Companion points and partially classical eigenvarieties.

1. p-adic automorphic forms

- p : prime number.
- $n \geq 2$, $n \in \mathbb{Z}$
- $F/\mathbb{Q}$ quadratic imaginary. p splits in F, choose $v|p$, $F_v \simeq \mathbb{Q}_p$

  [Rather, F/F^+ CM, F^+ totally real + technical assumptions]
- $\mathcal{G}$ definite unitary group $/\mathbb{Q}$, splits over F, i.e. $\mathcal{G} \times_{\mathbb{Q}} F = GL_{n/F}$. (fix the isomorphism)

  and $\mathcal{G}(\mathbb{R})$ compact.
- Tame level : $U^p = \prod_{\ell \neq p} U_\ell \subseteq \mathcal{G}(\mathbb{A}_f^p) = \prod'_{\ell \neq p} \mathcal{G}(\mathbb{Q}_\ell)$ open compact subgroup, small enough
- $G = \mathcal{G}(\mathbb{Q}_p) \simeq GL_n(F_v) = GL_n(\mathbb{Q}_p)$.
- $L/\mathbb{Q}_p$ $[L:\mathbb{Q}_p] < +\infty$, Large enough coefficient field.

  $\cup$

  $\mathcal{O}_L \leftarrow$ ring of integers $\quad k_L = \mathcal{O}_L/\varpi_L \leftarrow$ uniformizer
- p-adic automorphic forms:

  $\hat{S}(U^p, L) := \{ f : \mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}_f)/U^p \to L, \text{ continuous} \} \circlearrowleft GL_n(\mathbb{Q}_p) = G$.

  $\uparrow$

  Looks like $\mathcal{C}^{cont}(GL_n(\mathbb{Z}_p), L)^{\oplus s}$.
- $\mathbb{T}^{sph}$: some algebra of Hecke operators $/\mathcal{O}_L$

  $\circlearrowleft \hat{S}(U^p, L)$

  Hecke eigenvalues in $\hat{S}(U^p, L)$ $\overset{\text{match Satake parameters and Frobenius eigenvalues}}{\rightsquigarrow}$ $\rho : Gal(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_p)$ (semisimple)

  $\rightsquigarrow$ $m_\rho \in \operatorname{Spec} \mathbb{T}^{sph}$

  Hecke eigenvalues in $\hat{S}(U^p, k_L)$ $\rightsquigarrow$ $\bar{\rho} : Gal(\overline{F}/F) \to GL_n(\overline{k}_L)$.
- Fix $\bar{\rho} : Gal(\overline{F}/F) \to GL_n(k_L)$ absolutely irreducible, modular.

  Π $(= \Pi(U^p)) := \hat{S}(U^p, L)_{m_{\bar{\rho}}} \neq 0$.

Expectation: comes from global.

$$\Pi[\rho] := \Pi(U^p)[m_\rho] \xleftrightarrow[\text{p-adic Local Langlands}]{} \rho_p := \rho|_{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}.$$

$\circlearrowleft GL_n(\mathbb{Q}_p)$

- $\Pi[\rho]^{an}$ $\circlearrowleft G$ Locally analytic representation (Schneider-Teitelbaum)

  $\Pi[\rho]$ subspace of locally analytic vectors (w.r.t. G)

Example: $\mathcal{C}^{cont}(GL_n(\mathbb{Z}_p), L)^{an} = \mathcal{C}^{l\text{-}an}(GL_n(\mathbb{Z}_p), L)$

2. Companion points.

$T = \begin{pmatrix} \mathbb{Q}_p^\times & & 0 \\ & \ddots & \\ 0 & & \mathbb{Q}_p^\times \end{pmatrix} = (\mathbb{Q}_p^\times)^n \subset G$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset G$.

- Emerton's Jacquet module

$\Pi^{an} \circlearrowleft G$ Locally analytic representation. $\rightsquigarrow J_B(\Pi^{an}) \circlearrowleft T$ Locally analytic

- Eigenvariety

$$Y(U^p, \bar\rho) = \left\{ (\rho, \delta) \;\middle|\; \begin{array}{l} \rho: \text{ deformation of } \bar\rho\text{, n-dim'l p-adic cts rep of } Gal(\bar F/F) \\ \delta: T \to \bar{\mathbb{Q}}_p^\times \text{ continuous character} \\ \text{s.t. } Hom_T(\delta, J_B(\Pi[\rho]^{an})) \neq 0 \end{array} \right\}$$

$$\begin{array}{ccc} Y(U^p,\bar\rho) & & (\rho, \delta) \\ \Big\downarrow {\scriptstyle \text{Locally finite}} & & \Big\downarrow \\ \mathcal{W} & & \delta|_{(\mathbb{Z}_p^\times)^n} \end{array}$$

$\mathcal{W} := \{ \delta_0 : (\mathbb{Z}_p^\times)^n \longmapsto \bar{\mathbb{Q}}_p^\times, \text{continuous} \}$

$\delta(U_p) = \delta\left(\begin{pmatrix} p^{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}\right)$

- Companion forms

Fix $(\rho, \delta) \in Y(U^p, \bar\rho)$, in general, $\exists\ \delta' \neq \delta$ s.t. $(\rho, \delta') \in Y(U^p, \bar\rho)$

Conj (Breuil, Hansen) The set $W_H(\rho) := \{\delta' \mid (\rho, \delta') \in Y(U^p, \bar\rho)\}$ is "explicitly" determined by $\rho_p = \rho|_{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$

- Locally analytic socle conjecture.

Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$, $z^\lambda : (z_1, \cdots, z_n) \in (\mathbb{Q}_p^\times)^n \longmapsto z_1^{\lambda_1} \cdots z_n^{\lambda_n}$

locally algebraic characters

Assume $\delta = z^\lambda \delta_{sm}$ where δ_{sm} is a smooth character.

Breuil: $Hom_T(\delta, J_B(\Pi[\rho]^{an})) = Hom_G(\mathcal{F}(M(\lambda), \delta_{sm}), \Pi[\rho]^{an})$

Here $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$: Verma module.

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$, $U(-)$: universal enveloping algebra.

Orlik-Strauch: $\mathcal{F}(M(\lambda), \delta_{sm}) \circlearrowleft G$. Locally analytic.

Jordan Holder factors.

irreducible

$JH(\mathcal{F}(M(\lambda), \delta_{sm})) \xlongequal{\text{generically}} \{ \mathcal{F}(L(\lambda'), \delta_{sm}) \mid L(\lambda') \in JH(M(\lambda)) \}$

irreducible $U(\mathfrak{g})$-module of highest weight λ'.

Example: $n=3$, $\lambda = (1,1,2)$

$M\begin{pmatrix}1\\1\\2\end{pmatrix} = \left[L\begin{pmatrix}0\\1\\3\end{pmatrix} - L\begin{pmatrix}0\\2\\2\end{pmatrix} - L\begin{pmatrix}1\\1\\2\end{pmatrix} \right]$

Conj (Breuil) Assume $(\rho, \delta) \in Y(U^p, \bar\rho)$, $\delta = z^\lambda \delta_{sm}$, ρ_p de Rham, "δ_{sm} generic" then the set $W_B(\rho) = \{(\lambda', \delta_{sm}') \mid \mathcal{F}(L(\lambda'), \delta_{sm}') \hookrightarrow \Pi[\rho]^{an}\}$ is explicitly determined by ρ_p

"companion constituent"

Remark: Conjectures on companion forms / constituents should be viewed as "Locally analytic version of the weight part of Serre's modularity conjecture".

Thm (Regular case: Breuil-Hellmann-Schraen, non-regular: W.)
Assume the Taylor-Wiles assumptions (on $\bar\rho$, F/F^+, G, U^p, $p>2$)
Assume $(\rho, \delta = z^\lambda \delta_{sm}) \in Y(U^p, \bar\rho)$ and ρ_p is <u>generic</u> crystalline
Then $(\rho, z^{\lambda'}\delta_{sm}) \in Y(U^p, \delta_{sm})$ iff $\mathrm{Hom}_G(\mathcal{F}(L(\lambda'), \delta_{sm}), \Pi[\rho]^{an}) \neq 0$
iff $(\lambda', \delta_{sm}) \in W_B(\rho)$ iff $z^{\lambda'}\delta_{sm} \in W_H(\rho)$.

Rmk: ① ρ_p generic: $\varphi \curvearrowright D_{cris}(\rho_p)$ $(\varphi_1, \dots, \varphi_n)$ eigenvalues, then $\varphi_i \varphi_j^{-1} \notin \{1, p\}$ $i \neq j$

② regular: Hodge-Tate weights of ρ_p are pairwise distinct

③ regular: same result for δ_{sm}' (BHS)

non-regular: in progress (W.)

Idea of the proof:
Cycles/subset $\mathcal{L}(\lambda) := $ "$\{(\rho, z^\lambda \delta_{sm}) \mid \mathrm{Hom}_G(\mathcal{F}(L(\lambda), \delta_{sm}), \Pi[\rho]^{an}) \neq 0\}$ with δ_{sm}, ρ_p varying"
Need patching eigenvariety (BHS, Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin)

Local models of the trianguline variety
(BHS, W.)

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"Local Galois theoretic eigenvariety"
$= \{$ space of trianguline representations $\}$

Input: "$\mathcal{L}(\lambda)$ is partially de Rham if $\mathcal{F}(L(\lambda), \delta_{sm})$ are partially classical"

3. Partially classical constituents.

$n=3$, $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{Z}^3$.

① "Classical ⇒ de Rham" If $\lambda_1\geq\lambda_2\geq\lambda_3$, $L(\lambda)$ finite-dim $\circlearrowleft GL_3(\mathbb{Q}_p)$.

$\mathcal{F}(L(\lambda),\delta_{sm}) = L(\lambda)\otimes \overset{\text{smooth}}{\pi_{sm}}$ "Locally algebraic representation"

Then $\mathrm{Hom}_G(\mathcal{F}(L(\lambda),\delta_{sm}),\Pi[\mathfrak{p}]^{an})\neq 0 \Rightarrow \rho_p$ de Rham with regular HT weights.

② "finite slope ⇒ trianguline" If $\mathrm{Hom}_T(\delta, J_B(\Pi[\mathfrak{p}]^{an}))\neq 0$,

then ρ_p trianguline, i.e. $D_{rig}(\rho_p)$ ($\in (\varphi,\Gamma)$-module / Robba ring) admits a filtration $0\subset D_1\subset D_2\subset D_3=D_{rig}(\rho_p)$ s.t. $D_1, D_2/D_1, D_3/D_2$ rank one (φ,Γ)-modules

(Kisin, Colmez, Kedlaya-Pottharst-Xiao, R. Liu) (up to enlarge L)

(Galois representation attached to Hida's ordinary family are reducible at p)

Thm (W. based on Y. Ding) "Partially classical ⇒ Partially de Rham"

Assume $\mathrm{Hom}_G(\mathcal{F}(L(\lambda),\delta_{sm}),\Pi[\mathfrak{p}]^{an})\neq 0$, $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}$

($\Rightarrow$ $(\mathfrak{p}, z^{\lambda}\delta_{sm})\in Y(\mathcal{U}^p,\bar{\rho})$, $0\subset D_1\subset D_2\subset D_3=D_{rig}(\rho_p)$) associated with δ_{sm}

① If $\lambda_1\geq\lambda_2$, then D_2 de Rham

② If $\lambda_2\geq\lambda_3$, then D_3/D_1 de Rham.

Rmk: $\rho_p: V \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightsquigarrow D_{rig}(\rho_p) \overset{\text{Berger}}{\rightsquigarrow} W_{dR}(D_{rig}(\rho_p)) = B_{dR}\otimes_{\mathbb{Q}_p} V \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$D \overset{\text{Berger}}{\rightsquigarrow} W_{dR}(D)$$

D de Rham $\Leftrightarrow$ $W_{dR}(D) = W_{dR}(D)^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}\otimes_{\mathbb{Q}_p} B_{dR}$.

• Reason: $\lambda=(1,1,2)$ (or $\lambda_1\geq\lambda_2$)

$P=\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\subset G$. $P=MN$, $M=GL_2(\mathbb{Q}_p)\times\mathbb{Q}_p^{\times}$

$\mathfrak{p}=\mathrm{Lie}\,P$ $\quad\mathfrak{m}=\mathrm{Lie}\,M$

$L(\lambda)$ quotient of $\underbrace{U(\mathfrak{g})\otimes_{U(\mathfrak{p})} L_M(\lambda)}_{\text{parabolic Verma module}}$ (equal in this case)

• $L_M(\lambda)$ f.dim rep of $U(\mathfrak{m})$ of highest weight λ $\circlearrowleft M$

$\Rightarrow$ $\mathcal{O}^{\mathfrak{p}}\ni L(\lambda) \circlearrowleft M, U(\mathfrak{m})$.

Then $\mathrm{Hom}_G(\mathcal{F}(L(\lambda),\delta_{sm}),\Pi[\mathfrak{p}]^{an})\neq 0 \xRightarrow{\text{Breuil, Ding}}$

$$\mathrm{Hom}_M(L_M(\lambda), J_P(\Pi[\mathfrak{p}]^{an}))[\delta_{sm}]\neq 0.$$

$J_P(-)$: parabolic version Emerton's Jacquet module.

$J_P(\Pi[\rho]^{an}) \circlearrowleft M$ $\Rightarrow$ $0 \subset \underset{\substack{\uparrow \\ \text{rank } 2}}{D_2} \subset D_3 = D_{rig}(\rho_p)$ "paraboline filtration" Bergdall

$\cup$

non-zero locally algebraic rep of M $\Rightarrow$ D_2, D_3/D_2 de Rham

- Idea of the proof: $\lambda_0 = (1, 1, 2)$.

  Ding constructed a partial eigenvariety using $J_P(\Pi^{an})$ (previously Hill-Loeffler)

  $$(\rho, z^\lambda \delta_{sm}) \in Y(U^p, \bar{\rho})' \hookrightarrow Y(U^p, \bar{\rho})$$

  - $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, then $\lambda_1 = \lambda_2$.
  - $\mathrm{Hom}_M(L_M(\lambda), J_P(\Pi[\rho]^{an})) \neq 0$
  - Classical points / de Rham points dense in $Y(U^p, \bar{\rho})'$

Rmk: If $F_v \neq \mathbb{Q}_p$, $\rho_v = \rho|_{\mathrm{Gal}(\bar{F}_v/F_v)}$, $\Sigma = \{\sigma : F_v \hookrightarrow L\}$. L large enough.

$D_{dR}(\rho_v) = \bigoplus_{\sigma \in \Sigma} D_{dR}(\rho_v) \otimes_{F_v, \sigma} L =: \bigoplus_\sigma D_{dR,\sigma}(\rho_v)$

$\mathrm{Lie}\, G \otimes_{\mathbb{Q}_p} L = \mathfrak{gl}_{n/F_v} \otimes_{\mathbb{Q}_p} L = \bigoplus_\sigma \mathfrak{gl}_{n/L} =: \bigoplus_\sigma \mathfrak{g}_\sigma$

$L(\lambda_\sigma)$: f.dim alg rep of $\mathfrak{g}_\sigma$

Ding: $\mathrm{Hom}_{\mathfrak{g}_\sigma}(L(\lambda_\sigma), \Pi[\rho]^{an}) \neq 0$. $\rightsquigarrow$ $\dim_L D_{dR,\sigma} = n$