Physics based animation

Lecture 02 - Essential Mathematics

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Semester 1 - 2016/2017

Objectives

- Mathematics refresher

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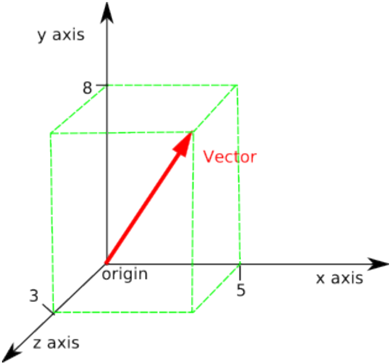
- Vectors and matrices

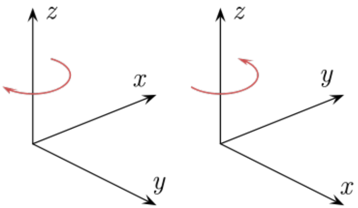
Outline

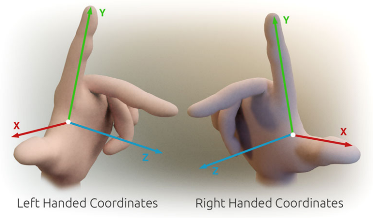
Vectors



- Vectors:
  - *direction* and *quantity*
- Computing vector:
  - collection of (3) values
- Notations:
  - vector: $\boldsymbol{v}$ or $\vec{v}$
  - norm: $\|vectv\|$
  - unit vector: $\hat{\boldsymbol{v}}$

Coordinate Systems





Vector operations

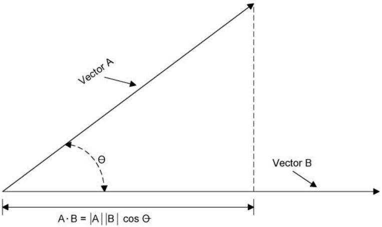
$\boldsymbol{a} = [a_x, a_y, a_z]$
$\boldsymbol{b} = [b_x, b_y, b_z]$

Addition and subtraction:
$\boldsymbol{a} + \boldsymbol{b} = [a_x + b_x, a_y + b_y, a_z + b_z]$
$\boldsymbol{a} - \boldsymbol{b} = [a_x - b_x, a_y - b_y, a_z - b_z]$

multiplication by scalar:
$s\boldsymbol{a} = [sa_x, sa_y, sa_z]$

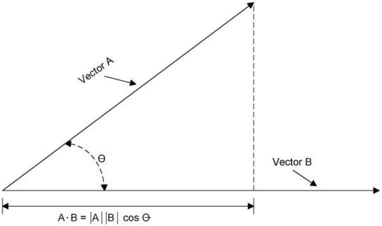
norm:
$\|\boldsymbol{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Dot product



$$\boldsymbol{a} \bullet \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$
$$\boldsymbol{a} \bullet \boldsymbol{b} = \|a\| \|b\| cos(\theta)$$

Dot product



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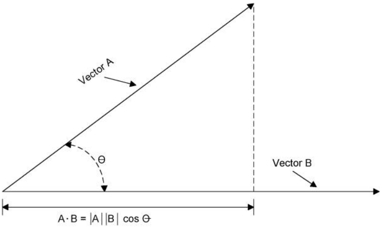
Example 1: find angle between two vectors ***a*** and ***b***

Dot product



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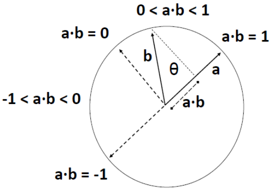
Example 1: find angle between two vectors ***a*** and ***b***

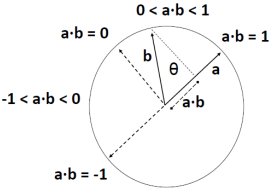
$$\theta = cos^{-1}\left(\frac{\boldsymbol{a} \bullet \boldsymbol{b}}{\|a\| \|b\|}\right)$$

Dot product



Dot product with unit vectors:
$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 1$

Dot product



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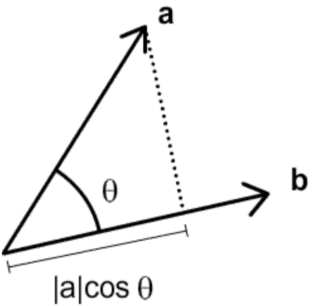

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Dot product with one unit vector:

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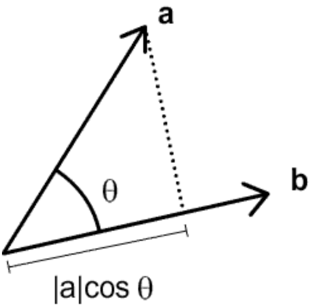
Dot product



Dot product with unit vectors:

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Dot product with one unit vector:

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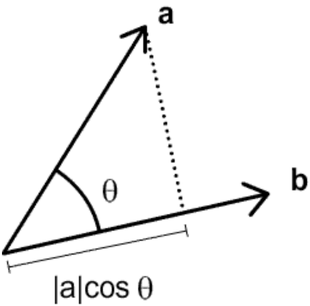
length of the projection of ***a*** onto ***b***

Dot product



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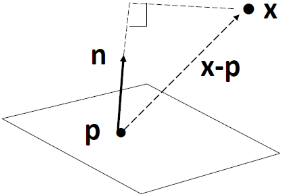
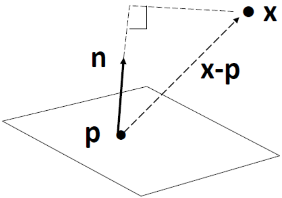
Dot product with no unit vector:
if $\theta < \pi/2$ then $\boldsymbol{a} \bullet \boldsymbol{b} > 0$
if $\theta = \pi/2$ then $\boldsymbol{a} \bullet \boldsymbol{b} = 0$
if $\theta > \pi/2$ then $\boldsymbol{a} \bullet \boldsymbol{b} < 0$

Dot product

Example 2: distance to plane:



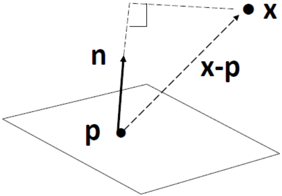
Dot product



Example 2: distance to plane:

Distance of the projection of $\boldsymbol{x} - \boldsymbol{p}$ onto $\boldsymbol{n}$

Dot product

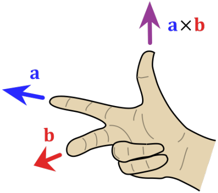


Example 2: distance to plane:

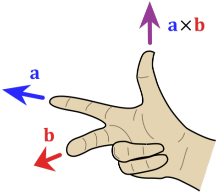
Distance of the projection of $\boldsymbol{x} - \boldsymbol{p}$ onto $\boldsymbol{n}$

$$d = \boldsymbol{x} - \boldsymbol{p} \bullet \boldsymbol{n}$$

Cross product

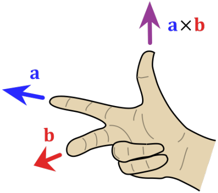


Cross product



$$\boldsymbol{a} \times \boldsymbol{b} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x]$$

Cross product



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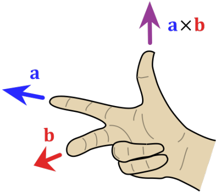
Cross product



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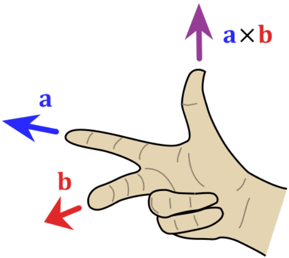
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

Cross product

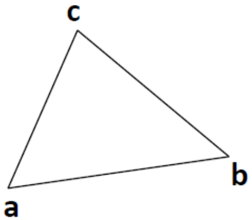


Properties of the Cross product:

- $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, in the direction defined by the right hand rule.
- $\|\boldsymbol{a} \times \boldsymbol{b}\| = 0$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
- $\|\boldsymbol{a} \times \boldsymbol{b}\|$ is the area of parallelogram $\boldsymbol{ab}$

Cross product

Example 1: Area of a triangle: Find the area of the triangle defined by the 3D points $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$



Cross product

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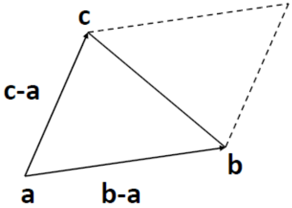
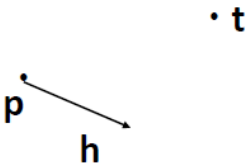


$$area = \frac{1}{2}\|(\boldsymbol{b}-\boldsymbol{a}) \times (\boldsymbol{c}-\boldsymbol{a})\|$$

Cross product

Example 2: Alignment to target: An object is at position $\boldsymbol{p}$ with a unit length heading of $\boldsymbol{h}$. We want to rotate it so that the heading is facing some target $\boldsymbol{t}$. Find a unit axis $\boldsymbol{a}$ and an angle θ to rotate around.



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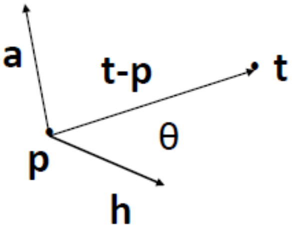

$$\boldsymbol{a} = \frac{\boldsymbol{h} \times (\boldsymbol{t} - \boldsymbol{p})}{\|\boldsymbol{h} \times (\boldsymbol{t} - \boldsymbol{p})\|}$$

$$\theta = cos^{-1}\left(\frac{\boldsymbol{h} \bullet (\boldsymbol{t} - \boldsymbol{p})}{\|(\boldsymbol{t} - \boldsymbol{p})\|}\right)$$

Homogeneous vectors

- 4D vector (added dimension) such that:
  $\boldsymbol{v} = [v_x, v_y, v_z]$
  $\boldsymbol{V} = [V_x, V_y, V_z, V_w]$

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  $\boldsymbol{V} = [v_x, v_y, v_z, w]$

  $\boldsymbol{v} = [V_x/w, V_y/w, V_z/w]$

Homogeneous vectors

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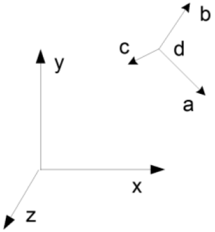
  $\boldsymbol{V} = [v_x, v_y, v_z, w]$
  $\boldsymbol{v} = [V_x/w, V_y/w, V_z/w]$

- Allows *all* transformation matrices to be used as matrix-vector multiplication
- Generally, $w = 1$, to represent points or vectors.
- $w = 0$ allows infinitely *far* vectors to be represented, which provides a concept of *direction*.

Outline

Matrices



- In Computer Graphics, 4x4 matrices are commonly used.
- A *rigid* 4x4 matrix transformation looks like this:

$$\begin{pmatrix} a_x & a_y & a_z & 0 \\ b_x & b_y & b_z & 0 \\ c_x & c_y & c_z & 0 \\ d_x & d_y & d_z & 1 \end{pmatrix}$$

- where ***a***, ***b*** and ***c*** are orthogonal unit length vectors representing orientation, and ***d*** is a vector representing position.

Transformations

- To transform a vector $\boldsymbol{v}$ by a matrix $\boldsymbol{M}$
  $\boldsymbol{v}' = \boldsymbol{v} \bullet \boldsymbol{M}$
- Multiple transformations (the order matters):
  $\boldsymbol{v}' = ((((\boldsymbol{v} \bullet \boldsymbol{M}_1) \bullet \boldsymbol{M}_2) \bullet \boldsymbol{M}_3) \bullet \boldsymbol{M}_4) \dots$
- Alternatively:
  $\boldsymbol{M} = \boldsymbol{M}_1 \bullet \boldsymbol{M}_2 \bullet \boldsymbol{M}_3 \bullet \boldsymbol{M}_4 \dots$
  $\boldsymbol{v}' = \boldsymbol{v} \bullet \boldsymbol{M}$

Transformations examples

- Translation by vector $\boldsymbol{t} = [t_x, t_y, t_z]$

$$\begin{pmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- Scale by vector $\boldsymbol{s} = [s_x, s_y, s_z]$

$$\begin{pmatrix} s_x x \\ s_y y \\ s_z z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Transformations examples

- Rotation of θ around $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ axes:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

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Orthonormality

- If all row and column vectors of a matrix are unit length, that matrix is *orthonormal*
- This also implies that all vectors are perpendicular to each other
- Useful property:
  - $M^{-1} = M^T$
- If a 4x4 matrix represents a rigid transformation, then the upper left 3x3 portion is orthonormal. Therefore:
  - $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{c}\|$
  - $\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c}$
  - $\boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}$
  - $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$

Determinants

- The determinant of a transformation matrix is a scalar value that represents the volume change that the transformation will cause
- An orthonormal matrix has a determinant of 1
- Non-orthonormal volume preserving matrices also have a determinant of 1
- A flattened or degenerate matrix has a determinant of 0
- A matrix that has been mirrored has a negative determinant

Determinants

- The determinant of a 4x4 matrix with no translation is equal to the determinant of the upper 3x3 portion

$$\begin{vmatrix} a_x & a_y & a_z & 0 \\ b_x & b_y & b_z & 0 \\ c_x & c_y & c_z & 0 \\ d_x & d_y & d_z & 1 \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

Outline

Summary

- Vector mathematics is fundamental
  - dot product
  - cross product

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- Matrix transformations