Grégory Leplâtre

Introductio

Rotationa

Summar

## Physics based animation Lecture 06 - Rigid bodies - Part 1

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Summar

## **Objectives**

- Introduction
- Rotational dynamics
  - Rotational momentum
  - Torque
  - Inertia

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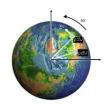
### **Outline**

- 1 Introduction
- 2 Rotational dynamics
- 3 Summary

#### Introduction

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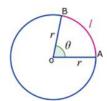
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- Kinematics: no forces considered
- Particle dynamics: no dimensions considered
- Rigid bodies: Object geometry taken into account
  - ▶ ⇒ translation
  - ▶ ⇒ rotation

Summar





Connection between rotational distance and angle:

$$\theta = \frac{I}{r}$$

if 
$$\theta > 2\pi$$
 then  $\theta - = 2\pi$  if  $\theta < -2\pi$  then  $\theta + = 2\pi$ 

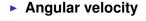
#### Introduction

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## Angular mathematics

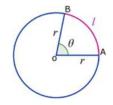
### One can define:



$$\omega = rac{ extstyle d heta}{ extstyle dt}$$

Angular acceleration

$$\alpha = \frac{d\omega}{dt}$$



# Rotational dynamics

Newton's second law:

$$F = ma$$

▶ For rotation:

$$au = I \alpha$$

- ightharpoonup is the **Torque**
- ▶ I is the Inertia
- $ightharpoonup \alpha$  is the angular acceleration

### Rotational dynamics

Cross product
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## Reminder: cross product

$$\boldsymbol{a}\times\boldsymbol{b}=[a_yb_z-a_zb_y,a_zb_x-a_xb_z,a_xb_y-a_yb_x]$$

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# Reminder: cross product

$$\boldsymbol{a}\times\boldsymbol{b}=[a_yb_z-a_zb_y,a_zb_x-a_xb_z,a_xb_y-a_yb_x]$$

$$\boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin(\theta)$$

Torque

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## Reminder: cross product

$$\boldsymbol{a} \times \boldsymbol{b} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x]$$

$$\boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin(\theta)$$

$$\mathbf{a} \times \mathbf{b} = \left[ \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{array} \right]$$

Cross product

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## Cross product

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

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## Cross product

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c_x = 0b_x - a_z b_y + a_y b_z$$

$$c_y = a_z b_x + 0b_y - a_x b_z$$

$$c_z = -a_z b_x + a_x b_y + 0b_z$$

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# Cross product

$$oldsymbol{a} imesoldsymbol{b}=oldsymbol{\hat{a}}\cdotoldsymbol{b}$$

Where:

$$\hat{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

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Angular velocity

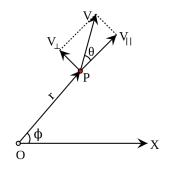
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$$v_{\perp} = r \frac{d\phi}{dt}$$



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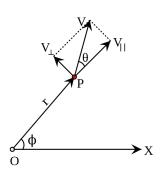
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$$egin{aligned} oldsymbol{v}_{oldsymbol{\perp}} &= r rac{oldsymbol{d} \phi}{oldsymbol{d} t} \ oldsymbol{v}_{oldsymbol{\perp}} &= \|oldsymbol{v}\| oldsymbol{sin}( heta) \end{aligned}$$

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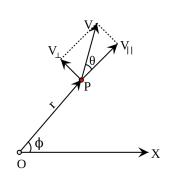
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If:
$$oldsymbol{\omega} = rac{ extsf{d}\phi}{ extsf{d}t}oldsymbol{u}$$

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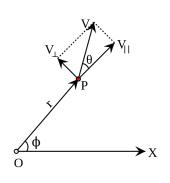
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# Angular velocity



$$egin{aligned} oldsymbol{v}_{oldsymbol{\perp}} &= r rac{oldsymbol{d} \phi}{oldsymbol{d} t} \ oldsymbol{v}_{oldsymbol{\perp}} &= \| oldsymbol{v} \| oldsymbol{sin}( heta) \end{aligned}$$

If:
$$oldsymbol{\omega} = rac{ extsf{d}\phi}{ extsf{d}t}oldsymbol{u}$$

Then we have:

$$oldsymbol{\omega} = rac{\|oldsymbol{v}\| oldsymbol{sin}( heta)}{\|oldsymbol{r}\|} oldsymbol{u}$$

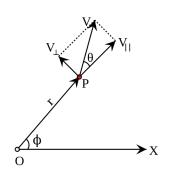
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#### Summary

# Angular velocity



$$egin{aligned} oldsymbol{v}_{oldsymbol{\perp}} &= r rac{oldsymbol{d} \phi}{oldsymbol{d} t} \ oldsymbol{v}_{oldsymbol{\perp}} &= \|oldsymbol{v}\| oldsymbol{sin}( heta) \end{aligned}$$

If: 
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Then we have:

$$oldsymbol{\omega} = rac{\|oldsymbol{v}\| oldsymbol{sin}( heta)}{\|oldsymbol{r}\|} oldsymbol{u}$$

$$\omega = \frac{r \times v}{\|r\|^2}$$

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## Derivative of a rotating vector



- Let's say that vector r is rotating around the origin, maintaining a fixed distance
- At any instant, its angular velocity is  $\omega$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

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## Rotating vector

Proof:

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### Proof:

#### Reminder:

$$\mathbf{1} \; \boldsymbol{\omega} = \frac{\boldsymbol{r} \times \boldsymbol{v}}{\|\boldsymbol{r}\|^2}$$

$$oxed{3} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \bullet \mathbf{a}) \mathbf{b} - (\mathbf{c} \bullet \mathbf{b}) \mathbf{a}$$

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#### Proof:

Reminder:

$$\mathbf{1} \omega = \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r}\|^2}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \bullet \mathbf{a})\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\mathbf{a}$$

(1) gives us: 
$$||r||^2 \omega = r \times v$$
  
 $\rightarrow (||r||^2 \omega) \times r) = (r \times v) \times r$   
 $\rightarrow ||r||^2 (\omega \times r) = (r \times v) \times r$   
 $\rightarrow ||r||^2 (\omega \times r) = (r \cdot r)v - (r \cdot v)r$   
 $\rightarrow ||r||^2 (\omega \times r) = ||r||^2 (v)$   
 $\rightarrow \omega \times r = v$ 

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## Derivative of rotating matrix

▶ If **A** is a rigid 3x3 matrix rotating with angular velocity  $\omega$ 

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## Derivative of rotating matrix

- ▶ If **A** is a rigid 3x3 matrix rotating with angular velocity  $\omega$
- This implies that its axes a, b and c must be rotating around ω

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## Derivative of rotating matrix

- ▶ If **A** is a rigid 3x3 matrix rotating with angular velocity  $\omega$
- ▶ This implies that its axes  ${\it a}$ ,  ${\it b}$  and  ${\it c}$  must be rotating around  $\omega$
- ▶ The derivatives of each axis are  $\omega \times \boldsymbol{a}$ ,  $\omega \times \boldsymbol{b}$  and  $\omega \times \boldsymbol{c}$

Rotational Ine

Summary

## Derivative of rotating matrix

- ▶ If **A** is a rigid 3x3 matrix rotating with angular velocity  $\omega$
- This implies that its axes a, b and c must be rotating around ω
- ▶ The derivatives of each axis are  $\omega \times a$ ,  $\omega \times b$  and  $\omega \times c$

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A} = \hat{\boldsymbol{\omega}} \cdot \mathbf{A}$$

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### Reminder: Particle kinematics

#### Reminder:

position:

X

velocity

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$

acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$$

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## Reminder: Particle dynamics

### Reminder:

mass:

m

momentum

$$p = mv$$

acceleration:

$$f = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$$

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### Moment of momentum

$$L = r \times p$$

- Moment of momentum also known as angular momentum
- Like linear momentum, angular momentum is conserved

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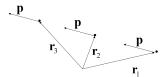
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## Moment of momentum - example



**L**  $\approx$  for all three particles

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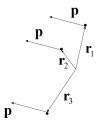
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## Moment of momentum - example



- $\boldsymbol{L} \approx$  for all three particles
- **L**  $\neq$  for all three particles

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Moment of force - torque

## Moment of force - Torque

$$oldsymbol{ au} = rac{doldsymbol{L}}{dt}$$

- Moment of force also known as torque about a point is the rate of change of the moment of momentum about that point
- we also have:

$$au = extbf{\textit{r}} imes extbf{\textit{f}}$$

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## Moment of force - Torque

#### Proof:

product rule also applies to vectors and matrices:

$$\frac{d(\mathbf{a} \bullet \mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt} \bullet \mathbf{b} + \mathbf{a} \bullet \frac{d\mathbf{b}}{dt}$$
$$\frac{d(\mathbf{a} \times \mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$
$$\frac{d(\mathbf{A} \cdot \mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}$$

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## Moment of force - Torque

Proof:

Torque

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# Moment of force - Torque

#### Proof:

$$L = r \times p$$

$$\tau = \frac{dL}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$

$$\tau = v \times p + r \times f$$

$$\tau = v \times (mv) + r \times f$$

$$\tau = r \times f$$

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- L = r × p is the general expression of moment of momentum, regardless of the trajectory of the particle.
- When a particle rotates around the origin, at a fixed distance, L can be expressed in function of ω

$$L = r \times p$$

$$L = r \times (mv) = mr \times v$$

$$L = mr \times (\omega \times r) = -mr \times (r \times \omega)$$

$$L = -m\hat{r} \cdot \hat{r} \cdot \omega$$

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

Where:

$$I = -m\hat{r} \cdot \hat{r}$$

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$$I = -m\hat{r}\cdot\hat{r}$$

$$\mathbf{I} = -m \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$I = -m \begin{bmatrix} -r_y^2 - r_z^2 & r_x r_y & r_x r_z \\ r_x r_y & -r_x^2 - r_z^2 & r_y r_z \\ r_x r_z & r_y r_z & -r_x^2 - r_y^2 \end{bmatrix}$$

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### Rotational Inertia

► The rotational inertia *I* can be seen as the **rotational** equivalent of mass

Torque Rotational Inertia

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► The rotational inertia *I* can be seen as the **rotational** equivalent of mass

Angular momentum and angular velocity linked by:

$$L = I \cdot \omega$$

linear momentum and linear velocity

$$p = mv$$

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#### What we have achieved:

- Rotational momentum: L
- Torque: τ
- Rotational Inertia: I

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### Summary

- What we have achieved:
  - Rotational momentum: L
  - Torque: τ
  - Rotational Inertia: I
- Still to do
  - From one particle to a system of particle
  - Force & torque ⇒ Position & rotation

Introduction

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Summary

- Wednesday: class test at 9:30am in H11
- Wednesday: will finish Tutorial 4
- Next week: Rigid bodies part 2
- Next week: project pitch