Physics-Based Animation (SET09119)

Tutorial 02 - Kinematics & Inverse Kinematics

1 Question

Alignment: An object is at position p with a unit length heading of h. We want to rotate it so that the heading is facing some target t. Find a unit axis (a) a and an angle θ to rotate around.

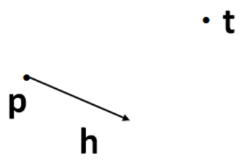

Figure 1: Question: Alignment.

Given $p =< 1, 2, 0 >$, $h =< 1, 0, 0 >$, and target of $t =< 3, 3, 0 >$ what is the unit axis (a) and angle (θ)?

$$a = \frac{h \times (t-p)}{||h \times (t-p)||}$$

$$\theta = cos^{-1}\left(\frac{h \bullet (t-p)}{||(t-p)||}\right)$$

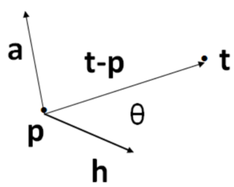

Figure 2: Question: Alignment Answer.

substituting values in:

$$a = < 0, 0, 1 >$$
$$\theta = 0.4646$$

2 Question

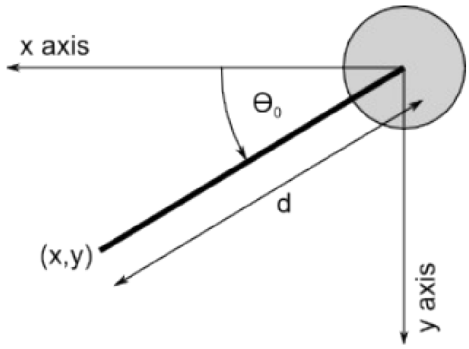

Figure 3: Planar manipulator with a single joint and extendible limb.

Part A

Forward Kinematics. Given the joint angle and length for the articulated geometry in Figure 3, find the orientation of the end-effector relative to the base frame (i.e., x and y in terms of θ_0 and d)

Given $d = 2$ metres and $\theta_0 = 0.2$ radians, what is (x, y)?

$$x = d \cos \theta_0$$
$$y = d \sin \theta_0$$

substituting values in:

$$x = (2)cos(0.2) = 1.960\ m$$
$$y = (2)sin(0.2) = 0.397\ m$$

Part B

Inverse Kinematics. Given the position and orientation of the end-effector relative to the base frame (see Figure 3), compute the joint angle and arm length for the articulated link geometry, which could be used to attain the given position and orientation of the end-effector (i.e., θ_0 and d in terms of x and y)

Given $(x, y) =< 5, 10 >$ what is θ_0 and d?

The angle θ_0 is fully defined by the coordinates x and y:

$$\theta_0 = atan2(y, x)$$

the distance d is the length from the origin to the end-effector:

$$d = \sqrt{x^2 + y^2}$$

substituting values in:

$$d = \sqrt{(5)(5) + (10)(10)} = 11.18$$
$$\theta = atan2(10, 5) = 1.107$$

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3 Question

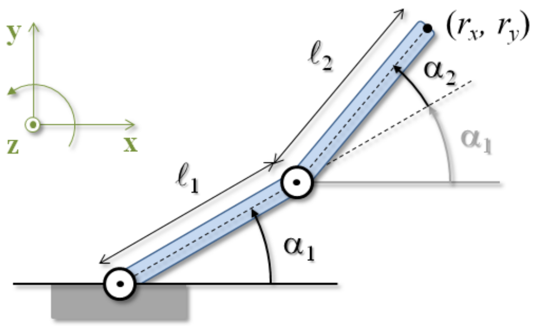

Figure 4: Two-link planar kinematic chain with revolute joints

Part A

Forward Kinematics. Given the joint angles and the linked geometry, compute the orientation of the end-effector relative to the base frame for Figure 4

Given $l_1 = 2$ and $l_2 = 3$ metres with $\alpha_1 = 0.4$ radians and $\alpha_2 = 0.2$ radians, what is the end-effector position (r_x, r_y)?

$$x = l_1 \cos\alpha_1 + l_2 \cos(\alpha_1 + \alpha_2)$$
$$y = l_1 \sin\alpha_1 + l_2 \sin(\alpha_1 + \alpha_2)$$

substituting in values:

$$x = (2)cos(0.4) + (3)cos(0.4 + 0.2) = 4.318$$
$$y = (2)sin(0.4) + (3)sin(0.4 + 0.2) = 2.473$$

Part B

Inverse Kinematics. Given the position and orientation of the end-effector relative to the base frame, compute all possible sets of joint angles and link geometries which could be used to attain the given position and orientation of the end-effector (see Figure 4)

Given $(r_x, r_y) =< 2, 3 >$, $l_1 = 2$ and $l_2 = 2$ metres, calculate α_1 and α_2?

Firstly, we find α_2:
See from the Figure 4 that, α_2 is a joint angle (referred as measured in the joint space); the angle of the segment 2 with respect to the horizontal is $\alpha_1 + \alpha_2$ in world space.

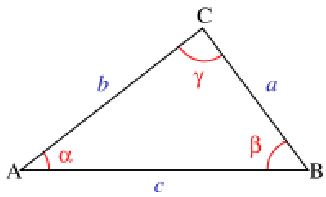

Figure 5: Cosine rule

From trigonometry, we have the 'cosine rule':

$$c^2 = a^2 + b^2 - 2\ a\ b\ \cos\gamma$$

where γ, denotes the angle contained between sides of lengths a and b and opposite the side of length c.
The inverse kinematics of this system is to specify the joint angles in terms of the endpoint position. Using the cosine rule, the angle α_2 is:

$$r_x^2 + r_y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\pi - \alpha_2)$$
$$\alpha_2 = acos\left(\frac{r_x^2 + r_y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right)$$

To find the angle α_1, we now look at the triangle in red in the Figure 6 below, its angle ϕ is:

$$\phi = atan\left(\frac{l_2\sin(\alpha_2)}{l_1 + l_2\cos(\alpha_2)}\right)$$

The angle of its hypotenuse with the horizontal is:

$$\alpha_1 + \phi = atan\left(\frac{r_y}{r_x}\right)$$

Then, the angle α_1 is:

$$\alpha_1 = atan\left(\frac{r_y}{r_x}\right) - atan\left(\frac{l_2 \sin(\alpha_2)}{l_1 + l_2 \cos(\alpha_2)}\right)$$

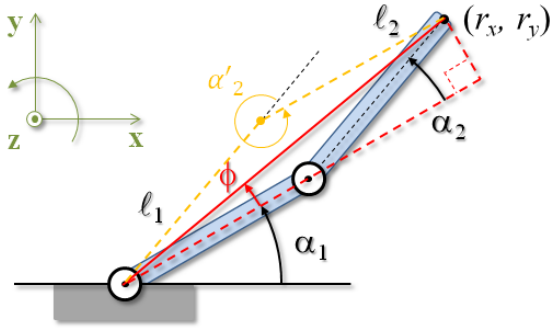

Figure 6: Indetermination in the inverse kinematics approach to determine one of the joint angles for a two-link chain with revolute joints.

- If x = y = 0 then the solution becomes undefined. In this case α_1 is arbitrary. This is only possible if both links have the same length and can fold back onto each other
- There are two possible sets of α_1, α_2 angles for the same r_x, r_y, coordinate that satisfy the equations above. Even for a simple two-link chain, we already have a problem of redundancy, there is more than one joint configuration for the same endpoint position, which becomes more problematic for chains with more links (more degrees of freedom).

substituting in values:

$$\alpha_2 = acos(\frac{(2)(2) + (3)(3) - (2)(2) - (2)(2)}{2(2)(2)}) = acos(0.625) = 0.896$$

$$\alpha_1 = atan(3,2) - atan(\frac{(2)sin(0.896)}{(2) + (2)cos(0.896)})$$

$$= 0.982 - atan\frac{1.527}{3.249} = 0.982 - 0.135 = 0.847$$