Chapter 2: Probability

MAS291 - STATISTICS & PROBABILITY

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Section 1.1

Sample spaces

Definition 1.1

Random experiment: An experiment that can result in different outcomes, even though it is repeated in the same manner every time.
Sample space (S) is the set of all possible outcomes of a **random experiment**.
An event is a subset of the sample space of a random experiment.

Example 1.1.

- Random experiment: flipping a coin three times (let H = heads, T = tails)
- The sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

- The event of getting at least two heads is $\{HHH, HHT, THH, HTH\}$.

Basis set operations

1. Union (Biến cố tổng): The union of two events $(E_1 \cup E_2)$ is the event that consists of all outcomes that are contained in either of the two events.
2. Intersection (Biến cố tích): The intersection of two events $(E_1 \cap E_2)$ is the event that consists of all outcomes that are contained in both of the two events.
3. Complement (Biến cố bù/đối lập): The complement an event E $(\bar{E}/E')$ is the set of outcomes in the sample space that are not in the event.
4. Mutually exclusive (Biến cố xung khắc): Two events, denoted as A and B are said to be mutually exclusive if that $E_1 \cap E_2 = \emptyset$.

Venn Diagram

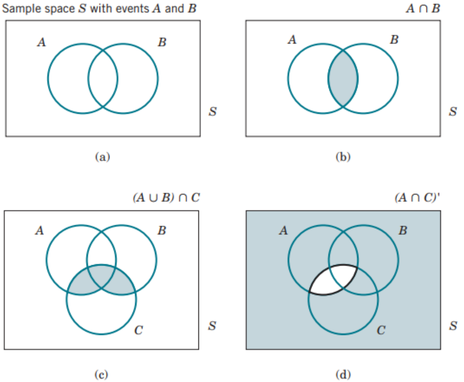


FIGURE 2.6

Venn diagrams.

FIGURE 2.7

Mutually exclusive events.

Important properties

The distributive law for set operations implies that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \quad \text{and} \quad (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

DeMorgan's laws imply that

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

Also, remember that

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A$$

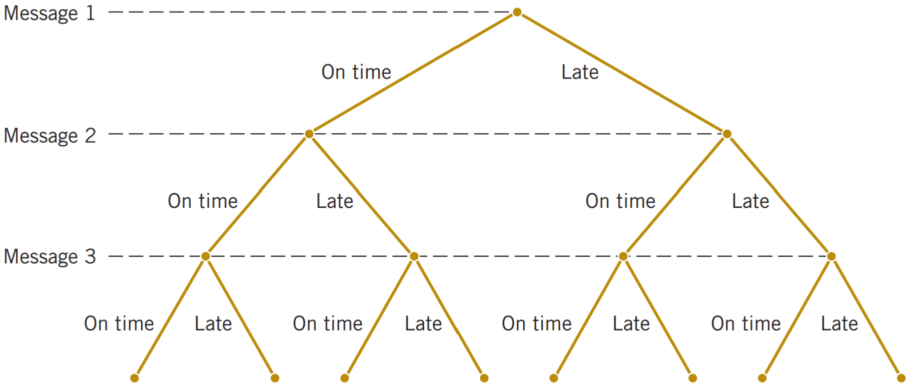
Tree diagram can be used to describe sample spaces graphically as follows:

- When a sample space can be constructed in several steps of stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree.
- Each of the ways of completing the second step can be represented as n_2 branches starting from the ends of the original branches, and so forth.

Example 1.2. Each message in a digital communication system is classified is as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.
Each message can either be received on time or late. The possible results for three messages can be displayed by eight branches in the three diagram.

Solution

1 Sample spaces and events



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Section 2.1

Axioms of probability

Definition 2.1

Equally Likely Outcomes: (đồng khả năng) Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is 1/N.
Probability of an Event: For a discrete sample space, the probability of an event E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

Example 2.1. A young family plans to have two children. What is the probability of two girls?

Definition 2.2

Equally Likely Outcomes: (đồng khả năng) Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is 1/N.
Probability of an Event: For a discrete sample space, the probability of an event E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

Example 2.2. A young family plans to have two children. What is the probability of two girls?

Solution.
Solution 1. (Wrong)

$$S = \{GG, BG, BB\} \Longrightarrow P(GG) = 1/3$$

Solution 2. (Right)

$$S = \{GG, BG, GB, BB\} \Longrightarrow P(GG) = 1/4$$

Example 2.3. A random experiment can result in one of the outcomes a, b, c, d with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event a, b, B the event b, c, d, and C the event d.

a/ $P(A)$, $P(B)$, $P(C)$.

b/ $P(A')$, $P(B')$, $P(C')$.

c/ $P(A \cap B)$, $P(A \cup B)$, $P(A \cap C)$.

Example 2.4. A random experiment can result in one of the outcomes a, b, c, d with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event a, b, B the event b, c, d, and C the event d.

a/ $P(A)$, $P(B)$, $P(C)$.

b/ $P(A')$, $P(B')$, $P(C')$.

c/ $P(A \cap B)$, $P(A \cup B)$, $P(A \cap C)$.

Solution.

a/ $P(A) = 0.1 + 0.3 = 0.4$, $P(B) = 0.3 + 0.5 + 0.1$, $P(C) = 0.1$.

b/ $P(A') = 0.6$, $P(B') = 0.1$, $P(C') = 0.9$.

c/ $P(A \cap B) = P(\{b\}) = 0.3$, $P(A \cup B) = P(\{a, b, c, d\}) = 1$,
$P(A \cap C) = P(\emptyset) = 0$.

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:
If S is the sample space and E is any event in a random experiment.

- P(S)=1
- $0 \leq P(E) \leq 1$
- For any two events E_1, E_2 with $E_1 \cap E_2 = \emptyset$ then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Probability of a Union

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

If A and B are **mutually exclusive** events: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Three or More Events

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = & P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\ & - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \end{aligned}$$

Example 2.5. Roll a six-sided die twice. The sample space is given by
$S = \{(1,1), (1,2), (1,3), \ldots, (4,6), (5,6), (6,6)\}$.
Let $A = \{(X,Y) | X = Y\}$ and $B = \{(X,Y) | X + Y > 7\}$.
Find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$.

Probability of a Union

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

If A and B are **mutually exclusive** events: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
Three or More Events

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = & P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) \\ & - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \end{aligned}$$

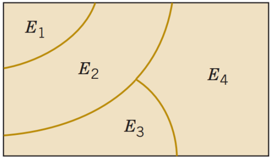
Example 2.6. Roll a six-sided die twice. The sample space is given by
$S = \{(1,1),(1,2),(1,3),\ldots,(4,6),(5,6),(6,6)\}$.
Let $A = \{(X,Y)|X = Y\}$ and $B = \{(X,Y)|X + Y > 7\}$.
Find $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$.

Solution. $P(A) = 1/6$, $P(B) = 15/36$, $P(A \cap B) = 3/36$, $P(A \cup B) = 1/2$.

Unions of Events and Addition Rules

A collection of events $E_1, E_2, ..., E_k$ is said to be mutually exclusive if for all pairs, $E_i \cap E_j = \emptyset$
For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup ... \cup E_k) = P(E_1) + P(E_2) + ... + P(E_k)$$



Example 2.7. If A, B, and C are mutually exclusive events with $P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.4$. Determine the probabilities

a/ $P(A \cap B)$

b/ $P(A \cap B \cap C)$

c/ $P(A \cup B \cup C)$

d/ $P[(A \cup B) \cap C]$

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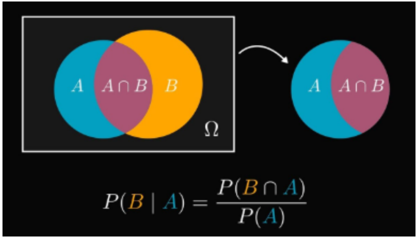
Definition 3.1

The conditional probability of an event B given an event A

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \ P(A) > 0$$



Example 3.1. Toss a coin twice.
The sample space is
$S = \{HH, HT, TH, TT\}$.
Let $A = \{at\ least\ one\ head\ occurs\}$ and
$B = \{a\ head\ and\ tail\ occur\}$.
$P(A) = 3/4, P(B) = 1/2, P(A \cap B) = 2/4$.
What are the probabilities $P(A|B)$ and $P(B|A)$?

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Multiplication rule

$$P(A \cap B) = P(B|A).P(A) = P(A|B).P(B)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_2 \cap A_3|A_1).P(A_1) = P(A_1).P(A_2|A_1).P(A_3|A_2 \cap A_1)$$

Example 4.1. Consider an urn with 10 balls inside, 7 of which are red and 3 of which are blue. Select 3 balls successively from the urn. Let A_1 = "1st ball is red", A_2 = " 2nd ball is red " and A_3= "3rd ball is red". Find $P(A_1 \cap A_2 \cap A_3)$.

Multiplication rule

$$P(A \cap B) = P(B|A).P(A) = P(A|B).P(B)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_2 \cap A_3|A_1).P(A_1) = P(A_1).P(A_2|A_1).P(A_3|A_2 \cap A_1)$$

Example 4.2. Consider an urn with 10 balls inside, 7 of which are red and 3 of which are blue. Select 3 balls successively from the urn. Let A_1 = "1st ball is red", A_2 = " 2nd ball is red " and A_3= "3rd ball is red". Find $P(A_1 \cap A_2 \cap A_3)$.

Solution. $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_2 \cap A_1) = \frac{7}{10}.\frac{6}{9}.\frac{5}{8}$

Total Probability Rules (Two events)

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$

Total Probability Rule (Multiple Events)) Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B \mid E_1) P(E_1) + P(B \mid E_2) P(E_2) + \cdots + P(B \mid E_k) P(E_k) \end{aligned}$$

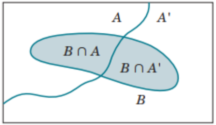
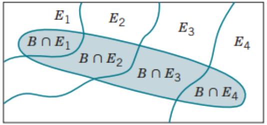


FIGURE 2.13



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$

FIGURE 2.14

Example 4.3. Suppose that in semiconductor manufacturing the probability is 0.10 that a chip that is subjected to high levels of contamination during manufacturing causes a product failure. The probability is 0.005 that a chip that is not subjected to high contamination levels during manufacturing causes a product failure. In a particular production run, 20% of the chip are subject to high levels of contamination. What is the probability that a product using one of these chips fails?

Example 4.4. Suppose that in semiconductor manufacturing the probability is 0.10 that a chip that is subjected to high levels of contamination during manufacturing causes a product failure. The probability is 0.005 that a chip that is not subjected to high contamination levels during manufacturing causes a product failure. In a particular production run, 20% of the chip are subject to high levels of contamination. What is the probability that a product using one of these chips fails?

Solution. Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination. The requested probability is P(F), and the information provided can be represented as

- P(F|H)=0.1, P(F|H')=0.005
- P(H)=0.2, P(H')=0.8

$\Longrightarrow P(F) = P(F|H)P(H) + P(F|H')P(H') = 0.1(0.2) + 0.005(0.8) = 0.024$

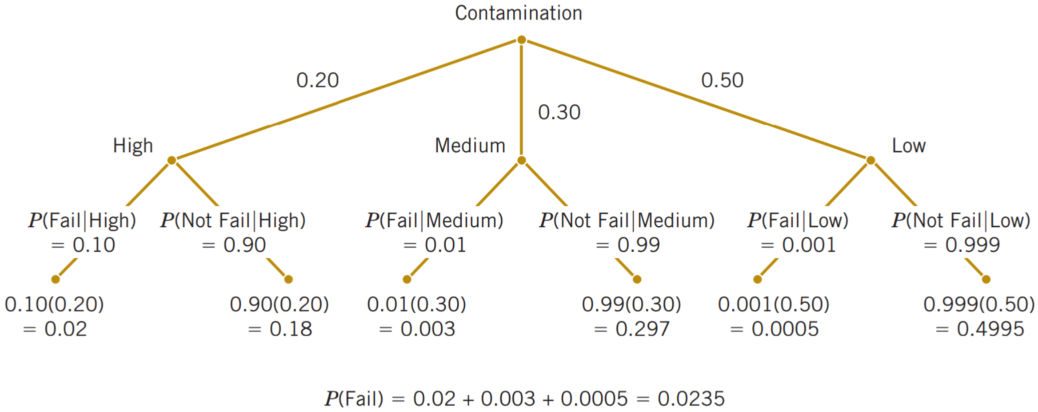
Example 4.5. Semiconductor Failures Continuing with semiconductor manufacturing, assume the following probabilities for product failure subject to levels of contamination in manufacturing

Probability of Failure	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails?

Solution

4 Multiplication and total probability rules



Section 4.1

Independence

Definition 4.1

Two events are independent if any one of the following equivalent statements is true

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A).P(B)$

Example 4.6. Toss a fair coin twice. The sample space is
$S = \{HH, HT, TH, TT\}$.
Consider events $A = \{HT, TH\}$, $B = \{HT, TT\}$, $C = \{HH\}$.
$P(A|B) = 1/2 = P(A) \Longrightarrow$ A and B are independent.
$P(C|A) = 1/2 \neq 1/4 = P(C) \Longrightarrow$ A and C are NOT independent.
$P(C \cap B) = 0 \neq 1/8 = P(C)P(B) \Longrightarrow$ B and C are NOT independent

Independence - Example: Series Circuit

Example 4.7. The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?

Independence - Example: Series Circuit

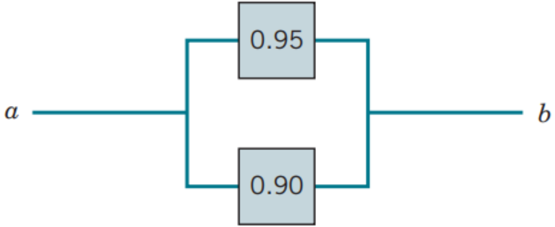
Example 4.8. The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?

Solution. Let L and R denote the events that the left and right devices operate, respectively. There is a path only if both operate. The probability that the circuit operates is

$$P(L \text{ and } R) = P(LR) = P(L)P(R) = 0.80(0.90) = 0.72$$

Independence - Example: Parallel Circuit

Example 4.9. The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Solution

Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. The probability that the circuit operates is

$$P(T \text{ or } B) = 1 - P\left[(T \text{ or } B)'\right] = 1 - P\left(T' \text{ and } B'\right)$$

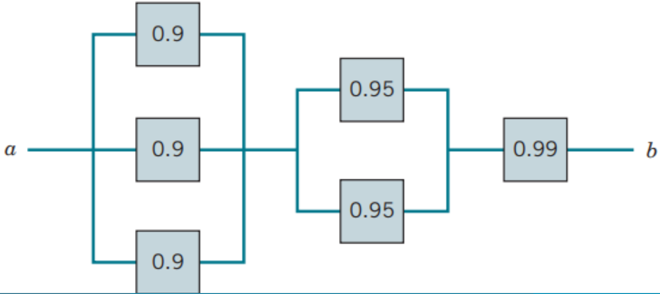
A simple formula for the solution can be derived from the complements T' and B'. From the independence assumption,

$$P\left(T' \text{ and } B'\right) = P\left(T'\right) P\left(B'\right) = (1 - 0.95)(1 - 0.90) = 0.005$$

so

$$P(T \text{ or } B) = 1 - 0.005 = 0.995$$

Example 4.10. The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Solution

The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left.
From the independence and based on the previous example,

$$P(L) = 1 - 0.1^3$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - 0.05^2$$

The probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.
Therefore, with the independence assumption used again, the solution is

$$(1 - 0.1^3)(1 - 0.05^2)(1 - 0.01) = 0.987$$

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From the definition of conditional probability,

$$P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)$$

$$\Longrightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)},\ P(B) > 0$$

Bayes's Theorem

If $E_1, E_2, ..., E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + ... + P(B|E_k)P(E_k)}$$

Example

Example 5.1. Bayesian networks are used on the Web sites of hight – technology manufactures to allow customers to quickly diagnose problems with products. An oversimplified example is presented here.
A printer manufacturer obtained the following probabilities from a database of test results. Printer failures are associated with three types of problems: hardware, software and other (such as connectors), with probabilities 0.1, 0.6, and 0.3, respectively. The probability of a printer failure given a hardware problem is 0.9, given software problem is 0.2 and given any other type of problem is 0.5. If a customer enters the manufacturer's Web site to diagnose, what is the most likely cause of the problem?

Solution

Let the events H, S and O denote a hardware, software and other problem, respectively and let F denote a printer failure. The most likely cause of the problem is the one that corresponds to the largest of P(H/F), P(S|F) and P(O|F). In Bayes'Theorem the denominatior is

$$P(F) = P(H)P(F|H)+P(S)P(F|S)+P(O)P(F|O) = 0.9(0.1)+0.2(0.6)+0.5(0.3) = 0.36$$

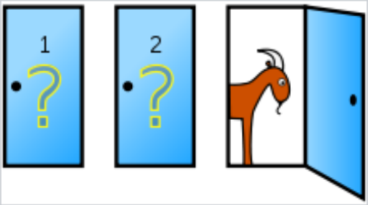
Then

$$P(H|F) = \frac{P(F|H)P(H)}{P(F)} = \frac{0.9(0.1)}{0.36} = 0.25$$

$$P(S|F) = \frac{P(F|S)P(S)}{P(F)} = \frac{0.2(0.6)}{0.36} = 0.333$$

$$P(O|F) = \frac{P(F|O)P(O)}{P(F)} = \frac{0.5(0.3)}{0.36} = 0.417$$

Example 5.2. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?



In search of a new car, the player picks a door, say *1*. The game host then opens one of the other doors, say *3*, to reveal a goat and offers to let the player switch from door *1* to door *2*.

Solution

5 Bayes's Theorem

Prior: P(car behind a door) **P(Car@...)**	**Event: you choose door 1, Monty Hall open door 3** **P(Open door 3 | Car@...)**	**Posterior: probability of car behind a door after the event** **P(Car@... | Opened door 3)**
P(Car@door 1) = 1/3	P(Open door 3 | Car@door 1) = 1/2	P(Car@ door 1 | Opened door 3) = $\frac{(1/3)(1/2)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 1/3$
P(Car@door 2) = 1/3	P(Open door 3 | Car@door 2) = 1	P(Car@ door 2 | Opened door 3) = $\frac{(1/3)(1)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 2/3$
P(Car@door 3) = 1/3	P(Open door 3 | Car@door 3) = 0	P(Car@ door 3 | Opened door 3) = $\frac{(1/3)(0)}{(1/3)(1/2) + (1/3)1 + (1/3)0} = 0$

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6 Random variables

Random variable

A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

$$X : S \to \mathbb{R}$$
$$X(\omega) \in \mathbb{R}$$

Example:

Flipping a coin twice. $\to$ The sample space is $S = \{HH, HT, TH, TT\}$.
$X : S \to \mathbb{R}$
$X(\omega) =$ number of heads in each outcome ω

$\omega \in S$	HH	HT	TH	TT
$X(\omega)$	2	1	1	0

Random variable

A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

A discrete random variable is a random variable with a finite (or countably infinite) range.

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Example

Continuous random variable: electrical current, length, pressure, temperature, time, voltage, voltage, weight,...

Discrete random variable: number of scratch on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error,...

Example

Flipping a coin twice. $\rightarrow$ The sample space is $S = \{HH, HT, TH, TT\}$.

$X : S \rightarrow \mathbb{R}$

$X(\omega)$ = number of heads in each outcome ω

$\omega \in S$	HH	HT	TH	TT
$X(\omega)$	2	1	1	0

Discrete random variable

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Counting techniques

- Multiplication rule
- Permutation
- Permutation of subsets
- Permutation of similar objects

Multiplication Rule

Assume an operation can be described as a sequence of k steps, and

- the number of ways of completing step 1 is n_1
- the number of ways of completing step 2 is n_2
- the number of ways of completing step 3 is n_3 and so forth

The total number of ways of completing the operation is

$$n_1 \times n_2 \times ... \times n_k$$

Permutations

A permutation of the elements is an ordered sequence of the elements.

The number of permutations of n different elements is

$$n! = 1 \times 2 \times 3 \times ... \times (n-1) \times n$$

Example

$S = \{a, b, c\}$ has six permutations: $abc, acb, bac, bca, cab, cba$.

Permutations of subsets

The number of permutations of subsets of r elements selected from a set of n different elements is

$$P_n^r = n \times (n-1) \times (n-2) \times ... \times (n-r+1) = \frac{n!}{(n-r)!}$$

Example

A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many different designs are possible?

Each design consists of selecting a location from the eight locations for the first component, a location from the remaining seven the second component, a location from the remaining six for the third component and a location for the remaining five for the fourth component.

Therefor $5.6.7.8 = \dfrac{8!}{4!} = 1680$ different designs are possible.

Permutations of similar objects

The number of permutations of $n = n_1 + n_2 + \ldots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, ... , and n_r are of an r^{th} type is

$$\frac{n!}{n_1!n_2!...n_r!}$$

Example - Hospital Schedule

A hospital operating room needs to schedule three knee surgeries and two hip surgeries in a day. We denote a knee and hip surgery as k and h, respectively. The number of possible sequences of three knee and two hip surgeries is

$$\frac{5!}{2!3!} = 10$$

The 10 sequences are easily summarized:
{kkkhh, kkhkh, kkhhk, khkkh, khkhk, khhkk, hkkkh, hkkhk, hkhkk, hhkkk}

Combinations

The number of combinations, subsets of r elements that can be selected from a set of n elements, is denoted as $\binom{n}{r}$ or C_n^r and

$$C_n^r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example - Printed Circuit Board Layout

A printed circuit board has eight different locations in which a component can be placed. If five identical components are to be placed on the board, how many different designs are possible?

Solution

Each design is a subset of size of five from the eight locations that are to contain the components. Therefore, the number of possible designs is

$$C_8^5 = \frac{8!}{5!3!} = 56$$

Q&A

Thank you for listening!