_{\mathcal{W}_{\natural,r}(\sigma^\vee)}$ induces an $\mathcal{H}$-map

$$\text{c-Ind}_I^G \mathcal{W}_{\natural,r}(\sigma^\vee) \to \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\sigma^\vee)$$

together with a G-map (see Proposition 2.7)

$$\Phi_{\mathcal{W}_{?,r}(\sigma^\vee)} : \text{c-Ind}_I^G \mathcal{W}_{?,r}(\sigma^\vee) \to \text{c-Ind}_I^G \mathcal{W}_{?,r}(\sigma^\vee) \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \to \mathrm{Ind}_B^G \sigma^\vee$$

for $? \in \{\natural,\sharp\}$. By the above lemma, the map restrict to an $\mathcal{H}$-map

$$\text{c-Ind}_I^G \mathcal{W}_{\natural,r}(\tau(M)^\vee) \to \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee)$$

and finally (use Corollary 2.3)

$$\Phi_{\mathcal{W}_{?,r}(\tau(M)^\vee)} : \text{c-Ind}_I^G \mathcal{W}_{?,r}(\tau(M)^\vee) \to \text{c-Ind}_I^G \mathcal{W}_{?,r}(\tau(M)^\vee) \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \to \mathcal{F}_B^G(M).$$

To simplify discussions, we will focus more on subquotients of principal series. Suppose that $\chi = z^\lambda \chi_{\mathrm{sm}}$ is a locally algebraic character of T where λ is the weight of χ and χ_{sm} is the smooth part. Then the principal series $\mathrm{Ind}_B^G \chi$ can be written as $\mathcal{F}_B^G(M)$ for $M = D(\mathfrak{g},B)\otimes_{D(B)}\chi^{-1} = (U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(-\lambda))\otimes_E \chi_{\mathrm{sm}}^{-1}$ where B acts diagonally (on the first factor by integration) and $\mathfrak{g}$

acts on the second factor trivially. In the notation of [OS15], $\mathcal{F}_B^G(M \otimes_E \chi_{\mathrm{sm}}^{-1}) = \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ for $M \in \mathcal{O}_{\mathrm{alg}}^{\flat}$ with the algebraic B-action. Note that $\tau(M \otimes_E \chi_{\mathrm{sm}}^{-1})^{\vee} = \tau(M)^{\vee} \otimes_E \chi_{\mathrm{sm}}$.

Proposition 3.9. *Suppose that $0 \to M_0 \to M \to M_1 \to 0$ is a short exact sequence of $U(\mathfrak{g})$-modules in $\mathcal{O}_{\mathrm{alg}}^{\flat}$ and let χ_{sm} be a smooth character of T. Then for r sufficiently close to 1, we have a short exact sequence of complexes for $? = (\natural/\sharp, r)$:*

$$0 \to \wedge^{\bullet} \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_?(\tau(M_1)^{\vee} \otimes \chi_{\mathrm{sm}}) \to \wedge^{\bullet} \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_?(\tau(M)^{\vee} \otimes \chi_{\mathrm{sm}})$$
$$\to \wedge^{\bullet} \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_?(\tau(M_0)^{\vee} \otimes \chi_{\mathrm{sm}}) \to 0.$$

Moreover, we have a commutative diagram

$$0 \longrightarrow \text{c-Ind}_I^G \mathcal{W}_?(\tau(M_1)^{\vee} \otimes \chi_{\mathrm{sm}})/\mathfrak{m} \longrightarrow \text{c-Ind}_I^G \mathcal{W}_?(\tau(M)^{\vee} \otimes \chi_{\mathrm{sm}})/\mathfrak{m} \longrightarrow \text{c-Ind}_I^G \mathcal{W}_?(\tau(M_0)^{\vee} \otimes \chi_{\mathrm{sm}})/\mathfrak{m} \longrightarrow 0$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$0 \longrightarrow \mathcal{F}_B^G(M_1, \chi_{\mathrm{sm}}) \longrightarrow \mathcal{F}_B^G(M, \chi_{\mathrm{sm}}) \longrightarrow \mathcal{F}_B^G(M_0, \chi_{\mathrm{sm}}) \longrightarrow 0$$

where the rows are exact and all vertical arrows are injective. Moreover if $? = (\sharp, r)$, the vertical maps are isomorphisms.

Proof. The functoriality of $\mathcal{W}_?$ is similar as in [OS15, Prop. 4.7] or by [Bre15, Prop. 3.4]. The exactness of taking $\mathcal{W}_?$ is the proof of [OS15, Prop. 4.2] or Lemma 3.2. By Proposition 2.2, the homologies of the Koszul complexes concentrate in degree 0. Hence we get the short exact sequence of degree 0 homologies. If M is a Verma module, then $\Phi_{\mathcal{W}_{\sharp,r}(\tau(M)^{\vee} \otimes \chi_{\mathrm{sm}})}$ induces an injection $\text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^{\vee} \otimes \chi_{\mathrm{sm}}) \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m} \hookrightarrow \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ by Proposition 3.6. As any irreducible M in $\mathcal{O}_{\mathrm{alg}}^{\flat}$ is a quotient of a Verma module, the injectivity of general M follows by the snake lemma and an induction on the length of M. If $? = (\sharp, r)$, the surjectivity follows similarly using the result of Kohlhaase-Schraen, i.e., Proposition 2.15. $\qquad\square$

3.4. The surjectivity.

We prove the surjectivity of the map

$$\Phi_{\mathcal{W}_{\sharp,r}} : \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\overline{M}(\lambda)^{\vee}) \to \text{Ind}_B^G \chi$$

constructed in §3.2 for locally algebraic characters χ using Proposition 4.11 that will be proved in §4.3 later. Then we can deduce from the surjectivity the main theorem on Kohlhaase-Schraen resolutions (Theorem 3.15).

We introduce some notations which serve only for the proof below. The exponential map $\exp : \overline{\mathfrak{n}} \to \overline{N}$ is an isomorphism of L-analytic manifolds. Choose basis of $\overline{\mathfrak{n}}$ in the root spaces compatible with the $\mathcal{O}_L$-lattice we obtain the L-analytification $\overline{\mathfrak{n}}^{\mathrm{an}} \simeq \mathbf{A}^{\dim_L N}$. We obtain an isomorphism of the analytic spaces $\overline{\mathbf{N}}^{\mathrm{an}} \simeq \mathbf{A}^{\dim_L N}$ with the ball of radius s denoted by $\overline{\mathbf{N}}_{\leq s}^{\mathrm{an}}$, satisfying that $\overline{\mathbf{N}}_{\leq 1}^{\mathrm{an}}(L) = \overline{N} \cap K$. For $h \in \mathbb{Z}$, let $\overline{N}_h = \overline{\mathbf{N}}_{\leq |\varpi_L|_p^h}^{\mathrm{an}}(L)$ which may or may not be a group. Using the Baker-Campbell-Hausdorff formula, $\overline{N}_h$ is a normal subgroup of $\overline{N} \cap I$ for h large enough. We say that a function $f \in \mathcal{C}_c^{\mathrm{la}}(\overline{N}, E)$ is h-analytic for some $h \geq 1$ large enough if f is rigid analytic on each left $\overline{N}_h$-coset. This is equivalent to that f is rigid analytic on each right $\overline{N}_h$-coset since $\mathcal{C}^{\mathrm{an}}(\overline{N}_h g, E) = \mathcal{C}^{\mathrm{an}}(g\overline{N}_h, E)$ for any $g \in \overline{N} \cap I$. Let $\mathcal{A}_h \subset (\text{Ind}_B^G \chi)(BI) \simeq \mathcal{C}^{\mathrm{la}}(I \cap \overline{N}, E)$ be the subspace of functions that are supported on $B \backslash BI$ and is h-analytic. For h large enough, $\mathcal{A}_h$ is the I-representation $\mathcal{W}_{\sharp,h}$ in §2.3. We recall again the result of Kohlhaase-Schraen.

Proposition 3.10 ([KS12, Prop. 2.4]). *The image $\text{c-Ind}_I^G \mathcal{A}_h \to \text{Ind}_B^G \chi$ is surjective for $h \geq 1$ large enough.*

Proposition 3.11. *The map $\Phi_{\mathcal{W}_{\natural,r}}$ above is surjective if χ has integral and antidominant (with respect to $\mathfrak{b}$) weights.*

Proof. Combine Proposition 3.10 and Lemma 3.12 below. The proof of Lemma 3.12 will use Proposition 4.11. □

Lemma 3.12. *The image of $\Phi_{\mathcal{W}_{\natural,r}}$ contains $\mathcal{A}_h \subset \text{Ind}_B^G \chi(BI)$ the space of all h-analytic functions on $\mathcal{C}^{\text{la}}(I \cap \overline{N}, E)$ for any $h \geq 1$.*

Proof. The image of $\mathcal{W}_{\natural,r} = (\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M}(\lambda)^\vee)$ under the map $i_{\mathcal{W}_{\natural,r}}$ contains the image of $U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M}(\lambda)^\vee$, especially the space $\overline{M}(\lambda)^\vee = \mathcal{C}^{\text{pol}}(I \cap \overline{N}, E) \subset (\text{Ind}_B^G \chi)(BI)$ of $\mathbb{Q}_p$-polynomial functions on $I \cap \overline{N}$.

For $h \geq 0$, we write $\mathcal{C}^{\text{an}}(\overline{N}_{-h}, E)\mathbf{1}_{I \cap \overline{N}}$ for the space of functions on $I \cap \overline{N}$ that are restrictions of rigid analytic functions on $\overline{N}_{-h}$ i.e., power series

$$\sum_{i \in \mathbb{N}^{\dim_{\mathbb{Q}_p} \overline{N}}} a_i \prod_{\alpha \in \Phi^-, j=1, \cdots, |L:\mathbb{Q}_p|} X_{\alpha,j}^{i_{\alpha,j}}, a_i \in E$$

that satisfy certain convergent condition where $X_{\alpha,j}$ are coordinates of the root group $\text{Res}_{L/\mathbb{Q}_p} \mathbf{G}_m$ corresponding to the negative roots $\alpha \in \Phi^-$. This is a Banach space.

Lemma 3.13. *The image of $U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \overline{M}(\lambda)^\vee$ under the map $i_{\mathcal{W}_{\natural,r}}$ contains $\mathcal{C}^{\text{an}}(\overline{N}_{-h}, E)\mathbf{1}_{I \cap \overline{N}}$ for some large enough h.*

Proof of Lemma 3.13. This is a reformulation of Proposition 4.11, which is proved for L-Lie algebras and L-analytic functions. We explain how to strengthen the result for $\mathbb{Q}_p$-Lie algebras. Write $\mathfrak{g} = \prod_{\sigma \in \text{Hom}(L, \overline{\mathbb{Q}}_p)} \mathfrak{g}_\sigma$. As in [Bre16, App. 9], there exists a factorization $U(\mathfrak{g}) = \otimes_\sigma U(\mathfrak{g}_\sigma) \to \widehat{\otimes}_\sigma U_{r'}(\mathfrak{g}_\sigma) \hookrightarrow U_r(\mathfrak{g})$. Similar thing happens for $\mathcal{C}^{\text{an}}(\overline{N}_{-h}, E)$ and $\widehat{\otimes}_\sigma \mathcal{C}^{\sigma-\text{an}}(\overline{N}_{-h'}, E)$ where σ-an means σ-rigid analytic functions: consider $\mathcal{O}_L \subset \text{Sp}(L\langle T \rangle)$, the rigid unit ball. By definition $\mathcal{C}^{L-\text{an}}(\mathcal{O}_L, E) = E\langle T \rangle$ while $\mathcal{C}^{\text{an}}(\mathcal{O}_L, E) = \mathcal{O}(\text{Res}_{L/\mathbb{Q}_p}(\text{Sp}(L\langle T \rangle)) \otimes_{\mathbb{Q}_p} E) = \mathcal{O}(\prod_\sigma \text{Sp}(E\langle T \rangle)) = \widehat{\otimes}_\sigma E\langle T \rangle$, see [Eme17, §2.3]. The decomposition of the space of analytic functions into those of σ-analytic functions is compatible with the actions of σ-Lie algebras. The result follows. □

We now prove that the image of $\Phi_{\mathcal{W}_{\natural,r}}$ contains $\mathcal{A}_{h_0}$ for any fixed large enough $h_0 \geq 1$. Take $t \in T^{--}$, i.e., take t such that for any $n \in \overline{N}$, $\varinjlim_{k \to \infty} t^k n t^{-k} = 0$. The action of $t^k \in T$ on $\text{Ind}_B^G \chi, f(-) \mapsto f(-t^k) = \chi(t)^k f(t^{-k} - t^k)$ induces an isomorphism $\mathcal{C}^{\text{an}}(\overline{N}_{-h}, E)\mathbf{1}_{I \cap \overline{N}} \simeq \mathcal{C}^{\text{an}}(t^k \overline{N}_{-h} t^{-k}, E)\mathbf{1}_{t^k(I \cap \overline{N})t^{-k}}$. Hence the image of $\Phi_{\mathcal{W}_{\natural,r}}$ contains $\mathcal{C}^{\text{an}}(t^k \overline{N}_{-h} t^{-k}, E)\mathbf{1}_{t^k(I \cap \overline{N})t^{-k}}$. Take k large enough such that $t^k \overline{N}_{-h} t^{-k} \subset \overline{N}_{h_0}$. The inclusion $t^k \overline{N}_{-h} t^{-k} \subset \overline{N}_{h_0}$ is induced by an inclusion of rigid analytic subsets. Then we see the image of $\Phi_{\mathcal{W}_{\natural,r}}$ contains $\mathcal{C}^{\text{an}}(\overline{N}_{h_0}, E)\mathbf{1}_{t^k(I \cap \overline{N})t^{-k}}$, all rigid analytic functions on $t^k(I \cap \overline{N})t^{-k}$ that converge over $\overline{N}_{h_0} \supset t^k \overline{N}_{-h} t^{-k}$.

For any rigid analytic function $f \in \mathcal{C}^{\mathrm{an}}(\overline{N}_{h_0}, E) \subset \mathcal{A}_{h_0}$, write f_g for the restriction of f on each coset $t^k(I \cap \overline{N})t^{-k}g, g \in t^k(I \cap \overline{N})t^{-k}\backslash \overline{N}_{h_0}$. The translation $g.f_g$ is in $\mathcal{C}^{\mathrm{an}}(\overline{N}_{h_0}, E)\mathbf{1}_{t^k(I\cap\overline{N})t^{-k}}$. Using translations by $\overline{N}_{h_0}$, we see that the image of $\Phi_{\mathcal{W}_{\mathfrak{h}, r}}$ contains $\mathcal{C}^{\mathrm{an}}(\overline{N}_{h_0}, E)$.

Any function in $\mathcal{A}_{h_0}$ is a linear combination of translations by elements in $I \cap \overline{N}$ of such functions in $\mathcal{C}^{\mathrm{an}}(\overline{N}_{h_0}, E)$, hence is also in the image of $\Phi_{\mathcal{W}_{\mathfrak{h}, r}}$. $\qquad\square$

Example 3.14. We explain the idea for Lemma 3.13 by calculating explicitly in the case $L = \mathbb{Q}_p$ and $\mathbf{G} = \mathrm{SL}_2$. In this case $\mathfrak{g} = E.\langle e, f, g \rangle$ is spanned by

$$e = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We take norm $|-|_n$ on $U(\mathfrak{g})$ such that $|e|_n = |f|_n = |h|_n = p^n$. And let $\widehat{U}(\mathfrak{g})_n$ be the Banach completion with respect to this norm.

Take x for the coordinate of $\overline{N} = \{\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p\}$ and $\overline{N} \cap I = \{\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \mid x \in p\mathbb{Z}_p\}$.

Let χ be a character with weight $\lambda \in \overline{\mathbb{Q}}_p$. Consider $\overline{M}(\lambda)^\vee = \mathcal{C}^{\mathrm{pol}}(\overline{N}, E) \simeq \mathcal{C}^{\mathrm{pol}}(\overline{N}, E)\mathbf{1}_{I\cap\overline{N}} \subset \mathrm{Ind}_B^G \chi$. Then $\mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$ is spanned linearly by $x^k\mathbf{1}_{I\cap\overline{N}}$ for $k \in \mathbb{N}$. Lemma 3.13 reduces to that the image of $\widehat{U}(\mathfrak{g})_n \otimes_{U(\mathfrak{g})} \mathcal{C}^{\mathrm{pol}}(I \cap \overline{N}, E)$ contains

$$\mathcal{C}^{\mathrm{an}}(\overline{N}_{-h}, E) = \{\sum_i a_i x^i \mid a_i \in E, \varprojlim_{i} |a_i| p^{hi} = 0\}$$

for some h. The element f acts on $\mathcal{C}^{\mathrm{pol}}(\overline{N}, E)$ by the derivative for x. We calculate that,

$$\begin{aligned}(e.x^k)(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}) &= \lim_{t\to 0} \frac{1}{t} x^k(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}) \\ &= \lim_{t\to 0} x^k(\begin{pmatrix} \frac{1}{1+xt} & t \\ & 1+xt \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{x}{1+xt} & 1 \end{pmatrix}) \\ &= \frac{d}{dt}|_{t=0}\chi(1+xt)^{-1}\frac{x^k}{(1+xt)^k} = -(\lambda+k)x^{k+1}.\end{aligned}$$

And similarly $h.x^k = (\lambda + 2k)x^k$. The $U(\mathfrak{g})$-module $\overline{M}(\lambda)^\vee$ is generated by the highest weight (with respect to $\overline{\mathfrak{b}}$) vector x^0 and possibly x^{k+1} if $\lambda = -k \in \mathbb{Z}_{\leq 0}$. Let $\alpha = 0$ if $\lambda \notin \mathbb{Z}_{\leq 0}$ and $\alpha = -\lambda + 1$ otherwise. The above formula shows that $x^{\alpha+i} = (-1)^i \prod_{j=0}^{i-1} \frac{1}{\lambda+\alpha+j} e^i.x^\alpha$ for $i \geq 1$.

For $g = \sum_{i\geq 1} a_i x^{\alpha+i} \in \mathcal{C}^{\mathrm{an}}(\overline{N}_{-h}, E)$, we can rewrite it as

$$g = (\sum_{i\geq 1}(-1)^i a_i \frac{1}{\prod_{j=0}^{i-1}(\lambda+\alpha+j)} e^i).x^\alpha.$$

The sum $u := \sum_{i\geq 1}(-1)^i a_i \frac{1}{\prod_{j=0}^{i-1}(\lambda+\alpha+j)} e^i$ converges in $\widehat{U}(\mathfrak{g})_n$ if and only if $\varprojlim_i \frac{p^{-ni}a_i}{\prod_{j=0}^{i-1}(\lambda+\alpha+j)} = 0$. Since $\varprojlim_i a_i p^{-hi} = 0$, we find that the element $g = u.x^\alpha$ lies in the image of $\widehat{U}(\mathfrak{g})_n \otimes_{U(\mathfrak{g})} \overline{M}(\lambda)^\vee$ if

$$\varprojlim_{i} \frac{p^{(h-n)i}}{\prod_{j=0}^{i-1}(\lambda+\alpha+j)} = 0.$$

Such h exists if $-(\lambda + \alpha)$ is of positive type (Definition 4.8). This holds at least for all $\lambda \in \mathbb{Z}$ using that $|k!|_p = p^{-\frac{k-s(k)}{p-1}} \geq p^{-\frac{k}{(p-1)}}$ for a positive integer k.

On the other hand, if $-(\lambda + \alpha)$ is not of positive type, we don't know whether Proposition 3.11 is still true.

Theorem 3.15. *Let $M \in \mathcal{O}_{\mathrm{alg}}^{\flat}$ and χ_{sm} be a smooth character of T, then for $? \in \{\natural, \sharp\}$, the resolution in Proposition 3.9 induces a quasi-isomorphism*

$$\wedge^{\bullet}\mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{?,r}(\tau(M)^{\vee} \otimes \chi_{\mathrm{sm}}) \to \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$$

of complexes of $\mathcal{D}(G, E)$-modules.

Proof. This follows from Proposition 3.11, Proposition 3.10, Proposition 3.9 and the snake lemma. $\qquad\square$

Remark 3.16. One can consider $\mathcal{W}_{\natural,n}(\overline{M}) := \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{B} \cap I)} \overline{M}$ instead of $\mathcal{W}_{\natural,r}(\overline{M})$. There are maps $\mathcal{W}_{\natural,n}(\overline{M}) \to \mathcal{W}_{\natural,r}(\overline{M}) \to \mathcal{W}_{\natural,n'}(\overline{M})$ for r sufficiently close to 1 and n large enough. These maps are injections at least if $\overline{M} = \mathcal{D}(\mathfrak{g}, \overline{B}) \otimes_{\mathcal{D}(\overline{B}, E)} \sigma$ for a finite-dimensional $\overline{B}$-representation σ and induce quasi-isomorphisms (using Proposition 2.2 and Corollary 2.3)

$$\wedge^{\bullet}\mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\natural,n}(\overline{M}) \simeq \wedge^{\bullet}\mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\natural,r}(\overline{M})$$

between n and r-versions. The quasi-isomorphisms also hold for general $\overline{M}$ in Theorem 3.15 by considering a presentation of $\overline{M}$ as in the proof of Lemma 3.3 and the right exactness of $- \otimes_{\mathcal{H}} \mathcal{H}/\mathfrak{m}$.

3.5. **Duality for complexes.** We can already observe certain duality between different Koszul resolutions in Theorem 3.15. At present, we will establish the duality only for complexes (Theorem 3.21). The duality between locally analytic representations will be discussed later in §5.

We continue with the notation in §3.3. The continuous linear E-dual $\mathcal{W}_{\natural,r}(\sigma^{\vee})^{\vee}$ of $\mathcal{W}_{\natural,r}(\sigma^{\vee})$ is $\mathcal{D}_r(I, E) \otimes_{D(B \cap I, E)} \sigma$ and the dual of $\mathcal{W}_{\sharp,r}(\tau(M)^{\vee})$ is $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M$. We adapt the convention that $g.f(-) = f(g^{-1}-)$ for $f \in \mathrm{Hom}_E(\mathcal{W}, E)$ for an I-representation $\mathcal{W}$ and $g \in I$. As in Lemma 3.3, for $t \in T^-$, there is a $\psi_{t^{-1}}$ operator on $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M$ extending the action of δ_t on M. We obtain also $U_{t^{-1}}$ operators on $\text{c-Ind}_I^G \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M$, as in §2.1.

Lemma 3.17. *Let $t \in T^-$. The transpose $\psi_t^{\vee}$ of ψ_t on $\mathcal{W}_{\sharp,r}(\tau(M)^{\vee})^{\vee} = \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M$ coincides with $\psi_{t^{-1}}$.*

Proof. Let H^m be the subgroup after Definition 3.1 such that $\mathcal{D}_r(I, E) = E[I] \otimes_{E[H^m]} U_r(\mathfrak{g})$. The canonical pairing $\mathcal{D}(I, E) \times \mathcal{C}^{\mathrm{la}}(I, E) \to E$ is I-equivariant for left I-modules (the left I-action on $\mathcal{D}(I, E)$ comes from the transpose of its left action on $\mathcal{C}^{\mathrm{la}}(I, E)$ composed with the inverse involution). The pairing refines to pairings $\delta_g D(H^m, E) \times g\mathcal{C}^{\mathrm{la}}(H^m, E) \to E$ for $g \in I/H^m$ where $\delta_g \in E[I]$ is the Dirac distribution. Hence the dual of $\delta_g U_r(\mathfrak{g})$ can be identified with the direct summand $g\mathcal{C}^r(H^m, E) = \mathcal{C}^r(H^m g, E) \subset \mathcal{C}^r(I, E)$ for $g \in I/H^m$. We see for $m \in M_r \subset \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M, g \in I$ and $f \in \mathcal{W}_{\sharp,r}(\tau(M)^{\vee}) \subset \mathrm{Ind}_B^G \sigma^{\vee}(BI)$, we have $m(f) = m(f\mathbf{1}_{BH^m})$ and $(\delta_g m)(f) = m(g^{-1}.f)$.

Hence for $m \in M \subset \mathcal{D}(G, E) \otimes_{\mathcal{D}(\mathfrak{g}, B)} M, g \in I$,

$$(\psi_t^{\vee}(\delta_g m))(f) = \delta_g m(\psi_t f) = \delta_g m((t^{-1}.f)\mathbf{1}_{BI}) = m(((g^{-1}t^{-1}).f)\mathbf{1}_{BI}) = m((g^{-1}t^{-1}).f),$$

where the last equality is calculated for $g^{-1}t^{-1}f \in \mathrm{Ind}_B^G \sigma^\vee$. By the construction (see Lemma 2.10), $\psi_{t^{-1}}(\delta_g m) = \delta_{tgt^{-1}} t.m$ for $g \in I \cap \overline{B}$. Hence

$$(\psi_{t^{-1}}(\delta_g m))(f) = (\delta_{tgt^{-1}} t.m)(f) = (t.m)(tg^{-1}t^{-1}f) = m(g^{-1}t^{-1}f).$$

We see $\psi_t^\vee = \psi_{t^{-1}}$ on $\delta_g.M$ for $g \in I \cap \overline{B}$. Since M is dense in M_r, $\sum_{g \in I \cap \overline{B}} \delta_g.M$ is dense in $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} M$ and both operators are continuous, the result follows. $\qquad \square$

We discuss some generality of Hecke operators. Let $\mathcal{W}^\vee$ be the (continuous) dual of an I-representation $\mathcal{W}$ (with the open compact topology). Suppose that there are ψ_t-actions on $\mathcal{W}$ which induce U_t-actions on c-$\mathrm{Ind}_I^G \mathcal{W}$. For $t \in T^-$, let $U_t^\vee$ be the transpose of U_t acting on c-$\mathrm{Ind}_I^G \mathcal{W}^\vee$ under the following G-invariant pairing

$$(3.2) \qquad \mathrm{c\text{-}Ind}_I^G \mathcal{W}^\vee \times \mathrm{c\text{-}Ind}_I^G \mathcal{W} \to E : \big(\sum_{g \in G/I} [g, f_g], \sum_{g \in G/I} [g', w_{g'}] \big) \mapsto \sum_{g \in G/I} f_g(w_g),$$

such that $(x, U_t y) = (U_t^\vee x, y)$ (see Lemma 3.18 or Lemma 3.19 below for the existence of the transpose). Let $\psi_{t^{-1}} := \psi_t^\vee$ be the transpose of ψ_t on $\mathcal{W}^\vee$ so that $\psi_{t^{-1}}(x.f) = (txt^{-1})\psi_{t^{-1}} f$ for $x \in I \cap t^{-1} It$. This allows us to define the corresponding Hecke operator $U_{t^{-1}}$ on c-$\mathrm{Ind}_I^G \mathcal{W}^\vee$.

Lemma 3.18. *The transpose $U_t^\vee$ of U_t is equal to $U_{t^{-1}}$.*

Proof. We calculate that for $f \in \mathcal{W}^\vee, g \in G$ and $w \in \mathcal{W}$,

$$U_t^\vee([1, f])([g, w]) = [1, f]\big(\sum_{x \in I/(I \cap tIt^{-1})} [gxt, \psi_t(x^{-1}w)] \big)$$

$$= \sum_{x \in I/(I \cap tIt^{-1}) \cap g^{-1}It^{-1}} f((gxt)\psi_t(x^{-1}w))$$

The intersection $I \cap g^{-1} I t^{-1} \neq \emptyset$ if and only if $g \in It^{-1}I$. And for $g = x't^{-1}$ where $x' \in I$ and $x \in I$, $gxt = x't^{-1}xt$ is in I if and only $x \in I \cap tIt^{-1}$. Thus $U_t^\vee([1, f]) = \sum_{x' \in I/(I \cap t^{-1}It)} [x't^{-1}, (w \mapsto f(x'\psi_t(w)))] = \sum_{x' \in I/(I \cap t^{-1}It)} [x't^{-1}, \psi_{t^{-1}}(x'^{-1}.f)] = U_{t^{-1}}([1, f])$. $\qquad \square$

We consider the left and right G-actions on the space $\mathcal{C}_c^{\mathrm{la}}(G, E)$ of compactly supported locally analytic functions on G given by right and left translations: $(l_g r_h f)(-) = f(g - h)$. For a locally analytic representation π, we write

$$\mathbb{D}_{\mathrm{BZ}}^0(\pi) := \mathrm{Hom}_G(\pi, \mathcal{C}_c^{\mathrm{la}}(G, E)) - \mathrm{Hom}_G(E, \mathrm{Hom}_E(\pi, \mathcal{C}_c^{\mathrm{la}}(G, E)))$$

where G acts on $f \in \mathrm{Hom}_E(\pi, \mathcal{C}_c^{\mathrm{la}}(G, E))$ by $g.f(-) = r_h(f(g^{-1}-))$ and we take the continuous Hom's. This is a right $\mathcal{D}(G, E)$-module induced by the left translation on the target $\mathcal{C}_c^{\mathrm{la}}(G, E)$ and we make it into a left $\mathcal{D}(G, E)$-module using the involution $g \mapsto g^{-1}$ of G, or consider the left action of G on $\mathcal{C}_c^{\mathrm{la}}(G, E)$ by $g.f = f(g^{-1}-)$. We will ignore the topology of $\mathbb{D}_{\mathrm{BZ}}^0(\pi)$ in this section (we refer to Theorem 5.3 for the discussion on "topology").

Lemma 3.19. *Let $\mathcal{W}$ be a locally analytic $\mathcal{D}(I, E)$-module over a Smith/Banach space (so that $\mathcal{W}^\vee$ is a Banach/Smith space).*

(1) *We have $\mathbb{D}_{\mathrm{BZ}}^0(\mathrm{c\text{-}Ind}_I^G \mathcal{W}) = \mathrm{c\text{-}Ind}_I^G \mathcal{W}^\vee$. And under this identification, $[g, f]([h, w]) = (g' \in gIh^{-1} \mapsto f(g^{-1}g'hw)) \in \mathcal{C}^{\mathrm{la}}(gIh^{-1}, E) \subset \mathcal{C}_c^{\mathrm{la}}(G, E)$.*

(2) If $U : \text{c-Ind}_I^G \mathcal{W} \to \text{c-Ind}_I^G \mathcal{W}$ is a G-equivariant map, then $\mathbb{D}_{\text{BZ}}^0(U) : \text{c-Ind}_I^G \mathcal{W}^\vee \to \text{c-Ind}_I^G \mathcal{W}^\vee$ under the identification in (1) is the transpose of U under the pairing (3.2).

Proof. (1) $\text{Hom}_G(\text{c-Ind}_I^G \mathcal{W}, \mathcal{C}_c^{\text{la}}(G, E)) = \text{Hom}_I(\mathcal{W}, \mathcal{C}^{\text{la}}(I, E) \otimes_{E[I]} E[G])$. Since $\mathcal{W}$ is a Banach or Smith over E and $\mathcal{C}_c^{\text{la}}(G, E) = \oplus_{g \in G/I} l_g \mathcal{C}^{\text{la}}(I, E)$, we have ([Sch13b, Cor. 8.9] or [RJRC22, Lem. 3.32])

$$\text{Hom}_E(\mathcal{W}, \mathcal{C}^{\text{la}}(I, E)) = \oplus_{g \in G/I} \text{Hom}_E(\mathcal{W}, \mathcal{C}^{\text{la}}(g^{-1}I, E)).$$

Taking I-invariants

$$\text{Hom}_I(W, \mathcal{C}^{\text{la}}(I, E) \otimes_{E[I]} E[G]) \quad = \text{Hom}_I(W, \mathcal{C}^{\text{la}}(I, E)) \otimes_{E[I]} E[G].$$

Using [RJRC22, Thm. 3.40], there is an I-isomorphism

$$\text{Hom}_I(\mathcal{W}, \mathcal{C}^{\text{la}}(I, E)) = \text{Hom}_{\mathcal{D}(I,E)}(\mathcal{D}(I, E), \mathcal{W}^\vee) = \mathcal{W}^\vee$$

which sends $f \in \mathcal{W}^\vee$ to $w \in \mathcal{W} \mapsto (g \mapsto f(gw))$. Finally, for $f \in \mathcal{W}^\vee$, we see $[g, f]([h, w]) = r_h.([g, f]([1, w])) = r_h.(l_{g^{-1}}.(g' \mapsto f(g'w))) = (g' \mapsto f(g^{-1}g'hw))$.

(2) By definition, for $f \in \text{c-Ind}_I^G \mathcal{W}^\vee, w \in \text{c-Ind}_I^G \mathcal{W}, f(w) \in \mathcal{C}_c^{\text{la}}(G, E)$, we have $(g.f)(w) = l_{g^{-1}} f(w), f(g.w) = r_g f(w)$. Hence the pairing below

$$\text{c-Ind}_I^G \mathcal{W}^\vee \times \text{c-Ind}_I^G \mathcal{W} \to \mathcal{C}_c^{\text{la}}(G, E) \to E, (f, w) \mapsto \langle f, w \rangle := (f(w))(1).$$

is equivariant for the left actions on $\text{c-Ind}_I^G \mathcal{W}^\vee$ and $\text{c-Ind}_I^G \mathcal{W}$ and the trivial action on E. Using that $[g, f]([h, w])(1)$ is not zero only if $h \in gI$, we can check that the pairing $\langle -, - \rangle$ coincides with (3.2).

On the other hand, the operator $\mathbb{D}_{\text{BZ}}^0(U)$ is characterized by for $g \in G, f \in \text{c-Ind}_I^G \mathcal{W}^\vee, w \in \text{c-Ind}_I^G \mathcal{W}$,

$$(\mathbb{D}_{\text{BZ}}^0(U).f)(w)(g) = f(U.w)(g),$$

and by the G-equivariances of U and $\mathbb{D}_{\text{BZ}}^0(U)$, characterized by the equality

$$\langle \mathbb{D}_{\text{BZ}}^0(U).f, w \rangle = \langle f, U.w \rangle$$

for all f, w. Hence $\mathbb{D}_{\text{BZ}}^0(U)$ is the transpose of U. $\qquad\square$

We return to Koszul complexes. Let $\mathfrak{m} \subset \mathcal{H}$ act on $\text{c-Ind}_I^G \mathcal{W}$ for $\mathcal{W} = \mathcal{W}_?(M)$ taken as in Proposition 3.9. Apply the functor $\mathbb{D}_{\text{BZ}}^0$ for the Koszul complex $\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}$, the "dual" $\mathbb{D}_{\text{BZ}}^0(\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W})$ is then a complex of G-modules in cohomological degrees $[0, d]$.

Lemma 3.20. *The complex $\mathbb{D}_{\text{BZ}}^0(\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W})$ is equal to the Koszul complex*

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}^\vee[-d]$$

where $\mathcal{H}$ acts on $\text{c-Ind}_I^G \mathcal{W}^\vee$ by the transpose of its action on $\text{c-Ind}_I^G \mathcal{W}$ and $[-d]$ denotes the cohomological degree shift.

Proof. Suppose that $\mathcal{H}$ is a commutative ring, A is a ring, M is an $(A, \mathcal{H})$-bimodule with commutative actions, and let N be an A-module. Let $\varphi : \mathcal{H}^m \to \mathcal{H}^n$ be an $\mathcal{H}$-linear map inducing $\varphi_M : \mathcal{H}^m \otimes_{\mathcal{H}} M \to \mathcal{H}^n \otimes_{\mathcal{H}} M$. For a perfect $\mathcal{H}$-module P, we have functorially $\text{Hom}_A(P \otimes_{\mathcal{H}} M, N) = \text{Hom}_{\mathcal{H}}(P, \text{Hom}_A(M, N))$ and $R\text{Hom}_{\mathcal{H}}(P, \text{Hom}_A(M, N)) = P^\vee \otimes_{\mathcal{H}}^L \text{Hom}_A(M, N)$, where $P^\vee := R\text{Hom}_{\mathcal{H}}(P, \mathcal{H})$ [Sta24, Tag 08JJ]. Take $P = \mathcal{H}^m, \mathcal{H}^n$, we see

$\mathrm{Hom}_A(-, N)(\varphi_M) : \mathrm{Hom}_A(\mathcal{H}^n \otimes_{\mathcal{H}} M, N) \to \mathrm{Hom}_A(\mathcal{H}^m \otimes_{\mathcal{H}} M, N)$ can be identified with the tensoring over $\mathcal{H}$ of the transpose $\varphi^\vee : (\mathcal{H}^n)^\vee \to (\mathcal{H}^m)^\vee$ with $\mathrm{Hom}_A(M, N)$.

Apply the same reasoning for connecting maps of $\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}$, we see the "dual" $\mathbb{D}_{\mathrm{BZ}}^0(\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W})$ is calculated by $\mathrm{Hom}_{\mathcal{H}}(\wedge^\bullet \mathcal{H}^d, \mathcal{H}) \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}^\vee$ where $\mathcal{H}$ acts on c-Ind$_I^G \mathcal{W}^\vee$ by the transpose in Lemma 3.19. We get the desired identification using that $\mathrm{Hom}_{\mathcal{H}}(\wedge^\bullet \mathcal{H}^d, \mathcal{H}) = \wedge^\bullet \mathcal{H}^d[-d]$ by the autoduality of the exterior algebras, cf. [Ser97, §A.2, Ch. IV]. $\qquad\square$

We can now prove our main theorem on the duality.

Theorem 3.21. *Let $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{b}}$ and χ_{sm} be a smooth character of T. Consider the resolution* c-Ind$_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes_E \chi_{\mathrm{sm}})$ *of $\mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ in Theorem 3.15. Then there is a quasi-isomorphism*

$$\mathbb{D}_{\mathrm{BZ}}^0(\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes_E \chi_{\mathrm{sm}})) \simeq \mathcal{F}_{\overline{B}}^G(\mathrm{Hom}_E(M, E)^{\overline{\mathfrak{n}}^\infty}, \mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}}))$$

of complexes of $E[G]$-modules.

Proof. By Lemma 3.20, the dual complex is $\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee[-d]$ for the action of $U_{t^{-1}}$ operators. By Lemma 3.17 and that $\mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee = \mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I \cap B)} (M \otimes \chi_{\mathrm{sm}}^{-1})$, the transpose $U_{t^{-1}}$ comes from the t-actions or $\psi_{t^{-1}}$-actions on $M \otimes \chi_{\mathrm{sm}}^{-1}$ in the way of Lemma 3.3.

Let $I' = I \cap \overline{I}$ where $\overline{I}$ is the opposite Iwahori. By Proposition 2.13,

$$(3.3) \qquad \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee \simeq \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{I'}^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee|_{I'}.$$

While $\mathcal{D}_r(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, B \cap I)} (M \otimes \chi_{\mathrm{sm}}^{-1})|_{I'} = \mathcal{D}_r(I', E) \otimes_{\mathcal{D}(\mathfrak{g}, I' \cap B)} (M \otimes \chi_{\mathrm{sm}}^{-1})$ and

$$\text{c-Ind}_{I'}^{\overline{I}} \mathcal{D}_r(I', E) \otimes_{\mathcal{D}(\mathfrak{g}, I' \cap B)} (M \otimes \chi_{\mathrm{sm}}^{-1}) = \mathcal{D}_r(\overline{I}, E) \otimes_{\mathcal{D}(\mathfrak{g}, \overline{I} \cap B)} (M \otimes \chi_{\mathrm{sm}}^{-1}) =: \overline{\mathcal{W}_{\sharp,r}}(M \otimes \chi_{\mathrm{sm}}^{-1}).$$

Hence by (2) of Proposition 2.11,

$$(3.4) \qquad \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{I'}^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee|_{I'} \simeq \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{\overline{I}}^G \overline{\mathcal{W}_{\sharp,r}}(M \otimes \chi_{\mathrm{sm}}^{-1}).$$

Combining (3.3), (3.4) and Theorem 3.15, we have

$$\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}_{\sharp,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})^\vee \simeq \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_{\overline{I}}^G \overline{\mathcal{W}}_{\sharp,r}(M \otimes \chi_{\mathrm{sm}}^{-1}) \simeq \mathcal{F}_{\overline{B}}^G(\tau(M)^\vee, \chi_{\mathrm{sm}}^{-1}).$$

Check the degree shift and we get the desired quasi-isomorphism. $\qquad\square$

4. Localization and completion

This section aims to prove Proposition 4.11 (used for Proposition 3.11) using the Beilinson-Bernstein localization of Ardakov-Wadsley [AW13].

4.1. **The localization functor.** We recall the settings and results in [AW13]. We change the notations from previous chapters. Let $\mathbf{G}$ be a connected split reductive group over $\mathcal{O}_L$. We assume that $\mathbf{G}$ is semisimple and simply-connected. Fix a Borel subalgebra $\mathbf{B}$ and the unipotent radical $\mathbf{N}$ and the opposite groups $\overline{\mathbf{B}}, \overline{\mathbf{N}}$. Write $\mathbf{X} = \mathbf{G}/\overline{\mathbf{B}}$ for the flag variety over $\mathcal{O}_L$. Let $G = \mathbf{G}(L), B = \mathbf{B}(L)$ as before. Let $\mathfrak{g}, \mathfrak{b}$, etc. be the Lie algebras over $\mathcal{O}_L$.

There is the moment map

$$\beta : \widetilde{T^*\mathbf{X}} = \mathbf{G} \times^{\overline{\mathbf{B}}} (\mathfrak{g}/\overline{\mathfrak{b}})^* \to \mathfrak{g}^*,$$

inducing a homomorphism $U(\mathfrak{g}) \to \widetilde{\mathcal{D}}$ as filtered $\mathcal{O}_L$-algebras. Here $\mathbf{H} = \overline{\mathbf{B}}/\overline{\mathbf{N}}$, $\widetilde{\mathcal{D}} = (\xi_* \mathcal{D}_{\widetilde{\mathbf{X}}})^{\mathbf{H}}$ for $\xi : \widetilde{\mathbf{X}} := \mathbf{G}/\overline{\mathbf{N}} \to \mathbf{G}/\overline{\mathbf{B}}$ and $\mathcal{D}_{\widetilde{\mathbf{X}}}$ denotes the sheaf of differential operators on $\widetilde{\mathbf{X}}$. The sheaf of algebras $\widetilde{\mathcal{D}} \simeq \mathcal{D} \otimes U(\mathfrak{h})$ is equipped with a natural increasing filtration $F_m \widetilde{\mathcal{D}}, m \geq 0$ (induced by order of differential operators) as in [AW13, §4.6] such that $\mathrm{gr}(\widetilde{\mathcal{D}}) \simeq \mathrm{Sym}_{\mathcal{O}_\mathbf{X}} \widetilde{\mathcal{T}}$ for the enhanced tangent sheaf $\widetilde{\mathcal{T}} = (\xi_* \mathcal{T}_{\widetilde{\mathbf{X}}})^{\mathbf{H}}$. The fiber of $\widetilde{\mathcal{D}}$ at the point $\overline{\mathbf{B}} \in \mathbf{X}$ with respect to the left $\mathcal{O}_\mathbf{X}$-action is equal to $U(\mathfrak{g})/\overline{\mathfrak{n}}U(\mathfrak{g})$.

Identify $\mathbf{H} = \overline{\mathbf{B}}/\overline{\mathbf{N}} = \mathbf{T}$, the maps $U(\mathfrak{g}) \to \widetilde{\mathcal{D}}$ and $U(\mathfrak{t}) \to \widetilde{\mathcal{D}}$ coincide on the center $Z(\mathfrak{g}) = U(\mathfrak{g})^\mathbf{G}$ of $U(\mathfrak{g})$ via the Harish-Chandra morphism $Z(\mathfrak{g}) \to U(\mathfrak{t})^{W,\cdot}$ ([AW13, §4.10]). And $U(\mathfrak{g})$ is also equipped with the Poincaré-Birkhoff-Witt filtration so that $\mathrm{gr}(U(\mathfrak{g})) = \mathrm{Sym}_{\mathcal{O}_L} \mathfrak{g}$.

There is a set of collections $\mathcal{S}_\mathbf{X}$ of affine Zariski opens of $\mathbf{X}$ that trivialise ξ as in [AW13, §4.3]. The n-th deformation $\widetilde{\mathcal{D}}_n$ is defined to be the subalgebra $\sum_i \varpi_L^{in} F_i \widetilde{\mathcal{D}}$ whose ϖ_L-adic completion is denoted by $\widehat{\widetilde{\mathcal{D}}_n}$ [AW13, Def. 5.9] with $\widehat{\widetilde{\mathcal{D}}_{n,L}} := \widehat{\widetilde{\mathcal{D}}_n} \otimes_{\mathcal{O}_L} L$ which are sheaves for the Zariski topology on $\mathbf{X}$. Note that $\widehat{\widetilde{\mathcal{D}}_n} := \varprojlim_a \widetilde{\mathcal{D}}_n/\varpi^a \widetilde{\mathcal{D}}_n$ is supported on the special fiber of $\mathbf{X}$. Similarly we define $U(\mathfrak{g})_n = U(\varpi_L^n \mathfrak{g})$ and $\widehat{U(\mathfrak{g})_{n,L}}$. The limit $\widehat{U(\mathfrak{g}_L)} = \varprojlim_{n \in \mathbb{N}} \widehat{U(\mathfrak{g})_{n,L}}$ is the Arens-Michael envelope of $\mathfrak{g}_L$.

Fix an $\mathcal{O}_L$-linear map $\lambda : \varpi_L^n \mathfrak{h} \to \mathcal{O}_L$ which induces a character $\lambda : U(\mathfrak{h})_n := U(\varpi^n \mathfrak{h}) \to \mathcal{O}_L$. The sheaf of deformed twisted differential operators is defined by ([AW13, Def. 6.4]) $\mathcal{D}_n^\lambda := \widetilde{\mathcal{D}}_n \otimes_{U(\mathfrak{h})_n} \lambda$ with the ϖ_L-adic completion $\widehat{\mathcal{D}_n^\lambda}$ and $\widehat{\mathcal{D}_{n,L}^\lambda} := \widehat{\mathcal{D}_n^\lambda} \otimes_{\mathcal{O}_L} L$. Let $\widehat{\mathcal{U}_n^\lambda}, \widehat{\mathcal{U}_{n,L}^\lambda}$ be corresponding completions for $\mathcal{U}_n^\lambda := U(\mathfrak{g})_n \otimes_{(U(\mathfrak{g})^\mathbf{G})_n} \lambda$ as in [AW13, §6.10]. And we write $\widehat{\mathcal{D}_{n,L}}, \widehat{\mathcal{U}_{n,L}}$, etc. for $\widehat{\mathcal{D}_{n,L}^\lambda}, \widehat{\mathcal{U}_{n,L}^\lambda}$, etc. if $\lambda = 0$. Then [AW13, Thm. 6.10] says that the map $\widehat{\mathcal{U}_{n,L}^\lambda} \to \widehat{\mathcal{D}_{n,L}^\lambda}$ induces an isomorphism

$$\widehat{\mathcal{U}_{n,L}^\lambda} \simeq \Gamma(\mathbf{X}, \widehat{\mathcal{D}_{n,L}^\lambda}).$$

Let $\mathrm{Loc}^\lambda : \mathrm{Mod}(\widehat{\mathcal{U}_{n,L}^\lambda}) \to \mathrm{Mod}(\widehat{\mathcal{D}_{n,L}^\lambda})$ be the localization functor

$$M \mapsto \widehat{\mathcal{D}_{n,L}^\lambda} \otimes_{\widehat{\mathcal{U}_{n,L}^\lambda}} M.$$

Let ρ be the half sum of all positive roots.

Theorem 4.1 ([AW13, Thm. 6.12]). *Let $\lambda \in \mathrm{Hom}_{\mathcal{O}_L}(\varpi_L^n \mathfrak{h}, \mathcal{O}_L)$.*

(1) If $\lambda + \rho$ is dominant ($\langle \lambda + \rho, \alpha^\vee \rangle \notin \{-1, -2, \cdots\}$ for any positive (with respect to $\mathfrak{b}$) coroot $\alpha^\vee$ of $\mathbf{H}$, [AW13, §6.7]) and regular (i.e., the stabilizer of $\lambda + \rho$ under the action of the Weyl group is trivial), then the functor Loc^λ and the functor $\Gamma(\mathbf{X}, -)$ of taking global sections induce an equivalence of the categories of coherent modules over $\widehat{\mathcal{U}_{n,L}^\lambda}$ and $\widehat{\mathcal{D}_{n,L}^\lambda}$.

(2) If $\lambda + \rho$ is dominant but not regular, then Loc^λ and $\Gamma(\mathbf{X}, -)$ still induce an equivalence between coherent $\widehat{\mathcal{U}_{n,L}^\lambda}$-modules and the quotient of the category of coherent $\widehat{\mathcal{D}_{n,L}^\lambda}$-modules by the full subcategory of modules that are in the kernel of Γ.

Remark 4.2. In [AW13], the prime number p needs to be very good for $\mathbf{G}$ [AW13, §6.8]. This assumption is removed in [Ard21, §5.3]

Let M be a coherent $U(\mathfrak{g}_L)$-module with infinitesimal character given by λ. The localization $\mathcal{M} := \mathcal{D}_{n,L}^\lambda \otimes_{\mathcal{U}_{n,L}^\lambda} M$ is a coherent module of $\mathcal{D}_L^\lambda = \mathcal{D}_{n,L}^\lambda$ and $\Gamma(\mathbf{X}, \mathcal{M}) = M$ (this is essentially the classical localization). Let $\widehat{M_{n,L}} = \widehat{\mathcal{U}_{n,L}^\lambda} \otimes_{\mathcal{U}_{n,L}^\lambda} M$ and $\widehat{\mathcal{M}_{n,L}} := \mathrm{Loc}^\lambda(\widehat{M_{n,L}})$.

Lemma 4.3. *For $\mathbf{V} \in \mathcal{S}_{\mathbf{X}}$,*

$$\widehat{\mathcal{M}_{n,L}}(\mathbf{V}) = \widehat{\mathcal{D}_{n,L}^\lambda}(\mathbf{V}) \otimes_{\mathcal{D}_{n,L}^\lambda(\mathbf{V})} \mathcal{M}(\mathbf{V}).$$

Proof. Since $\mathcal{S}_{\mathbf{X}}$ is coherently acyclic (even affine) for the sheaves of rings $\widehat{\widetilde{\mathcal{D}}_{n,L}}$ and $\widetilde{\mathcal{D}}_{n,L}$ in the sense of [AW13, Def. 5.1] by [AW13, Cor. 5.9, Thm. 5.13], so is for their quotients $\mathcal{D}_{n,L}^\lambda, \widehat{\mathcal{D}_{n,L}^\lambda}$ ([AW13, Lem. 6.11]) by the same proof. For any $\mathbf{V} \in \mathcal{S}_{\mathbf{X}}$, we get $\mathcal{M}(\mathbf{V}) = \mathcal{D}_{n,L}^\lambda(\mathbf{V}) \otimes_{U_{n,L}^\lambda} M$ and $\widehat{\mathcal{M}_{n,L}}(\mathbf{V}) = \widehat{\mathcal{D}_{n,L}^\lambda}(\mathbf{V}) \otimes_{\widehat{\mathcal{U}_{n,L}^\lambda}} \widehat{M_{n,L}}$ (writing a finite presentation of M using free $\mathcal{U}_{n,L}^\lambda$-modules). The result follows. $\square$

Moreover $\widehat{\mathcal{D}_{n,L}^\lambda}(\mathbf{V}) = \widehat{\mathcal{D}_n^\lambda(\mathbf{V})} \otimes_{\mathcal{O}_L} L$ by [AW13, Prop. 6.5]. Hence $\widehat{\mathcal{M}_{n,L}}(\mathbf{V})$ is the completion of the coherent module $\mathcal{M}(\mathbf{V})$ over $\mathcal{D}_{n,L}^\lambda(\mathbf{V})$ with respect to the p-adic topology induced from the lattice $\mathcal{D}_n^\lambda(\mathbf{V}) \subset \mathcal{D}_{n,L}^\lambda(\mathbf{V})$.

4.2. Dual Verma modules for dominant weights.
We consider the localization of dual Verma modules with dominant weights.

Suppose that λ is integral (for the group $\mathbf{H}$) and $\lambda + \rho$ is dominant with respect to $\mathfrak{b}$ (equivalently, $w_0(\lambda) - \rho = w_0(\lambda + \rho)$ is dominant with respect to $\overline{\mathfrak{b}}$, where w_0 is the longest element in the Weyl group). Take $\overline{M} = \overline{M}(w_0(\lambda))^\vee$ the BGG dual of the Verma module $\overline{M}(w_0(\lambda)) = U(\mathfrak{g}_L) \otimes_{U(\overline{\mathfrak{b}})} w_0(\lambda)$ in the category $\mathcal{O}^{\overline{\mathfrak{b}}_L}$ of the dominant (for $\overline{\mathfrak{b}}$) highest weight $w_0(\lambda)$. Consider the Bruhat cells $\mathbf{X}_w^\circ = \overline{\mathbf{B}} w \overline{\mathbf{B}}/\overline{\mathbf{B}} \subset \mathbf{X}$. The inclusion $j_{w_0} : \mathbf{X}_{w_0}^\circ \hookrightarrow \mathbf{X}$ is an open embedding.

Lemma 4.4. *There is an isomorphism of $U(\mathfrak{g}_L)$-modules,*

$$\overline{M}(w_0(\lambda))^\vee \simeq \Gamma(\mathbf{X}, ((j_{w_0,*}\mathcal{O}_{\mathbf{X}_{w_0}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}(\lambda)) \otimes_{\mathcal{O}_L} L).$$

Here $j_{w_0,*}$ is the usual direct image functor for $\mathcal{O}_{\mathbf{X}}$-modules and $\mathcal{O}(\lambda)$ is the line bundle $\mathbf{G} \times^{\overline{\mathbf{B}}} \lambda$ over $\mathbf{X}$ consisting of functions f on $\mathbf{G}$ such that $f(gb) = \lambda^{-1}(b)f(g)$ for $b \in \overline{\mathbf{B}}$. And,

$$\mathcal{M}_{n,L} \simeq ((j_{w_0,*}\mathcal{O}_{\mathbf{X}_{w_0}}) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}(\lambda)) \otimes_{\mathcal{O}_L} L$$

as $\mathcal{D}_{n,L}^\lambda$-modules under the equivalences in Theorem 4.1.

Proof. This is [BG99, Prop. 4.4]. We give some explanation. The sheaf $(j_{w_0,*}\mathcal{O}_{\mathbf{X}_{w_0}}) \otimes_{\mathcal{O}_L} L$ is a $\mathcal{D}_L = \mathcal{D}_{n,L}$-module for the trivial character 0 and the translation $(j_{w_0,*}\mathcal{O}_{\mathbf{X}_{w_0}})_L \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}(\lambda)$ becomes a $\mathcal{D}_L^\lambda$-module via the identification $\mathcal{D}_L^\lambda \simeq \mathcal{O}(\lambda) \otimes \mathcal{D}_L \otimes \mathcal{O}(\lambda)^{-1}$. The action of $\widetilde{\mathcal{D}} = (\xi_*\mathcal{D}_{\mathbf{G}/\overline{\mathbf{N}}})^{\mathbf{H}}$ on $\mathcal{O}(\lambda) \subset (\xi_*\mathcal{O}_{\mathbf{G}/\overline{\mathbf{N}}})^{\mathbf{H}=\lambda}$ factors through the quotient $\widetilde{\mathcal{D}}_0^\lambda$ since $U(\mathfrak{h}) \subset \widetilde{\mathcal{D}}$ in [AW13, Prop. 4.6] acts via the character λ, see [Kas88, §6]. If $\lambda = 0, \overline{M}(w_0(0))^\vee = \overline{M}(ww_0 \cdot 0)^\vee$

for $w = w_0$ (here $w \cdot \lambda = w(\lambda - \rho) + \rho$), the result is [BK81, Cor. 5.8], or Lemma 4.6 below. In general, if λ is dominant for $\mathfrak{b}$, $\Gamma(\mathbf{X}, \mathcal{O}(\lambda))$ has B-highest weight λ, $\overline{B}$-highest weight $w_0(\lambda)$, see [Jan03, Prop. II.2.2]. The global section of the geometric translation $\mathrm{Loc}_0(\overline{M}(0)) \otimes \mathcal{O}(\lambda)$ when $\lambda + \rho$ is $\mathfrak{b}$-dominant is the translation $T_{w_0 \cdot 0}^{w_0 \cdot w_0(\lambda)} \overline{M}(0)^\vee = \overline{M}(w_0(\lambda))^\vee$ by [Hum08, Thm. 7.6] and the proof of [BG99, Prop. 2.8]. $\qquad\square$

Taking global sections, the $U(\mathfrak{g}_L)$-module $\overline{M}(w_0(\lambda))^\vee = \Gamma(\mathbf{X}, \mathcal{M}_{n,L}) = \mathcal{O}_{\mathbf{X}}(\mathbf{X}_{w_0})(\lambda)_L$ can be identified with $\mathcal{O}(\overline{\mathbf{N}}_L)$ after trivializing the line bundle $\mathcal{O}(\lambda)_L$ over $\mathbf{X}_{w_0}^\circ = \overline{\mathbf{N}} w_0 \overline{\mathbf{B}}/\mathbf{B}$. Let Φ^- be the set of roots of $\overline{N}$. We fix a decomposition $\overline{\mathbf{N}} \simeq \prod_{\alpha \in \Phi^-} \mathbf{G}_a$ into root subgroups with coordinates $X_\alpha, \alpha \in \Phi^-$ so that $\overline{\mathbf{N}} \simeq \mathrm{Spec}(\mathcal{O}_E[X_\alpha, \alpha \in \Phi^-])$.

Lemma 4.5. *Fix a trivialization of $\mathcal{O}(\lambda)$ on $\mathbf{X}_{w_0}^\circ$. The section $\widehat{\mathcal{M}_{n,L}}(\mathbf{X}_{w_0}^\circ)$ is equal to the Tate algebra $E\langle X_\alpha, \alpha \in \Phi^- \rangle$.*

Proof. Without the completion, $\mathcal{M}_{n,L}(\mathbf{X}_{w_0}^\circ)$ is equal to the space $\mathcal{C}^{\mathrm{pol}}(\overline{N}, L) = \mathcal{O}(\mathbf{N}_L)$. By Lemma 4.3, $\widehat{\mathcal{M}_{n,L}}(\mathbf{X}_{w_0}^\circ)$ is the completion of $\mathcal{C}^{\mathrm{pol}}(\overline{N}, L)$ for the p-adic topology on $\mathcal{D}_{n,L}^\lambda(\mathbf{X}_{w_0}) \simeq \mathcal{D}_{n,L}(\overline{\mathbf{N}}) = L[X_\alpha, \partial_\alpha, \alpha \in \Phi^-]$ [AW13, Lem. 6.4] is the Weyl algebra by the pinning $\overline{\mathbf{N}} \simeq \mathbf{A}^{|\Phi^-|}$. By definition [AW13, §3.5, §5.7] and by [AW13, Prop. 5.7], $\widehat{\mathcal{D}_{n,L}}(\overline{\mathbf{N}})$ is the completion of $\mathcal{D}_{n,L}(\overline{\mathbf{N}})$ for the p-adic norm

$$\left| \sum_{i=(i_\alpha), j=(j_\alpha) \in \mathbb{N}^{\Phi^-}} \lambda_{i,j} \prod_\alpha X_\alpha^{i_\alpha} \prod_\alpha \partial_\alpha^{j_\alpha} \right| = \sup_{i,j} |\lambda_{i,j}| \|\varpi_L\|_p^{-n \sum_\alpha j_\alpha}.$$

Since $\mathcal{M}_{n,L}(\overline{\mathbf{N}}) \simeq \mathcal{D}_{n,L}(\overline{\mathbf{N}})/(\partial_\alpha, \alpha \in \Phi^-)$ as left $\mathcal{D}_{n,L}(\overline{\mathbf{N}})$-modules (up to a twist), we conclude that

$$\widehat{\mathcal{M}_{n,L}}(\overline{\mathbf{N}}) \simeq \widehat{\mathcal{D}_{n,L}}(\overline{\mathbf{N}})/(\partial_\alpha, \alpha \in \Phi^-) = E\langle X_\alpha, \alpha \in \Phi^- \rangle$$

is the Tate algebra $\qquad\square$

On the other hand, let $\chi_\lambda : T \to L^\times$ be an L-analytic character with the weight λ. Consider the locally analytic principal series

$$\mathrm{Ind}_{\overline{B}}^G \chi_\lambda = \{f : G \to L \text{ locally analytic } \mid f(gb) = \chi_\lambda^{-1}(b) f(g), \forall b \in \overline{B}\}$$

with the usual left action of G given by $g.f(-) = f(g^{-1}-)$. Fix an isomorphism $N \simeq N\overline{B}/\overline{B}$. Consider the subspace $\mathcal{C}^{\mathrm{pol}}(N, L) = \mathcal{O}(\mathbf{N})_L$ of polynomial functions on N, which is equal to the dual Verma module $M(\lambda)^\vee = (U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} \lambda)^\vee$ in $\mathcal{O}^{\mathfrak{b}_L}$ as in Remark 3.4. Via the isomorphism

$$\mathcal{C}^{\mathrm{pol}}(N\overline{B}/\overline{B}, L) \simeq \mathcal{C}^{\mathrm{pol}}(\overline{N} w_0 \overline{B}/\overline{B}, L) : f(-) \mapsto f(w_0 -),$$

we see again that the $U(\mathfrak{g}_L)$-module $\mathcal{C}^{\mathrm{pol}}(\overline{N} w_0 \overline{B}/\overline{B}, L)$ is isomorphic to $\overline{M}(w_0(\lambda))^\vee$ the twist by w_0 of $M(\lambda)^\vee$.

Lemma 4.6. *The two $U(\mathfrak{g}_L)$-module structures on $\mathcal{C}^{\mathrm{pol}}(\overline{N}, L) \simeq \overline{M}(w_0(\lambda))^\vee$ (via $\mathcal{C}^{\mathrm{pol}}(\overline{N}, L) w_0 \subset \mathrm{Ind}_{\overline{B}}^G \chi_\lambda$ and $\mathcal{C}^{\mathrm{pol}}(\overline{N}, L) \simeq \Gamma(\mathbf{X}, \mathcal{M}_{n,L}))$ coincide.*

Proof. The $U(\mathfrak{g})$-module structures on $\mathcal{O}_{\mathbf{X}_{w_0}}(\lambda)$ both come from the infinitesimal action of $G = \mathbf{G}(L)$ on $\mathcal{O}(\overline{\mathbf{N}} w_0 \overline{\mathbf{B}})^{\overline{\mathbf{B}} = \lambda}$ given by $(x.f)(g) = \frac{d}{dt} f(\exp(-tx)g)$ for $g \in \mathbf{G}, x \in \mathfrak{g}$ and f regular near a neighborhood of g. $\qquad\square$

4.3. **Completion near the boundary.** We will prove Proposition 4.11 by studying the completion $\widehat{\mathcal{M}_{n,L}}$ of the coadmissible $\mathcal{D}_{n,L}^{\lambda}$-module $\mathcal{M}_{n,L}$ in Lemma 4.4.

We take a finite covering

$$S = \{\mathbf{V}_w := w\overline{\mathbf{N}}w_0\overline{B}/\overline{B}, w \in W\}$$

of the flag variety $\mathbf{X}$ where W denotes the Weyl group of $\mathbf{G}$. Let $\partial\mathbf{X} = \mathbf{X} - \mathbf{X}_{w_0}^{\circ}$ be the boundary (Weil or Cartier) divisor which is also the union $\cup_s \mathbf{X}_{w_0 s}$ of codimension one Schubert varieties where s runs over all simple reflections [BB05, Prop. 2.3.2]. For all $\mathbf{V} \in S$, let $f_{\mathbf{V}} \in \mathcal{O}(\mathbf{V}_L)$ be an element cutting out the boundary $\mathbf{V}_L \cap \partial\mathbf{X}_L$ (note that the Picard group of $\mathbf{V}_L$ is trivial [Sta24, Tag 0BDA]).

Take one $\mathbf{V} \in S$. Let $A = \mathcal{O}(\mathbf{V}_L) = L[Y_1, \cdots, Y_l]$ where $l = |\Phi^-|$, $\widehat{A} = L\langle Y_1, \cdots, Y_d\rangle$, $\mathcal{D} := \mathcal{D}_{n,L}(\mathbf{V}) = A[\varpi_L^n \partial_1, \cdots, \varpi_L^n \partial_l]$ and $\widehat{\mathcal{D}}_{n,L} = \widehat{\mathcal{D}_{n,L}}(\mathbf{V})$. Write $|\cdot|_{\widehat{A}}$ for the norm

$$|\sum_{i=(i_1,\cdots,i_l)\in\mathbb{Z}^l} \lambda_i \prod_{j=1}^l Y_j^{i_j}|_{\widehat{A}} = \sup_i |\lambda_i|_p$$

and $f = f_{\mathbf{V}} \in A$.

The reason that $A[\frac{1}{f}]$ is a finitely generated $\mathcal{D}$-module lies in the theory of b-functions ([MNM91, Cor. 3.1.2]).

Theorem 4.7 ([MNM91, Thm. 3.1.1]). *For any $u \in A$, there exist non-zero polynomials $b(s) \in L[s]$ and $P(s) \in \mathcal{D}[s]$ such that*

$$b(s)f^{-s-1}u = P(s)f^{-s}u$$

after specializing $s \in \mathbb{Z}$.

The behavior of the $\mathcal{D}$-module $A[\frac{1}{f}]$ under completion is studied in [BB21]. We adapt the arguments in [BB21] to our case. We recall the following definition.

Definition 4.8 ([BB21, Def. 3.1]). Let $\lambda \in \overline{\mathbb{Q}}_p$. We say that λ is of positive type if $\lambda \in \mathbb{Z}_{\geq 0}$ or there exists $M > 0$ such that $\varprojlim_{i \to \infty} \frac{p^{iM}}{\prod_{j=0}^{i-1}(\lambda-j)} = 0$, equivalently ([BB21, Lem. B.2]) the type of λ which is, by [Ked22, Def. 13.1.1], the radius of convergence of the formal power series $\sum_{i\geq 0, i\neq \lambda} \frac{x^i}{\lambda - i}$, is positive.

Remark 4.9. Recall that a number $\lambda \in \overline{\mathbb{Q}}_p$ is p-adic non-Liouville if $\pm\lambda$ have type 1 [Ked22, Def. 13.1.2].

Lemma 4.10. *Let u be an element in A such that the coefficients of the polynomial $b(s)$ in Theorem 4.7 are in $\overline{\mathbb{Q}}[s]$. Then there exists $M > 0$ such that the series $\sum_{i\geq 0} a_i p^{Mi} \frac{1}{f^i} u$ for $a_i \in \widehat{A}$ converges in $\widehat{\mathcal{D}}_{n,L} \otimes_{\mathcal{D}} (A[\frac{1}{f}])$ provided that $\lim_{i\to\infty} |a_i|_{\widehat{A}} = 0$.*

Proof. Let $s_0 \geq 0$ such that $b(s)$ has no zero in $\mathbb{Z}_{\geq s_0}$, we get

$$f^{-s}u = \frac{P(s-1)\cdots P(s_0)}{b(s-1)\cdots b(s_0)} f^{-s_0}u$$

for $s > s_0$.

By the assumption, all the roots of $b(s)$ are of positive type by [DGS16, Prop. VI.1.1]. By [BB21, Lem. 4.1], there exists an integer M such that for all $i \geq 0$, $p^{Mi}\frac{P(i-1)\cdots P(s_0)}{\prod_{s=s_0}^{i-1} b(s)}$ lies in the unit ball of $\widehat{\mathcal{D}}_{n,L}$: the p-adic completion of $\mathcal{O}_L[Y_1, \cdots, Y_l, \varpi_L^n \partial_1, \cdots, \varpi_L^n \partial_l]$. We take one such M.

Let $g = \sum_{i \geq 0} a_i p^{Mi} \frac{1}{f^i} u \in \widehat{A}\langle \frac{p^M}{f} \rangle$ for $a_i \in \widehat{A}$ with $\lim_i |a_i|_{\widehat{A}} = 0$. Then g can also be written as $\sum_{i \geq s_0} a_i p^{Mi} \frac{P(i-1)\cdots P(s_0)}{\prod_{s=s_0}^{i-1} b(s)} f^{-s_0} u$ up to finitely many terms. Since $\varinjlim_{i \to \infty} a_i = 0$ and $p^{Mi} \frac{P(i-1)\cdots P(s_0)}{\prod_{s=s_0}^{i-1} b(s)}$ are bounded, the element $\sum_{i \geq s_0} a_i p^{Mi} \frac{P(i-1)\cdots P(s_0)}{\prod_{s=s_0}^{i-1} b(s)}$ converges in $\widehat{\mathcal{D}}_{n,L}$. Hence the series $g = \sum_{i \geq 0} a_i p^{Mi} \frac{1}{f^i} u$ converges in $\widehat{\mathcal{D}}_{n,L} \otimes_{\mathcal{D}} A[\frac{1}{f}]$. $\qquad\square$

Proposition 4.11. *Take $M = \overline{M}(w_0(\lambda))^\vee$ for $\lambda + \rho$ dominant with respect to $\mathfrak{b}$ and integral for* **H**. *Let $\widehat{\mathcal{M}_{n,L}} = \mathrm{Loc}^\lambda(\widehat{M_{n,L}})$. Then the restriction map $\widehat{\mathcal{M}_{n,L}}(\mathbf{X}) = \widehat{\mathcal{U}_{n,L}^\lambda} \otimes_{\mathcal{U}_{n,L}^\lambda} \overline{M}(w_0(\lambda))^\vee \to \widehat{\mathcal{M}_{n,L}}(\mathbf{X}_{w_0}^\circ)$ contains (in the notation of Lemma 4.5)*

$$\mathcal{O}(\overline{\mathbf{N}}^{\mathrm{an}}(\leq p^M)) := \{ \sum_{i=(i_\alpha)_\alpha \in \mathbb{N}^{\Phi^-}} \lambda_i p^{M(\sum_\alpha i_\alpha)} \prod_\alpha X_\alpha^{i_\alpha} \mid \lambda_i \in L, \varinjlim_{|i| \to \infty} |\lambda_i|_p = 0 \}$$

for some $M \geq 0$.

Proof. By Lemma 4.4, the localization of $\overline{M}(w_0(\lambda))^\vee$ is the pushforward of $\mathcal{O}_{\mathbf{X}}(\lambda)|_{X_{w_0}^\circ}$. Hence for $\mathbf{V} \in S$, $\mathcal{M}_{n,L}(\mathbf{V}) = \mathcal{O}_{\mathbf{X}}(\lambda)(\mathbf{V} - \partial \mathbf{X}) \otimes_{\mathcal{O}_L} L = \mathcal{O}_{\mathbf{X}_L}(\mathbf{V}_L)[\frac{1}{f_{\mathbf{V}}}](\lambda)$. Here $\mathcal{O}_{\mathbf{X}_L}(\mathbf{V}_L)[\frac{1}{f_{\mathbf{V}}}](\lambda)$ denotes the twist by $\mathcal{O}_{\mathbf{X}_L}(\lambda)$ for the action of $\mathcal{D}_{n,L}^\lambda$. Denote also by $X_\alpha \in \mathcal{M}_{n,L}(\mathbf{X}_{w_0}^\circ)$ the image of the coordinates of $\overline{\mathbf{N}} \simeq \overline{\mathbf{N}} w_0 \overline{\mathbf{B}}/\overline{\mathbf{B}}$ in $\mathcal{O}_{\mathbf{X}}(\mathbf{V}_L)[\frac{1}{f_{\mathbf{V}}}] = \mathcal{O}_{\mathbf{X}}(\mathbf{X}_{w_0}^\circ \cap \mathbf{V})[\frac{1}{p}]$.

We fix a trivialization of $\iota_{\mathbf{X}_{w_0}^\circ} : \mathcal{M}_{n,L}|_{\mathbf{X}_{w_0}^\circ} = \mathcal{O}_{\mathbf{X}}(\lambda)|_{\mathbf{X}_{w_0}^\circ}[\frac{1}{p}] \simeq \mathcal{O}_{\mathbf{X}}|_{\mathbf{X}_{w_0}^\circ}[\frac{1}{p}]$. Since $\widehat{\mathcal{M}_{n,L}}$ is a sheaf, to prove the proposition, we only need take M such that the sequences

$$\sum_i \lambda_i p^{M|i|} X^i := \sum_{i \in \mathbb{N}^{\Phi^-}} \lambda_i p^{M(\sum_\alpha i_\alpha)} \prod_\alpha X_\alpha^{i_\alpha}$$

converge in the completions $\widehat{\mathcal{M}_{n,L}}(\mathbf{V}) = \widehat{\mathcal{D}_{n,L}^\lambda}(\mathbf{V}) \otimes_{\mathcal{D}_{n,L}^\lambda(\mathbf{V})} (\mathcal{O}_{\mathbf{X}_L}(\mathbf{V}_L)[\frac{1}{f_{\mathbf{V}}}](\lambda))$ (Lemma 4.3) for all $\mathbf{V} \in S$ provided that $\varinjlim_i \lambda_i = 0$.

There are only finitely many $\mathbf{V} \in S$, thus we can focus on one $\mathbf{V}$. On $\mathbf{V}$ the line bundle $\mathcal{O}(\lambda)$ can be trivialized and there are isomorphisms $\widehat{\mathcal{D}_{n,L}^\lambda}(\mathbf{V}) \simeq \widehat{\mathcal{D}_{n,L}}(\mathbf{V})$ [AW13, Prop. 6.5]. Under this isomorphism, we can identify $\mathcal{M}_{n,L}(\mathbf{V})$ with the $\mathcal{D}_{n,L}(\mathbf{V})$-module $\mathcal{O}_{\mathbf{X}_L}(\mathbf{V}_L)[\frac{1}{f_{\mathbf{V}}}]$ via $\iota_{\mathbf{V}} : \mathcal{O}_{\mathbf{X}}(\lambda)|_{\mathbf{V}}[\frac{1}{p}] \simeq \mathcal{O}_{\mathbf{X}}|_{\mathbf{V}}[\frac{1}{p}]$, i.e., we can assume $\lambda = 0$ if we focus on the completion over $\mathbf{V}$. Write $f = f_{\mathbf{V}}$.

The trivializations $\iota_{\mathbf{X}_{w_0}^\circ}$ and $\iota_{\mathbf{V}}$ of $\mathcal{O}_{\mathbf{X}}(\lambda)|_{\mathbf{X}_{w_0}^\circ}$ and $\mathcal{O}_{\mathbf{X}}(\lambda)|_{\mathbf{V}}$ may not be compatible over $\mathbf{V} \cap \mathbf{X}_{w_0}^\circ$. The composite $\iota_{\mathbf{V}} \circ \iota_{\mathbf{X}_{w_0}^\circ}^{-1}|_{\mathbf{V} \cap \mathbf{X}_{w_0}^\circ} : \mathcal{O}_{\mathbf{X}}(\mathbf{V} - \partial \mathbf{X})[\frac{1}{p}] \simeq \mathcal{O}_{\mathbf{X}}(\mathbf{V} - \partial \mathbf{X})[\frac{1}{p}] = \mathcal{O}_{\mathbf{X}}(\mathbf{V}_L)[\frac{1}{f}]$ is $\mathcal{O}_{\mathbf{X}}(\mathbf{V} - \partial \mathbf{X})$-linear and is determined by an element $\iota_{\mathbf{V}} \circ \iota_{\mathbf{X}_{w_0}^\circ}^{-1}|_{\mathbf{V} \cap \mathbf{X}_{w_0}^\circ}(1) = \frac{u}{f^{n_u}}$ in $\mathcal{O}_{\mathbf{X}}(\mathbf{X}_{w_0}^\circ \cap V)^\times$ for some $u \in \mathcal{O}_{\mathbf{X}}(\mathbf{V}), n_u \in \mathbb{Z}$. Nevertheless, we can always write in $\mathcal{M}_{n,L}(\mathbf{V}_L)[\frac{1}{f}]$ that $X_\alpha = \frac{g_\alpha}{f^{n_\alpha}} \frac{u}{f^{n_u}}$ for $\alpha \in \Phi^-$ and for some $g_\alpha \in A := \mathcal{O}_{\mathbf{X}_L}(\mathbf{V}_L)$ and $n_\alpha \geq 0$ under the trivialization $\iota_{\mathbf{V}}$. Then $X_\alpha^i = (\frac{g_\alpha}{f^{n_\alpha}})^i \frac{u}{f^{n_u}}$ in $A[\frac{1}{f}]$.

The split reductive group $\mathbf{G}$ with $\mathbf{B} \subset \mathbf{G}$, as well as the inclusion $\mathbf{V} \subset \mathbf{G}/\overline{\mathbf{B}}$ and the line bundle $\mathcal{O}_{\mathbf{X}}(\lambda)$ for λ integral, can be defined over $\mathbb{Q}$. Hence we can take u such that $b(s) \in \mathbb{Q}[s]$ satisfying Theorem 4.7 for our A, u, f.

Take M_0 such that $|g_\alpha|_{\widehat{A}} < p^{M_0}$ for all α. Let $M_1 \geq 0$ such that Lemma 4.10 holds for $f = f_{\mathbf{V}}, M = M_1$ and u. Then the series

$$\sum_{i \in \mathbb{N}^{\Phi^-}} \lambda_i p^{M|i|} X^i = \sum_{i \in \mathbb{N}^{\Phi^-}} \lambda_i p^{M|i|} \frac{\prod_\alpha g_\alpha^{i_\alpha}}{f^{n_u + \sum_\alpha i_\alpha n_\alpha}} u = \frac{1}{f^{n_u}} \sum_{j \geq 0} \left(\sum_{\sum_\alpha i_\alpha n_\alpha = j} \lambda_i p^{M|i| - M_1 j} \prod_\alpha g_\alpha^{i_\alpha} \right) \frac{p^{M_1 j}}{f^j} u$$

converges in $\widehat{\mathcal{D}_{n,L}}(\mathbf{V}) \otimes_{\mathcal{D}_{n,L}(\mathbf{V})} A[\frac{1}{f}]$ if $\varinjlim_i \lambda_i = 0$ and $M > M_0 + M_1(\sum_\alpha (1 + n_\alpha))$ by Lemma 4.10. $\qquad \square$

5. Duality of patching modules

Using solid formalism of locally analytic representations in [RJRC22, RJRC23], we will define the Bernstein-Zelevinsky duality functor $\mathbb{D}_{\mathrm{BZ}}(-)$ (Definition 5.4) and then discuss coherent sheaves arising from the patching functors (Definition 5.9).

Our notations follow largely [RJRC23]. We consider the p-adic local field E as a solid condensed ring. A solid E-space is roughly the condensed version of a complete locally convex topological E-space. Write $\mathrm{Mod}_{E_\blacksquare}$ for the (∞-)derived category of solid E-modules, associated to the analytic ring $E_\blacksquare := (E, \mathcal{O}_E)_\blacksquare$, and write $\mathrm{Mod}_{E_\blacksquare}^\heartsuit$ for its heart with respect to the natural t-structure.

If D is an associative solid E-algebra (we will only consider non-derived/static condensed rings), we write $\mathrm{Mod}_{E_\blacksquare}(D)$ for the stable ∞-category of left D-modules on $E_\blacksquare$-spaces. Let G be a split reductive p-adic Lie group as before. We will take for example $D = E_\blacksquare[G]$ the solid Iwasawa algebra or $\mathcal{D}(G, E)$ the locally analytic distribution algebra [RJRC23, §2.1]. Note that if $D = E_\blacksquare[G]$ or $\mathcal{D}(G, E)$, there is a canonical involution of D induced by the inverse map of G which induces an equivalence of the categories of left and right D-modules, denoted by ι. We write $\iota(-) \otimes_{D,\blacksquare}^L -$, or $- \otimes_{D,\blacksquare}^L -$, for the relative tensor product of $\mathrm{Mod}_{E_\blacksquare}(D)$, an analog of complete tensor products. There are condensed non-derived or derived Hom's for D-modules $\underline{\mathrm{Hom}}_D(-, -), R\underline{\mathrm{Hom}}_D(-, -) \in \mathrm{Mod}_{E_\blacksquare}$. If $M, N \in \mathrm{Mod}_{E_\blacksquare}(D)$ and S is a profinite set, then $R\underline{\mathrm{Hom}}_D(M, N)(S) = R\mathrm{Hom}_E(E[S], R\underline{\mathrm{Hom}}_D(M, N)) = R\mathrm{Hom}_D(E[S] \otimes_E M, N)$.

We will focus on the category $\mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E))$ of solid $\mathcal{D}(G, E)$-modules. There is a full subcategory $\mathrm{Rep}_{E_\blacksquare}^{\mathrm{la}}(G)$ of (derived) solid locally analytic representations of G, which is the derived category of its heart $\mathrm{Rep}_{E_\blacksquare}^{\mathrm{la}, \heartsuit}(G)$ [RJRC23, Prop. 3.2.6].

5.1. Cohomological duality.
Let $I \subset G$ be the Iwahori subgroup as in §3. Suppose that $\mathcal{W} \in \mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(I, E))$, we define the compact induction

$$\text{c-Ind}_I^G \mathcal{W} := E_\blacksquare[G] \otimes_{E_\blacksquare[I]}^L \mathcal{W} = \mathcal{D}(G, E) \otimes_{\mathcal{D}(I,E)}^L \mathcal{W}.$$

If $\mathcal{W} \in \mathrm{Mod}_{E_\blacksquare}^\heartsuit(\mathcal{D}(I, E))$, then $\text{c-Ind}_I^G \mathcal{W} = \mathcal{D}(G, E) \otimes_{\mathcal{D}(I,E)} \mathcal{W}$ is in $\mathrm{Mod}_{E_\blacksquare}^\heartsuit(\mathcal{D}(G, E))$ (since $\mathcal{D}(G, E)$ is free over $\mathcal{D}(I, E)$ and $\mathrm{Mod}_{E_\blacksquare}^\heartsuit$ is stable under direct sums which are exact).

Remark 5.1. If $\mathcal{W} = \underline{\mathcal{W}(*)}$ (in the notation of [SC19, Prop. 1.7]) arises as a classical LS space of compact type, then $(\text{c-Ind}_I^G \mathcal{W})(*) = \mathrm{Hom}_E(E, \text{c-Ind}_I^G \mathcal{W}) = \text{c-Ind}_I^G \mathcal{W}(*)$ since E is compact in $\mathrm{Mod}_{E_\blacksquare}$. One may equip $(\text{c-Ind}_I^G \mathcal{W})(*)$ with certain topology as in *loc. cit.* However,

this topology may not coincide with the locally convex direct sum topology on c-Ind$_I^G\mathcal{W}(*)$. If $V = \varinjlim_i V_i$ is a locally convex inductive limit of Banach spaces with compact injective transition maps, then $\underline{V} = \varinjlim_i \underline{V_i}$ [CGN23, Lem. 2.19], which is not true in general.

Suppose that $\mathcal{W} = \underline{\mathcal{W}(*)}$ for $\mathcal{W}(*) = \mathcal{W}_?(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})$ with c-Ind$_I^G\mathcal{W}(*) \to \pi(*) = \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ as in Theorem 3.15 (notation in §3.3) where $\pi = \underline{\mathcal{F}_B^G(M, \chi_{\mathrm{sm}})}$. Then $\mathcal{W}(*)$ is a classical $\mathcal{D}(I, E)$-module whose underlying topological space is a Fréchet space or a Smith space with elements in $\mathcal{H}$ acting continuously on c-Ind$_I^G\mathcal{W}(*)$. If $\mathcal{W}$ is a Banach space, c-Ind$_I^G\mathcal{W}(*)$ with the locally convex inductive limit topology is an LF space. The augmented complex

$$\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G\mathcal{W}(*) \to \pi(*)$$

is an exact complex of LF-spaces where all morphisms are strict by the open mapping theorem [Eme17, Thm. 1.1.17].

Proposition 5.2. *Let $\mathcal{W}, \pi$ be as above, then the map $\mathcal{W} \to \pi$ induces an acyclic augmented complex*

$$\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G\mathcal{W} \to \pi$$

in $\mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E))$.

Proof. The map $\mathcal{W}(*) \to \pi(*)$ induces $\mathcal{W} \to \pi$ in $\mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E))$ and $\psi_t : \mathcal{W}(*) \to \mathcal{W}(*)$ induces $\psi_t : \mathcal{W} \to \mathcal{W}$ via the fully faithful embedding of [SC19, Prop. 1.7] (note that Fréchet and Smith spaces are compactly generated). The general construction in §2.1 applies formally to give the desired complex. We remain to show the complex is exact.

We show that $\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} \text{c-Ind}_I^G\mathcal{W} \simeq \text{c-Ind}_I^G\mathcal{W} \otimes_\mathcal{H} \mathcal{H}/\mathfrak{m}$. We check that for any extremally disconnected set S, the complex $\wedge^\bullet \mathcal{H}^d \otimes_\mathcal{H} (\text{c-Ind}_I^G\mathcal{W})(S)$ concentrates in degree 0. While $(\text{c-Ind}_I^G\mathcal{W})(S) = \mathrm{Hom}_E(E_\blacksquare[S], \text{c-Ind}_I^G\mathcal{W}) = \text{c-Ind}_I^G\mathrm{Hom}_E(E_\blacksquare[S], \mathcal{W})$ since $E_\blacksquare[S]$ is compact. Hence the complex of S-points concentrates in degree 0 by Proposition 2.2 replacing $\mathcal{W}$ by $\mathrm{Hom}_E(E_\blacksquare[S], \mathcal{W})$.

We assume first that $\mathcal{W}$ is a (condensed) Banach space so that c-Ind$_I^G\mathcal{W}$ is an LF space.

We prove the surjectivity of c-Ind$_I^G\mathcal{W} \to \pi$. We write K for the kernel of c-Ind$_I^G\mathcal{W} \to \pi$, and write c-Ind$_I^G\mathcal{W} = \varinjlim_n F_n$ where F_n are Fréchet spaces and $K_n = F_n \cap K$, the kernel of $F_n \to \pi$. Since π is LF (π is LS of compact type [RJRC22, Cor. 3.38]), the map $F_n \to \pi$ factors through some Fréchet subspace of π [RJRC22, Lem. 3.32]. By the fully faithfulness of [RJRC22, Prop. 3.7], the map $F_n \to \pi$ in $\mathrm{Hom}_E(F_n, \pi)$ arises from some $F_n(*) \to \pi(*)$. The kernel is a Fréchet space denoted by $K_n(*)$ closed in $F_n(*)$. We let $\pi_n = F_n/K_n$. Then $(F_n/K_n)(*) = F_n(*)/K_n(*)$ as Fréchet spaces by [Bos21, Lem. A.33]. By the surjectivity of c-Ind$_I^G\mathcal{W}(*) \to \pi(*)$, we know that $\pi(*) = \varinjlim_n \pi_n(*)$ as sets. Equip $\varinjlim_n \pi_n(*)$ with the locally convex inductive limit topology, then the map $\varinjlim_n \pi_n(*) \to \pi(*)$ is a topological isomorphism if $\pi(*)$ is equipped with the classical locally convex topology as an LF space by the open mapping theorem. Since filtered colimits are exact [SC19, Thm. 1.10], it remains to show that $\pi = \varinjlim_n \pi_n$. Write $\pi = \varinjlim_n B_n$ as a colimit of Fréchet spaces with injective transition maps. Then $\pi(*) = \varinjlim_n B_n(*)$ if we equip $\pi(*)$ with the locally convex inductive limit topology. As $\mathrm{Hom}_E(\pi(*), \pi(*)) = \varprojlim_n \varinjlim_m \mathrm{Hom}_E(\pi_n(*), B_m(*))$ by [Sch13b, Cor. 8.9], we see the two direct systems $(\pi_n(*))_n, (B_n(*))_n$ are equivalent. Then the same holds for $(\pi_n)_n, (B_m)_m$ by the fully faithful embedding of Fréchet spaces to solid E-spaces. Hence $\pi = \varinjlim_n B_n = \varinjlim_n \pi_n$.

We need also show that the map $\mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W} \to \text{c-Ind}_I^G \mathcal{W}$ maps surjectively on the kernel $K = \varinjlim_n K_n$ of $\text{c-Ind}_I^G \mathcal{W} \to \pi$. This can be proved in the same way as in the preceding paragraph using the surjectivity of $\mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}(*) \to K(*)$.

Finally, we suppose that $\mathcal{W}$ is a Smith space, for example, $\mathcal{W}$ has the form $\mathcal{W}_{\natural,r}(\overline{M})$. Then there exist Banach spaces W, W' with injections $\mathcal{W}'' \subset \mathcal{W} \subset \mathcal{W}'$ such that the result holds for $\mathcal{W}', \mathcal{W}''$. Using Proposition 2.2 and Corollary 2.3, $\wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}$ concentrates in degree 0 and there is an injection $\mathcal{H}/\mathfrak{m} \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W} \hookrightarrow \mathcal{H}/\mathfrak{m} \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}'$. Since the surjection $\text{c-Ind}_I^G \mathcal{W}'' \twoheadrightarrow \pi$ factors through the injection $\mathcal{H}/\mathfrak{m} \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W} \hookrightarrow \pi \simeq \mathcal{H}/\mathfrak{m} \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}'$, the map $\text{c-Ind}_I^G \mathcal{W} \to \pi$ is also surjective. $\qquad\square$

We consider the $\mathcal{D}(G,E)$-bimodule $\mathcal{C}_c^{\text{la}}(G,E) = \mathcal{C}^{\text{la}}(I,E) \otimes_{\mathcal{D}(I,E)} \mathcal{D}(G,E)$ as in §3.5.

Theorem 5.3. *Suppose that $\mathcal{W} \in \text{Mod}_{E_\blacksquare}^\heartsuit(\mathcal{D}(I,E))$ is a $\mathcal{D}(I,E)$-module over a Smith E-space. Then there is an isomorphism of solid $\mathcal{D}(G,E)$-modules*

$$R\underline{\text{Hom}}_{E_\blacksquare[G]}(\text{c-Ind}_I^G\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) \simeq \text{c-Ind}_I^G\mathcal{W}^\vee$$

where $\mathcal{W}^\vee = \underline{\text{Hom}}_E(\mathcal{W}, E)$ with the usual dual action of I.

Proof. Since $\mathcal{C}_c^{\text{la}}(G,E)$ is nuclear being LB ([Bos21, Thm. A.43]) and $\mathcal{W}$ is Smith, we have $R\underline{\text{Hom}}_E(\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) = \underline{\text{Hom}}_E(\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) = \mathcal{W}^\vee \otimes_{E,\blacksquare}^L \mathcal{C}_c^{\text{la}}(G,E)$ (see [RJRC22, Lem. 3.8 & Cor. 3.17]). We have (use [RJRC23, Lem. 3.1.7])

$$\begin{aligned}
R\underline{\text{Hom}}_{E_\blacksquare[G]}(\text{c-Ind}_I^G\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) &= R\underline{\text{Hom}}_{E_\blacksquare[I]}(\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) \\
&= R\underline{\text{Hom}}_{E_\blacksquare[I]}(E, R\underline{\text{Hom}}_E(\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E))) \\
&= R\underline{\text{Hom}}_{E_\blacksquare[I]}(E, \mathcal{W}^\vee \otimes_{E,\blacksquare}^L \mathcal{C}_c^{\text{la}}(G,E))) \\
&= (\iota(\mathcal{W}^\vee) \otimes \chi) \otimes_{E_\blacksquare[I]}^L \mathcal{C}_c^{\text{la}}(G,E)[-\dim G]
\end{aligned}$$

by [RJRC23, Prop. 3.1.12] where $\chi = \det(\mathfrak{g})^{-1}$ is a right G-module as in *loc. cit.* Then one sees that

$$R\underline{\text{Hom}}_{E_\blacksquare[G]}(\text{c-Ind}_I^G\mathcal{W}, \mathcal{C}_c^{\text{la}}(G,E)) \simeq ((\iota(\mathcal{W}^\vee) \otimes \chi) \otimes_{E_\blacksquare[I]}^L \mathcal{C}^{\text{la}}(I,E)) \otimes_{E_\blacksquare[I]}^L E_\blacksquare[G][-\dim G].$$

By [RJRC22, Prop. 4.41], $\mathcal{W}$ is a $\mathcal{D}^h(I,E)$-module (in the notation of [RJRC23, §3.1]) for h large enough and is locally analytic. By the equivalence in [RJRC23, Thm. 4.1.7] (we use also the notation in *loc. cit.* below),

$$\begin{aligned}
(\iota(\mathcal{W}^\vee) \otimes \chi) \otimes_{E_\blacksquare[I]}^L \mathcal{C}^{\text{la}}(I,E)[-\dim G] &\simeq R\underline{\text{Hom}}_{E_\blacksquare[I]}(\mathcal{W}, \mathcal{C}^{\text{la}}(I,E)) \\
&= R\underline{\text{Hom}}_{\text{Mod}_{E_\blacksquare}^{\text{qc}}(\mathcal{D}(I,E))}(j^*\mathcal{W}, j^*\mathcal{C}^{\text{la}}(I,E)).
\end{aligned}$$

since both $\mathcal{W}$ and $\mathcal{C}^{\text{la}}(I,E)$ are (derived) locally analytic. Note that $j^*\mathcal{C}^{\text{la}}(I,E) = (\mathcal{D}^h(I,E) \otimes \chi^{-1}[\dim G])_h$ (here χ^{-1} is viewed as a bimodule over $\mathcal{D}(G,E)$ where G acts trivially on the right and via χ^{-1} on the left) and $j^*\mathcal{W} = (\mathcal{W})_{h \text{ large}}$ by [RJRC23, Exam. 4.1.9]. Use the trick as

before we get (see also [RJRC23, Prop. 4.1.13])

$$R\underline{\mathrm{Hom}}_{\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(\mathcal{D}(I,E))}(j^*\mathcal{W}, j^*\mathcal{C}^{\mathrm{la}}(I,E))$$

$$=R\varprojlim_h R\underline{\mathrm{Hom}}_{E_\blacksquare[I]}(\mathcal{W}, \mathcal{D}^h(I,E) \otimes \chi^{-1}[\dim G])$$

$$=R\varprojlim_h (\iota(\mathcal{W}^\vee) \otimes \chi) \otimes^L_{E_\blacksquare[I]} \mathcal{D}^h(I,E) \otimes \chi^{-1}[\dim G]$$

$$=\iota(\mathcal{W}^\vee)[\dim G].$$

where we used that $\mathcal{D}^h(I,E)$ are idempotent over $E_\blacksquare[I]$ [RJRC22, Cor. 5.11] and that $\mathcal{W}^\vee$ is a $\mathcal{D}^h(I,E)$-module for h large. The result follows by returning the right module to a left module via the involution ι. $\qquad\square$

Definition 5.4. For $V \in \mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G,E))$, we define

$$\mathbb{D}_{\mathrm{BZ}}(V) := R\underline{\mathrm{Hom}}_{E_\blacksquare[G]}(V, \mathcal{C}^{\mathrm{la}}_c(G,E))$$

which is in $\mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G,E))$ (a priori not in $\mathrm{Rep}^{\mathrm{la}}_{E_\blacksquare}(G)$!) via the left translations of $\mathcal{C}^{\mathrm{la}}_c(G,E)$ by G.

Theorem 5.5. *Let $M \in \mathcal{O}^{\flat}_{\mathrm{alg}}$ and χ_{sm} be a smooth character of T. Then there exists an isomorphism of locally analytic representations*

$$\mathbb{D}_{\mathrm{BZ}}(\mathcal{F}^G_B(M, \chi_{\mathrm{sm}})) \simeq \mathcal{F}^G_{\overline{B}}(\mathrm{Hom}_E(M,E)^{\overline{\mathfrak{n}}^\infty}, \mathbb{D}_{\mathrm{BZ}}(\chi_{\mathrm{sm}})).$$

Proof. The arguments for Theorem 3.21 together with Proposition 5.2 and Proposition 5.3 give the duality. $\qquad\square$

5.2. **Stein spaces.** For the discussions on patching functors later, we make some preparation about coherent sheaves on Stein spaces, following [RJRC23].

We let $s \geq 1$ and $S = \mathcal{O}_E[[X_1, \cdots, X_s]] = \mathcal{O}_E[[\mathbb{Z}_p^s]]$. Let $S \to R$ be a local morphism of complete Noetherian local rings. We choose a presentation of R over S: let $A = S \otimes_{\mathcal{O}_E,\blacksquare} B$ for some $B = \mathcal{O}_E[[Y_1, \cdots, Y_t]]$ and suppose that $R = A/I$ for an ideal $I \subset A$. For positive integers h, k, we let $S^+_h = \mathcal{O}_E\langle \frac{X_1^h}{p}, \cdots, \frac{X_s^h}{p}\rangle$, $B^+_k = \mathcal{O}_E\langle \frac{Y_1^k}{p}, \cdots, \frac{Y_t^k}{p}\rangle$, $S_h = S^+_h[\frac{1}{p}]$, and similarly $A^+_{h,k} := S^+_h \otimes_{\mathcal{O}_E,\blacksquare} B^+_k$. Let $R^+_{h,k} := A^+_{h,k}/I$. We write $A_h := A_{h,h}, R_h := R_{h,h}$ for short. Finally, define $S^{\mathrm{rig}} := \varprojlim_h S_h, R^{\mathrm{rig}} := \varprojlim_h R_h, A^{\mathrm{rig}} := \varprojlim_h A_h$. Then $A^{\mathrm{rig}} = S^{\mathrm{rig}} \otimes^L_{E,\blacksquare} B^{\mathrm{rig}}$ ([Bos21, Cor. A.65, Cor. A.67, Prop. A.68]).

The rigid generic fiber $\mathrm{Spf}(R)^{\mathrm{rig}}$ of the formal scheme $\mathrm{Spf}(R)$ over $\mathrm{Spf}(\mathcal{O}_E)$ admits a covering by affinoids $\mathrm{Spa}(R_h, R^+_h)$. From any complete Huber pair as (R_h, R^+_h) we obtain an analytic ring $(R_h, R^+_h)_\blacksquare$ by [And21, Thm. 3.28]. We write $\mathrm{Mod}_{(R_h, R^+_h)_\blacksquare}$ for the $(\infty$-)derived category of $(R_h, R^+_h)_\blacksquare$-modules and write $\mathrm{Mod}^{\heartsuit}_{(R_h, R^+_h)_\blacksquare}$ for its heart [SC19, Prop. 7.5]. For $h' > h$, by [And21, Prop. 3.34, Lem. 3.31] (cf. [RJRC23, Lem. 2.1.9]), the map $(R_{h'}, R^+_{h'})_\blacksquare \to (R_h, R^+_h)_\blacksquare$ of analytic rings factors through $(R_{h'}, R^+_{h'})_\blacksquare \to (R_h, \mathcal{O}_E)_\blacksquare = (R_h, \mathcal{O}_E + R^{00}_h)_\blacksquare \to (R_h, R^+_h)_\blacksquare$ where R^{00}_h denotes the subset of topologically nilpotent elements. Here $(R_h, \mathcal{O}_E)_\blacksquare$ denotes the analytic ring induced from $\mathcal{O}_{E,\blacksquare}$ [And21, Prop. 2.16], and $\mathrm{Mod}_{(R_h, \mathcal{O}_E)_\blacksquare} = \mathrm{Mod}_{E_\blacksquare}(R_h)$ is the category of condensed R_h-modules whose underlying condensed E-vector spaces are solid. The

category of quasi-coherent sheaves on $\mathrm{Spf}(R)^{\mathrm{rig}}$ is then equivalent to the limit (cf. [RJRC23, §2.1.2, §4.1])

$$\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}}) := \varprojlim_h \mathrm{Mod}_{(R_h, R_h^+)_\blacksquare} = \varprojlim_h \mathrm{Mod}_{E_\blacksquare}(R_h).$$

Thus a quasi-coherent sheaf on $\mathrm{Spf}(R)^{\mathrm{rig}}$ can be given by a sequence $(M_h)_h$ where M_h are solid R_h-modules together with $M_{h'} \otimes^L_{R_{h'},\blacksquare} R_h = M_h$ for $h' > h$. If $(M_h)_h, (N_h)_h \in \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}})$, then

$$R\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}})}((M_h)_h, (N_h)_h) = R\varprojlim_h R\mathrm{Hom}_{R_h}(M_h, N_h).$$

There is a "global section" functor

$$(5.1) \qquad j_* : \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}}) \to \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}}) :$$
$$(M_h)_h \mapsto R\varprojlim_h M_h.$$

Conversely, we have the "localization":

$$(5.2) \qquad j^* : \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}}) \to \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}}) :$$
$$M \mapsto (M \otimes^L_{R^{\mathrm{rig}},\blacksquare} R_h)_h.$$

Recall by [RJRC22, Prop. 5.10], A_h and A^{rig} are idempotent algebras over A, namely $A_h \otimes^L_{A,\blacksquare} A_h = A_h$ and $A^{\mathrm{rig}} \otimes^L_{A,\blacksquare} A^{\mathrm{rig}} = A^{\mathrm{rig}}$. Moreover, A^{rig} and A_h are flat over A [ST03, Prop. 4.7]. Since $R = A/I$, similar statements hold for R_h, R^{rig}. We see $R_h = A_h \otimes^L_A A/I$ and $R^{\mathrm{rig}} = A^{\mathrm{rig}} \otimes^L_A A/I$. Hence $R_h = A_h \otimes^L_{A^{\mathrm{rig}},\blacksquare} R^{\mathrm{rig}}$ and $R_h \otimes^L_{R,\blacksquare} R_h = (A_h \otimes^L_{A,\blacksquare} A_h) \otimes^L_{A,\blacksquare} A/I = R_h$. And $R_h \otimes^L_{R^{\mathrm{rig}},\blacksquare} R_h = R_h \otimes^L_{R^{\mathrm{rig}},\blacksquare} R^{\mathrm{rig}} \otimes^L_{R^{\mathrm{rig}},\blacksquare} R_h = R_h \otimes^L_{R,\blacksquare} R_h = R_h$.

Lemma 5.6. *The functor j_* in (5.1) is fully faithful with the left adjoint j^*.*

Proof. The statement is proved in [RJRC23, §4.1] if $R = A = \mathcal{O}_E[[\mathbb{Z}_p^d]]$ is the Iwasawa algebra of a compact p-adic Lie group. In this case $A^{\mathrm{rig}} = \mathcal{D}(\mathbb{Z}_p^d, E)$. The fully faithfulness and the adjunction property of j_* is [RJRC23, Cor. 4.1.8]. In general, $R = A/I$ for $A = \mathcal{O}_E[[\mathbb{Z}_p^d]], d = t + s$. There are functors $i_* : \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}}) \to \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(A^{\mathrm{rig}}), \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}}) \to \mathrm{Mod}_{E_\blacksquare}(A^{\mathrm{rig}})$ given by restrictions via the ring maps $A^{\mathrm{rig}} \to R^{\mathrm{rig}}$ and $A_h \to R_h$. The functors j_* and j^* defined above (we use the same notation for A and R) commute with i_*: since $R_h = A_h \otimes^L_{A^{\mathrm{rig}},\blacksquare} R^{\mathrm{rig}}$, we have $j^*M = (R_h \otimes^L_{R^{\mathrm{rig}},\blacksquare} M)_h = (A_h \otimes^L_{A^{\mathrm{rig}},\blacksquare} M)_h$ for $M \in \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}})$. Using the statement for A, for any $\mathcal{M} = (M_h)_h \in \mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(R^{\mathrm{rig}})$, the natural morphism $i_*j^*j_*\mathcal{M} = j^*j_*i_*\mathcal{M} \to i_*\mathcal{M}$ is an isomorphism. Notice that the functor i_* is conservative, namely a morphism $M \to N$ of R^{rig}-modules (or R_h-modules) is an isomorphism if and only if it is an isomorphism of A^{rig}-modules (this can be checked on the abelian level by taking cohomologies). Hence the counit maps $j^*j_*\mathcal{M} \to \mathcal{M}$ are also isomorphisms which implies the fully-faithfulness of the functor j_* for R. □

The ring maps $S \to A \to R$ induce morphisms

$$\mathrm{Spf}(R)^{\mathrm{rig}} \xrightarrow{\ i\ } \mathrm{Spf}(A)^{\mathrm{rig}} = \mathrm{Spf}(S)^{\mathrm{rig}} \times \mathrm{Spf}(B)^{\mathrm{rig}}$$

with maps f and g to $\mathrm{Spf}(S)^{\mathrm{rig}}$.

of rigid analytic spaces. These maps admit !-functors: $i^!, i_! = i_*, g^!, g_!$, etc., for the six functor formalism in [Cam24, §3] (building on [Man22]) of quasi-coherent modules using, for example, [Cam24, Prop. 3.3.6]. We will not essentially need the general machinery as we will treat only coherent sheaves later as in Lemma 5.8 below.

Lemma 5.7. *Suppose that $\mathcal{M} = (M_{h,k})_{h,k} \in \mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(A^{\mathrm{rig}})$ and $M = j_*\mathcal{M} \in \mathrm{Mod}_{E_\blacksquare}(A^{\mathrm{rig}})$. Then the natural map $M \to R\varprojlim_h (S_h \otimes_{S,\blacksquare}^L M)$ is an isomorphism and $g_*\mathcal{M}$ is the quasi-coherent sheaf attached to the S^{rig}-module M via the localization.*

Proof. By [RJRC23, Cor. 4.1.5], the inverse systems $(S_h \otimes_{S,\blacksquare}^L M)_h$ is equivalent to (note that $\underline{\mathrm{Hom}}_E(S_h, E)$ is Smith)

$$(R\underline{\mathrm{Hom}}_S(\underline{\mathrm{Hom}}_E(S_h, E)[-s], M))_h.$$

Taking inverse limit

$$
\begin{aligned}
R\varprojlim_h S_h \otimes_{S,\blacksquare}^L M &= R\varprojlim_h R\underline{\mathrm{Hom}}_S(\underline{\mathrm{Hom}}_E(S_h, E)[-s], M) \\
&= R\varprojlim_{h,h',k} R\underline{\mathrm{Hom}}_S(\underline{\mathrm{Hom}}_E(S_h, E)[-s], M_{h',k}) \\
&= R\varprojlim_{k,h',h} S_h \otimes_{S,\blacksquare}^L M_{h',k} \\
&= R\varprojlim_{k,h} M_{h,k} = M
\end{aligned}
$$

where we applied [RJRC23, Cor. 4.1.5] again for the inverse system over h to get the third equality. □

For an affinoid algebra like S_h, an object $M \in \mathrm{Mod}_{E_\blacksquare}(S_h)$ is said to be perfect if M is quasi-isomorphic to a finite complex of finite projective S_h-modules (cf. [And21, Prop. 5.12]).

Lemma 5.8. *Suppose that $M \in \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}})$ such that $M \otimes_{S^{\mathrm{rig}},\blacksquare}^L S_h$ is a perfect S_h-module for all h. Then $M = j_*\mathcal{M} \in \mathrm{Mod}_{E_\blacksquare}(R^{\mathrm{rig}})$ is in the essential image of j_* where $\mathcal{M} = j^*M$. Moreover, there exists an isomorphism $f_*\mathcal{M} = f_!\mathcal{M}$ in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S^{\mathrm{rig}})$.*

Proof. Fix h. We first prove that there exists k such that $M_{S_h} := M \otimes_{S^{\mathrm{rig}},\blacksquare}^L S_h$ is an $R_{h,k}$-module (extending the R-module structure). In other words, we find k such that the natural R-map $M_{S_h} \to R_{h,k} \otimes_{R,\blacksquare}^L M_{S_h} = A_{h,k} \otimes_{A,\blacksquare}^L M_{S_h}$ is an isomorphism. Since $A_{h,k} = S_h \otimes_{E,\blacksquare} B_k$ and $M_{S_h} = M_{S_h} \otimes_{S,\blacksquare}^L S_h$, we see $A_{h,k} \otimes_{A,\blacksquare}^L M_{S_h} = (S_h \otimes_{E,\blacksquare} B_k) \otimes_{S \otimes_{E,\blacksquare} B,\blacksquare}^L M_{S_h} = B_k \otimes_{B,\blacksquare}^L (S \otimes_{E,\blacksquare} B) \otimes_{S \otimes_{E,\blacksquare} B,\blacksquare}^L M_{S_h} = B_k \otimes_{B,\blacksquare}^L M_{S_h}$. Hence it is enough to show that there exists k such that the B-action on M_{S_h} extends to B_k (recall that $B_k \otimes_{B,\blacksquare}^L B_k = B_k$). By [RJRC22, Thm. 4.36], it suffices to show that the cohomology groups $H^n(M_{S_h})$ for all n are B_k-modules for some k.

Since M_{S_h} is a perfect S_h-module, there exist finitely many n such that $H^n(M_{S_h}) \neq 0$ and these cohomology groups are finite S_h-modules, in particular Banach spaces. The desired action of some B_k follows from [RJRC22, Prop. 4.41].

Now we prove $f_* \mathcal{M} = f_! \mathcal{M}$. Since i is a closed embedding, $i_! \mathcal{M} = i_* \mathcal{M}$. We may assume $R = A$. Let $\mathcal{M}_{S_h} := (M_{h,k} := M_{S_h} \otimes^L_{A,\blacksquare} A_{h,k})_k \in \varprojlim_k \mathrm{Mod}_{E_\blacksquare}(A_{h,k})$. Then $M_{h,k} = M_{S_h} \otimes^L_{A,\blacksquare} A_{h,k} \otimes^L_{A,\blacksquare} A_{h,k'} = M_{h,k'}$ for $k' \geq k$. By Lemma 5.7, $g_* \mathcal{M} = (M_{S_h})_h$. We calculate that for $N_h \in \mathrm{Mod}_{E_\blacksquare}(S_h)$,

$$
\begin{aligned}
R\mathrm{Hom}_{S_h}(M_{S_h}, N_h) &= R\mathrm{Hom}_S(M_{S_h} \otimes^L_{A,\blacksquare} A_{h,k}, N_h) \\
&= \varinjlim_{k'} R\mathrm{Hom}_A(M_{S_h}, R\underline{\mathrm{Hom}}_S(A_{h,k'}, N_h)) \\
&= \varinjlim_{k'} R\mathrm{Hom}_B(E, R\underline{\mathrm{Hom}}_{S_h}(M_{S_h}, R\underline{\mathrm{Hom}}_S(A_{h,k'}, N_h))) \\
&= R\mathrm{Hom}_B(E, R\underline{\mathrm{Hom}}_{S_h}(M_{S_h}, \varinjlim_{k'} R\underline{\mathrm{Hom}}_S(A_{h,k'}, N_h)))
\end{aligned}
$$

where for the last equality we used that E is a compact B-module (cf. [RJRC22, Thm. 5.7]) and M_{S_h} is a compact S_h-module being perfect ([And21, Lem. 5.46, Cor. 5.51.1]). The map $R\underline{\mathrm{Hom}}_S(A_{h,k'}, N_h) = R\underline{\mathrm{Hom}}_E(B_{k'}, N_h) \to R\underline{\mathrm{Hom}}_S(A_{h,k''}, N_h)$ for $k'' \geq k'$ factors through $R\underline{\mathrm{Hom}}_E(B_{k'}, E) \otimes^L_{E,\blacksquare} N_h$ by [RJRC23, Lem. 4.1.4]. Hence $\varinjlim_{k'} R\underline{\mathrm{Hom}}_S(A_{h,k'}, N_h) = \varinjlim_{k'} R\underline{\mathrm{Hom}}_E(B_{k'}, E) \otimes^L_{E,\blacksquare} N_h = \underline{\mathrm{Hom}}_E(B^{\mathrm{rig}}, E) \otimes^L_{E,\blacksquare} N_h$. And by [RJRC23, Exam. 4.1.9],
(5.3)
$$
(A_{h,k} \otimes^L_{A,\blacksquare} (\underline{\mathrm{Hom}}_E(B^{\mathrm{rig}}, E) \otimes^L_{E,\blacksquare} N_h))_k = (B_k \otimes^L_{B,\blacksquare} \underline{\mathrm{Hom}}_E(B^{\mathrm{rig}}, E) \otimes^L_{E,\blacksquare} N_h)_k = (B_k \otimes^L_{E,\blacksquare} N_h[t])_k.
$$

Hence

$$
\begin{aligned}
R\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(S^{\mathrm{rig}})}(g_* \mathcal{M}, (N_h)_h) &= \varprojlim_h R\mathrm{Hom}_S(M_{S_h}, N_h) \\
&= \varprojlim_h R\mathrm{Hom}_A(M_{S_h}, \varinjlim_{k'} R\underline{\mathrm{Hom}}_E(B_{k'}, E) \otimes^L_{E,\blacksquare} N_h) \\
&= \varprojlim_h R\mathrm{Hom}_A(M_{S_h}, \underline{\mathrm{Hom}}_E(B^{\mathrm{rig}}, E) \otimes^L_{E,\blacksquare} N_h) \\
&= \varprojlim_{h,k'} R\mathrm{Hom}_A(M_{S_h}, A_{h,k'} \otimes^L_{S,\blacksquare} N_h[t]) \\
&= R\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(A^{\mathrm{rig}})}(\mathcal{M}, g^!(N_h)_h) \\
&= R\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{qc}}_{E_\blacksquare}(S^{\mathrm{rig}})}(g_! \mathcal{M}, (N_h)_h)
\end{aligned}
$$

where for the fourth equality we used (5.3) and the fully faithfulness of $j_!$ [RJRC23, Thm. 4.1.7] since M_{S_h} and $\underline{\mathrm{Hom}}_E(B^{\mathrm{rig}}, E) \otimes^L_{E,\blacksquare} N_h$ are derived locally analytic for the action of $\mathbb{Z}_p^{t+s}$. We used that g is cohomologically smooth of relative dimension t for the fifth equality. We conclude that $g_* \mathcal{M} = g_! \mathcal{M}$ by the Yoneda lemma. $\square$

5.3. **Patching functors and patching modules.** The Taylor-Wiles-Kisin patching method has been indispensible for p-adic Langlands program and motivated Emerton-Gee-Hellmann's

categorical p-adic local Langlands conjecture [EGH23, §3]. The abstract formalism of the patching functor for $\mathrm{GL}_n(\mathcal{O}_L)$-representations was proposed in [EGS15, §6] and the patching of completed cohomologies or $\mathrm{GL}_n(L)$-representations was carried out in [CEG$^+$16]. The method was applied to locally analytic settings by Breuil-Hellmann-Schraen in [BHS19], etc. and more recently in [HHS24]. We will work with the abstract patching modules reviewed below. The Serre duality of patching modules in this subsection is well-known at least in modular settings (e.g., [Man21]). Our aim is to explain its direct relationship with the Bernstein-Zelevinsky duality for locally analytic representations.

We assume the existence of the following abstract patching data. We suppose that $G = \mathrm{GL}_d(L), d \geq 2$ with the standard Iwahori subgroup I.

Let $S_\infty \to R_\infty$ be a local morphism of complete Noetherian local rings over $\mathcal{O}_E$ with the residue fields $\mathcal{O}_E/\varpi_E$. Assume that there exists $s \geq 1$ such that $S_\infty = \mathcal{O}_E[[X_1, \cdots, X_s]] \simeq \mathcal{O}_E[[J]]$ for $J = \mathbb{Z}_p^s$. Then the Iwasawa algebra $S_\infty[[I]] \simeq \mathcal{O}_L[[\widetilde{I}]]$ where $\widetilde{I} = I \times J = I \times \mathbb{Z}_p^s$. Similarly, write $\widetilde{G} = G \times J$.

Suppose that there is a (big patching) module M_∞ over the ring $R_\infty[\mathrm{GL}_n(L)]$ such that there exists an isomorphism $M_\infty|_H \simeq S_\infty[[H]]^a$ as topological H-modules for an open normal pro-p subgroup $H \subset I$. Hence M_∞ is finite projective over $S_\infty[[I]]$.

We assume the existence of the Poincaré dual of M_∞ consisting of the following data. Suppose that there are isomorphisms $\eta : S_\infty \to S'_\infty, R_\infty \to R'_\infty$ of local $\mathcal{O}_E$-algebras and M'_∞ is a big patching module over R'_∞. We assume that there exists an $R_\infty \times \mathrm{GL}_n(L)$-equivalent S_∞-linear isomorphism

$$\mathrm{Hom}_{S_\infty[[I]]}(M_\infty, S_\infty[[I]]) \simeq M'_\infty$$

where R_∞ acts on M'_∞ via η. The existence of such isomorphism is provided in [Din24, Cor. D.9] which is a patched version of the Poincaré duality of completed cohomologies ([CE12, §1.3]). The map η in [Din24] is essentially induced by $\rho \mapsto \rho^\vee \otimes \epsilon^{1-d}$ of Galois representations where $(-)^\vee$ means the dual representation and ϵ denotes the cyclotomic character (see [Zhu20, (3.9)] in terms of the Cartan involution of the C-group).

With the big patching modules, we can define patching functors for locally analytic representations. Let $M_\infty^{\mathrm{rig}} = \mathcal{D}(\widetilde{I}, E) \otimes_{S_\infty[[I]], \blacksquare} M_\infty$ and define similarly $M_\infty^{',\mathrm{rig}}$. Using that M_∞ is finite projective over $S_\infty[[I]]$, we obtain an R_∞-linear $\mathcal{D}(\widetilde{I}, E)$-isomorphism

$$R\underline{\mathrm{Hom}}_{\mathcal{D}(\widetilde{G}, E)}(M_\infty^{\mathrm{rig}}, \overline{\mathcal{D}}(\widetilde{G}, E)) = R\underline{\mathrm{Hom}}_{\mathcal{D}(\widetilde{I}, E)}(M_\infty^{\mathrm{rig}}, \mathcal{D}(\widetilde{I}, E)) \simeq M_\infty^{',\mathrm{rig}}$$

where $\overline{\mathcal{D}}(\widetilde{G}, E) = R\underline{\mathrm{Hom}}_{E_\blacksquare[\widetilde{G}]}(\mathcal{C}_c^{\mathrm{la}}(\widetilde{G}, E), E)$ with two left $\mathcal{D}(\widetilde{G}, E)$-module structures given by left and right multiplications (cf. [ST05] or [RJRC23, §4.2]).

Set $R_\infty^{\mathrm{rig}} = \varprojlim_h R_h, S_\infty^{\mathrm{rig}} = \varprojlim_h S_h$ as projective limits of affinoid E-algebras as in §5.2, which are Fréchet-Stein algebras. Recall we have a localization functor j^* (5.2) for solid R_∞^{rig}-modules. Let $\mathcal{M}_\infty^{\mathrm{rig}} := j^* M_\infty^{\mathrm{rig}} \in \mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$ and similarly $\mathcal{M}_\infty^{',\mathrm{rig}}$.

Definition 5.9. We define the patching functor $\mathfrak{A}_\infty^{\mathrm{rig}} : \mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E)) \to \mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$ (following the notation of [EGH23]) by

$$\mathfrak{A}_\infty^{\mathrm{rig}}(\pi) := j^* M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(G, E), \blacksquare}^L \pi = j^*(M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(G, E), \blacksquare}^L \pi)$$

for $\pi \in \mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E))$. Similarly $\mathfrak{A}_\infty^{',\mathrm{rig}}(\pi) := \mathcal{M}_\infty^{',\mathrm{rig}} \otimes_{\mathcal{D}(G, E), \blacksquare}^L \pi$.

For convenience, we also define the functor $A_\infty^{\mathrm{rig}} : \mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E)) \to \mathrm{Mod}_{E_\blacksquare}(R_\infty^{\mathrm{rig}})$ by

$$\pi \mapsto A_\infty^{\mathrm{rig}}(\pi) := M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(G,E),\blacksquare}^L \pi$$

so that $\mathfrak{A}_\infty^{\mathrm{rig}} = j^* \circ A_\infty^{\mathrm{rig}}$.

If $\mathcal{W}$ is a locally analytic I-representation concentrated in degree 0, then

$$A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) = M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(G,E),\blacksquare}^L \text{c-Ind}_I^G \mathcal{W} = M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E),\blacksquare} \mathcal{W}$$

since M_∞^{rig} is finite projective over $\mathcal{D}(I, E) \otimes_{E,\blacksquare} \mathcal{D}(J, E)$. We see $A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G(-))$ is exact on $\mathrm{Mod}_{E_\blacksquare}^\heartsuit(\mathcal{D}(I, E))$. Suppose that the Hecke algebra $\mathcal{H}$ acts on c-Ind$_I^G \mathcal{W}$ as in §2.1 and $\mathfrak{m}$ is the corresponding maximal ideal of $\mathcal{H}$, then $\mathcal{H}$ acts on c-Ind$_I^G \mathcal{W}$ as homomorphism of $\mathcal{D}(G, E)$-modules. Apply A_∞^{rig} to the complex c-Ind$_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m} := \wedge^\bullet \mathcal{H}^d \otimes_{\mathcal{H}} \text{c-Ind}_I^G \mathcal{W}$ in $\mathrm{Mod}_{E_\blacksquare}(\mathcal{D}(G, E))$, we see

$$A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}) \simeq A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}.$$

And the isomorphism holds after pulling back via j^*.

Lemma 5.10. *Let $\mathcal{W} \in \mathrm{Mod}_{E_\blacksquare}^\heartsuit(\mathcal{D}(I, E))$ be a solid locally analytic representation over a Smith E-space. There exist isomorphisms of solid S_∞^{rig}-modules*

$$M_\infty^{',\mathrm{rig}} \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee \simeq R\underline{\mathrm{Hom}}_{S_\infty^{\mathrm{rig}}}(M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}, S_\infty^{\mathrm{rig}}) = \underline{\mathrm{Hom}}_{S_\infty^{\mathrm{rig}}}(M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}, S_\infty^{\mathrm{rig}}).$$

Proof. We calculate that

$$M_\infty^{',\mathrm{rig}} \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee$$
$$\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}(\widetilde{I},E)}(M_\infty^{\mathrm{rig}}, \mathcal{D}(\widetilde{I}, E)) \otimes_{\mathcal{D}(\widetilde{I},E),\blacksquare}^L (\mathcal{D}(\widetilde{I}, E) \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee)$$
$$\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}(\widetilde{I},E)}(M_\infty^{\mathrm{rig}}, \mathcal{D}(\widetilde{I}, E) \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee)$$

where for the last equality we used that M_∞^{rig} is fintie projective over $\mathcal{D}(\widetilde{I}, E)$. While $\mathcal{D}(\widetilde{I}, E) \otimes_{\mathcal{D}(I,E),\blacksquare}^L$ $\mathcal{W}^\vee = (S_\infty^{\mathrm{rig}} \otimes_{E,\blacksquare}^L \mathcal{D}(I, E)) \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee = S_\infty^{\mathrm{rig}} \otimes_{E,\blacksquare}^L \mathcal{W}^\vee = S_\infty^{\mathrm{rig}} \otimes_{E,\blacksquare} \mathcal{W}^\vee$ since $\mathcal{W}^\vee$ is flat over E [Bos21, Cor. A.65]. As S_∞^{rig} is nuclear [RJRC22, Prop. 3.29] and $\mathcal{W}^\vee = R\underline{\mathrm{Hom}}_E(\mathcal{W}, E)$ for $\mathcal{W}$ Smith, we have $S_\infty^{\mathrm{rig}} \otimes_{E,\blacksquare}^L \mathcal{W}^\vee = R\underline{\mathrm{Hom}}_E(\mathcal{W}, S_\infty^{\mathrm{rig}})$ (cf. [Bos21, Prop. A.55]). Take the adjunction (cf. [RJRC23, Lem. 3.1.7])

$$R\underline{\mathrm{Hom}}_{\mathcal{D}(\widetilde{I},E)}(M_\infty^{\mathrm{rig}}, \mathcal{D}(\widetilde{I}, E) \otimes_{\mathcal{D}(I,E),\blacksquare}^L \mathcal{W}^\vee) = R\underline{\mathrm{Hom}}_{E_\blacksquare[\widetilde{I}]}(M_\infty^{\mathrm{rig}}, R\underline{\mathrm{Hom}}_E(\mathcal{W}, S_\infty^{\mathrm{rig}}))$$
$$= R\underline{\mathrm{Hom}}_{E_\blacksquare[J]}(E, R\underline{\mathrm{Hom}}_{E_\blacksquare[I]}(M_\infty^{\mathrm{rig}}, R\underline{\mathrm{Hom}}_E(\mathcal{W}, S_\infty^{\mathrm{rig}})))$$
$$= R\underline{\mathrm{Hom}}_{E_\blacksquare[J]}(E, R\underline{\mathrm{Hom}}_E(M_\infty^{\mathrm{rig}} \otimes_{E_\blacksquare[I]}^L \mathcal{W}, S_\infty^{\mathrm{rig}}))$$
$$= R\underline{\mathrm{Hom}}_{E_\blacksquare[J]}(M_\infty^{\mathrm{rig}} \otimes_{E_\blacksquare[I]}^L \mathcal{W}, S_\infty^{\mathrm{rig}}),$$

we get the result since $S_\infty^{\mathrm{rig}} = \mathcal{D}(J, E)$ is an idempotent algebra over $E_\blacksquare[J]$. $\square$

The following proposition is a standard result of the eigenvariety machinery.

Proposition 5.11. *Let $\mathcal{W} = \mathcal{W}_{\natural,r}(\tau(M)^\vee \otimes \chi_{\mathrm{sm}})$ as in Theorem 3.15. Then the solid S_∞^{rig}-modules $A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}$ is a perfect object in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})$.*

Proof. We show that $(S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W})) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}$ is a perfect S_h-complex for each h. We would like to write a proof for $\mathcal{W} = \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}$ for some n in Remark 3.16 and $\overline{M} = \tau(M)^\vee \otimes \chi_{\mathrm{sm}}$. Then $M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{W} = M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}$. This will not change $A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m})$.

Since M_∞^{rig} is finite projective over $\mathcal{D}(\widetilde{I}, E)$, we can write $M_\infty^{\mathrm{rig}} = e.\mathcal{D}(\widetilde{I}, E)^a \subset \mathcal{D}(\widetilde{I}, E)^a$ for some integer a and a $\mathcal{D}(\widetilde{I}, E)$-linear projection $e : \mathcal{D}(\widetilde{I}, E)^a \to \mathcal{D}(\widetilde{I}, E)^a$. Then $S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) = e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a$. For $t \in T^-$, $U_t \in \mathcal{H}$ induces $e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a \to e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a$. Since $\mathcal{W}$ is Smith, $e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a$ is a compact object in $\mathrm{Mod}_{E_\blacksquare}(S_h)$. We claim that if t is sufficiently regular $(t^{-1}(I_n \cap N)t \subset I_{n+1} \cap N)$, then U_t is a trace class map [SC20, Def. 13.11].

By the construction (Lemma 3.3 and Lemma 2.10), $\psi_t : \mathcal{W} \to \mathcal{W}$, induced from the multiplication of $\times t^{-1}$ on

$$\mathcal{D}^{I_n-\mathrm{an}}(I_n(I \cap \overline{B}), E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} = \mathcal{D}^{\mathrm{an}}(I_n, E) \otimes_{\mathcal{D}(\mathfrak{g}, I_n\cap\overline{B})} \overline{M}.$$

Hence ψ_t factors through the E-map

$$\iota : \mathcal{D}^{(t^{-1}I_n t\cap I)-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} \to \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}.$$

We show that ι is trace-class. By [RJRC22, Lem. 3.36], we need to show that the map factors through the Banach space $\mathcal{D}^{I_n-\mathrm{an}}(I, E)^B \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}$ attached to the target ([RJRC22, Def. 3.34]). If $\overline{M} = \mathcal{D}(\mathfrak{g}, I \cap \overline{B}) \otimes_{\mathcal{D}(I\cap B)} \sigma = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \sigma$ for a finite-dimensional representation σ of B, then $\mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} = \mathcal{D}^{(I_n\cap N)-\mathrm{an}}(I \cap N, E) \otimes_E \sigma$ and ι factors through $\mathcal{D}^{(I_n\cap N)-\mathrm{an}}(I \cap N, E)^B \otimes_E \sigma$ since $t^{-1}(I_n \cap N)t \subset I_{n+1} \cap N$ is relatively compact in $I_n \cap N$ (cf. [Eme06, Prop. 4.2.22]). In general, $\overline{M}$ admits a presentation as in Lemma 3.3

$$U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \sigma' \to U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} \sigma \to \overline{M}$$

for some σ, σ', which can be used to show that ι factors through $\mathcal{D}^{I_n-\mathrm{an}}(I, E)^B \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}$, the cokernel of $\mathcal{D}^{(I_n\cap N)-\mathrm{an}}(I \cap N, E)^B \otimes_E \sigma' \to \mathcal{D}^{(I_n\cap N)-\mathrm{an}}(I \cap N, E)^B \otimes_E \sigma$.

The base change to S_h of ι is still of trace-class (cf. [And21, Rem. 5.31]), as well as the following map (by checking the definition or using [Bos23, Lem. A.14])

(5.4)
$$(S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{D}^{(tI_n t^{-1}\cap I)-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} = e.(S_h \otimes_{E, \blacksquare} \mathcal{D}^{(tI_n t^{-1}\cap I)-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M})^a$$
$$\to (S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} = e.(S_h \otimes_{E, \blacksquare} \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M})^a.$$

Recall by definition (2.1), U_t is given by

(5.5)
$$M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{W} \to M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(G,E), \blacksquare} \text{c-Ind}_I^G \mathcal{W} = M_\infty^{\mathrm{rig}} \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{W}$$
$$m \otimes w \mapsto m \otimes \sum_{x \in I/(tIt^{-1}\cap I)} [xt, \psi_t(x^{-1}w)] = \sum_{x \in (I\cap\overline{N})/t(I\cap\overline{N})t^{-1}} t^{-1}x^{-1}.m \otimes \psi_t(x^{-1}w).$$

The formula for (5.5) defines a map

$$(S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{E, \blacksquare} \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M} \to (S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{E, \blacksquare} \mathcal{D}^{(tI_n t^{-1}\cap I)-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I\cap\overline{B})} \overline{M}.$$

The above map descends to the following map (cf. [Eme06, Lem. 4.2.11]).

$$(5.6) \qquad (S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{D}^{I_n-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I \cap \overline{B})} \overline{M}$$

$$\to (S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} M_\infty^{\mathrm{rig}}) \otimes_{\mathcal{D}(I,E), \blacksquare} \mathcal{D}^{(tI_n t^{-1} \cap I)-\mathrm{an}}(I, E) \otimes_{\mathcal{D}(\mathfrak{g}, I \cap \overline{B})} \overline{M}.$$

Hence $U_t : e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a \to e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a$ is trace-class being the composite of (5.6) and the trace class map (5.4) by [CS22, Lem. 8.2]. By discussions in [CS22, §9] and the proof of [CS22, Prop. 9.11] or [And21, Lem. 5.51], $\mathrm{cone}(1 - U_t) = S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) \otimes_{\mathcal{H}}^L \mathcal{H}/(U_t - 1)$ is a perfect S_h-complex. In a more classical language, the map U_t can be factored as $e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a \xrightarrow{g} V \xrightarrow{f} e.(S_h \otimes_{E, \blacksquare} \mathcal{W})^a$ for some Banach S_h-module V (that is a direct summand of an orthonormalizable Banach S_h-module). Then $\mathrm{cone}(1 - U_t) \simeq \mathrm{cone}(1 - g \circ f)$ with $g \circ f$ an S_h-compact operator on V and is perfect over S_h by the classical Fredholm theory (see [Eme06, Prop. 2.2.6]).

Finally, $\mathcal{H}/(U_t - 1)$ is a Noetherian regular ring (finite étale over a polynomial ring, of the form $E[U_{t_1}, \cdots, U_{t_d}]/(U_{t_1}^{s_1} \cdots U_{t_d}^{s_d} - 1)$), hence has finite global projective dimension [Sta24, Tag 00OE]. We get that $\mathcal{H}/\mathfrak{m}$ admits a finite projective resolution over $\mathcal{H}/(U_t - 1)$. Hence $S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}^\vee) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m} = (S_h \otimes_{S_\infty^{\mathrm{rig}}, \blacksquare} A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}) \otimes_{\mathcal{H}}^L \mathcal{H}/(U_t - 1)) \otimes_{\mathcal{H}/(U_t - 1)}^L \mathcal{H}/\mathfrak{m}$ is also a perfect S_h-complex ([Sta24, Tag 066R]). We finished the proof. $\qquad \square$

For $\mathcal{M} = (M_h)_h \in \mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$, we let

$$\mathbb{D}_{\mathrm{GS}}(\mathcal{M}) := R\underline{\mathrm{Hom}}_{\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})}(M, f^!(S_h)_h)$$

where $f : \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \to \mathrm{Spf}(S_\infty)^{\mathrm{rig}}$ is induced by $S_\infty \to R_\infty$ as in §5.2 and $(S_h)_h$ denotes the structure sheaf of $\mathrm{Spf}(S_\infty)^{\mathrm{rig}}$. The following theorem should be compared with [Zhu20, Conj. 4.5.1 (1)] and [EGH23, Conj. 6.1.14 (3) & Rem. 6.2.22].

Theorem 5.12. *Let $M \in \mathcal{O}_{\mathrm{alg}}^{\flat}$ and χ_{sm} be a smooth character of T. Let $\pi = \mathcal{F}_B^G(M, \chi_{\mathrm{sm}})$ as in Theorem 5.5. Then there exists an isomorphism*

$$\mathbb{D}_{\mathrm{GS}}(\mathfrak{A}_\infty^{\mathrm{rig}}(\pi)) \simeq \eta^* \mathfrak{A}_\infty^{',\mathrm{rig}}(\mathbb{D}_{\mathrm{BZ}}(\pi)).$$

in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$.

Proof. Write $\mathcal{F}_B^G(M, \chi_{\mathrm{sm}}) = \text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}$ as in Proposition 5.2 using Theorem 3.15. By Proposition 5.11, $A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m})$ is a perfect object in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}}) \subset \mathrm{Mod}_{E_\blacksquare}(S_\infty^{\mathrm{rig}})$. By Lemma 5.8, $A_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}) = j_* \mathfrak{A}_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m})$ and

$$f_! \mathfrak{A}_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}) = f_* \mathfrak{A}_\infty^{\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}).$$

One can check formally that the isomorphism in Lemma 5.10 is compatible with the actions of $\mathcal{H}$, where $\mathcal{H}$ acts on $A_\infty^{',\mathrm{rig}}(\text{c-Ind}_I^G \mathcal{W}^\vee)$ via the transpose in Lemma 3.18. By Theorem 5.3 and

the Koszul duality [Ser97, Chap. IV.A.2] (we omit the pullback η^* in the following)

$$R\underline{\mathrm{Hom}}_{S_\infty^{\mathrm{rig}}}(A_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W}) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}, S_\infty^{\mathrm{rig}})$$

$$\simeq R\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}/\mathfrak{m}, R\underline{\mathrm{Hom}}_{S_\infty^{\mathrm{rig}}}(A_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W}), S_\infty^{\mathrm{rig}}))$$

$$\simeq A_\infty^{'\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W}^\vee) \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}[-d]$$

$$= A_\infty^{'\mathrm{rig}}(\mathbb{D}_{\mathrm{BZ}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m})).$$

Applying the adjunctions, we get identifications of R_∞^{rig}-module objects in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})$:

$$f_* R\underline{\mathrm{Hom}}_{\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})}(\mathfrak{A}_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}), f^!(S_h)_h)$$

$$= R\underline{\mathrm{Hom}}_{\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})}(f_! \mathfrak{A}_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}), (S_h)_h)$$

$$= R\underline{\mathrm{Hom}}_{\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})}(f_* \mathfrak{A}_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}), (S_h)_h)$$

$$= j^* R\underline{\mathrm{Hom}}_{S_\infty^{\mathrm{rig}}}(A_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}), S_\infty^{\mathrm{rig}})$$

$$= f_* \mathfrak{A}_\infty^{'\mathrm{rig}}(\mathbb{D}_{\mathrm{BZ}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}))$$

where for the third equality, we used the fully faithfulness of j_* (for S_∞) and Lemma 5.7. The functor f_* sending $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$ to R_∞^{rig}-modules in $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})$ is fully faithful (composed with the global section functor for $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(S_\infty^{\mathrm{rig}})$ it gives that for $\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})$ in Lemma 5.6). We conclude that

$$R\underline{\mathrm{Hom}}_{\mathrm{Mod}_{E_\blacksquare}^{\mathrm{qc}}(R_\infty^{\mathrm{rig}})}(\mathfrak{A}_\infty^{\mathrm{rig}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}), f^!(S_h)_h) \simeq \mathfrak{A}_\infty^{'\mathrm{rig}}(\mathbb{D}_{\mathrm{BZ}}(\mathrm{c\text{-}Ind}_I^G\mathcal{W} \otimes_{\mathcal{H}}^L \mathcal{H}/\mathfrak{m}))$$

which gives the desired isomorphism. $\qquad\square$

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