Math136 - January 6'th, 2016

Vector Spans

Proof

We will prove Theorem 1 Property 4.

Recall Property 4: There exists $\vec{0} \in \mathbb{R}^n$ such that $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ for any $\vec{x} \in \mathbb{R}^n$

Because this is a 'there exists' proof, we will show that the vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ has these properties.

Now for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, we have $\vec{x} + \vec{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ (By the definition of vector additions).

Similarly, $\vec{0} + \vec{x} = \vec{x}$ $\square$

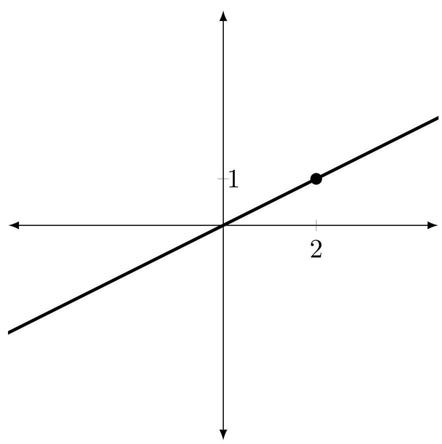
Span

For a set $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors from $\mathbb{R}^n$, $\text{span}(B) = \{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$ is the set of all linear combinations of the vectors in B.

E.g. Give a geometric description of $\text{span}(\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\})$

$= \{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} | \in \mathbb{R}\}$

If we look at this vector graphically on the 2D plane, its a line from the origin to (2, 1). The span is the set of all vectors that are a scalar multiple of the original vector(s). In this example, it means the span of this vector is any point on the line shown below: (OH BOY GRAPH TIME)



E.g. Describe $\text{span}(\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\})$ in $\mathbb{R}^3$)

$= \{c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\} = \{\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix}\}$ (By expanding and adding)

This is the set of vectors $\vec{x} \in \mathbb{R}^3$ such that $x_2 = 0$ (The x_1x_3 plane).

E.g. Consider: span($\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\}$)

Note that $\begin{bmatrix}2\\1\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}$

As we will soon see, it follows that span($\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\}$) = span($\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\}$)

We will prove this soon. For now, we'll do an illustrative example.

Consider the vector $\vec{v} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - 3\begin{bmatrix}2\\1\\3\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}$ in $\mathbb{R}^3$

Lets show that $\vec{v} \in$ span($\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\}$)

$$\begin{aligned}\vec{v} &= 2\begin{bmatrix}1\\1\\1\end{bmatrix} - 3(2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}) + \begin{bmatrix}0\\-1\\1\end{bmatrix}\\ &= 2\begin{bmatrix}1\\1\\1\end{bmatrix} - 6\begin{bmatrix}1\\1\\1\end{bmatrix} - 3\begin{bmatrix}0\\-1\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\1\end{bmatrix}\\ &= -4\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}0\\-1\\1\end{bmatrix}\end{aligned}$$

Theorem 2

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then, for an index $i \in \{1, \ldots, k\}$ we have $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) = \text{span}(\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\})$ iff $\vec{v}_i$ can be written as a linear combinations of $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k$

Proof:

$\Longleftarrow$ Suppose $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_k\vec{v}_k$

Then, suppose $\vec{v} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k \in \text{span}(\{\vec{v}_1, \ldots, \vec{v}_k\})$

Then, $\vec{v} = d_1\vec{v}_1 + \cdots + d_{i-1}\vec{v}_{i-1} + d_i\vec{v}_i + d_{i+1}\vec{v}_{i+1} + \cdots + d_k + \vec{v}_k$
$= d_1\vec{v}_1 + \cdots + d_{i-1}\vec{v}_{i-1} + d_{i+1}\vec{v}_{i+1} + \cdots + d_k\vec{v}_k + d_i(c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_k\vec{v}_k)$
Distribute and Group!
$= (d_1 + d_ic_1)\vec{v}_1 + \cdots + (d_{i-1} + d_ic_{i-1})\vec{v}_{i-1} + (d_{1+1} + d_ic_{i+1})\vec{v}_{i+1} + \cdots + (d_k + d_ic_k)\vec{v}_k \in$ $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\})$

So, we have showed the left hand side of the equality is a subset of the right hand side.
Now we will show the right hand side is a subset of the left hand side.

$\vec{v} = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + v_k\vec{v}_k$
$= c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + (0)\vec{v}_i + c_{i+1}\vec{v}_{i+1} + \cdots + v_k\vec{v}_k$

TO BE CONTINUED!!