GPGN409/509 homework | **Bayesian yacht positioning**

your name: | November 3, 2024

You are sailing your new yacht in an area surrounding by dangerous land masses. One night, you find that your GPS navigation is broken while you are surrounded by fog. In order to seek help and avoid crashing into land, you decide to find your position by solving an INVERSE PROBLEM using information from the many light beacons surrounding you.

Using a catalog you carry onboard, you know precisely the positions x_b, y_b of all light beacons in the area, Figure 1, and you can identify them using their light patterns documented in the catalog. You also know that the light beacons generate light of the same intensity $i_R = 1$ a.u. (arbitrary units). You also have an idea of your intended sailing path (illustrated by the dashed line in Figure 1), and you are pretty sure that you did not deviate a lot from this path while drifting without accurate positioning, although you do not know how far off course you might be.

You also know that the beacons are not colimated, i.e., their light spreads evenly in all directions in a plane, and the observed light intensity i is related to the reference light intensity i_R by

$$i = \frac{r_o}{r_o + r} i_R \ , \tag{1}$$

where r is the distance from the light beacon to your location, and r_o is a reference distance at which you know the light intensity halves, e.g., $r_o = 1$ km. Measuring i, you can find r, and having access to multiple light beacons allows you to find your position. The problem is that the weather is pretty foggy, and therefore the observed light intensity is likely lower than what you would observe in clear weather, and thus your distance calculation is most likely inaccurate. Also, you are observing the light intensity using a crude device that gives you measurements with variable uncertainty. You take light intensity measurements i from N beacons, and estimate their uncertainty s on the fly. The data are summarized in the attached datafile, with positions reported in km and light intensity in a.u.

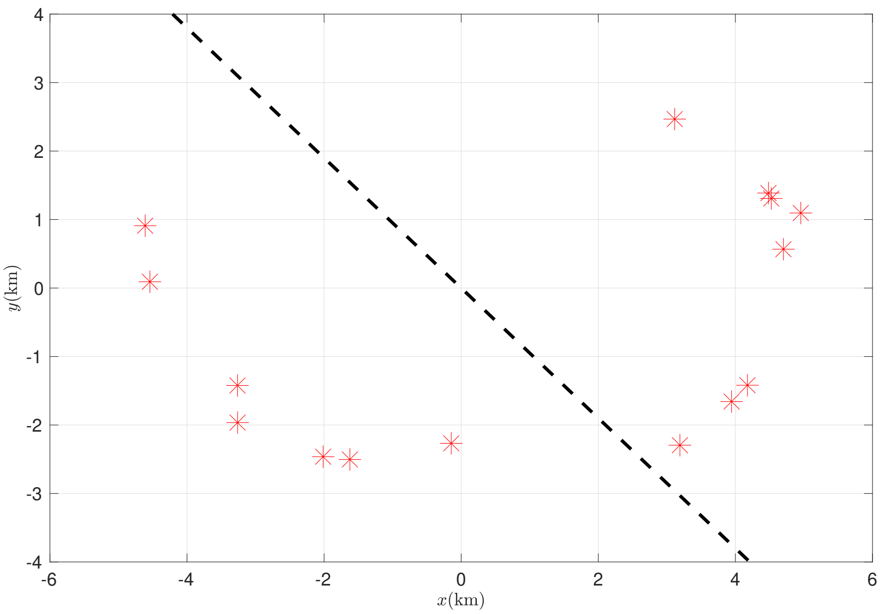

Figure 1: Geometry of the yacht location problem. The stars indicate the light beacons; the dashed line indicates the intended sailing trajectory direction.

You decide that given the uncertainties, the best strategy you could use to solve this problem is through a Bayesian INVERSE PROBLEM. You also decide that a good assumption for your theoretical prediction of light intensity at a given position could be described by exponential decay relative to the light intensity that you would predict in clear weather, i_T, i.e., the probability density function is

$$f(i) = e^{-k|i - i_T|} \tag{2}$$

for all values of light intensity $i \leq i_T$, and zero otherwise. You do not know the exponential decay constant, but you assume that it is approximately $k = 20$ a.u.$^{-1}$.

Your goal is to use all available information to locate the position of your yacht and to characterize its location uncertainty. Your assignment is to solve the problem in two different ways:

1. **Individual inversion**: Find the yacht position by solving a Bayesian INVERSE PROBLEM based on light intensity measurements from each beacon independently.
2. **Cummulative inversion**: Find the yacht position by solving a cascading Bayesian INVERSE PROBLEM, i.e., progressively refine the position of the yacht using light from an increasing number of beacons.

Include plots to illustrate your prior, theoretical and posterior probability density functions for a representative case and evaluate the center and dispersion of the model posterior marginal probability density function. Make assumptions whenever necessary, but motivate your choices.

Extra credit:

1. Redo the Bayesian inversion using other distributions for the various PDFs. Motivate your choices and compare your inversion results for the different distributions.
2. Generate an animation illustrating how the posterior marginal model probability density function changes as you include more light beacons in the Bayesian INVERSE PROBLEM.

N.B. This is an individual assignment – your work is subject to the Mines Academic Integrity policy.

INSTRUCTIONS

FORMAT

- Submit the assignment to Canvas as a standalone **Jupyter notebook**.
- Make sure to run **Kernel/Restart & Run All** in Jupyter before submission.

CLARITY

- Include text documenting your reasoning and how you approached the solution.
- Show all intermediate mathematical derivation steps, if applicable.
- Include figures demonstrating the solution and explain their meaning.

PROGRAMMING

- Include detailed comments documenting the functionality of your codes.
- Organize your programs in clear functional blocks.
- Isolate repeated code in functions. Provide unit tests for all defined functions.
- Define and initialize all variables; indicate in comments their physical units.

POLICIES

- Incomplete or incorrect answers receive partial credit at the discretion of the grader.
- Submissions lose 25%/day if late for two days and are not graded afterward.
- Multiple submissions to Canvas are allowed, but only the last one is graded.

GRADING RUBRIC

Individual inversion (50 pts)

- Explain in detail your inversion procedure. List and justify all assumptions. (10 pts)
- For one beacon:
  - Plot the prior marginal model probability density function. (5 pts)
  - Plot the prior joint probability density function. (5 pts)
  - Plot the theoretical joint probability density function. (5 pts)
  - Plot the posterior joint probability density function. (5 pts)
  - Plot the posterior marginal model probability density function. (5 pts)
  - For the model posterior marginal probability density function, find the mean, standard deviation, and correlation coefficient. (5 pts)
- Repeat the inversion and plot the model posterior marginal probability density function for 3 different light beacons. Discuss your results and explain the shape of the posterior marginal model probability density functions. Plot the corresponding beacon location used for each inversion. (10 pts)

Cumulative inversion (30 pts)

- Explain in detail your inversion procedure. List and justify all assumptions. (5 pts)
- Plot the posterior marginal model probability density function using 5, 10, and 15 light beacons. (15 pts)
- For each marginal model probability density function, find the mean, standard deviation, and correlation coefficient. Plot the corresponding beacon locations used for each inversion. (10 pts)

Code (20 pts)

Include all codes used with comments to explain their functionality.

Extra credit (25 pts)

- Solve the INVERSE PROBLEMwith different assumptions about the distributions. Motivate your choices and compare your results. (15 pts)
- Generate an animation illustrating the posterior marginal model probability density functionas a function of the number of beacons. (10 pts)