Multi-particle Basis States:

→ Position not physical, but expectation value of operator at (x,t) is

→ p can be measured in arbitrarily small volume.

Single particle basis state $\{|\vec{k}_1\rangle\}$, 2: $\{|\vec{k}_1, \vec{k}_2\rangle\}$

→ since particles spinless, $|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$

and $\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle = \delta(\vec{k}_1 - \vec{k}_1')\delta(\vec{k}_2 - \vec{k}_2') + \delta(\vec{k}_1 - \vec{k}_2')\delta(\vec{k}_2 - \vec{k}_1')$

$H|\vec{k}_1, \vec{k}_2\rangle = (\omega_{k_1} + \omega_{k_2})|\vec{k}_1, \vec{k}_2\rangle$, $\vec{P}|\vec{k}_1, \vec{k}_2\rangle = (\vec{k}_1 + \vec{k}_2)|\vec{k}_1, \vec{k}_2\rangle$

And vacuum: $\langle 0|0\rangle = 0$, $H|0\rangle = \vec{P}|0\rangle = 0$

Completeness: $I = |0\rangle\langle 0| + \int d^3k |\vec{k}\rangle\langle\vec{k}| + \frac{1}{2!}\int d^3k_1 d^3k_2 |k_1, k_2\rangle\langle k_1, k_2|$

$+ \cdots \Rightarrow |\psi\rangle = \psi_0 |0\rangle + \underbrace{\int d^3k\, \psi(k)|\vec{k}\rangle}_{\text{func of 3 var}} + \frac{1}{2!}\int d^3k_1 d^3k_2 \underbrace{\psi(k_1, k_2)}_{\text{function of 6 var.}} |k_1, k_2\rangle + \cdots$

... interaction term will connect wave function over 2 particle basis to 3, 3 to 4 ... unwieldy

→ Consider periodic box of length L

⇒ discretizes momenta: $\vec{k} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L}\right)$

⇒ write states as occupation representation:

$|\ldots n(\vec{k}), n(\vec{k}') \ldots\rangle$, where $n(k) \equiv$ # particles w/ momentum $\vec{k}$.

$= |n(\cdot)\rangle$

→ <u>Number operator</u> $N(\vec{k})$ such that

$N(\vec{k})|n(\cdot)\rangle = n(\vec{k})|n(\cdot)\rangle \Rightarrow H = \sum_k \omega_k N(\vec{k})$,

$\vec{P} = \sum_k \vec{k} N(\vec{k})$ →

→ taking continuum limit:

$$[a_k, a^\dagger_{k'}] = \delta(\vec{k}-\vec{k}'), \quad [a_k, a_{k'}] = [a^\dagger_k, a^\dagger_{k'}] = 0$$

$$\cancel{\langle k|k\rangle = \langle k'|a_k|0\rangle}$$

$$\cancel{\langle k'|[a_k, a^\dagger_{k'}]|k\rangle = \langle k'|a_k a^\dagger_{k'} - a^\dagger_{k'} a_k|k\rangle}$$

$$\cancel{= \langle k'|a_k a^\dagger_{k'}|k\rangle - \langle k'|a^\dagger_{k'} a_k|k\rangle}$$

$$\cancel{\langle k'|k\rangle = \langle 0|a_{k'} a^\dagger_k|0\rangle = \langle 0|[a_{k'}, a^\dagger_k]|0\rangle}$$

$$\langle k'|k\rangle = \langle 0|a_{k'} a^\dagger_k|0\rangle = \langle 0|[a_{k'}, a^\dagger_k]|0\rangle - \langle 0|a^\dagger_k a_{k'}|0\rangle$$

(since $a_k, a_k^\dagger$ are Hermitian conjugates)

$$= \langle 0|\delta^{(3)}(\vec{k}'-\vec{k})|0\rangle + 0 = \delta^{(3)}(k'-k)\langle 0|0\rangle = \boxed{\delta^{(3)}(k'-k)}$$

$$a^\dagger_k|0\rangle = |\vec{k}\rangle$$

$$H = \sum_k \omega_k a_k^\dagger a_k \to \int d^3k\, \omega_k a_k^\dagger a_k$$

$$\Rightarrow H|\vec{k}\rangle = H a^\dagger_k|0\rangle = \int d^3k'\, \omega_{k'} a^\dagger_{k'} a_{k'} a^\dagger_k|0\rangle$$

$$\cancel{\to a_k a_k a^\dagger_{k'} = a_k[a_k, a^\dagger_{k'}] - a^\dagger_k a^\dagger_{k'} a_k}$$

$$\cancel{= [a^\dagger_k a_k, a^\dagger_{k'}] + a^\dagger_{k'} a^\dagger_k a_k}$$

$$a^\dagger_{k'} a_{k'} a^\dagger_k = [a^\dagger_{k'} a_{k'}, a^\dagger_k] + a^\dagger_k a^\dagger_{k'} a_{k'}$$

$$= a^\dagger_k H$$

$$\Rightarrow H|\vec{k}\rangle = \int d^3p' \left([a^\dagger_{k'} a_{k'}, a^\dagger_k] + a^\dagger_k a^\dagger_{k'} \underbrace{a_{k'}|0\rangle}_{=0}\right)$$

$$\to [a^\dagger_{k'} a_{k'}, a^\dagger_k] = a^\dagger_{k'}[a_{k'}, a^\dagger_k] + \overbrace{[a^\dagger_{k'}, a^\dagger_k]}^{0} a_{k'} \to$$

$$= a^\dagger_{k'}\delta(k'-k) = a^\dagger_{k'}\delta(k-k')$$

$$H|\vec{k}\rangle = \int d^3k' \omega_{k'} a^\dagger_{k'} \delta^{(3)}(\vec{k}-\vec{k}')|0\rangle$$

$$= \omega_k a^\dagger_k |0\rangle = \omega_k |\vec{k}\rangle \Rightarrow \boxed{H|\vec{k}\rangle = \omega_k |\vec{k}\rangle}$$

→ similar to show $\hat{\vec{P}}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$

(verification that continuum limit of Fock space basis works)

→ Worthy to note that any observable can be expressed in terms of creation + annihilation operators.

Canonical Quantization

→ Given classical system w/ generalized coordinates q_a, p_a, we obtain quantum theory by replacing $q_a(t)$, $p_a(t)$ w/ operator-valued functions $\hat{q}_a(t)$, $\hat{p}_a(t)$, with the conditions that:

$$[\hat{q}_a(t), \hat{q}_b(t)] = [\hat{p}_a(t), \hat{p}_b(t)] = 0, \; [\hat{p}_a(t), \hat{q}_b(t)] = -i\hbar\delta_{ab}$$

Heisenberg picture: (time dependence in operators, not states)

$$|\psi(t)\rangle_H = e^{iH(t-t_0)}|\psi(t)\rangle_S$$

Consistency of physical observations preserves matrix elements:

$${}_S\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle_S = {}_H\langle\psi(t)|\mathcal{O}_H|\psi(t)\rangle_H$$

$$= {}_S\langle\psi(0)|e^{iHt}\mathcal{O}_S e^{-iHt}|\psi(0)\rangle$$

$$\Rightarrow \mathcal{O}_H(t) = e^{iHt}\mathcal{O}_S e^{-iHt}$$

$$= e^{iHt}\mathcal{O}_H(0)\, e^{-iHt}$$

→ Heisenberg equation of motion (which $O_H(t)$ solves)

$$i \frac{d}{dt} O_H(t) = [O_H(t), H] \Leftrightarrow \frac{d}{dt} O_H(t) = i[H, O_H(t)]$$

Classical Field Theory

In CFT (not conformal), observables defined at points in ST: generalized coordinates are simply components of field as function of x:

$$\phi_a(x) \qquad \sum_a \to \int d^3x \sum_a \,, \quad \delta_{ab} \to \delta_{ab}\, \delta^{(3)}(\vec{x} - \vec{x}')$$

Since Lagrangian in particle mechanics can couple coordinates w/ different labels a, most general lagrangian could couple fields at different coordinates. To make theory causal, dont want action at a distance, dynamics of field should be local in spacetime. Also, since we want CFT & ultimately QFT to be Lorentz invariant, & $L \propto d/dt$, can only introduce spatial derivatives of first order:

$$L(t) = \sum_a \int d^3x\, \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$$

$$, S = \int_{t_1}^{t_1} dt\, L(t) = \int d^4x\, \underbrace{\mathcal{L}(t, \vec{x})}_{\text{Lagrange density}}$$

* $\mathcal{L}$, S Lorentz invariant, but $L(t)$ not.

Euler Lagrange:

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

$$H = \sum_a \int d^3x \left(\Pi^0_a \partial_0 \phi_a - \mathcal{L} \right) \equiv \int d^3x \, \underbrace{\mathcal{H}(x)}_{\text{Hamiltonian Density}}$$

Quantization

- Replace $\phi(x)$, $\Pi^\mu_a(x)$ w/ operator-valued functions st

$$[\phi_a(\vec{x},t), \phi_b(\vec{y},t)] = [\Pi^0_a(\vec{x},t), \Pi^0_b(\vec{y},t)] = 0$$

$$, \; [\phi_a(\vec{x},t), \Pi^0_b(\vec{y},t)] = i\hbar \delta_{ab} \underbrace{\delta^{(3)}(\vec{x}-\vec{y})}_{\text{causality-preserving}}$$

$\phi_a(\vec{x},t)$, $\Pi_a(\vec{y},t)$ are Heisenberg operators:

$$\frac{d\phi_a(\vec{x},t)}{dt} = i[H, \phi_a(\vec{x},t)], \quad \frac{d\Pi_a(\vec{x},t)}{dt} = i[H, \phi_a(\vec{x},t)]$$

Klein-Gordon Hamiltonian: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$

$$H = \frac{1}{2}\int d^3x \left[\Pi^2 + (\nabla\phi)^2 + \mu^2\phi^2 \right] \rightarrow \text{verify EOM!} \rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \mu^2\phi, \quad \Pi^0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \ddot{\phi} - (\nabla\phi)^2 = \cancel{\ddot{\phi} - (\nabla\phi)^2} - \mu^2\phi = 0$$

$$= \ddot{\phi}$$

$\phi_a(\vec{x},t)$:

$$i\frac{\partial\phi}{\partial t} = [\phi(x), H] = \left[\phi(\vec{x},t), \int d^3x\,(\Pi^2 + (\nabla\phi)^2 + \mu^2\phi^2)\right]$$

$$= \left[\phi(\vec{x},t), \tfrac{1}{2}\int d^3x\,\Pi^2\right] = \frac{1}{2}\left[\phi_a(\vec{x},t), \int d^3x(\Pi_a^0)^2\right]$$

$$= \int d^3x' \left[\phi(\vec{x},t), \tfrac{1}{2}\Pi(\vec{x}',t)\right] = -\int d^3x' \left[\tfrac{1}{2}\Pi(\vec{x}',t), \phi(\vec{x},t)\right]$$

$$= \cancel{-\frac{1}{2}\int d^3x' ([} = -\frac{1}{2}\int d^3x' \left(\Pi_a^0(\vec{x}',t)[\Pi(\vec{x}',t), \phi(\vec{x},t)] + [\Pi(\vec{x}',t), \phi(\vec{x},t)]\Pi\right)$$

$$= -\frac{1}{2}\int d^3x' \left(\Pi_a^0(\vec{x}',t)(-i\hbar\,\delta^{(3)}(\vec{x}'-\vec{x}) - i\hbar\,\delta^{(3)}(\vec{x}'-\vec{x}))\right)$$

$$= \frac{1}{2}\int d^3x'\,\Pi_a^0(\vec{x}',t)(2i\hbar\,\delta^{(3)}(\vec{x}'-\vec{x})) = \int d^3x'\,\Pi(\vec{x}',t)(i\hbar\,\delta^{(3)}(\vec{x}'-\vec{x}))$$

$$= i\int d^3x'\,\delta^{(3)}(\vec{x}'-\vec{x})\,\Pi(\vec{x}',t) = \boxed{i\,\Pi(\vec{x},t)}$$

$$\Rightarrow \frac{\partial\phi(\vec{x},t)}{\partial t} = \Pi(\vec{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\vec{x},t))}$$

In classical free field theory, KG has $H = \frac{1}{2}\int d^3x\,(\Pi^2 + (\nabla\phi)^2 + \mu^2\phi^2)$

$\Rightarrow \dfrac{\partial H}{\partial \Pi} = \Pi = \dot{\phi}$ ✓ What about and $-\cancel{\dfrac{\partial H}{\partial\phi}}$ $\ddot{\phi} = \nabla^2\phi - \mu^2\phi$ (From $E\mathcal{L}$)

$= \dot{\Pi}$

verify...

→

$$\frac{\partial \Pi(\vec{x},t)}{\partial t} = i[H, \Pi(\vec{x},t)]$$

$$= i\left[\int d^3x' \left(\tfrac{1}{2}\Pi^2(x',t) + \tfrac{1}{2}(\nabla\phi)^2 + \mu^2\phi^2\right), \Pi(x,t)\right]$$

$$= \frac{i}{2}\left[d^3x'\left(\Pi^2(x') + (\nabla\phi(x)\cdot\nabla\phi(x')) + \mu^2\phi(x')\right), \Pi(x)\right]$$

$$= \frac{i}{2}\int d^3x'\left([(\nabla\phi(x')\cdot\nabla\phi(x'), \Pi(x)] + [\mu^2\phi^2(x'), \Pi(x)]\right)$$

$$\rightarrow [AB, C] = A[B,C] + [A,C]B$$

$$= \frac{i}{2}\int_V d^3x'\left(\nabla\phi(x')[\nabla\phi(x'), \Pi(x)] + [\nabla\phi(x'), \Pi(x)]\nabla\phi(x') + [\mu^2\phi(x'), \Pi(x)]\right)$$

$\rightarrow$ Use fact that $[\nabla\phi(x'), \Pi(x)] = \nabla(i\delta^{(3)}(x'-x))$

remember $\rightarrow [\nabla\phi(x'), \Pi(x)] \text{"="} \nabla[\phi(x'), \Pi(x)] \text{"="} \nabla(i\delta^{(3)}(x'-x))$

quotes

$$= \frac{i}{2}\int_V d^3x'\left(2\nabla\phi(x')\nabla(i\delta^{(3)}(x'-x)) + 2\mu^2\phi(x') i\delta^{(3)}(x'-x)\right)$$

$\uparrow [\phi^2(x'), \Pi(x)] = \phi(x')[\phi(x'), \Pi(x)] + [\phi(x'), \Pi(x)]\phi(x') = 2\phi(x')\delta^{(3)}(x'-x)$

Integration by parts:

$$d^3v = \nabla(i\delta^{(3)}(x'-x))d^3x' \Rightarrow v = i\delta^{(3)}(x'-x)$$

$$u = \nabla\phi(x') \Rightarrow \cancel{\frac{\partial u}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(\nabla\phi(x'))}$$

$$\Rightarrow d_i u = \frac{\partial}{\partial x'_i}(\nabla\phi(x'))dx^i \Rightarrow \underline{d^3u = \nabla^2\phi(x')(dx')^3}$$

$$\rightarrow uv - \int v\,du = i\nabla\phi(x')\delta^{(3)}(x'-x)\Big|_{-\infty}^{\infty} - i\int_V \delta^{(3)}(x'-x)\nabla^2\phi(x')d^3x'$$

$\rightarrow$

Now, for physicality, we assert that all quantities that are functions of quantum fields go to 0 at infinity:

$$\Rightarrow i \nabla\phi(\vec{x}') \delta^{(3)}(\vec{x}'-\vec{x}) \Big|_{-\infty}^{\infty} = 0$$

so $$\frac{d\Pi(\vec{x},t)}{dt} = i\left[\left(-i \iiint_V \nabla^2\phi(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x})\, d^3\vec{x}'\right) + \mu^2 \iiint_V i\,\phi(x')\delta^{(3)}(x'-x)\, d^3x'\right]$$

$$= \boxed{\nabla^2\phi(x) - \mu^2\phi(x) = \dot{\Pi}(x)} \checkmark$$

Thus, quantum fields obey KG equation.

→ From $\dot{\phi}(x) = \Pi(x)$ and $\dot{\Pi}(x) = \nabla^2\phi(x) - \mu^2\phi(x)$,

we have $\ddot{\phi}(x) - \nabla^2\phi(x) + \mu^2\phi(x) = 0$

$$= \boxed{(\partial_\mu\partial^\mu + \mu^2)\phi(x) = 0}$$

→ Can solve PDE normally: General solution is sum of plane waves!

→ $e^{ik\cdot x}$ st $k^2 = \mu^2$ (k, x 4 vectors)

Pf:

$\Rightarrow (\partial_\mu\partial^\mu + \mu^2) e^{ik\cdot x} = (ik_0 \; ik$

$$= (-k_0^2 + k_1^2 + k_2^2 + k_3^2) e^{ik\cdot x} + \mu^2 e^{ik\cdot x}$$

$$= -\mu^2 e^{ik\cdot x} + \mu^2 e^{ik\cdot x} = 0$$

$$\Rightarrow \phi(x) = \int d^3\vec{k}\left[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right]$$

$k_0^2 = \mu^2 - \vec{k}^2$

$\Rightarrow k_0^2 = \vec{k}^2 - \mu^2$

$= (E_k)^2$

→

$k^2 = \mu^2 = (k^0)^2 - \vec{k}^2$

$\Rightarrow \boxed{(k^0)^2 = \mu^2 + \vec{k}^2}$

$E = mc^2$

Want to solve for $a_k, a_k^\dagger$, $k^0 = \omega_k$

Remark: $\displaystyle\int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$

› Evaluating ϕ at $\phi(\vec{x},0)$ and similarly $\partial_0\phi(\vec{x},0)$:

$$\phi(\vec{x},0) = \int d^3k\,[a_k e^{-i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{+i\vec{k}\cdot\vec{x}}]$$

$$\partial_0\phi(\vec{x},0) = \int d^3k\,[a_k \frac{\partial}{\partial t}(e^{i(\omega_k t - k_x x - k_y y} \ldots$$

$$= \int d^3k\,(\text{-}i\omega_k)[a_k e^{-i\vec{k}\cdot\vec{x}} - a_k^\dagger e^{+i\vec{k}\cdot\vec{x}}]$$

Using integral of Fourier transform:

$$\int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta^{(3)}(\vec{k}-\vec{k}')$$

we get:

$$\int \frac{d^3x}{(2\pi)^3}\phi(\vec{x},0)e^{-i\vec{k}'\cdot\vec{x}} = \int \frac{d^3x\, d^3\vec{k}}{(2\pi)^3}[a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger e^{+i\vec{k}\cdot\vec{x}}]\, e^{-i\vec{k}'\cdot\vec{x}}$$

$$= \int \frac{d^3x\, d^3\vec{k}}{(2\pi)^3}[a_{\vec{k}} \ldots$$

$$= \int d^3\vec{k}\left(\frac{d^3x}{(2\pi)^3}[a_k e^{-i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{+i\vec{k}\cdot\vec{x}}]\right)e^{-i\vec{k}'\cdot\vec{x}}$$

$$\cancel{= \int d^3\vec{k}\,(a_{\vec{k}}\,\delta^{(3)}(-\vec{k}) + a_{\vec{k}}^\dagger\,\delta^{(3)}(\vec{k}))\,e^{-i\vec{k}\cdot\vec{x}}}$$

$$-i(\vec{k}'-\vec{k})$$

$$= \int d^3\vec{k} \left(\frac{d^3x}{(2\pi)^3} \left[\alpha_{\vec{k}} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} + \alpha^\dagger_{\vec{k}} e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}} \right] \right)$$

$$-i(\vec{k}'-\vec{k})$$

$$= \int d^3\vec{k} \left(\frac{d^3x}{(2\pi)^3} \left[\alpha_{\vec{k}} e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}} + \alpha^\dagger_{\vec{k}} e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} \right] \right)$$

$$= \int d^3\vec{k} \left[\alpha_{\vec{k}}\, \delta^{(3)}(\vec{k}'+\vec{k}) + \alpha^\dagger_{\vec{k}}\, \delta^{(3)}(\vec{k}'-\vec{k}) \right]$$

$$= \boxed{\alpha_{-\vec{k}'} + \alpha^\dagger_{+\vec{k}'}} = \cancel{\alpha_{\vec{k}} + \alpha_{\vec{k}}} \qquad = \alpha_{\vec{k}} + \alpha^\dagger_{-\vec{k}}$$

if $\vec{k} \to \vec{k}'$, $\vec{k}' \to \vec{k}$

Similarly:

$$\int \frac{d^3x}{(2\pi)^3} \dot{\phi}(\vec{x},0) e^{-i\vec{k}\cdot\vec{x}} = \int \frac{d^3x\, d^3k'}{(2\pi)^3} (-i\omega_{k'}) \left[\alpha_{k'} e^{-i\vec{k}'\cdot\vec{x}} + \alpha^\dagger_{\vec{k}'} e^{+i\vec{k}'\cdot\vec{x}} \right] \cdot e^{-i\vec{k}\cdot\vec{x}}$$

$$= -i \int d^3k' \left(\frac{d^3x}{(2\pi)^3} \left[\alpha_{\vec{k}'} e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}} - \alpha^\dagger_{\vec{k}'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} \right] \right) \omega_{k'}$$

$$= -i \int d^3k' \left[\alpha_{\vec{k}'}\, \delta^{(3)}(\vec{k}'+\vec{k}) - \alpha^\dagger_{\vec{k}'}\, \delta^{(3)}(\vec{k}-\vec{k}') \right] \omega_{k'}$$

$\omega_k = \sqrt{|k|^2+\mu^2}$, so $\omega_{-\vec{k}} = \omega_{+\vec{k}}$

$$= \boxed{-i\omega_k(\alpha_{-\vec{k}} - \alpha^\dagger_{\vec{k}})}$$

(intuitevely, direction of momenta dnea energy)

So together, we have:

$$\int \frac{d^3x}{(2\pi)^3} \phi(\vec{x},0) e^{-i\vec{k}\cdot\vec{x}} = \alpha_{-\vec{k}} + \alpha^\dagger_{\vec{k}} \longrightarrow$$

$$\int \frac{d^3x}{(2\pi)^3} \dot{\phi}(\vec{x},0) e^{-i\vec{k}\cdot\vec{x}} = -i\omega_k(\alpha_{-\vec{k}} - \alpha^\dagger_{\vec{k}})$$

$$\cancel{2a^\dagger_{\vec{k}}} = \Rightarrow 2a_{-\vec{k}} = \int \frac{d^3x}{(2\pi)^3}\left[\phi(\vec{x},0) + \frac{i}{\omega_k}\partial_0\phi(\vec{x},0)\right]e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \boxed{a_{\vec{k}} = \frac{1}{2}\int \frac{d^3x}{(2\pi)^3}\left[\phi(\vec{x},0) + \frac{i}{\omega_k}\partial_0\phi(\vec{x},0)\right]e^{i\vec{k}\cdot\vec{x}}}$$

$$2a^\dagger_{\vec{k}} = \int \frac{d^3x}{(2\pi)^3}\left[\phi(\vec{x},0) - \frac{i}{\omega_k}\partial_0\phi(\vec{x},0)\right]e^{-i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \boxed{a^\dagger_{\vec{k}} = \frac{1}{2}\int \frac{d^3x}{(2\pi)^3}\left[\phi(\vec{x},0) - \frac{i}{\omega_k}\partial_0\phi(\vec{x},0)\right]e^{-i\vec{k}\cdot\vec{x}}}$$

Using equal time commutation relations:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \frac{1}{4}\left[\frac{d^3x}{(2\pi)^3}\left(\phi(\vec{x}) + \frac{i}{\omega_k}\dot{\phi}(\vec{x})\right)e^{i\vec{k}\cdot\vec{x}}, \frac{d^3y}{(2\pi)^3}\left(\phi(\vec{y}) - \frac{i}{\omega_{k'}}\dot{\phi}(\vec{y})\right)e^{-i\vec{k}'\cdot\vec{y}}\right]$$

$$= \frac{1}{4}\int \frac{d^3x\, d^3y}{(2\pi)^6}\left[-\frac{i}{\omega_{k'}}[\phi(\vec{x}), \dot{\phi}(\vec{y})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} + \frac{i}{\omega_k}[\dot{\phi}(\vec{x}), \phi(\vec{y})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}}\right]$$

$$= -\frac{1}{4}\int \frac{d^3x\, d^3y}{(2\pi)^6}\left[\frac{i}{\omega_{k'}}[\phi(\vec{x}), \dot{\phi}(\vec{y})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} + \frac{i}{\omega_k}[\phi(\vec{y}), \dot{\phi}(\vec{x})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}}\right]$$

→ in free field Lag. $\dot{\phi}(\vec{x},0) = \Pi(\vec{x},0)$

$$= -\frac{1}{4}\int \frac{d^3x\, d^3y}{(2\pi)^6}\left[\frac{i}{\omega_{k'}}[i\delta^{(3)}(\vec{x}-\vec{y})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} + \frac{i}{\omega_k}[i\delta^{(3)}(\vec{y}-\vec{x})]e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}}\right]$$

$$= -\frac{1}{4}\int \frac{d^3x}{(2\pi)^6}\left[-\frac{1}{\omega_{k'}}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} - \frac{1}{\omega_k}e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\right]$$

$$= \frac{1}{4}\int \frac{d^3x}{(2\pi)^6}\left[\frac{1}{\omega_{k'}} + \frac{1}{\omega_k}\right]e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} \longrightarrow$$

$$= \frac{1}{(2\pi)^3}\frac{1}{4}\left(\frac{1}{\omega_{k'}}+\frac{1}{\omega_k}\right)\int\frac{d^3x}{(2\pi)^3}e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}$$

$$= \frac{1}{(2\pi)^3}\cdot\frac{1}{4}\left(\frac{1}{\omega_{k'}}+\frac{1}{\omega_k}\right)\delta^{(3)}(\vec{k}'-\vec{k})$$

$$= \frac{1}{(2\pi)^3}\cdot\frac{1}{4}\cdot\frac{2}{\omega_k}\delta^{(3)}(\vec{k}'-\vec{k}) = \boxed{\frac{1}{(2\pi)^3 2\omega_k}\delta^{(3)}(\vec{k}'-\vec{k})}$$

If we define $a_k \equiv (2\pi)^{3/2}\sqrt{2\omega_k}\,\alpha_k$, then

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 2\omega_k[\alpha_k, \alpha^\dagger_{k'}]$$

$$= \frac{(2\pi)^3 2\omega_k}{(2\pi)^3 2\omega_k}\delta^{(3)}(\vec{k}'-\vec{k}) = \boxed{\delta^{(3)}(\vec{k}'-\vec{k})}$$

$\Rightarrow [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}') \to$ commutation relation for creation/annihilation operators!

*If $a_k, a_k^\dagger$ are in fact annihilation/creation ops, we better have:

$$[H, a^\dagger_{\vec{k}}] = \omega_k a_k^\dagger \quad \text{and} \quad [H, a_{\vec{k}}] = -\omega_k a_k$$

...g so:

$\to H = \frac{1}{2}\int d^3x[\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2]$, $\pi^2 = \dot{\phi}^2$, and

$$\phi(x) = \int d^3k[\alpha_{\vec{k}}e^{-ik\cdot x} + \alpha^\dagger_{\vec{k}}e^{ik\cdot x}] = \boxed{\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}[a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x}]}$$

$$\partial_0\phi(\vec{x},t) = \int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)[a_{\vec{k}}e^{-ik\cdot x} - a^\dagger_{\vec{k}}e^{ik\cdot x}]$$

$$= \int\frac{d^3k}{(2\pi)^{3/2}}$$

$\rightarrow$

$$\rightarrow \frac{\partial\phi(\vec{x},t)}{\partial x_j} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(ik_j)(a_{\vec{k}}e^{-ik\cdot x} - a^{\dagger}_{\vec{k}}e^{ik\cdot x})$$

Recall that for the Klein-Gordan field, we have the Hamiltonian:

$$H = \frac{1}{2}\int d^3x\,[(\Pi^2 + (\nabla\phi)^2 + \mu^2\phi^2], \text{ where } \Pi^2(\vec{x},t) = \dot{\phi}(\vec{x},t)$$

Lets express H in terms of creation and annihilation operators:

→ Compute each contribution:

$$\frac{1}{2}\int d^3x\,\Pi(\vec{x},t)^2, \text{ w/ } \Pi(\vec{x},t) = \partial_0\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)(a_{\vec{k}}e^{-ik\cdot x} - a^{\dagger}_{\vec{k}}e^{ik\cdot x})$$

$$\Rightarrow \Pi^2(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)[a_{\vec{k}}e^{-ik\cdot x} - a^{\dagger}_{\vec{k}}e^{ik\cdot x}]\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(-i\omega_p)[a_{\vec{p}}e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}e^{ip\cdot x}]$$

$$\Rightarrow \frac{1}{2}\int d^3x\,\Pi^2(\vec{x},t) = \frac{1}{2}\iiint \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}\underbrace{-}_{\text{since } -1\cdot -1\cdot i^2 = -1}(\omega_k\omega_p)[(a_{\vec{k}}e^{-ik\cdot x} - a^{\dagger}_{\vec{k}}e^{ik\cdot x})(a_p e^{-ip\cdot x} - a^{\dagger}_p e^{ip\cdot x})]$$

$$= \frac{1}{2}\iiint \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(\omega_k\omega_p)[a_{\vec{k}}a_{\vec{p}}e^{-i(k\cdot x + p\cdot x)} + a_{\vec{k}}a^{\dagger}_{p}e^{-i(k\cdot x - p\cdot x)} + a^{\dagger}_{\vec{k}}a_{\vec{p}}e^{-i(p\cdot x - k\cdot x)} - a^{\dagger}_{\vec{k}}a^{\dagger}_{\vec{p}}e^{i(k\cdot x + p\cdot x)}$$

$$= \frac{1}{2}\iiint \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3}\sqrt{\omega_k\omega_p}\,[a_{\vec{k}}a_{\vec{p}}e^{-i(k+p)\cdot x} + a_{\vec{k}}a^{\dagger}_{\vec{p}}e^{-i(k-p)\cdot x} + a^{\dagger}_{\vec{k}}a_{\vec{p}}e^{-i(p-k)\cdot x} - a^{\dagger}_{\vec{k}}a^{\dagger}_{\vec{p}}e^{i(k+p)\cdot x}] \rightarrow$$

Recalling integral of a Fourier transform:

$$\int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} = \delta^{(3)}(\vec{k}'-\vec{k}) :$$

$$= \cancel{\frac{1}{2}\iint \frac{d^3k\,d^3p}{2(2\pi)^3}\sqrt{\omega_k\omega_p}\Big[a_{\vec{k}}a_{\vec{p}}}$$

$$= \frac{1}{2}\iiint \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3}\sqrt{\omega_k\omega_p}\Big[a_{\vec{k}}a_{\vec{p}}e^{-i(\omega_k+\omega_p)t}e^{i(\vec{k}+\vec{p})\cdot\vec{x}} + a_{\vec{k}}a_{\vec{p}}^\dagger e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}e^{i(\vec{k}-\vec{p})\cdot\vec{x}}$$

$$+ a_{\vec{k}}^\dagger a_{\vec{p}} e^{-i(\omega_{\vec{p}}-\omega_{\vec{k}})t}e^{i(\vec{p}-\vec{k})\cdot\vec{x}} + a_{\vec{k}}^\dagger a_{\vec{p}}^\dagger e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}e^{-i(\vec{k}+\vec{p})\cdot\vec{x}}\Big]$$

$$= \frac{1}{2}\iiint \frac{d^3k\,d^3p}{2(2\pi)^{3/2}}\sqrt{\omega_k\omega_p}\Big[\int\frac{d^3x}{(2\pi)^3}\Big(a_{\vec{k}}a_{\vec{p}}e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t}e^{-i(-\vec{k}-\vec{p})\cdot\vec{x}} \ldots$$

$$- a_{\vec{k}}a_{\vec{p}}^\dagger e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} - a_{\vec{k}}^\dagger a_{\vec{p}} e^{-i(\omega_{\vec{p}}-\omega_{\vec{k}})t}e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} + a_{\vec{k}}^\dagger a_{\vec{p}}^\dagger e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}e^{-i(\vec{k}+\vec{p})\cdot\vec{x}}\Big)\Big]$$

Performing d^3x integral:

$$= \frac{1}{2}\iint \frac{d^3k\,d^3p}{2(2\pi)^{3/2}} - \sqrt{\omega_k\omega_p}\Big(a_{\vec{k}}a_{\vec{p}}e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\delta^{(3)}(-\vec{k}-\vec{p}) - a_{\vec{k}}a_{\vec{p}}^\dagger e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\underbrace{\delta^{(3)}(\vec{p}-\vec{k})}_{\to 1}$$

$$- a_{\vec{k}}^\dagger a_{\vec{p}} e^{-i(\omega_{\vec{p}}-\omega_{\vec{k}})t}\underbrace{\delta^{(3)}(\vec{k}-\vec{p})}_{\to 1} + a_{\vec{k}}^\dagger a_{\vec{p}}^\dagger e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\delta^{(3)}(\vec{k}+\vec{p})\Big)$$

Performing d^3p integral

$$= \cancel{\frac{1}{2}\iint \frac{d^3k\,d^3p}{2(2\pi)^{3/2}}\sqrt{\omega_k\omega_p}\Big[-(a_{\vec{k}}a_{\vec{p}}^\dagger + a_{\vec{k}}^\dagger a_{\vec{p}})}$$

$$= \cancel{\frac{1}{2}\int \frac{d^3k\,d^3p}{2(2\pi)^{3/2}}\sqrt{\omega_k\omega_p}\Big[}$$

$$= \frac{1}{2}\int \frac{d^3k}{2(2\pi)^{3}} - \omega_k\Big(a_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_{\vec{k}}t} - a_{\vec{k}}a_{\vec{k}}^\dagger - a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger e^{2i\omega_{\vec{k}}t}\Big) \rightarrow$$

$$= \boxed{\frac{1}{2}\int \frac{d^3k}{2\omega_k}\Big[\omega_k^2\left(a_{\vec k}a_{-\vec k}e^{-2i\omega_k t} + a^\dagger_{\vec k}a^\dagger_{-\vec k}e^{2i\omega_k t}\right) + \omega_k^2\left(a_{\vec k}a^\dagger_{\vec k} + a^\dagger_{\vec k}a_{\vec k}\right)\Big] = \int d^3x\, \Pi^2(\vec x, t)}$$

Now:

$$\frac{1}{2}\int d^3x (\nabla\phi)^2$$

$$\Rightarrow \frac{\partial\phi(\vec x,t)}{\partial x_j} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(ik_j)\left[a_{\vec k}e^{-ik\cdot x} - a^\dagger_{\vec k}e^{ik\cdot x}\right]$$

$$\Rightarrow \left(\frac{\partial\phi(\vec x,t)}{\partial x_j}\right)^2 = \cancel{\int \frac{d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}k_j^2\left(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\right)}$$

$$\Rightarrow \left(\frac{\partial\phi(\vec x,t)}{\partial x_j}\right)^2 = \iint -\frac{d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(k_jp_j)\left(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\right)$$

~~Identically, this integral evaluates to~~

$$= \frac{1}{2}\int d^3x(\nabla\phi)^2 = \frac{1}{2}\int \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(-\vec k_j)^2\left(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\right)$$

$$= -\frac{1}{2}\int \frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(\vec k\cdot\vec p)\left(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\right)$$

Which evaluates to:

$$\cancel{\frac{1}{2}\int \frac{d^3k}{2\omega_k}\left[-\vec k^2\right.}$$

$\rightarrow$

$$= \boxed{\frac{1}{2}\int \frac{d^3k}{2\omega_k}\Big[\omega_k^2\left(a_{\vec k}a_{-\vec k}e^{-2i\omega_{\vec k}t}+a^\dagger_{\vec k}a^\dagger_{-\vec k}e^{2i\omega_{\vec k}t}\right) + \omega_k^2\left(a_{\vec k}a^\dagger_{\vec k}+a^\dagger_{\vec k}a_{\vec k}\right)\Big] = \int d^3x\, \Pi^2(\vec x,t)}$$

Now:

$$\frac{1}{2}\int d^3x (\nabla\phi)^2$$

$$\Rightarrow \frac{\partial\phi(\vec x,t)}{\partial x^j} = \int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(ik_j)\left[a_{\vec k}e^{-ik\cdot x} - a^\dagger_{\vec k}e^{ik\cdot x}\right]$$

$$\cancel{\Rightarrow \left(\frac{\partial\phi(\vec x,t)}{\partial x_j}\right)^2 = \int\frac{d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}k_j^2\Big(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\Big)}$$

$$\Rightarrow \left(\frac{\partial\phi(\vec x,t)}{\partial x_j}\right)^2 = \iint -\frac{d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(k_j p_j)\Big(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\Big)$$

~~Identically, this integral evaluates to~~

$$\Rightarrow \frac{1}{2}\int d^3x(\nabla\phi)^2 = \frac{1}{2}\int\frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(-\vec k_j)^2\Big(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-\vec k-p)\cdot x}\Big)$$

which evaluates to:

$$\cancel{\frac{1}{2}\int\frac{d^3k}{2\omega_k}\Big[-\vec k^2}$$

$$= -\frac{1}{2}\int\frac{d^3x\,d^3k\,d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}(\vec k\cdot\vec p)\Big(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x} - a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} - a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\Big)$$

$\rightarrow$

$$= -\frac{1}{2}\iint \frac{d^3k\, d^3p}{2\sqrt{2\omega_k\omega_p}}\left(\int \frac{d^3x}{(2\pi)^3}\Big[a_{\vec k}a_{\vec p}\, e^{-i(\omega_{\vec k}+\omega_{\vec p})t}\, e^{-i(\vec k+\vec p)\cdot\vec x}\right.$$

$$-a_{\vec k}a^\dagger_{\vec p}\, e^{-i(\omega_{\vec k}-\omega_{\vec p})t}\, e^{-i(\vec p-\vec k)\cdot\vec x} - a^\dagger_{\vec k}a_{\vec p}\, e^{-i(\omega_{\vec p}-\omega_{\vec k})}\, e^{-i(\vec k-\vec p)\cdot\vec x}$$

$$+a^\dagger_{\vec k}a^\dagger_{\vec p}\, e^{+i(\omega_{\vec k}+\omega_{\vec p})t}\, e^{-i(\vec k+\vec p)\cdot\vec x}\Big]$$

$\vec p - \vec k$

$$= \frac{1}{2}\iint \frac{d^3k\, d^3p}{2\sqrt{\omega_k\omega_p}}(\vec k\cdot\vec p)\Big[a_{\vec k}a_{\vec p}\, e^{-i(\omega_{\vec k}+\omega_{\vec p})t}\,\delta^{(3)}(\vec k+\vec p) - a_{\vec k}a^\dagger_{\vec p}\, e^{-i(\omega_{\vec k}-\omega_{\vec p})t}\,\delta^{(3)}(\vec p-\vec k)$$

$$-a^\dagger_{\vec k}a_{\vec p}\, e^{-i(\omega_{\vec p}-\omega_{\vec k})t}\,\delta^{(3)}(\vec k-\vec p) + a^\dagger_{\vec k}a^\dagger_{\vec p}\, e^{i(\omega_{\vec k}+\omega_{\vec p})t}\,\delta^{(3)}(\vec k+\vec p)\Big]$$

→ Now, note that $\vec k\cdot\vec p = \vec k^2$ ~~when $\vec k$~~ or $\vec k\cdot\vec p = -\vec k^2$, depending on whether or not $\vec p = \vec k$ or $\vec p = -\vec k$.

$$\cancel{= \frac{1}{2}\int\frac{d^3k}{2\sqrt{\omega_k\omega_p}}\Big[}$$

$$= \frac{1}{2}\int \frac{d^3k}{2\omega_k}\Big[-\vec k^2 a_{\vec k}a_{-\vec k}e^{-2i\omega_{\vec k}t} - \vec k^2 a_{\vec k}a^\dagger_{\vec k} - \vec k^2 a^\dagger_{\vec k}a_{\vec k} - \vec k^2 a^\dagger_{\vec k}a^\dagger_{-\vec k}\, e^{2i\omega_{\vec k}t}\Big]$$

$$= \boxed{\frac{1}{2}\int \frac{d^3k}{2\omega_k}\vec k^2\Big[a_{\vec k}a_{-\vec k}e^{-2i\omega_k t} + a_{\vec k}a^\dagger_{\vec k} + a^\dagger_{\vec k}a_{\vec k} + a^\dagger_{\vec k}a^\dagger_{-\vec k}\, e^{2i\omega_k t}\Big]}$$

$$\frac{1}{2}\int d^3x\, \mu^2\phi^2(\vec x, t) = \frac{\mu^2}{2}\int \frac{d^3x\, d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}\Big(a_{\vec k}a_{\vec p}e^{-i(k+p)\cdot x}$$

$$+a_{\vec k}a^\dagger_{\vec p}e^{-i(k-p)\cdot x} + a^\dagger_{\vec k}a_{\vec p}e^{-i(p-k)\cdot x} + a^\dagger_{\vec k}a^\dagger_{\vec p}e^{-i(-k-p)\cdot x}\Big)$$

This just evaluates to

$$\boxed{\frac{1}{2}\int \frac{d^3k}{2\omega_k}\mu^2\left(a_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_k t} + a_{\vec{k}}a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}}a_{\vec{k}} + a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}}e^{2i\omega_k t}\right)}$$

Now, combining all:

$$H = \frac{1}{2}\int d^3x\left(\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2\right)$$

$$= \frac{1}{2}\int \frac{d^3k}{2\omega_k}\Big(a_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_k t}\left[-\omega_k^2 + \vec{k}^2 + \mu^2\right] + a_{\vec{k}}a^\dagger_{\vec{k}}\left[\omega_k^2 + \vec{k}^2 + \mu^2\right]$$

$$+ a^\dagger_{\vec{k}}a_{\vec{k}}\left[\omega_k^2 + \vec{k}^2 + \mu^2\right] + a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}}\left[-\omega_k^2 + \vec{k}^2 + \mu^2\right]\Big)$$

But from relativistic dispersion relation:

$\omega_k^2 = \vec{k}^2 + \mu^2 \Rightarrow -\omega_k^2 + \vec{k}^2 + \mu^2 = 0$, so time-dependent terms drop out (as expected, since KG Hamiltonian time independent): and

$$\Rightarrow H = \frac{1}{2}\int \frac{d^3k}{2\omega_k}\left[a_{\vec{k}}a^\dagger_{\vec{k}}(2\omega_k^2) + a^\dagger_{\vec{k}}a_{\vec{k}}(2\omega_k^2)\right]$$

$$\boxed{= \frac{1}{2}\int d^3k\,\omega_k\left[a_{\vec{k}}a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}}a_{\vec{k}}\right]}$$

In Fock space, $H = \int d^3k\,\omega_k N = \int d^3k\,\omega_k a^\dagger_{\vec{k}}a_{\vec{k}}$ ... almost...

From commutation relation $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}')$

$\Rightarrow a_{\vec{k}}a^\dagger_{\vec{k}} - a^\dagger_{\vec{k}}a_{\vec{k}} = 1 \Rightarrow a_{\vec{k}}a^\dagger_{\vec{k}} = 1 + a^\dagger_{\vec{k}}a_{\vec{k}}$:

$\Rightarrow$

$$H = \frac{1}{2}\int d^3k\,\omega_k\left[\overset{\delta^{(3)}(0)}{\cancel{1}} + a_k^\dagger a_k + a_k^\dagger a_k\right]$$

$$= \cancel{\int d^3k\,\omega_k\left[\frac{\delta^{(3)}(0)}{2} + a_k^\dagger a_k\right]} = \boxed{\int d^3k\,\omega_k\left[a_k^\dagger a_k + \frac{1}{2}\delta^{(3)}(0)\right]}$$

$$= \int d^3k\,\omega_k\left[\frac{1}{2}\delta^{(3)}(0) + a_k^\dagger a_k\right]$$

Now, $\delta^{(3)}(0)$ is BIG ("very very infinite" - Michael Luke, 2023 Lecture 4)

Whats going on?

→ In periodic box w/ discrete $\vec{k}$, $\delta^{(3)}$ goes to 1 through Kronecker Delta

$$H = \frac{1}{2}\sum_k \omega_k\left[a_k^\dagger a_k + a_k a_k^\dagger\right] = \cancel{\frac{1}{2}}\sum_k \omega_k\left[a_k^\dagger a_k + \underbrace{\overset{(n+1/2)\omega_k}{}\tfrac{1}{2}}_{\text{ground-state energy!}}\right]$$

→ So in discrete case, each oscillator contributes a ground state energy.

→ In continuous case, uncountably many qhos, so sum of uncountable ground state energies $\frac{1}{2}\hbar\omega_k$

→ Cool, we know origin, but how to resolve...

1. Can set ground state energy to 0.

ie, absolute energies aren't measured, only care about relative energies.

Resolution is to define zero-point energy more usefully.

2) Ordering ambiguity:

$$\text{SHO}: H = p^2 + q^2 = \frac{\omega}{2}(\underbrace{p+iq}_{a^\dagger})(\underbrace{p-iq}_{a}) = (\underbrace{p-iq}_{a})(\underbrace{p+iq}_{a^\dagger})$$

(with $q-ip$ over $p+iq$, $q+ip$ over $p-iq$, $q+ip$ over the third factor and $q-ip$ over the fourth)

$$= \omega a^\dagger a \quad :=: \omega a a^\dagger$$

3) $H \to H - E_0$ ($\mathcal{L} \to \mathcal{L} + E_0$, $E_0 = \frac{1}{2}\sum_k \omega_k$, set E_0 st energy of vacuum state is 0 (vacuum energy counterterm)

Formally, E_0 is formally divergent but constant, so we "cutoff" sum at some k where we know physics, take limit $\omega \to \infty$. (Regularization) (divergence came from assuming theory valid at arbitrarily small distance scales).

$$a = \frac{q+ip}{\sqrt{2}}, \quad a^\dagger = \frac{q-ip}{\sqrt{2}} \quad ; \quad H = \frac{\omega}{2}(\overbrace{p-iq}^{a^\dagger})(\overbrace{p+iq}^{a})$$

$$= 2\frac{\omega}{2}(\sqrt{2}^{-1}a^\dagger)(\sqrt{2}^{-1}a) = \omega a^\dagger a$$

$$; p - iq = -ia\sqrt{2}, \quad p + iq = ia^\dagger\sqrt{2}$$

$$= \frac{\omega}{2}(p+iq)(p-iq)$$

$$H = \frac{\omega}{2}(-ia)(ia^\dagger) = \frac{\omega}{2} a a^\dagger$$

$\leftarrow \omega a^\dagger a$ instead of $\omega(a^\dagger a + \frac{1}{2})$

~~In general~~, this def. eliminates infinite zero-point energy. In general $\rightarrow$

for a set of free fields $\phi_1(x_1), \phi_2(x_2)$. $\cdots \phi_n(x_n)$, define normal-ordered product

$:\phi_1(x_1) \cdots \phi_n(x_n):$

as usual product w/ caveat that all creation operators on the left and all annihilation on the right.

→ For KG field, I think this means

going from $\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} (a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^{\dagger} e^{ik\cdot x})$

to $\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} (a_{\vec{k}}^{\dagger} e^{ik\cdot x} + a_{\vec{k}} e^{-ik\cdot x})$?

Define: Normal Ordered Product

$:\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n):$ w/ creation on left, annihilation on right ⇒ uniquely determines ordering.

$H \to :H:$ is defined as energy as system.

Since annihilation on right, $a_{\vec{k}}|0\rangle = 0$, ie ground state energy 0, ∞ gone!

$$\boxed{\langle 0|:H:|0\rangle = 0}$$

Aside: $\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}$ not evidently Lorentz Invariant lets introduce a Lorentz-invariant measure:

- d^4k is manifestly Lorentz invariant. (4-volume)
- $k^2 = \mu^2$, k is 4 momentum
- $k^0 > 0$ under <u>proper</u> Lorentz transforms

$\Rightarrow \underbrace{d^4k\,\delta(k^2-\mu^2)\,\Theta(k^0)}_{\text{Lorentz invariant}}$ (Θ step function, 0 if $k^0<0$, 1 if $k^0>0$)

But $\delta(k^2-\mu^2) = \delta((k^0)^2 - \vec{k}^2 - \mu^2)$

$= \frac{d^4k}{k^0}\delta(k^2-\omega_k)\Theta(k^0) \Rightarrow \frac{d^3\vec{k}}{2\omega_k}$ after k^0 integral

$= \delta($ $\quad = \cancel{\delta([k^0]^2-\omega_k^2)\Theta(k^0)}$

$= d^4k\,\delta(k^0-\omega_k)\Theta(k^0)$ (since $\Theta(k^0)$ keeps non-zero measure only for $k^0>0$)

$\Rightarrow \cancel{d^4k} \quad = \Theta(k^0)\,\delta([k_0-\sqrt{\vec{k}^2+\mu^2}\,][k_0+\sqrt{\vec{k}^2+\mu^2}\,])$

Now, identity that $\delta(f(x)) = \sum_{x_i:\,f(x_i)=0} \frac{\delta(x-x_i)}{|f'(x_i)|}$

$$\Rightarrow \delta(k^2-\mu^2) = \frac{\delta(k^0-\sqrt{\vec{k}^2+\mu^2})}{\left|\frac{\partial}{\partial k^0}(k_0)\Big|_{k^0=\sqrt{\vec{k}^2+\mu^2}}\right|} + \frac{\delta(k^0+\sqrt{\vec{k}^2+\mu^2})}{\left|\frac{\partial}{\partial k_0}k_0\Big|_{k^0=-\sqrt{\vec{k}^2+\mu^2}}\right|}$$

$\therefore \Theta(k^0)\,\delta(k^2-\mu^2) = \delta(k^0-\sqrt{\vec{k}^2+\mu^2}) \rightarrow$ deal w/ later...

In short, $\boxed{\frac{d^3\vec{k}}{2\omega_{\vec{k}}}}$ turns out to be Lorentz invariant

---

→ intuitively, under $\mathcal{L}$ transformations d^3k gets 1D Lorentz contracted, but $\omega_{\vec{k}}$ transforms in the same way.

→ Define relativistically normalized states:

$$|k\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}\,|\vec{k}\rangle, \quad \langle k'|k\rangle = (2\pi)^3 2\omega_{\vec{k}}\,\delta(\vec{k}-\vec{k}'),$$

which is $\mathcal{L}$ invariant.

$$a(k) = (2\pi)^{3/2}\sqrt{2\omega_k}\,a_k, \quad |k\rangle = a^\dagger(k)|0\rangle$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(k)e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right]$$ → relativistically-normalized field

$\mathcal{L}$-invariant $\quad$ $\mathcal{L}$ invariant (why?...)

Causality:

→ Canonical quantization not Lorentz invariant, since equal time commutation is frame-dependent

→ Check observable commutation for spacelike separation:

→ Spacelike-separated fields:

$$[\phi(x),\phi(y)], \quad (x-y)^2<0$$

Define $\phi(x) = \underbrace{\phi^+(x)}_{a\text{ piece},\ e^{-ik\cdot x}} + \underbrace{\phi^-(x)}_{a^\dagger\text{ piece},\ e^{ik\cdot x}}$, similar for $\phi(y)$

then only $\phi^+(x)$, $\phi^-(y)$ won't commute:

$$\Rightarrow [\phi(x),\phi(y)] = [\phi^+(x),\phi^-(y)] + [\phi^-(x),\phi^+(y)]$$

$$= \left[\int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_{\vec k}e^{-ik\cdot x} \cancel{+ a^\dagger_{\vec k} e^{ik\cdot x}}\right), \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\left(\cancel{a_{\vec p}e^{-ip\cdot y}} + a^\dagger_{\vec p}e^{ip\cdot y}\right)\right]$$

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}[a_{\vec k}, a^\dagger_{\vec p}]e^{-i(k\cdot x-p\cdot y)}$$

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}\delta^{(3)}(\vec k-\vec p)e^{-i(k\cdot x-p\cdot y)}$$

$$= \cancel{\int \frac{d^3k}{2\sqrt{\omega_k\omega_p}}\int\frac{d^3p}{(2\pi)^3}} = \boxed{\int \frac{d^3k}{2(2\pi)^3\omega_k}e^{-ik\cdot(x-y)}}$$

and $[\phi^-(x),\phi^+(y)]$ →

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}[a^\dagger_{\vec k}, a_{\vec p}]e^{-i(p\cdot y-k\cdot x)}$$

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}}-\delta^{(3)}(\vec p-\vec k)e^{-i(p\cdot y-k\cdot x)}$$

$$[\phi^-(x), \phi^+(y)] = \left[\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{\omega_k}} a^\dagger_{\vec{k}} e^{ik\cdot x}, \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{\omega_p}} a_{\vec{p}} e^{-ip\cdot y}\right]$$

$$= \int \frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}} [a^\dagger_{\vec{k}}, a_{\vec{p}}] e^{-i(p\cdot y - k\cdot x)}$$

$$= \int -\frac{d^3k\, d^3p}{2(2\pi)^3\sqrt{\omega_k\omega_p}} \delta^{(3)}(\vec{k}-\vec{p}) e^{-i(p\cdot y - k\cdot x)}$$

$$= -\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(y-x)}$$

→ Now, both $[\phi^+(x), \phi^-(y)]$ and $[\phi^-(x), \phi^+(y)]$ are manifestly Lorentz invariant (measure invariant + $e^{k_\mu(y-x)^\mu}$ invariant)

$$\Rightarrow \underbrace{[\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)]}_{=[\phi(x), \phi(y)]} = \boxed{D(x-y) - D(y-x)}$$

is Lorentz invariant.

*"Cute argument" → answer is same in all frames, what frame makes this trivial?

→ Since x, y spacelike separated, exists frame such that $x^0 = y^0$, ie events are simultaneous! But ~~this~~ in this frame, equal time commutation relations hold!

ETCR tell us $[\phi(\vec{x},0), \phi(\vec{y},0)] = 0$, so $D(x-y) - D(y-x) = 0$, so $[\phi(x), \phi(y)] = 0$ in <u>all frames</u>, so fields commute for all spacelike separations

→ QFT is necessary for this result, .. Causality is violated if we attempt to calculate propagator for particle to go from $\vec{x}$ to $\vec{y}$, spacelike in QM.

* For timelike separations, in general fields won't commute.

Interpretation: $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$ is probability amplitude for particle to be created at $\vec{y}$, destroyed at $\vec{x}$ - the reverse, and the composite commutators are prob. amps, which cancel for spacelike separations!

To see,

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \langle 0 | (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) | 0 \rangle$$

$$= \langle 0 | \phi^+(x) \overset{\phi^+(y)}{} | 0 \rangle + \langle 0 | \phi^+(x) \phi^-(y) | 0 \rangle + \langle 0 | \phi^-(x) \phi^+(y) | 0 \rangle + \langle 0 | \phi^-(x) \phi^-(y) | 0 \rangle$$

since ϕ^+ is annihilation piece + ϕ^- is creation; on $|0\rangle$ (converse for dual $\langle 0|$), only non-zero term:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \langle 0 | \phi^+(x) \phi^-(y) | 0 \rangle$$

$$= \langle 0 | [\phi^+(x), \phi^-(y)] | 0 \rangle \quad (\text{since } \langle 0 | \phi^-(y) \phi^+(x) | 0 \rangle = 0$$

$$= D(x-y)$$

for creation at $\vec{x}$, annihilation at $\vec{y}$.

Similarly $\langle 0 | \phi(y) \phi(x) | 0 \rangle = \langle 0 | \phi^+(y) \phi^-(x) | 0 \rangle$

$$= \langle 0 | [\phi^+(y), \phi^-(x)] | 0 \rangle = -D(y-x)$$

So total probability amplitude is sum $= [\phi^+(x), \phi^-(y)] + [\phi^+(y), \phi^-(y)]$

$= [\phi(x), \phi(y)]$ as shown

$= D(x-y) - D(y-x)$

i $D(x-y) - D(y-x)$ shows L invariance and $[\phi(x), \phi(y)]$ indicates use of ETCR.
↖ Equal time commutation relations

What does quantum scalar field $\phi(x)$ actually do? Well,

consider action of $\phi(\vec{x}, 0)$ on vacuum $|0\rangle$:

$$\Rightarrow \phi(\vec{x}, 0)|0\rangle = \int \frac{d^3k}{2(2\pi)^3 \omega_k} (a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^\dagger e^{ik\cdot x})|0\rangle$$

$$= \int \frac{d^3k}{2(2\pi)^3 \omega_k} a_k e^{-ik\cdot x} |0\rangle = \int \frac{d^3k}{2(2\pi)^3 \omega_k} e^{-i\vec{k}\cdot\vec{x}} |k\rangle$$

$$\Rightarrow \langle p|\phi(\vec{x}, 0)|0\rangle = \langle \vec{p}| \int \frac{d^3k}{2(2\pi)^3 \omega_k} e^{-i\vec{k}\cdot\vec{x}} |k\rangle$$

$$= \int \frac{d^3k}{2(2\pi)^3 \omega_k} \langle p|\vec{k}\rangle e^{-i\vec{p}\cdot\vec{x}} = \int \frac{d^3k}{2(2\pi)^3 \omega_k} \delta^{(3)}(\vec{p}-\vec{k}) e^{-i\vec{p}\cdot\vec{x}}$$

$= e^{-i\vec{p}\cdot\vec{x}} \Rightarrow \phi(\vec{x}, 0)|0\rangle$ plays role of $|\vec{x}\rangle$, since $\langle \vec{p}|\vec{x}\rangle = e^{-i\vec{p}\cdot\vec{x}}$, ie, $\phi(\vec{x}, 0)$ creates a particle $t=0$ at $\vec{x}$.

Lorentz transformation:

$\varphi(x) \to \varphi'(x')$, $\quad x' = \Lambda x$

$\to = \varphi(\Lambda^{-1} x')$ (new field at new coord = old field at old coord).

-or- $x \to \Lambda x$, $\phi(x) \to \phi(\Lambda^{-1} x)$ (scalar field)

For vector fields, fields rotate into one another (for instance, Electric field in one frame looks like $\vec{E} + \vec{B}$ in another)

$\to$ $$\phi^{\mu}(x) \to \Lambda^{\mu}{}_{\nu}\phi^{\nu}(\Lambda^{-1}x)$$

In general, $$\underset{1\ldots n}{\varphi_a(x)} \to \underbrace{D_{ab}(\Lambda)}_{n\times n}\varphi_b(\Lambda^{-1}x).$$

where D_{ab} has same multiplication properties as Lorentz transformations:

- $D_{ab}(\Lambda_1)D_{bc}(\Lambda_2) = D_{ac}(\Lambda_1\Lambda_2)$

  $D(\Lambda^{-1}) = D(\Lambda)^{-1}$

$\to$ looks like groups! D is an isomorphism

$\to$ Set of D's form n-dimensional representation of Lorentz group.

Scalar fields: $D = \mathbb{1}$

4-vector fields $D_{ab}(\Lambda) = \Lambda_{ab}$ $\to$ Lorentz transformation

$\to$ $\Lambda^{\mu}{}_{n}$ forms "4D" representation of L. group

* How do we know $\phi(x)$ acts on elements of Fock space?

Lagrangian for a 4-vector field $A^\mu(x)$

$\to \cancel{\Lambda^\mu{}_\nu A^\nu(x)}$ $\quad \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$

Most general Lagrangian s.t.

- $\mathcal{L}$ quadratic in fields
- ≤ 2 derivatives
- $\mathcal{L}$ invariant

0 derivatives: $A^\mu A_\mu$

1 derivative: No terms!

2 derivatives: $\partial_\mu A^\nu \partial_\nu A^\mu$, $\partial^\mu A_\nu \partial^\nu A_\mu$, $\partial_\mu A^\mu \partial_\nu A^\nu = (\partial_\mu A^\mu)^2$, $[\partial_\mu \partial^\mu A_\nu] A^\nu$, $\partial_\mu A^\nu \partial^\mu A_\nu$.

$\to \partial_\mu A^\nu \partial_\nu A^\mu = \partial_\mu (A^\nu \partial_\nu A^\mu) - A^\nu \partial_\mu \partial_\nu A^\mu$

$\downarrow$

total derivative, no effect on Euler-Lagrange equations

$= \cancel{\partial_\nu (A^\nu \partial_\mu A^\mu)} + \boxed{\partial_\nu A^\nu \partial_\mu A^\mu}$

in short, only 2 independent terms contributing to EOM, namely $\partial_\mu A^\nu \partial^\mu A_\nu$ and $\partial_\mu A^\mu \partial_\nu A^\nu$ $\to$

$$\boxed{\mathcal{L} = \pm\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu + a\,\partial_\mu A^\mu \partial_\nu A^\nu + b A_\mu A^\mu\right]}$$

EOM:

$$\frac{\partial A_\nu}{\partial A_\mu} = \delta^\mu_\nu$$

$$\frac{\partial A_\nu}{\partial A^\mu} = \delta_{\nu\mu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$$

→ For free scalar field Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \Rightarrow \underline{\partial\mathcal{L}}\quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$

$$= \partial_\mu\left(\frac{\partial(\partial_\mu\phi\,\partial^\mu\phi)}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\left(\frac{\partial(\partial_\nu\phi\,\partial^\nu\phi)}{\partial(\partial_\mu\phi)}\right)$$

$$\rightarrow \frac{\partial A^\nu}{\partial A_\mu} = \delta^{\nu\mu},\quad \frac{\partial A_\nu}{\partial A_\mu} = \delta^\mu_\nu$$

$$= \partial_\mu\left(\partial_\nu\phi\frac{\partial(\partial^\nu\phi)}{\partial(\partial_\mu\phi)} + \partial^\nu\phi\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\mu\phi)}\right)$$

KD

$$\frac{\partial A_\mu}{\partial A_\nu} = \delta^\nu_\mu \Rightarrow \partial_\mu\left(\partial_\nu\phi\,\delta^{\mu\nu} + \partial^\nu\phi\,\delta^\mu_\nu\right)$$

metric tensor

$$= \frac{1}{2}\partial_\mu\left(\partial^\mu\phi + \partial^\mu\phi\right) = \frac{1}{2}\partial_\mu(2\partial^\mu\phi) = \boxed{\partial_\mu\partial^\mu\phi}$$

4 vector field:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu + a\,\partial_\mu A^\mu\partial_\nu A^\nu + b A_\mu A^\mu\right)$$

$$\rightarrow \frac{\partial\mathcal{L}}{\partial A_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_i)}\right) = 0 \rightarrow i^{th}\text{ EL equation} \rightarrow$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial A_i} = \frac{\partial}{\partial A_i}\left(\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu + a\partial_\mu A^\mu \partial_\nu A^\nu + b A_\mu A^\mu\right]\right)$$

$$= \frac{1}{2}\left(\partial_\mu A_\nu \frac{\partial(\partial^\mu A^\nu)}{\partial A_i} + \partial^\mu A^\nu \frac{\partial(\partial_\mu A_\nu)}{\partial A_i} + a\partial_\nu A^\nu \frac{\partial(\partial_\mu A^\mu)}{\partial A_i}\right.$$
$$+ a\partial_\mu A^\mu \frac{\partial(\partial_\nu A^\nu)}{\partial A_i} + bA^\mu \frac{\partial A_\mu}{\partial A_i}$$
$$\left. + bA_\mu \frac{\partial A^\mu}{\partial A_i}\right)$$

$$= \frac{1}{2}\left(\partial_\mu A_\nu (0) + \partial^\mu A^\nu (0) + a\partial_\nu A^\nu (0) + a\partial_\mu A^\mu (0)\right.$$
$$\left. + bA^\mu \delta^i_\mu + bA_\mu \delta^{i\mu}\right)$$

$$= \frac{1}{2}\left(bA^\mu \delta^i_\mu + bA_\mu \delta^{\mu i}\right) = \frac{1}{2}\left(bA^i + bA^i\right) = bA^i \text{ (contravariant comp)}$$
$$= bA^\nu$$

Kinectic:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\frac{1}{2}\left\{\partial_i A_j \partial^i A^j + a\partial_i A^i \partial_j A^j + bA_j A^j\right\}\right]\right)$$

$$= \frac{\partial_\mu}{2}\left[\partial_i A_j \frac{\partial(\partial^i A^j)}{\partial(\partial_\mu A_\nu)} + \partial^i A^j \frac{\partial(\partial_i A_j)}{\partial(\partial_\mu A_\nu)} + a\partial_i A^i \frac{\partial(\partial_j A^j)}{\partial(\partial_\mu A_\nu)}\right.$$
$$\left. + a\partial_j A^j \frac{\partial(\partial_i A^i)}{\partial(\partial_\mu A_\nu)} + \cancel{bA_j \frac{\partial(A^j)}{\partial(\partial_\mu A_\nu)}} + \cancel{bA^j \frac{\partial(A_j)}{\partial(\partial_\mu A_\nu)}}\right]$$

$$= \frac{1}{2}\partial_\mu\left(\partial_i A_j \delta^{j\nu}\delta^{i\mu} + \partial^i A^j \delta^\nu_j \delta^\mu_i + a\partial_i A^i \delta^{j\nu}\delta^\mu_j\right.$$
$$\left. + a\partial_j A^j \delta^{i\nu}\delta^\mu_i\right) \longrightarrow$$

$$= \frac{1}{2}\partial_\mu\left(\partial^\mu A^\nu + \partial^\mu A^\nu + a\partial_i A^i \delta^{\nu\mu} + a\partial_j A^j \delta^{\nu\mu}\right)$$

$$= \partial_\mu\left(\partial^\mu A^\nu + a\partial_i A^i \delta^{\mu\nu}\right) = \partial_\mu\partial^\mu A^\nu + a\partial_\mu\partial_i A^i \delta^{\mu\nu}$$
$$= \partial_\mu\partial^\mu A^\nu + a\partial^\nu\partial_\mu A^\mu$$

$\therefore$ EOM: $\boxed{bA^\nu - \partial_\mu\partial^\mu A^\nu - a\partial^\nu\partial_\mu A^\mu = 0}$

$\boxed{\text{(In notes } -\Box A_\nu - a\partial_\nu\partial_\mu A^\mu + bA_\nu = 0)}$ → ~~think error in notation, process right.~~

→ ~~$\delta_{\mu\nu}(bA^\nu - \partial_\mu\partial^\mu$~~

$\delta_{i\nu}(bA^\nu - \partial_\mu\partial^\mu A^\nu - a\partial^\nu\partial_\mu A^\mu) = 0$

$= bA_i - \partial_\mu\partial^\mu A_i - a\partial_i\partial_\mu A^\mu) = 0$

$= bA_\nu - \partial_\mu\partial^\mu A_\nu - a\partial_\nu\partial_\mu A^\mu = 0$

$=$ ~~bA_ν~~ $= \boxed{-\Box A_\nu - a\partial_\nu\partial_\mu A^\mu + bA_\nu = 0}$

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \partial_\mu\partial^\mu$$

→ Looks similar to free, check plane wave solutions!

Guess: $A_\nu(x) = \varepsilon_\nu e^{-ik\cdot x}$

→ Solution if $k^2\varepsilon_\nu + ak_\nu k\cdot\varepsilon + b\varepsilon_\nu = 0$

Pf: $-\Box\varepsilon_\nu e^{-ik\cdot x} = \frac{-\partial^2}{\partial t^2}\varepsilon_\nu e^{-ik\cdot x} + \nabla^2\varepsilon_\nu e^{-ik\cdot x} \rightarrow = k^2\varepsilon_\nu e^{-ik\cdot x}$ →
$$= -(-\varepsilon_\nu k_0^2 e^{-ik\cdot x}) + -\varepsilon_\nu \vec{k}^2 e^{-ik\cdot x}$$

$$\to -a\partial_\nu\partial_\mu A^\mu = -a\partial_\nu\partial_\mu\left(\varepsilon^\mu e^{-ik\cdot x}\right) = -a\partial_\nu\,\varepsilon^\mu\partial_\mu e^{-ik_\nu x^\nu}$$

$$= \varepsilon^\mu a\partial_\nu e^{-ik\cdot x}\partial_\mu(-ik_\alpha x^\alpha) = \varepsilon^\mu i a\partial_\nu e^{-ik\cdot x}(\underbrace{\cancel{x^\alpha\partial_\mu k_\alpha}}_{0} + k_\alpha\partial_\mu x^\alpha)$$

$$= ia\partial_\nu e^{-ik\cdot x}(k_\alpha\delta^\alpha_\mu) = ia\partial_\nu e^{-ik\cdot x}k_\mu\varepsilon^\mu$$

$$= iak_\mu\varepsilon^\mu e^{-ik\cdot x}\partial_\nu(-ik_\alpha x^\alpha) = \varepsilon^\mu ak_\mu e^{-ik\cdot x}(k_\alpha\partial_\nu x^\alpha)$$

$$= ak_\mu e^{-ik\cdot x}(k_\alpha\delta^\alpha_\nu)\varepsilon^\mu = ak_\mu\varepsilon^\mu e^{-ik\cdot x}k_\nu$$

$$= \boxed{ak_\nu k\cdot\varepsilon\, e^{-ik\cdot x}}$$

$$\to -\Box A_\nu = -\partial_\mu\partial^\mu A_\nu = -\partial_\mu\partial^\mu\varepsilon_\nu e^{-ik\cdot x}$$

$$= -\partial_\mu\varepsilon_\nu\partial^\mu(e^{-ik\cdot x}) = -\varepsilon_\nu\partial_\mu e^{-ik\cdot x}\partial^\mu(-ik_\alpha x^\alpha)$$

$$= +i\varepsilon_\nu\partial_\mu e^{-ik\cdot x}(k^\alpha\partial^\mu x_\alpha) = i\varepsilon_\nu\partial_\mu e^{-ik\cdot x}k^\alpha\delta^\mu_\alpha$$

$$= i\varepsilon_\nu\partial_\mu e^{-ik\cdot x}k^\mu = i\varepsilon_\nu k^\mu e^{-ik\cdot x}\partial_\mu(-ik_\alpha x^\alpha)$$

$$= \varepsilon_\nu k^\mu e^{-ik\cdot x}k_\alpha\partial_\mu x^\alpha = \varepsilon_\nu k^\mu e^{-ik\cdot x}k_\alpha\delta^\alpha_\mu$$

$$= \varepsilon_\nu k^\mu e^{-ik\cdot x}k_\mu = \varepsilon_\nu k_\mu k^\mu e^{-ik\cdot x} = \boxed{k^2\varepsilon_\nu e^{-ik\cdot x}}$$

$$\to bA_\nu = \boxed{b\varepsilon_\nu e^{-ik\cdot x}}$$

☆ In EOM calculations, some δ are wrong, but gave right EOM

Substituting into equation of motion, we find that if $A_\nu(x) = \varepsilon_\nu e^{-ik\cdot x}$ solves, then the following constraint:

$$\boxed{k^2\varepsilon_\nu + ak_\nu k\cdot\varepsilon + b\varepsilon_\nu = 0}$$

We can classify solutions in a Lorentz-invariant manner:

1. $\varepsilon \propto k$ (4-D longitudinal) 2. $\varepsilon\cdot k = 0$ (4-D transverse)

→ In the rest frame of the field, K only has a temporal component, so 4-D longitudinal solution is $(\varepsilon_0, 0)$, and 4-D transverse solution is $(0, \vec{\varepsilon})$ (in rest frame, then calculate $\varepsilon \cdot K$)

These lead to following EOMs:

long:

$$K^2 \varepsilon_\nu + a K_\nu K \cdot \varepsilon + b \varepsilon_\nu = 0 \Rightarrow K^2 \varepsilon_\nu + a K_\nu K^2 + b \varepsilon_\nu$$
$$= K^2 + aK^2 + b = K^2(1+a) + b = 0$$

$$\Rightarrow K^2 = -\frac{b}{1+a} \equiv \underline{\mu_L^2}$$ → dispersion relation for particle of mass μ_L^2 $(p_\mu p^\mu = +m^2c^2)$

trans:

$$K^2 \varepsilon_\nu + a K_\nu K \cdot \varepsilon + b \varepsilon_\nu = K^2 \varepsilon_\nu + b \varepsilon_\nu \Rightarrow K^2 + b = 0$$
$$\Rightarrow K^2 = -b \equiv \mu_T^2$$

→ transverse solution has 3 polarizations (like spin 1) w/ $m = 1, 0, -1$ (spin $1 \equiv l = 1$)

→ To describe spin 1 particle, it would be nice to get rid of longitudinal solution, which resembles a scalar theory (looks same under rotations since no spatial comp)

Is there choice of Lagrangian parameters that eliminates "scalar" solution?

→ $K \propto \varepsilon$, well $\mu_L^2 = -\frac{b}{1+a}$, so $\boxed{\text{set } b = -1!}$

$\rightarrow \mu_L^2 \rightarrow \infty$, "cant make it" ie disappears from spectrum:

-or- $a=-1 \Rightarrow (K^2 + aK^2 + b)K_\nu = (K^2 - K^2 + b)K_\nu = bK_\nu = 0 \Rightarrow b = 0$, no solutions to EOM!

$\Rightarrow \mu_L$ particle vanishes from universe...
except when $\underline{b=0}$, <u>every</u> K_ν is a solution (for long. mode).

Something strange about massless spin 1 particles... $\Rightarrow$ longitudinal <u>comes back</u> unconstrained, anything is a solution!

, so lets not look at $b=0$ for now, (look at massive spin 1 field)
* unconstrained solutions for $b=0$ related to gauge invariance.

$$a=-1 \Rightarrow \mathcal{L} = \pm \frac{1}{2}\left[(\partial_\mu A_\nu)^2 - (\partial_\mu A^\mu)^2 - \mu^2 A_\mu A^\mu\right]$$
with $\mu^2 = \mu_T^2$

Equation of motion:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}\left(-\mu^2 A_\mu A^\mu\right) = -\mu^2\left(A_\mu \frac{\partial (A^\mu)}{\partial (A_\nu)} + A^\mu \frac{\partial (A_\mu)}{\partial (A_\nu)}\right)$$

$$= -\frac{\mu^2}{2}\left(A_\mu \delta^\mu_\nu + A^\mu \delta_{\mu\nu}\right) = -\frac{\mu^2}{2}(A_\nu + A_\nu) = \boxed{-\mu^2 A_\nu}$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial_\mu}{2}\left(\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A^\mu \partial_\nu A^\nu\right]\right)$$

$$= \frac{1}{2}\partial\left(\partial_\mu A_\nu \frac{\partial(\partial^\mu A^\nu)}{\partial(\partial_i A_j)} + \partial^\mu A^\nu \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_i A_j)} - \partial_\mu A^\mu \frac{\partial(\partial_\nu A^\nu)}{\partial(\partial_i A_j)} - \partial_\nu A^\nu \frac{\partial(\partial_\mu A^\mu)}{\partial(\partial_i A_j)}\right) \Rightarrow$$

$$= \frac{1}{2}\partial_i\left(\partial_\mu A_\nu \delta^{\mu i}\delta^{\nu j} + \partial^\mu A^\nu \delta_\mu^{\ i}\delta_\nu^{\ j} - \partial_\mu A^\mu \delta_\nu^{\ i}\delta^{\nu j} - \partial_\nu A^\nu \delta_\mu^{\ i}\delta^{\mu j}\right)$$

$$= \frac{1}{2}\partial_i\left(\partial^i A^j + \partial^i A^j - \partial_\mu A^\mu \delta^{ij} - \partial_\nu A^\nu \delta^{ij}\right)$$

$$= \partial_i\left(\partial^i A^j - \partial_\mu A^\mu \delta^{ij}\right)$$

$$= \partial_i \partial^i A^j - \partial_i \partial_\mu A^\mu \delta^{ij}$$

$$= \partial_\mu \partial^\mu A^j - \partial^j \partial_\mu A^\mu$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu + \mu^2 A^\nu = 0$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu A_\nu - \partial_\nu \partial_\mu A^\mu + \mu^2 A_\nu = 0} \quad (1)$$

This Lagrangian can be rewritten in terms of the field strength tensor:

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$\rightarrow F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu$$

$$= 2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu$$

→ but $\partial_\mu A_\nu \partial^\nu A^\mu \sim \partial_\mu A^\mu \partial_\nu A^\nu$ up to ~~tim~~ total time derivative

$$\Rightarrow \frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left((\partial_\mu A_\nu)^2 - (A_\mu A^\mu)^2 \right.$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu}$$

Equations of motion:

$$\frac{\partial \mathcal{L}}{\partial A_i} = \frac{\partial}{\partial A_i}\left(-\frac{1}{2}\mu^2 A_\mu A^\mu\right) = -\frac{1}{2}\mu^2 \frac{\partial}{\partial A_i}(A_\mu A^\mu)$$

$$= -\frac{1}{2}\mu^2\left(A_\mu \frac{\partial A^\mu}{\partial A_i} + A^\mu \frac{\partial A_\mu}{\partial A_i}\right)$$

$$= -\frac{1}{2}\mu^2\left(A_\mu \delta^{i\mu} + A^\mu \delta^i_\mu\right) = -\frac{1}{2}\mu^2(A^i + A^i) = \underline{-\mu^2 A^i}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_j\left(\frac{\partial}{\partial(\partial_\mu A_i)}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\right)$$

$$= \frac{\partial_j}{4}\left(\frac{\partial}{\partial(\partial_\mu A_i)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right]\right)$$

$$= \frac{1}{4}\partial_j\left((\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{\partial(\partial^\mu A^\nu - \partial^\nu A^\mu)}{\partial(\partial_j A_i)} + (\partial^\mu A^\nu - \partial^\nu A^\mu)\frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_j A_i)}\right)$$

$$= \frac{1}{4}\partial_j\left(F_{\mu\nu}(\delta^{\mu j}\delta^{\nu i} - \delta^{\nu j}\delta^{\mu i}) + F^{\mu\nu}(\delta^j_\mu \delta^i_\nu - \delta^j_\nu \delta^i_\mu)\right)$$

$$= \frac{1}{4}\partial_j\left(F_{\mu\nu}\delta^{\mu j}\delta^{\nu i} - F_{\mu\nu}\delta^{\nu j}\delta^{\mu i} + F_i^{\mu\nu}\delta^j_\mu\delta^i_\nu - F^{\mu\nu}\delta^j_\nu\delta^i_\mu\right)$$

$$= \frac{1}{4}\partial_j\left(F^{ji} - F^{ij} + F^{ji} - F^{ij}\right)$$

using antisymmetry of $F^{\mu\nu}$ ($F_{\mu\nu}$):

$$= \frac{1}{4}\partial_j\left(2F^{ji} + 2F^{ji}\right) = \partial_j F^{ji} = \partial_\mu F^{\mu i}$$

$$\Rightarrow \partial_\mu F^{\mu i} + \mu^2 A^i = 0$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0}$$ (same EOM as (1))

This is the Proca Equation.

Using antisymmetry of field strength tensor:

$$\partial_\nu \partial_\mu F^{\mu\nu} + \partial_\nu \mu^2 A^\nu = 0$$

$$= \partial_\nu\partial_\mu F^{\mu\nu} + \mu^2\partial_\nu A^\nu \underset{\text{relabel}}{=} \partial_\mu\partial_\nu F^{\nu\mu} + \mu^2\partial_\nu A^\nu$$

$$= -\partial_\mu\partial_\nu F^{\mu\nu} + \mu^2\partial_\nu A^\nu \underset{\text{swap partials}}{=} -\partial_\nu\partial_\mu F^{\mu\nu} + \mu^2\partial_\nu A^\nu$$

so $\partial_\nu\partial_\mu F^{\mu\nu} + \mu^2\partial_\nu A^\nu = -\partial_\nu\partial_\mu F^{\mu\nu} + \mu^2\partial_\nu A^\nu$

$$\Rightarrow \underline{\partial_\nu\partial_\mu F^{\mu\nu} = 0} \Rightarrow \mu^2\partial_\nu A^\nu = 0$$

$$\Rightarrow \boxed{\partial_\nu A^\nu = 0}$$ (Lorentz Condition)

→ this is just the requirement that the field is transverse! →

$\to A^{\nu}(x) = \varepsilon^{\nu} e^{-ik\cdot x}$

$\Rightarrow \partial_\mu A^\mu = \partial_\mu \varepsilon^\mu e^{-ik\cdot x} = \varepsilon^\mu e^{-ik\cdot x}\partial_\mu(-ik_\alpha x^\alpha)$

$= -ik_\alpha \varepsilon^\mu e^{-ik\cdot x}\partial_\mu x^\alpha = -ik_\alpha \varepsilon^\mu e^{-ik\cdot x}\delta^\alpha_\mu$

$= -ik_\mu \varepsilon^\mu e^{-ik\cdot x} = 0 \Rightarrow k_\mu \varepsilon^\mu = \boxed{\varepsilon\cdot k = 0}$ (Lorentz condition)

→ This is expected as we constructed our Lagrangian to only permit a transverse solution!

Substituting $\partial_\mu A^\mu = 0$ into Proca equation:

$\partial_\nu \partial_\mu F^{\mu\nu} + \mu^2 \partial_\nu A^\nu = \partial_\nu \partial_\mu F^{\mu\nu}$

$\to \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$= \cancel{\partial_\mu} = \partial_\mu \partial^\mu A^\nu - \cancel{\partial^\nu \partial_\mu A^\mu}$

0 from Lorentz

$\Rightarrow \partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \partial_\mu \partial^\mu A^\nu + \mu^2 A^\nu = 0$

$\Rightarrow \delta_{i\nu}(\partial_\mu \partial^\mu A^\nu + \mu^2 A^\nu) = \partial_\mu \partial^\mu A_i + \mu^2 A_i = 0$

$\Rightarrow \partial_\mu \partial^\mu A_\nu + \mu^2 A_\nu = \boxed{(\partial_\mu \partial^\mu + \mu^2)A_\nu = 0}$ * $= (\partial_\mu \partial^\mu + \mu^2)A^\nu$

equation w/ $\partial_\mu A^\mu = 0$

↓ each comp. satisfies KG equation!

* This ~~form of the Proca equation~~ is equivalent to the Proca, however unclear how to get from Lagrangian.

(expect theory to describe free particles)

Promising → $\mu^2 \to 0$ limit is smooth: $\cancel{\partial_\mu \partial^\mu}\ \partial_\mu A^\mu = 0$,

$\Box A^\nu = 0$ →

But these are just Maxwells equations in free space!

EM: $\to$ $A^\mu = (\varphi, \vec{A})$, $\partial_\mu A^\mu = 0 \Rightarrow \dot{\varphi} + \vec{\nabla}\cdot\vec{A} = 0$
(Lorenz Gauge)

$\Rightarrow$ Maxwells Equations reduce to $\Box\varphi = 0$, $\Box\vec{A} = 0$,
solutions are EM waves!

Recall: $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla}\times\vec{A}$

Then $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

EOM: (massless) : $\partial_\mu F^{\mu\nu} = 0$

Say $\nu = 0$: $\partial_\mu F^{\mu 0} = 0 - \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{E} = 0$, Gauss's Law in free space!

$\nu = 1$: $\partial_\mu F^{\mu 1} = \frac{\partial(E_x)}{\partial t} - \frac{\partial(B_z)}{\partial y} + \frac{\partial(B_y)}{\partial z} = 0$

$\Rightarrow \frac{\partial(E_x)}{\partial t} = \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} = \cancel{\vec{\nabla}\times\vec{B}_x}$

$\Rightarrow \frac{\partial(\vec{E})}{\partial t} = \vec{\nabla}\times\vec{B}$, Maxwell-Ampere!
(Free space)

(other 2 equations follow from definitions of $\vec{A}, \phi$) $\to$

• $\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$, $\nabla \cdot \vec{B} = 0$ follow from def. of $\vec{E}$ & $\vec{B}$.

+ The slowww development of electromagnetism from first principles is complex & off putting, but we just showed that they are inevitable and fall out from Lorentz invariance.

* $\partial_\mu A^\mu = 0$ only derived from $\mu^2 \neq 0$

Back to Proca:

+ solutions are $A_\mu = \varepsilon_\mu e^{\pm i k \cdot x}$ w/ $K^2 = \mu^2$

As showed before, $\partial_\mu A^\mu \Rightarrow \varepsilon \cdot k = \varepsilon_\mu K^\mu = 0$

$\Rightarrow$ 3 linearly independent polarization vectors $\varepsilon_\mu^{(r)}$, $r = 1, 2, 3$:

In rest frame of $\varepsilon_\mu^{(r)}$:

Choose basis $\varepsilon^{(1)} = (0,1,0,0)$, $\varepsilon^{(2)} = (0,0,1,0)$, $\varepsilon^{(3)} = (0,0,0,1)$

Often, however, it will be convenient to choose linear combinations as basis vectors: (to be eigenstates of J_z):

$\varepsilon^{(1)} = \frac{1}{\sqrt{2}}(0,1,i)$, $\varepsilon^{(2)} = \frac{1}{\sqrt{2}}(0,1,-i,0)$, $\varepsilon^{(3)} = \frac{1}{\sqrt{2}}(0,0,0,1)$

$\rightarrow$

Properties (useful for later):

- In any basis, orthonormal spacelike:

$$\varepsilon_\mu^{(r)} \varepsilon^{\mu *(s)} = -\delta^{rs}$$

- Completeness:

$$\sum_{r=1}^{3} \varepsilon_\mu^{(r)} \varepsilon_\nu^{(r)*} = -g_{\mu\nu} + \frac{K_\mu K_\nu}{\mu^2}$$

→ relations are covariant, so are true in any frame.

Per usual, the overall sign of the Lagrangian may be determined by demanding that energy is bound below (I implicitly used ± for simplification):

$$\mathcal{L} = \pm\left[\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} - \frac{\mu^2}{2} A_i A^i - \frac{\mu^2}{2} A_0 A^0\right]$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \frac{\partial}{\partial(\partial_0 A_i)}\left(\pm\left[\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij}\right]\right) \cdot \frac{1}{2}$$

$$= \frac{\partial}{\partial(\partial_0 A_i)}\left(\pm\left[F_{0i} F^{0i}\right]\right) = \pm \frac{\partial}{\partial(\partial_0 A_i)}\left([\partial_0 A_i - \partial_i A_0][\partial^0 A^i - \partial^i A^0]\right)$$

$$= \pm\left([\partial_0 A_i - \partial_i A_0](\delta^{00}\delta^{ii} - \delta^{i0}\delta^{0i}) + [\partial^0 A^i - \partial^i A^0](\delta_0^0 \delta_i^i - \delta_i^0 \delta_0^i)\right.$$

$$= \pm\left([\partial_0 A_i - \partial_i A_0](1 \cdot -1 - 0 \cdot 0) + [\partial^0 A^i - \partial^i A^0](1 \cdot 1 - 0 \cdot 0)\right)/2$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \pm F^{0i} \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0$$

→ Momentum conjugate of A_0 vanishing is fine. Since $\partial_\mu A^\mu = 0$, there are fewer degrees of freedom than naively expected, and spatial A_i's w/ canonical momentum sufficiently define system.

$$\Pi^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \quad , \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

$$\rightarrow \Pi^0_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \frac{\partial}{\partial(\partial_0 A_i)} \left(-\frac{1}{4} [\partial_\mu A_\nu - \partial_\nu A_\mu][\partial^\mu A^\nu - \partial^\nu A^\mu] \right)$$

$$= -\frac{1}{4} \left([\partial_\mu A_\nu - \partial_\nu A_\mu] \frac{\partial(\partial^\mu A^\nu - \partial^\nu A^\mu)}{\partial(\partial_0 A_i)} + [\partial^\mu A^\nu - \partial^\nu A^\mu] \frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_0 A_i)} \right)$$

$$= -\frac{1}{4} \left([\partial_\mu A_\nu - \partial_\nu A_\mu](\delta^{\mu 0}\delta^{\nu i} - \delta^{\nu 0}\delta^{\mu i}) + [\partial^\mu A^\nu - \partial^\nu A^\mu](\delta^0_\mu \delta^i_\nu - \delta^0_\nu \delta^i_\mu) \right] \rightarrow$$

$$= -\frac{1}{4}\Big[\partial_\mu A_\nu \delta^{\mu 0}\delta^{\nu i} - \partial_\mu A_\nu \delta^{\nu 0}\delta^{\mu i} - \partial_\nu A_\mu \delta^{\mu 0}\delta^{\nu i}$$
$$+ \partial_\nu A_\mu \delta^{\nu 0}\delta^{\mu i} + \partial^\mu A^\nu \delta^0_\mu \delta^i_\nu - \partial^\mu A^\nu \delta^0_\nu \delta^i_\mu$$
$$- \partial^\nu A^\mu \delta^0_\mu \delta^i_\nu + \partial^\nu A^\mu \delta^0_\nu \delta^i_\mu \Big]$$

$$= -\frac{1}{4}\Big[\partial^0 A^i - \partial^i A^0 - \partial^i A^0 + \partial^0 A^i + \partial^0 A^i - \partial^i A^0$$
$$- \partial^i A^0 + \partial^0 A^i \Big]$$

$$= -\frac{1}{4}\left[4\partial^0 A^i - 4\partial^i A^0\right] = \partial^i A^0 - \partial^0 A^i$$

$$= -(\partial^0 A^i - \partial^i A^0)$$

$$= \boxed{-F^{0i}} \quad (i \in \{1,2,3\})$$

-or-

$$= \boxed{0} \quad \{i = 0\}$$

Quantization:

→ Generalization of scalar field quantization

*Since spatial components A_i + their conjugate momenta form complete set of initial conditions, commutation relations only imposed on these fields.

Equal Time Commutation Relations:

$$\boxed{[A_i(\vec{x},t), F^{j0}(\vec{y},t)] = i\delta^j_i\, \delta^{(3)}(\vec{x}-\vec{y})}$$

$$\boxed{[A_i(\vec{x},t), A_j(\vec{y},t)] = [F^{i0}(\vec{x},t), F^{j0}(\vec{y},t)] = 0}$$

$$H = -\left[\frac{1}{2} F_{0i} F^{0i} - \frac{1}{4} F_{ij} F^{ij} + \frac{\mu^2}{2} A_i A^i - \frac{\mu^2}{2} A_0 A^0 \right]$$

→ Quantum fields $\phi_a(x)$ will obey massive Klein Gordon equation $(\Box + \mu^2) A_\mu = 0$ with plane wave solutions $\cancel{\varepsilon e^{-ik \cdot x}}$ $\varepsilon_r e^{-ik \cdot x}$

$$\Rightarrow A_\mu(x) = \sum_{r=1}^{3} \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[a_k^{(r)} \varepsilon_\mu^{(r)}(k) e^{-ik \cdot x} + a_k^{\dagger(r)} \varepsilon_\mu^{(r)*}(k) e^{ik \cdot x} \right]$$

w/ $a, a^\dagger$ satisfying

$$\Rightarrow [a_k^{(r)}, a_{k'}^{\dagger(s)}] = \delta^{rs} \delta^{(3)}(\vec{k} - \vec{k}')$$

$$[a_k^{(r)}, a_{k'}^{(s)}] = [a_k^{\dagger(r)}, a_{k'}^{\dagger(s)}] = 0$$

w/ $(r), (s)$ labeling polarization of the particle (sum over r represents superposition of polarization states/particles in vacuum?)

$$:H: = \sum_r \int d^3k \, \omega_k a_k^{\dagger(r)} a_k^{(r)}$$

→ we can interpret $a_k^{\dagger(r)}$ and $a_k^{(r)}$ as creation and annihilation operators for spin-1 particles w/ polarization r!

→

Symmetries and Conservation Laws

→ Symmetry arguments will prove vital in QFT since anything outside of free field theory is not analytically solvable (EOMs) in 4D

→ Symmetry allows us to extract information without solving anything exactly.

Formalism:

Given transformation $q_a(t) \to q_a(t, \lambda)$ such that $q_a(t, 0) = q_a(t)$ (no variation at endpoints), define:

$$Dq_a \equiv \left.\frac{\partial q_a}{\partial \lambda}\right|_{\lambda=0}$$

eg. $\vec{r} \to \vec{r} + \lambda \hat{e}$; $D\vec{r} = \hat{e}$

Time translation:

$$q_a(t) \to q_a(t+\lambda) = q_a(t) + \lambda \frac{dq_a}{dt} + \mathcal{O}(\lambda^2) + \cdots$$

$$\Rightarrow Dq_a = dq_a/dt$$

→ We may define a symmetry as a transformation that leaves the Lagrangian invariant, however this is too restrictive:

$$L(t, \lambda) = L(q_a(t+\lambda), \dot{q}_a(t+\lambda)) = L(0) + \lambda \frac{dL}{dt} + \cdots$$

$$\sim L\left(q_a(t) + \lambda \frac{dq_a}{dt}, \dot{q}_a(t) + \lambda \frac{d\dot{q}_a}{dt}\right)$$

$\to DL = \frac{\partial L}{\partial t} \to$ better definition if

$DL = dF/dt$ since:

$$DS = \int_{t_1}^{t_2} dt\, DL = \int_{t_1}^{t_2} dt\, \frac{\partial F}{\partial t} = F(q_a(t_2), \dot{q}_a(t_2), t_2) - F(q_a(t_1), \dot{q}_a(t_1), t_1)$$

or rather:

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt\, DL\, \delta\lambda$$

$$= \int_{t_1}^{t_2} dt\, \frac{dF}{dt}\delta\lambda = \delta\lambda\left[F(q_a(t_2), \dot{q}_a(t_2), t_2) - F(q_a(t_1), \dot{q}_a(t_1), t_1)\right]$$

$$= \lambda[\delta F(\ldots) - \delta F(\ldots)]$$

$$\propto \lambda\left(\frac{\partial F}{\partial q_a}\delta q_a\Big|_{t_2} + \frac{\partial F}{\partial \dot{q}_a}\delta \dot{q}_a\right) = 0$$ since we do not vary path at endpoints

Noethers Thm:

$$DL = \sum_a \left[\frac{\partial L}{\partial q_a} Dq_a + \frac{\partial L}{\partial \dot{q}_a} D\dot{q}_a\right]$$

$$= \sum_a \left[\dot{p}_a Dq_a + p_a D\dot{q}_a\right] = \sum_a \left[\dot{p}_a Dq_a + p_a (\dot{Dq_a})\right] >$$

$$= \frac{d}{dt} \sum_a p_a D q_a = \frac{dF}{dt}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\sum_a p_a D q_a - F\right) = 0} \to \text{conserved}$$

Energy conservation:

$$t \to t+\lambda \Rightarrow D q_a = \frac{\partial q_a}{\partial \lambda}\Big|_{\lambda=0}, \quad q_a(t,\lambda) = q_a(t) + \lambda \frac{\partial q_a}{\partial t}$$

$$= \frac{\partial\left(q_a(t) + \lambda \frac{\partial q_a(t)}{\partial t}\right)}{\partial \lambda} = \frac{\partial q_a}{\partial t} = \dot{q}_a$$

$$DL = \frac{\partial \mathcal{L}}{\partial \lambda}\Big|_{\lambda=0}, \quad L = L(q_a(t,\lambda), \dot{q}_a(t,\lambda), t)$$

$$\sim L(\lambda=0) + \frac{\partial q_a}{\partial q_a}\mathcal{L} \cdot \delta q_a$$

$$DL = \frac{\partial \mathcal{L}}{\partial q_a} D q_a + \frac{\partial \mathcal{L}}{\partial \dot{q}_a} D\dot{q}_a = \frac{\partial \mathcal{L}}{\partial q_a}\dot{q}_a + \frac{\partial \mathcal{L}}{\partial \dot{q}_a}\ddot{q}_a$$

why?

$$\to \cancel{\frac{\partial \mathcal{L}}{\partial q_a} \sim \frac{\partial \mathcal{L}}{\partial t} \cdot \frac{\partial t}{\partial q_a} = \frac{\partial \mathcal{L}}{\partial t}}. \quad = \frac{dL}{dt} - \frac{\partial \mathcal{L}}{\partial t} = \frac{dL}{dt}$$

$\Rightarrow$ Conserved q is $\boxed{\sum_a p_a \dot{q}_a - L = H}$

Symmetries in Field Theory

$\rightarrow$ Field theory is "continuum limit" of classical particle mechanics, however quantities are not just conserved globally, but locally.

eg: $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \rightarrow$ local conservation of current.

$$\Rightarrow \frac{dQ_V}{dt} = -\int_V \nabla \cdot \vec{j} \, d^3 = -\int_S dS \cdot \vec{j}$$

Taking $S \rightarrow \infty$, $\frac{dQ_V}{dt} = 0$, total charge conserved.

But current conservation is stronger, so conservation laws in field theory will be of form $\partial_\mu J^\mu = 0$ for some 4-current J^μ.

$\rightarrow \phi_a(x) \rightarrow \phi_a(x, \lambda)$, symmetry if $D\mathcal{L} = \partial_\mu F^\mu$

$$\rightarrow \delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)$$

$$+ \text{Add } 0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \rightarrow \delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right)$$

→ Brackets 0 since equations of motion are satisfied

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \cancel{X}^\mu$$

Let $\delta\phi$ be sum function of ϕ $X(\phi)$'

$$\rightarrow \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} X(\phi) - F \right) = 0}$$

-or-

$$D\mathcal{L} = \sum_a \frac{\partial \mathcal{L}}{\partial \phi_a} D\phi_a + \Pi_a^\mu D(\partial_\mu \phi_a)$$

$$\overset{\checkmark EL}{=} \sum_a \left[\partial_\mu \Pi_a^\mu D\phi_a + \Pi_a^\mu \partial_\mu D\phi_a \right]$$

$$= \partial_\mu \sum_a (\Pi_a^\mu D\phi_a) = \partial_\mu F^\mu$$

$$\Rightarrow \partial_\mu \left(\sum_a (\Pi_a^\mu D\phi_a - F^\mu) \right) = 0$$

$$\Rightarrow \boxed{j^\mu = \sum_a (\Pi_a^\mu D\phi_a - F^\mu)}, \text{ where } \partial_\mu j^\mu = 0$$

Integrate over all space:

$$\Rightarrow \frac{dQ}{dt} = \frac{d}{dt} \int d^3x\, J^0(x) = 0$$

↑ intepret as density of "stuff"

$$\text{since } \partial_0 \int_V d^3x\, J^0 = -\int_V d^3x\, \nabla \cdot \vec{J} = -\int_S d^2s (\hat{n} \cdot \vec{J})$$

density ⇒ total integral is <u>charge</u>.

Space-time translations

→ Can compute conserved quantity corresponding to space-time translations!

$x \to x + \lambda e$, where e is a 4-vector:

$$\phi_a(x) \to \phi_a(x+\lambda e) = \phi_a(x) + \underbrace{\lambda e_\mu \partial^\mu \phi_a(x)} + \cdots$$

$$\delta\phi_a = \frac{\partial \phi_a}{\partial x^\mu}\delta x^\mu, \quad \delta x^\mu = \lambda e^\mu$$

$$\Rightarrow \delta\phi_a = \partial_\mu \phi_a(x) \lambda e^\mu = \underline{\lambda e_\mu \partial^\mu \phi_a(x)}$$

$$\Rightarrow D\phi_a = \cancel{\lambda} \; e_\mu \partial^\mu \phi_a(x)$$

$$\Rightarrow D\phi_a(x) = e_\mu \partial^\mu \phi_a(x)$$

$$\mathcal{D}\mathcal{L} = \sum_a \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \mathcal{D}\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \mathcal{D}(\partial_\mu \phi_a) \right]$$

$$= \sum_a \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \mathcal{D}\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu (\mathcal{D}\phi_a) \right] \quad \rightarrow$$

$$= \sum_a \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} D\phi_a \right] = \partial_\mu \sum_a \Pi^\mu_a e_\nu \partial^\nu \phi_a$$

But what is $D\mathcal{L}$?

$D\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} D\phi_a$, $\cancel{\mathcal{L}(x)}$

$$\mathcal{L}(\phi(x)) \to \mathcal{L}(\phi(x + \lambda e)) = \mathcal{L}(\phi_a(x) + \lambda e_\mu \partial^\mu \phi_a(x))$$

$$\sim \mathcal{L}(\phi_a(x)) + \frac{\partial \mathcal{L}}{\partial \phi_a(x)} (\lambda e_\mu \partial^\mu \phi_a(x))$$

$$\to D\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} D\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} D(\partial_\mu \phi_a)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_a} e_\mu \partial^\mu \phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu D\phi_a$$

$$= \frac{\partial \mathcal{L}}{\partial \phi_a} e_\mu \partial^\mu \phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu (e_\nu \partial^\nu \phi_a(x))$$

$$= e_\nu \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \partial^\nu \phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu (\partial^\nu \phi_a(x)) \right]$$

$$= e_\nu \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \cdot \partial^\nu \phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu (\partial_\mu \phi_a(x)) \right]$$

$$= \boxed{e_\nu \partial^\nu \mathcal{L}}$$

→ change in Lagrangian is total derivative → invoke Noethers

$$= \partial_\nu e^\nu \mathcal{L} = \partial_\mu (e^\mu \mathcal{L})$$

$\rightarrow J^\mu = \sum_a [\Pi_a^\mu D\phi - F]$

$= \sum_a \Pi_a^\mu e_\nu \partial^\nu \phi_a(x) - e_\nu \partial^\nu \mathcal{L}]$

but $e_\nu \partial^\nu \mathcal{L} = \partial_\mu (g^{\mu\rho} e_\rho \mathcal{L})$ $\xrightarrow{}$ Why? $(\partial_\mu g^{\mu\rho} = \partial^\rho)$

and e_ρ commutes w ∂_μ since e_ρ constant

$\Rightarrow \partial_\mu (g^{\mu\rho} e_\rho \mathcal{L}) = e_\rho \partial_\mu (g^{\mu\rho} \mathcal{L}) = \cancel{e_\rho (\mathcal{L} \partial_\mu g}$

$= e_\rho (\mathcal{L} \partial_\mu g^{\mu\nu} + g^{\mu\rho} \partial_\mu \mathcal{L})$

$= e_\rho (\mathcal{L} \partial^\rho + g^{\mu\rho} \partial_\mu \mathcal{L}) = e_\rho \mathcal{L} \partial^\rho + e_\rho g^{\mu\rho} \partial_\mu \mathcal{L}$

$\cancel{= \mathcal{L} \partial_\rho}$ $= \mathcal{L} \partial^\rho e_\rho + e_\rho \partial^\rho \mathcal{L} = \boxed{\partial^\rho (e_\rho \mathcal{L})}$ as before

$\uparrow$ since $e_\rho \partial = \partial e_\rho$

$\Rightarrow$ We set $F = g^{\mu\rho} e_\rho \mathcal{L}$

$\Rightarrow J^\mu = \sum_a [\Pi_a^\mu e_\nu \partial^\nu \phi_a(x) - g^{\mu\rho} e_\rho \mathcal{L}]$

$= \boxed{e_\nu \sum_a [\Pi_a^\mu \partial^\nu \phi_a(x) - g^{\mu\rho} \mathcal{L}]} \equiv \boxed{T^{\mu\nu}\text{, energy-momentum Tensor}}$

or: $= \sum_a [\Pi_a^\mu e_\nu \partial^\nu \phi_a(x) - e^\mu \mathcal{L}]$

$= e_\nu \sum_a [\Pi_a^\mu \partial^\nu \phi_a(x) - g^{\mu\nu} \mathcal{L}]$ $\quad$ $, J^\mu = e_\nu T^{\mu\nu}$

Now, from $\epsilon_\nu T^{\mu\nu}$ being a conserved current, we have:

$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = 0 = \epsilon_\nu \partial_\mu T^{\mu\nu}$. Since this was irrespective of ϵ_ν, it must follow that $\boxed{\partial_\mu T^{\mu\nu} = 0}$

→ For time translation, the corresponding transformation is $\epsilon_\nu = (1, \vec{0})$. This gives $T^{\mu 0}$ a suggestive interpretation as energy current, with the corresponding conserved quantity:

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30} = 0$$

$$= \frac{\partial T^{00}}{\partial t} + \partial_i T^{i0} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_V d^3x\, T^{00} = \cancel{\int_V \nabla \cdot T} = \cancel{-\partial_i \int_V d^3x\, T^{i0}}$$

$$= -\cancel{\int_V \partial_i T^{i0}\, d^3x}$$

$$= -\cancel{\int_S}$$

$$\cancel{\Rightarrow \int d^3x} \Rightarrow Q = \int_V d^3x\, T^{00} = \int_V d^3x \left(\Pi^0_a \partial^0 \phi_a - g^{00}\mathcal{L}\right)$$

$$\boxed{= \int d^3x \left(\Pi^0_a \dot{\phi}_a - \mathcal{L}\right) = \int d^3x\, \mathcal{H}}$$

→ Thus, the Hamiltonian as previously defined is the energy of the system!

(∵ is the energy? ⟺ is the conserved quantity ~~thex~~ resulting from integration over all space associated with time-translation invariance)

→ $\mu = 0 \Rightarrow$ ~~density~~ ~~T?~~ $T^{0\nu}$ is a density
if $\mu = 1,2,3$, ~~T~~ $T^{i\nu}$ is a current (density?) → pretty sure its density 90% (~60%?)

(As seen in $\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$)

ν tells us direction of translation
time translation → energy conservation
$i = 1,2,3 \Rightarrow i^{th}$ component of momentum conservation.

Similarly, $T^{0i} \equiv$ density of i^{th} component of momentum in field

→ $T^{ji} \equiv j^{th}$ component of the current of the i^{th} component of momentum!

→ Total momentum in i^{th} direction is
$\int d^3x \, T^{0i}$

→ In quantum theory, fields go to operators:
→ Lets calculate the total x-momentum: (for KG field)

$$\int d^3x \underbrace{T^{01}}_{\text{density of } i^{th} \text{ component of momentum}} = \int d^3x \left(\Pi^0 \partial^1 \phi(x) - g^{01} \mathcal{L} \right) \longrightarrow$$

$$= \int d^3x \left(\Pi_a^0 \partial^1 \phi_a(x) \right) = - \int d^3x \left(\Pi_a^0(x) \partial_1 \phi_a(x) \right)$$

$$\to \phi_a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right)$$

$$, \Pi_a^0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} = \partial^0 \phi_a(x)$$

$$\to \partial_0^0 \phi_a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} (-i\omega_k) \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right) = \Pi_a^0(x)$$

$$\cancel{\Rightarrow \int d^3x \left(\Pi_a^0(x) \partial_1 \phi_a(x) \right)}$$

and $\cancel{\partial \phi_a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}}$

$$\partial_1 \phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} (ik_1) \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right)$$

$$\Rightarrow - \int d^3x \left(\Pi_a^0(x) \partial_1 \phi_a(x) \right)$$

$$= - \iiint \frac{d^3x\, d^3k\, d^3p}{2(2\pi)^3 \sqrt{\omega_k \omega_p}} (\omega_k p_1) \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right) \left(a_p e^{-ip\cdot x} - a_p^\dagger e^{ip\cdot x} \right)$$

$$= - \iiint \frac{d^3x\, d^3k\, d^3p}{2(2\pi)^3 \sqrt{\omega_k \omega_p}} (\omega_k p_1) \Big(a_k a_p e^{-i(k+p)\cdot x} - a_k a_p^\dagger e^{-i(k-p)\cdot x}$$
$$- a_k^\dagger a_p e^{-i(p-k)\cdot x} + a_k^\dagger a_p^\dagger e^{-i(-k-p)\cdot x} \Big)$$

$$= - \iint \frac{d^3k\, d^3p}{2\sqrt{\omega_k \omega_p}} (\omega_k p_1) \int \frac{d^3x}{(2\pi)^3} \Big(a_k a_p e^{-i(k+p)\cdot x} - a_k a_p^\dagger e^{-i(k-p)\cdot x}$$
$$- a_k^\dagger a_p e^{-i(p-k)\cdot x} + a_k^\dagger a_p^\dagger e^{-i(-k-p)\cdot x} \Big)$$

$$= -\iint \frac{d^3k\, d^3p}{2\sqrt{\omega_k \omega_p}} (\omega_k p_1) \cancel{(\ldots)} (a_{\vec{k}} p.$$

Fourier Transform of e is δ: $\int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta^{(3)}(\vec{k}-\vec{k}')$

Seperation space-time inner product

$$= -\iint \frac{d^3k\, d^3p}{2\sqrt{\omega_k \omega_p}} (\omega_k p_1) \Big(a_{\vec{k}} a_{\vec{p}} e^{-i(\omega_k+\omega_p)t} \, e^{-i(-\vec{k}-\vec{p})\cdot\vec{x}}$$

$\int \frac{d^3x}{(2\pi)^3}$

$$- a_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\omega_k-\omega_p)t} e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} - a^\dagger_{\vec{k}} a_{\vec{p}} e^{-i(\omega_p-\omega_k)t} e^{-i(\vec{k}-\vec{p})\cdot\vec{x}}$$

$$+ a^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(-\omega_k-\omega_p)} e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} \Big)$$

$$= -\iint \frac{d^3k\, d^3p}{2\sqrt{\omega_k \omega_p}} (\omega_k p_1) \Big(a_{\vec{k}} a_{\vec{p}} e^{-i(\omega_k+\omega_p)t} \delta^{(3)}(-\vec{k}-\vec{p})$$

$$- a_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\omega_k-\omega_p)t} \delta^{(3)}(\vec{p}-\vec{k}) - a^\dagger_{\vec{k}} a_{\vec{p}} e^{-i(\omega_p-\omega_k)t} \delta^{(3)}(\vec{k}-\vec{p})$$

$$+ a^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{-i(\omega_k+\omega_p)t} \delta^{(3)}(\vec{k}+\vec{p}) \Big)$$

$$= -\int \frac{d^3p}{2\sqrt{\omega_k \omega_p}} (\omega_k p_1) \int d^3k \ldots$$

$$= -\int \frac{d^3p}{2\sqrt{\omega_p}} p_1 \int d^3k \sqrt{\omega_k} (\ldots)$$

$$= -\int \frac{d^3p}{2\sqrt{\omega_p}} p_1 \Big(\sqrt{\omega_{-\vec{p}}}\, a_{-\vec{p}} a_{\vec{p}} e^{-i2\omega_{\vec{p}}t} - a_{\vec{p}} a^\dagger_{\vec{p}} - a^\dagger_{\vec{p}} a_{\vec{p}} + a^\dagger_{-\vec{p}} a_{\vec{p}} e^{+i2\omega_{\vec{p}}t} \Big)$$

$$= -\int \frac{d^3p}{2\sqrt{\omega_{\vec{p}}}} p_1 \Big(\sqrt{\omega_{\vec{p}}}\, a_{-\vec{p}} a_{\vec{p}} e^{-i2\omega_{\vec{p}}t} - \sqrt{\omega_{\vec{p}}}\, a_{\vec{p}} a^\dagger_{\vec{p}} - \sqrt{\omega_{\vec{p}}}\, a^\dagger_{\vec{p}} a_{\vec{p}} + \sqrt{\omega_{\vec{p}}}\, a^\dagger_{-\vec{p}} a^\dagger_{\vec{p}} e^{i2\omega_{\vec{p}}t} \Big)$$

$$= -\int \frac{d^3p}{2} p_1 \Big(a_{-\vec{p}} a_{\vec{p}} e^{-i2\omega_{\vec{p}}t} - a_{\vec{p}} a^\dagger_{\vec{p}} - a^\dagger_{\vec{p}} a_{\vec{p}} + a^\dagger_{-\vec{p}} a^\dagger_{\vec{p}} e^{i2\omega_{\vec{p}}t} \Big) \rightarrow$$

Now what the hell do we do with this? This thing (namely $P_1 = + \int d^3x\, T^{01}$) looks time-dependent? Well...

$a_{\vec{p}} a_{-\vec{p}} e^{-2i\omega_{\vec{p}} t}$ is manifestly even:

$f(p) = a_{\vec{p}} a_{-\vec{p}}\, e^{-2i\omega_{\vec{p}} t} \Rightarrow f(-\vec{p}) = a_{-\vec{p}} a_{\vec{p}} e^{-2i\omega_{-\vec{p}} t}$

but $[a_p(\vec{x}), a'_p(\vec{y})] = 0 \Rightarrow f(-p) = a_{\vec{p}} a_{-\vec{p}} e^{-2i\omega_{-\vec{p}} t}$

but $\omega_{-\vec{p}} = \omega_{\vec{p}}$ (since this is energy comp. of K_μ)

$= a_{\vec{p}} a_{-\vec{p}} e^{-2i\omega_{\vec{p}} t} \Rightarrow f(\vec{p}) = f(-\vec{p})$

$\Rightarrow -\frac{1}{2} \int d^3p\, p_1\, a_{-\vec{p}} a_{\vec{p}}\, e^{-2i\omega_{\vec{p}} t}$ is the integral over even $\cdot$ odd functions of p (p_1 odd)

$\Rightarrow \int$ odd, but this vanishes!

$\Rightarrow$ identical argument for $a^\dagger_{-\vec{p}} a^\dagger_{\vec{p}} e^{2i\omega_{\vec{p}} t}$ term

$$\Rightarrow P_x = -\frac{1}{2} \int d^3p\, p_x \left(-a_{\vec{p}} a_{\vec{p}}^\dagger - a_{\vec{p}}^\dagger a_{\vec{p}}\right)$$

$$= \frac{1}{2} \int d^3p\, p_x \left(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}\right)$$

→ we should normal order this

$$= \boxed{\int d^3p \left(a^\dagger_{\vec{p}} a_{\vec{p}}\right) p_x}$$

Note

→ ~~Notice~~ that $\vec{p}$ is the physical momentum, ie the conserved charge associated with space-time translations, but has nothing to do with the conjugate momentum, π_a^0 of the field $\phi_a(x)$.

Lorentz Transformations (Noethers Theorem)

- Under a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, a 4-vector transforms as $a^\mu \to \Lambda^\mu{}_\nu a^\nu$.

→ Scalar fields transform as $\phi(x) \to \phi(\Lambda^{-1}x)$, ie, the new field at the new point is the old field at the old point. , $x \to \delta(x+vt)$

Meanwhile, a vector field transforms as

$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$, since fields rotate into each other

We restrict ourselves to scalar fields.

→ Consider a 1-parameter subgroup of Lorentz transformations by λ, $\Lambda(\lambda)^\mu{}_\nu$, from which we wish to get $D\phi = \partial\phi/\partial\lambda|_{\lambda=0}$

→ Define $e^\mu{}_\nu \equiv D\Lambda^\mu{}_\nu$. Then under a Lorentz transformation $a^\mu \to \Lambda^\mu{}_\nu a^\nu$:

$$Da^\mu = \left.\frac{\partial a^\mu}{\partial\lambda}\right|_{\lambda=0} = D\Lambda^\mu{}_\nu a^\nu \quad \left(\text{since } D\Lambda^\mu{}_\nu = \left.\frac{\partial(\Lambda^\mu{}_\nu(\lambda))}{\partial\lambda}\right|_{\lambda=0}\right)$$

$$= \frac{\partial}{\partial\lambda}(\Lambda^\mu{}_\nu(\lambda)a^\nu) = \cancel{D} = e^\mu{}_\nu a^\nu$$

(since a^μ has no λ dependence)

But it is easy to show $\epsilon_{\mu\nu}$ is antisymmetric:

$$0 = D(a^\mu b_\mu) = (Da^\mu)b_\mu + a^\mu(Db_\mu)$$

$$= \epsilon^\mu{}_\nu a^\nu b_\mu + a^\mu \epsilon_\mu{}^\nu b_\nu$$

$$\Rightarrow 0 = g_{\lambda\mu}(\epsilon^\mu{}_\nu a^\nu b_\mu + a^\mu \epsilon_\mu{}^\nu b_\nu)$$

$$= \epsilon_{\mu\nu} a^\nu b^\mu + \epsilon_{\nu\mu} a^\nu b^\mu \text{ (dummy exchange)}$$

$$= (\epsilon_{\mu\nu} + \epsilon_{\nu\mu}) a^\nu b^\mu = 0 \Rightarrow \boxed{\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}}$$

(alternatively, consider $x^\mu \to x^\mu + \epsilon^{\mu\nu} x_\nu d\lambda$,
$y_\mu \to y_\mu + \epsilon_{\mu\nu} y^\nu d\lambda$

$$\to (x^\mu y_\mu)' = (x^\mu + \epsilon^{\mu\nu} x_\nu d\lambda)(y_\mu + \epsilon_{\mu\nu} y^\nu d\lambda)$$

$$= x^\mu y_\mu + \epsilon^{\mu\nu} x_\nu y_\mu d\lambda + \epsilon_{\mu\nu} y^\nu x^\mu d\lambda + O(\,^2)$$

and since λ is Lorentz, $(x^\mu y_\mu)' = x^\mu y_\mu$, same result)

$\to \epsilon_{\mu\nu}$ has 6 independent components, which is good since there are 6 independent Lorentz transformations: 3 rotations + 3 boosts.

Ex: $\epsilon_{12} = -\epsilon_{21} = 1$, all other components 0

$$\to Da^1 = \epsilon^1{}_\nu a^\nu = \epsilon^1{}_2 a^2 = -\epsilon_{12} a^2 = -a^2$$

$$Da^2 = \epsilon^2{}_\nu a^\nu = \epsilon^2{}_1 a^1 = -\epsilon_{21} a^1 = +a_1$$

$\to$ This corresponds to $\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \to \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \to$

at $\lambda = 0$, since $a_1 \to a_1 \cos\lambda - a_2 \sin\lambda$
$a_2 \to a_1 \sin\lambda + a_2 \cos\lambda$

$$\Rightarrow Da_i = \frac{d}{d\lambda}\left(a_1\cos\lambda - a_2\sin\lambda\right) = -a_1\sin\lambda - a_2\cos\lambda$$

$$\Rightarrow Da_i\Big|_{\lambda=0} = -a^2 \text{ as shown}$$

Take $\varepsilon_{01} = -\varepsilon_{10} = +1$. Then $Da^0 = \varepsilon^0{}_1 a^1 = \varepsilon_{01} a^1 = +a^1$
$Da^1 = \varepsilon^1{}_0 a^1 = -\varepsilon_{10} a^1 = +a^0$

Equivalent to infinitesimal form of

$$\begin{pmatrix} a^0 \\ a^1 \end{pmatrix} \to \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}\begin{pmatrix} a^0 \\ a^1 \end{pmatrix}$$

Now we can construct the 6 conserved currents of each L transformation:

$$D\phi(x) = \frac{\partial}{\partial\lambda}\phi\left(\Lambda^{-1}(\lambda)^\mu{}_\nu\, x^\nu\right)\Big|_{\lambda=0}$$

$$= \partial_\alpha \phi(x) \frac{\partial}{\partial\lambda}\left(\Lambda^{-1}(\lambda)(x)^\alpha\right)\Big|_{\lambda=0}$$

↓ why?

$$= \partial_\alpha \phi(x)\, D\left(\Lambda^{-1}(\lambda)^\alpha{}_\beta\, x^\beta\right)$$

Since $\frac{\partial}{\partial\lambda}\phi\left(M^\mu_0(\lambda)x^0 + M_1^\mu(\lambda)x^1 + \cdots\right)$

$$= \underline{\partial\phi}$$

→

or + $\phi(x) \to \phi(\Lambda^{-1}x)$, $(\Lambda^{-1}x)^\rho = x^\rho - \varepsilon^{\rho\sigma} x_\sigma d\lambda$

$= x^\rho + \varepsilon^{\sigma\rho} x_\sigma d\lambda \Rightarrow \phi(x) \to \phi(x^\rho + \varepsilon^{\sigma\rho} x_\sigma d\lambda)$

$\sim \phi(x^\rho) + \frac{\partial\phi}{\partial x^\rho} \varepsilon^{\sigma\rho} x_\sigma d\lambda \Rightarrow \underline{D\phi = \varepsilon^{\sigma\rho} x_\sigma \partial_\rho \phi}$

$\Rightarrow \boxed{D\phi = \varepsilon_{\sigma\rho} x^\sigma \partial^\rho \phi}$

→ Since $\mathcal{L}$ is a Lorentz scalar, it depends only on x through dependence on field + its derivatives:

$D\mathcal{L} = \varepsilon_{\sigma\rho} x^\sigma \partial^\rho \mathcal{L} = \partial^\rho(\varepsilon_{\sigma\rho} x^\sigma \mathcal{L})$ ~~$\partial_\mu(\varepsilon$~~

$= \partial_\mu(g^{\mu\rho} \varepsilon_{\sigma\rho} x^\sigma \mathcal{L}) = \partial_\mu F^\mu$

$\Rightarrow J^\mu = \Pi^\mu D\phi - F^\mu = \Pi^\mu \varepsilon_{\sigma\rho} x^\sigma \partial^\rho \phi - g^{\mu\rho} \varepsilon_{\sigma\rho} x^\sigma \mathcal{L}$

$= \boxed{\varepsilon_{\sigma\rho} x^\sigma (\Pi^\mu \partial^\rho \phi - g^{\mu\rho} \mathcal{L})}$ $=$ ~~$\varepsilon_{\sigma\rho}(\Pi^\mu x^\sigma \partial^\rho \phi$~~

→ Since current must be conserved for all 6 components of $\varepsilon_{\sigma\rho}$, the part of that is antisymmetric in σ + ρ must be conserved

$=$ ~~$\varepsilon_{\sigma\rho}(\Pi^\mu x^\sigma \partial^\rho \phi -$~~ $\to \partial_\mu M^{\mu\sigma\rho} = 0$

(antisymmetrization)

where $M^{\mu\sigma\rho} = \Pi^\mu x^\sigma \partial^\rho \phi - x^\sigma g^{\mu\rho} \mathcal{L} - \alpha \leftrightarrow \beta$

$= x^\mu(\Pi^\mu \partial^\rho \phi - g^{\mu\rho} \mathcal{L}) - \alpha \leftrightarrow \beta = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma}$

$\to J^{\mu\sigma} = \int d^3x\, M^{0\sigma\rho} = \int d^3x (x^\sigma T^{0\rho} - x^\rho T^{0\sigma})$ (6 conserved charges)

→ We will call quantities conserved due to rotations angular momentum (ε_{ij}). For instance,

$$J^{12} = \int d^3x \left(x^1 T^{02} - x^2 T^{01} \right)$$ is conserved from

invariance of rotation about x^3 axis.

What about boosts?

$$J^{0i} = \int d^3x \left(x^0 T^{0i} - x^i T^{00} \right)$$

→ This has explicit time dependenc (x^0), something we're not encountered in conservation laws:

From definition of J, $\frac{d}{dt} J^{0i} = 0$

$$= \frac{d}{dt}\left[t \int d^3x\, T^{0i} - \int d^3x\, x^i T^{00} \right]$$

$$= t \frac{d}{dt} \int d^3x\, T^{0i} + \int d^3x\, T^{0i} - \frac{d}{dt}\int d^3x\, x^i T^{00}$$

$$= \cancel{t \frac{d}{dt}(p^i)} + p^i - \frac{d}{dt}\int d^3x\, x^i T^{00}$$

$=0$ from momentum conservation

$$\Rightarrow p^i = \frac{d}{dt}\int d^3x\, x^i T^{00} = \text{constant}$$

→ This is just field-theoretic + relativistic generalization that "center of mass moves w/ constant →

velocity: but now w/ center of energy.

We say J^{0i} ~~is~~ are the Lorentz partner to the ~~quantity J~~ angular momentum.

Internal Symmetries

→ There are observed conserved quantities (charge, Baryon # etc) that DON'T come from space-time transformations. Such transformations are internal symmetries.

Example:

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{2} \partial_\mu \phi_a \partial^\mu \phi_a - \mu^2 \phi_a \phi_a - g\Big(\sum_a (\phi_a)^2\Big)^2$$

→ This is a theory of 2 scalar-fields, w/ a common mass μ + potential

$$g\Big(\sum_a (\phi_a)^2\Big)^2 = g\big((\phi_1)^2 + (\phi_2)^2\big)^2$$

→ This Lagrangian is invariant under the following transformation:

$$\phi_1 \to \phi_1 \cos(\lambda) + \phi_2 \sin(\lambda)$$
$$\phi_2 \to -\phi_1 \sin(\lambda) + \phi_2 \cos(\lambda)$$

→ This is just a rotation of ϕ_1 into ϕ_2 in "field space"!

→

$\rightarrow (\phi_1 \cos(\lambda) + \phi_2 \sin(\lambda))^2 + (-\phi_1 \sin(\lambda) + \phi_2 \cos(\lambda))^2$

$= \phi_1^2 \cos^2\lambda + \cancel{2\phi_1\phi_2 \cos\lambda \sin\lambda} + \phi_2^2 \sin^2\lambda + \phi_1^2 \sin^2\lambda$

$\cancel{-2\phi_1\phi_2 \sin\lambda \cos\lambda} + \phi_2^2 \cos^2\lambda = \underline{\phi_1^2 + \phi_2^2}$, so potential term invariant under this internal transformation.

(Similar for $(\partial_\mu \phi_1)^2$, $(\partial_\mu \phi_2)^2$).

In matrix form, the transformation can be read off as:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos(\lambda) & \sin(\lambda) \\ -\sin(\lambda) & \cos(\lambda) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$\rightarrow$ This is known as an SO(2) transformation.
(S ≡ "special", O ≡ "orthogonal", 2 ≡ 2x2 matrix)

"$\mathcal{L}$ has an SO(2) symmetry"

Corresponding conserved charge:

$D\phi_1 = \left.\frac{\partial \phi_1'}{\partial \lambda}\right|_{\lambda=0} = \phi_2$, $D\phi_2 = \left.\frac{\partial \phi_2'}{\partial \lambda}\right|_{\lambda=0} = -\phi_1$

$D\mathcal{L} = 0 \rightarrow F^\mu = \text{constant}$, so we ignore it.

$\Rightarrow J^\mu = \Pi_1^\mu D\phi_1 + \Pi_2^\mu D\phi_2 = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1$

Charge: $Q = \int d^3x\, J^0 = \int d^3x\, (\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1) \rightarrow$

This conserved quantity, however, is not very interesting on the classical level, however upon second quantization, it has a very nice interpretation in terms of particles and anti-particles.

→ Lets forget about the potential term:

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{2} \partial_\mu \phi_a \partial^\mu \phi_a + \mu^2 \phi_a \phi_a$$

→ Denote creation + annihilation operators $a^\dagger_{ki}$, a_{ki}, $i=1,2$:

$$a^\dagger_{k1}|0\rangle = |k,1\rangle, \quad a^\dagger_{k2}|0\rangle = |k,2\rangle$$

→ Substituting the free field expansion

$$\phi_i = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[a_{ki}e^{-ik\cdot x} + a^\dagger_{ki}e^{ik\cdot x}\right]$$

$$\Rightarrow \dot{\phi}_i = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}(-i\omega_k)\left[a_{ki}e^{-ik\cdot x} - a^\dagger_{ki}e^{ik\cdot x}\right]$$

$$\Rightarrow \dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1 = \int \frac{d^3k\, d^3p}{(2\pi)^3 2\omega_k}(-i\omega_k)\left(a_{k1}e^{-ik\cdot x} + a^\dagger_{k1}e^{ik\cdot x}\right)\left(a_{p2}e^{-ip\cdot x} + a^\dagger_{p2}e^{ip\cdot x}\right)$$

$- \ldots$

$$= \int \frac{d^3k\, d^3p}{(2\pi)^3 2\omega_k}(-i\omega_k)\left[a_{k1}a_{p2}e^{-i(k+p)\cdot x} + a_{k1}a^\dagger_{k2}e^{-i(k-p)\cdot x} - a^\dagger_{k1}a_{k2}e^{-i(p-k)\cdot x} - a^\dagger_{k1}a^\dagger_{k2}e^{-i(-k-p)\cdot x}\right]$$

$$\Rightarrow Q = \int \frac{d^3k\, d^3p\, d^3x}{(2\pi)^3\, 2\omega_k} (-i\omega_k)(\cdots)$$

$$= \int \frac{d^3k\, d^3p\, i}{2} \left(\frac{d^3x}{(2\pi)^3} \left[a_{k1} a_{p2} e^{-i(k+p)\cdot x} + a_{k1} a_{p2}^\dagger e^{-i(k-p)\cdot x} - a_{k1}^\dagger a_{p2} e^{-i(p-k)\cdot x} - a_{k1}^\dagger a_{p2}^\dagger e^{-i(-k-p)\cdot x} \right] \right)$$

$$= -i \int \frac{d^3k\, d^3p}{2} \left(a_{k1} a_{p2}\, \delta^{(3)}(\vec{k}+\vec{p}) + a_{k1} a_{p2}^\dagger\, \delta^{(3)}(\vec{k}-\vec{p}) - a_{k1}^\dagger a_{p2}\, \delta^{(3)}(\vec{p}-\vec{k}) - a_{k1}^\dagger a_{p2}^\dagger\, \delta^{(3)}(-\vec{k}-\vec{p}) \right)$$

$$= -i \int \frac{d^3k}{2} \left(a_{k1} a_{-k2} + a_{k1} a_{k2}^\dagger - a_{k1}^\dagger a_{k2} - a_{k1}^\dagger a_{-k2}^\dagger \right)$$

these will cancel w/ terms in $-\dot{\phi}_2 \phi_1$, leaving:

$$\boxed{Q = i \int d^3k \left(a_{k1}^\dagger a_{k2} - a_{k2}^\dagger a_{k1} \right)}$$

→ This LOOKS like a number operator, except the terms are off-diagonal.

→ Can fix be redefining creation + annihilation operators (finding eigenstates of operator)

$$\rightarrow b_k \equiv \frac{a_{k1} + i a_{k2}}{\sqrt{2}}, \quad b_k^\dagger = \frac{a_{k1}^\dagger - i a_{k2}^\dagger}{\sqrt{2}}$$

$$c_k \equiv \frac{a_{k1} - i a_{k2}}{\sqrt{2}}, \quad c_k^\dagger \equiv \frac{a_{k1}^\dagger + i a_{k2}^\dagger}{\sqrt{2}}$$

(it turns out these obey commutation relations for creation / annihilation operators) →

Now, $b_k^\dagger|0\rangle = \frac{|\vec{k}(1)\rangle + i|\vec{k}(2)\rangle}{\sqrt{2}} \equiv |\vec{k}(b)\rangle$

(b particle w/ momentum $\vec{k}$)

+ $c_k^\dagger|0\rangle = |\vec{k}(c)\rangle$

In terms of b's + c's:

$$Q = i\int d^3k\,(a_{k1}^\dagger a_{k2} - a_{k2}^\dagger a_{k1})$$

$$= \int d^3k\,(b_k^\dagger b_k - c_k^\dagger c_k) = N_b - N_c$$

$$= \int d^3k\, b_k^\dagger b_k - \int d^3k\, c_k^\dagger c_k$$

So $N_b - N_c$ is a globally-conserved quantity due to internal symmetry!

→ Conserved charge $N_b - N_c$ means that each b-particle has +1 charge, c-particles have -1 conserved charge, but $\mu_1 = \mu_2$, so b + c only vary by charge, anti-particles!

→ call b "particle", c "antiparticle"

→ In general, the existence of a particle w/ a conserved charge implies existence of anti-particle w/ opposite conserved charge!

→

→ We constructed b's + c's because the original particles constituted a bad basis. With this realization, we can redefine our Lagrangian:

Complex (Non-Hermitian) Field!

$$\psi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

and then $\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - \mu^2 \psi^\dagger \psi$

w/ expansions $\psi = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(b_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x} \right)$

$$\psi^\dagger = \int \frac{d^3k}{(2\pi)^{3/2} 2\omega_k} \left(c_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x} \right)$$

→ This ψ field annihilates b's + creates c's (destroys particles, creates anti-particles)

⇒ ψ always reduces charge by 1, $\psi^\dagger$ increases charge by 1.

From now on, charged fields will be described by complex fields.

→ The internal symmetry in terms of action on ψ is

$$\psi \to e^{-i\lambda}\psi, \text{ as } e^{i\lambda}\psi = \cancel{(\cos(\lambda) + i\sin(\lambda))}\,\psi$$

$$= (\cos(\lambda) - i\sin(\lambda)) \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$= \cos(\lambda)\phi_1 + i\cos(\lambda)\phi_2 - i\sin(\lambda)\phi_1 - i^2\sin(\lambda)\phi_2$$

$\Rightarrow \psi^{+} \rightarrow (\psi^{+})' = e^{i\lambda}\psi \quad \Rightarrow (\psi^{+})'\psi' = e^{i\lambda}\psi e^{-i\lambda}\psi = \underline{\psi\psi}$

To find Euler-Lagrange equations for a complex field theory, treat ψ & ψ^{+} <u>independently</u> (independent fields wrt variations)

$$\rightarrow \Pi^{\mu}_{\psi} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}, \quad \Pi^{\mu}_{\psi^{+}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^{+})}$$

$$, \mathcal{L} = \partial_\mu \psi^{*} \partial^{\mu} \psi - \mu^{2} \psi^{*} \psi$$

$$\Rightarrow \Pi^{\mu}_{\psi} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)}(\partial_\mu \psi^{*} \partial^{\mu} \psi)$$

$$= \partial_\mu \psi^{*} \frac{\partial}{\partial(\partial_\mu \psi)}(\partial^{\mu} \psi) = \partial_\nu \psi^{*} \delta^{\mu\nu} = \cancel{\partial_\mu \psi^{*}} = \partial^{\mu} \psi^{*}$$

(similarly, $\Pi^{\mu}_{\psi^{*}} = \partial^{\mu} \psi$)

$$\Rightarrow \partial_\mu \Pi^{\mu}_{\psi} = \frac{\partial \mathcal{L}}{\partial \psi} \Rightarrow \cancel{\partial_\mu \partial^{\mu} \Pi} \quad \partial_\mu \partial^{\mu} \psi^{*} + \mu^{2} \psi^{*} = 0$$

$$= (\partial_\mu \partial^{\mu} + \mu^{2}) \psi^{*} = \underline{(\Box + \mu^{2}) \psi^{*} = 0}$$

$$(\text{and } (\Box + \mu^{2}) \psi = 0)$$

Adding + subtracting equations recovers ϕ_1 + ϕ_2 c.o.m.

→ Can also quantize the theory by imposing commutation relations:

$$[\psi(\vec{x},t), \Pi_\psi^0(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}), \; [\psi^\dagger(\vec{x},t), \Pi_{\psi^\dagger}^0(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

...

→ Can be shown to yield correct commutation relations for ϕ_1, ϕ_2 fields.

Now, ψ + ψ^* are obviously not independent, so why treat them as such in EL? It works because there are 2 real DOF in $\phi_1 + \phi_2$ and 2 real DOF in ψ which may be independently varied. Consider the EL equations for a general theory of a complex field ψ:

$$\Rightarrow \delta S = \int d^4x \left(A\delta\psi + A^*\delta\psi^* \right) = 0$$

where A is some function of the fields.

To obtain EOM, first peform $\delta\psi$, which is real, $\delta\psi = \delta\psi^*$:

$$\Rightarrow A + A^* = 0$$

Then imaginary (purely): $\delta\psi = -\delta\psi^* \Rightarrow A - A^* = 0$

Combining, $A = A^* = 0$, ~~which is no good.~~

Instead, imagine ψ + ψ^* unrelated so we can vary independently. First take $\delta\psi^* = 0$

$\Rightarrow A = 0$. Similarly take $\delta\psi = 0 \Rightarrow A^* = 0$
so $A = A^* = 0$, so we get same equations of motion.

→ Refer to complex fields as charged fields..., fields charged in that they carry U(1) symmetry (U(1) conserved quantum #)

Non-Abelian Internal Symmetries

→ order, unlike U(1), of transformation matters

$$\mathcal{L} = \sum_{a=1}^{3} \frac{1}{2}(\partial_\mu \phi_a \partial^\mu \phi_a - \mu^2 \phi_a) + \mathcal{L}_{int}(\varphi_a^2)$$

→ has SO(3) symmetry (can rotate fields into each other)

In general, can have n fields w/ symmetry group of n-rotations (SO(n)) w/ $\frac{n(n-1)}{2}$ independent rotations. (n choices for axis, $n-1$ for next, avoid double count...)

→ ie SO(n) has $\frac{n(n-1)}{2}$ generators.

, $\frac{n(n-1)}{2}$ conserved charges...

non-abelianess ⇒ charge operators won't commute, ie states cannot be eigenstates of all charges.

Free field theory w/ a source

Emag: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ (Free field theory for photons)

Source term $A_\mu J^\mu$:

$\rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu(x) J^\mu(x) \Rightarrow \partial_\mu F^{\mu\nu} = J^\nu$
(Maxwells eqs. w/ source)

In general, adding term to $\mathcal{L}$ linear in field will give source term w/ factor

→ In general, J^μ & $F^{\mu\nu}$ backreact, since soures are charges which affect charges which move charges...

Introduce classical source, ie source independent of field induced.

→ $J^\mu(x) = (\rho/c(x), \vec{j}(x))$ given (no dynamics)

General source Lagrangian:

→ $\mathcal{L} = \mathcal{L}_0 - \rho(x)\varphi(x)$

→ we can solve this exactly (analogous to quantum driven harmonic oscillator)

→ We also assume $\rho(x)$ is turned off in far past + far future, ie source has finite extent in spacetime. We want to find the state of the system.

EOM:

$$(\Box + \mu^2)\phi(x) = -\rho(x)$$ → need to solve this

Claim: $$\phi(x) = \underbrace{\phi_0(x)}_{\text{KG solution}} + i\int d^4y\, D_R(x-y)\,\rho(y)$$

Free theory w/ source continued

$\rightarrow \phi(x) = \phi_0(x) + (-i)\int d^4y\, D_R(x-y)\rho(y)$

where $D_R(x-y)$ satisfies

- $(\partial_\mu\partial^\mu + \mu^2) D_R(x-y) = -i\delta^{(4)}(x-y)$
- $D_R(x-y) = 0,\ x^0 < y^0$

as $\int d^4y\, D_R(x-y)\rho(y) = \cancel{\int d^4}$

$$(\partial_\mu\partial^\mu + \mu^2)\left(-i\int d^4y\, D_R(x-y)\rho(y)\right) =$$

$$-i\int d^4y\,(\partial_\mu\partial^\mu + \mu^2) D_R(x-y)\rho(y)$$

$$= -i\int d^4y\,(-i\delta^{(4)}(x-y))\rho(y) = -\int d^4y\,\delta^{(4)}(x-y)\rho(y)$$

$$= \underline{-\rho(x)} \text{ as desired}$$

$\rightarrow D_R(x-y) = 0,\ x^0 < y^0$ so boundary condition $\phi(x) \rightarrow \phi_0(x)$ as $x^0 \rightarrow -\infty$ is satisfied
(look at particular spacetime coordinate $x = (x^0, \vec{x})$ and integral vanishes for $x^0 \rightarrow -\infty$)

$\rightarrow$ To find $D_R(x-y)$, it is convenient to work in momentum space:

$$D_R(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}\,\tilde{D}_R(k)$$

Plug into definition:

shouldn't exp. be positive?

$$\to (\Box+\mu^2) D_r(x-y) = (\Box+\mu^2)\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{D}_R(k)$$

$$= \int \frac{d^4k}{(2\pi)^4} (\partial_\mu\partial^\mu + \mu^2)\tilde{D}_R(k) e^{-ik\cdot(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} (\partial_\mu\partial^\mu e^{-ik\cdot(x-y)} + \mu^2 e^{-ik\cdot(x-y)})\tilde{D}_R(k)$$

$$\to \partial_\mu\partial^\mu e^{-ik\cdot(x-y)} \propto \partial_\mu\partial^\mu e^{-ik_\nu x_\nu}$$

$$\to \cancel{\partial_\mu(k_\nu^\mu x_\nu^\mu) = k_\nu \delta^\nu_\mu = k_\mu}$$

$$\partial_\mu(k_\nu x^\nu) = k_\nu \delta^\nu_\mu = k_\mu \Rightarrow \partial^\mu\partial_\mu(k_\nu x^\nu) = \partial^\mu k_\mu$$

$$= k_\mu \partial^\mu (k_\nu x^\nu) = k_\mu k_\nu g^{\mu\nu} = k_\mu k^\mu = +k^2$$

$$\Rightarrow \partial_\mu\partial^\mu e^{-ik\cdot(x-y)} = (-i)^2 k^2 e^{-ik\cdot(x-y)} = -k^2 e^{-ik\cdot(x-y)}$$

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} (-k^2+\mu^2) e^{-ik\cdot(x-y)} \tilde{D}_R(k) \stackrel{?}{=} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \cdot (-k^2+\mu^2)\tilde{D}_R(k)$$

WTH is going on here??

$$= (-k^2+\mu^2)\tilde{D}_R(k)\,\delta^{(4)}(x-y) = -i\,\delta^{(4)}(x-y)$$

$$\Rightarrow \underline{(-k^2+\mu^2)\tilde{D}_R(k) = -i}$$

$\to$

$$\int \frac{d^4k}{(2\pi)^4} (-k^2+\mu^2) e^{-ik\cdot(x-y)} \tilde{D}_R(k) = -i\delta^{(4)}(x-y)$$

integrate over spacetime:

$$\iint \frac{d^4x\, d^4k}{(2\pi)^4} (-k^2+\mu^2) e^{-ik\cdot(x-y)} \tilde{D}_R(k) = -i \int d^4x\, \delta^{(4)}(x-y)$$

$$= \int d^4k (-k^2+\mu^2) \tilde{D}_R(k) \int \frac{d^4x}{(2\pi)^4} e^{-ik\cdot(x-y)}$$

$$= \int d^4k (-k^2+\mu^2) \tilde{D}_R(k)\, \delta^{(4)}(k) = \underline{(-k^2+\mu^2)\tilde{D}_R(k) = -i}$$

(not very rigorous ... but works)
definitely ask...)

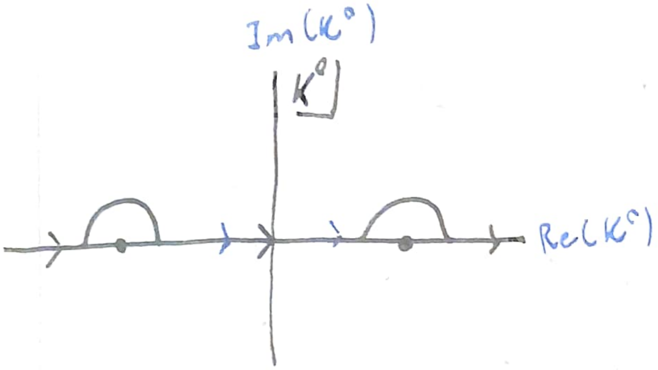
$$\Rightarrow \tilde{D}_R(k) = \frac{-i}{-k^2+\mu^2} = \frac{i}{k^2-\mu^2} \Rightarrow \boxed{D_R(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2} e^{-ik\cdot(x-y)}}$$

→ k^0 integrand has poles at $k^0 = \pm\omega_k$ as

$$k^2-\mu^2 = k^{0\,2} - \vec{k}^2 - \mu^2 = (k^0)^2 - \underbrace{(\vec{k}^2+\mu^2)}_{\text{dispersion relation } \omega_k^2}$$

$$= (k^0)^2 - \omega_k^2$$

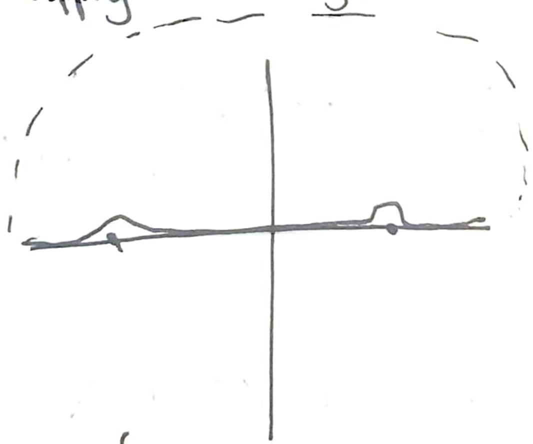
→ To define this integral, we must perform contour integration around the poles: →



→ Redefine integral in terms of a limiting procedure (various ways to do)

Now for $y^0 > x^0$ (in which we want the Green's func. to vanish to preserve causality)

→ If we consider adding the contour of a large semicircle, $e^{-ik\cdot(x-y)}$ will vanish, so we can close the contour w/out affecting $D_R(x-y)$ & apply Cauchy's Thm: (since contour contains no singularities)



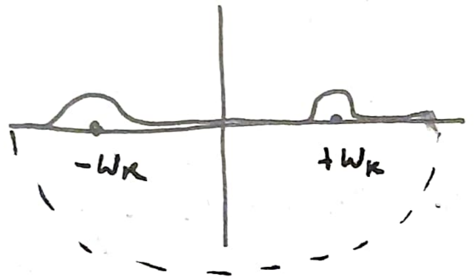
$$\rightarrow \oint_{C} \frac{d^4k}{(2\pi)^4} \lim_{\epsilon\to 0} \cdots = 0 \Rightarrow D_R(x-y) + 0 \therefore D_R(x-y) = 0$$

for $y^0 > x^0$

For $y^0 < x^0$, close in bottom half plane (since exp. large in top half) but contour contains 2 poles.

$$x^0 > y^0: \quad D_R(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2} e^{-ik\cdot(x-y)}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi i} \left(-\frac{1}{k^2-\mu^2} e^{-ik\cdot(x-y)} \right)$$



<u>Residue Theorem:</u> $\oint_\gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$

$$R(\omega_k) = \lim_{k^0 \to \omega_k} (k^0 - \omega_k) \frac{1}{k^{0^2} - \vec{k}^2 - \mu^2} e^{-ik\cdot(x-y)}$$

$$= \lim_{k^0 \to \omega_k} (k^0 - \omega_k) \frac{1}{k^{0^2} - \omega_k^2} e^{-ik\cdot(x-y)}$$

$$= \lim_{k^0 \to \omega_k} \frac{1}{k^0 + \omega_k} e^{-ik\cdot(x-y)} = \frac{1}{2\omega_k} e^{-ik\cdot(x-y)} \Big|_{k^0 = \omega_k}$$

Similarly,

$$R(-\omega_k) = \lim_{k^0 \to -\omega_k} (k^0 + \omega_k) \frac{1}{(k^0-\omega_k)(k^0+\omega_k)} e^{-ik\cdot(x-y)}$$

$$= -\frac{1}{2\omega_k} e^{-ik\cdot(x-y)} \Big|_{k^0 = -\omega_k} \longrightarrow$$

$$\Rightarrow D_R(x-y) = \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2\pi i}\right)\cancel{(2\pi i)}\left(\frac{1}{2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=\omega_k}\right.$$

$$\left. + \left(-\frac{1}{2\omega_k}\right)e^{-ik\cdot(x-y)}\Big|_{k^0=-\omega_k}\right)$$

$(k^0 = \pm\sqrt{(\vec{k}^2+\mu^2)}$

why does $k^0 = -\omega_k$ introduce whole sign flip? & not just on x^0-y^0 component?

$$= \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2\omega_k}\right)\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)$$

$= D(x-y) - D(y-x) = [\phi(x), \phi(y)]$ (commutator of free fields)

$$\Rightarrow D_R(x-y) = \Theta(x^0-y^0)[\phi(x),\phi(y)]$$

$$= \Theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle$$

Inserting into solution:

$$\Rightarrow \phi(x) = \phi_0(x) + i\int d^4y \int \frac{d^3k}{(2\pi)^3 2\omega_k}\Theta(x^0-y^0)\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)\rho(y)$$

Evaluating in the limit $x^0 \to \infty$ so that all of $\rho(x)$ is in the past.

$$\to \Theta(x^0-y^0) = 1 \Rightarrow \phi(x) = \phi_0(x) + i\int d^4y \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)\rho(y)$$

$$= \phi_0(x) + i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\int d^4y \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)\rho(y)$$

$\to$

Defining the Fourier transform of $\rho(y)$:

$$\tilde{\rho}(K) = \int d^4y \, e^{iK\cdot y} \rho(y)$$

$$\rightarrow \phi(x) = \phi_0(x) + i \int \frac{d^3K}{(2\pi)^3 (2\omega_K)} \left(e^{-iK\cdot x} \tilde{\rho}(K) - e^{iK\cdot y} \tilde{\rho}(-K) \right) \quad (w/\ K^0 = \omega_K)$$

→ this almost looks like the expression for the free field $\phi_0(x)$ but w/ a function multiplying exponentials

$$\rightarrow \boxed{\phi(x) = \int \frac{d^3K}{(2\pi)^{3/2}\sqrt{2\omega_K}} \left\{ \left(a_K + \frac{i}{(2\pi)^{3/2}\sqrt{2\omega_K}} \right) e^{-iK\cdot x} + h.c. \right\}}$$

→ Since all observables are constructed from fields, our theory is solved!

Finding the Hamiltonian:

$$H = \int d^3x \, T^{00} = \int d^3x \left(\Pi^0 \dot{\phi} - \mathcal{L} \right), \quad \Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

(or simply replace $a_K \rightarrow a_K + \frac{i}{(2\pi)^{3/2}\sqrt{2\omega_K}} e^{-iK\cdot x}$ + other)

$$\rightarrow H_{free} = \int d^3K \, a_K^\dagger a_K \omega_K \rightarrow H = \int d^3K \, \omega_K \left(a_K^\dagger - \frac{i}{(2\pi)^{3/2}\sqrt{2\omega_K}} \tilde{\rho}^*(K) \right) \left(a_K + \frac{i}{(2\pi)^{3/2}\sqrt{2\omega_K}} \tilde{\rho}(K) \right)$$

$$\left(\tilde{\rho}^*(K) = \tilde{\rho}(-K) \right)$$

→ The energy in the Future is simply $\langle 0|H|0\rangle$ since the states don't evolve in the Heisenberg picture.

$$\langle 0|:H:|0\rangle = \langle 0| \int d^3k\, \omega_k \left(a_k^\dagger - \frac{i}{(2\pi)^{3/2}\sqrt{2\omega_k}}\tilde{\rho}(k)\right)\left(a_k + \frac{i}{(2\pi)^{3}\sqrt{2\omega_k}}\tilde{\rho}(k)\right) |0\rangle$$

→ Notice that the only contributing term will be product of $\tilde{\rho}(k)$ as there is always a annihilation operator acting on a vacuum bra/ket

$$\rightarrow \langle 0|H|0\rangle = \langle 0| \int \frac{d^3k}{2(2\pi)^3} \tilde{\rho}(k)^2 |0\rangle$$

$$= \boxed{\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} |\tilde{\rho}(k)|^2}$$ (Vacuum energy expectation value)

→ The only way to add energy to a free theory is add particles, so we have particle production! How many of each?

$$dN(\vec{k}) = \frac{|\tilde{\rho}(k)|^2}{(2\pi)^3 2\omega_k} d^3k$$ (integrand w/ measure is total energy contribution from particles of energy ω_k, so divide by ω_k to yield # of particles.

→ Each Fourier component of ρ produces particles w/ corresponding 4-momentum w/ probability proportional to $|\tilde{\rho}(k)|^2$.

→ Expectation value for total # of particles is:

$$\int dN = \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{\rho}(k)|^2$$

More on Green Functions

→ We studied the retarded Green function w/ path of integration passing above poles in k^0 ℂ space, but we have other GF for other boundary conditions:

- advanced Green function, $G_A(x-y) = 0$ for $x^0 > y^0$ (opposed to $x^0 < y^0$ for $D_R(x-y)$)

  → useful for knowing value of field in the far future + interested in value of field before the source is switched on.

- below $-\omega_k$, above ω_k (Feynman propogator)

  → For $x^0 > y^0$, peform k^0 integral by closing contour below. For $x^0 < y^0$, close above

→ $$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$ →

$= \Theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$

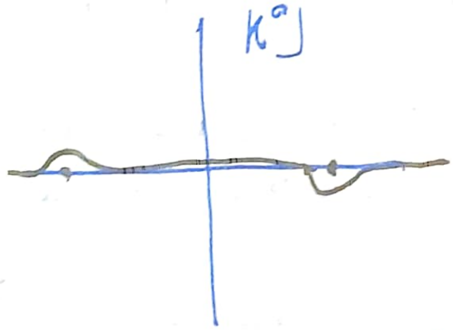
$= \underline{\langle 0|T\phi(x)\phi(y)|0\rangle}$ → in notes, but in lecture is ~~$\langle 0|T$~~ $\langle 0|T[\phi(x),\phi(y)]|0\rangle$...
ask

where T is the time-ordering symbol, which tells us to place the operators that follow in order w/ latest on the left.

$\cancel{\langle 0|T[\phi(x),\phi(y)]|0\rangle = \langle 0|T\phi(x)\phi(y)|0\rangle + \langle 0|T\phi(y)\phi(x)|0\rangle}$

note that time-ordering is well defined

$= \cancel{\langle 0|}$



$\rightarrow \lim_{\varepsilon\to 0^+}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon}e^{-ik\cdot(x-y)}$

as poles occur at $(k^0)^2-(\vec{k}^2+\mu^2)+i\varepsilon=0$

$\therefore (k^0)^2 = \omega_k^2 - i\varepsilon$

$\therefore k^0 = \pm\sqrt{\omega_k^2 - i\varepsilon}$

(notes say $k^0 = \pm(\omega_k - i\varepsilon)$...)

Q. How to get this propagator from first principles? ie solving for EOM + BC? (as done for retarded Greens)

→ Note → all propagators solve same PDE but satisfy different BC!

Mesons coupled to dynamic source

→ Previous Lagrangian has a fixed source, but in the real world the current itself interacts w/ the field, resulting in complicated dynamics. Lets couple 2 fields

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + (-g\psi^\dagger\psi\phi)$$

→ note interaction term does not break $U(1)$ symmetry $\psi \to e^{i\theta}\psi$, so interacting theory will conserve charge.

→ In analogy w/ source problem, this theory describes the creation of ϕ fields (photons) by ψ fields (electrons)

Equations of Motion:

$$\phi: \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}(\mathcal{L}_\phi + \mathcal{L}_\psi - g\psi^\dagger\psi\phi)$$

$$\cancel{= \partial^\mu\phi \to \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi}$$

where $\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2)$, $\mathcal{L}_\psi = \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi$

$$\to \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left(\frac{1}{2}(\partial_\nu\phi\partial^\nu\phi - \mu^2\phi^2)\right)$$

$$= \frac{1}{2}\left(\partial_\nu\phi\frac{\partial(\partial^\nu\phi)}{\partial(\partial_\mu\phi)} + \partial^\nu\phi\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\mu\phi)}\right) \to$$

$$= \frac{1}{2}\left(\partial_\nu\phi\, g^{\nu\mu} + \partial^\nu\phi\, \delta^\mu_\nu\right) = \frac{1}{2}\left(\partial^\mu\phi + \partial^\mu\phi\right) = \partial^\mu\phi$$

$$\Rightarrow \partial_\mu \Pi^\mu_\phi = \partial_\mu\partial^\mu\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial}{\partial\phi}\left(-\frac{1}{2}\mu^2\phi^2 - g\psi^\dagger\psi\phi\right) = -\mu^2\phi - g\psi^\dagger\psi$$

$$\Rightarrow \partial_\mu\partial^\mu\phi = -\mu^2\phi - g\psi^\dagger\psi \Rightarrow \boxed{\partial_\mu\partial^\mu\phi + \mu^2\phi = -g\psi^\dagger\psi}$$

ψ / $\psi^\dagger$: $\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial\psi}\left(-m^2\psi^\dagger\psi - g\psi^\dagger\psi\phi\right)$

$$= -m^2\psi^\dagger - g\psi^\dagger\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \frac{\partial}{\partial(\partial_\mu\psi)}\left(\partial_\nu\psi^\dagger\partial^\nu\psi\right)$$

$$= \partial_\nu\psi^\dagger \frac{\partial(\partial^\nu\psi)}{\partial(\partial_\mu\psi)} = \partial_\nu\psi^\dagger g^{\nu\mu} = \partial^\mu\psi^\dagger$$

$$\Rightarrow \partial_\mu\Pi^\mu_\psi = \partial_\mu\partial^\mu\psi^\dagger \Rightarrow \partial_\mu\partial^\mu\psi^\dagger + m^2\psi^\dagger + g\psi^\dagger\phi = 0$$

$$\Rightarrow \partial_\mu\partial^\mu\psi^\dagger + m^2\psi^\dagger = -g\psi^\dagger\phi$$

$$\Rightarrow \boxed{\partial_\mu\partial^\mu\psi + m^2\psi = -g\psi\phi}$$

→ System of coupled PDEs which are in fact unsolvable!

= "Nucleon-meson field theory"

→ We know theory for $g=0$ (free field), so our →

solution will be a power series in g, solved perturbatively in g:

$$a = A + gB + g^2 C + \cdots$$

; since g has dimensions, expansion in terms of g/E, where E is some characteristic energy of the problem. (here, $g \ll \mu$ should be sufficient)

Interaction Picture

Recalling Schrödinger + Heisenberg pictures:

$$|\psi(0)\rangle_S = |\psi(0)\rangle_H = |\psi(0)\rangle_I \,;\; O_S(0) = O_H(0) = O_I(0)$$

; In Schrödinger picture, states carry time-dependence:

$$O_S(t) = O_S(0), \quad i\frac{d}{dt}|\psi(t)\rangle_S = H|\psi(t)\rangle_S$$

vs. Heisenberg:

$$|\psi(t)\rangle_H = |\psi(0)\rangle_H, \quad i\frac{d}{dt}|O_H(t)\rangle = [O_H(t), H]$$

→ Split Hamiltonian into 2 parts:

$$H = H_0 + H_I,$$ where H_0 is the free Hamiltonian

→

$\rightarrow H = \int d^3x \sum_a \Pi_a^0 \dot{\phi}_a - L = \int d^3x \sum_a \Pi_a^0 \dot{\phi}_a - L_0 - L_I$

$\rightarrow$ if $\mathcal{L}_I$ contains no derivatives of fields so it doesn't change conjugate momenta of free theory, then we get!

$$H_I = -L_I, \text{ since}$$

$H = H_I + H_0 = \int d^3x \sum_a \Pi_a^0 \dot{\phi}_a - L_0 - L_I$ w/ Π_a^0 independent of $L_I \Rightarrow H_0 = \int d^3x \sum_a \Pi_a^0 \dot{\phi}_a - L_0$, $\rightarrow$ H of free theory since $\mathcal{L}_I$ left out.

$$H_I = -L_I$$

(If $\mathcal{L}_i$ has derivatives of ϕ, then conjugate p will be related to both L_0 & L_I, not separated).

$\rightarrow$ In nucleon-meson theory, we have

$$\mathcal{H}_I = -\mathcal{L}_I = g\psi^\dagger\psi\phi$$

$\rightarrow$ states in I.P. are defined by:

$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$$

$\rightarrow$ In a free field theory, $H_I = 0$

$$\rightarrow e^{iH_0 t}|\psi(t)\rangle_S = e^{iHt}|\psi(t)\rangle_S = |\psi(0)\rangle_S = |\psi(0)\rangle_H$$

which is precisely the Heisenberg picture!

→ constrain equality of matrix elements:

$${}_S\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle_S = {}_I\langle\psi(t)|\mathcal{O}_I|\psi(t)\rangle_I = {}_S\langle\psi(t)|e^{-iH_0t}\,\mathcal{O}_I(t)\,e^{iH_0t}|\psi(t)\rangle_S$$

$$= {}_S\langle\psi(t)|e^{-iH_0t}\,\mathcal{O}_I\,e^{iH_0t}|\psi(t)\rangle_S$$

so operators in the interaction picture evolve as:

$$\mathcal{O}_I(t) = e^{iH_0t}\,\mathcal{O}_I(0)\,e^{-iH_0t} \quad \text{(using alignment at } t=0\text{)}$$

$$\mathcal{O}_S(0) = e^{iH_0t}\,\mathcal{O}_I(t)\,e^{-iH_0t} = \mathcal{O}_I(0) \Rightarrow \underline{\mathcal{O}_I(t) = e^{iH_0t}\,\mathcal{O}_I(0)\,e^{-iH_0t}}$$

→ But this is the solution to the Heisenberg EOM!

$$i\frac{d}{dt}\mathcal{O}_I(t) = [\mathcal{O}_I(t), H]$$

This is useful as now fields in the interaction picture evolve just like free fields in the Heisenberg picture, so we can use our results from free field theory.

(since Dirac reduces to Heisenberg when $H = H_0$, all time dependency in states themselves are encoded in the interactions!)

From Schrödinger Equation:

$$i\frac{d}{dt}e^{-iH_0t}|\psi(t)\rangle_I = H_S\,e^{-iH_0t}|\psi(t)\rangle_I \quad \rightarrow$$

$\rightarrow H_0 e^{-iH_0 t}|\psi(t)\rangle_I + ie^{-iH_0 t}\frac{d}{dt}|\psi(t)\rangle_I = (H_0(0) + H_I(0))e^{-iH_0 t}|\psi(t)\rangle_I$

$\Rightarrow i\frac{d}{dt}|\psi(t)\rangle_I = e^{iH_0 t}H_I(0)e^{-iH_0 t}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I$

($H_0(0) = H_0$ due to energy conservation for free theory?)

where $H_I(t) = e^{iH_0 t}H_I(0)e^{-iH_0 t}$ as expected. When $H_I = 0$:

$i\frac{d}{dt}|\psi(t)\rangle_I = 0$, so fields in I.P. are time-independent. (should fields say states?)

Dysan's Formula:

Lets quantify finite extent into scattering:

$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + (-gf(t)\psi^\dagger\psi\phi)$

where $f(t)=0$ for large $|t|$, $f(t)=1$ for t near 0.

→ When the interaction does not generate bound states (of which Hamiltonian?), adding $f(t)$ to the interaction won't change scattering amplitute.

→ after long time scattering, interaction turned off slowly over period Δ, simple states in the real theory will turn into eigenstates of the free theory! (1-1 corresponsence between simple eigenstates and free eigenstates)

→ We want to solve $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$

with the boundary condition ~~$|\psi(\infty)\rangle = S|\psi(-\infty)\rangle = S|i\rangle$~~

$$|\psi(-\infty)\rangle = |i\rangle$$

→ we want to connect the description in the far past to that in the far future by defining the scattering operator:

$$|\psi(\infty)\rangle = S|\psi(-\infty)\rangle = S|i\rangle$$

→ then the amplitude to find the system in state $|f\rangle$ is:

$$\langle f|S|i\rangle \equiv S_{fi}$$

→ this is the S-matrix element, which we can solve for iteratively:

$$\rightarrow i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle \Rightarrow i\int_{t_1=-\infty}^{t}\frac{d}{dt}|\psi(t)\rangle = \int_{t_1=-\infty}^{t} H_I(t_1)|\psi(t_1)\rangle dt_1$$

$$= i(|\psi(t)\rangle - |\psi(-\infty)\rangle) = i(|\psi(t)\rangle - |i\rangle)$$

$$\Rightarrow |\psi(t)\rangle - |i\rangle = -i\int_{t_1=-\infty}^{t} dt_1\, H_I(t)|\psi(t_1)\rangle \Rightarrow |\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1\, H_I(t)|\psi(t_1)\rangle$$

Sub in $|\psi(t)\rangle$

$$= |i\rangle + (-i)\int_{-\infty}^{t} dt_1\, H_I(t_1)\left(|i\rangle + (-i)\int_{-\infty}^{t_1} dt_2\, H_I(t_2)|\psi(t_2)\rangle\right)$$

$$= |i\rangle + (-i)\int_{-\infty}^{t} dt_1\, H_I(t_1)|i\rangle + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} H_I(t_1) H_I(t_2)|\psi(t_2)\rangle dt_2$$

→

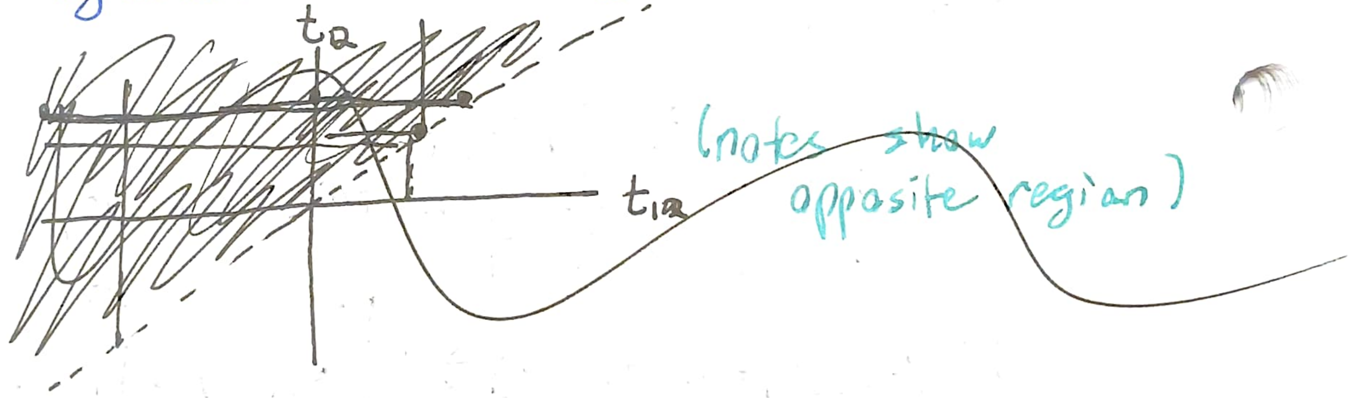
Repeating indefinitely & taking $t \to \infty$:

$$\boxed{S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \, H_I(t_1) \cdots H_I(t_n)}$$

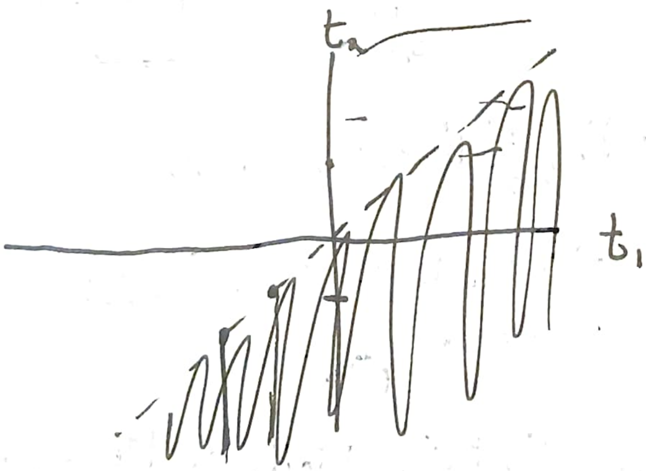
Looking at $n=2$ term:

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, H_I(t_1) H_I(t_2) \qquad (1)$$

→ This integration is over the region:



(notes show opposite region)



→ We can rewrite (1) as:

$$\int_{-\infty}^{\infty} dt_2 \int_{t_2}^{\infty} dt_1 \, H_I(t_1) H_I(t_2) = \int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \, H_I(t_2) H_I(t_1)$$

(relabel of integration variables)

Thus we can rewrite the $n=2$ term as.

$$\frac{1}{2!}\left[\int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2\, H_I(t_2) H_I(t_1) + \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, H_I(t_1) H_I(t_2)\right)$$

→ In the first term, $t_2 > t_1$, and in the second term, $t_1 > t_2$, so the H_I's are always ordered with earlier on the right. We define the time-ordered product:

$$T(O_1 O_2): T(O_1(x_1) O_2(x_2)) = \begin{cases} O_1(x_1) O_2(x_2) & t_1 > t_2 \\ O_2(x_2) O_1(x_1) & t_1 < t_2 \end{cases}$$

→ We can rewrite the second term in terms of the time-ordered product:

$$\frac{1}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, T(H_I(t_1) H_I(t_2))$$

☆ (If we fix a t_1, the dt_2 integral up to t_1 from $-\infty$ is time-ordered w/ $H(t_2)$ on the right, and the integral from t_1 to ∞ is ordered with $H(t_1)$ on the right. Expanding the integral as a sum, we recover the original expression at the top)

→ To generalize for n operators, we define such that earliest on right & latest on left. Since H_I commutes w/ itself at equal times, there is no ambiguity in the definition.

→

→ the n^{th} term can be written as:

$$\frac{1}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \, T(H_I(t_1)\ldots H_I(t_n))$$

$$\Rightarrow S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \, T(H_I(t_1)\ldots H_I(t_n))$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int d^4x_1 \cdots \int d^4x_n \, T(\mathcal{H}_I(x_1)\ldots \mathcal{H}_I(x_n))}$$

or slickly:

$$\boxed{S = Te^{-i\int d^4x \, \mathcal{H}_I(x)}}$$ → Dyson's Formula!

Wick's Theorem

→ Our whole problem now is basically to evaluate S.

→ Ex: Elastic nucleon N-N scattering (Ψ)

$N(\vec{k}_1) + N(\vec{k}_2) \rightarrow N(\vec{k}_3) + N(\vec{k}_4)$

Then $|i\rangle = |N(\vec{k}_1), N(\vec{k}_2)\rangle$, $|f\rangle = |N(\vec{k}_3), N(\vec{k}_4)\rangle$

Want: $\langle f|S|i\rangle$

$$\to \langle f|S|i\rangle = \langle f|i\rangle - i\int d^4x \langle f|H_I|i\rangle - i\int d^4x_1 \int d^4x_2 \langle f|T(H_I(x_1)H_I(x_2))|i\rangle + \cdots$$

$, H_i = -\mathcal{L}_i = g\psi^\dagger\psi\phi$

$\to \langle f|i\rangle = 0$ since states are orthogonal.
(ie in 0^{th} order perturbation, no scattering as free field theory)

What about $\langle f|H_I|i\rangle$? This is also 0. Why? Note that fields are superpositions of creation & annihilation operators, and the ϕ creation will annihilate bra to 0 & annihilation will annihilate ket to 0 as the states have no mesons.

(Since no mesons present, can we say $|i\rangle = |N(\vec{k}_1), N(\vec{k}_2), 0\rangle$?) so that $a_\phi|N_1, N_2, 0\rangle = 0$?)

$\langle f|T(H_I(x_1), H_I(x_2))|i\rangle$. Well the creation piece of $H(x_2)$ can create a meson on $|i\rangle$ while the annihilation piece of $H_I(x_1)$ can annihilate ~~create~~ a meson on $|f\rangle$. For nucleons:

$$\langle f| T[\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2))|i\rangle$$

$\to \psi^\dagger(x_2)\psi(x_2)$ annihilate $N(\vec{k}_1)$
create $N(\vec{k}_3)$

$\psi^\dagger(x_1)\psi(x_1)$ annihilate $N(\vec{k}_2)$
create $N(\vec{k}_4)$

→ Now we have to stick fields into a time-ordered product, yuck. But what is easy is normal-ordered products of fields. This is where Wick's Theorem comes in

Contraction

$$\overline{\varphi_a(x)\varphi_b(y)} = T(\varphi_a(x)\varphi_b(y)) - :\varphi_a(x)\varphi_b(y):$$

→ Obviously if we know the difference, we can find $::$, but what do we know about contractions? Well it turns out the contraction is a NUMBER, not an operator →.

If $x^0 > y^0$, $T[\phi(x)\phi(y)] = \phi(x)\phi(y)$

$$= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y))$$

→ only term not normal ordered is $\phi^+(x)\phi^-(y)$ (since NO has creation on left)

$$= :\phi(x)\phi(y): + [\phi^+(x), \phi^-(y)] \quad \text{since}$$

$::$ has $\phi^-(y)\phi^+(x)$ + $[\phi^+(x)\ \phi^-(y)] = \phi^+(x)\phi^-(y) - \phi^-(y)\phi^+(x)$, so terms cancel.

→ $[\phi^+(x), \phi^-(y)]$ is just a number + is the contraction.

→ Numbers are equal to their own expectation values!

$$\rightarrow \overbracket{A(x)\,B(y)} = \langle 0|\, A(x)\, B(y)\,|0\rangle$$

$$= \langle 0|\,T(A(x)\,B(y))\,|0\rangle - \langle 0|\,{:}A(x)\,B(y){:}\,|0\rangle$$

$\rightarrow \langle 0|\,{:}A(x)B(y){:}\,|0\rangle$ vanishes as annihilation operators are placed on the right and annihilate the vacuum.

→ Expectation value of contraction is just the expectation value of the time-ordered product:

$$\langle 0|\,T(A(x)\,B(y))\,|0\rangle = \langle 0|\,\overbracket{A(x)\,B(y)}\,|0\rangle$$
$$= \cancel{A(x)\,B(y)}$$
$$= \overbracket{A(x)\,B(y)}$$

But $\langle 0|\,T(A(x)B(y))\,|0\rangle$ is the Feynman propagator!

$$\overbracket{\phi(x)\,\phi(y)} = D_F(x-y) = \langle 0|\,T\phi(x)\phi(y)\,|0\rangle$$

$$= \lim_{\varepsilon\to 0^+} \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}\, \frac{i}{k^2-m^2+i\varepsilon}$$

For charged fields:

$$\overbracket{\psi(x)\,\psi^\dagger(y)} = \langle 0|\,T(\psi(x)\,\psi^\dagger(y))\,|0\rangle$$

$$= T(\psi(x)\,\psi^\dagger(y)) - {:}\psi(x)\,\psi^\dagger(y){:}$$

Why?

$$\rightarrow \overbracket{\psi(x)\,\psi^\dagger(y)} = \overbracket{\psi^\dagger(x)\,\psi(y)} = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)}\, \frac{i}{k^2-m^2+i\varepsilon}$$

In general, $T(\phi_1 \cdots \phi_n) = :\phi_1 \cdots \phi_n:$

$$+ :\overbrace{\phi_1\phi_2}\phi_3 \cdots \phi_n: + :\overbrace{\phi_1\phi_2\phi_3} \cdots \phi_n: + : \cdots :$$

$$+ :\phi_1\phi_2 \cdots \overbrace{\phi_{n-1}\phi_n}: + :\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} \cdots \phi_n:$$

$$+ \cdots + :\phi_1\phi_2 \cdots \overbrace{\phi_{n-3}\phi_{n-2}}\,\overbrace{\phi_{n-1}\phi_n}: + \cdots$$

→ better to pull contractions out of normal order.

(all possible field contractions)

For $n=3$ fields:

$\rightarrow \langle 0|T(\phi_a(x)\phi_b(y)\phi_c(z))|0\rangle$

For $x^0 > y^0 > z^0$: $T(\phi \cdots) = \phi_a(x)\phi_b(y)\phi_c(z)$

$$= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y))(\phi^+(z) + \phi^-(z))$$

$$\begin{aligned}
&= \phi^+(x)\phi^+(y)\phi^+(z) + \phi^+(x)\phi^+(y)\phi^-(z) \\
&+ \phi^+(x)\phi^-(y)\phi^+(z) + \phi^+(x)\phi^-(y)\phi^-(z) \\
&+ \phi^-(x)\phi^+(y)\phi^+(z) + \phi^-(x)\phi^+(y)\phi^-(z) + \phi^-(x)\phi^-(y)\phi^+(z) \\
&+ \phi^-(x)\phi^-(y)\phi^-(z)
\end{aligned}$$

→ To get annihilation ϕ^+ on right, we invoke the various commutators of fields, and when done for other time orderings, produces Feynman propogators.

→ Lets apply this for the $n=2$ perturbation term:

$$H_I = -\mathcal{L}_i = g\psi^\dagger\psi\phi$$

~~$S_2 = \frac{-ig}{}$~~

$$S_2 = \frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2\, T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right)$$

→ Since $T(\phi_1 \cdots \phi_n) = :\phi_1 \cdots \phi_n: + :\text{all possible contractions}:$, we only care about the contraction piece since the expectation value of the normal ordered product vanishes. (verify this) ~~as: S_2 is an exp. value, not an oper~~ (might only be true since $|i\rangle$ is a vacuum term for ϕ particles).

→ One term is given by:

$$\frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2 :\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2):$$

$$= :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\ \phi(x_1)\phi(x_2)$$

→ This term can contribute to nucleon-nucleon scattering due to the :: term, ie.

$$\langle \vec{k}_3(N); \vec{k}_4(N)| :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2): \phi(x_1)\phi(x_2) |\vec{k}_1(N)\vec{k}_2(N)\rangle$$
$$\neq 0$$

(Note that even though we have a :: product, the S-matrix element does not vanish, as they are not waching a vacuum nucleon state!)

→ other combinations can contribute to processes $\bar{N}+\bar{N} \to \bar{N}+\bar{N}$, $N+\bar{N} \to N+\bar{N}$, however there is no combination that can contribute to $N+N \to \bar{N}+\bar{N}$, as this would require 2 ψ to annihilate 2 nucleons and 2 more to create anti, but ψ^+ are present... but we already knew this, as the theory has a conserved U(1) charge which is not conserved in such a process.

→ Another term in S_2 is given by:

$$\frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2 :\psi^+(x_1)\phi(x_1)\psi(x_2)\phi(x_2):\ \overbrace{\psi(x_1)\psi^+(x_2)}$$

→ This term can contribute to the processes

$N+\phi \to N+\phi$, $\bar{N}+\phi \to \bar{N}+\phi$, $N+\bar{N} \to \phi+\phi$,

$\phi+\phi \to N+\bar{N}$

To see $N+\phi \to N+\phi$, we consider the transition amplitude from the state $|\vec{k}_1(\phi), \vec{k}_1(\psi)\rangle$ to $|\vec{k}_2(\phi), \vec{k}_2(\psi)\rangle$:

$$\langle f|S_2|i\rangle = \langle \vec{k}_2(\phi), \vec{k}_2(\psi)| :\psi^+(x_1)\phi(x_1)\psi(x_2)\phi(x_2): |\vec{k}_1(\phi), \vec{k}_1(\psi)\rangle$$

→ ϕ consists of creation & annihilation:

$$\psi(x_2)\phi(x_2) = \psi(x_2)\left(\phi^+(x_2)+\phi^-(x_2)\right)$$

$$\overset{::}{=} \phi^-(x_2)\psi(x_2) + \psi(x_2)\phi^+(x_2)$$

$$\Rightarrow \psi^+(x_1)\phi(x_1)\psi(x_2)\phi(x_2) = \psi^+(x_1)(\phi^-(x_1)+\phi^+(x_1))\psi(x_2)$$
$$(\phi^-(x_2)+\phi^+(x_2))$$

$$= (\psi^+(x_1)\phi^-(x_1) + \psi^+(x_1)\phi^+(x_1))\psi(x_2)(\phi^-(x_2)+\phi^+(x_2))$$

$$= (\psi^+(x_1)\phi^-(x_1)\psi(x_2) + \psi^+(x_1)\phi^+(x_1)\psi(x_2))(\phi^-(x_2)+\phi^+(x_2))$$

$$= \underbrace{\psi^+(x_1)\phi^-(x_1)\phi^-(x_2)\psi(x_2)}_{1} + \underbrace{\psi^+(x_1)\phi^-(x_1)\psi(x_2)\phi^+(x_2)}_{2}$$

$$+ \underbrace{\psi^+(x_1)\phi^-(x_2)\phi^+(x_1)\psi(x_1)}_{3} + \underbrace{\psi^+(x_1)\phi^+(x_1)\psi(x_2)\phi^+(x_2)}_{4}$$

$$\Rightarrow \langle \vec{k}_2(\phi), \vec{k}_2(\psi) | :\;: | \vec{k}_1(\phi), \vec{k}_1(\psi)\rangle = \dots$$

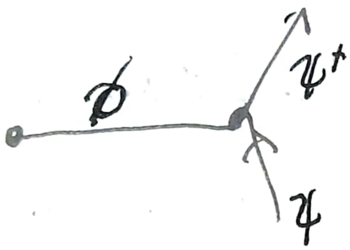
1 annihilates ψ particle on right, annihilates on left, ultimately 2+3 contribute

→ the other contractions in the 2^{nd} order term contribute to the various scattering processes.

→ the process $N+N \rightarrow \bar{N} + \bar{N}$ is only possible through 4 ψ operators, but S_2 has no such normal-ordered string of operators. This is good, as our interacting theory obeys a $U(1)$ symmetry, & this process does not conserve charge.

→ We can compactly show each contraction in the form of a Wick Diagram:

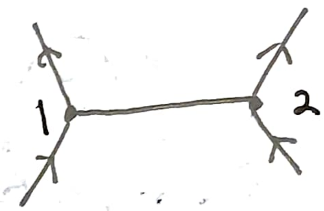
→ we draw n vertices x_i for the n^{th} term in the expansion:



w/ contractions connected

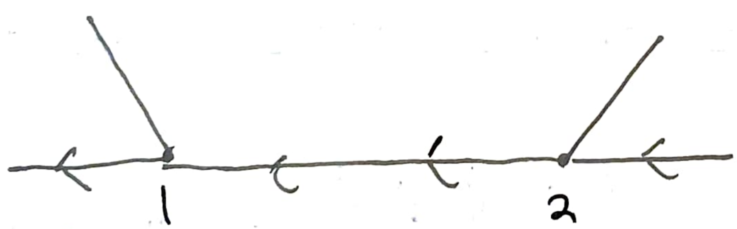
Ex: $\frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2 \, :\psi_1^\dagger \psi_1 \phi_1 \psi_2^\dagger \psi_2 \phi_2:$ (with ϕ_1 and ϕ_2 contracted)

→ corresponds to the diagram



Ex: $\frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2 \, :\psi_1^\dagger \psi_1 \phi_1 \, \psi_2^\dagger \psi_2 \phi_2:$ (with $\psi_1^\dagger$ and ψ_2 contracted)

has the diagram



(second-order diagram model)

→ here we have an operator $:\psi_1 \phi_1 \psi_2^\dagger \phi_2:$

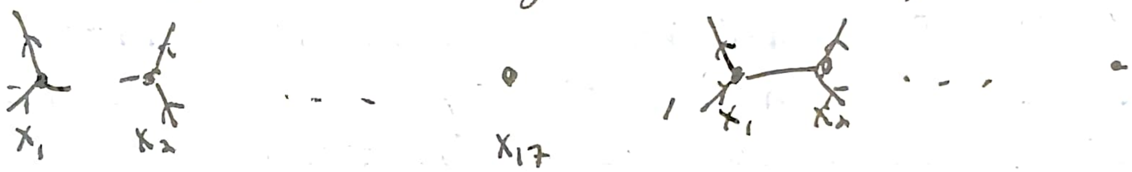
What if we work backwards? That is given a diagram, construct the Wick expansion term: consider the first diagram

- second-order since 2-vertices
- each vertex contributes a factor of $(-ig)$ & $2!$
- $d^4x_1 d^4x_2$ since 2 vertices
- $\phi_1 \phi_2$ contraction due to 
- remainder corresponds to normal-ordered product of fields, $:\psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2:$

$\boxed{\text{The Wick diagrams are in 1-1 correspondence with terms in the Wick expansion}}$

... and so the entire Wick expansion may be represented by a series (although some may evaluate to 0)

For instance, S_{17} diagrams consist of diagrams ranging from $: :$ of $17 \cdot 3 = 51$ fields (no lines connecting dots). The second term consists of all single-contractions, and so on:



→ In first-order perturbation theory, we have:

I think they vanish, have to verify

$$W(\psi, \psi^\dagger, \phi) = :\psi^\dagger \psi \phi: + \overline{\psi^\dagger \psi}\, \phi$$

(why not all possible contractions? p. 161, Coleman)

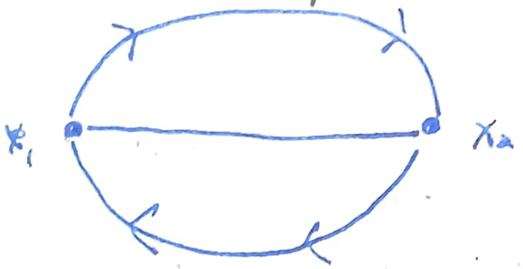
→ 1st term vanishes as it is normally ordered + $\psi^\dagger\psi$ leaves nucleon state invariant, same with contraction. (dont understand Coleman's argument via energy-momentum conservation)

(why can't $:\psi^\dagger\psi\phi:$ contribute to a process like $|N\rangle \to |N, \phi\rangle$ other than energy violation, ie how to see just through matrix element?)

→ some terms have no external lines, ie all operators are contracted →

→ consider $\frac{(-ig)^2}{2!}\int d^4x_1 \int d^4x_2 : \psi_1^\dagger \psi_1 \phi_1 \psi_2^\dagger \psi_2 \phi_2 :$

→ This corresponds to the Wick diagram:



→ Note that alternating nucleon contractions leaves diagram invariant, as switching labels gives same diagram (rotation of 180°)

→ some diagrams are equivalent, ie yield via a permutation of indices

$S(D) \equiv$ # of permutations that leave diagram invariant

$\Rightarrow \left(\begin{array}{c} \text{sum of all} \\ \text{diagrams of} \\ \text{pattern} \end{array} \right) = \frac{n(D)!}{S(D)} \cdot \frac{:O(D):}{n(D)!} = \frac{:O(D):}{S(D)}$

→ a diagram contributes $\frac{:O(D):}{n(D)!}$ to perturbative expansion, and $\frac{n(D)!}{S(D)}$ is the number of distinct diagrams yielded by index permutation.

Connected diagram - diagram that is in "one piece": all parts of diagram are contigous to at least one other part at a vertex

Ex:  first-order connected

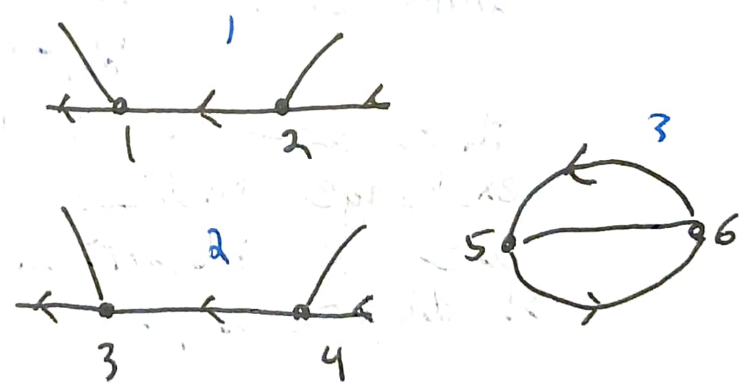
 fourth-order disconnected.

Now for a theorem:

$$\boxed{\sum \text{all wick diagrams} = :\exp\left(\sum \text{connected wick diagrams}\right):}$$

→ Let $D_r^{(c)}$, $r=1,2,\ldots,\infty$ be a complete set of connected diagrams for a theory, one of each pattern. In general, a diagram D will have some integer n_r components of the patterns of $D_r^{(c)}$.

Consider the graph



→ 1 has integral over $d^4x_1\, d^4x_2$ with only functions of $x_1 + x_2$ in the integrand. Similar for 2, integral is over $d^4x_3 d^4x_4$ w/ integrand only a function of $x_3 + x_4$, and same for 3 wrt x_5, x_6, ie the expression for the diagram splits into 3 factors...

→ the diagram yields an operator that is a product of operators (mod combinatoric factor)

→ In general, this is a characteristic of disconnected diagrams – their operators are normal ordered products of operators associated w/ the individual connected components!

$$:O(D^{(d)}): = :\prod_{r=1}^{\infty}\left[O(D_r^{(c)})\right]^{n_r}:$$

→ note that this holds not only for disconnected diagrams $D^{(d)}$, but for a general diagram $D_i^{(c)}$, as $n_r = 1$ for $i=r$, 0 for $j \neq i$, so $:O(D^{(d)}): = :O(D_i^{(c)}):$

→ For $S(D)$, we can permute the indices of each individual connected diagram, contributing $\prod_{r=1}^{\infty}\left[\frac{1}{S(D_r^{(c)})^{n_r}}\right]$. In addition, if there are 2 identical components, we can exchange indices between $D_r^{(c)}$'s. If there are 3 identical diagrams, we have 3! diagrams, so

$$\frac{:O(D):}{S(D)} = :\prod_{r=1}^{\infty}\frac{1}{n_r!}\left[\frac{O(D_r^{(c)})}{S(D_r^{(c)})}\right]^{n_r}$$ for all diagrams of a particular pattern

→ This puts us in a position to find an expression for sum of all diagrams, ie S, by noting S is the sum over all patterns!

→ $S = \sum$ all Wick diagrams

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots : \prod_{r=1}^{\infty} \frac{1}{n_r!} \left[\frac{O(D_r^{(c)})}{S(D_r^{(c)})} \right]^{n_r} :$$

since $\frac{:O(D):}{S(D)}$ is the contribution from one pattern & summing over the complete set of connected diagrams, and for distinct choices of n_i, we get a distinct pattern.

$$\Rightarrow = : \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{n_r!} \left[\frac{O(D_r^{(c)})}{S(D_r^{(c)})} \right]^{n_r}$$

but this is just $: \prod_{r=1}^{\infty} \exp\left(\frac{O(D_r^{(c)})}{S(D_r^{(c)})} \right) :$

$$= \exp \sum_{r=1}^{\infty} \left(\frac{O(D_r^{(c)})}{S(D_r^{(c)})} \right) :$$

→ We can use this to solve the theory ~~$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi - g\psi^\dagger\psi\phi$~~, $H_I = g\rho(x)\phi(x)$

, where ρ goes to 0 in all directions

→ The primitive vertex is just a single line with a vertex since only 1 ϕ field is present (ie our $n=1$ term in S is $T(\phi_1) = :\phi_1:$, which corresponds to •——)

→ a second diagram looks like this:

D_2 $\quad 1 \bullet\!\!-\!\!-\!\!-\!\!\bullet 2$ in 2^{nd} order

→ a pattern of vertices such that only one line can come out of any vertex yields only 2 connected diagrams, D_1 + D_2 (since 2 contractions cannot be performed on the same field operator)

→ Now, D_1 is only 1 vertex, so trivially $S(D_1)=1$, so all diagrams of D_1 correspond to the operator:

$$O_1 = -ig\int d^4x_1\, \rho(x_1)\phi(x_1)$$

For D_2, $S(D_2)=2$, since permuting 1+2 leave the diagram invariant, so

$$O_2 = (-ig)^2\int d^4x_1\, d^4x_2\, \phi(x_1)\phi(x_2)\, \rho(x_1)\rho(x_2)$$

There are no connected diagrams left, so we can evaluate the sum over all digarms:

$$S = :\exp\left(\frac{O_1}{1!} + \frac{O_2}{2!}\right) = :e^{O_2/2}\, e^{O_1}$$

Now $O_2 = -a+ib$, $a\geq 0$:

$$\Rightarrow S = e^{\frac{1}{2}(-a+ib)} :e^{O_1}:$$

→

Since ϕ is a free field we have

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x})$$

and define the Fourier transform of $\rho(x)$:

$$\tilde{\rho}(p) = \int d^4x\, e^{ip\cdot x} \rho(x) \quad , \quad \tilde{\rho}(-p) = \tilde{\rho}(p)^*$$

$$\Rightarrow \mathcal{O}_1 = -ig \int d^4x_1\, \rho(x_1)\phi(x_1) = -ig \int d^4x_1\, \rho(x_1) \cdots (e^{-ip\cdot x} + e^{ip\cdot x})$$

$$= -ig \int \frac{d^3\vec{p}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}} [a_{\vec{p}}\, \tilde{\rho}(\vec{p}, \omega_{\vec{p}})^* + a^\dagger_{\vec{p}}\, \tilde{\rho}(\vec{p}, \omega_{\vec{p}})]$$

$$= \int d^3\vec{p}\, [-h(\vec{p})^* a_{\vec{p}} + h(\vec{p}) a^\dagger_{\vec{p}}]$$

$\Rightarrow$ note if $\tilde{\rho}(p)$ vanishes on the mass shell ($p^0 = \omega_p$), then nothing happens! (but why distinguish on ~~mass~~ shell from off shell here when p^0 is not free and restricted to $p^0 = \omega_{\vec{p}}$? We only integrate over $d^3\vec{p}$...)

Back to NN scattering:

We want to compute $\langle p_1'(N), p_2'(N) | (S-1) | p_1(N), p_2(N) \rangle$ w/ state relativistically normalized:

$\rightarrow$

$|K\rangle = (2\pi)^{3/2}\sqrt{2\omega_k}\,|\vec{k}\rangle = a^\dagger(k)|0\rangle$, with

$$a^\dagger(k) = (2\pi)^{3/2}\sqrt{2\omega_k}\,a^\dagger_{\vec{k}}$$

→ Considering the scalar free field expansion:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right]$$

+ $a(k')|K\rangle = a(k')a^\dagger(k)|0\rangle$
$= [a(k'), a^\dagger(k)]|0\rangle$ (since $a(k')$ annihilates)
$= (2\pi)^3 2\omega_k\,\delta^{(3)}(k-k')|0\rangle$

$$\Rightarrow \int \frac{d^3k'}{(2\pi)^3 2\omega_k} a(k')|K\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega_k}(2\pi)^3 2\omega_k\delta^{(3)}(k-k')|0\rangle$$

$$= \int d^3k'\,\delta^{(3)}(k-k')|0\rangle = |0\rangle$$ (relativistic normalization for ground state?)

→ similar relations hold for nucleon + antinucleon states + operators:

$$|p_1(N), p_2(N)\rangle = b^\dagger(p_1)\,b^\dagger(p_2)|0\rangle.$$

We evaluate $\langle p_1', p_2'|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):|p_1, p_2\rangle$

→ insert the nucleonic completion relation:

make sure this is whats happening

$$= \langle p_1'; p_2'|\psi_i^\dagger(x_1)\psi^\dagger(x_2)\left(|0\rangle\langle 0| + \int d^3k|\vec{k}\rangle\langle\vec{k}| + \cdots\right)\psi(x_1)\psi(x_2)|p_1; p_2\rangle$$

→

→ the contributions from the non-vacuum projectors is 0, as for the projector $|\vec{k}\rangle\langle\vec{k}|$, the annihilation piece on $|i\rangle$ gives $|0\rangle$, and the creation piece on $|\vec{k}\rangle$ gives 3 particles, so 0 contribution!

$$= \langle p_1' ; p_2' | \psi^+(x_1)\psi^+(x_2) | 0\rangle\langle 0| \psi(x_1)\psi(x_2) | p_1 ; p_2\rangle$$

$$\to \langle 0|\psi(x_1)\psi(x_2)|p_1;p_2\rangle = \langle 0| \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} (b_k e^{-ik\cdot x_1} + c_k^+ e^{ik\cdot x_1})$$

$$\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} (b_p e^{-ip\cdot x_2} + c_k^+ e^{ip\cdot x_2}) |p_1;p_2\rangle$$

$$= \langle 0| \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} b_k e^{-ik\cdot x_1} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} b_p e^{-ip\cdot x_2} |p_1,p_2\rangle$$

(since creation fields annihilate $\langle 0|$)

$$= \cancel{\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}}$$

→ square roots should be gone since we are working w/ relativistically normalized states + operators

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3p}{(2\pi)^3 2\omega_p} \langle 0| b_k e^{-ik\cdot x_1} b_p e^{-ip\cdot x_2} |p_1;p_2\rangle$$

$$= \int \frac{d^3k\, d^3k}{(2\pi)^6 (2\omega_k)(2\omega_p)} \langle 0| e^{-ik\cdot x_1 - ip\cdot x_2} b_k b_p |p_1;p_2\rangle$$

→ from relativistic normalization, we saw that to

$$\int \frac{d^3k'}{(2\pi)^3 2\omega_k} a(k')|k\rangle = |0\rangle \quad \text{, so :}$$

$$\int \frac{d^3k \, d^3p}{(2\pi)^6 (2\omega_p)(2\omega_k)} \langle 0| e^{-ik\cdot x_1 - ip\cdot x_2} b_k b_p |p_1, p_2\rangle = e^{-ip_1\cdot x_1 - ip_2\cdot x_2}$$

(delta functions)

~~Similarly, we have the piece of the term~~

$$\cancel{\text{to} \int \frac{d^3k \, d^3p}{(2\pi)^6 (2\omega_k)(2\omega_p)}}$$

→ But now the funny bit – we got that part by assuming b_k acts on $|p_1\rangle$ and ~~b_p~~ b_p on $|p_2\rangle$, but we can easily do the reverse! This contributes $e^{-ip_1\cdot x_2 - ip_2\cdot x_1}$! (But how do we know contributions add...?)

similarly, $\langle p_1', p_2' | \psi^\dagger(x_1) \psi^\dagger(x_2) | 0\rangle$ gives us

$e^{ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1}$, so in total, we get: $\langle p_1', p_2' | : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : | p_1, p_2\rangle$

$$= \left(e^{ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1}\right)\left(e^{-ip_1\cdot x_1 - ip_2\cdot x_2} + e^{-ip_1\cdot x_2 - ip_2\cdot x_1}\right)$$

$$= e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_1 - ip_2\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_2 - ip_2\cdot x_1}$$

$$+ e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_1 - ip_2\cdot x_2} + e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_2 - ip_2\cdot x_1}$$

→ Notice that the first 2 terms differ by the last 2 via interchange of x_1 & x_2, and since we are integrating over x_1 & x_2 symmetrically, & since $\phi(x_1)\phi(x_2)$ is symmetric, the terms give identical contributions. →

→ so our λ^{nd} order term, explicitly, is

$$(-ig)^2 \int d^4x_1\, d^4x_2\, \phi(x_1)\phi(x_2) \left(e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_1 - ip_2\cdot x_2}\right.$$

$$\left. + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_2 - ip_2\cdot x_1}\right)$$ (Since $\frac{1}{2!}$ + 2 equal contributions cancel)

Substituting our free field ϕ :

$$= (-ig)^2 \int d^4x_1\, d^4x_2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} e^{+ik\cdot(x_1-x_2)} \cdots$$

$$= (-ig)^2 \int d^4x_1\, d^4x_2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \cdots$$

$$\cancel{\left(e^{i(p_1'-p_1+k)\cdot x_1 + i(p_2'-p_2+k)\cdot x_2} + e^{i(}\right.}$$

$$= e^{ip_1'\cdot x_1 - ip_1\cdot x_1 + ik\cdot x_1 + ip_2'\cdot x_2 - ip_2\cdot x_2 + ik\cdot x_2}$$

$$+ e^{ip_2'\cdot x_1 - ip_2\cdot x_1 + ik\cdot x_1 + ip_1'\cdot x_2 - ip_1\cdot x_2 + ik\cdot x_2}$$

$$= e^{i(p_1'-p_1+k)\cdot x_1 + i(p_2'-p_2\mp k)\cdot x_2} + e^{i(p_2'-p_2+k)\cdot x_1 + i(p_1'-p_1\mp k)\cdot x_2}$$

(this is all wrong, term should be

$$e^{ix_1\cdot(p_1'-p_1) + ix_2\cdot(p_2'-p_2)} + e^{ix_1\cdot(p_2'-p_1) + ix_2\cdot(p_1'-p_2)} + (x_1 \leftrightarrow x_2))$$

$$= (-ig)^2 \int d^4x_1\, d^4x_2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \left(e^{i(p_1'-p_1+k)\cdot x_1 + i(p_2'-p_2-k)\cdot x_2}\right.$$

$$\left. + e^{i(p_2'-p_1+k_1)\cdot x_1 + i(p_1'-p_2-k)\cdot x_2}\right)$$

→

→ since $\int \frac{d^4x}{(2\pi)^4} e^{-i(k-k')\cdot x} = \delta^{(4)}(k-k')$:

$$=(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \Big[(2\pi)^4\delta^{(4)}(p_1'-p_1+k)(2\pi)^4\delta^{(4)}(p_2'-p_2-k)$$
$$+(2\pi)^4\delta^{(4)}(p_2'-p_1+k)(2\pi)^4\delta^{(4)}(p_1'-p_2-k)\Big]$$

Performing integral over delta functions:

$$=i(-ig)^2\left[\frac{1}{(p_1-p_1')^2-\mu^2+i\varepsilon}+\frac{1}{(p_1-p_2')^2-\mu^2+i\varepsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')$$

(To see this, note that to satisfy either product of delta functions w/ 0 argument, $p_1+p_2-p_1'-p_2'=0$. Then, i guess you can just choose a delta function from each product to integrate).

→ ask if this is the case.

→ the delta function is just energy-momentum conservation, w/ $(2\pi)^4$ factor. Also, we can actually drop the $i\varepsilon$ term. To see, consider the COM frame where $\vec{p}_1=-\vec{p}_2$ and $|\vec{p}_1|=|\vec{p}_1'|$: (ask about Tong's reasoning behind dropping $i\varepsilon$) + why $E=0$)

$$p_1=(\sqrt{p^2+m^2}, p\hat{e}),\quad p_2=(\sqrt{p^2+m^2}, -p\hat{e})$$

$$p_1'=(\sqrt{p^2+m^2}, p\hat{e}'),\quad p_2'=(\sqrt{p^2+m^2}, -p\hat{e}')$$

$$\Rightarrow (p_1-p_1') = \cancel{2p(\hat{e}-\hat{e}') \Rightarrow (p_1-p_1')^2} = (0, p(\hat{e}-\hat{e}'))$$

$$\Rightarrow (p_1-p_1')^2 = \cancel{0}\ \cancel{-2p^2}\ 0-p^2(\hat{e}-\hat{e}')(\hat{e}-\hat{e}')$$

$$=-p^2(1-2\hat{e}\cdot\hat{e}'+1) = -2p^2(1-2\hat{e}\cdot\hat{e}')$$
$$=-2ps(1-\cos\theta)$$

$$+\ (p_1-p_2')^2 = -2p^2(1+\cos\theta) \quad\rightarrow$$

$$\Rightarrow A = g^2 \left[\frac{1}{2p^2(1-\cos\theta)+\mu^2} + \frac{1}{2p^2(1+\cos\theta)+\mu^2} \right]$$

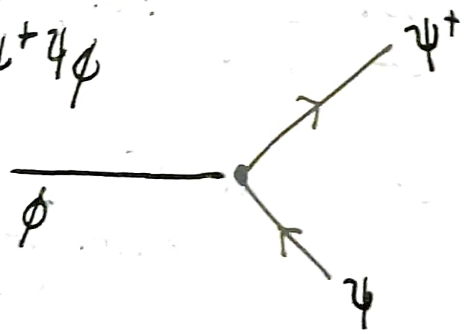
where A is the matrix element without $(2\pi)^4 \delta^{(4)}(p_f - p_i)$ (Feyman Amplitude)

$$\langle f | (S-1) | i \rangle = i A_{fi} (2\pi)^4 \delta^{(4)} (p_f - p_i)$$

Now, it turns out we don't need Dyson's Formula or Wick's Theorem as there is a nice diagrammatic approach with all formalism built in. These are Feynman Diagrams.

→ At n^{th} order perturbation theory, the interaction Hamiltonian wacks n times, so we have n interaction vertices, with fields in the interaction Hamiltonian are lines eminating from the vertex.

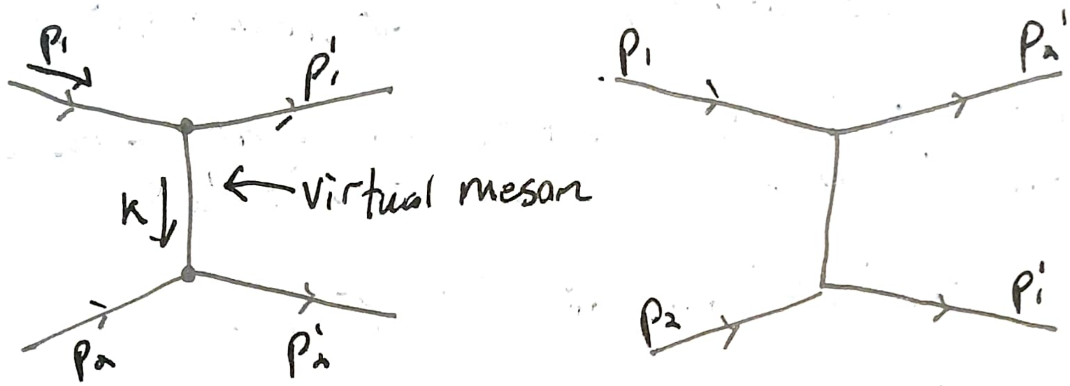
$H_I = \psi^\dagger \psi \phi$



→ contractions are simply connections between vertices, and this is nice since all "not nice" lines give 0 contribution, ie $\psi^\dagger(x) \psi^\dagger(y)$

→

NN scattering has 2 Feynman Diagrams.



→ Since incoming nuclei are identical, it is impossible to say which carries p_1 + which carries p_2, so amplitude sums over both. (shouldn't this say p_1' + p_2'? Why couldn't we just measure initial momenta?)

(ask about connection between Wick + Feynman diagrams)

→ We know rules to evaluate iA_{fi}, the <u>Feynman Rules</u>: (for our theory)

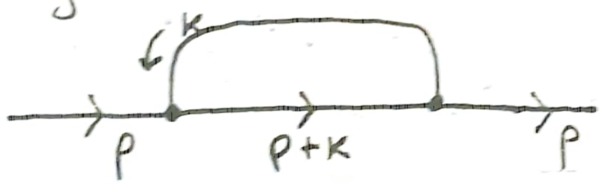
- At a vertex, write $(-ig)(2\pi)^4\delta^{(4)}(\sum_i k_i)$ (sum of momenta flowing into vertex)

- For each internal line w/ momentum k, write $\int \frac{d^4k}{(2\pi)^4} D(k^2)$, where $D(k^2)$ is the appropriate propagator

- Divide final result by overall E-p conserving δ function, $(2\pi)^4\delta(p_f - p_i)$

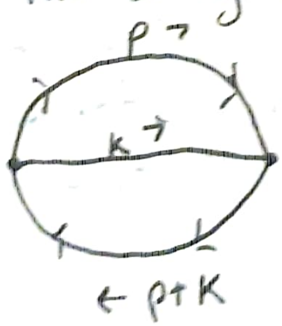
alternatively, enforce E-P conservation at each vertex, then for each vertex write $(-ig)$ and for each contracted line, write the propogator.

→ Consider the matrix element given by $\langle p | : \psi^\dagger(x_1)\psi(x_2)\psi(x_1)\psi^\dagger(x_2)\phi(x_1)\phi(x_2) | p\rangle$. The Feynman diagram looks like:



→ enforcing E-P conservation does not fix the momentum k, as it is not constrained by a known exitting momentum, so we must keep the factor $\int \frac{d^4k}{(2\pi)^4}$

→ Similarly, the term $\langle 0 | : \psi^\dagger(x_1)\psi(x_2)\psi(x_1)\psi^\dagger(x_2)\phi(x_1)\phi(x_2) : | 0\rangle$ has the diagram:

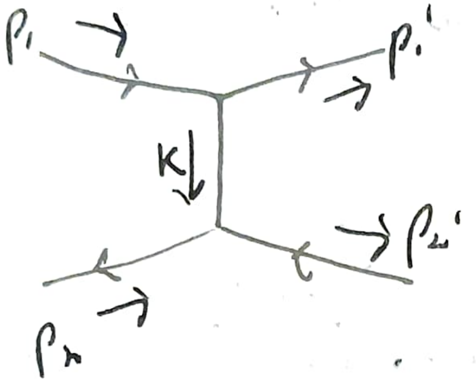


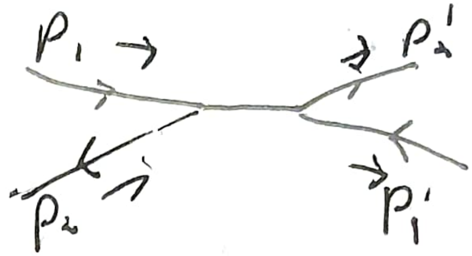
→ neither p nor k is constrained. This inspires an additional Feynman rule:

- for each internal loop w/ momentum k unconstrained by E-P conservation, write a factor $\int \frac{d^4k}{(2\pi)^4}$

→

Consider $\bar{N}N \to \bar{N}N$ scattering, w/ 2 Feynman Diagrams:





From Feynman rules, we get $(-ig)^2$, each one Feynman ~~propogator~~ vertex. For diagram 1:

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \cdot \frac{1}{(2\pi)^4} \cdot \frac{1}{\delta^{(4)}(p_1'+p_2'-p_1-p_2)}$$

$$(-ig)^2 (2\pi)^8 \delta^{(4)}(p_1'-p_1)\, \delta^{(4)}(p_2'-p_2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon}$$

(Prof states Feynman diagrams have 1-1 correspondence to terms in Wick expansion, but I thought this was true for Wick diagrams... the contribution to S from one Wick diagram corresponds to <u>multiple</u> Feynman diagrams)

Applying Feynman rules:

diagram 1

$$\to (-ig)(2\pi)^4\delta^{(4)}(p_1-p_1'-k)(-ig)(2\pi)^4\delta^{(4)}(p_2-p_2'+k)\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon} + (-ig)(2\pi)^4\cancel{\delta^{(4)}(p_1-p_2'-k)}(-ig)(2\pi)^4\cancel{\delta^{(4)}(p_2-p_1'+k)}\int\frac{d^4k}{(2\pi)^4}$$

$\delta^{(4)}(p_1+p_2-k)$ $\quad\delta^{(4)}(-p_2'+p_1'+k)$

diagram 2

$$= (-ig)^2(2\pi)^8\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon}\Big[\delta^{(4)}(p_1-p_1'-k)\delta^{(4)}(p_2-p_2'+k) + \delta^{(4)}(\cancel{p_1-p_2'-k})\,\delta^{(4)}(\cancel{p_2-p_1'+k})\Big]$$

(p_1+p_2-k) $\quad (k-p_1'-p_2')$

→ can perform integral in either delta function in each term (we'll do first)

$$= (-ig)^2(2\pi)^4\left[\frac{i}{(p_1-p_1')^2-\mu^2+i\varepsilon}\delta^{(4)}(p_2-p_2'+k) + \frac{i}{(p_1+p_2')^2-\mu^2+i\varepsilon}\delta^{(4)}(p_2-p_1'+k)\right]$$

$$= (-ig)^2(2\pi)^4\left[\frac{i}{(p_1-p_1')^2-\mu^2+i\varepsilon}\delta^{(4)}(p_2-p_2'+p_1-p_1') + \frac{i}{(p_1-p_2')^2-\mu^2+i\varepsilon}\delta^{(4)}(p_2-p_1'+p_1-p_2')\right]$$

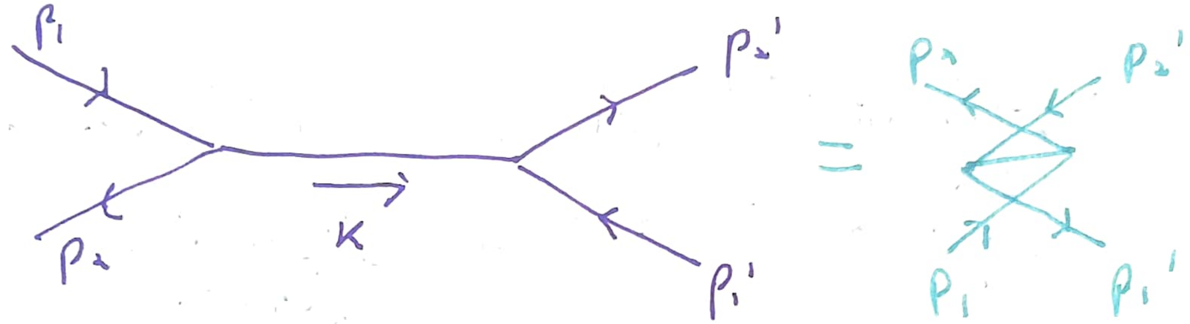
$(p_1+p_2)^2$

$$= (-ig)^2(2\pi)^4\left(\frac{i}{(p_1-p_1')^2-\mu^2+i\varepsilon} + \frac{i}{(p_1+p_2)^2-\mu^2+i\varepsilon}\right)\delta^{(4)}(p_1+p_2-p_1'-p_2')$$

Applying 3rd rule which says to divide by $(2\pi)^4\delta^{(4)}(p_f-p_i)$, we get:

$$\boxed{iA = (-ig)^2\left(\frac{i}{(p_1-p_1')^2-\mu^2+i\varepsilon} + \frac{i}{(p_1+p_2)^2-\mu^2+i\varepsilon}\right)}$$

→ It's important to note that the same Feynman diagram can have various orientations. For instance:



→ We can also find the Feynman amplitude for Nucleon-antinucleon annihilation via the Feynman rules. To show that this process can even occur at second order:

$$:\psi^{+}(x_1)\,\psi(x_1)\,\phi(x_1)\,\psi^{+}(x_2)\,\psi(x_2)\,\phi(x_2):$$

$$= :\psi(x_1)\,\phi(x_1)\,\psi^{+}(x_2)\,\phi(x_2):\,\psi^{+}(x_1)\,\psi(x_2)$$

→ the normal ordering will destroy both particle states & ϕs will create 2 mesons.

(Q: When normal ordered products of fields w/ "separate" time variables occur, do times matter wrt action on states?) A: I think it does, but only manifests in expectation values, ie $\langle 0|\psi(x_1)\psi(x_2)|p_1,p_2\rangle \propto e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}$

Potentials & Resonance:

→ Lets look at non-relativistic limit of NN scattering & understand it in terms of NRQM: →

Recall Born approximation - at first order perturbation, amplitude for incoming state w/ momentum $\vec{k}$ to scatter off a potential $U(\vec{r})$ w/ momentum $\vec{k}'$ $\propto$ FT of potential:

$$A_{NR}(\vec{k} \to \vec{k}') = -i \int d^3r \, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}} \, U(\vec{r})$$

→ We can imagine that non-relativistic scattering is the incoming particle scattering off the potential of the second particle:

$$iA = \frac{-ig^2}{(p_1-p_1')^2-\mu^2} = \frac{ig^2}{|\vec{p}_1-\vec{p}_1'|^2+\mu^2}$$ (since in COM frame, energies of initial + scattered nucleons is same)

how does $-\mu^2 \to +\mu^2$?

→ we also must divide the relativistic amplitude by $(2m)^2$ to account for difference in normalization:

$$\Rightarrow \int d^3\vec{r} \, U(\vec{r}) e^{-(\vec{p}_1'-\vec{p}_1)\cdot\vec{r}} = \frac{-\lambda^2}{|\vec{p}_1-\vec{p}_1'|^2+\mu^2}, \quad \lambda = g/2m$$

(also, what is m here? The rest mass energy?)

→ Inverting the Fourier transform:

$$U(\vec{r}) = -\lambda^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2} = -\frac{\lambda^2}{4\pi^2}\int_0^\infty dq \, \frac{q^2}{q^2+\mu^2} \, \frac{e^{iqr}-e^{-iqr}}{iqr}$$

(since integrand only prop. to magnitude of $\vec{k}^2$)

⇒ after some residue stuff:

$$\boxed{U(r) = -\frac{\lambda^2}{4\pi r} e^{-\mu r}}$$ ↪ Yukawa Potential

→ The sign of the Yukawa term for processes mediated by a scalar meson is always − the force is attractive!

What about $N\bar{N}$ annihilation?

$$ia = \frac{(-ig)^2}{(p_1+p_2')^2 - \mu^2 + i\varepsilon}$$

In NR limit, $(p_1+p_2)^2 \sim 4M^2$, but suppressed by $\frac{\vec{p}^2}{m^2}$ compared to virtual exchange, as expected, since particle creation/annihilation is purely relativistic.

→ In the relativistic limit, annihilation can occur: (COM frame):

$$\vec{p}_1 = -\vec{p}_2 = \vec{p}, \quad E_1 = E_2 = \sqrt{\vec{p}^2 + m^2}$$

$$\Rightarrow a \propto \frac{1}{(p_1+p_2)^2 - \mu^2}, \quad (p_1+p_2)^2 = p_1^2 + 2p_1\cdot p_2 + p_2^2 = 2M^2 + 2\underbrace{E_1E_2}_{E_1=E_2} - 2\vec{p}_1\cdot\vec{p}_2$$

$$= 2M^2 + 2(\vec{p}^2 + M^2) + 2\vec{p}^2 \quad (\text{since } \vec{p}_1 = -\vec{p}_2)$$

$$= 4(M^2 + \vec{p}^2) \therefore a \propto \frac{1}{4M^2 + 4\vec{p}^2 - \mu^2 + i\varepsilon}$$

How does this behave? There are 2 cases: →

(always virtual)

$\not$ (never 0)

- $\mu < 2m \rightarrow$ denom. positive, ie ϕ particle never on-shell
- $\mu > 2m \rightarrow$ meson Not virtual $\rightarrow$ pole.

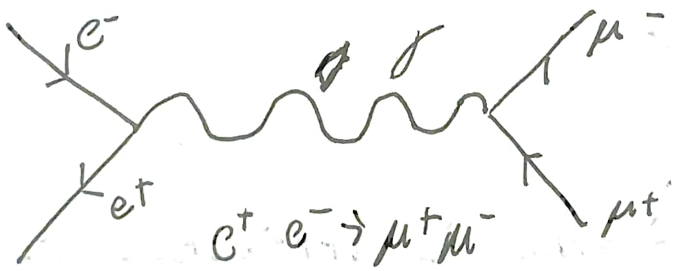
$\rightarrow$ probability drops with $\vec{p}^2$ as ϕ becomes "more virtual"

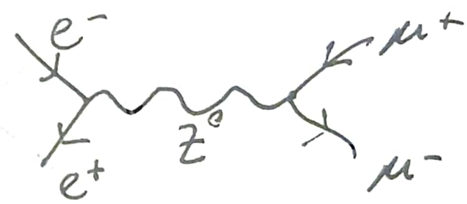
$\rightarrow$ pole: $4M^2 + 4\vec{p}^2 - \mu^2 = 0 = (p_1+p_2)^2 - \mu^2 \Rightarrow (p_1+p_2)^2 = k^2 = \mu^2$
(meson satisfies dispersion relation)

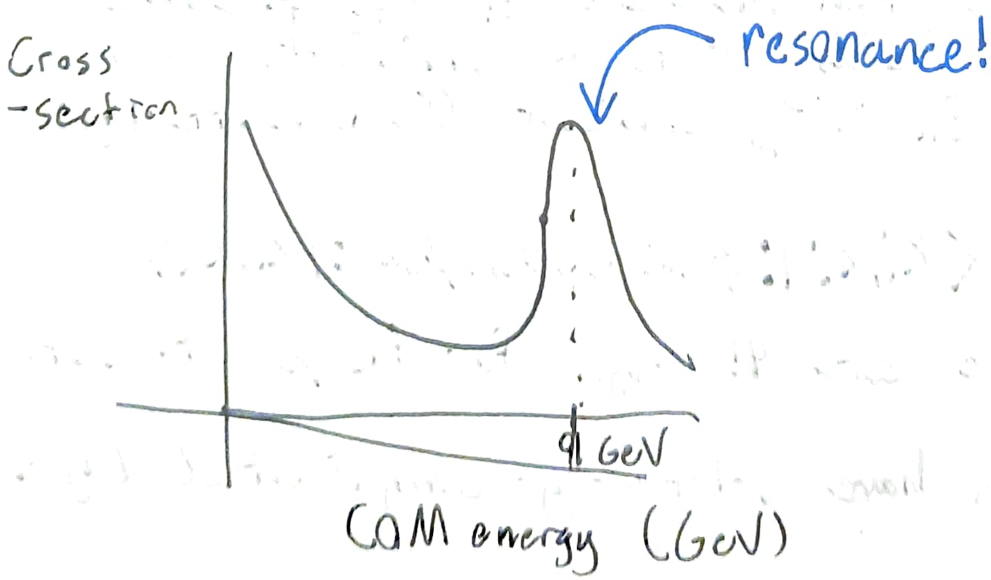
$\rightarrow$ experimentally, formation of real particle is a resonance. (This is how particles are searched for)

Ex: Z^0 Boson, $M_Z \sim 91.2$ GeV (analogous to meson)

$e^- e^+$ scattering:







How high does this peak go? Surely its not infinite...

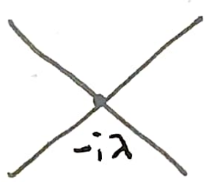
→ It would be ∞ if Z^0 could propagate forever, but it decays (finite lifetime)
(it turns out higher-order graphs are connections to the propogator) stops on-shell)

Interaction terms: Relevant, Marginal, + Irrelevant Operators.

How do we generalize Feynman rules for other interactions? Consider the following interaction:

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4$$

→ The Feyman rule has 4 fields per vertex:



→ The 4! difference is attributed to the fact that there are 4! ways to choose which fields create + annihilate each line. Explicitly, consider $\mathcal{O}(\lambda)$ scattering:

$$-\frac{\lambda}{4!}\langle k_1', k_2' | :\phi(x)\phi(x)\phi(x)\phi(x): | k_1, k_2\rangle$$

→ there are 4! ways for the ϕs to act.

We can have arbitrarily-complicated Lagrangians:

$$\mathcal{L} = \mathcal{L}_0 - \sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n$$, where ϕ^n interaction has n legs.

→ We can, however, divide interactions into 3 distinct categories:

• $[\mathcal{L}] = 4$, $[\phi] = 1 \Rightarrow$ ~~$[\lambda_n \phi^n] = [\lambda_n]^n = 4$~~

$[\lambda_n \phi^n] = [\mathcal{L}] = 4 = [\lambda_n][\phi^n]$ (in mass dimension)

$\Rightarrow [M]^4 = [\lambda_n][M]^n \Rightarrow [\lambda_n] = [M]^{4-n} \equiv 4-n$

→ <u>3 classes</u>: $\lambda_n < 0$, $\lambda_n = 0$, $\lambda_n > 0$ I

1.) $[\lambda_3] = 4-3 = 1$ → λ_3 has units, so to say if it's "big" or "small", the ratio λ_3/M tells us effects of interaction are strong at low energies + weak at high energies. (since $[E]=1$, can compare as dimensionless #)
→ such a term is a relevant operator or super-renormalizable (as long as $\lambda_3 <$ masses)

2) $[\lambda_4] = 0$ (marginal/renormalizable operator)
→ Doesn't <u>care</u> about energy since λ_4 is dimensionless, theory perturbative if $\lambda_4 \ll 1$ <u>independent</u> of energy

3) (most operators): $[\lambda_n] < 0$ for $n \geq 5$. → What is dimensionless?

$$\to \lambda_n = \frac{\hat{\lambda}_n}{M^{n-4}} \quad \text{where } [\hat{\lambda}_n] = 0$$

ie $\underline{\hat{\lambda}_n E^{n-4}}$ is dimensionless → small at <u>low</u> energies, strong at <u>high</u> energies
↳ (irrelevant operators/non-renormalizable at low energies)

→ Perturbation theory will be expansion of powers of $(E/M)^{n-4}$, where E is characteristic. so the effects of such a term will be small at low energies. (at sufficiently low energies, ignore operators)

→ At low energies, only relevant and marginal operators are kept!

For energies $E \sim M$, <u>all</u> operators are important, which is bad as neither perturbation theory nor expansion of $\mathcal{L}$ in powers of ϕ are well-behaved, so there are probably <u>new</u> physics at this scale, that $\mathcal{L}$ does <u>not</u> account for.

Ex: strong coupling for 4D FT leads to 5D QG. This is the AdS/CFT correspondence.

We will stick to weak coupling...

Decay Widths, Cross Sections, + Phase Space

→ We've learned how to calculate amplitudes, want to apply this to measurements...

→ From NN scattering:

$$P_{scatter} = |S_p|^2 = |a|^2 (2\pi)^8 |\delta^{(4)}(p_f - p_i)|^2$$

WTF is this... scary...

manifestly nonsense... why does this happen? It's because we're scattering non-normalizable plane-wave eigenstates! Squaring them squares a delta function. ($\langle \vec{k} | \vec{k}' \rangle = \delta^{(3)}(\vec{k}-\vec{k}')$)

→ Plane wave states are <u>not</u> "particles," collisions are happening everywhere anytime, we should localize the states... scatter wave <u>packets</u>!

→ Also, we can't actually measure outgoing momenta precisely, so we actually are measuring a <u>range</u> of momenta (integrate over range).

How many states are in range $(\Delta p)^3$? Infinite! We need a sensible limiting procedure...

→ Back to the box! (To quantize momenta)

→ System in box of volume V w/ interaction time t (free in past + future w/ adiabatic change)

→ We calculate transition probability per unit time + take $V, T \to \infty$.

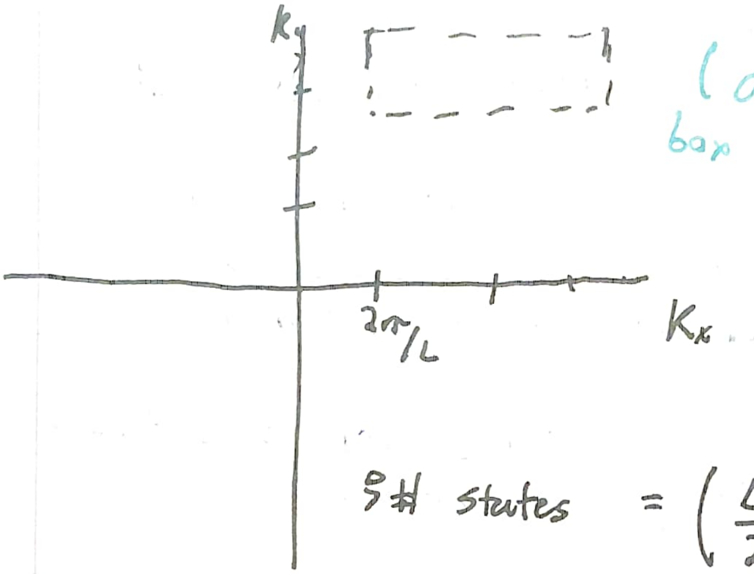
→ Box quantization: $K_{x,y,z} = \dfrac{2\pi n_{x,y,z}}{L}$

→ Integral over momentum for field expansions becomes a sum over discrete momenta:

$$\phi(x) = \sum_{K} \left[\frac{a_k e^{-ik\cdot x}}{\sqrt{2\omega_k}\sqrt{V}} + \frac{a_k^{+} e^{ik\cdot x}}{\sqrt{2\omega_k}\sqrt{V}} \right]$$

→ $1/\sqrt{V}$ is needed so that fields satisfy equal time commutation relations.

If we measure states over a range $\Delta p_x, \Delta p_y, \Delta p_z$ (our detector resolution), we can find the # of states:



(do we assume resolution box is centered at a state?)

$$\#\text{ states} = \left(\frac{\Delta p_x}{2\pi/L}\right)\left(\frac{\Delta p_y}{2\pi/L}\right)\left(\frac{\Delta p_z}{2\pi/L}\right)$$

$$= \left(\frac{L}{2\pi}\right)^3 \Delta p_x \Delta p_y \Delta p_z$$

→ The $1/\sqrt{V}$ is important because Feynman rules will have this extra factor.

What about the EP delta function? We'll define it as:

$$(2\pi)^4 \delta^{(4)}_{VT}(p) = \int_{-T/2}^{T/2} \int_V d^3\vec{x}\, dt\, e^{ip\cdot x}$$

→ this turns out to be something we can square.

$$\rightarrow \lim_{V,T\to\infty} |(2\pi)^4 \delta^{(4)}_{VT}(p)|^2 = VT(2\pi)^4 \delta^{(4)}(p)$$

→ In total, we get:

$$\langle f|(S-1)|i\rangle_{VT} = i\mathcal{A}_{fi}^{VT}(2\pi)^4\delta_{VT}^{(4)}(p_F-p_i)\prod_f \frac{1}{\sqrt{2\omega_k}\sqrt{V}}\prod_i \frac{1}{\sqrt{2\omega_k}\sqrt{V}}$$

→ If there are N particles in the final state, then in momentum space, there are:

$\prod_{f=1}^{N} \frac{V}{(2\pi)^3} d^3p_f$ states in the region of size $d^3p_1 \cdots d^3p_N$

After squaring, summing over all final states & dividing by T:

sum over amplitudes of each final state?

$$\frac{W_{VT}}{T} = \frac{1}{T}|\mathcal{A}_{fi}^{VT}|^2(2\pi)^8|\delta_{VT}^{(4)}(p_F-p_i)|^2\prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_p}\prod_i \frac{1}{2\omega_i V}$$

ask about how to get this

(Coleman, p. 242-244)

Explicitly squaring the box-normalized delta:

$$\int d^4p\,|\delta_{VT}^{(4)}(p)|^2 = \frac{1}{(2\pi)^8}\int d^4p\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\int_V d^3\vec{x}\int_V d^3\vec{x}'e^{ip\cdot x-ip\cdot x'}$$

$$= \frac{1}{(2\pi)^4}\int_V d^3\vec{x}\int d^3\vec{x}'\delta^{(4)}(x-x')\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'$$

$$= \frac{1}{(2\pi)^4}\int d^3\vec{x}\int_{-T/2}^{T/2}dt = \frac{VT}{(2\pi)^4}$$

, so $|\delta^{(4)}(p)|^2$ is proportional to a delta function.

$$\lim_{V,T\to\infty}|(2\pi)^4\delta_{VT}^{(4)}(p)|^2 = VT(2\pi)^4\delta^{(4)}(p)$$

→ substituting the limit into the transition probability:

$$\frac{W}{T} = |A_{fi}|^2 (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{\text{final part.}} \frac{d^3 p_f}{(2\pi)^3 2E_f} \prod_{\text{initial}} \frac{1}{2E_i V}$$

$$\equiv |A_{fi}|^2 V D \prod_{\text{initial}, i} \frac{1}{2E_i V}, \quad D \equiv (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{\text{final } f} \frac{d^3 p_f}{(2\pi)^3 2E_f}$$

→ D is manifestly Lorentz invariant, and all $(2\pi)^n$ have a $\delta^{(n)}$ + $(2\pi)^{-n}$ a $\int d^n k$

Decays:

→ We have a single-particle initial state!

so we look for $\frac{W}{T} = \frac{1}{2E} |A_{fi}|^2 D$

$$\left(|A_{fi}|^2 V D \prod_{\substack{\text{initial} \\ \text{particles}, i}} \frac{1}{2E_i V} = \frac{|A_{fi}|^2 V D}{2EV} = \frac{|A_{fi}|^2 D}{2E}! \right)$$

good that Vs cancel in $V \to \infty$ limit!

→ In particle's rest frame, define $d\Gamma$ as the differential decay probability / unit time:

$$d\Gamma \equiv \frac{1}{2M} |A_{fi}|^2 D \Rightarrow \Gamma = \frac{1}{2M} \int_{\substack{\text{all final} \\ \text{states}}} |A_{fi}|^2 D$$

→ Probability particle does not decay after time t is $e^{-\lambda t}$ (how?)

Cross Sections:

We now consider smashing beams of particles and measuring # of incident particles on a detector. We consider incident flux F:

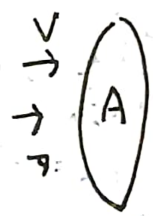
F = # of particles/ unit time/ unit area

$\Rightarrow dN = F\,d\sigma$ ← differential cross section

/ unit time

$\Rightarrow d\sigma = \dfrac{dN}{F} = \dfrac{\text{differential probability}}{\text{unit time} \times \text{unit flux}}$

$= \dfrac{A_{fi}^{2}}{4E_1E_2V} D \cdot \dfrac{1}{\text{flux}}$ ⟶



→ $N = |\vec{v}|Atd$, d particle density

$\Rightarrow \dfrac{N}{At} = |\vec{v}|d$

, $d = 1/V$ by normalization

$\Rightarrow F = |\vec{v}|/V$ → For 2 beam, probability of finding particle is $1/V$ in unit volume, but collision can happen anywhere, so V's cancel, and flux is $|\vec{v}_1 - \vec{v}_2|/V$

$\Rightarrow d\sigma = \dfrac{A_{fi}^{2}}{4E_1E_2V} D \dfrac{V}{|\vec{v}_1-\vec{v}_2|} = \dfrac{A_{fi}^{2}}{4E_1E_2} \dfrac{D}{|\vec{v}_1-\vec{v}_2|}$

$$\Rightarrow \sigma = \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1-\vec{v}_2|}\int_{\text{all final states}} |A_{fi}|^2 D$$

→

2- Body final states:

→ For 2 particle final states, we must do 6 integrals, $d^3\vec{p}_1\, d^3\vec{p}_2$, but 4 variables are constrained by the EM delta, $\delta^{(4)}(p_1+p_2-p_I)$:

$$D = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_I)$$

(really $\int D$, but call it D)

→ In the COM frame, $\vec{p}_I = 0$ and $E_I = E_F$:

$$\Rightarrow D = \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^3 \delta^{(3)}(\vec{p}_1+\vec{p}_2)(2\pi)\delta(E_1+E_2-E_I)$$

~~(Since $p_1 = E_1$, $p_2 = E_2$, $E_I = E_F$, $\vec{p}_1 = -\vec{p}_2$)~~

(Since $p_1 = E_1 - \vec{p}_1$, $p_2 = E_2 - \vec{p}_2$, $E_I = E_I$.)

$$\Rightarrow \delta^{(4)}(p_1+p_2-p_I) = \delta^{(4)}(E_1-\vec{p}_1+E_2-\vec{p}_2-E_I)$$
$$= \delta^{(3)}(\vec{p}_1+\vec{p}_2)\,\delta(E_1+E_2-E_I)$$

$$\Rightarrow \int \frac{d^3p_1\, d^3p_2}{(2\pi)^3 2E_1 (2\pi)^3 2E_2} (2\pi)^3 \delta^{(3)}(\vec{p}_1+\vec{p}_2)(2\pi)\delta(E_1+E_2-E_I)$$

$$= \int \frac{d^3\vec{p}_1}{(2\pi)^3 4E_1E_2} (2\pi)\delta(E_1+E_2-E_I) = \int \frac{1}{(2\pi)^3 4E_1E_2} p_1^2\, dp_1\, d\Omega_1 (2\pi)\,\delta(E_1+E_2-E_I)$$

$$(d^3\vec{p}_1 = p_1^2\, dp_1\, d\cos\theta_1\, d\phi_1 = p_1^2\, dp_1\, d\Omega_1$$

To get rid of the energy delta, we invoke

$$\delta[f(x)] = \sum_{x_0 \in \text{zeroes}} \frac{1}{|f'(x_0)|}\delta(x-x_0)$$

where we differentiate wrt p_1:

$$\left|\frac{\partial(E_1+E_2)}{\partial p_1}\right|^{-1} = ? \quad \to E_1^2 = p_1^2 + m^2, \; E_2^2 = p_2^2 + m^2 = p_1^2 + m^2$$
$$(\vec{p}_1 = -\vec{p}_2)$$

$$\Rightarrow \frac{\partial(E_1^2)}{\partial p_1} = \frac{\partial}{\partial p_1}(p_1^2 + m^2) = 2p_1 = 2E_1\frac{\partial E_1}{\partial p_1} \Rightarrow \frac{\partial E_1}{\partial p_1} = \frac{p_1}{E_1}$$

$$, \frac{\partial(E_2^2)}{\partial p_1} = 2E_2\frac{\partial E_2}{\partial p_1} = 2p_1 \Rightarrow \frac{\partial E_2}{\partial p_1} = \frac{p_1}{E_2}$$

$$\Rightarrow \left|\frac{\partial(E_1+E_2)}{\partial p_1}\right| = \frac{p_1}{E_1} + \frac{p_1}{E_2} = \frac{p_1(E_1+E_2)}{E_1 E_2} = \frac{p_1 E_T}{E_1 E_2}$$

$$\Rightarrow \delta(E_1 + E_2 - E_T) = \frac{E_1 E_2}{p_1 E_T}\delta(p_1 -$$

$$\Rightarrow D = \frac{p_1^2\, dp_1\, d\Omega_1}{16\pi^2 E_1 E_2} \cdot \frac{E_1 E_2}{p_1 E_T}\delta(p_1 - 0) = \int \frac{p_1\, dp_1\, d\Omega_1}{16\pi^2 E_T}\delta(p_1)$$

$$= \boxed{\frac{1}{16\pi^2}\frac{p_1\, d\Omega_1}{E_T}}$$

(should be $\delta(p_1 - \sqrt{\frac{E_T}{2} - m^2}$, but then p_1 would be integrated out, and notes / Coleman not explicit.)

Spin 1/2 fields

→ We have so far only had theories of interacting scalar fields, and under Lorentz transformations:

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

If ϕ instead comprised components of a 4-vector ϕ^μ, we have $\phi^\mu(x) \to \phi'^\mu(x) = \Lambda^\mu{}_\nu \phi^\nu(\Lambda^{-1}x)$

→ In general, a field transforms in a well-defined way under the Lorentz group,

$$\phi_a(x) \to D_{ab}(\Lambda)\phi_b(\Lambda^{-1}x)$$ → If ϕ_a has n-comps, D_{ab} is $n \times n$ matrix

→ The matrices $D(\Lambda)$ form an n-dimensional representation of the Lorentz group:

- $D_{ab}(\Lambda_1)D_{bc}(\Lambda_2) = D_{ac}(\Lambda_1\Lambda_2)$
- $D(\Lambda^{-1}) = D(\Lambda)^{-1}$ (Lorentz transforms rep inverse is inverse of rep)
- $D(\mathbb{1}) = \mathbb{1}$

→ We want to describe particles of spin 1/2, which we know how they transform under rotations.

$$|\psi\rangle = \begin{pmatrix} |\psi_\uparrow\rangle \\ |\psi_\downarrow\rangle \end{pmatrix}, \cancel{S_x = \tfrac{1}{2}\sigma_x}\ S_i = \tfrac{1}{2}\sigma_i, \ \sigma_i \text{: Pauli matrices.}$$

* For a particle with no orbital angular momentum, total is given by spin operators.

under rotations about the $\hat{e}$ axis, a state $|\psi\rangle$ transforms as $|\psi\rangle \to U_R(\hat{e},\theta)|\psi\rangle$,

$$U_R(\hat{e},\theta) = e^{-i\vec{J}\cdot\hat{e}\theta}$$

For a state w/ total angular momentum 1/2:

$|\psi\rangle \to e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}|\psi\rangle$, so the matrices $e^{-i\vec{\sigma}\cdot\theta/2}$ form the spinor representation of the rotation group.

→ spin 1/2 fields transform as $u(x) = \begin{pmatrix} u_\uparrow(x) \\ u_\downarrow(x) \end{pmatrix} \to e^{-i\vec{\sigma}\cdot\hat{n}\theta/2} u(R^{-1}x)$

, but this does not specify transformation under Lorentz.

→ There IS a 2D representation of the Lorentz group which gives spin 1/2 rep for rotations:

Fact: Group of 2×2 matrices w/ det = 1 form a 2D representation of the Lorentz group. (we know subgroup for rotations, $e^{-i\vec{\sigma}\cdot\hat{n}\theta/2}$).

Boosts: $Q = e^{-\vec{\sigma}\cdot\hat{n}\varphi/2}$ → NOT unitary, but Hermitian, det = 1.
, $\varphi \equiv$ rapidity, $\cosh(\varphi) = \gamma$

$$(t,z) \to (\cosh(\varphi)t + \sinh(\varphi)z,\ \cosh(\varphi)z + \sinh(\varphi)t)$$

* It's convenient to think about ~~think~~ the inequivalency of representations as reps. related by a change of basis of states:

→ Given 2 reps. $Q(\Lambda)$, $\tilde{Q}(\Lambda)$ + a matrix T st $Q(\Lambda) = T\tilde{Q}(\Lambda)T^{\dagger}$ $\forall \Lambda$, then given state $|\psi\rangle$, it transforms as

$|\psi\rangle \to Q(\Lambda)|\psi\rangle$. Performing a change of basis: $|\tilde{\psi}\rangle = T|\psi\rangle \Rightarrow |\tilde{\psi}\rangle$ transforms as $|\tilde{\psi}'\rangle = T|\psi'\rangle = TQ(\Lambda)|\psi\rangle = TQ(\Lambda)T^{\dagger}|\tilde{\psi}\rangle$, so $Q(\Lambda)$ + $TQ(\Lambda)T^{\dagger}$ are the same transformation under a change of basis!

But if no such T exists, then representations are inequivalent (like scalars vs. 4-vectors)

→ There are 2 inequivalent representations of the Lorentz group for spin 1/2 particles, weird... namely Q and Q^*:

	$Q(\Lambda)$	$Q^*(\Lambda)$
rotations	$e^{-i\vec{\theta}\cdot\hat{n}/2}$	$e^{-i\vec{\theta}\cdot\hat{n}/2}$
boosts	$e^{\vec{\theta}\cdot\hat{n}/2}$	$e^{-i\vec{\theta}\cdot\hat{n}/2}$

How do we choose? We choose both! (2 spin 1/2 fields)

→ $U_+(x)$ will transform under Q and $U_-(x)$ as Q^*.

$\Rightarrow U_{\pm} \to e^{-i\vec{\theta}\cdot\hat{n}/2} U_{\pm}$ for rotation subgroup →

Boosts: $u_\pm \to e^{\pm\vec{\sigma}\cdot\hat{n}/2} u_\pm$

Lets look at u_+ and write a Lagrangian with desirable properties... we know how u_+ transforms, but $(u_+)^2$? How do products of fields transform?

→ $u_+^\dagger u_+ \longrightarrow u_+^\dagger u$ under rotations, since
$u_+^\dagger u_+ \to u_+^\dagger Q^\dagger Q u_+ = u_+^\dagger Q^{-1} Q u_+ = u_+^\dagger u_+$, so
$u_+^\dagger u_+$ is a rotational scalar, but <u>not</u> under boosts:
(since $Q = Q^\dagger$ for boosts).

→ Since u's have 2 components, there are 4 linearly-independent bilinears:

- $u_+^\dagger u_+$
- $u_+^\dagger \vec{\sigma} u_+$

Pset: show these form components of a 4-vector! (p.395, Coleman)
ie, $V^\mu = (u_+^\dagger u_+, u_+^\dagger \vec{\sigma} u_+)$, which allows construction of Lorentz-invariant quantities. ($W^\mu = (u_-^\dagger u_-, -u_-^\dagger \sigma_z u_-)$ similarly)

What is the difference between the spinor fields?
First we describe a desired Lagrangian:

- Bilinear in fields
- Real action
- $U(1)$ symmetry

→ only way to maintain bilinearity is contraction with partials:

$\mathcal{L} = i[u_+^\dagger \partial_0 u_+ + u_+^\dagger \vec{\sigma}\cdot\vec{\nabla} u_+]$ → integration by parts reveals i necessary for real action →

$$\rightarrow i\int d^4x\,(u_+^\dagger\partial_0 u_+ + u_+^\dagger\vec{\sigma}\cdot\vec{\nabla}u_+) = i\int d^4x\, u_+^\dagger\partial_0 u_+ + i\int d^4x\, u_+^\dagger\vec{\sigma}\cdot\nabla u_+$$

$dv = \partial_0 u_+ \, dt \; d^3x \; \& \; v = u_+, \; u = u_+^\dagger \; \& \; \frac{du}{dt} = \frac{\partial(u_+^\dagger)}{\partial t} = \partial_0 u_+^\dagger$

$$\Rightarrow i\int d^4x\, u_+^\dagger\partial_0 u_+ = i\left(\cancel{u_+^\dagger u_+} - \int d^4x\, u_+ d(u_+^\dagger)\right) = -i\int d^4x\, \partial_0 u_+^\dagger u_+$$

Similarly, $i\int d^4x\, u_+^\dagger\vec{\sigma}\cdot\vec{\nabla}u_+ = -i\int(\nabla u_+^\dagger)\cdot\vec{\sigma}u_+$

but the new integrand is the negative complex conjugate of the original, ie $z^* = -z$, ie the integrand is purely imaginary, so iz yields a real number!

$\rightarrow \mathcal{L} = i(u_+^\dagger\partial_0 u_+ + u_+^\dagger\vec{\sigma}\cdot\vec{\nabla}u_+)$ is the Weyl Equation.

Equations of motion:

$$\Pi^\mu_{u_+^\dagger} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu u_+^\dagger)} = 0 \quad \text{(all derivatives act on } u_+)$$

$$\Rightarrow \boxed{\frac{\partial\mathcal{L}}{\partial u_+^\dagger} = (\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ = 0} \quad \text{(matrix equation)}$$

w/ $\partial_0^{ij} = \partial_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Is this free field theory? We have a quadratic Lagrangian and want KG:

→ Multiply by $\partial_0 - \vec{\sigma}\cdot\vec{\nabla}$:

$$(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) = (\partial_0^2 - [\vec{\sigma}\cdot\vec{\nabla}]^2) \rightarrow$$

$$(\vec{\sigma}\cdot\nabla)^2 = \sigma_i \sigma_j \partial_i \partial_j = (i\varepsilon_{ijk}\sigma_k + \delta_{ij})\partial_i\partial_j$$

$$= i\varepsilon_{ijk}\sigma_k \partial_i \partial_j + \delta_{ij}\partial_i\partial_j = \nabla^2 +$$

$$\hookrightarrow \varepsilon_{ijk}\sigma_k\partial_i\partial_j = \varepsilon_{jik}\sigma_k\partial_j\partial_i = -\varepsilon_{ijk}\sigma_k\partial_i\partial_j = 0$$

$$\Rightarrow (\vec{\sigma}\cdot\vec{\nabla})^2 = \nabla^2 \rightarrow \text{so } (\partial_0 - \vec{\sigma}\cdot\nabla)(\partial_0 + \vec{\sigma}\cdot\nabla)\, u_+(x)$$

$= (\partial_0^2 - \nabla^2)\, u_+(x) \rightarrow$ This is Klein Gordon! So u_+ fields are free fields (for $m=0$), ie a massless spin $1/2$ particle.

→ This won't be a theory for an electron, which has mass, but we press on...

Plane Wave Solutions:

- $u_+(x) = u_+ e^{-ik\cdot x}$, $k^0 > 0$ ①

constant 2-comp. spinor

- $u_+(x) = v_+ e^{ik\cdot x}$ ②

→ Sub ① into EOM:

$$(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\, u_+(x) = (\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\, u_+ e^{-ik\cdot x} = (-ik_0 + i\vec{\sigma}\cdot\vec{k})\, u_+ e^{-ik\cdot x}$$

$$(\vec{\sigma}\cdot\nabla u_+ e^{-ik\cdot x} = \vec{\sigma}\cdot(u_+ i\vec{k} e^{-ik\cdot x}))$$

$$\Rightarrow \boxed{(-ik^0 + i\vec{\sigma}\cdot\vec{k})\, u_+ = 0} \Rightarrow \boxed{(k^0 - \vec{\sigma}\cdot\vec{k})\, u_+ = 0}$$

→ This matrix equation is easy to solve in a reference frame where $\vec{k} = k^0 \hat{z}$ (since $m=0$)

$\Rightarrow (K^0 - \vec{\sigma}\cdot(K^0\hat{z}))\, u_+ = (K^0 - \sigma_z K^0)\, u_+ \Rightarrow \underline{(\mathbb{1} - \sigma_z)\, u_+ = 0}$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} u_+ = 0$$

$\Rightarrow \boxed{u_+ \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ → weird, this says only spin up obeys EOM, but we expects spin 1/2 particles to have 2 spin states!

Definitely <u>not</u> an electron.

(IF $\vec{K} = -K^0\hat{z}$, $u_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$) → spin <u>up</u> in the direction of motion, weird.

→ The quanta will be massless particles w/ helicity +1/2, antiparticles w/ helicity −1/2

(helicity is NOT Lorentz invariant for massive particles $v < c$, as for frame w/ $v' = v + \varepsilon$, the spin is unchanged but direction of motion flips)

* To see the quanta explicitly, consider $|K\rangle$ w/ $\vec{K} = (0, 0, K_z)$, $K_0 > 0$:

$$\langle 0 | u_+(x) | K\rangle \propto e^{-ik\cdot x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ -or- } \langle 0 | u_+(x) | K\rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

→ this state turns out to be an eigenstate of z-angular momentum J_z!

$$J_z|K\rangle = \lambda|K\rangle \Rightarrow e^{-iJ_z\cdot\theta}|K\rangle = e^{-i\lambda\theta}|K\rangle$$

$$\Rightarrow e^{-i\lambda\theta}\langle 0 | u_+(0) | K\rangle = \langle 0 | u_+(0) e^{-iJ_z\theta} | K\rangle$$

$$= \langle 0 | e^{iJ_z\theta} u_+(0) e^{-iJ_z\theta} | 0\rangle$$ (since vacuum rotationally invariant)

but $e^{iJ_z\theta} u_+(0) e^{-iJ_z\theta} = e^{-\frac{1}{2}i\sigma_z\theta} u_+(0)$

$\Rightarrow e^{-i\lambda\theta}\langle 0|u_+(0)|k\rangle \propto e^{i\lambda\theta} u_p$, $\langle 0|e^{iJ_z\theta} u_+(0) e^{-iJ_z\theta}|k\rangle$

$\propto e^{-\frac{1}{2}i\theta\sigma_z} u_p = e^{-\frac{1}{2}i\theta} u_p$ (since $\sigma_z u_p = u_p$)

and $e^{-i\lambda\theta} = e^{-\frac{1}{2}i\theta} \Rightarrow \boxed{\lambda = \frac{1}{2}}$ $\rightarrow$ helicity $1/2$