Types of Data

Categorical data

Is non-numeric, can be observed but not measured

E.g. Favorite color, Place of Birth

Quantitative data

Is numerical data which can be measured

Discrete

Random variable which takes only isolated values in its range of variation. For example number of heads in 10 tosses of a coin

Continuous

Random variable which takes any value in its range of variation. For example, height of a person

Nominal

- Values do not have ordering
- Example categorical variables like color, nationality and so on

Ordinal

- Values are ordered
- Example Satisfaction scores

Types of Statistics

Statistics is the science of collecting, organizing, presenting, analyzing, and interpreting numerical data to assist in making more effective decisions.

Descriptive Statistics describes the data set that's being analyzed, but doesn't allow us to draw any conclusions or make any inferences about the data

- Measures of Central Tendency
- Measures of Dispersion
- Tables and Graphs

Inferential statistics is a set of methods that is used to draw conclusions or inferences about the characteristics of populations based on data from a sample

- Estimation
- Hypothesis Testing

Measures of Central Tendency

MEAN

It is just the average of the data, computed as the sum of the data points divided by the number of points

- (+) It is the easiest metric to understand and communicate
- (−) Mean is prone to presence of outliers

Example: What is a typical student in the class doing?

MEDIAN

It is the value in the middle of the data set, when the data points are arranged from smallest to largest.

Tricky circumstances:

If there is an even number of data points, you will need to take the average of the two middle values.

- (+) Median is a more “robust” to presence of outliers
- (−) It is more complicated to communicate

Example: To compare performance of any single student against group

MODE

It is the most common value in the data set.

Tricky circumstances:

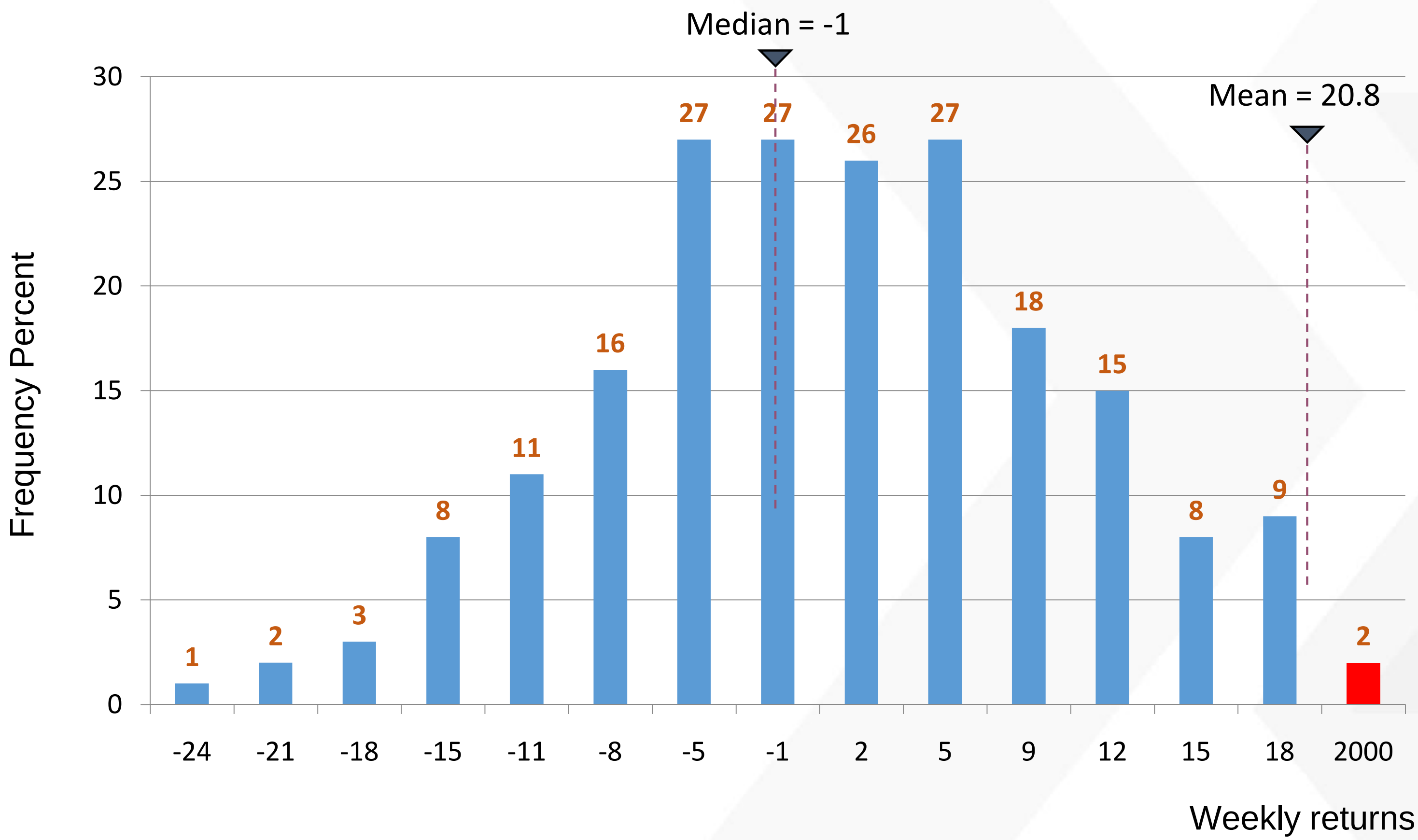
If no value occurs more than once, then there is no mode

If two, or more, values occur as frequently as each other and more frequently than any other, then there are two, or more, modes.

- (+) Not very practical since it is affected by skewness
- (−) Most real life distributions are multimodal

Example: A parent wanting to know whether their child is better or worse than typical child at his grade level

An example – Histogram of weekly returns of XYZ share prices



And Mode?

But are these sufficient?

- There is the man who drowned crossing a stream with an average depth of six inches.
- Say you were standing with one foot in the oven and one foot in an ice bucket. According to the averages, you should be perfectly comfortable.
- Time taken by different modes of transport

	Auto	Office Transport	Own Car
	7	9	1
	6	9	3
	3	9	5
	8	9	7
	12	9	9
	9	9	9
	9	9	9
	13	9	11
	13	9	13
	9	9	15
	10	9	17
Mean	9	9	9
Median	9	9	9
Mode	9	9	9

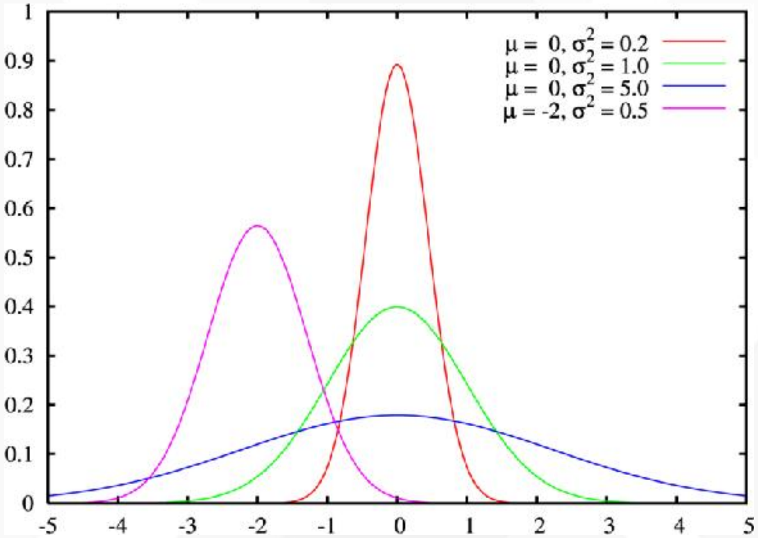
Measures of Variation/Dispersion

Dispersion refers to the spread or variability in the data. It determines how spread out are the scores around the mean.

Why is Dispersion important?

- It gives additional information that enables to judge the reliability of the measure of central tendency
- If data are widely spread the central location is less representative of data as a whole than it would be for data more closely centered around Mean
- Since problems are peculiar to widely dispersed data, dispersion enables to identify and tackle problems accordingly
- This enables to compare dispersions of various samples
- For eg. If a wide spread of values are away from center, this may be
- undesirable or presents a risk, one may avoid choosing that distribution

Distributions with different dispersions



Common measures of dispersion

Standard Deviation is a measure of how spread out numbers are

Variance is defined as the average of the squared differences from the Mean

$$\sigma^2 = \frac{\Sigma\,(X - \mu)^2}{N} = \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots}{N}$$

Where **X is the value of an observation in the population**

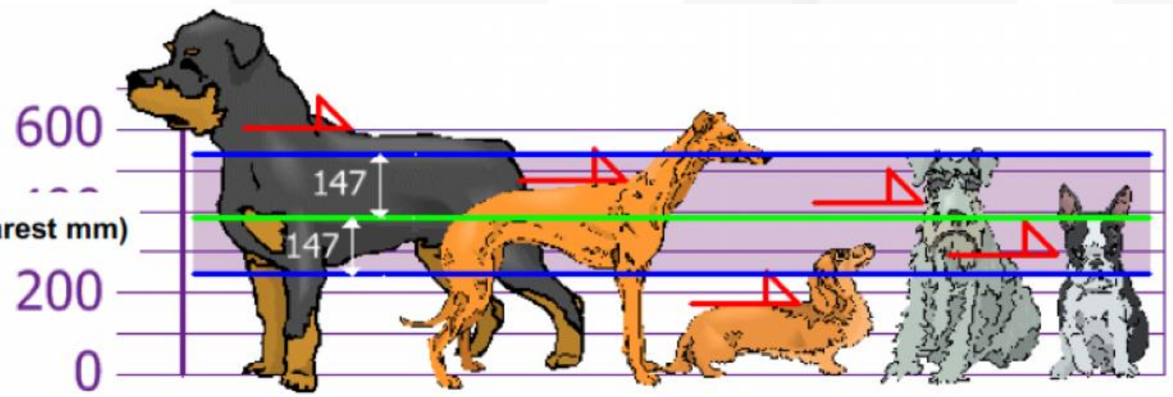
μ is the arithmetic mean of the population

N is the number of observations in the population

Example: You have just measured the heights of your dogs (in millimeters). The heights are:
600mm, 470mm, 170mm, 430mm and 300mm.
Mean = 394mm

$$\text{Variance: } \sigma^2 = \frac{206^2 + 76^2 + (-224)^2 + 36^2 + (-94)^2}{5} = 21{,}704$$

Standard Deviation: $\sigma = \sqrt{21{,}704} = 147.32\ldots = 147$ (to the nearest mm)



Using the Standard Deviation we have a "standard" way of knowing what is normal,
and what is extra large or extra small.

Now have a look….

	Auto	Office Transport	Own Car
	7	9	1
	6	9	3
	3	9	5
	8	9	7
	12	9	9
	9	9	9
	9	9	9
	13	9	11
	13	9	13
	9	9	15
	10	9	17
Mean	9	9	9
Median	9	9	9
Mode	9	9	9

Std Dev	**3.0**	**0.0**	**4.9**
Variance	**9.2**	**0.0**	**24.0**

Coefficient of Variation (CV)

It is a normalized measure of dispersion of a probability distribution. It is calculated as the ratio of the standard deviation to the mean.

- Measure of relative dispersion
- Always a %
- Shows variation relative to mean
- Used to compare 2 or more groups

Which Cricketer do you like? Who is more consistent?

Dravid	150	150	130	125	145	110	100	152	120	50	128
Sehwag	230	240	150	50	173	23	20	300	45	1	128

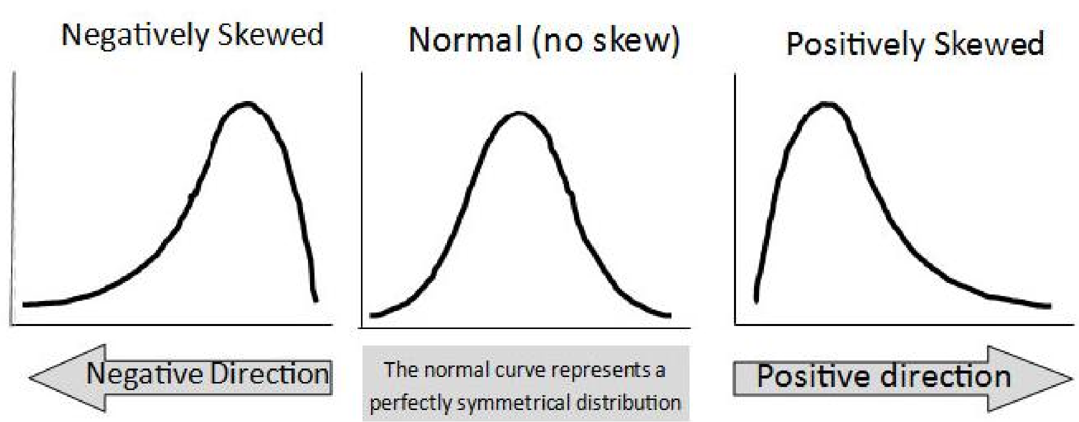
	Dravid	Sehwag
Mean	123.636	123.636
Median	128	128
CV	24%	84%

Descriptive Statistics

Central Tendency: is the middle point of distribution. Measures of Central Tendency include Mean, Median and Mode

Dispersion: is the spread of the data in a distribution, or the extent to which the observations are scattered.

Skewness: When the data is asymmetrical ie the values are not distributed equally on both sides. In this case, values are either concentrated on low end or on high end of scale on horizontal axis.



If the trail is to the right or positive end of the scale, the distribution is said to be "positively skewed".
If the distribution trails off to the left or negative side of the scale, it is said to be "negatively skewed".

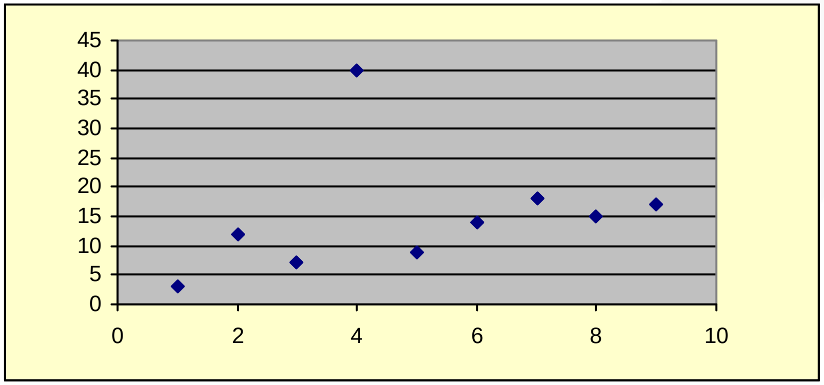
Outliers

An outlier is an observation that is numerically distant from the rest of the data.

An outlying observation, or outlier, is one that appears to deviate markedly from other members of the sample in which it occurs. Outliers can occur by chance in any distribution, but they are often indicative either of measurement error or that the population has a heavy-tailed distribution.

Example: Bill Gates makes $500 million a year. He's in a room with 9 teachers, 4 of whom make $40k, 3 make $45k, and 2 make $55k a year. What is the mean salary of everyone in the room? What would be the mean salary if Gates wasn't included? Mean With Gates: **$50,040,500** Mean Without Gates: **$45,000**

A **Scatterplot** is useful for "eyeballing" the presence of **outliers**.



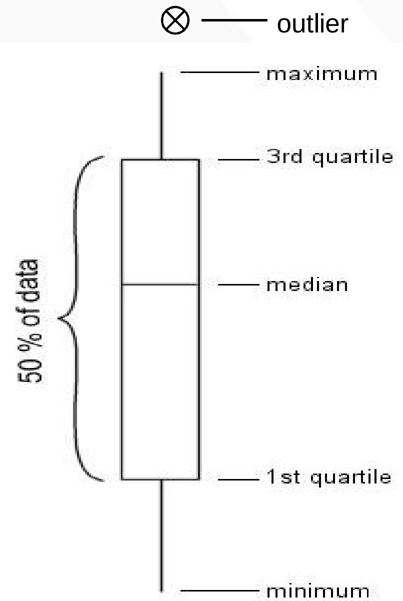
We can also use Standard Deviation to identify Outliers!

Other Measures of Dispersion

Box-plot

- Reveals the spread of the data
- Outliers defined using the

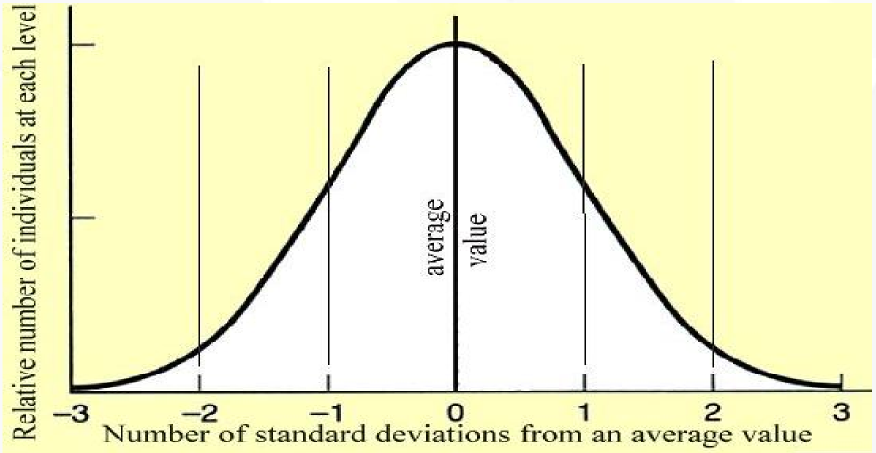
  Q1 - 1.5(Q3-Q1) and Q3 + 1.5(Q3-Q1)



Normal Distribution

Normal distribution is a pattern for the distribution of a set of data which follows a ***bell shaped curve***. This also called the *Gaussian distribution*.
The normal distribution is a theoretical ideal distribution. Real-life empirical distributions never match this model perfectly. However, many things in life do approximate the normal distribution, and are said to be "normally distributed."

- Normal Distribution has the mean, the median, and the mode all coinciding at its peak
- The curve is concentrated in the center and decreases on either side ie most observations are close to the mean
- The bell shaped curve is symmetric and Unimodal
- It can be determined entirely by the values of mean and std dev
- Area under the curve = 1
- **The empirical *68-95-99.7 rule* states that for a normal distribution:**
  - **68.3% of the data will fall within 1 SD of mean**
  - **95.4% of the data will fall within 2 SD's of the mean**
  - **Almost all (99.7%) of the data will fall within 3 SD's of the mean**



Standard Normal Distribution

Standard Normal distribution is a special case of the Normal distribution which has a mean of 0 and a standard deviation of 1

Any normal distribution can be converted to a Standard normal distribution through:

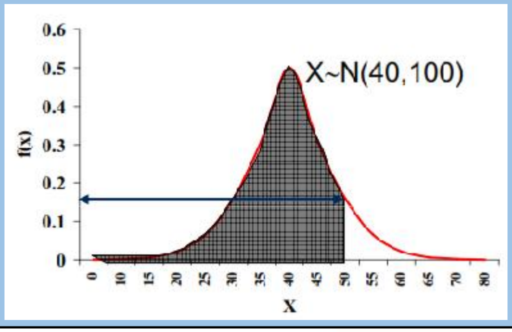
$$Z = \frac{X - \mu}{\sigma}$$

Example: If X is a continuous random variable with a mean of 40 and a standard deviation of 10, what proportion of observations are a) Less than 50 b) Less than 20 c) Between 20 and 50

a) P(x<50)?

$$Z = \frac{50-40}{10} = 1$$

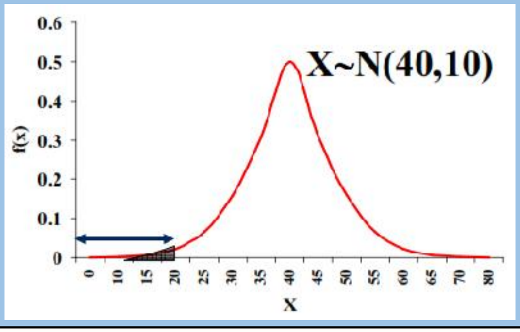
P(x<50) = P(z<1) = .8413



b) P(x<20)?

$$Z = \frac{20-40}{10} = -2$$

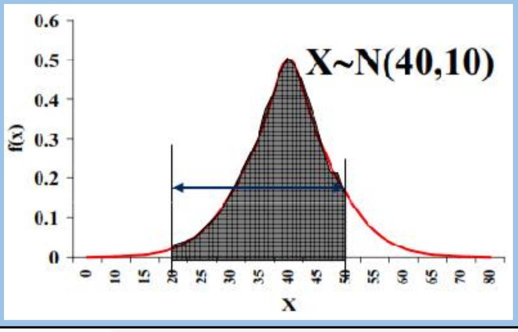
P(x<20) = P(x<-2) = .0228



c) P(20<x<50)?

$$\frac{20-40}{10} < Z < \frac{50-40}{10}$$

P(-2 < Z < 1) = .8185



Standard scores

A standard score (also called Z score) is the number of standard deviations that a given raw score is above or below the mean.

- All normal distributions can be converted into the standard normal curve by subtracting the mean and dividing by the standard deviation.
- Standardizing variables helps to get variables to the same scale, or makes the variable unit free For eg: If we want to input Income (in INR) and Number of calling minutes into the same analysis, we will have to standardize both variables to get them to the same scale

$$Z = \frac{X - \text{«}}{\pm}$$

Central Limit Theorem

It is always not possible to get the true information about the population. In this case we have to live with samples. For eg: we don't know the actual average income for India, but can estimate it based on a random sample picked from the Indian population

In this case, the average we have is not the population average μ but an estimate $\overline{X}$

If we take a similar second sample, it is extremely unlikely that the average calculated for the second sample will be the same as the average calculated for the first sample. In fact, statisticians know that repeated samples from the same population give different sample means.

They have also proven that the distribution of these sample means will always be normally distributed, regardless of the shape of the parent population. This is known as the Central Limit Theorem.

A distribution with a mean μ and variance σ^2, the sampling distribution of the mean approaches a normal distribution with a mean (μ) and a variance σ^2/N as N, the sample size increases.

The amazing and counter-intuitive thing about the central limit theorem is that *The distribution of an average tends to be Normal, even when the distribution from which the average is computed is decidedly non-Normal distribution from which the average is computed is decidedly non-Normal.*
As the sample size n increases, the variance of the sampling distribution decreases. This is logical, because the larger the sample size, the closer we are to measuring the true population sample size, the closer we are to measuring the true population parameters.

Standard Error

Since all samples drawn from a population are similar BUT NOT the same as population, we calculate a Standard Error.

Standard Error is the standard deviation of the sample means from the population mean

Also, Standard Error ultimately converges to the Standard Deviation of the population.

$$\text{Standard Error} = \frac{\sigma}{\sqrt{N}}$$

Populations and Samples

So far we have determined the results associated with individual observations or sample means when the true population parameters are known. In reality, the true population parameters are seldom known. We now learn how to infer **levels of confidence**, or a measure of accuracy on parameters, estimated using samples

POINT ESTIMATOR

- If we take a sample from a population, we can estimate parameters from the population, using sample statistics
- Example: Sample mean (x) is our best estimate of the population mean (μ)
- Whereas, we really don’t know how close the estimate is to the true parameter
- The mean annual rainfall of Melbourne is 620mm per year

INTERVAL ESTIMATOR

- If we estimate a range or interval within which the true population parameter lies, then we are using an interval estimation method
- This is the most common method of estimation. We can also apply a level of how confident we are in the estimate
- In 80% of all years Melbourne receives between 440 and 800 mm rain

Sampling methodologies

Sampling is required because it is seldom possible to measure all the individuals in a population. Researchers hence, use samples and infer their results to the population of interest

Eg: Election polls, market research surveys, etc

For a sample to be a "good sample", it is imperative that there is a good sample size and there is no biasness in the sample.

Simple Random Sample

is one in which every member of the population is equally likely to be measured

Eg: Allocate a number to each member of the population and use a random number generator to determine which individuals will be measured

Stratified Sampling

separates the population into mutually exclusive groups and randomly samples within the groups

Eg: Randomly select a number of people within each demographic cell, while maintaining overall proportions like gender ratio, income ratio, etc

Other methodology

Cluster sampling: is used when there is a considerable variation within each group but the groups are essentially similar to each other. Here we divide the population into groups, or clusters, and then select a random sample of these clusters.

Confidence Intervals

Because we know the properties of the normal distribution so well, we can use these properties to assist us in applying confidence intervals to estimates. This is essentially the interval estimator range.

For example, we know that 68.3 % of sample means lie within one standard error of the true population mean.

Therefore, if we know the true population variance, we can infer the range within which we can be 68.3% confident that the true population mean lies

Example:

X ~ N(μ, 3.62) grams, n = 36, x = 25.5

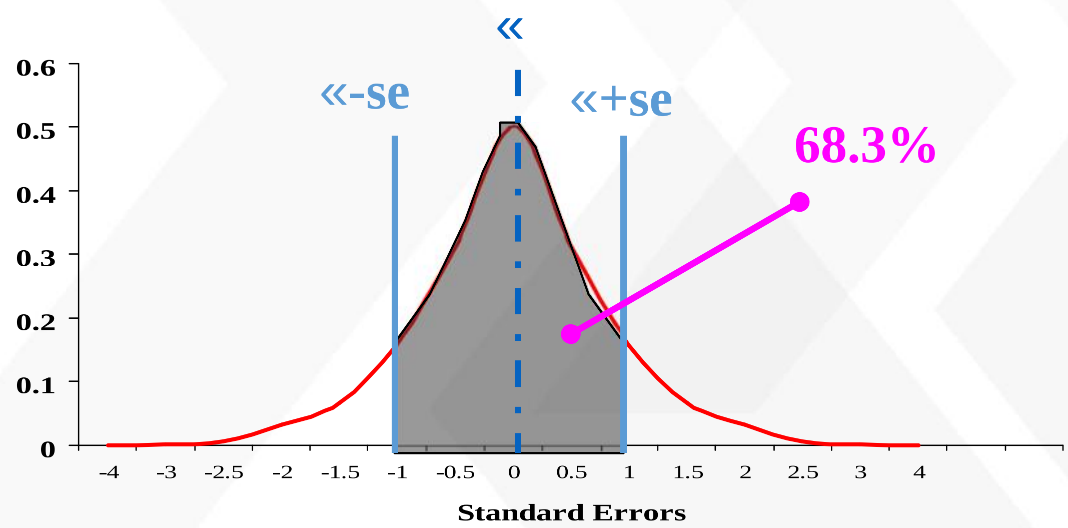
We can be 68.3% confident that μ is within the interval

x ± SE

= 25.5 ± 3.6/ √ 36

= 25.5 ± 0.6

= (24.9 to 26.1) grams



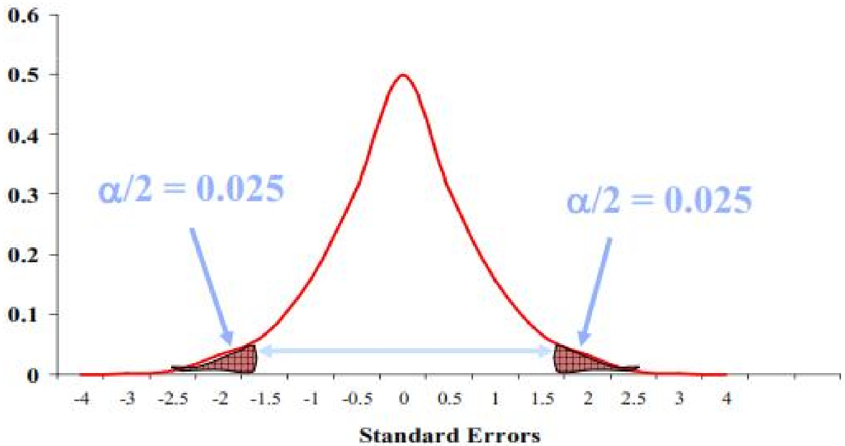
Confidence Intervals

We can extend this principle further:

- We can be *90% confident* that the true population mean lies within $x \pm 1.645(SE)$
- We can be *95% confident* that the true population mean lies within $x \pm 1.960(SE)$
- We can be *99% confident* that the true population mean lies within $x \pm 2.576(SE)$

P - value

- Furthermore, the area outside the confidence interval is cumulatively known as α (alpha)
- Confidence Interval = 1 - α
- Example: for 95% confidence interval, ∝=0.05
- α is also known as p-value.
- Hence, p-value is the probability that a randomly picked sample will have the mean **lying outside the confidence interval**.

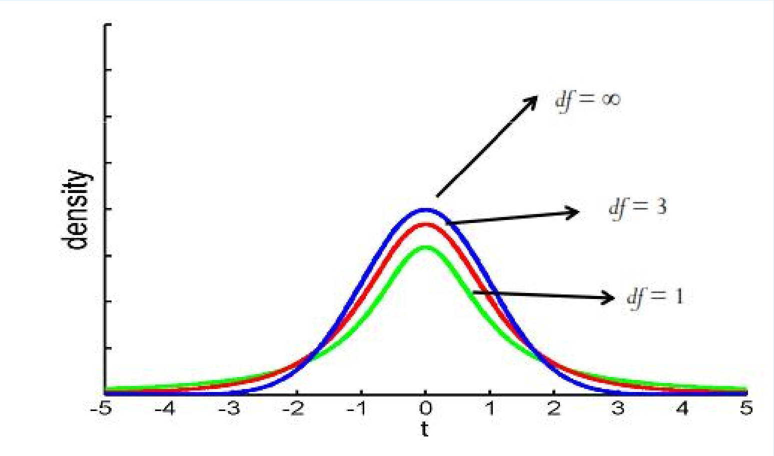


Student's t - distribution

- While z-distribution is for the population, t-distribution is for the sample distribution.
- Hence, the shape of 't' sampling distribution is similar to that of the 'z' sampling distribution in that it is
  - a) Symmetrical
  - b) Centered over a mean of zero
  - c) Variance depends on the sample size, more specifically on the degrees of freedom (abbreviated as df)
- As the number of degrees of freedom increases , variance of the t distribution approaches more closely to that of z
- For n ≥ 30, shapes are almost similar
- For n of 30 taken as dividing point between small & large samples
- t-test for population mean is:

$$\frac{\overline{X} - \mu_0}{s/\sqrt{n}}$$

When n < 30



When to use

z-test:

- σ is known and the population is normal
- σ is known and the sample size is at least 30. (The population need not be normal)

t-test:

- Whenever σ is not known
- The population is assumed to be normal
- And $n<30$
- The correct distribution to use is the ‘t’ distribution with n-1 df

Hypothesis testing

- In a Test Procedure, to start with, a hypothesis is made.
- The validity of the hypothesis is tested.
- If the hypothesis is found to be true, it is accepted.
- If it is found to be untrue, it is rejected.
- The hypothesis which is being tested for possible rejection is called null hypothesis
- Null hypothesis is denoted by H_0
- The hypothesis which is accepted when null hypothesis is rejected is called Alternate Hypothesis H_a
- Ex. H_o : The drug works –it has a real effect.

  H_a : The drug doesn't work - Any effect you saw was due to chance.

Hypothesis testing

Hypothesis tests consist of the following steps:

- Null Hypothesis
- Alternative Hypothesis
- Confidence Level
- Decision Rule
- Test statistic
- Decision

Hypothesis testing

- Null hypothesis - We always assume the null hypothesis is true, or at least is the most plausible explanation before we do the test. The test can only ***disprove*** the null hypothesis.
- Alternative hypothesis - The alternative hypothesis is the hypothesis that we set out to test for. It is the hypothesis that we wish to ***prove***.
- Decision Rule - After we know the null and alternative hypotheses and the level of confidence associated with the test, we determine the points on the distribution of the test statistic where we will decide when the null hypothesis should be rejected in favor of the alternative hypothesis
- Use the terminology " **Reject H_o**" or "**Do not reject H_o**". Never say "Accept Ho"

Type I and Type II Error

Process of testing a hypothesis indicates that there is a possibility of making an error. There are two types of errors:

Type I error: The error of rejecting the null hypothesis H_0 even though H_0 was true.

Type II error: The error of accepting the null hypothesis H_0 even though H_0 was false.

Example 1

Suppose that we have been told that the price of petrol in Melbourne is normally distributed with a mean of 92 cents per litre, and a standard deviation of 3.1 cents/litre. To test whether this price is in fact true, we sample 50 service stations and obtain a mean of 93.6 cents/litre

Solution:

- Step 1: State the null and alternative hypotheses
  - Ho: $\mu = 92$
  - Ha: $\mu \neq 92$

- Step2: Determine the appropriate test statistic and it's distribution

  Because we know the population standard deviation, we can use the z distribution

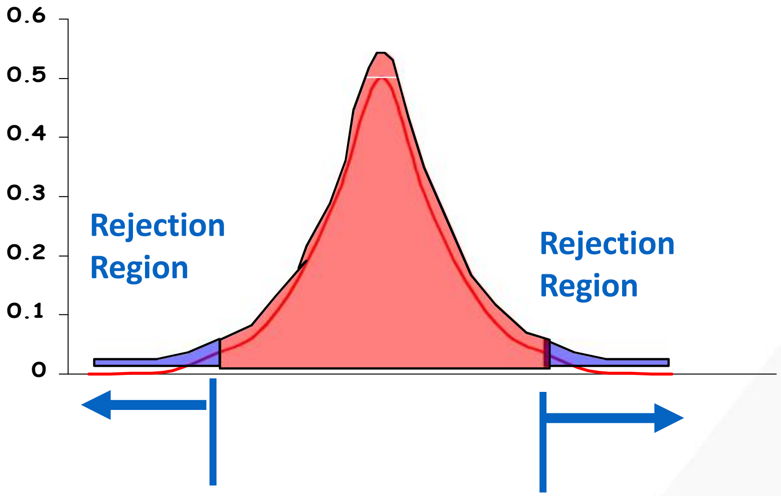
- Step3: Specify the significance level, Say $\alpha = 0.05$

Example 1 (contd.)

- Step 4: Define the decision rule.

  Using a z distribution (from tables), if α = 0.05,

  the rejection region is > +1.96 and < -1.96



i.e., if the test statistic is

> 1.96 or < -1.96,

we will reject H_o and

accept H_a

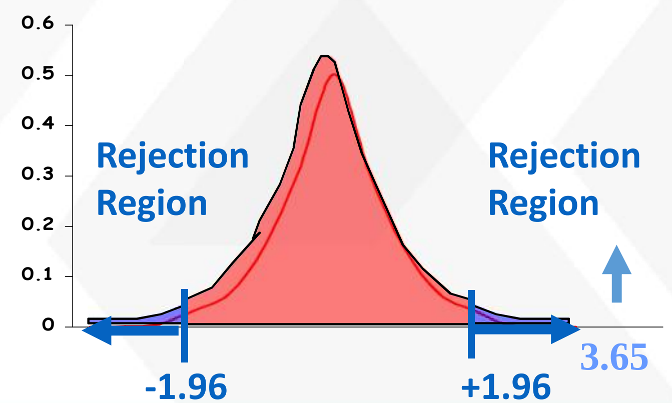
Example 1 (contd.)

- Step 5: Calculate the test statistic

$$Z = \frac{\bar{X} - \text{«}_{\bar{x}}}{SE} = \frac{\bar{X} - \text{«}_{o}}{\pm / \sqrt{n}} \qquad Z = \frac{93.6 - 92}{3.1/\sqrt{50}} = 3.65$$

- Step 6: Make a decision and answer the question: As, 3.65 > 1.96, the test statistic is in the rejection region
  Reject H_o, accept H_a as a more plausible explanation

- Step 7: Write your conclusion in the context of the aims of the study.
  “The average price of petrol in Melbourne was significantly different to 92 cents/litre”



Example 1 (contd.) – Importance of sample sizes

- Consider the petrol prices in Melbourne example. If the sample size we had used was only 10, rather than 50, the test statistic would have been;

$$Z = \frac{93.6 - 92}{3.1/\sqrt{10}} = 1.63$$

In which case we would not have rejected the H_o

Example 2

A company pays production workers \$630 per week. The union claims that these workers are paid below the industry average for their work. A sample of 15 workers from other sites gives a mean wage of \$670/week with a standard deviation of \$58/week. Is the unions claim justified?

Solution:

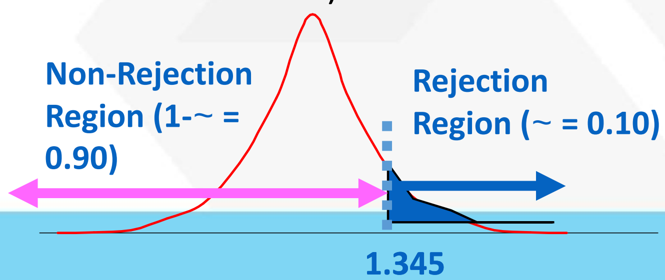
Step 1: Ho: μ =< \$630 (industry weekly average is not significantly different to \$630)

Ha: μ > \$630 (The industry weekly average is greater than \$630)

Step2: Test Statistic - As we don't know the population variance, and the sample size is < 30, we shall use the t test.

Step3: Significance level - We will use α = 0.10 (as we want to be liberal rather than conservative)

Step 4 : Decision rule - From 't' table, $t_{(0.1, 14df)} = 1.345$



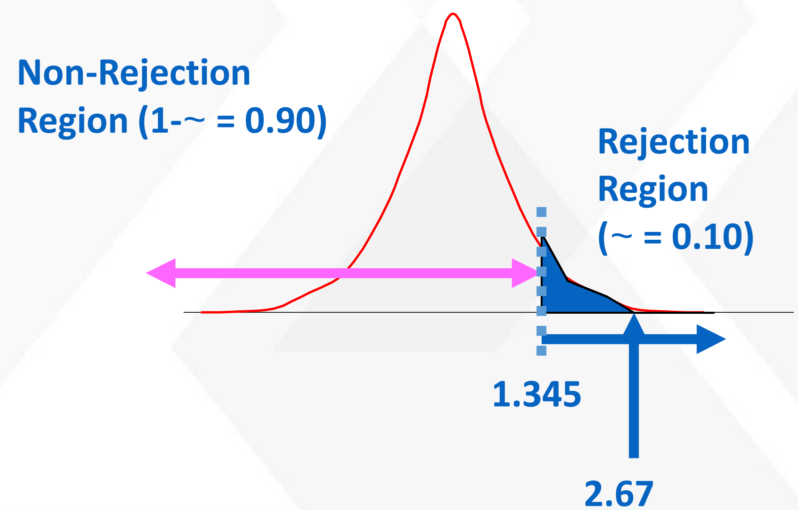
Example 2 (contd.)

Step 5: Calculate test statistic;

$$t = \frac{\bar{X} - \mu_{\bar{x}}}{SE} = \frac{\bar{X} - \mu_{\bar{x}}}{s/\sqrt{n}} \qquad t = \frac{670 - 630}{58/\sqrt{15}} = 2.67$$

Step 6: <u>Make a decision</u> - As 2.67 is > 1.345, we will reject the H_o

Step 7: <u>Conclusion</u> - "Production workers at the company earn an average of $40 per week less than the industry standard (t = 2.67, df = 14, p < 0.1)"

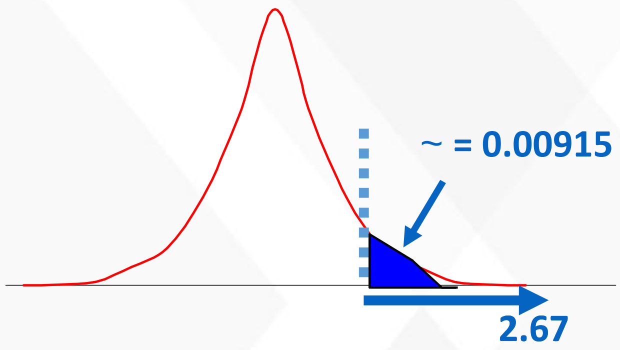


P-value

In fact, with the advent of computers it is simple to calculate the exact probability of a test statistics.

Example: $P(t \geq 2.67) = 0.00915$

i.e. The area under the curve is 0.00915



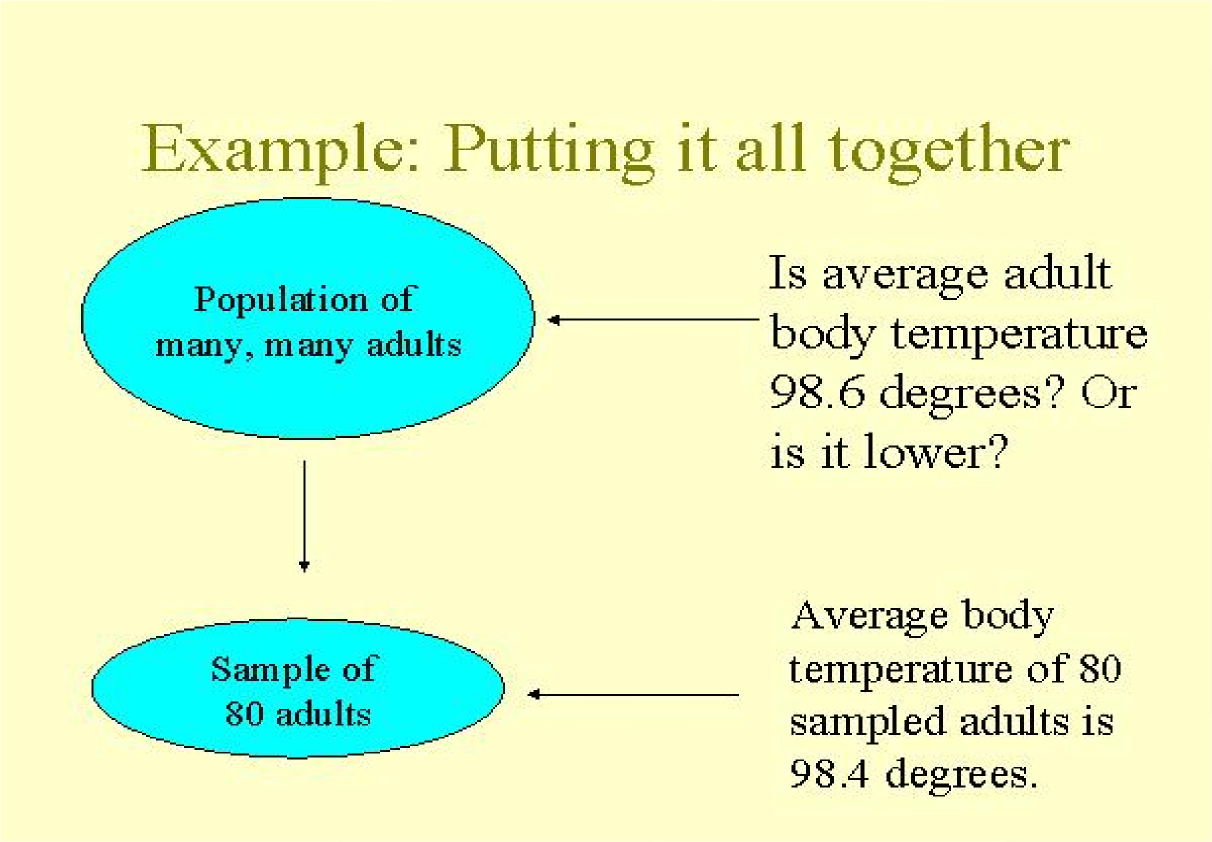
With a z test, it is possible to determine the exact probabilities using the table in the text book

Ex.: What is the exact probability of obtaining a value of $z \leq 1.5$?

From table, when $z = 1.5$, $p = 0.5 + 0.4332$,

$$P(z \leq 1.5) = 0.9332$$

A Practical Example



A Practical Example

Example (continued)

- Specify hypotheses.
  - $H_0: \mu = 98.6$ degrees
  - $H_A: \mu < 98.6$ degrees
- Make initial assumption: $\mu = 98.6$ degrees
- Collect data: Average body temp of 80 sampled adults is 98.4 degrees. How likely is it that a sample of 80 adults would have an average body temp as low as 98.4 if the average body temp of population was 98.6?

A Practical Example

Using the **p-value** to make the decision

- The **p-value** represents how likely we would be to observe such an extreme sample if the null hypothesis were true.
- The p-value is a probability, so it is a number between 0 and 1.
- Close to 0 means "unlikely."
- So if p-value is "small," (typically, less than 0.05), then reject the null hypothesis.

A Practical Example

Example (continued)

The p-value can easily be obtained from statistical software like MINITAB.

```
Test of mu = 98.6000 vs mu < 98.6000
The assumed sigma = 0.600

Variable   N   Mean   StDev   SE Mean    Z        P
Temp      80   98.4   0.67    0.0671   -2.80   0.0026
```

(Generally, the p-value is labeled as "P")

A Practical Example

Example (continued)

- The p-value, 0.0026, indicates that, if the average body temperature in the population is 98.6 degrees, it is unlikely that a sample of 80 adults would have an average body temperature as extreme as 98.4 degrees.
- Decision: Reject the null hypothesis.
- Conclude that the average body temperature is lower than 98.6 degrees.

Comparison of two populations

Hypothesis testing for two samples:

- Difference between independent samples & dependent samples
- Two sample z test for means using independent samples
- Two sample t test for means using independent Samples
- Two sample t tests for means using dependent Samples

Chi-square test

Two properties are associated if the probability of having one property affects the probability of having another. Sometimes it is not known whether two properties are associated or not. What is required is a test of association, or, what is equivalent, a test of independence.
The Chi-square (χ 2) distribution can be used as a test of independence.

Example:
A psychologist conducted a survey into the relationship between the way in which a calculator was held and the speed with which 10 arithmetical operations were performed. The calculator could be either placed on a table or held in the hand; the sums could be performed in either less than 2 minutes, between 2 and 3 minutes or more than 3 minutes.
The following results were obtained for a sample of 150 children between 12 and 13 years old.

		Mode of Computation	
		On Table	**Hand Held**
Speed Of Computation	<2	28	12
	2-3	25	35
	>3	21	29

Example (contd.)

Solution:

Step 1: Ho: Speed and mode are independent

Ha: Speed and mode are associated

Step2: In order to determine whether the two variables are associated it is necessary to calculate what the frequencies would be if there was absolutely no connection between them, or as we call them “Expected Frequencies”

	Table	**Hand**
<2	40*74/150 = 19.73	40*76/150 = 20.27
2-3	60*74/150 = 29.60	60*76/150 =30.40
>3	50*74/150 =24.67	50*76/150 = 25.33

Example (contd.)

Step 3: Now we have to use the expected and observed frequencies to calculate a test statistic.

The χ 2 test statistic is determined by $$\sum \frac{(O_i - E_i)^2}{E_i}$$

Step 4: In order to compare this with a critical value, we need to know the degrees of freedom of statistic.

v=degrees of freedom=(row number −1)· (column number −1)

$$Then\ \chi^2_{test} = 9.329 > \chi^2_{critical} = 5.992$$

Step 5: Therefore, we reject H_O and accept H_1 .

The result is significant at the 0.05 or 5% level. This means that there is a 5 in 100 probability that the difference between the two conditions could have arisen by chance.

According to these results the way you use your calculator does affect the speed with which you do a calculation.

ANOVA

- Analysis of variance is a statistical technique used for comparing the means of different samples and deciding whether they are drawn from the same population or different populations.
- Main Question: Do the (means of) the quantitative variables depend on which group (given by categorical variable) the individual is in?
- The ANOVA F-statistic is a ratio of the Between Group Variation divided by the Within Group Variation:

$$F = \frac{Between}{Within} = \frac{MSTR}{MSE}$$

- A large F is evidence against H0, since it indicates that there is more difference between groups than within groups.

Non - Parametric tests

- Deals with enumerative data
- Does not deal with specific population parameters
- Does not require assumptions about specific population distributions (in particular , the assumption of normality)
- Non-Parametric tests ignores the magnitude of information contained in observations
- Use either frequencies or ranks (categorical or ordinal)
- Non-parametric tests are called ''non-parametric'' because they do not make any assumption about a population parameter.
- In other words, when we apply a non-parametric test we do not have to make assumptions about mean of a population, its variance or background probability distribution.
- Thus, non-parametric tests are not as powerful as parametric tests they are of more general application and are available when the parametric tests fail.
- Basically, there is at least one non-parametric equivalent for each parametric general type of test
- Non-Parametric tests broadly fall into the following categories:
- Tests of differences between independent samples: The Mann-Whitney U test (t-test for independent samples), The Kruskal-Wallis H test (ANOVA), The Kolmogorov-Smirnov test (t-test for independent samples)
- Tests of differences between dependent samples: Wilcoxon Mann-Whitney Test (t-test for independent samples)

Some Commonly Used Normal Statistical & NonParametric		
Normal theory based test	**Corresponding nonparametric test**	**Purpose of test**
t test for independent samples	Mann-Whitney U test; Kolmogorov-Smirnov 2 sample test	Compares two independent samples
Paired t test	Wilcoxon matched pairs signed-rank test	Examines a set of differences
One way analysis of variance (F test)	Kruskal-Wallis analysis of variance by ranks	Compares three or more groups
Two way analysis of variance	Friedman Two way analysis of variance	Compares groups classified by two different factors