Logic recap

Propositional logic consists of atomic formulae p, q, r and connectives

$\neg p$ not p

$p \wedge q$ p and q

$p \vee q$ p or q

$p \Rightarrow q$

$p \rightarrow q$ p implies q

$p \supset q$

$p \Leftrightarrow q$ p iff q

$p \leftrightarrow q$ p equivalent to q

$p \equiv q$

$\perp$ bottom / False

$\top$ top / True

Set theory recap

Sets are collections of objects.
They have neither multiplicity nor order.

$$\{1,1,2,3\} = \{1,2,3\} = \{3,3,1,2,3\}$$

$$\{x \mid x \in \mathbb{N} \wedge x \geq 10\}$$
$$= \{10, 11, 12, \dots\}$$

A, B

$A \cup B$ A union B
$\{x \mid x \in A \text{ or } x \in B\}$

$A \cap B$ A intersection B
$\{x \mid x \in A \text{ and } x \in B\}$

A^c A - complement

$\emptyset$ empty set $\{\}$

$A \subseteq B$ A is a subset of B
everything in A is in B.

Semantics

Semantics is about <u>meaning</u>.

How can we do this?

We use <u>models</u> or <u>possible worlds</u>.

Example

α : $0 \leq p \leq 10$ and p is divisible by 4

We need to assign a <u>set</u> to this sentence.

First off, what about $0 \leq p \leq 10$?

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

What about p is divisible by 4?

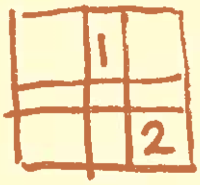
$$\{n \mid n \in \mathbb{N} \wedge n = 4m\}$$

$$\{0, 4, 8, 12, 16, \ldots\}$$

Then, a model for α is

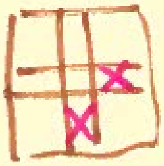
$$\{0, 1, \ldots, 10\} \cap \{0, 4, 8, \ldots\}$$
$$= \{0, 4, 8\}.$$

EXAMPLE

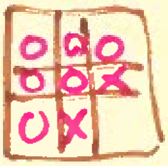


This is a conjunction

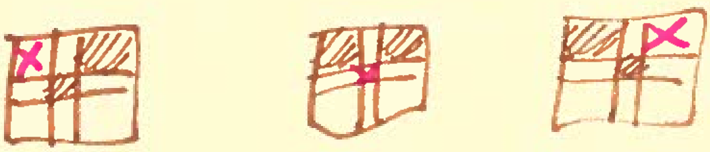
A possible world for the left is:



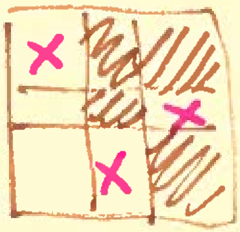
The set of possible worlds is:

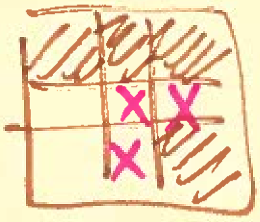
$\{$  $\mid$ where $o \in \{x, \ \} \}$

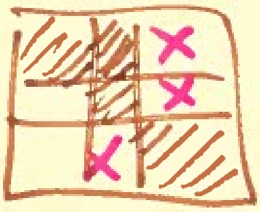
Similarly, for the second part, the models will look like:



We can then take the intersection :





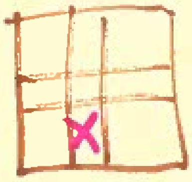


with the possibility of extra mines.

Next, we use this to answer questions.

eg $p_{1,2}$ means there is a mine
is square (1,2).

We create a set of possible worlds
for this sentence :



We check whether one is a subset of the other. This relationship is called <u>entailment</u>.

$$\alpha \models \beta$$

iff

$$M(\alpha) \subseteq M(\beta)$$

models/ possible worlds for α

models/ possible worlds for β

When we have a collection of sentences we call them a knowledge base.

$$M(KB) \subseteq M(\beta)$$

Does this hold?

Yes.

Every model for KB is a model for β. (Every board in $M(KB)$ has a mine in $(1,2)$)

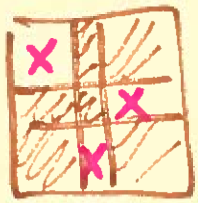
Eg

γ: $P_{1,1}$

Does $KB \models \gamma$? This is true whenever

$$M(KB) \subseteq M(\gamma)$$

We suspect this is false. Why?



This is a possible world for KB

However, this is <u>not</u> a possible world for γ, since there is no mine in (1,1).

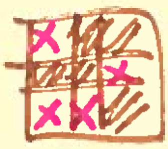
Hence

$$KB \not\models \gamma$$

In an ideal world:

$$KB \models \neg\gamma$$

However this is not true either



We say the KB is consistent with γ and $\neg\gamma$. γ is independent of the KB.

eg

KB is $p \geq 10$ and $p < 4$

α: p is the least ~~even~~ even integer not expressible as the sum of two primes.

$$M(KB) = \{10, 11, 12, \ldots\} \cap \{0, 1, 2, 3\}$$
$$= \phi$$

Since $\phi \subseteq A$ for any A, without working out $M(\alpha)$ we have

$$M(KB) \subseteq M(\alpha).$$

We have seen this before:

$$\perp \models p$$

This entailment holds for any p.

Other ways to find truth

Inference rules hide away the models/semantics. They operate on syntax only. They can be quick.

They need to be <u>sound</u>.

ie you cannot derive something that is false.

eg
$$\frac{p \wedge q}{p}$$
natural deduction style

is truth-preserving / sound.

$$\frac{p \vee q}{p}$$

is not. It is <u>unsound</u>.

Rules should also be <u>complete</u>, if possible.

Completeness says that every true statement is provable.

Propositional logic has sets of rules which are both <u>sound</u> and <u>complete</u>.

semantics

Truth is the same as derivability

syntax

There is a minimal set of inference rules.

They introduce and eliminate connectives in a sound fashion.

$(\rightarrow E)$

$$\frac{\alpha \qquad \alpha \rightarrow \beta}{\beta} \text{ (modus ponens)}$$

$(\wedge E)$

$$\frac{\alpha \wedge \beta}{\alpha}, \quad \frac{\alpha \wedge \beta}{\beta}$$

$(\vee E)$

$$\frac{\alpha \vee \beta \qquad \begin{array}{c}[\alpha]\\ \vdots \\ \gamma\end{array} \qquad \begin{array}{c}[\beta]\\ \vdots \\ \gamma\end{array}}{\gamma}$$

$(\wedge I)$

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}$$

$(\vee I)$

$$\frac{\alpha}{\alpha \vee \beta}, \quad \frac{\beta}{\alpha \vee \beta}$$

$(\rightarrow I)$

$$\frac{\begin{array}{c}[\alpha]\\ \vdots \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

We deal with negation as

$$\neg p \equiv p \to \bot$$

$$(\bot I) \qquad \begin{array}{c} [p] \\ \vdots \\ p \to \bot \\ \hline \bot \end{array}$$

$$(\bot E) \qquad \frac{\bot}{C} \qquad C \text{ is unrestricted.}$$

This is called <u>natural deduction</u>

It can be shown to be sound & complete.

We can then use it to derive true formulae.

For small examples, we can use truth tables:

If p and q is true, then $p \wedge q$ is true.

Similarly

$p \vee q$ is true iff p or q is true.

p	q	$p \vee q$
$\perp$	$\perp$	$\perp$
T	$\perp$	T
$\perp$	T	T
T	T	T

It is the same process, merely presented slightly differently.

eg

$$(((p \wedge q) \vee r) \wedge q) \to r$$

Using sequent calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\times}{p, q, q \Rightarrow r}}{p \wedge q, q \Rightarrow r}\,L\wedge \qquad \dfrac{}{r, q \Rightarrow r}\,Ax}{(p \wedge q) \vee r, q \Rightarrow r}\,L\vee}{((p \wedge q) \vee r) \wedge q \Rightarrow r}\,L\wedge$$

Since this tree cannot be properly finished, we can conclude $((p \wedge q) \vee r) \wedge q \Rightarrow r$ is not derivable. By soundness & completeness, it is therefore not true.