Chapter 9:
Tests of Hypothesis for a Single Sample

MAS291 - STATISTICS & PROBABILITY

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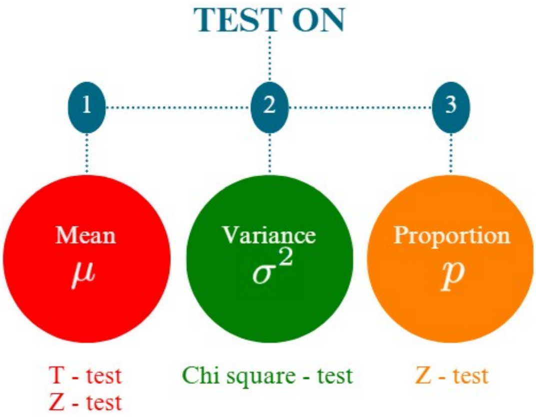
Common Parameters in Hypothesis Testing



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2 Hypothesis Testing

Definition: A statistical hypothesis is a statement about the parameters of one or more populations.

Example:

Null hypothesis H_0: $\mu = 0.3$ seconds (authorized use).

Alternative hypothesis H_1: $\mu \neq 0.3$ (unauthorized use).

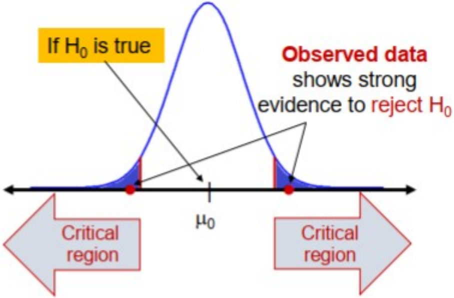
Observed data: 0.29, 0.35, 0.27, 0.28, 0.37, 0.32, 0.32, 0.34, 0.28, 0.34, 0.32, 0.29, 0.30, 0.28, 0.41, 0.31, 0.36, 0.32, 0.29, 0.35.

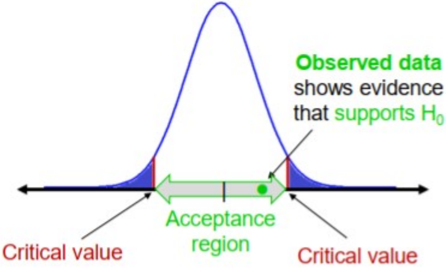
- $\bar{x}$ closes to 0.3 $\Longrightarrow$ Does not conflict with H_0.
- $\bar{x}$ is considerably different from 0.3 $\Longrightarrow$ An evidence in support of H_1.

Statistical hypothesis

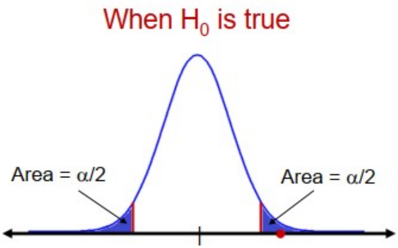
2 Hypothesis Testing

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$





Significance level α and Types of errors



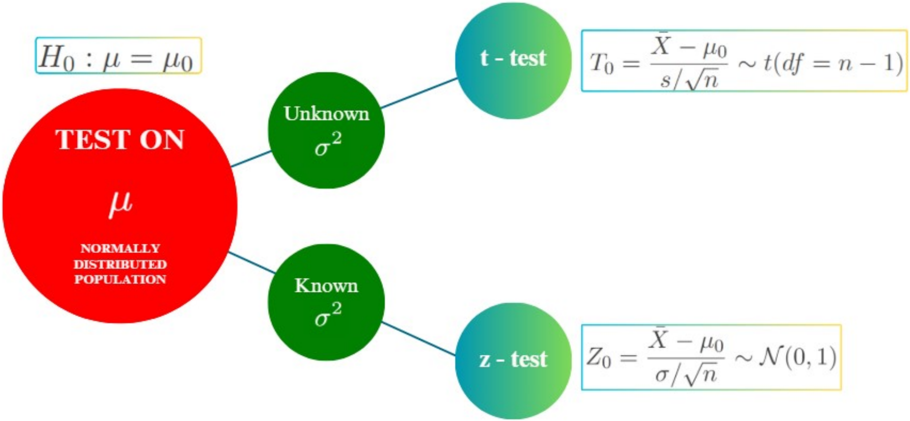
If observed data falls in the critical region (probability $= \alpha/2 + \alpha/2 = \alpha$)
$\Longrightarrow$ Reject H_0
α: significance level, or the α-error (usual α is in $[0.01, 0.1]$)

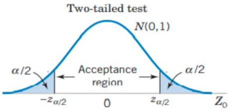
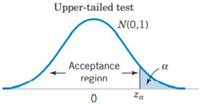
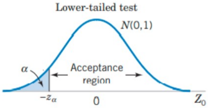
TABLE 9.1 Decisions in Hypothesis Testing

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	No error	Type II error
Reject H_0	Type I error	No error

$$\begin{aligned}\alpha &= \mathrm{P}(\text{type I error})\\ &= \mathrm{P}(\text{reject } H_0 \text{ when } H_0 \text{ is true}).\\ \beta &= \mathrm{P}(\text{type II error})\\ &= \mathrm{P}(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}).\end{aligned}$$

Hypothesis Tests on the Mean



Two-tailed Hypotheses	One-tailed Hypotheses	
$H_0 : \mu = \mu_0$ $\mathrm{H}_1 : \mu \neq \mu_0$	$H_0 : \mu = \mu_0$ $\mathrm{H}_1 : \mu \geq \mu_0$	$H_0 : \mu = \mu_0$ $\mathrm{H}_1 : \mu \leq \mu_0$
		
$\lvert z_0 \rvert > z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Reject H_0

Two-Sided Test (Variance known) - Traditional method

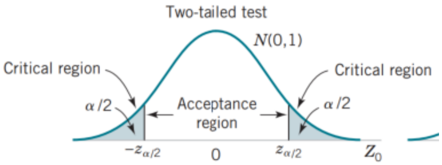
- Step 1: Establishing

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

- Step 2: Compute test-statistic

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

- Step 3: Conclusion
  1. Accept H_0 if $-z_{\alpha/2} \leq z_0 \leq z_{\alpha_2}$
  2. Reject H_0 if $z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$



Exercise

2 Hypothesis Testing

Air crew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. The experimenter decides to specify a type I error probability or significance level of $\alpha = 0.05$ and selects a random sample of $n = 25$ and obtains a sample average burning rate of $\overline{x} = 51.3$ centimeters per second. What conclusions should be drawn?

$H_0 : \mu = \mu_0$

$$T_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t(df = n-1)$$

$$Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

Solution

Establishing

$$H_0 : \mu = 50$$
$$H_1 : \mu \neq 50$$

Compute test-statistic

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

We reject H_0 since $z_0 > z_{\alpha/2} (3.25 > 1.96)$

Two-Sided Test - P-value method

Definition: The P-value (computed from observed data) is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data (Usual significance levels $\alpha \in [0.01, 0.1]$).

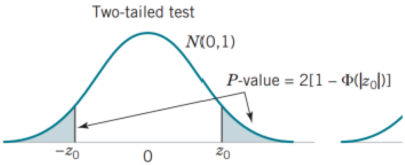
- Step 1: Establishing

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

- Step 2: Compute test-statistic

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

- Step 3: Find P-value and conclusion
  P-value $= 2(1 - \phi(|z_0|))$
  1. Accept H_0 if P-value $\geq \alpha$
  2. Reject H_0 if P-value $< \alpha$



Exercise

2 Hypothesis Testing

Air crew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. The experimenter decides to specify a type I error probability or significance level of $\alpha = 0.05$ and selects a random sample of $n = 25$ and obtains a sample average burning rate of $\overline{x} = 51.3$ centimeters per second. What conclusions should be drawn?

Solution

2 Hypothesis Testing

Establishing

$$H_0 : \mu = 50$$
$$H_1 : \mu \neq 50$$

Compute test-statistic

$$z_0 = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

P-value=$2(1 - \phi(3.25)) = 0.0012$
We reject H_0 since P-value $< \alpha$ $(0.0012 < 0.05)$.

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T-test on μ of a normal dist., σ^2 unknown

Null hypothesis:	$H_0 : \mu = \mu_0$	
Test statistic:	$T_0 = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$	
Alternative Hypothesis	**P-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1 : \mu \neq \mu_0$	Probability above $\|t_0\|$ & below $-\|t_0\|$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1 : \mu > \mu_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1 : \mu < \mu_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

Z-test on μ, σ^2 Known

Null hypothesis:	$H_0 : \mu = \mu_0$	
Test statistic:	$Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$	
Alternative Hypotheses	**P-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1 : \mu \neq \mu_0$	Probability above $\lvert z_0 \rvert$ and below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(\lvert z_0 \rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1 : \mu > \mu_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1 : \mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

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4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Null hypothesis:	$H_0 : \sigma^2 = \sigma_0^2$
Test statistic:	$\chi_0^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$
Alternative Hypothesis	**Rejection Criteria**
$H_1 : \sigma^2 \neq \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$
$H_1 : \sigma^2 > \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha,n-1}$
$H_1 : \sigma^2 < \sigma_0^2$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$

χ^2-test on σ^2 and σ of a Normal Distribution

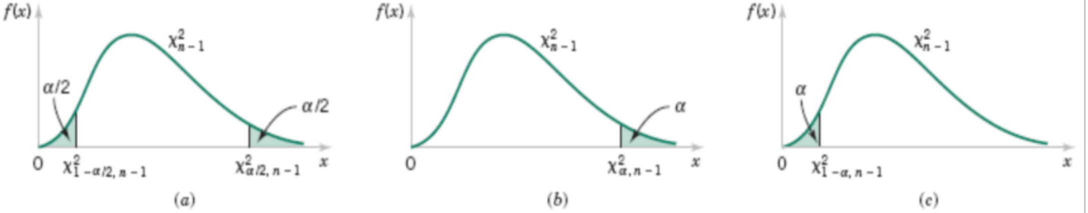

Figure 9-11 Reference distribution for the test of H_0: $\sigma^2 = \sigma_0^2$ with critical region values for (a) H_1: $\sigma^2 \neq \sigma_0^2$, (b) H_1: $\sigma^2 > \sigma_0^2$, and (c) H_1: $\sigma^2 < \sigma_0^2$.

An automated filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume exceeds 0.01 (fluid ounces)^2, an unacceptable proportion of bottles will be underfilled or overfilled. Is there evidence in the sample data to suggest that the manufacturer has a problem with underfilled or overfilled bottles? Use $\alpha = 0.05$, and assume that fill volume has a normal distribution.

Solution

4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

1. Establishing

$$H_0 : \sigma^2 = 0.01$$
$$H_1 : \sigma^2 > 0.01$$

2. Test statistic

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{19(0.0153)}{0.01} = 29.07$$

3. Conclusion

   $\chi_0^2 = 29.07 < \chi^2_{0.05,19} = 30.14 \Rightarrow$ accept H_0.

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5 Tests on a Population Proportion

Z-test on a Population Proportion

We wish to test:

$$H_0 : p = p_0$$
$$H_1 : p \neq p_0$$

X: X be the number of observations in a random sample of size n that belongs to the class associated with p $\Longrightarrow X \sim \mathcal{N}[np, np(1-p)]$,
Suppose H_0 is true, we use the test statistic:

$$Z_0 = \frac{X - np_0}{\sqrt{np_0(1-p_0)}}$$

Summary of Approximate Tests on a Binomial Proportion

Null hypotheses	$H_0 : p = p_0$	
Test statistic	$Z_0 = \dfrac{X - np_0}{\sqrt{np_0(1-p_0)}}$	
Alternative Hypotheses	**P-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1 : p \neq p_0$	Probability above $\lvert z_0 \rvert$ and below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(\lvert z_0 \rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1 : p > p_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1 : p < p_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Exercise

A semiconductor manufacturer produces controllers used in automobile engine applications. The customer requires that the process fallout or fraction defective at a critical manufacturing step not exceed 0.05 and that the manufacturer demonstrate process capability at this level of quality using $\alpha = 0.05$. The semiconductor manufacturer takes a random sample of 200 devices and finds that four of them are defective. Can the manufacturer demonstrate process capability for the customer?

Solution

5 Tests on a Population Proportion

1. Establishing

$$H_0 : p = 0.05$$
$$H_1 : p < 0.05$$

2. Test statistic

$$z_0 = \frac{x - np_0}{\sqrt{np_0(1 - p_0)}} = \frac{4 - 200(0.05)}{\sqrt{200(0.05)(0.95)}} = -1.95$$

3. Conclusion
   $z_0 = -1.95 < -z_{0.05} = -1.645 \Rightarrow$ reject H_0

Q&A

Thank you for listening!