Various Quantum Field Theory Problems

Jay Epstein

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1 Lectures of Sidney Coleman, 2.1

In d space-time dimensions, determine the dimension (in mass units) of the canonical free scalar field ϕ. What about the dimensions of a Lagrangian density with self-interactions? The action? The coefficients a_n? (as will be shown).

Solution:
Since we are dealing in natural units, we should first determine what the relevant remaining dimensions are. By definition of natural units, we have

$$[c] = \frac{[L]}{[T]} = 1 \implies [L] = [T]$$
$$[\hbar] = [S] = [E][T] = \frac{[M][L^2]}{[T^2]}[T] = \frac{[M][L^2]}{[T]} = [M][L] = 1 \implies [M] = \frac{1}{[L]}$$

Since we do not yet know the dimensions of the Lagrangian density for a free scalar field (and likewise neither the action) in mass dimensions, we refer to the equal time commutation relation between the scalar field and its conjugate momentum:

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = \phi(\vec{x},t)\dot{\phi}(\vec{y},t) - \dot{\phi}(\vec{y},t)\phi(\vec{x},t) = i\hbar\delta^{(d-1)}(\vec{x}-\vec{y})$$

This is exactly what we need, as we can express the dimensions of either term on the left hand side purely in terms of $[\phi]$, and we can determine the dimensions of the right hand side using the definition of the $d-1$ delta function:

$$[\phi][\dot{\phi}] = [\phi]\frac{[\phi]}{[T]} = [\phi]^2[M]$$
$$\int d^{d-1}x\delta^{(d-1)}(x) = 1 \implies [\delta^{(d-1)}(x)] = \frac{1}{[L]^{d-1}} = [M]^{d-1}$$

where we used the fact that $[d^n x] = [L]^n$. Equation the dimensions of the equal time commutation relation, and recalling that $[\hbar] = 1$:

$$[\phi]^2[M] = [M]^{d-1} \implies [\phi]^2 = [M]^{d-2} \implies \boxed{[\phi] = [M]^{\frac{d}{2}-1}}$$

To determine the dimensions of a self-interacting Lagrangian density, we recall the fact that the Lagrangian density is quadratic in its derivative:

$$\mathcal{L} \propto \partial_\mu\phi\partial^u\phi \implies [\mathcal{L}] = [\partial_\mu\phi\partial^u\phi] = [\phi]^2[M]^2 = \boxed{[M]^d}$$

since $[\partial_\mu\phi] = \frac{[\phi]}{[L]} = [\phi][M]$. To find the dimension of the action, we simply integrate the Lagrangian density over all spacetime:

$$S = \int d^d x\mathcal{L} \implies [S] = [d^d x][\mathcal{L}] = [M]^d[L]^d = \frac{[M]^d}{[M]^d} = \boxed{1}$$

Now consider a Lagrangian density of the following form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - \sum_{n\geq 2} a_n\phi^n$$

To find the dimensions of the coefficients a_n, simply apply the fact that each term in the sum must have the same units and the units of the Lagrangian density:

$$[a_n\phi^n] = [a_n][\phi]^n = [a_n]([M]^{\frac{d}{2}-1})^n = [\mathcal{L}] = [M]^d \implies \boxed{[a_n] = [M]^{n+d-\frac{nd}{2}}}$$

As a sanity check, we look at the dimensions of the term a_2, as in the Klein Gordon Lagrangian, the recall that the coefficient of the ϕ^2 is m^2 and describes the mass m of the free particle:

$$[a_2] = [M]^{2+d-\frac{2d}{2}} = [M]^{2+d-d} = [M]^2$$

and this is true for any spacetime dimension d. This is good, as for higher dimensional generalization of the Klein Gordon Lagrangian, the once self-interaction term should always be scaled by the squared-mass of the particle.

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First, computing $\frac{\partial y(x,t)}{\partial t}$ and $\frac{\partial y(x,t)}{\partial x}$:

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)q_n(t)\right) = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\dot{q}_n(t)$$
$$\frac{\partial y(x,t)}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)q_n(t)\right) = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)q_n(t)$$

Substituting these expressions into the Lagrangian:

$$L = \int_0^a dx\left[\frac{\sigma}{2}\left(\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\dot{q}_n^2(t)\right)^2 - \frac{T}{2}\left(\sqrt{\frac{2}{a}}\sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)q_n(t)\right)^2\right]$$

We can rewrite the Lagrangian by noting that $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i,j}^{n} a_i a_j$ and pulling the summation outside of the integral:

$$L = \frac{1}{a}\sum_{n,m}^{\infty}\int_0^a dx\left[\sigma\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\dot{q}_n(t)\dot{q}_m(t) - T\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\frac{nm\pi^2}{a^2}q_n(t)q_m(t)\right]$$

where $n, m = 0, 1, 2, \ldots$. We can then apply the orthogonality of the sines and cosines on the interval $[0, a]$:

$$\frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \delta_{nm}$$
$$\frac{2}{a}\int_0^a \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right) = \delta_{nm}$$
$$\implies L = \sum_{n,m}^{\infty}\left(\frac{\sigma}{2}\delta_{nm}\dot{q}_n(t)\dot{q}_m(t) - \frac{T}{2}\delta_{nm}\frac{nm\pi^2}{a^2}q_n(t)q_m(t)\right)$$
$$\boxed{= \sum_{n=1}^{\infty}\left(\frac{\sigma}{2}\left(\frac{\partial q_n(t)}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{n\pi}{a}\right)^2 q_n^2(t)\right)}$$

To derive the equations of motion for the system, we must first establish the degrees of freedom, which are simple the Fourier modes of $q(t)$check if this is the right terminology, so the Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0$$

$$\longrightarrow \frac{\partial L}{\partial \dot{q}_n} = \frac{\partial}{\partial \dot{q}_n}\left(\sum_{m=1}^{\infty} \frac{\sigma}{2}\left(\frac{\partial q_m(t)}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{m\pi}{a}\right)^2 q_m^2(t)\right)$$

Since the set of functions $q(t) = \{q_1(t), q_2(t), \dots\}$ are independently-varying functions, we have that $\frac{\partial \dot{q}_m}{\partial \dot{q}_n} = \delta_{nm}$

$$\implies \frac{\partial L}{\partial \dot{q}_n} = \sigma \dot{q}_n$$
$$\implies \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = \sigma \ddot{q}_n$$

From a similar argument for the independence of the constituent of the constituent $q_n(t)$ functions, we can easily evaluate $\frac{\partial L}{\partial q_n}$:

$$\begin{aligned} \frac{\partial L}{\partial q_n} &= \frac{\partial}{\partial q_n}\left(\sum_{m=1}^{\infty} \frac{\sigma}{2}\left(\frac{\partial q_m(t)}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{m\pi}{a}\right)^2 q_m^2(t)\right) \\ &= -T\left(\frac{n\pi}{a}\right)^2 q_n \end{aligned}$$

Thus, the resulting equations of motion are:

$$\sigma \ddot{q}_n + T\left(\frac{n\pi}{a}\right)^2 q_n = 0$$

It is convenient to rewrite this equation as:

$$\ddot{q}_n(t) = -\frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n(t)$$

as this is clearly the equation of motion for a simple harmonic oscillator. Thus, our equations of motion are an infinite set of decoupled harmonic oscillators.

To calculate the frequency of each oscillator, we simply recall the simple harmonic equation of motion, $\ddot{x} = -\omega^2 x$. Thus, the frequency of oscillation for the n^{th} oscillator is:

$$\boxed{\omega_n = \sqrt{\frac{T}{\sigma}}\left(\frac{n\pi}{a}\right)}$$

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Computing the change in the Lagrangian density due to the internal transformation as defined:

$$\begin{aligned}\phi_a\phi_a &\longrightarrow \left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)\left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)\\ &= \phi_a\phi_a + 2\theta\epsilon_{abc}n_b\phi_a\phi_c + \mathcal{O}(\theta^2)\end{aligned}$$

The anti-symmetry of the Levi-Cevita tensor can be exploited:

$$\begin{aligned}2\theta\epsilon_{abc}n_b\phi_a\phi_c &= 2\theta\epsilon_{cba}\phi_c\phi_a\\ &= -2\theta\epsilon_{abc}\phi_a\phi_c \implies 2\theta\epsilon_{abc}n_b\phi_a\phi_c = 0\end{aligned}$$

where first the dummy indices were relabelled, following by applying $\epsilon_{cab} = -\epsilon_{abc}$ and commuting the fields. Thus we have $\phi_a\phi_a \longrightarrow \phi_a\phi_a$, so the potential term is left invariant under the transformation.

Considering the kinetic term:

$$\begin{aligned}\partial_\mu\phi_a\partial^\mu\phi_a &\longrightarrow \left(\partial_\mu\phi_a + \theta\epsilon_{abc}n_b\partial_u\phi_c\right)\left(\partial^\mu\phi_a + \theta\epsilon_{abc}n_b\partial^u\phi_c\right)\\ &= \partial_\mu\phi_a\partial^\mu\phi_a + \theta\epsilon_{abc}n_b\partial_\mu\phi_a\partial^\mu\phi_c + \theta\epsilon_{abc}n_b\partial_\mu\phi_c\partial^\mu\phi_a + \mathcal{O}(\theta^2)\end{aligned}$$

Applying anti-symmetry again:

$$\theta\epsilon_{abc}n_b\partial_\mu\phi_c\partial^\mu\phi_a = \theta\epsilon_{cba}n_b\partial_\mu\phi_a\partial^\mu\phi_c = -\theta\epsilon_{abc}n_b\partial_\mu\phi_a\partial^\mu\phi_c$$

This term precisely adds out the second term in the sum, so we have:

$$\partial_\mu\phi_a\partial^\mu\phi_a \longrightarrow \partial_\mu\phi_a\partial^\mu\phi_a$$

Thus the Lagrangian is invariant under the transformation $\phi_a \longrightarrow \phi_a + \theta\epsilon_{abc}n_b\phi_c$. Calculating the Noether current:

$$
\begin{aligned}
j^\mu &= \sum_{a=1}^{3} \pi_a^\mu D\phi_a - F^\mu \\
\pi_a^\mu &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} = \frac{\partial}{\partial\left(\partial_\mu \phi_a\right)}\left(\frac{1}{2}\partial_{nu}\phi_a\partial^\nu\phi_a - \frac{1}{2}m^2\phi_a\phi_a\right) = \frac{1}{2}\left(\frac{\partial\left(\partial_{nu}\phi_a\right)}{\partial\left(\partial_\mu\phi_a\right)}\partial^\nu\phi_a + \frac{\partial\left(\partial^\nu\phi_a\right)}{\partial\left(\partial_\mu\phi_a\right)}\partial_\nu\phi_a\right) \\
&= \frac{1}{2}\left(\partial^\nu\phi_a\delta_\nu^\mu + \partial_\nu\phi_a g^{\mu\nu}\right) = \frac{1}{2}\left(\partial^\mu\phi_a + \partial^\mu\phi_a\right) = \boxed{\partial^\mu\phi_a} \\
D\phi_a &= \frac{\partial\left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)}{\partial\theta} = \epsilon_{abc}n_b\phi_c
\end{aligned}
$$

$$
\Longrightarrow \boxed{j^\mu = \sum_{a=1}^{3} \partial^\mu\phi_a\epsilon_{abc}n_b\phi_c = \partial^\mu\phi_a\epsilon_{abc}n_b\phi_c}
$$

(the sum was dropped due to implicit summation) If we take the $\mu = 0$ component of the Noether current and integrate over all space, we will recover a globally-conserved charge:

$$
Q = \int d^3x j^0 = \int d^3x \left(\epsilon_{abc}n_b\dot{\phi}_a\phi_c\right)
$$

Since Q is conserved , it is conserved for any choice of the components of the rotation vector n, so in particular we can choose n such that it is one of the standard basis vectors, ie $n_b = \delta_{bd}$ for a fixed $d \in \{1, 2, 3\}$.

$$
\Longrightarrow \; Q = \int d^3x\epsilon_{adc}\dot{\phi}_a\phi_c = \int d^3x\epsilon_{bdc}\dot{\phi}_b\phi_c = -\int d^3x\epsilon_{abc}\dot{\phi}_b\phi_c
$$

The second equality follows after the relabelling of dummy indices $a \longrightarrow b$ and the third equality after relabelling d with a and using the anti-symmetry of the Levi-Cevita tensor. Since Q is conserved, $-Q$ is conserved as well. We can then label this quantity Q_a according to the free index:

$$
\boxed{Q_a = \int d^3x\epsilon_{abc}\dot{\phi}_b\phi_c}
$$

Since we let n arbitrarily be one of the standard basis vectors, Q_a is actually 3 independent conserved quantities, each corresponding to a different choice of d in $n_b = \delta_{bd}$ (In the relabelling from d to a, the conserved charge is really Q_d as a was not chosen to avoid a triple index contraction).

Finding the equations of motion for the theory:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} = \partial^\mu \phi_a \implies \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) = \partial_\mu \partial^\mu \phi_a$$
$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \frac{\partial}{\partial \phi_a}\left(\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a \phi_a\right) = -\frac{m^2}{2}\left(2\phi_b \frac{\partial \phi_b}{\partial \phi_a}\right) = -m^2 \phi_b \delta_{ab} = -m^2 \phi_a$$
$$\implies \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) = \left(-m^2 - \partial_\mu \partial^\mu\right)\phi_a = 0 \implies \ddot{\phi}_a = \nabla^2 \phi_a - m^2 \phi_a$$

Computing the time derivative of Q_a:

$$\begin{aligned}
\frac{d}{dt}Q_a = \frac{d}{dt}\int d^3x \epsilon_{abc}\dot{\phi}_b \phi_c &= \int d^3x \epsilon_{abc}\left(\ddot{\phi}_b \phi_c + \dot{\phi}_b \dot{\phi}_c\right) \\
&= \int d^3x \epsilon_{abc}\ddot{\phi}_b \phi_c && \left(\text{since } \epsilon_{abc}\dot{\phi}_b\dot{\phi}_c = -\epsilon_{abc}\dot{\phi}_b\dot{\phi}_c = 0\right) \\
&= \int d^3x \epsilon_{abc}\left(\nabla^2 \phi_b - m^2 \phi_b\right)\phi_c \\
&= \int d^3x \epsilon_{abc}\nabla^2 \phi_b && \left(\text{since } \epsilon_{abc}\phi_b\phi_c = -\epsilon_{abc}\phi_b\phi_c = 0\right)
\end{aligned}$$

Integrating by parts:

$$\int d^3x \epsilon_{abc}\nabla^2 \phi_b = \int_{\partial \mathcal{M}} \phi_b \left(\nabla \phi_b \cdot \hat{n}\right) d\left(\partial \mathcal{M}\right) + \int_{\mathcal{M}} d^3x \epsilon_{abc} \nabla \phi_b \cdot \nabla \phi_c$$

where $\partial \mathcal{M}$ denotes the boundary of the region of integration and $\mathcal{M}$ is the region itself (in this case, all of space), and $\hat{n}$ is the outward normal to the boundary. Since we assume our fields to be physical, they vanish sufficiently fast at the boundaries so that the contribution from the first integral is 0. Similarly, due to the anti-symmetry of ϵ_{abc}, $\epsilon_{abc}\nabla\phi_b \cdot \nabla\phi_c = \epsilon_{cba}\nabla\phi_c \cdot \nabla\phi_b = -\epsilon_{abc}\nabla\phi_b \cdot \nabla\phi_c = 0$. So we have:

$$\frac{dQ_a}{dt} = 0$$

from the fields directly.

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Under Lorentz transformations, the Minkowski metric is preserved:

$$\begin{aligned}\eta_{\mu u}x^{\mu}x^{\nu} &= \eta_{\mu\nu}x'^{\mu}x'^{\nu} \\ &= \eta_{\mu\nu}x^{\sigma}x^{\tau}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}\end{aligned}$$

Performing a relabelling of dummy indices in the unprimed line interval $\mu \longrightarrow \sigma$, $\nu \longrightarrow \tau$:

$$\eta_{\sigma\tau}x^{\sigma}x^{\tau} = \eta_{\mu\nu}x^{\sigma}x^{\tau}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau} \implies \boxed{\eta_{\sigma\tau} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}}$$

as desired. (the coordinates were "divided out" as suggested by the identification of the same dummy indices, but this really just says that the equivalency is true for arbitrary 4-vectors)

Consider the infinitesimal transformation defined by $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$. For $\Lambda^{\mu}{}_{\nu}$ to be a Lorentz transformation, it must preserve the Minkowski metric:

$$\begin{aligned}\eta_{\mu\nu}\Lambda^{\mu}{}_{\tau}\Lambda^{\nu}{}_{\sigma} &= \eta_{\mu\nu}\left(\delta^{\mu}{}_{\sigma} + \omega^{\mu}{}_{\sigma}\right)\left(\delta^{\nu}{}_{\tau} + \omega^{\nu}{}_{\tau}\right) \\ &= \eta_{\nu\mu}\left(\delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\tau} + \delta^{\mu}{}_{\sigma}\omega^{\nu}{}_{\tau} + \omega^{\mu}{}_{\sigma}\delta^{\nu}{}_{\tau} + \mathcal{O}(w^2)\right) \qquad \text{(since } w \text{ is an infinitesimal transformation)} \\ &= \eta_{\sigma\tau} + \eta_{\nu\sigma}\omega^{\nu}{}_{\tau} + \eta_{\tau\mu}\omega^{\mu}{}_{\sigma} = \eta_{\sigma\tau} + \omega_{\sigma\tau} + \omega_{\tau\sigma}\end{aligned}$$

For the Minkowski metric to be preserved, it must follow that $\omega_{\sigma\tau} = -\omega_{\tau\sigma}$. Raising the indices on the equality via the Minkowski metric yields $\omega^{\sigma\tau} = -\omega^{\tau\sigma}$ as desired.

To construct the matrix $\omega^{\mu}{}_{\nu}$ corresponding to a rotation through an infinitesimal angle θ about the x^3 axis, it suffices to construct a 4 x 4 matrix that leaves the coordinates ct and x^3 unchanged while performing a 2-dimensional rotation of the coordinates x^1 and x^2 into each other:

$$
\begin{aligned}
\Lambda \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} &= \begin{pmatrix} ct \\ x^1 \cos\theta - x^2 \sin\theta \\ x^1 \sin\theta + x^2 \cos\theta \\ x^3 \end{pmatrix} \\
\implies \Lambda &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta - 1 & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Since the rotation is infinitesimal, the trigonometric terms can be expanded to first order:

$$
\boxed{\omega^{\mu}{}_{\nu} \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}
$$

To construct the matrix Λ corresponding to a Lorentz boost along the x^1-axis by an infinitesimal velocity v, we simply recall the 1D Lorentz transformations (in natural units):

$$
\begin{aligned}
t' &= \gamma(t - vx^1) \\
x^{1'} &= \gamma(x^1 - vt) \\
x^{2'} &= x^2 \\
x^{3'} &= x^3 \\
\implies \Lambda &= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma - 1 & -\gamma v & 0 & 0 \\ -\gamma v & \gamma - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Since v is infinitesimal, we can expand the matrix elements to first order:

$$\gamma \approx 1$$
$$\gamma v \approx v$$
$$\implies \boxed{\omega^{\mu}{}_{\nu} \approx \begin{pmatrix} 0 & -v & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

5 Peskin & Schroeder 2.2

Consider the field theory of a complex-valued scalar field theory obeying the Klein-Gordon equation. The action of the theory is

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right)$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi\right).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

c) Rewrite the conserved charge

$$Q = \int d^3x \, \frac{i}{2} \left(\phi^* \pi^* - \pi \phi\right)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q^i = \int d^3x \; \frac{i}{2} \left(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b \right)$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

a) Given the action $S = \int d^4x \, (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)$, the Lagrangian is immediately given by $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^*$. Computing the conjugate momenta:

$$\begin{aligned}
\Pi_\phi^\mu &= \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} = \frac{\partial}{\partial \left(\partial_\mu \phi \right)} \left(\partial_\nu \phi^* \partial^\nu \phi - m^2 \phi^* \phi \right) \\
&= \partial_\nu \phi^* \frac{\partial \left(\partial^\nu \phi \right)}{\partial \left(\partial_\mu \phi \right)} = \partial_\nu \phi^* g^{\mu\nu} = \boxed{\partial^\mu \phi^*} \\
\Pi_{\phi^*}^\mu &= \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^* \right)} = \frac{\partial}{\partial \left(\partial_\mu \phi^* \right)} \left(\partial_\nu \phi^* \partial^\nu \phi - m^2 \phi^* \phi \right) \\
&= \partial^\nu \phi \frac{\partial \left(\partial_\nu \phi^* \right)}{\partial \left(\partial_\mu \phi^* \right)} = \partial^\nu \phi \delta_\nu^\mu = \boxed{\partial^\mu \phi}
\end{aligned}$$

The canonical commutation relations are

$$\begin{aligned}
&[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\phi(\vec{x}, t) \phi^*(\vec{y}, t)] = [\phi^*(\vec{x}, t) \phi^*(\vec{y}, t)] = 0 \\
&\left[\Pi_\phi^0(\vec{x}, t), \Pi_\phi^0(\vec{y}, t) \right] = \left[\Pi_\phi^0(\vec{x}, t), \Pi_{\phi^*}^0(\vec{y}, t) \right] = \left[\Pi_{\phi^*}^0(\vec{x}, t), \Pi_{\phi^*}^0(\vec{y}, t) \right] = 0 \\
&\left[\phi(\vec{x}, t), \Pi_{\phi *}^0(\vec{y}, t) \right] = \left[\phi^*(\vec{x}, t), \Pi_\phi^0(\vec{y}, t) \right] = 0 \\
&\left[\phi(\vec{x}, t), \Pi_\phi^0(\vec{y}, t) \right] = \left[\phi^*(\vec{x}, t), \Pi_{\phi^8}^0(\vec{y}, t) \right] = i\delta^{(3)}(\vec{x} - \vec{y})
\end{aligned}$$

The Hamiltonian (density) is

$$\begin{aligned}
\mathcal{H} &= \Pi_\mu^0 \partial_0 \phi_\mu - \mathcal{L} = \Pi_\phi^0 \partial_0 \phi + \Pi_{\phi^*}^0 \partial_0 \phi^* - \mathcal{L} \\
&= \partial^0 \phi^* \partial_0 \phi + \partial^0 \phi \partial_0 \phi^* - \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi \\
&= \partial^0 \phi \partial_0 \phi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi = \partial^0 \phi \partial^0 \phi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi
\end{aligned}$$

In terms of the conjugate momenta, the Hamiltonian is

$$\begin{aligned}
\mathcal{H} = \partial^0 \phi \partial^0 \phi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi &= \Pi_{\phi^*}^0 \Pi_\phi^0 + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \\
&= \boxed{\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi}
\end{aligned}$$

Where $\pi = \Pi^0_\phi$ and $\pi^* = \Pi^0_{\phi^*}$. Integrating over all space:

$$H = \int d^3x \mathcal{H}(x) = \boxed{\int d^3x \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right)}$$

Computing the Heisenberg equation of motion for $\phi(x)$:

$$\begin{aligned}\frac{\partial\phi}{\partial t} = i\left[H(\vec{y},t), \phi(\vec{x},t)\right] &= i\left[\int d^3y \left(\pi^*(\vec{y},t)\pi(\vec{y},t) + \nabla\phi^*(\vec{y},t)\cdot\nabla\phi(\vec{y},t) + m^2\phi(\vec{y},t)\phi^*(\vec{y},t)\right), \phi(\vec{x},t)\right] \\ &= i\int d^3y \left[\pi^*\left(\vec{y},t)\pi(\vec{y},t) + \nabla\phi^*(\vec{y},t)\cdot\nabla\phi(\vec{y},t) + m^2\phi(\vec{y},t)\phi^*(\vec{y},t)\right), \phi(\vec{x},t)\right]\end{aligned}$$

Since fields and their spatial derivatives commute, we can restrict the commutator to the first term:

$$\begin{aligned}\int d^3y \left[\pi^*(\vec{y},t)\pi(\vec{y},t), \phi(\vec{x},t)\right] = \int d^3y \pi^*(\vec{y},t)\left[\pi(\vec{y},t), \phi(\vec{x},t)\right] &= -\int d^3y \pi^*(\vec{y},t)(i\delta^{(3)(\vec{x}-\vec{y})}) \\ &= -i\int d^3y \delta^{(3)}(\vec{x}-\vec{y})\pi^*(\vec{y},t) = -i\pi^*(\vec{x},t) \\ \implies i\left[H(\vec{y},t), \phi(\vec{x},t)\right] &= \frac{\partial\phi(\vec{x},t)}{\partial t} = \boxed{\pi^*(\vec{x},t)}\end{aligned}$$

Computing the equation of motion for $\pi(x)$:

$$\frac{\partial\pi(\vec{x},t)}{\partial t} = i\left[H(\vec{y},t), \pi(\vec{x},t)\right] = i\int d^3y \left[\pi^*\left(\vec{y},t)\pi(\vec{y},t) + \nabla\phi^*(\vec{y},t)\cdot\nabla\phi(\vec{y},t) + m^2\phi(\vec{y},t)\phi^*(\vec{y},t)\right), \pi(\vec{x},t)\right]$$

In this case we can only neglect the commutator of $\pi(\vec{x},t)$ with the first term:

$$\int d^3y \left[\nabla\phi^*(\vec{y},t)\cdot\nabla\phi(\vec{y},t), \pi(\vec{x},t)\right] = \int d^3y \left(\nabla\phi^*(\vec{y},t)\cdot\left[\nabla\phi(\vec{y},t), \pi(\vec{x},t)\right] + \left[\nabla\phi^*(\vec{y},t), \pi(\vec{x},t)\right]\cdot\nabla\phi(\vec{y},t)\right)$$

We can "pull out" the gradient from the commutator (Have yet to find a rigorous justification for this, see Coleman p.70):

$$\int d^3y \nabla\phi^*(\vec{y},t)\cdot\left[\nabla\phi(\vec{y},t), \pi(\vec{x},t)\right] = \int d^3y \nabla\phi^*(\vec{y},t)\cdot\nabla\left[\phi(\vec{y},t), \pi(\vec{x},t)\right] = i\int d^3y \nabla\phi^*(\vec{y},t)\cdot\nabla\delta^{(3)}(\vec{y}-\vec{x})$$

Integrating by parts:

$$d^3v = \nabla\delta^{(3)}(\vec{y}-\vec{x})d^3y \implies v = \delta^{(3)}(\vec{y}-\vec{x})$$
$$u = \nabla\phi^*(\vec{y},t) \implies d^iu = \frac{\partial}{\partial y^j}(\nabla_i\phi^*(\vec{y},t))dy^j \implies d^3u = \nabla^2\phi^* d^3y$$
$$\implies i\int d^3y\nabla\phi^*(\vec{y},t)\cdot\nabla\delta^{(3)}(\vec{y}-\vec{x}) = i\left(\nabla\phi^*(\vec{y},t)\delta^{(3)}(\vec{y}-\vec{x})|_{\partial\mathcal{M}} - \int d^3y\nabla^2\phi^*(\vec{y},t)\delta^{(3)}(\vec{y}-\vec{x})\right)$$
$$\boxed{= -i\nabla^2\phi^*(\vec{x},t)}$$

(Integration by parts justification for v is scuffed) Computing the final commutator:

$$\int d^3y\left[m^2\phi(\vec{y},t)\phi^*(\vec{y},t),\pi(\vec{x},t)\right] = \int d^3y\left(m^2\phi^*(\vec{y},t)\left[\phi(\vec{y},t),\pi(\vec{x},t)\right]\right)\boxed{= im^2\phi^*(\vec{x},t)}$$
$$\implies \frac{\partial\pi(\vec{x},t)}{\partial t} = i\left(im^2\phi^*(\vec{x},t) - i\nabla^2\phi^*(\vec{x},t)\right)$$
$$\boxed{= \left(\nabla^2 - m^2\right)\phi^*(\vec{x},t)}$$

It immediately follows from the equations of motion that the ϕ^* fields obey the Klein-Gordon equation for a massive scalar field of mass m, and since the Hamiltonian is symmetric up to commutation in ϕ and ϕ^*, the ϕ fields will also obey the Klein-Gordon equation.

b) To introduce creation and annihilation operators, we solve for the fields explicitly. Since both $\phi(x)$ and $\phi^*(x)$ obey the Klein Gordon equation, ie

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = \left(\partial_\mu\partial^\mu + m^2\right)\phi^* = 0$$

the fields $\phi(x)$ and $\phi^*(x)$ can be expressed as a superposition of plane wave solutions, with the caveat that ϕ (and naturally ϕ^*) no longer must satisfy Hermicity:

$$\phi(x) = \int\frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{-ik\cdot x} + b^\dagger_{\vec{k}}e^{ik\cdot x}\right)$$
$$\phi^*(x) = \int\frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left(a^\dagger_{\vec{k}}e^{ik\cdot x} + b_{\vec{k}}e^{-ik\cdot x}\right)$$

Where $a_{\vec{k}}$ and $b_{\vec{k}}$ are independent annihilation operators. Evaluating the constituents of the

Hamiltonian:

$$\pi = \partial^0\phi^*(x) = \frac{\partial}{\partial(ct)}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left(a^\dagger_{\vec{k}}e^{ik\cdot x} + b_{\vec{k}}e^{-ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left((i\omega_{\vec{k}})a^\dagger_{\vec{k}}e^{ik\cdot x} + (-i\omega_{\vec{k}})b_{\vec{k}}e^{-ik\cdot x}\right)$$

$$= i\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(a^\dagger_{\vec{k}}e^{ik\cdot x} - b_{\vec{k}}e^{-ik\cdot x}\right)$$

$$\pi^* = \partial^0\phi(x) = \frac{\partial}{\partial(ct)}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}e^{-ik\cdot x} + b^\dagger_{\vec{k}}e^{ik\cdot x}\right) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}}\left((-i\omega_{\vec{k}})a_{\vec{k}}e^{-ik\cdot x} + (i\omega_{\vec{k}})b^\dagger_{\vec{k}}e^{ik\cdot x}\right)$$

$$= i\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(b^\dagger_{\vec{k}}e^{ik\cdot x} - a_{\vec{k}}e^{-ik\cdot x}\right)$$

$$\implies \pi^*\pi = (i)^2\int \frac{d^3kd^3p}{(2\pi)^3}\frac{\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}}{2}\left(a^\dagger_{\vec{p}}e^{ip\cdot x} - b_{\vec{p}}e^{-ip\cdot x}\right)\left(b^\dagger_{\vec{k}}e^{ik\cdot x} - a_{\vec{k}}e^{-ik\cdot x}\right)$$

$$= -\int \frac{d^3kd^3p}{(2\pi)^3}\frac{\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}}{2}\left(a^\dagger_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(-p-k)\cdot x} - b_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(p-k)\cdot x} - a^\dagger_{\vec{p}}a_{\vec{k}}e^{-i(k-p)} + a_{\vec{k}}b_{\vec{p}}e^{-i(p+k)\cdot x}\right)$$

(it's worth noting that I actually computed $\pi\pi^*$ and not $\pi^*\pi$, but we know these quantities are the same due to the second quantization conditions already established)

$$\int d^3x(\pi^*\pi) = -\int \frac{d^3xd^3kd^3p}{(2\pi)^3}\frac{\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}}{2}\left(a^\dagger_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(-p-k)\cdot x} - b_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(p-k)\cdot x} - a^\dagger_{\vec{p}}a_{\vec{k}}e^{-i(k-p)} + a_{\vec{k}}b_{\vec{p}}e^{-i(p+k)\cdot x}\right)$$

Recalling that the Fourier transform of a complex exponential:

$$\int \frac{d^3x}{(2\pi)^3}e^{-i(\vec{k}-\vec{k}')\cdot x} = \delta^{(3)}(\vec{k}-\vec{k}')$$

$$\implies \int d^3x(\pi^*\pi) = -\int \frac{d^3kd^3p}{2}\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}\int \frac{d^3x}{(2\pi)^3}\left(a^\dagger_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(-\omega_{\vec{p}}-\omega_{\vec{k}})t}e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} - b_{\vec{p}}b^\dagger_{\vec{k}}e^{-i(\omega_{\vec{p}}-\omega_{\vec{k}})t}e^{-i(\vec{k}-\vec{p})\cdot\vec{x}}\dots\right)$$

$$-a^\dagger_{\vec{p}}a_{\vec{k}}e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} + a_{\vec{k}}b_{\vec{p}}e^{-i(\omega_{\vec{p}}+\omega_{\vec{k}})t}e^{-i(-\vec{p}-\vec{k})\cdot\vec{x}})$$

$$= -\int \frac{d^3kd^3p}{2}\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}\left(a^\dagger_{\vec{p}}b^\dagger_{\vec{k}}e^{i(\omega_{\vec{p}}+\omega_{\vec{k}})t}\delta^{(3)}(\vec{k}+\vec{p}) - b_{\vec{p}}b^\dagger_{\vec{k}}e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\delta^{(3)}(\vec{k}-\vec{p}) - a^\dagger_{\vec{p}}a_{\vec{k}}e^{i(\omega_{\vec{p}}-\omega_{\vec{k}})t}\delta^{(3)}(\vec{p}-\vec{k})\right)$$

$$+a_{\vec{k}}b_{\vec{p}}e^{-i(\omega_{\vec{p}}+\omega_{\vec{k}})t}\delta^{(3)}(-\vec{p}-\vec{k}))$$

$$\boxed{= -\int \frac{d^3k}{2}\omega_{\vec{k}} \left(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} - b_{\vec{k}} b^\dagger_{\vec{k}} - a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \right)}$$

$$\nabla\phi = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}} \left((i\vec{k}) a_{\vec{k}} e^{-ik\cdot x} + (-i\vec{k}) b^\dagger_{\vec{k}} e^{ik\cdot x} \right) = i \int \frac{\vec{k} d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{k}}}} \left(-b^\dagger_{\vec{k}} e^{ik\cdot x} + a_{\vec{k}} e^{-ik\cdot x} \right)$$

$$\nabla\phi^* = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{p}}}} \left((-i\vec{p}) a^\dagger_{\vec{p}} e^{ip\cdot x} + (i\vec{p}) b_{\vec{p}} e^{-ip\cdot x} \right) = i \int \frac{\vec{p} d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{p}}}} \left(-a^\dagger_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{=ip\cdot x} \right)$$

$$\implies \int d^3x \nabla\phi^* \cdot \nabla\phi = -\int \frac{d^3x d^3k d^3p}{2(2\pi)^3\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \left(\vec{k}\cdot\vec{p} \right) \left(a^\dagger_{\vec{p}} b^\dagger_{\vec{k}} e^{-i(-\omega_{\vec{p}}-\omega_{\vec{k}})t} e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} - b_{\vec{p}} b^\dagger_{\vec{k}} e^{-i(\omega_{\vec{p}}-\omega_{\vec{k}})t} e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} \dots \right.$$

$$\left. - a^\dagger_{\vec{p}} a_{\vec{k}} e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t} e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} + a_{\vec{k}} b_{\vec{p}} e^{-i(\omega_{\vec{p}}+\omega_{\vec{k}})t} e^{-i(-\vec{p}-\vec{k})\cdot\vec{x}} \right)$$

$$= -\int \frac{d^3k d^3p}{2\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \left(\vec{k}\cdot\vec{p} \right) (a^\dagger_{\vec{p}} b^\dagger_{\vec{k}} e^{i(\omega_{\vec{p}}+\omega_{\vec{k}})t} \delta^{(3)}(\vec{k}+\vec{p}) - b_{\vec{p}} b^\dagger_{\vec{k}} e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t} \delta^{(3)}(\vec{k}-\vec{p}) - a^\dagger_{\vec{p}} a_{\vec{k}} e^{i(\omega_{\vec{p}}-\omega_{\vec{k}})t} \delta^{(3)}(\vec{p}-\vec{k})$$

$$+ a_{\vec{k}} b_{\vec{p}} e^{-i(\omega_{\vec{p}}+\omega_{\vec{k}})t} \delta^{(3)}(-\vec{p}-\vec{k}))$$

$$\boxed{= \int \frac{d^3k}{2\omega_{\vec{k}}} \vec{k}^2 \left(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} + b_{\vec{k}} b^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \right)}$$

since $\vec{k}\cdot\vec{p} = \pm\vec{k}^2$ depending on whether $\vec{p} = \pm\vec{k}$:

$$m^2\phi^*\phi = \int \frac{m^2 d^3k d^3p}{2(2\pi)^3\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \left(a^\dagger_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x} \right) \left(a_{\vec{k}} e^{-ik\cdot x} + b^\dagger_{\vec{k}} e^{ik\cdot x} \right)$$

$$= \int \frac{m^2 d^3k d^3p}{2(2\pi)^3\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \Big(a^\dagger_{\vec{p}} a^{e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}}_{\vec{k}} e^{-i(-\vec{k}+\vec{p})\cdot\vec{x}} + a_{\vec{k}} b_{\vec{p}} e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t} e^{-i(-\vec{k}-\vec{p})\cdot x} + b^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t} e^{-i(\vec{k}+\vec{p})\cdot\vec{x}}$$
$$+ b_{\vec{p}} b^\dagger_{\vec{k}} e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t} e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} \Big)$$

$$\implies \int d^3x \; m^2 \phi^* \phi = \int \frac{m^2 \; d^3k d^3p}{2\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \int \frac{d^3x}{(2\pi)^3} \Big(a^\dagger_{\vec{p}} a_{\vec{k}} e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t} e^{-i(-\vec{k}+\vec{p})\cdot\vec{x}} + a_{\vec{k}} b_{\vec{p}} e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t} e^{-i(-\vec{k}-\vec{p})\cdot x}$$
$$+ b^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t} e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} + b_{\vec{p}} b^\dagger_{\vec{k}} e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t} e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} \Big)$$
$$= \int \frac{m^2 \; d^3k d^3p}{2\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}} \Big(a^\dagger_{\vec{p}} a_{\vec{k}} e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t} \delta^{(3)}(-\vec{k}+\vec{p}) + a_{\vec{k}} b_{\vec{p}} e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t} \delta^{(3)}(-\vec{k}-\vec{p}) + b^\dagger_{\vec{k}} a^\dagger_{\vec{p}} e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t} \delta^{(3)}(\vec{k}+\vec{p})$$
$$+ b_{\vec{p}} b^\dagger_{\vec{k}} e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t} \delta^{(3)}(\vec{k}-\vec{p}) \Big)$$
$$\boxed{= \int \frac{m^2 \; d^3k}{2\omega_{\vec{k}}} \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} + b^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} e^{2i\omega_{\vec{k}}t} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big)}$$

Thus the explicit form of the Hamiltonian is then

$$\int \frac{d^3k}{2\omega_{\vec{k}}} \Big[-\omega^2_{\vec{k}} \Big(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} - b_{\vec{k}} b^\dagger_{\vec{k}} - a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \Big) + \vec{k}^2 \Big(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} + b_{\vec{k}} b^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \Big)$$
$$+ m^2 \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} + b^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} e^{2i\omega_{\vec{k}}t} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big) \Big]$$

From the relativistic dispersion relation $\omega^2_{\vec{k}} = \vec{k}^2 + m^2$, the Hamiltonian simplifies greatly:

$$H = \int \frac{d^3k}{2\omega_{\vec{k}}} \Big[\omega^2_{\vec{k}} \Big(b_{\vec{k}} b^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} \Big) + \vec{k}^2 \Big(b_{\vec{k}} b^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} \Big) + m^2 \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big)$$
$$- \omega^2_{\vec{k}} \Big(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \Big) + \vec{k}^2 \Big(a^\dagger_{-\vec{k}} b^\dagger_{\vec{k}} e^{2i\omega_{\vec{k}}t} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} \Big)$$
$$+ m^2 \Big(a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega_{\vec{k}}t} + b^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} e^{2i\omega_{\vec{k}}t} \Big)$$
$$= \int \frac{d^3k}{2\omega_{\vec{k}}} 2\omega^2_{\vec{k}} \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big) = \boxed{\int d^3k \; \omega_{\vec{k}} \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big)}$$

Applying commutation relations (since ϕ and $\phi*$ are free fields, we know that the constituent Fourier modes obey creation/annihilation operator algebra):

$$\int d^3k \; \omega_{\vec{k}} \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + b_{\vec{k}} b^\dagger_{\vec{k}} \Big) = \boxed{\int d^3k \; \omega_{\vec{k}} \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}} + \delta^{(3)}(0) \Big)}$$

From this form of the Hamiltonian, we see that our theory contains *two* free particles, as the total Hamiltonian is the sum of the individual Hamiltonians for independent free particles (we can systematically ignore the infinite delta function under the integral, as this is effectively just setting the "zero" of the energy, similar to how what altitude you measure gravitational potential energy from is arbitrary).

It's also worth noting why we are even dealing with a Lagrangian of this form in the first place - it turns out that if you construct a theory of 2 free non-interacting particles that is only dependent on the norm of the sum of the fields, then the theory will exhibit an $SU(2)$ symmetry, ie a rotation of the fields in "field space." This can also be expressed in terms of complex-valued fields as shown in this problem, where the Lagrangian now exhibits a $U(1)$ symmetry, except now we are dealing with a transformation that is a number, $e^{i\theta}$, as opposed to a rotation matrix.

c) Rewriting the conserved charge $Q = \int d^3x\ \frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right)$ in terms of creation and annihilation operators (which will be $a^\dagger/b^\dagger$ and a/b respectively):

$$\begin{aligned}\int\ d^3x\phi^*\pi^* &= i\int \frac{d^3x d^3k d^3p}{2(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{k}}}}\Big(a^\dagger_{\vec{k}}b^\dagger_{\vec{p}}e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} - a^\dagger_{\vec{k}}a_{\vec{p}}e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t}e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} \\ &+ b_{\vec{k}}b^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}-\omega_{\vec{k}})t}e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} - b_{\vec{k}}a_{\vec{p}}e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})}e^{-i(-\vec{k}-\vec{p})\cdot\vec{x}}\Big) \\ &= i\int\frac{d^3kd^3p}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{k}}}}\Big(a^\dagger_{\vec{k}}b^\dagger_{\vec{p}}e^{i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\delta^{(3)}(\vec{k}+\vec{p}) - a^\dagger_{\vec{k}}a_{\vec{p}}e^{i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\delta^{(3)}(\vec{k}-\vec{p}) \\ &+ b_{\vec{k}}b^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}-\omega_{\vec{k}})t}\delta^{(3)}(\vec{p}-\vec{k}) - b_{\vec{k}}a_{\vec{p}}e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\delta^{(3)}(-\vec{k}-\vec{p})\Big) \\ &= i\int\frac{d^3k}{2}\left(a^\dagger_{\vec{k}}b^\dagger_{-\vec{k}}e^{2i\omega_{\vec{k}}t} - a^\dagger_{\vec{k}}a_{\vec{k}} + b_{\vec{k}}b^\dagger_{\vec{k}} - b_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_{\vec{k}}t}\right)\end{aligned}$$

From a similar calculation:

$$\int d^3x\ \pi\phi = i\int\frac{d^3k}{2}\left(a^\dagger_{\vec{k}}a_{\vec{k}} + a^\dagger_{\vec{k}}b_{-\vec{k}}e^{2i\omega_{\vec{k}}t} - b_{\vec{k}}a_{-\vec{k}}e^{-2i\omega_{\vec{k}}t} - b^\dagger_{\vec{k}}b_{\vec{k}}\right)$$

In all we have:

$$\int d^3x\left(\phi^*\pi^* \ - \pi\phi\right) = i\int\frac{d^3k}{2}\left(-2a^\dagger_{\vec{k}}a_{\vec{k}} + 2b^\dagger_{\vec{k}}b_{\vec{k}}\right)$$
$$\Longrightarrow \boxed{Q = \int d^3k\left(a^\dagger_{\vec{k}}a_{\vec{k}} - b^\dagger_{\vec{k}}b_{\vec{k}}\right)}$$

This quantity is precisely the number of a particles minus the number of b particles, and from the sign convention, a particles have a charge of $+1$ and b particles have a charge of -1!

(to compute $\int d^3x\ \pi\phi$, one could take the conjugate of the previous quantity $\phi^*\pi^*$,

however my own attempt of this did not work)

6 Unknown source

Show that replacing the Lagrange density $\mathcal{L} = \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right)$ by

$$\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu(x)$$

where $\Lambda^\mu(x), \mu = 0, \ldots, 3$, are arbitrary functions of the fields $\phi_a(x)$(i.e the coordinate dependence of Λ is implicit in its dependence on the fields), does *not* alter the equations of motion. Thus, when constructing the most general Lagrange density for a field, we do not have to include terms which are total derivatives.

Solution - Given that the equations of motion are yielded by constraining the variation of the action to vanish, it would suffice to simply integrate the Lagrange density $\mathcal{L}'$ over all spacetime and argue that the contribution to the action from the total derivative term contributes only a boundary term and thus vanishes, but this is not very enlightening, at least to me. Instead, we compute the equations of motion for the Lagrange density $\mathcal{L}'$ and show that these are the equations of motion for $\mathcal{L}$:

$$\frac{\partial \mathcal{L}'}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \phi_a)} = \frac{\partial(\mathcal{L} + \partial_\mu \Lambda^\mu)}{\partial \phi_a} - \partial_\mu \frac{\partial(\mathcal{L} + \partial_\nu \Lambda^\nu)}{\partial(\partial_\mu \phi_a)} = 0$$

From linearity of the partial derivative, we get:

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \Pi^\mu_{\phi_a} + \frac{\partial(\partial_\mu \Lambda^\mu)}{\partial \phi_a} - \partial_\nu \frac{\partial(\partial_\mu \Lambda^\mu)}{\partial(\partial_\nu \phi_a)} = 0$$

This equation is suggestive - the left-hand side is the sum of the LHS of the equations of motion for the Lagrange density $\mathcal{L}$ and 2 extra terms. If the sum of these last 2 terms is 0, then we precisely recover the equation of motion for $\mathcal{L}$, $\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \Pi^\mu_{\phi_a} = 0$. Thus, our problem has been reduced to showing that $\frac{\partial(\partial_\mu \Lambda^\mu)}{\partial \phi_a} - \partial_\nu \frac{\partial(\partial_\mu \Lambda^\mu)}{\partial(\partial_\nu \phi_a)} = 0$. (You can think of this as showing that the variation in the equations of motion due to the transformation of the Lagrange density vanishes):

$$\frac{\partial(\partial_\mu \Lambda^\mu)}{\partial \phi_a} = \frac{\partial}{\partial \phi_a}\left(\frac{\partial \Lambda^\mu}{\partial \phi_a}\frac{\partial \phi_a}{\partial x^\mu}\right) = \frac{\partial \phi_a}{\partial x^\mu}\frac{\partial}{\partial \phi_a}\left(\frac{\partial \Lambda^\mu}{\partial \phi_a}\right) + \frac{\partial \Lambda^\mu}{\partial \phi_a}\frac{\partial}{\partial \phi_a}\left(\frac{\partial \phi_a}{\partial x^\mu}\right)$$

The last term $\frac{\partial}{\partial \phi_a}\left(\frac{\partial \phi_a}{\partial x^\mu}\right)$ vanishes by construction of the Euler-Lagrange equations - recall that in classical mechanics, generalized coordinates and their velocities are treated as independent variables, ie $\frac{\partial q_a}{\partial \dot{q}_a} = 0$. The field theoretic analog is that fields and their *spacetime* derivatives are treated as independent variables, ie $\frac{\partial \phi_a}{\partial(\partial_\mu \phi_a)} = 0$ (and thus the reciprocal derivative vanishes, which is the term of interest):

$$\implies \frac{\partial \phi_a}{\partial x^\mu} \frac{\partial}{\partial \phi_a} \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \right) + \frac{\partial \Lambda^\mu}{\partial \phi_a} \frac{\partial}{\partial \phi_a} \left(\frac{\partial \phi_a}{\partial x^\mu} \right) = \frac{\partial \phi_a}{\partial x^\mu} \frac{\partial}{\partial \phi_a} \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \right) = \partial_\mu \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \right)$$

where the chain rule is applied in the last equality:

$$\partial_\nu \left(\frac{\partial \left(\partial_\mu \Lambda^\mu \right)}{\partial \left(\partial_\nu \phi_a \right)} \right) = \partial_\nu \left(\frac{\partial \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \frac{\partial \phi_a}{\partial x^\mu} \right)}{\partial \left(\partial_\nu \phi_a \right)} \right) = \partial_\nu \left(\partial_\mu \phi_a \frac{\partial \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \right)}{\partial \left(\partial_\nu \phi_a \right)} + \frac{\partial \Lambda^\mu}{\partial \phi_a} \frac{\partial \left(\frac{\partial \phi_a}{\partial x^\mu} \right)}{\partial \left(\partial_\nu \phi_a \right)} \right)$$

Recalling that $\Lambda^\mu(\phi_a)$ is purely a function of the fields ϕ_a (i.e. not a function of spacetime derivatives of the fields $\partial_\mu \phi_a$), we then see that $\frac{\partial \Lambda^\mu}{\partial \phi_a}$ also is independent of the spacetime derivatives of the fields, as $\frac{\partial \Lambda^\mu}{\partial \phi_a} = 0 \implies \frac{\partial}{\partial \phi_a} \left(\frac{\partial \Lambda^\mu}{\partial(\partial_\nu \phi_a)} \right) = \frac{\partial}{\partial(\partial_\nu \phi_a)} \left(\frac{\partial \Lambda^\mu}{\partial \phi_{[a]}} \right) = 0$, so we are left with:

$$\partial_\nu \left(\frac{\partial \left(\partial_\mu \Lambda^\mu \right)}{\partial \left(\partial_\nu \phi_a \right)} \right) = \partial_\nu \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \frac{\partial \left(\partial_\mu \phi_a \right)}{\partial \left(\partial_\nu \phi_a \right)} \right) = \partial_\nu \frac{\partial \Lambda^\mu}{\partial \phi_a} \delta^\nu{}_\mu = \partial_\nu \left(\frac{\partial \Lambda^\nu}{\partial \phi_a} \right)$$

In all, we have:

$$\frac{\partial(\partial_\mu \Lambda^\mu)}{\partial \phi_a} - \partial_\nu \frac{\partial(\partial_\mu \Lambda^\mu)}{\partial(\partial_\nu \phi_a)} = \partial_\mu \left(\frac{\partial \Lambda^\mu}{\partial \phi_a} \right) - \partial_\nu \left(\frac{\partial \Lambda^\nu}{\partial \phi_a} \right) = 0$$

And thus $\frac{\partial \mathcal{L}'}{\partial \phi_a} - \partial_\mu {\Pi'}^\mu{}_a = \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \Pi^\mu{}_a = 0$, i.e. the equations of motion for the tranformed Lagrange density are the same equations of motion for that of the original Lagrange density.