Problem 7.2 (First-Order Semantics) 30 pt

Let $= \in \Sigma^2_p$, $P \in \Sigma^p_1$ and $+ \in \Sigma^f_2$. We use the semantics of first-order logic without equality. Prove or refute the following formulas semantically. That means you show that $I_\varphi(A) = T$ for all models I and assignments φ (without using a proof calculus) or to give some I, φ such that $I_\varphi(A) = F$.

1. $\forall X.\forall Y. = (+(X, Y), +(Y, X))$
2. $\exists X.(P(X) \Rightarrow \forall Y.P(Y))$
3. $P(Y) \Rightarrow \exists X.P(X)$

We make φ be any value function.

Question 1

We use Not valid - Counter-model:
We make $I\varphi(=)$ be the empty relation on an arbitrary domain.

Question 2

$I\varphi(\exists X.(P(X) \Rightarrow \forall Y.P(Y))) = \top$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi((P(a) \Rightarrow \forall Y.P(Y))) = \top$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi(\neg(P(a) \wedge \neg\forall Y.P(Y))) = \top$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi(P(a) \wedge \neg\forall Y.P(Y)) = \perp$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi(P(a)) = \perp$ or $I\varphi(\neg\forall Y.P(Y)) = \perp$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi(P(a)) = \perp$ or $I\varphi(\forall Y.P(Y)) = \top$

$\Leftrightarrow \exists a \in D_I$ s.t. $I\varphi(P(a)) = \perp$ or $\forall b \in D_I : I\varphi(P(b)) = \top$

Question 3

$I_\varphi(P(Y) \Rightarrow \exists X.P(X)) = \top$

$\Leftrightarrow I_\varphi(\neg(P(Y) \land \neg\exists X.P(X))) = \top$

$\Leftrightarrow I_\varphi(P(Y) \land \neg\exists X.P(X)) = \bot$

$\Leftrightarrow I_\varphi(P(Y)) = \bot$ or $I_\varphi(\neg\exists X.P(X)) = \bot$

$\Leftrightarrow I_\varphi(P(Y)) = \bot$ or $I_\varphi(\exists X.P(X)) = \top$

$\Leftrightarrow I_\varphi(P)(\varphi(Y)) = \bot$ or $\exists a \in D_I$ s.t. $I_\varphi(P(a)) = \top$