Artificial Intelligence 1 – WS 2020/2021
Assignment 7: First Order Logic
– Given Jan. 14., Due Jan. 24. –

Problem 7.1 (Induction) 30 pt

Use structural induction on terms and formulas to define a function C that maps every term/formula to the number of occurrences of free variables. For example, $C(\forall x.P(x,x,y,y,z)) = 3$ because the argument has 2 free occurrences of y and 1 of z.

Hint: Use an auxiliary function $C'(V,A)$ that takes the set V of bound variables and a term/formula A. Define C' by structural induction on A. Then define $C(A) = C'(\emptyset, A)$.

Problem 7.2 (First-Order Semantics) 30 pt

Let $=\in \Sigma_2^p$, $P \in \Sigma_1^p$ and $+ \in \Sigma_2^f$. We use the semantics of first-order logic without equality.

Prove or refute the following formulas semantically. That means you show that $I_\varphi(A) = T$ for all models I and assignments φ (without using a proof calculus) or to give some I, φ such that $I_\varphi(A) = F$.

1. $\forall X.\forall Y. = (+(X,Y), +(Y,X))$
2. $\exists X.(P(X) \Rightarrow \forall Y.P(Y))$
3. $P(Y) \Rightarrow \exists X.P(X)$

Problem 7.3 (Natural Deduction) 40 pt

Let $\leq, \in \Sigma_2^p$, $P \in \Sigma_1^p$, $+ \in \Sigma_2^f$ and $- \in \Sigma_1^f$. We use the semantics and natural deduction of first-order logic.

Prove the following formulas in Natural Deduction:

1. $\forall X.(X \leq -X \Rightarrow \exists Y.X \leq Y)$
2. $\exists X.(P(X) \Rightarrow \forall Y.P(Y))$

Hint: The second formula requires the law of the excluded middle. Try proving the lemma $(\neg\forall Y.P(Y)) \Rightarrow \exists Y.\neg P(Y)$ first and use that in your actual proof.
