Physics based animation

Lecture 06 - Rigid bodies - Part 1

Grégory Leplâtre

g.leplatre@napier.ac.uk, room D32
School of Computing
Edinburgh Napier University

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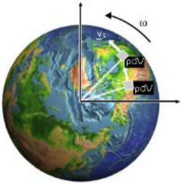
Objectives

- Introduction
- Rotational dynamics
  - Rotational momentum
  - Torque
  - Inertia

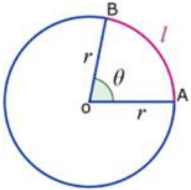
Outline

Context



- Kinematics: no forces considered
- Particle dynamics: no dimensions considered
- Rigid bodies: Object geometry taken into account
  - $\Rightarrow$ **translation**
  - $\Rightarrow$ **rotation**

Angular mathematics

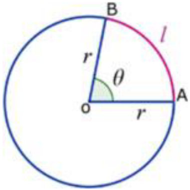


- Connection between rotational distance and angle:

$$\theta = \frac{l}{r}$$

if $\theta > 2\pi$ then $\theta - = 2\pi$ if $\theta < -2\pi$ then $\theta + = 2\pi$

Angular mathematics



One can define:

- **Angular velocity**

$$\omega = \frac{d\theta}{dt}$$

- **Angular acceleration**

$$\alpha = \frac{d\omega}{dt}$$

Rotational dynamics

- **Newton's second law**:

$$F = ma$$

- **For rotation**:

$$\tau = I\alpha$$

- τ is the **Torque**
- I is the **Inertia**
- α is the **angular acceleration**

Outline

Cross product

Reminder: cross product

$$\boldsymbol{a} \times \boldsymbol{b} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x]$$

Reminder: cross product

$$\boldsymbol{a} \times \boldsymbol{b} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x]$$

$$\boldsymbol{a} \times \boldsymbol{b} = \|a\|\|b\| sin(\theta)$$

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$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Cross product

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product

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$$c_x = 0b_x - a_z b_y + a_y b_z$$

$$c_y = a_z b_x + 0b_y - a_x b_z$$

$$c_z = -a_z b_x + a_x b_y + 0b_z$$

Cross product

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{a}} \cdot \boldsymbol{b}}$$

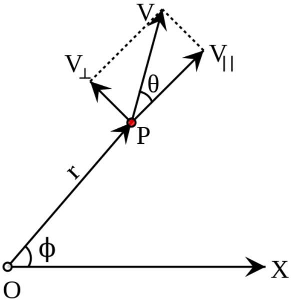
Where:

$$\hat{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

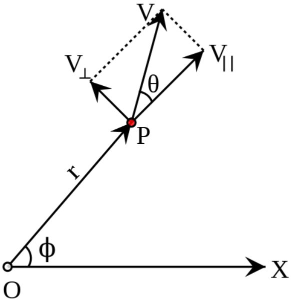
Angular velocity

Angular velocity

$$v_{\perp} = r\frac{d\phi}{dt}$$



Angular velocity



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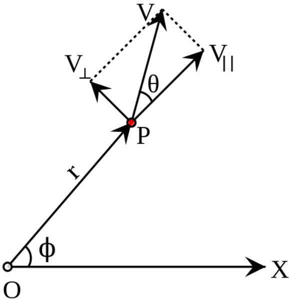

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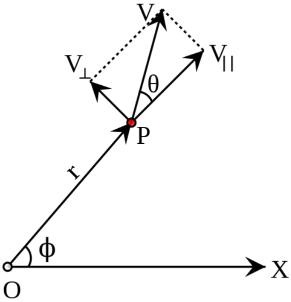
Angular velocity



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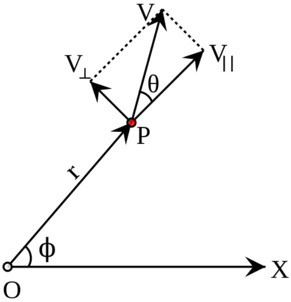
Then we have:

$$\omega = \frac{\|\boldsymbol{v}\| sin(\theta)}{\|\boldsymbol{r}\|}\boldsymbol{u}$$

Angular velocity



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If:

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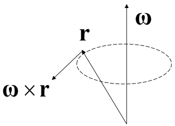
Then we have:

$$\omega = \frac{\|\boldsymbol{v}\| sin(\theta)}{\|\boldsymbol{r}\|}\boldsymbol{u}$$

$$\boxed{\omega = \frac{\boldsymbol{r} \times \boldsymbol{v}}{\|\boldsymbol{r}\|^2}}$$

Derivative of a rotating vector



- Let's say that vector $\boldsymbol{r}$ is rotating around the origin, maintaining a fixed distance
- At any instant, its angular velocity is $\boldsymbol{\omega}$

$$\boxed{\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times \boldsymbol{r}}$$

Rotating vector

Proof:

Rotating vector

Proof:
Reminder:

1. $\omega = \dfrac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r}\|^2}$
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\mathbf{a}$
3. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \bullet \mathbf{a})\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\mathbf{a}$

Rotating vector

Proof:

Reminder:

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(1) gives us: $\|r\|^2\omega = \boldsymbol{r} \times \boldsymbol{v}$

$\rightarrow (\|\boldsymbol{r}\|^2\omega) \times \boldsymbol{r}) = (\boldsymbol{r} \times \boldsymbol{v}) \times \boldsymbol{r}$

$\rightarrow \|\boldsymbol{r}\|^2(\omega \times \boldsymbol{r}) = (\boldsymbol{r} \times \boldsymbol{v}) \times \boldsymbol{r}$

$\rightarrow \|\boldsymbol{r}\|^2(\omega \times \boldsymbol{r}) = (\boldsymbol{r} \bullet \boldsymbol{r})\boldsymbol{v} - (\boldsymbol{r} \bullet \boldsymbol{v})\boldsymbol{r}$

$\rightarrow \|\boldsymbol{r}\|^2(\omega \times \boldsymbol{r}) = \|\boldsymbol{r}\|^2(\boldsymbol{v})$

$\rightarrow \omega \times \boldsymbol{r} = \boldsymbol{v}$

Derivative of rotating matrix

- If $\boldsymbol{A}$ is a rigid 3x3 matrix rotating with angular velocity ω

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$$\boxed{\frac{d\boldsymbol{A}}{dt} = \omega \times \boldsymbol{A} = \hat{\omega} \cdot \boldsymbol{A}}$$

Rotational momentum

Reminder: Particle kinematics

Reminder:

- position:

$$\mathbf{x}$$

- velocity

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$

- acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$$

Reminder: Particle dynamics

Reminder:

- mass:

$$m$$

- momentum

$$\mathbf{p} = m\mathbf{v}$$

- acceleration:

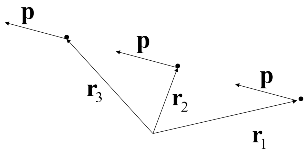
$$\mathbf{f} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$$

Moment of momentum

$$\boxed{\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}}$$

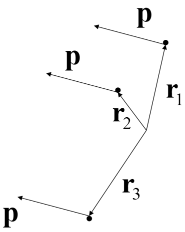
- **Moment of momentum** also known as **angular momentum**
- Like linear momentum, angular momentum is **conserved**

Moment of momentum - example



- $L \approx$ for all three particles

Moment of momentum - example



- $L \approx$ for all three particles
- $L \neq$ for all three particles

Moment of force - torque

Moment of force - Torque

$$\boxed{\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

- **Moment of force** also known as **torque** about a point is the rate of change of the moment of momentum about that point
- we also have:

$$\boxed{\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}}$$

Moment of force - Torque

Proof:

- product rule also applies to vectors and matrices:

$$\frac{d(\boldsymbol{a} \bullet \boldsymbol{b})}{dt} = \frac{d\boldsymbol{a}}{dt} \bullet \boldsymbol{b} + \boldsymbol{a} \bullet \frac{d\boldsymbol{b}}{dt}$$

$$\frac{d(\boldsymbol{a} \times \boldsymbol{b})}{dt} = \frac{d\boldsymbol{a}}{dt} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{d\boldsymbol{b}}{dt}$$

$$\frac{d(\boldsymbol{A} \cdot \boldsymbol{B})}{dt} = \frac{d\boldsymbol{A}}{dt} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \frac{d\boldsymbol{B}}{dt}$$

Moment of force - Torque

Proof:

Moment of force - Torque

Proof:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$
$$\tau = \frac{d\boldsymbol{L}}{dt} = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$
$$\tau = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{f}$$
$$\tau = \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \boldsymbol{f}$$
$$\tau = \boldsymbol{r} \times \boldsymbol{f}$$

Rotational Inertia

Rotational inertia

- $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the general expression of moment of momentum, regardless of the trajectory of the particle.
- When a particle rotates **around the origin**, at a **fixed distance**, $\boldsymbol{L}$ can be expressed in function of ω

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$
$$\boldsymbol{L} = \boldsymbol{r} \times (m\boldsymbol{v}) = m\boldsymbol{r} \times \boldsymbol{v}$$
$$\boldsymbol{L} = m\boldsymbol{r} \times (\omega \times \boldsymbol{r}) = -m\boldsymbol{r} \times (\boldsymbol{r} \times \omega)$$
$$\boldsymbol{L} = -m\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} \cdot \omega$$

$$\boxed{\boldsymbol{L} = \boldsymbol{I} \cdot \omega}$$

Where:

$$\boldsymbol{I} = -m\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}$$

Rotational Inertia

$$\boldsymbol{I} = -m\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}$$

$$\boldsymbol{I} = -m \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$\boldsymbol{I} = -m \begin{bmatrix} -r_y^2 - r_z^2 & r_x r_y & r_x r_z \\ r_x r_y & -r_x^2 - r_z^2 & r_y r_z \\ r_x r_z & r_y r_z & -r_x^2 - r_y^2 \end{bmatrix}$$

$$\boxed{\boldsymbol{I} = \begin{bmatrix} m(r_y^2 + r_z^2) & -m r_x r_y & -m r_x r_z \\ -m r_x r_y & m(r_x^2 + r_z^2) & -m r_y r_z \\ -m r_x r_z & -m r_y r_z & m(r_x^2 - r_y^2) \end{bmatrix}}$$

Rotational Inertia

- The rotational inertia ***I*** can be seen as the **rotational equivalent of mass**

Rotational Inertia

- The rotational inertia $\boldsymbol{I}$ can be seen as the **rotational equivalent of mass**

- Angular momentum and angular velocity linked by:
$$\boldsymbol{L} = \boldsymbol{I} \cdot \omega$$

- linear momentum and linear velocity
$$\boldsymbol{p} = m\boldsymbol{v}$$

Outline

Summary

- What we have achieved:
  - Rotational momentum: $\boldsymbol{L}$
  - Torque: $\boldsymbol{\tau}$
  - Rotational Inertia: $\boldsymbol{I}$

Summary

- What we have achieved:
  - Rotational momentum: $\boldsymbol{L}$
  - Torque: $\boldsymbol{\tau}$
  - Rotational Inertia: $\boldsymbol{I}$
- Still to do
  - From one particle to a system of particle
  - Force & torque $\Rightarrow$ Position & rotation

Coming up

- Wednesday: class test at **9:30am** in **H11**
- Wednesday: will finish Tutorial 4
- Next week: Rigid bodies part 2
- Next week: **project pitch**