Physics based animation

Lecture 07 - Rigid bodies - Part 2

Grégory Leplâtre

g.leplatre@napier.ac.uk, room D32
School of Computing
Edinburgh Napier University

Semester 1 - 2016/2017

Objectives

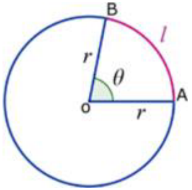
- From **one particle** to a **system of particles**

Objectives

- From **one particle** to a **system of particles**
- Force & torque $\Rightarrow$ Position & rotation of rigid body

Outline

Reminder



- **Angular velocity**

$$\omega = \frac{d\theta}{dt}$$

- **Angular acceleration**

$$\alpha = \frac{d\omega}{dt}$$

- **Angular momentum**

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{I} \cdot \omega$$

- **torque**

$$\tau = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{f}$$

Outline

Particle system

- **Total mass** of all particles:

$$m_{total} = \sum_{i=1}^{n} m_i$$

Particle system

- **Total mass** of all particles:

$$m_{total} = \sum_{i=1}^{n} m_i$$

- **position** of center of mass (CoM or cm)

$$\boldsymbol{x}_{cm} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{x_i}}{\sum_{i=1}^{n} m_i}$$

Particle system

- **Total mass** of all particles:

$$m_{total} = \sum_{i=1}^{n} m_i$$

- **position** of center of mass (CoM or cm)

$$\boldsymbol{x}_{cm} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{x_i}}{\sum_{i=1}^{n} m_i}$$

- **total momentum**:

$$\boldsymbol{p}_{cm} = \sum_{i=1}^{n} \boldsymbol{p_i} = \sum_{i=1}^{n} m_i \boldsymbol{v_i}$$

Velocity of center of mass

$$\boldsymbol{v}_{cm} = \frac{d\boldsymbol{x}_{cm}}{dt}$$

$$\boldsymbol{v}_{cm} = \frac{d}{dt}\frac{\sum_{i=1}^{n} m_i \boldsymbol{x}_i}{\sum_{i=1}^{n} m_i}$$

$$\boldsymbol{v}_{cm} = \frac{\sum_{i=1}^{n} m_i \frac{d\boldsymbol{x_i}}{dt}}{\sum_{i=1}^{n} m_i}$$

$$\boldsymbol{v}_{cm} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{v}_i}{\sum_{i=1}^{n} m_i}$$

$$\boldsymbol{v}_{cm} = \frac{\boldsymbol{p}_{cm}}{m_{total}}$$

$$\boxed{\boldsymbol{p}_{cm} = m_{total}\boldsymbol{v}_{cm}}$$

Forces

$$\mathbf{p}_{cm} = \sum \mathbf{p}_i$$

$$\frac{d\mathbf{p}_{cm}}{dt} = \frac{\sum \mathbf{p}_i}{dt} = \sum \frac{\mathbf{p}_i}{dt} = \sum \mathbf{f}_i$$

- The change of momentum of the CoM is equal to the change of momentum of all particles: $\mathbf{p}_{cm} = \sum \mathbf{p}_i$
- Therefore the resulting change in the total momentum is independent from the location of the applied force

Rotational momentum

- The total moment of momentum (or rotational momentum) around the CoM is:

$$\boldsymbol{L}_{cm} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i$$

$$\boldsymbol{L}_{cm} = \sum (\boldsymbol{x}_i - \boldsymbol{x}_{cm}) \times \boldsymbol{p}_i$$

Torque

$$\boldsymbol{L}_{cm} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i$$

$$\boldsymbol{\tau}_{cm} = \frac{d\boldsymbol{L}_{cm}}{dt} = d\frac{\sum \boldsymbol{r}_i \times \boldsymbol{p}_i}{dt}$$

$$\boldsymbol{\tau}_{cm} = \sum \frac{d(\boldsymbol{r}_i \times \boldsymbol{p}_i)}{dt}$$

$$\boldsymbol{\tau}_{cm} = \sum \boldsymbol{r}_i \times \boldsymbol{f}_i$$

$$\boldsymbol{\tau}_{cm} = \sum \boldsymbol{\tau}_i$$

Internal forces?

- Newton’s third law (action has equal reaction) means that internal forces cancel each other out
- $\Rightarrow$ **no incidence on the total momentum of the system.**

recap

- A system of particles behaves a lot like a single particle
- It has a **mass**, **position** (CoM), **velocity**, **momentum** and **acceleration**
- It responds to **forces**:

$$\boldsymbol{f}_{cm} = \sum \boldsymbol{f}_i$$

- Its **torque** (or change in the system's angular momentum) relates to the forces applied to particles of the system:

$$\boldsymbol{\tau}_{cm} = \sum (\boldsymbol{r}_i \times \boldsymbol{f}_i)$$

Outline

Rigid body kinematics

rigid body kinematics

- position:

$$\boldsymbol{x}_{cm}$$

- velocity

$$\boldsymbol{v}_{cm} = \frac{d\boldsymbol{x}_{cm}}{dt}$$

- acceleration:

$$\boldsymbol{a}_{cm} = \frac{d\boldsymbol{v}_{cm}}{dt} = \frac{d^2\boldsymbol{x}_{cm}}{dt^2}$$

rigid body kinematics

Orientation of the rigid body:

- 3x3 orientation matrix

$$\boldsymbol{A}$$

- Angular velocity

$$\boldsymbol{\omega}$$

- Angular acceleration:

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$$

Offset position

- Let's consider a point on a rigid body
- If $\boldsymbol{r}$ is the offset of the point relatice to the CoM of the rigid body, the position $\boldsymbol{x}$ of that point is:

$$\boldsymbol{x} = \boldsymbol{x}_{cm} + \boldsymbol{r}$$

Offset velocity

- Velocity $\boldsymbol{v}$ of the offset point:

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}_{cm}}{dt} + \frac{d\boldsymbol{r}}{dt}$$

$$\boldsymbol{v} = \boldsymbol{v}_{cm} + \boldsymbol{\omega} \times \boldsymbol{r}$$

Offset acceleration

- Acceleration $\boldsymbol{a}$ of the offset point:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}_{cm}}{dt} + \frac{d\omega}{dt} \times \boldsymbol{r} + \omega \times \frac{d\boldsymbol{r}}{dt}$$

$$\boldsymbol{a} = \boldsymbol{a}_{cm} + \alpha \times \boldsymbol{r} + \omega \times (\omega \times \boldsymbol{r})$$

Offset acceleration

- Acceleration $\boldsymbol{a}$ of the offset point:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}_{cm}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + \boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt}$$

$$\boldsymbol{a} = \boldsymbol{a}_{cm} + \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{a} = \boldsymbol{a}_{cm} + \boldsymbol{a}_{tan} + \boldsymbol{a}_{cen}$$

Where:

- $\boldsymbol{a}_{tan} = \boldsymbol{\alpha} \times \boldsymbol{r}$: **tangential** acceleration
- $\boldsymbol{a}_{cen} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$: **centripetal** acceleration

Offset point kinematics

Recap:

- Position:

$$\boldsymbol{x} = \boldsymbol{x}_{cm} + \boldsymbol{r}$$

- Velocity:

$$\boldsymbol{v} = \boldsymbol{v}_{cm} + \omega \times \boldsymbol{r}$$

- acceleration:

$$\boldsymbol{a} = \boldsymbol{a}_{cm} + \alpha \times \boldsymbol{r} + \omega \times (\omega \times \boldsymbol{r})$$

Rigid body dynamics

Rigid body mass

- In the physical world:

$$m = \int_V \rho dV$$

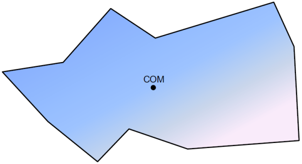
If the density is uniform:

$$m = \rho * V$$

- In the digital world:

$$m = \sum_{i=1}^{n} m_i$$

Center of mass



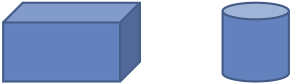
- In the physical world:

$$x_{cm} = \frac{\int_V \rho \boldsymbol{x} dV}{\int_V \rho dV}$$

- In the digital world:

$$m = \frac{1}{m} \sum_{i=1}^{n} m_i \boldsymbol{x}_i$$

Center of mass

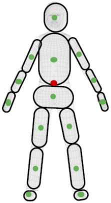


- Easy to determine for primitive shapes

Center of mass

- Easy to determine for primitive shapes

Center of mass



- Easy to determine for primitive shapes
- Complex models can be simplified

Rotational Intertia

Rotational Intertia

$$\boldsymbol{I} = \begin{bmatrix} \int \rho(r_y^2 + r_z^2)dV & -\int \rho r_x r_y dV & -\int \rho r_x r_z dV \\ -\int \rho r_x r_y dV & \int \rho(r_x^2 + r_z^2)dV & -\int \rho r_y r_z dV \\ -\int \rho r_x r_z dV & -\int \rho r_y r_z dV & \int \rho(r_x^2 - r_y^2)dV \end{bmatrix}$$

Rotational Intertia

$$\boldsymbol{I} = \begin{bmatrix} \int \rho(r_y^2 + r_z^2)dV & -\int \rho r_x r_y dV & -\int \rho r_x r_z dV \\ -\int \rho r_x r_y dV & \int \rho(r_x^2 + r_z^2)dV & -\int \rho r_y r_z dV \\ -\int \rho r_x r_z dV & -\int \rho r_y r_z dV & \int \rho(r_x^2 - r_y^2)dV \end{bmatrix}$$

$$\boldsymbol{I} = \int_V \rho(x, y, z) \begin{bmatrix} (x^2 + z^2) & xy & xz \\ xy & (x^2 + z^2) & yz \\ xz & yz & (x^2 - y^2) \end{bmatrix} dxdydz$$

Rotational Intertia

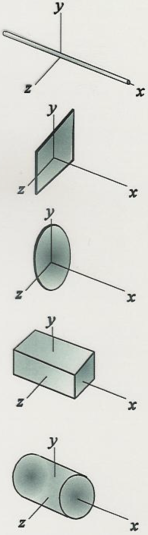
$$\boldsymbol{I} = \begin{bmatrix} \int \rho(r_y^2 + r_z^2)dV & -\int \rho r_x r_y dV & -\int \rho r_x r_z dV \\ -\int \rho r_x r_y dV & \int \rho(r_x^2 + r_z^2)dV & -\int \rho r_y r_z dV \\ -\int \rho r_x r_z dV & -\int \rho r_y r_z dV & \int \rho(r_x^2 - r_y^2)dV \end{bmatrix}$$

$$\boldsymbol{I} = \int_V \rho(x, y, z) \begin{bmatrix} (x^2 + z^2) & xy & xz \\ xy & (x^2 + z^2) & yz \\ xz & yz & (x^2 - y^2) \end{bmatrix} dxdydz$$

- Diagonal elements: Distances to the **principal axes**
- Non-diagonal elements: products of the perpendicular distances to the respective planes

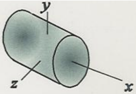
Inertia of primitive shapes

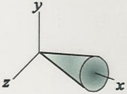
	I_x	I_y	I_z
Slender Rod m = mass, l = length of rod	0	$1/12\ ml^2$	$1/12\ ml^2$
Rectangular Plate m = mass, b = height of plate, c = width of plate	$1/12\ m(b^2+c^2)$	$1/12\ mc^2$	$1/12\ mb^2$
Thin Disk m = mass, r = radius of disk	$½\ mr^2$	$¼\ mr^2$	$¼\ mr^2$
Rectangular Prism m = mass, a = depth (x), b = height (y), c = width (z)	$1/12\ m(b^2+c^2)$	$1/12\ m(a^2+c^2)$	$1/12\ m(a^2+b^2)$
Circular Cylinder m = mass, l = length of cylinder, r = radius	$½\ mr^2$	$1/12\ m(3r^2+l^2)$	$1/12\ m(3r^2+l^2)$

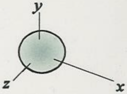


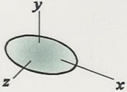
Inertia of primitive shapes

Elliptical Cylinder	$1/12\, m(3c^2 + l^2)$	$1/12\, m(3b^2 + l^2)$	$\frac{1}{4}\, m(b^2 + c^2)$
m = mass, l = length of cylinder (x), b = height/2 (y), c = width/2 (z)			
Circular Cone	$3/10\, mr^2$	$3/5\, m(\frac{1}{4} r^2 + l^2)$	$3/5\, m(\frac{1}{4} r^2 + l^2)$
m = mass, l = length of cone, r = radius at base			
Sphere	$2/5\, mr^2$	$2/5\, mr^2$	$2/5\, mr^2$
m = mass, r = radius			
Ellipsoid	$1/5\, m(b^2+c^2)$	$1/5\, m(a^2+c^2)$	$1/5\, m(a^2+b^2)$
m = mass, a = depth (x), b = height (y), c = width (z)			









diagonalisation of Rotational Inertia

- $\boldsymbol{I}$ can be expressed as:

$$\boldsymbol{I} = \boldsymbol{A} \cdot \boldsymbol{I}_O \cdot \boldsymbol{A}^T$$

Where: $\boldsymbol{I}_O = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$

diagonalisation of Rotational Inertia

- $\boldsymbol{I}$ can be expressed as:

$$\boldsymbol{I} = \boldsymbol{A} \cdot \boldsymbol{I}_O \cdot \boldsymbol{A}^T$$

diagonalisation of Rotational Inertia

- $\boldsymbol{I}$ can be expressed as:

$$\boldsymbol{I} = \boldsymbol{A} \cdot \boldsymbol{I}_O \cdot \boldsymbol{A}^T$$

- The matrix $\boldsymbol{A}$ rotates the object from an orientation where the principal axes line up with the x, y, and z axes

diagonalisation of Rotational Inertia

- $\boldsymbol{I}$ can be expressed as:

$$\boldsymbol{I} = \boldsymbol{A} \cdot \boldsymbol{I}_O \cdot \boldsymbol{A}^T$$

- The matrix $\boldsymbol{A}$ rotates the object from an orientation where the principal axes line up with the x, y, and z axes
- The three values in $\boldsymbol{I}_O$, (namely I_x, I_y, and I_z) are the **principal inertias.** They represent the resistance to torque around the corresponding principal axis (in a similar way that mass represents the resistance to force)

Derivative of angular momentum

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{I}}{dt} \cdot \boldsymbol{\omega} + \mathbf{I} \times \frac{d\boldsymbol{\omega}}{dt}$$

$$\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \boldsymbol{\alpha}$$

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \boldsymbol{\alpha}}$$

Newton-Euler equations

$$\boxed{\boldsymbol{f} = m\boldsymbol{a}}$$

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\omega} \times \boldsymbol{I} \cdot \boldsymbol{\omega} + \boldsymbol{I} \cdot \boldsymbol{\alpha}}$$

Forces and torques

$$\boldsymbol{f} = m \sum \boldsymbol{f}_i$$

$$\boldsymbol{\tau} = \sum (\boldsymbol{r}_i \times \boldsymbol{f}_i)$$

This gives us the **linear** and **rotational** accelerations:

$$\boxed{\boldsymbol{a} = \frac{1}{m} \boldsymbol{f}}$$

$$\boxed{\alpha = \boldsymbol{I}^{-1} \cdot (\boldsymbol{\tau} - \omega \times \boldsymbol{I} \cdot \omega)}$$

recap

Linear:	Rotational:
m	$\boldsymbol{I}$
$\boldsymbol{x}$	$\boldsymbol{A}$
$\boldsymbol{v}$	$\boldsymbol{\omega}$
$\boldsymbol{a}$	$\boldsymbol{\alpha}$
$\boldsymbol{p} = m\boldsymbol{v}$	$\boldsymbol{L} = \boldsymbol{I} \cdot \omega$
$\boldsymbol{f} = m\boldsymbol{a}$	$\boldsymbol{\tau} = \boldsymbol{\omega} \times \boldsymbol{I} \cdot \boldsymbol{\omega} + \boldsymbol{I} \cdot \boldsymbol{\alpha}$

Rigid body simulation

1. Compute all **forces** and **torques** acting within the system. $\boldsymbol{f} = \sum_i \boldsymbol{f}_i \quad \boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$

Rigid body simulation

1. Compute all **forces** and **torques** acting within the system. $\boldsymbol{f} = \sum_i \boldsymbol{f}_i \quad \boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$
2. **Integrate** to calculate:

Rigid body simulation

1. Compute all **forces** and **torques** acting within the system. $\boldsymbol{f} = \sum_i \boldsymbol{f}_i \quad \boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$
2. **Integrate** to calculate:
   1. the **velocity** and **position** of the CoM:
      $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \frac{1}{m}\boldsymbol{f}\Delta t$
      $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \boldsymbol{v}_{n+1}\Delta t$

Rigid body simulation

1. Compute all **forces** and **torques** acting within the system. $\boldsymbol{f} = \sum_i \boldsymbol{f}_i \quad \boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$
2. **Integrate** to calculate:
   1. the **velocity** and **position** of the CoM:
      $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \frac{1}{m}\boldsymbol{f}\Delta t$
      $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \boldsymbol{v}_{n+1}\Delta t$
   2. the **rotational momentum**, **angular velocity** and **rotation angle** of the rigid body:
      $\boldsymbol{L}_{n+1} = \boldsymbol{L}_n + \boldsymbol{\tau}\Delta t$
      $\boldsymbol{\omega}_{n+1} = \boldsymbol{I}^{-1} \cdot \boldsymbol{L}_{n+1}$
      $\theta_{n+1} = \|\omega_{n+1}\|\Delta t$

Rigid body simulation

1. Compute all **forces** and **torques** acting within the system. $\boldsymbol{f} = \sum_i \boldsymbol{f}_i \quad \boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$
2. **Integrate** to calculate:
   1. the **velocity** and **position** of the CoM:
      $\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \frac{1}{m}\boldsymbol{f}\Delta t$
      $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \boldsymbol{v}_{n+1}\Delta t$
   2. the **rotational momentum**, **angular velocity** and **rotation angle** of the rigid body:
      $\boldsymbol{L}_{n+1} = \boldsymbol{L}_n + \boldsymbol{\tau}\Delta t$
      $\boldsymbol{\omega}_{n+1} = \boldsymbol{I}^{-1} \cdot \boldsymbol{L}_{n+1}$
      $\theta_{n+1} = \|\omega_{n+1}\|\Delta t$
3. **translate** and **rotate** rigid body

Outline

Summary

- What we have achieved:
  - Simulation of translation and rotation of a rigid body

Summary

- What we have achieved:
  - Simulation of translation and rotation of a rigid body
- Still to do
  - Collisions (detection and response (impulse))
  - Implementation details/subtleties: see SIGGRAPH 2001 course notes ...
  - Tutorials

Coming up

- Wednesday: Tutorial 5: Centre of Mass
- Friday: **Pitch**
- Next week: **Collision**