Vectors

Computer Graphics - SET08116

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Outline

What is a Vector?

A vector can be thought of as an *n*-tuple of real numbers.

OK, what do we mean by an *n*-tuple of real numbers. Firstly, an *n*-tuple is just a string of *n* values seperated by commas. For example, a *5*-tuple of students in the class could be:

$$< Jason, Connor, Daniel, Mark, Chris >$$

Real numbers, as you should all know, are any non-complex numbers. This includes integers, rational and irrational (e.g. whole and fractional numbers). Therefore, a vector is just a string of *n* real numbers:

$$\mathbf{V} =< V_1, V_2, \ldots, V_n >$$

For example, for a *3*-dimensional vector:

$$\mathbf{V}_3 =< 4.56, 2.45, 1.12 >$$

What is a Vector?

Typically, we are interested in 2, 3, or 4 dimensional vectors. The are obvious for representing 2-dimensional and 3-dimensional positions in space.

$$\mathbf{V}_2 =< x, y >$$

$$\mathbf{V}_3 =< x, y, z >$$

4-dimensional vectors have their uses in matrix transformations, so it is worth remembering that you may come across them too.

Although we can refer to the individual components of a vector using an index number, it is more usual for our work to use x, y and z notation. For example:

$$\mathbf{V}_3 =< V_x, V_y, V_z >$$

Vector Representation

We have already shown how a vector can be represented in tuple

$$\mathbf{V} = < V_1, V_2, \ldots, V_n >$$

It is also common in graphics work to think of a vector in its matrix form, which is a single column with n rows:

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

However, to save space, it is common to write a vector as a single row. This is actually the transpose of the column vector, and should be written:

$$\mathbf{V}^T = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$$

Vector Operations

Vector addition (and subtraction) works by adding each individual component of the vectors together:

$$\mathbf{P} + \mathbf{Q} =< P_1 + Q_1, P_2 + Q_2, \ldots, P_n + Q_n >$$

So, for example:

$$< 2, 3, 4 > + < 1, 5, 7 >=< 2, 15, 28 >$$

For scaling a vector by a real number, we simply scale each individual value:

$$a\mathbf{V} =< aV_1, aV_2, \ldots, aV_n >$$

So, for example:

$$5 < 1, 2, 3 >=< 5, 10, 15 >$$

Vector Operations

Another common operation is calculating the length of a vector.

The length is considered the distance from the end of the vector, to the origin $(0, 0, \ldots, 0)$.

The length (magnitude) of the vector is denoted as follows:

$$\|\mathbf{V}\| = \sqrt{\sum_{i=1}^{n} V_i^2}$$

The $\|\mathbf{V}\|$ being the magnitude operator (you will see this operator in many places).

Magnitude Algorithm

So some of you might have just gone what does that even mean. Well, let us take a look at this one from a code point of view. The definition was as follows:

$$\|\mathbf{V}\| = \sqrt{\sum_{i=1}^{n} V_i^2}$$

The Σ operator is called a summation operator, and means that we add all the values together. We are essentially defining a for loop:

$total \leftarrow 0$
for $i = 1 \rightarrow n$ **do**
  $total \leftarrow total + (V_i \times V_i)$
end for
$magnitude \leftarrow \sqrt{total}$

Vector Operations

It is most likely that we will only get lengths of 3-dimensional vecto means that we can simplify the previous definition to the following:

$$\|\mathbf{V}\| = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$$

We are also usually interested in creating a vector that only has unit length (that is, its magnitude is 1). This operation is called normalization, and is performed as follows:

$$\hat{\mathbf{V}} = \frac{\mathbf{V}}{\|\mathbf{V}\|}$$

That is, we divide the individual components of the vector by the length of the vector. You can also write the equation as follows:

$$\hat{\mathbf{V}} = \frac{1}{\|\mathbf{V}\|}\mathbf{V}$$

Some Questions

You should be able to answer the following questions. If not, then help (it's what I'm here for).

Given the values $\mathbf{V} =< 12, 18, 24 >$, $\mathbf{U} =< 2, 10, 5 >$ and $a = 3$ calculate the following:

- $\mathbf{U} + \mathbf{V}$
- $\mathbf{U} - \mathbf{V}$
- $a\mathbf{U} - \frac{\mathbf{V}}{a}$
- $||\mathbf{U}||$
- $||a\mathbf{U}||$
- $||\mathbf{V}||$
- $\frac{\mathbf{V}}{||\mathbf{V}||}$

Dot Product

We have seen how to multiply a vector by a scalar, but what abou. multiplying vectors together. Well, there are two methods to do this:

- Dot Product
- Cross Product

The dot product is also known as the *scalar product* or *inner product*. It is useful as a measure between the difference in directions of two vectors.

For any two n-dimensional vectors, the dot product is defined as follows:

$$\mathbf{P} \cdot \mathbf{Q} = \sum_{i=1}^{n} P_i Q_i$$

Spot Quiz

What is the algorithm for the dot product of an *n*-dimensional vector?

$total \leftarrow 0$
for $i = 1 \rightarrow n$ **do**
    $total \leftarrow total + (P_i \times Q_i)$
end for
$dotproduct \leftarrow total$

Dot Product

For a 3-dimensional vector, we essentially have the following equation:

$$\mathbf{P} \cdot \mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z$$

The dot product also has another property, which allows us to measure the angle between two vectors. The dot product $\mathbf{P} \cdot \mathbf{Q}$ satisfies the following equation:

$$\mathbf{P} \cdot \mathbf{Q} = \|\mathbf{P}\| \|\mathbf{Q}\| \cos \alpha$$

Where α is the planar angle between the lines connecting the origin to the points represented by $\mathbf{P}$ and $\mathbf{Q}$. There is a proof of why this is the case in the maths text.

Dot Product

The relation between the dot product and the cosine of the angle them provides us with a useful observation. If the dot product is equal to 0, then the vectors must be perpendicular. That is:

$$\mathbf{P} \cdot \mathbf{Q} = 0 \Rightarrow \mathbf{P} \perp \mathbf{Q}$$

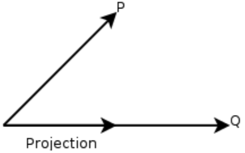
We can also use the sign of the dot product to determine how close two vectors are to pointing in the same direction. We can consider **P** to be orthogonal to a plane passing through the origin. Then, any vector lying on the same side of the plane has a positive dot product, and any vector lying on the other side of the plane has a negative dot product. That is:

$$\mathbf{P} \cdot \mathbf{Q} > 0 \Rightarrow \angle \mathbf{PQ} < \frac{\pi}{2} \mathit{rad}$$

$$\mathbf{P} \cdot \mathbf{Q} < 0 \Rightarrow \angle \mathbf{PQ} > \frac{\pi}{2} \mathit{rad}$$

Projection Vector

Sometimes we wish to work out the projection of one vector onto
A vector projected onto another has the same direction as the second vector, but has a length equal to the distance along the second that the first vector reaches. The following diagram should help:



The equation for working out the projection of one vector onto another is as follows:

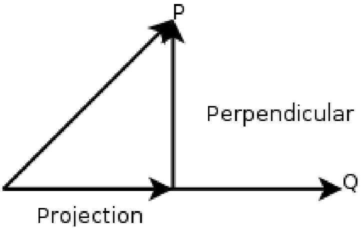
$$proj_q \mathbf{P} = \frac{\mathbf{P} \cdot \mathbf{Q}}{\|\mathbf{Q}\|^2} \mathbf{Q}$$

Perpendicular Component

The perpendicular component of **P** in regards to **Q** is the vector le
we subtract the projection vector from the original vector. That is:

$$\begin{aligned} perp_q\mathbf{P} &= \mathbf{P} - proj_q\mathbf{Q} \\ &= \mathbf{P} - \frac{\mathbf{P}\cdot\mathbf{Q}}{\|\mathbf{Q}\|^2}\mathbf{Q} \end{aligned}$$

The following diagram indicates the two values we have just calculated:



Projection as a Linear Transformation

Projecting **P** onto **Q** is a linear transformation, and thus can be expressed as a matrix. The following is the matrix representation of a projection:

$$proj_q\mathbf{P} = \frac{1}{\|\mathbf{Q}\|^2} \begin{bmatrix} {Q_x}^2 & Q_xQ_y & Q_xQ_z \\ Q_xQ_y & {Q_y}^2 & Q_yQ_z \\ Q_xQ_z & Q_yQ_z & {Q_z}^2 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

Don't worry - I don't expect you to remember this. However, we are essentially creating a projection matrix here.

Cross Product

The cross product only holds properties of interest for 3-dimensio vectors. The cross product, sometimes called the *vector product*, is defined as follows:

$$\mathbf{P} \times \mathbf{Q} = < P_y Q_z - P_z Q_y, P_z Q_x - P_x Q_z, P_x Q_y - P_y Q_x >$$

There is a simple way to remember this:

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

The cross product generates a vector that is perpendicular to the original two vectors. This also allows us to make the following observation:

$$(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{P} = 0$$
$$(\mathbf{P} \times \mathbf{Q}) \cdot \mathbf{Q} = 0$$

Cross Product

As the dot product, the cross product has a relationship to a trigonometric value:

$$\|\mathbf{P} \times \mathbf{Q}\| = \|\mathbf{P}\|\|\mathbf{Q}\| \sin \alpha$$

Where α is the planar angle between the two lines connecting the origin to the two points.

We can also use the cross product to calculate the area of a triangle. Given a triangle defined by the points $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$, we can use the following equation to calculate the area, A:

$$A = \frac{1}{2}\|(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)\|$$

The maths text illustrates why this is the case.

Questions

You should be able to answer the following questions. If not, then ask for help (it's what I'm here for).

Given the values $\mathbf{V} =< 12, 18, 24 >$ and $\mathbf{U} =< 2, 10, 5 >$

- $\mathbf{U} \cdot \mathbf{V}$
- $\mathbf{U} \times \mathbf{V}$
- $(\mathbf{V} \times \mathbf{U}) \cdot \mathbf{V}$
- $proj_Q\mathbf{P}$
- $perp_Q\mathbf{Q}$
- Area of the triangle defined by **P** and **Q** using the origin as the third vertex

Recommended Reading

Mathematics for 3D Game Programming and Computer Graphics, chapter 2

- Try and understand the proofs
- Look at vector spaces if you want to know the theory. I can discuss this individually if you are interested.