Inference in FOL

Inference rules for the quantifiers :

$$\frac{[t/x]A}{\exists x.\, A} \quad \exists\text{-introduction}$$

(particular) substitution of t for x in A

e.g. $R(x)$ is "x is red".

$\underbrace{\boxed{\text{This ink}}}_{t}$ is red.

Then, we can ~~in~~ infer

"there exists an x such that x is red."

$$\frac{[y/x]A}{\forall x. A} \qquad \forall\text{-introduction}$$

(general) substitution of y for x in A.

eg $S(x)$ is "is a student".

Whoever we substitute into this formula will make it true;

"Alex is a student"

"Davide is a student"

⋮

Then, we can ~~iffer~~ infer

"every x is a student"

We have just described propositionalisation.

We instantiate/substitute all variables with elements from the model, making a (big) propositional formula. We then check that formula:

- disjunction for existentials
- conjunction for universals

All true $\longrightarrow$ $\forall x$

At least one true $\longrightarrow$ $\exists x$

At least one false $\longrightarrow$ $\neg \forall x$

All false $\longrightarrow$ $\neg \exists x$

EXAMPLES

$N(x)$ is a unary predicate

$E(x,y)$ is a binary predicate

Sentences to determine

$\forall x.\ N(x)$

Is $N(1)$ true? Yes, so
$\forall x.\ N(x)$ is true

$\exists x,y.\ E(x,y)$

Is $E(1,1)$ true? Yes, so
$\exists x,y.\ E(x,y)$ is true

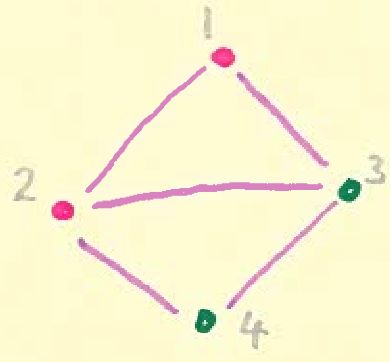
$\exists x,y.\ \neg(x=y)$

Is $\neg(1=1)$ true? No, so
$\exists x,y.\ \neg(x=y)$ is false.

$\forall x,y.\ \neg(x=y) \rightarrow E(x,y)$

Is $\left(\neg(1=1) \rightarrow E(1,1)\right)$ true?
Yes: $\neg(1=1)$ is false, so the implication is true.

EXAMPLE 2



Pink is a unary predicate

Green is a unary predicate

Edge is a binary predicate

Green(3) is true

Edge(1, 4) is false

$$\forall x.\ Pink(x) \longrightarrow (\exists y.\ Green(y) \wedge E(x,y))$$

Propositionalisation

$x = 1,\ y = 1$

$x = 1,\ y = 2$

$x = 1,\ y = 3$

$x = 1,\ y = 4$

$\vdots$

$x = 4,\ y = 4$

Check all of these sentences. We'd need all to be true. There are 16 of them... time-consuming.

However, whenever $x=3$ or $x=4$ we have to check:

$$\text{Pink}(3) \rightarrow \phi'$$

$$\text{Pink}(4) \rightarrow \phi''$$

Since both Pink(3) and Pink(4) are false, we can conclude <u>any</u> sentence matching the above patterns is true.

We need only focus on those sentences where Pink(x) is true. ie $x=1$ and $x=2$

Similarly, we only care about $y=3$ and $y=4$, since that is where Green(x) is true.

Check

$$A \begin{cases} \text{Pink}(1) \rightarrow (\text{Green}(3) \wedge E(1,3)) \checkmark \\ \text{Pink}(1) \rightarrow (\text{Green}(4) \wedge E(1,4)) \times \end{cases}$$

$$B \begin{cases} \text{Pink}(2) \rightarrow (\text{Green}(3) \wedge E(2,3)) \checkmark \\ \text{Pink}(2) \rightarrow (\text{Green}(4) \wedge E(2,4)) \checkmark \end{cases}$$

At least one of A, and at least one of B, needs to be true.

So, we can conclude the sentence is true.

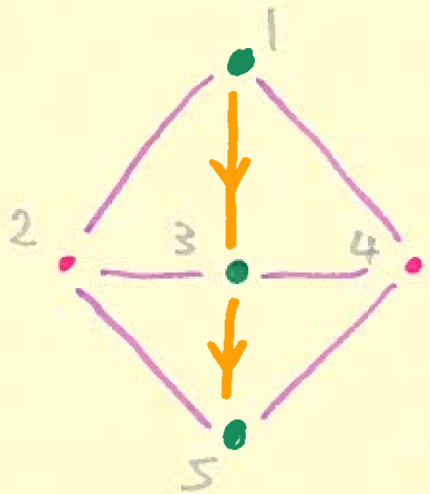
What about:

$$\forall x, y. \left(\mathrm{Pink}(x) \wedge \mathrm{Green}(y)\right) \to E(x, y)$$

This is false. Just give the <u>counterexample</u>: $x = 1$ and $y = 4$.

$$\underbrace{\left(\underset{\checkmark}{\mathrm{Pink}(1)} \wedge \underset{\checkmark}{\mathrm{Green}(4)}\right) \to \underset{\times}{E(1, 4)}}_{\times}$$

EXAMPLE 3



Green is unary
Pink is unary
E is binary
D is binary

$$\forall x, y, z. \left(E(x,y) \wedge E(y,z) \right) \rightarrow D(x,z)$$

This is false: counter-examples include

$x=1,\ y=2,\ z=5$

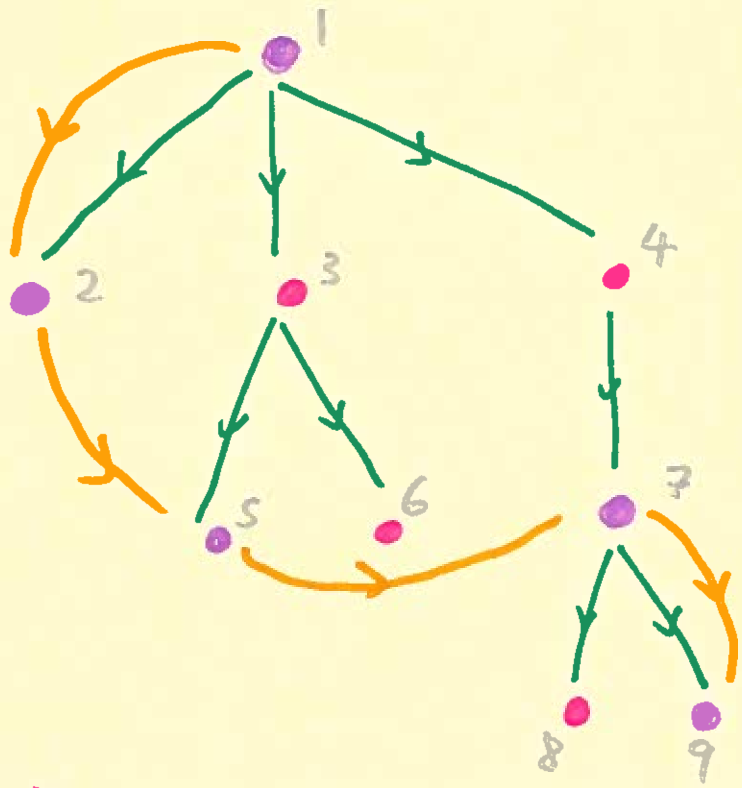
$x=5,\ y=2,\ z=1$

$\vdots$

$x=1,\ y=2,\ z=1$

$\vdots$

EXAMPLE 4



Predicates

King is unary

Spare is unary

hasChild is binary

succeeds is binary

hasOffspring is unary

Sentences

$$\forall x. (\mathrm{king}(x) \wedge \mathrm{hasOffspring}(x)) \longrightarrow (\exists y. \mathrm{king}(y) \wedge \mathrm{hasChild}(x,y))$$

"Every king with children has some child who is king."

Only 1 and 7 satisfy the first part of the implication:

$king(1) \wedge hasOffspring(1)$

$king(7) \wedge hasOffspring(7)$

are true.

For 1; we need one of

$king(2) \wedge hasChild(1,2)$ ✓

$king(3) \wedge hasChild(1,3)$ ✗

$king(4) \wedge hasChild(1,4)$ ✗

to be true.

For 7; we need one of

$king(8) \wedge hasChild(7,8)$ ✗

$king(9) \wedge hasChild(7,9)$ ✓

to be true.

Therefore, the sentence is true.

Sentence 2:

$$\exists x, y.\ King(x) \wedge \neg King(y) \wedge hasChild(x,y)$$

"There is a father-son pair where the father is king but the son is not."

This is true:

$$King(7) \wedge \neg King(8) \wedge hasChild(7,8)$$

Sentence 3:

$$\forall x.\ hasOffspring(x) \rightarrow (\exists y.\ King(y) \wedge hasChild(x,y))$$

"Every father has a son who is king."

Check 1, 3, 4, 7 as only they satisfy hasOffspring(x). Each has a child who is king (3, 5, 7, 9 respectively).

So the sentence is true.