Gödel's Incompleteness Theorems

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Kurt Gödel

A quick reminder of Axioms

Definition (Axiom)

Statements that are true without a formal proof of them.
For example:

$$x = y \land y = z \implies x = z$$

"It is possible to draw a straight line from any point to any other point"

- Any mathematical system starts out with a set of axioms

Completeness

Definition (Complete)

A set of axioms is (syntactically, or negation-) complete if, for any statement in the axioms' language, that statement or its negation is provable from the axioms. [1]

Gödel's First Incompleteness Theorem

- If axioms do not contradict each other and are computably enumerable some statements are true, but cannot be proofed.

Definition (Computably Enumerable Language)

A recursively enumerable language is a formal language for which there exists a Turing machine which will enumerate all valid strings of the language.

Gödel's First Incompleteness Theorem

- **The goal is to have a set of axioms that is powerful enough to proof everything in mathematics.**
- The status quo is that we have some axioms that are unproofable. Wouldn't it just make sense to add these axioms to our system, that we have a complete system?
- To answer this question we actually have to take a look at the Gödel numbering.

Gödel Numbering

- Gödel encoded every axiom with a unique natural number.
- He basically allowed mathematics to talk about itself.
- The so called "Gödelisierung" is not limited to axioms. You can encode every word ω in a language L.
- You can compare this to a modern computer, every text you type is encoded, for example in ASCII. In the end it comes even down to an encoding of 0s and 1s.

A short side statement: those numbers can be absolutely huge.

Gödel's First Incompleteness Theorem

- Now back to our first question.
- We encode now the statement "This statement cannot be proved from the axioms".
- Since we can work with numbers now, we can set up an equation.
- **Remember** An equation is **always** true or false, in mathematics.

Gödel's First Incompleteness Theorem

- We start now by saying this equation is false.
- This means that "This statement is provable from the axioms" is true, but a provable statement must be true.
- So now we have started with something which we assumed was false and now we have deduced that it was actually true.
- We have got a contradiction.

Gödel's First Incompleteness Theorem

- Since we are assuming that mathematics is consistent we cannot have contradictions.
- That means it cannot be false.
- We now can conclude that it must be true, since an equation must always be true or false.
- Now we reinterpret what it says "This statement cannot be proved from the axioms". We have now a statement that cannot be proofed with the axioms of mathematics.

Gödel's First Incompleteness Theorem

- This is now really **important** Within a system of mathematics with certain axioms we found a true statement within there which cannot be proved true with that system. We have proofed by working outside the system and looking in.
- Since it is true we can add that as an axiom. It is a true statement, so it will not make something which is consistent inconsistent.

Gödel's First Incompleteness Theorem

- Now back to our question. "Wouldn't it just make sense to add these axioms to our system, that we have a complete system?"
- We have just shown that it would be possible to add new axioms to the system. On the other hand we end up adding endless new axioms to our system. We are stuck in an endless loop.

A Modern Approach

- Now we are getting more formal and take a look at modern proofs and approaches about the theorem.

Parameters of the Proofs

All of the modern proofs of Gödel's theorem do have the following five parameters:

1. the choice of a specific basic formal system T;
2. the choice of a universal computation model on some family U of objects of T, where by such a model I mean any mathematically rigorous definition of the notion of a $c.e.$ [1] set of elements of U (or of a computable function from U to U);

[1] $c.e.$ ... computably enumerable

Parameters of the Proofs

3. the choice of a Gödel numbering, that is, an encoding of the syntax of the theory T by objects in U;
4. a proof of the enumerability of the system T (in the sense of the chosen computation model and the Gödel numbering);
5. a presentation of an example of an expressible non-c.e. set (together with the proof of its expressibility and non-enumerability).

[2]

Parameters of the Proofs - Examples

To prove Gödel's theorem, various authors have considered the following computation models:

1. *c.e.* sets as projections of primitive recursive relations (Gödel);
2. the Herbrand–Gödel computable functions and partial recursive functions (Kleene);
3. elementary formal systems (Smullyan);
4. Turing machines;
5. Σ-definable relations (Ershov) and others.

[2]

Simplified Proofs

We note immediately that many authors of simplified proofs of Gödel's theorem neglect some of these points due to their intuitive clearness, using, as a rule, an informal concept of algorithm and some of the forms of the Church–Turing thesis. For example, the enumerability of the arithmetic PA is intuitively clear. At the same time, an "honest" proof of this statement needs programming in the framework of the chosen computation model, that is, significant technical work in general.

[2]

What did Gödel actually do?

- Choosing the apparatus of primitive recursive functions, Gödel managed quite effectively with the problem.
- In our opinion there are still no complete proofs of Gödel's theorem that are essentially simpler than his own proof.

[2]

Gödel's Plan

This proof is based on the technical notion of *representability* of a *function* in a theory T and in the construction of an arithmetical formula asserting its own unprovability. The plan of Gödel's proof can be described as follows: [2]

Gödel's Plan

1. the proof of the fact that the proof predicate $Prf_T(x, y)$ for the theory T is primitive recursive;
2. the proof of the fact that every primitive recursive function is representable in T, which implies the decidability in T of the predicate $Prf_T(x, y)$;

Gödel's Plan

3. the construction of a formula ψ such that

$$T \vdash \psi \leftrightarrow \neg \mathrm{Pr}_T \left(\underline{\ulcorner \psi \urcorner} \right)$$

, where $Prf_T(x)$ stands for the provability formula $\exists y Prf_T(x, y)$, and where we use the representability in T of the substitution function.

The proof is completed with the following argument, which shows that if T is ω-consistent, then the formula ψ is unprovable and irrefutable in T.

[2]

Dive Deeper

If you are interested to see the complete proofs i would suggest to check out the following resources.

- Gödel incompleteness theorems and the limits of their applicability. I by L. D. Beklemishev. Pages 29 - 33. [2]
- Or you can even take a look a the original paper of Gödel himself. [3]

Gödel's Second Incompleteness Theorem

- Now we are going to a look on the mostly overlooked second theorem of Kurt Gödel.

Gödel's Second Incompleteness Theorem

- The first theorem asserts that there is a sentence which is neither provable nor refutable in the theory P under consideration.
- The second theorem asserts that for this sentence one can take a formalization in P of the statement that the theory P itself is consistent.

A not so Formal Explanation

- A consistent math system cannot prove its own consistency

The Parameters of the Proof

1. Recursive arithmetization of the syntax of the language of T.
2. Σ_1-definability of the recursively enumerable predicates.
3. Provability in T of true Σ_1 sentences of the language of arithmetic.
4. Diagonalization.
5. Provability in T of some of the properties of the *Provability* predicate Bew_T.

[4]

The two Take Home Messages are

- **First theorem:** Any sufficiently expressive math system must be either incomplete or inconsistent.
- **Second theorem:** A consistent math system cannot prove its own consistency.

References

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