Physics-Based Animation (SET09119)

Tutorial 05 - Notes - Centre of Mass & Moments

1 Centre of Mass & Momentum

1.1 What is the centre of mass?

In physics, the centre of mass of a distribution of mass in space is the unique point where the weighted relative position of the distributed mass sums to zero.

In this tutorial, we explain how to find the centre of mass for a number of regular shapes, including cases where the shapes are added to or removed from others.

Figure 1-1: Balancing a dish on a stick by placing the point of contact under the dish's centre of mass.

When you have finished this tutorial, you should be able to:

1. work out the turning effect of a force (i.e., its moment)
2. use moments to solve problems
3. tackle equilibrium problems involving objects with mass and density
4. find the position of the centre of mass of one, two, and three dimensional shapes

2 Moment of a Force

Torque, moment, or moment of a force is the tendency of a force to rotate an object about an axis,or pivot. Just as a force is a push or a pull, a torque can be thought of as a twist to an

object. Mathematically, torque is defined as the 'cross product' of the lever-arm distance vector and the force vector, which tends to produce rotation.
The symbol for torque is typically τ, the Greek letter tau. When it is called *moment*, it is commonly denoted M.
The magnitude of torque depends on three quantities:

- the force applied,
- the length of the lever arm connecting the axis to the point of force application
- the angle between the force vector and the lever arm.

The mathematical formula for torque is:

$$\begin{aligned} \text{torque vector}: \ \vec{\tau} &= \vec{r} \times \vec{F} \\ \text{torque magnitude}: \ \tau &= |\vec{r}||\vec{F}| \sin(\theta) \end{aligned} \tag{1}$$

where $\vec{\tau}$ is the torque vector, τ is the length or magnitude of the torque, $\vec{r}$ is the displacement vector (a vector from the point from which torque is measured to the point where force is applied), $\vec{F}$ is the force vector, $\times$ denotes the cross product, and θ is the angle between the force vector and the lever arm vector.

2.1 'Torque' or 'Moment'

The words 'torque' and 'moment' (of force) mean the same. However, 'torque' tends to be used when there is an axle or pivot to be turned around, while 'moment' tends to be used in essentially non-rotational situations, such as analysis of forces on a beam. So be aware, if you are asked to calculate the 'moment' or 'torque' magnitude for simple cases in this tutorial - they are the same.

2.2 Perpendicular

From Equation 1 above, we can simplify the calculation for situations when the force is perpendicular (i.e., at a right angle):

$$\tau = (r)(F_\perp)$$

where $F_\perp$ is the amount of force directed perpendicularly to the position of the particle. Importantly, any force directed parallel to the particle's position vector does not produce a torque. It follows from the properties of the cross product that the torque vector is perpendicular to both the position and force vectors.

2.3 Moment of an Arm

In general, as shown in Figure 2-2 below, for the perpendicular case, the moment of a force (F) about a point (O) is defined as:

$$\begin{aligned} \tau &= (\text{distance to pivot point})(force) \\ &= (d)(F) \end{aligned}$$

where d is the perpendicular distance of (O) from the line of action of (F). If d is measured in metres and F in newtons, then Moment (L) (i.e., (F)(d)) is measured in newton-metres or Nm.

The construction of the 'moment arm' is shown in the Figure 2-2, along with the vectors d and F. The problem with this definition is that it does not give the direction of the torque but only the magnitude, and hence it is difficult to use in three-dimensional cases. However, for this tutorial and examples, we can use the simplified case for numerical exercises.

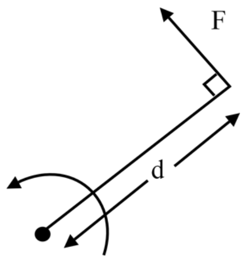

Figure 2-2: For a perpendicular force applied to a point offset by a distance (d) the moment is defined as L = (F)(d).

2.3.1 Example

If a person places a force of 20 N at the terminal end of a wrench 0.5 m long (or a force of 20 N exactly 0.5 m from the twist point of a wrench of any length), the torque will be 10 N-m, i.e., assuming that the person moves the wrench by applying force in the plane of movement of and perpendicular to the wrench.

2.3.2 Example

A force (F) of 28N acts as shown below in Figure 2-3. Given the distance (d) is 3 metres, and the angle is 30 degrees. What is the torque about the point (P).

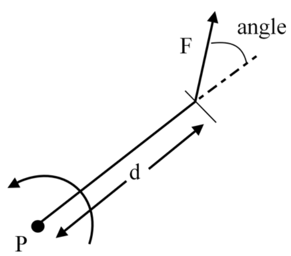

Figure 2-3:

Solution:
We resolve this problem by splitting the Force into the perpendicular and parallel components. Any force directed parallel to the position vector direction does not produce a torque. Hence, we are only concerned with the perpendicular magnitude.

$$F_{\perp} = 28 sin(30) = 14N$$

$\therefore$

$$\tau = (3)(14) = 42Nm$$

Note, the torque rotates in an anti-clockwise direction.

3 Equilibrium

We say a body is in equilibrium under a system of forces, if the body doesn't move, i.e., it must not move upwards, sideways, or rotate. Hence, we review how to calculate the point of equilibrium for horizontal, vertical, and angular movement.

3.1 Example

In Figure 3-4, we show a uniform rod PQ of mass 5kg. The rod is 8m long and is supported horizontally by two symmetrically placed strings attached to A and B. Given $d_1 = 3m$, $d_2 = 2m$, and $d_3 = 3m$, what is the tension on the strings (i.e., F)?

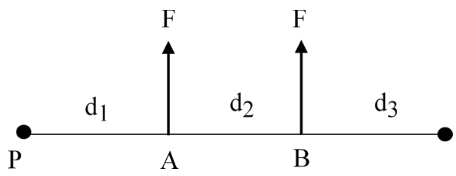

Figure 3-4:

Solution:
Rod PQ is uniform and so its centre of mass is the middle of the rod. We therefore take the weight of the rod as acting through this centre. This gives:

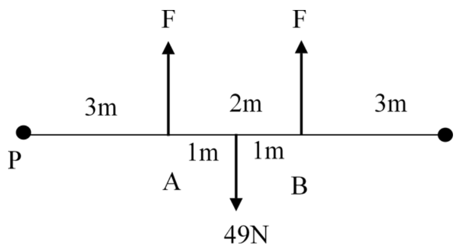

Figure 3-5:

(The tension force F will be the same for each side, since the system is symmetrical).
The weight of the rod = (5)(9.8) = 49N

$$2F = 49N$$

$\therefore$

$$F = 24.5N$$

The tension in each string is 24.5N

3.2 Example

In Figure 3-6 below, we show a uniform plank AE of length 6m and weight 90N. Then an elephant who weighs 500N sits at A, and a hippo who weighs 500N sits at B. The lengths of the plank are defined as: $d_1 = 1m$, $d_2 = 1m$, $d_3 = 1m$, $d_4 = 2m$, and $d_5 = 1m$. If a very large frog sits at D and the plank balances horizontally (i.e., in equilibrium). How much does the frog weigh and what is the upward normal reaction force F? (assume gravity is $9.8ms^{-2}$)

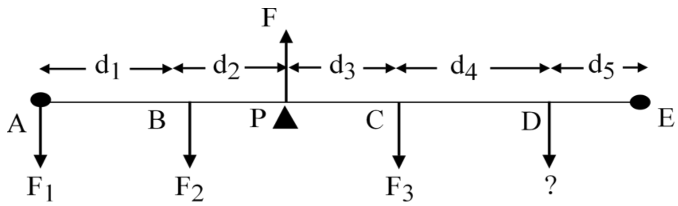

Figure 3-6:

Solution:
We substitute in the values for the diagram to visually see what is going on. Note, the 90N for the 'centre' of the rod due to its weight.

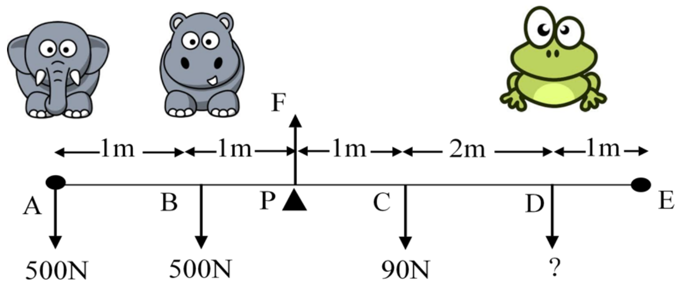

Figure 3-7:

Let the frog weigh 'W' Newtons.
Therefore, the upward force $F = 1090 + W$.
The moment (i.e., torque) around P is given by:

$$(500)(2) + (500)(1) = (90)(1) + (W)(3)$$

Hence, we can solve for W:

$$W = 470N$$

If we substitute into $F = 1090 + W$, then $F = 1560N$.
$\therefore$ The frog weighs 470N - that makes his mass 470/9.8 = 48kg. Which is a big frog.

3.3 Example

A rod shown below in Figure 3-8 is hinged at A. The distance $d_1 = 2m$, the angle is 30 degrees, and the downward force $F = 50N$. Calculate the distance AB, if the smooth support point at B exerts a force of 30N on the rod.

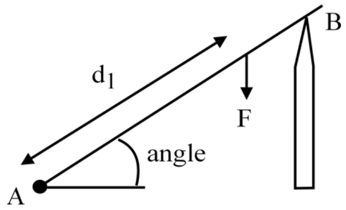

Figure 3-8:

Solution:
We split the force into the parallel and perpendicular components.

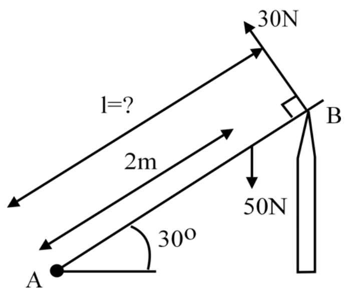

Figure 3-9:

The component of the 50N weight perpendicular to AB is:

$$50cos30^o = 43.3N$$

The counter-clockwise torque around A from the force at d_1 is (43.3)(2), while the force relative at B is: $(30)(l)$. Therefore, we equate them:

$$(30)(l) = (43)(2)$$
$$\therefore \; l = 2.9$$

Hence, AB = 2.9m.

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4 Centre of Mass for Multiple Particles

For a collection of N particles we can calculate the overall centre of mass given the equation:

$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \quad y_{cm} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i} \quad z_{cm} = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i} \tag{2}$$

where x_i, y_x and z_i are the locations of the masses relative to the origin, m_i is the mass of particle i, and N is the total number of particles.
This approach applies to discrete masses even if they are not point masses (i.e., shapes as we show in the next section). If the position x_i is taken to be the position of the centre of mass of the ith mass. It also points the way toward the calculation of the centre of mass of an extended object.

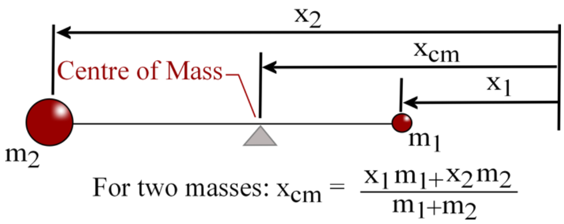

Figure 4-10:

The centre of mass of any shape is the point about which it balances. We are able to represent a shape as a point of particles.

4.1 Example

A light rod AB, shown below in Figure 4-11, has masses 4kg, 5kg, and 6kg. What is the centre of mass so a pivot point P will cause the rod to balance?

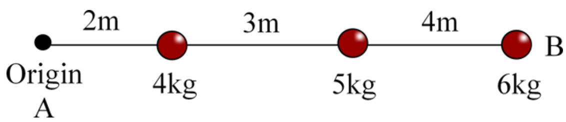

Figure 4-11:

Solution:

$$P = x_{cm} = \frac{(2)(4) + (5)(5) + (9)(6)}{4 + 5 + 6} = 5.8$$

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5 Centre of Mass for Primitive Shapes

For shapes with uniform density, we can work out the area and the centre of mass.

5.1 Example

A 4x4 piece of card of uniform density with a 2x2 square removed from the top right hand corner. Find the coordinates of the centre of mass of the remaining card?

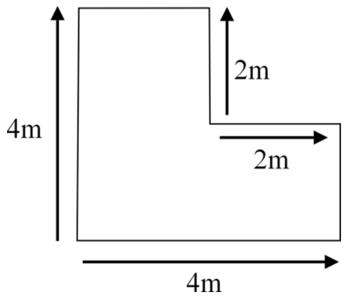

Figure 5-12:

Solution:
Find the centre of mass for the smaller rectangle regions and sum them.
You can work out the centre of mass for the four sub regions (i.e., the smaller 2x2 squares), and treat them as point masses.

$$\begin{aligned}
cm_0 &= (1,3) \\
cm_1 &= (1,1) \\
cm_2 &= (3,1) \\
\text{mass of each point is: } 2 \times 2 &= 4 \\
overall\ cm_x &= \frac{(4)(1)+(4)(1)+(4)(3)}{12} = \frac{5}{3}m \\
overall\ cm_y &= \frac{(4)(3)+(4)(1)+(4)(1)}{12} = \frac{5}{3}m
\end{aligned}$$

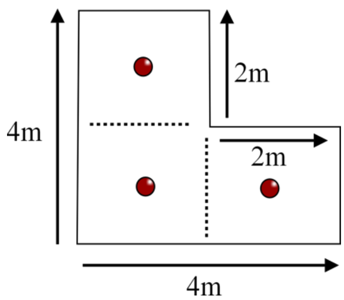

Figure 5-13:
