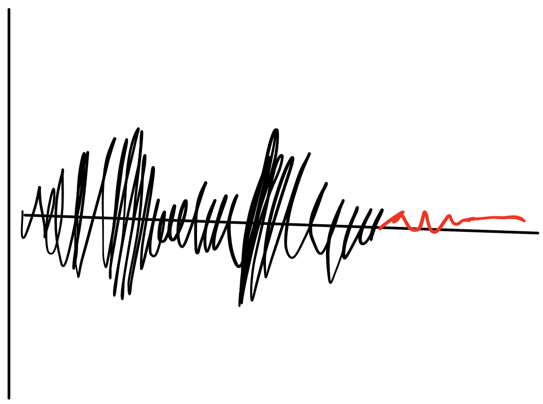
GARCH

Generalized ARCH

Auto-Regressive Conditional Heteroscedasticity



r_t - log return

nonlinearity

$$r_t = \underbrace{\mu_t} + \underbrace{a_t}$$

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} a_{t-i}$$

$$\sigma_t^2 = \underbrace{Var(r_t \mid F_{t-1})} = Var(a_t \mid F_{t-1})$$

1) Exact function of σ_t^2
2) Stochastic models

ARCH-model

Assumptions:

1) a_t - serial uncorrelated, but dependent
2) dependence of a_t - lagged quadratic function

$$a_t = \sigma_t \varepsilon_t \qquad \sigma_t^2 = \underbrace{\alpha_0 + \alpha_1 a_{t-1}^2 + \dots + \alpha_m a_{t-m}^2}$$

ε_t - iid, $E(\varepsilon_t)=0$, $Var(\varepsilon_t)=1$ $\quad \alpha_0 > 0$

$\alpha_i \geq 0 \ \forall i > 0$

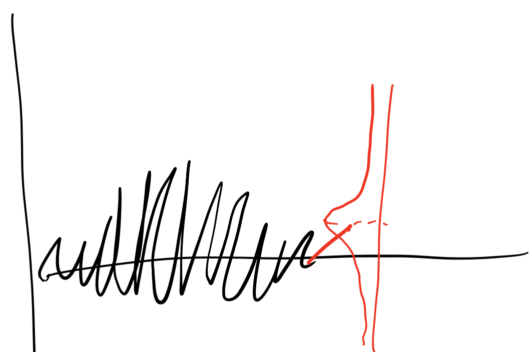
ε_t - N, t,

ARCH(1) $\quad a^2 = \sigma_t^2 \overset{\sim N_{\varepsilon t}\,?}{\varepsilon_t^2}$ $\quad \boxed{a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2}$

$$\underline{a_t = \sigma_t \varepsilon_t}, \quad \sigma_t^2 = \underbrace{\hat{\alpha}_0 + \hat{\alpha}_1 a_{t-1}^2}, \quad \alpha_0 > 0, \alpha_1 \geq 0$$

$$\varepsilon_t \sim N(0, \sigma^2)$$

$$r_t = \mu_t + \sigma_t \underline{\varepsilon_t}$$



$$E(r_t | F_{t-1}) =$$

$$= E(\mu_t | F_{t-1}) + E(\sigma_t | F_{t-1}) \cdot \cancel{E(\varepsilon_t)}$$

$$E(a_t) = E\big(E(a_t | F_{t-1})\big) = E[\sigma_t \underset{=0}{E(\varepsilon_t)}] = 0$$

$$Var(a_t) = E(a_t^2) = E\big[E(a_t^2 | F_{t-1})\big] =$$

$$= E[\alpha_0 + \alpha_1 a_{t-1}^2] = \alpha_0 + \alpha_1 E(a_{t-1}^2)$$

$$Var(a_t) = \alpha_0 + \alpha_1 Var(a_t) \Rightarrow Var(a_t) = \frac{\alpha_0}{1 - \alpha_1}$$

Weaknesses

1) Сложное правдоподобие
2) Positive and negative shocks have same effect.
3) Constraints: $\alpha_1^2 \in [0; \frac{1}{3}]$, complicated for higher orders
4) Not good interpretation
5) likely to overpredict volatility

Building

1) Build mean-model (like ARIMA)
2) Specify ARCH order and fit
3) Check and reestimate

$$a_t = \sigma_t \varepsilon_t, \quad \sigma^2_t = \alpha_0 + \alpha_1 a^2_{t-1}$$

$$\tilde{a}_t = \frac{a_t}{\hat{\sigma}_t} \approx \varepsilon_t$$

$\{\tilde{a}_t\}$ 1) BG
2) Check dist: hist, QQ, skew, kurt ...

GARCH

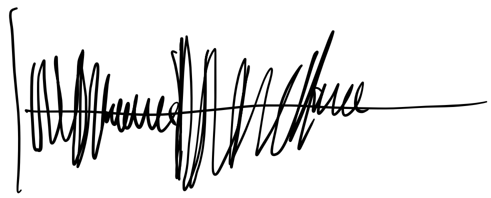
$a_t = \sigma_t \varepsilon_t \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^2$

GARCH(1,1) / (1,2) / (2/1)

$\alpha_0 > 0, \alpha_i \geq 0 \quad \beta_j \geq 0$

Integrated

IGARCH



$a_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1-\beta_1) a_{t-1}$

GARCH-M

$r_t = \mu + \boxed{c}\,\sigma_t^2 + a_t, \quad a_t = \sigma_t \varepsilon_t$

$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \dots$

1) $c > 0$

2) $c < 0$

EGARCH