Geometry Part 1

Computer Graphics - SET08116

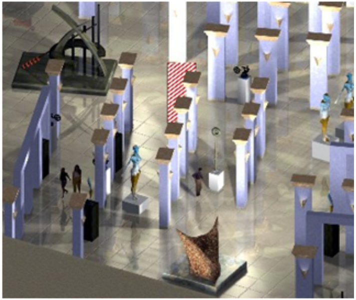
EDINBURGH NAPIER UNIVERSITY

Outline

1. Computer Graphics & Geometry
2. Review Mathematics
3. Summary

Why is it important to understand the mathematics of geometry in computer graphics?

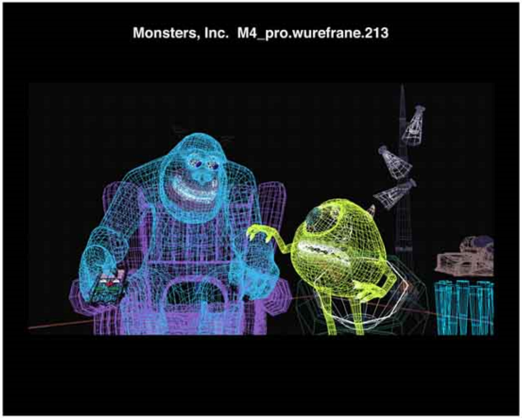

Unreal 2 © Legend Entertainment / Epic Games
GameWallpapers.com



Manipulation of geometry and colour...

Monsters, Inc

Manipulation of geometry and colour...



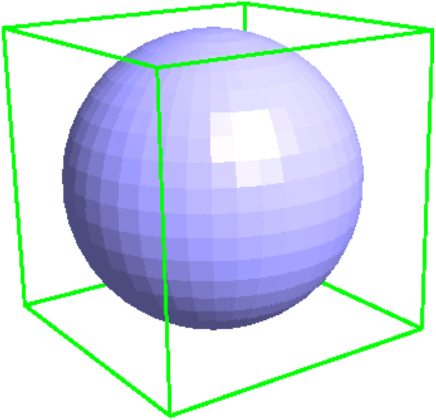
Manipulation of geometry and colour...

Manipulation of geometry and colour...

Manipulation of geometry and colour...

How do we represent geometry?

- We represent objects using mainly linear primitives:
  - points
  - lines, segments
  - planes, polygons
- Need to know how to compute distances, transformations, projections?



How to approach geometric problems (why formal methods?)

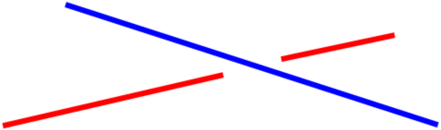
We have two ways:

- Employ our geometric intuition
- Formalize everything and employ our algebra skills

Often we first do No.1 and then solve with No.2 For complex problems No.1 is not always easy?

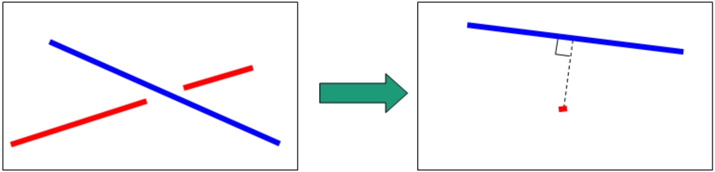
Example: distance between 2 lines in 3D

- Geometric problem: we have two lines (or segments) in 3D, need to find the distance between them



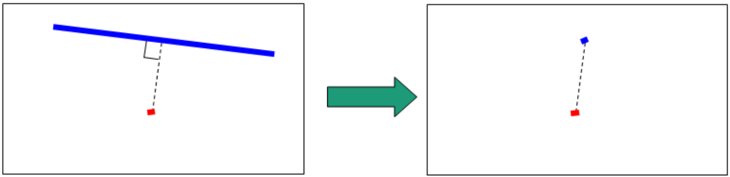
Example: distance between 2 lines in 3D

- Geometric approach:
  - If we look from the direction of one of the lines, that line reduces to a point
  - So all we need is point-line distance in 2D (the projection plane)
  - By projecting, we reduced the problem from 3D to 2D



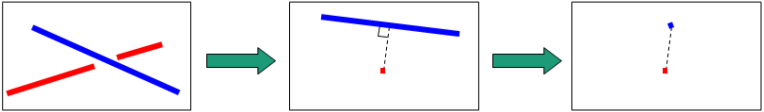
Example: distance between 2 lines in 3D

- Geometric approach:
  - We can continue reducing the dimensions
  - Project the red point and the blue line on the plane perpendicular to the blue line
  - Now we get point-point distance



Example: distance between 2 lines in 3D

- But how do we get the actual number?
  - Need to represent the lines **mathematically**
    - OK...
  - Write down the **projection formulae**
    - Have to wipe the dust off our algebra...
  - Compute the point-point distance
    - Easy



Example: distance between 2 lines in 3D

- Alternatively:
  - (Almost) skip the geometric intuition step?
  - Represent the lines mathematically
  - We know that the distance is achieved at a segment that is perpendicular to both lines
  - Write down the equation for that segment and solve
  - Formal proof! (no graphical)

Mathematics

With or without geometric intuition and good 3D orientation, in any case we need to

review our algebra..

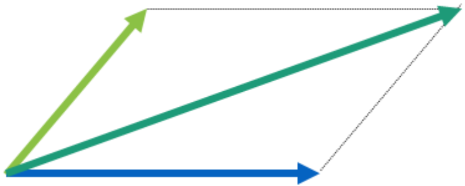
Basic definitions

- Points specify *location* in space (or in the plane)
- Vectors have *magnitude* and *direction* (like velocity)

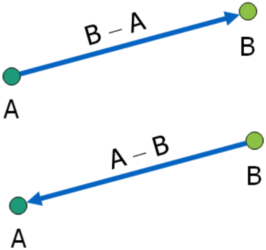
$$\boxed{\text{Points} \neq \text{Vectors}}$$

point + vector = point

Parallelogram rule



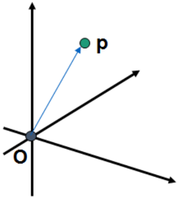
point - point = vector



point + point: not defined!!

Mapping points to vectors

- If we have a coordinate system with origin at point **O**
- We can define correspondence between points and vectors:

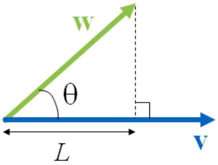


$$\mathbf{p} \rightarrow \vec{\mathbf{p}} = \mathbf{p} - \mathbf{O}$$
$$\mathbf{v} \rightarrow \mathbf{O} + \mathbf{v}$$

Inner (dot) product

- Defined for vectors:

$$\langle \mathbf{v},\ \mathbf{w} \rangle = \| \mathbf{v} \| \cdot \| \mathbf{w} \| \cdot \cos\theta$$

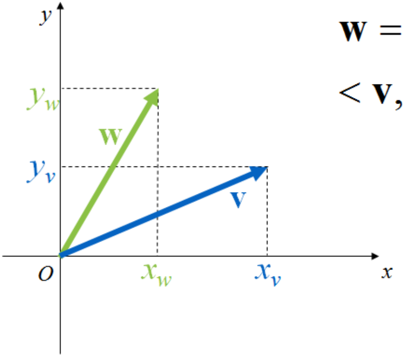


$$\cos\theta = \frac{L}{\|\mathbf{w}\|}$$

$$\boxed{L = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\| \mathbf{v} \|}}$$

Projection of $\mathbf{w}$ onto $\mathbf{v}$

Dot product in coordinates (2D)



$$\mathbf{v} = (x_v, y_v)$$

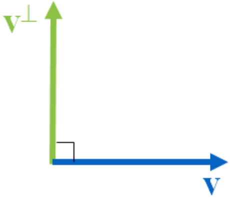
$$\mathbf{w} = (x_w, y_w)$$

$$< \mathbf{v}, \mathbf{w} > = x_v x_w + y_v y_w$$

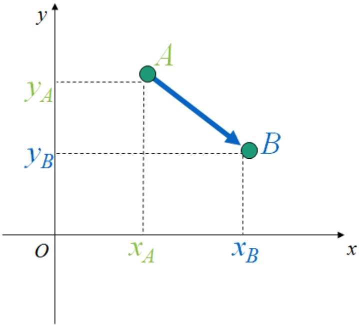
Perpendicular vectors

$$< \mathbf{v}, \mathbf{w} > = 0$$

In 2D only: $\mathbf{v} = (x_v, y_v) \Rightarrow \mathbf{v}^{\perp} = \pm(-y_v, x_v)$



Distance between two points



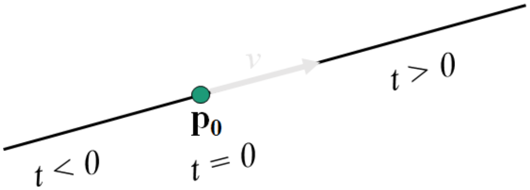
$$\mathrm{dist}(A,B) = \| B - A \| =$$

$$= \sqrt{< B - A, B - A >} =$$

$$= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

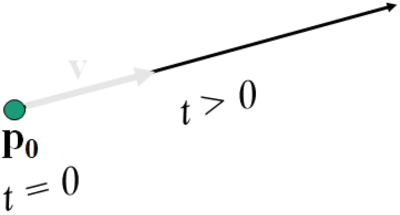
Parametric equation of a line

$$\ell(t) = \mathbf{p}_0 + t\mathbf{v}, \quad t \in (-\infty, \infty)$$



Parametric equation of a ray

$$\ell(t) = \mathbf{p}_0 + t\mathbf{v}, \quad t \in (0, \infty)$$



Distance between point and line

Find a point $\mathbf{q'}$ such that $(\mathbf{q} - \mathbf{q'}) \perp \mathbf{v}$

dist(q, *l*) = || q − q' ||

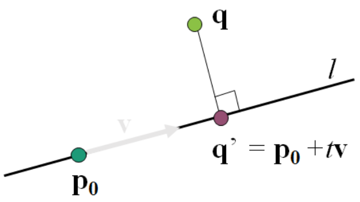
$$< \mathbf{q} - \mathbf{q}', \ \mathbf{v} > = 0$$

$$< \mathbf{q} - (\mathbf{p}_0 + t\mathbf{v}), \ \mathbf{v} > = 0$$

$$< \mathbf{q} - \mathbf{p}_0, \ \mathbf{v} > - \ t < \mathbf{v}, \mathbf{v} > = 0$$

$$t = \frac{< \mathbf{q} - \mathbf{p}_0, \ \mathbf{v} >}{< \mathbf{v}, \mathbf{v} >} = \frac{< \mathbf{q} - \mathbf{p}_0, \ \mathbf{v} >}{\| \mathbf{v} \|^2}$$

$$\boxed{\mathrm{dist}^2(\mathbf{q}, \ l) = \| \mathbf{q} - \mathbf{q}' \|^2 = \| \mathbf{q} - \mathbf{p}_0 \|^2 - \frac{< \mathbf{q} - \mathbf{p}_0, \ \mathbf{v} >^2}{\| \mathbf{v} \|^2}}$$



Easy geometric interpretation

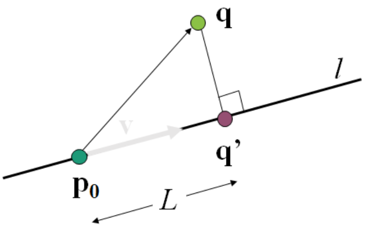
Pythagoras :

$$(1)\quad L^2 + \mathrm{dist}(\mathbf{q},\ \mathbf{q}')^2 = \|\mathbf{q} - \mathbf{p}_0\|^2$$

$$(2)\quad L = \frac{<\mathbf{q} - \mathbf{p}_0,\ \mathbf{v}>}{\|\mathbf{v}\|}$$

$$\Rightarrow \quad \mathrm{dist}(\mathbf{q},\ \mathbf{q}')^2 = \|\mathbf{q} - \mathbf{p}_0\|^2 - L^2 =$$

$$= \|\mathbf{q} - \mathbf{p}_0\|^2 - \frac{<\mathbf{q} - \mathbf{p}_0,\ \mathbf{v}>^2}{\|\mathbf{v}\|^2}.$$



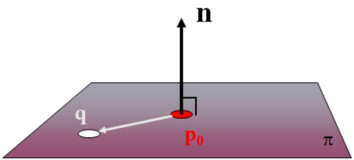
2D & 3D

Distance between point and line ? also works in 3D

- The parametric representation of the line is coordinates-independent
- v and p_0 and the checked point q can be in 2D or in 3D or in any dimensions

Representation of a plane in 3D space

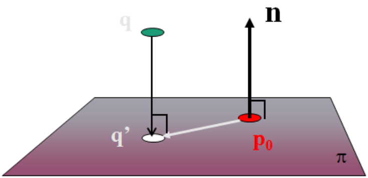
- A plane π is defined by a normal $\mathbf{n}$ and one point in the plane $\mathbf{p}_0$.
- A point $\mathbf{q}$ belongs to the plane $\Leftrightarrow$ $< \mathbf{q} - \mathbf{p}_0, \mathbf{n} > = 0$
- The normal $\mathbf{n}$ is perpendicular to all vectors in the plane



Distance between point and plane

- Project the point onto the plane in the direction of the normal:

$$\mathrm{dist}(\mathbf{q},\ \pi) = \|\mathbf{q}' - \mathbf{q}\|$$



Distance between point and plane

$$(\mathbf{q}'-\mathbf{q}) \parallel \mathbf{n} \quad \Rightarrow \quad \mathbf{q}'-\mathbf{q} = \alpha\mathbf{n}, \ \alpha \in \mathrm{R} \quad \Rightarrow \quad \mathbf{q}' = \mathbf{q} + \alpha\mathbf{n}$$

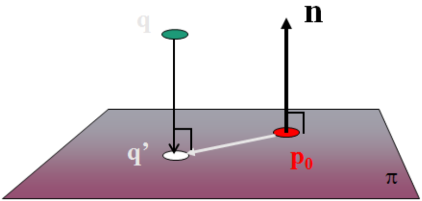
$$< \mathbf{q}'-\mathbf{p}_0, \ \mathbf{n} >= 0 \quad \text{(because } \mathbf{q}' \text{ is in the plane } \pi\text{)}$$

$$< \mathbf{q} + \alpha\mathbf{n} - \mathbf{p}_0, \ \mathbf{n} >= 0$$

$$< \mathbf{q} - \mathbf{p}_0, \mathbf{n} > + \ \alpha < \mathbf{n}, \mathbf{n} >= 0$$

$$\alpha = \frac{< \mathbf{p}_0 - \mathbf{q}, \ \mathbf{n} >}{\| \mathbf{n} \|^2}$$

$$\boxed{\mathrm{dist}^2(\mathbf{q}, \ \pi) = \| \mathbf{q}' - \mathbf{q} \|^2 = \alpha^2 \| \mathbf{n} \|^2 = \frac{< \mathbf{q} - \mathbf{p}_0, \ \mathbf{n} >^2}{\| \mathbf{n} \|^2}.}$$

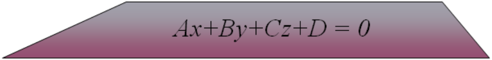


Implicit representation of planes in 3D

$$\boxed{Ax + By + Cz + D = 0,\ \ A, B, C, D \in \mathrm{R},\ \ (A, B, C) \neq (0,0,0)}$$

- (x, y, z) are coordinates of a point on the plane
- (A, B, C) are the coordinates of a normal vector to the plane

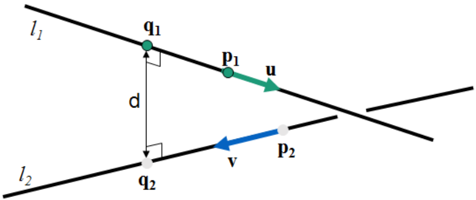
$$Ax+By+Cz+D > 0$$



$$Ax+By+Cz+D < 0$$

Distance between two lines in 3D

$$l_1(s) = \mathbf{p}_1 + s\mathbf{u}$$
$$l_2(t) = \mathbf{p}_2 + t\mathbf{v}$$

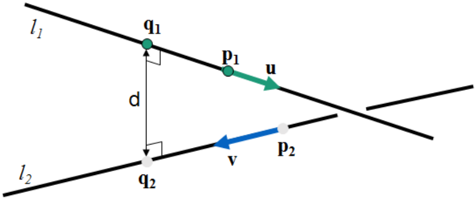


$$\tilde{s} = \frac{<\mathbf{u},\ \mathbf{v}><\mathbf{v},\ \mathbf{p}_1-\mathbf{p}_2> - \|\mathbf{v}\|^2<\mathbf{u},\ \mathbf{p}_1-\mathbf{p}_2>}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - <\mathbf{u},\ \mathbf{v}>^2}$$

$$\tilde{t} = \frac{\|\mathbf{u}\|^2<\mathbf{u},\ \mathbf{p}_1-\mathbf{p}_2> - <\mathbf{u},\ \mathbf{v}><\mathbf{u},\ \mathbf{p}_1-\mathbf{p}_2>}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - <\mathbf{u},\ \mathbf{v}>^2}$$

Distance between two lines in 3D

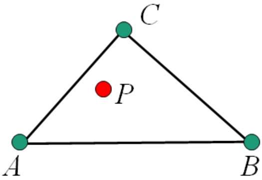
$$l_1(s) = \mathbf{p}_1 + s\mathbf{u}$$
$$l_2(t) = \mathbf{p}_2 + t\mathbf{v}$$



$$\boxed{\operatorname{dist}(l_1, l_2) = \| l_1(\tilde{s}) - l_2(\tilde{t}) \|}$$

Barycentric coordinates

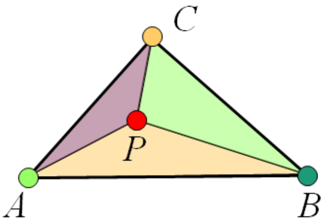
- Define a point's position relatively to some fixed points.
- $\vec{P} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$, where A, B, C are not on one line, and $\alpha, \beta, \gamma \in \mathrm{R}$.
- (α, β, γ) are called Barycentric coordinates of P with respect to A, B, C (unique!)
- If P is inside the triangle, then $\alpha+\beta+\gamma=1$, $\alpha, \beta, \gamma > 0$



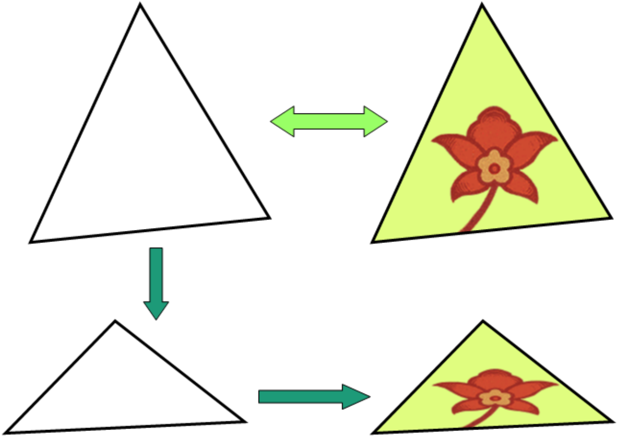
Barycentric coordinates (2D-plane)

$$\vec{P} = \frac{\langle P, B, C \rangle}{\langle A, B, C \rangle} \vec{A} + \frac{\langle P, C, A \rangle}{\langle A, B, C \rangle} \vec{B} + \frac{\langle P, A, B \rangle}{\langle A, B, C \rangle} \vec{C}$$

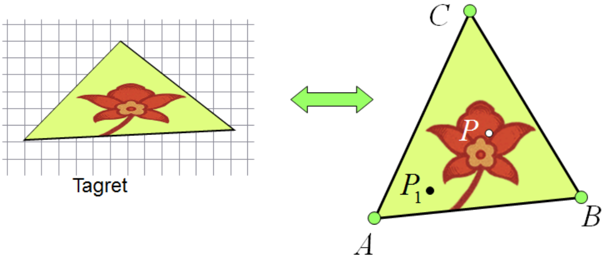
$\langle \cdot, \cdot, \cdot \rangle$ denotes the area of the triangle



Example of usage: warping



Example of usage: warping



We take the barycentric coordinates α, β, γ of P' with respect to A', B', C'.

Color(P) = Color($\alpha A + \beta B + \gamma C$)

Summary

We have now covered the core geometric concepts

- Importance of Geometry
- Mathematics Concepts
- (lines, rays, intersections, triangles, ...)

Recommended Reading

Interactive Computer Graphics, chapter 7

Real-Time Rendering, chapter 6