

Companion points and partially classical eigenvarieties.

1. p-adic automorphic forms

- p : prime number.
- $n \geq 2, n \in \mathbb{Z}$
- \bar{F}/\mathbb{Q} quadratic imaginary. p splits in \bar{F} , choose $v|p$, $F_v \cong \mathbb{Q}_p$
[Rather, \bar{F}/\bar{F}^+ CM, \bar{F}^+ totally real + technical assumptions]
- G definite unitary group $/\mathbb{Q}$, splits over \bar{F} , i.e. $G \times_{\mathbb{Q}} \bar{F} = \text{Gal}(\bar{F}/F)$. (fix the isomorphism)
and $G(\mathbb{R})$ compact.
- Tame level: $U^p = \prod_{\ell \neq p} U_\ell \subset G(\mathbb{A}_{f,\ell}^p) = \prod'_{\ell \neq p} G(\mathbb{Q}_\ell)$ open compact subgroup, small enough
- $G = G(\mathbb{Q}_p) \cong \text{Gal}(F_v) = \text{Gal}(\bar{F}/\mathbb{Q}_p)$.
- L/\mathbb{Q}_p $[\mathbb{Z}/\mathbb{Z}_p] < +\infty$, large enough coefficient field.
 \mathcal{O}_L ring of integers
 $k_L = \mathcal{O}_L/\mathfrak{m}_L$ uniformizer
- p-adic automorphic forms:
 $\hat{S}(U^p, L) := \{ f: G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/U^p \rightarrow L, \text{continuous} \} \subset \text{Gal}(\bar{F}/\mathbb{Q}_p) = G$.
 Looks like $\mathcal{C}^{\text{cont}}(\text{Gal}(\mathbb{Z}_p), L)^{\otimes s}$.
- $\tilde{\Pi}^{\text{sp}}$: some algebra of Hecke operators / \mathcal{O}_L
 $\hat{S}(U^p, L)$ match Satake parameters and Frobenius eigenvalues.
- Hecke eigenvalues in $\hat{S}(U^p, L)$ \rightsquigarrow $\rho: \text{Gal}(\bar{F}/F) \rightarrow \text{Gal}(\bar{k}_L)$ (semisimple)
 $\rightsquigarrow m_p \in \text{Spec } \tilde{\Pi}^{\text{sp}}$
 Hecke eigenvalues in $\hat{S}(U^p, k_L)$ $\rightsquigarrow \bar{\rho}: \text{Gal}(\bar{F}/F) \rightarrow \text{Gal}(\bar{k}_L)$.
- Fix $\bar{\rho}: \text{Gal}(\bar{F}/F) \rightarrow \text{Gal}(\bar{k}_L)$ absolutely irreducible, modular.
 $\Pi (= \Pi(U^p)) := \hat{S}(U^p, L)_{m_p} \neq 0$.
- Expectation: $\Pi[\rho] := \Pi(U^p)[m_p]$ \rightsquigarrow $\rho_p := \rho|_{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$.
 comes from global.
- $\Pi[\rho]^{\text{an}}$ \hookrightarrow locally analytic representation (Schneider - Teitelbaum)
 $\Pi[\rho]^{\text{an}}$ subspace of locally analytic vectors (w.r.t. \mathcal{O})
- Example: $\mathcal{C}^{\text{cont}}(\text{Gal}(\mathbb{Z}_p), L)^{\text{an}} = \mathcal{C}^{\text{an}}(\text{Gal}(\mathbb{Z}_p), L)$

2. Companion points.

$$T = \begin{pmatrix} \mathbb{Q}_p^\times & 0 \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} = (\mathbb{Q}_p^\times)^n \subset G, \quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset G.$$

• Emerton's Jacquet module

$\pi^{\text{an}} \subset G$ Locally analytic representation. $\rightsquigarrow J_B(\pi^{\text{an}}) \subset T$ Locally analytic

• Eigenvariety

$$\Upsilon(U_p, \bar{p}) = \left\{ (\rho, \delta) \mid \begin{array}{l} p: \text{deformation of } \bar{p}, n\text{-dim'l } p\text{-adic cts rep of } \text{Gal}(\bar{F}/F) \\ \delta: T \rightarrow \mathbb{Q}_p^\times \text{ continuous character} \\ \text{s.t. } \text{Hom}_T(\delta, J_B(\pi|_{T(\bar{p})}^{\text{an}})) \neq 0 \end{array} \right\}$$

\downarrow

Locally finite

\downarrow

$\delta|_{(\mathbb{Z}_p^\times)^n}$

$$W = \{ \delta_0: (\mathbb{Z}_p^\times)^n \rightarrow \mathbb{Q}_p^\times, \text{continuous} \}$$

$$\delta(U_p) = \delta\left(\begin{smallmatrix} p^n & \\ & 1 \end{smallmatrix}\right)$$

• Companion forms

Fix $(\rho, \delta) \in \Upsilon(U_p, \bar{p})$, in general, $\exists \delta' \neq \delta$ s.t. $(\rho, \delta') \in \Upsilon(U_p, \bar{p})$

Conj (Borel, Hansen) The set $W_H(\rho) := \{ \delta' \mid (\rho, \delta') \in \Upsilon(U_p, \bar{p}) \}$ is "explicitly" determined by $P_p = P \cap \text{Gal}(\bar{F}/F_p)$

• Locally analytic socle conjecture.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, $z^\lambda: (z_1, \dots, z_n) \in (\mathbb{Q}_p^\times)^n \mapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}$

Assume $\delta = z^\lambda \delta_{sm}$ where δ_{sm} is a smooth character.

Borel: $\text{Hom}_T(\delta, J_B(\pi|_{T(\bar{p})}^{\text{an}})) = \text{Hom}_G(\mathcal{F}(M(\lambda), \delta_{sm}), \pi|_{T(\bar{p})}^{\text{an}})$

Here $M(\lambda) = U(g) \otimes_{U(b)} \lambda$: Verma module.

$g = \text{Lie } G$, $b = \text{Lie } B$, $U(-)$: universal enveloping algebra.

Orlik-Strickland: $\mathcal{F}(M(\lambda), \delta_{sm}) \subset G$. Locally analytic.

Jordan-Hölder factors.

$$JH(\mathcal{F}(M(\lambda), \delta_{sm})) \xrightarrow{\text{generically}} \left\{ \mathcal{F}(L(\lambda'), \delta_{sm}) \mid L(\lambda') \in JH(M(\lambda)) \right\}$$

↑
irreducible $U(g)$ -module of highest weight λ' .

Example: $n=3$ $\lambda = (1, 1, 2)$

$$M\left(\begin{smallmatrix} 1 & \\ & 2 \end{smallmatrix}\right) = \left[L\left(\begin{smallmatrix} 0 & \\ 3 & \end{smallmatrix}\right) - L\left(\begin{smallmatrix} 0 & \\ 2 & \end{smallmatrix}\right) - L\left(\begin{smallmatrix} 1 & \\ 2 & \end{smallmatrix}\right) \right]$$

Conj (Breuil) Assume $(p, s) \in Y(U^p, \bar{p})$, $s = z^\lambda s_{sm}$, p de Rham, "sm generic."
 then the set $W_B(p) = \{(x', s'_{sm}) \mid F(L(x'), s'_{sm}) \hookrightarrow \overline{\pi[p]}^{an}\}$ is explicitly
 determined by p
 "companion constituent"

Remark: Conjectures on companion forms / constituents should
 be viewed as "locally analytic version of the weight part of
 Serre's modularity conjecture".

Theorem (Regular case: Breuil-Hellmann-Schraen, non-regular: W.)

Assume the Taylor-Wiles assumptions (on \bar{p} , F/F^+ , g , U^p , $p \geq 2$)

Assume $(p, s = z^\lambda s_{sm}) \in Y(U^p, \bar{p})$. and p_p is generic crystalline

Then $L(p, z^\lambda s_{sm}) \in Y(U^p, s_{sm})$ iff $\text{Hom}_G(F(L(x')), s_{sm}), \overline{\pi[p]}^{an} \neq 0$
 iff $(x', s_{sm}) \in W_B(p)$ iff $z^\lambda s_{sm} \in W_H(p)$

Rank: ① p_p generic: $\{e_i\}_{i=1}^n$ eigenvalues,
 then $e_i e_j^{-1} \in \{1, p\}$ $i \neq j$

② regular: Hodge-Tate weights of p_p are pairwise distinct

③ regular: same result for s'_{sm} (BHS)
 non-regular: in progress (W.)

Idea of the proof:

Cycles/subset $L(\lambda) := \{(p, z^\lambda s_{sm}) \mid \text{Hom}_G(F(L(x)), s_{sm}), \overline{\pi[p]}^{an} \neq 0\}$ with s_{sm}, p varying

Need patching eigenvariety (BHS, Caraiani-Gerberon-Gee-Horngtay-Paskunas-Shin)
 ↓ component above p

trianguline variety = local Galois theoretic eigenvariety

local model
 ↓
 (BHS, W.)
 = {space of trianguline representations}
 not "partially de Rham"

exists cycles $L(\lambda) \cup L(\lambda')$
 ↓ $\#$ \nearrow \uparrow \nearrow
 "partially de Rham" \Leftarrow "partially classical"
 companion cycle.
 Ding's partial eigenvariety.

3. Partially classical constituents.

$n=3, \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$

① "Classical \Rightarrow de Rham" If $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $L(\lambda)$ finite-dim $\hookrightarrow \text{GL}_3(\mathbb{Q}_p)$.
 $\mathcal{F}(LL(\lambda), s_{sm}) = L(\lambda) \otimes \tilde{\pi}_{sm}^{\text{smooth}}$ "Locally algebraic representation".

Then $\text{Hom}_n(\mathcal{F}(LL(\lambda), s_{sm}), \tilde{\pi}[\mathbb{P}]^{\text{an}}) \neq 0 \Rightarrow p_p$ de Rham with regular HT weights.

② "finite slope \Rightarrow trianguline" If $\text{Hom}_n(\mathcal{F}, J_{\mathcal{B}}(\tilde{\pi}[\mathbb{P}]^{\text{an}})) \neq 0$,
then p_p trianguline, i.e. $\text{Drig}(p_p)^{\text{(\mathbb{Q}, \mathbb{P})-module / Robba ring}}$ admits a
filtration $0 \subset D_1 \subset D_2 \subset D_3 = \text{Drig}(p_p)$ s.t. $D_1, D_2/D_1, D_3/D_2$ rank one (\mathbb{Q}, \mathbb{P}) -modules
(Kisin, Colmez, Kedlaya-Pottharst-Xiao, R. Liu) (up to enlarge L)

(Galois representation attached to Hida's ordinary family are reducible at p)

Theorem (W. based on Y. Ding) "Partially classical \Rightarrow Partially de Rham"

Assume $\text{Hom}_n(\mathcal{F}(LL(\lambda), s_{sm}), \tilde{\pi}[\mathbb{P}]^{\text{an}}) \neq 0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$

$(\Rightarrow (p, z^\lambda s_{sm}) \in \mathcal{Y}(\mathbb{W}_p, \bar{p}))$, $0 \subset D \subset D_2 \subset D_3 = \text{Drig}(p_p)$ associated with s_{sm}

① If $\lambda_1 \geq \lambda_2$, then D_2 de Rham

② If $\lambda_2 \geq \lambda_3$, then D_3/D_2 de Rham.

Rmk: $p_p : V \hookrightarrow \text{Aut}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightsquigarrow \text{Drig}(p_p) \xrightarrow{\text{Berger}} W_{dR}(\text{Drig}(p_p)) = B_{dR} \otimes_{\mathbb{Q}_p} V \hookrightarrow \text{Aut}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$
 $D \xrightarrow{\text{Berger}} W_{dR}(D)$

D de Rham $\Leftrightarrow W_{dR}(D) = W_{dR}(D) \text{Aut}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}$.

Reason: $\lambda = (1, 1, 2)$ ($\oplus_2 \lambda_1 \geq \lambda_2$)

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset \mathfrak{g}, P = MN, M = \text{GL}_2(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$$

$$\mathfrak{p} = \text{Lie } P \quad m = \text{Lie } M \quad \text{parabolic Verma module.}$$

$LL(\lambda)$ quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_m(\lambda)$ (equal in this case)

$L_m(\lambda)$ f. dim rep of $U(\mathfrak{m})$ of highest weight $\lambda \in M$

$\Rightarrow (\mathfrak{g}^P \ni LL(\lambda) \subset M, U(\mathfrak{m})$).

Then $\text{Hom}_n(\mathcal{F}(LL(\lambda), s_{sm}), \tilde{\pi}[\mathbb{P}]^{\text{an}}) \neq 0 \xrightarrow{\text{Berger, Ding}}$

$$\text{Hom}_m(L_m(\lambda), J_p(\tilde{\pi}[\mathbb{P}]^{\text{an}})) [s_{sm}] \neq 0.$$

$J_p(-)$: parabolic version of Emerton's Jacquet module.

$$J_p(\pi[\rho]^{\text{an}}) \otimes M \Rightarrow \begin{matrix} 0 \subset D_2 \subset D_3 = D_{\text{rig}}(\rho) \\ \uparrow \text{rank 2} \end{matrix} \quad \text{"parabolic filtration"} \quad \text{Berndtsson}$$

non-zero locally algebraic $\Rightarrow D_2, D_3/D_2$ de Rham
rep of M

- Idea of the proof: $\lambda = (1, 1, 2)$.

Ding constructed a partial eigenvariety using $J_p(\pi^{\text{an}})$
(previously Hill-Loeffler)

$$(\rho, \mathbb{Z}^n \otimes_M) \in Y(U^\circ, \bar{\rho})' \hookrightarrow Y(U^\circ, \bar{\rho})$$

- $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, then $\lambda_1 = \lambda_2$.
- $\text{Hom}_M(L_m(\lambda), J_p(\pi[\rho]^{\text{an}})) \neq 0$
- Classical points / de Rham points dense in $Y(U^\circ, \bar{\rho})'$

Rmk: If $F_v \neq \mathbb{Q}_p$, $P_v = P|_{\text{Gal}(\bar{F}_v/F_v)}$, $\Sigma = \{g: F_v \hookrightarrow L\}$. L large enough.

$$D_{dR}(P_v) = \bigoplus_{g \in \Sigma} D_{dR}(P_v) \otimes_{F_v, g} L =: \bigoplus_g D_{dR, g}(P_v)$$

$$\text{Lie } G \otimes_{\mathbb{Q}_p} L = \text{gl}_n/F_v \otimes_{\mathbb{Q}_p} L = \bigoplus_g \text{gl}_n/L =: \bigoplus_g g_L$$

$L(\lambda_6)$: f-dim alg rep of g_L

Ding: $\text{Hom}_{g_L}(L(\lambda_6), \pi[\rho]^{\text{an}}) \neq 0 \Rightarrow \dim_L D_{dR, 6} = n$