Camera Transformations

Computer Graphics - SET08116

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Outline

Line Definition

We can define a line using two points. Let us take this definition and see how we can define a line computationaly.

Given two 3D points, $\mathbf{P}_1$ and $\mathbf{P}_2$, we can define the line that passes through these points parametrically as

$$\mathbf{P}(t) = (1 - t)\mathbf{P}_1 + \mathbf{P}_2$$

where t ranges over all the real numbers. For $t = 0$ we have $\mathbf{P}_1$ and at $t = 1$ we have $\mathbf{P}_2$.

Ray Definition

A ray can be defined as a point and a direction. Again, let us look at a computational definition.

Given an endpoint, **S**, and a direction vector, **V**, we can define a ray parametrically as:

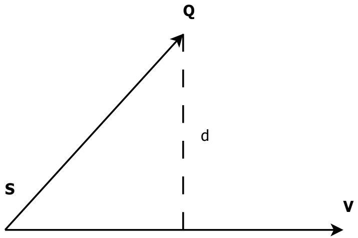
$$\mathbf{P}(t) = \mathbf{S} + t\mathbf{V}$$

where t is a real number greater than or equal to zero.

Distance Between a Point and a Line

One of the most important calculations we can perform is the distance, *d*, between a point, **Q**, and a line.

Given the line defined by an endpoint, **S**, and direction, **V**, we need to calculate the magnitude of $\mathbf{Q} - \mathbf{S}$ that is perpendicular to the line. The following image should help:



Distance Between a Point and a Line

Pythagorean theorem lets us now calculate the distance from the point to the line:

$$
\begin{aligned}
d^2 &= \|\mathbf{Q} - \mathbf{S}\|^2 - (proj_v(\mathbf{Q} - \mathbf{S}))^2 \\
&= \|\mathbf{Q} - \mathbf{S}\|^2 - \left(\frac{(\mathbf{Q} - \mathbf{S}) \cdot \mathbf{V}}{V^2}\mathbf{V}\right)^2 \\
d &= \sqrt{\|\mathbf{Q} - \mathbf{S}\|^2 - \frac{((\mathbf{Q} - \mathbf{S}) \cdot \mathbf{V})^2}{V^2}}
\end{aligned}
$$

Plane Definition

Like a line, we want a method where we can computationally define a plane. A plane is just a 2D area that expands infinitely along its axes.

Given a 3D point, **P** and a normal vector, **N**, the plane passing through **P** and perpendicular to **N** can be defined as the set of points **Q** such that the following equation holds:

$$\mathbf{N} \cdot (\mathbf{Q} - \mathbf{P}) = 0$$

The equation for a plane is sometimes written as:

$$Ax + By + Cz + D = 0$$

where A, B and C are the x, y and z components of the normal vector, and $D = -\mathbf{N} \cdot \mathbf{P}$.

Points and Planes

It is common to try and determine if a point exists on the positive or negative side of a plane. We can do this fairly trivially.

If **N** is unit length (which it should be), then we can calculate the signed distance of a point to the plane using the following equation:

$$d = \mathbf{N} \cdot \mathbf{Q} + D$$

where d is the signed distance. If $d = 0$ then **Q** lies on the plane. If $d > 0$ then **Q** is on the positive side of the plane. If $d < 0$ then **Q** is on the negative side of the plane. This is a useful technique for visibility determination.

Plane Definition

Normally we store a plane definition as a 4D vector. The vector $< \mathbf{N}, D >$ denotes the plane consisting of points $\mathbf{Q}$ satisfying $\mathbf{N} \cdot \mathbf{Q} + D$

This does allow us to simplify our previous equation $d = \mathbf{N} \cdot \mathbf{Q} + D$. If we consider our points to be 4D homogenous ones, with the w component set to 1, we can rewrite the equation as follows:

$$d = \mathbf{L} \cdot \mathbf{Q}$$

where $\mathbf{L} =< \mathbf{N}, D >$

Plane Line Intersection

Another common operation is determining the point where a line intersects a plane.

Remember that we have defined a line computationaly as:

$$\mathbf{P}(t) = \mathbf{S} + t\mathbf{V}$$

We also have our plane definition:

$$\mathbf{N} \cdot \mathbf{Q} + D = 0$$

What we have to do is find the point where the line intersects the plane. To do this, we must solve the equation:

$$\mathbf{N} \cdot \mathbf{P}(t) + D = 0$$

for t.

Plane Line Intersection

If we substitute our line definition into this equation, we get the following:

$$\mathbf{N} \cdot \mathbf{S} + (\mathbf{N} \cdot \mathbf{V})t + D = 0$$

Solving for t, we get the following equation:

$$t = \frac{-(\mathbf{N} \cdot \mathbf{S} + D)}{\mathbf{N} \cdot \mathbf{V}}$$

If we put this value for t into our line equation, we get the point of intersection. If $\mathbf{N} \cdot \mathbf{V} = 0$ then the line is parallel to the plane. If $\mathbf{N} \cdot \mathbf{S} + D = 0$ the line lies on the plane.

Plane Line Intersection

If we use homogenous coordinates, we can simplify this calculation further:

$$t = -\frac{\mathbf{L} \cdot \mathbf{S}}{\mathbf{L} \cdot \mathbf{V}}$$

As we are dealing with homogeonous coordinates, the w component of **S** is set to 1 (it it a point), whereas the w component of V is set to 0 (it is a vector). This is the equation we would normally use.

View Frustum

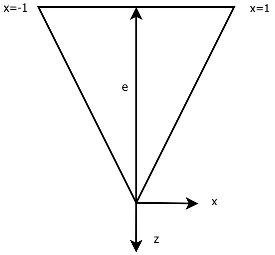
Remember that we can define a camera, or a view frustum, as six planes:

- Left and Right
- Top and Bottom
- Near and Far

Camera space or eye space is the coordinate space where the camera location is centred at the origin, the x axis to the right, and y axis upward. The z axis depends on the library used (left handed and right handed space).

Field of View

We can view illustrate our field of view as follows:



What we want to do is determine the value *e*, the near plane distance, given the horizontal field of view, α. We can do this using the following equation:

$$e = \frac{1}{\tan \frac{\alpha}{2}}$$

which will give the correct focal length for our view. Are we setting this correctly in our applications?

Field of View

Our vertical field of view is determined by our aspect ratio - the screen height divided by the width. We denote this value as a. We also need our actual near plane distance n and far plane distance f.

We want to find the 4D vector $< \mathbf{N}, D >$ for each plane:

$$
\begin{aligned}
Near &= < 0, 0, -1, -n > \\
Far &= < 0, 0, 1, f > \\
Left &= < \frac{e}{\sqrt{e^2+1}}, 0, -\frac{1}{\sqrt{e^2+2}}, 0 > \\
Right &= < -\frac{e}{\sqrt{e^2+1}}, 0, -\frac{1}{\sqrt{e^2+2}}, 0 > \\
Bottom &= < 0, \frac{e}{\sqrt{e^2+a^2}}, -\frac{a}{\sqrt{e^2+a^2}}, 0 > \\
Top &= < 0, -\frac{e}{\sqrt{e^2+a^2}}, -\frac{a}{\sqrt{e^2+a^2}}, 0 >
\end{aligned}
$$

Projection

The final main part of the rendering process is our final projection of our rendered scene.

The maths text shows the steps to build the projection matrix. Here, we will just give it:

$$\mathbf{M}_{frustum} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2nf}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

where r, l, t, b, f and n are the right, left, top, and bottom values for the screen and the far and near values for the frustum.

Projection

We transform a camera space coordinate to screen space by multiplying by this matrix:

$$\mathbf{P}' = \mathbf{M}_{frustum} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2nf}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

We then divide the x, y and z components of $\mathbf{P}'$ by the w component to produce the final screen space coordinate. For the point to be visible, it must lie in the cube defined by the minimum and maximum points $< -1, -1, -1 >$ and $< 1, 1, 1 >$

Recommended Reading

Chapter 5 of the maths text covers this area in more detail, including line - line intersection, and perspective correction.