Physics-Based Animation (SET09119)

Tutorial 01 - Mathematics for Physics

1 Question

If $p = \sqrt{3}$ and $s = \sqrt{2}$ evaluate:

$$\sqrt{(5p-4s)^2-(4p-5s)^2}$$

$$\sqrt{(5p-4s)^2-(4p-5s)^2}$$
$$= [(5p-4s)^2-(4p-5s)^2]^{\frac{1}{2}}$$
$$= [(25p^2-20ps-20ps+16s^2)^2-(16p^2-20ps-20ps-25s^2)^2]^{\frac{1}{2}}$$
$$= [9p^2-9s^2]^{\frac{1}{2}}$$
$$= 3[p^2-s^2]^{\frac{1}{2}}$$
where $p^2 = 3$ and $s^2 = 2$
$$= 3[3-2]^{\frac{1}{2}}$$
$$= 3[1]^{\frac{1}{2}}$$
$$= 3$$

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2 Question

Find the set of real numbers λ for which the quadratic equation:

$$x^2-(\lambda-3)x+\lambda = 0$$

has distinct, real roots for x.

(Remember: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

$$x = \frac{3-\lambda\pm\sqrt{(\lambda-3)^2-4\lambda}}{2}$$

therefore distinct real roots if $(\lambda - 3)^2 > 4\lambda$ (i.e., the number must be positive)
hence, we solve for: $(\lambda - 3)^2 = 4\lambda$

$$\lambda^2 - 6\lambda + 9 = 4\lambda$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$(\lambda - 1)(\lambda - 9) = 0$$

so:

$$\lambda = 1, 9$$

which requires:

$$(\lambda - 3)^2 > 4\lambda$$

$\therefore$

$$\lambda < 1 \quad \text{or} \quad \lambda > 9$$

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3 Question

Let $\mathbf{u} =< -4, -1, 2 >$, $\mathbf{v} =< 1, 3, 2 >$ and $\mathbf{w} =< 0, -2, -4 >$,
Find:

1. $\mathbf{u} + \mathbf{v}$

   $< u_x, u_y, u_z > + < v_x, v_y, v_z > =< (u_x + v_x), (u_y + v_y), (u_z + v_z) > = < -3, 2, 4 >$

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2. $-3\mathbf{u}$

   $a < u_x, u_y, u_z > =< (a\ u_x), (a\ u_y), (a\ u_z) > = < 12, 3, -6 >$

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3. $3\mathbf{u} - 4\mathbf{v}$

   $a < u_x, u_y, u_z > -b < v_x, v_y, v_z > =$
   $< (a\ u_x), (a\ u_y), (a\ u_z) > - < (b\ v_x), (b\ v_y), (b\ v_z) > = < -16, -15, -2 >$

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4. $2\mathbf{u} + 4\mathbf{v} - 5\mathbf{w}$

   $< -4, 20, 32 >$

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4 Question

Let $\mathbf{u} =< 0, 1, 5 >$, $\mathbf{v} =< 1, 1, 5 >$ and $\mathbf{w} =< -0, -1, 2 >$,
Find:

1. $\mathbf{u} \cdot \mathbf{v}$

   26

2. $\mathbf{u} \cdot \mathbf{w}$

   9

3. $\mathbf{u} \times \mathbf{w}$

   $< 7, 0, 0 >$

4. $\mathbf{v} \times \mathbf{w}$

   $< 7, -2, -1 >$

5. $||\mathbf{u}||$ (i.e., the length/magnitude of $\mathbf{u}$)

   $< 5.099 >$

5 Question

1. Show that if $\vec{a}$ and $\vec{b}$ are two parallel vectors, then $\vec{a} \times \vec{b} = 0$

   knowing

$$\begin{aligned}\vec{a} \times \vec{b} &=< (a_y b_z - a_z b_y), (a_x b_z - a_z b_x), (a_x b_y - a_y b_x) > \\ &= ||\vec{a}||\ ||\vec{b}|| sin(\theta)\end{aligned}$$

   e.g., $\theta = 0$ if the vectors parallel and $sin(0) = 0$

where θ is the angle between the vectors. For the vectors to be parallel, the angle between them must be 0 or a multiple of π. That means the sin of the angle will be 0, so the cross product will be 0.

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2. Show that if $\vec{a}$ and $\vec{b}$ are two orthogonal vectors where $\vec{c} = \vec{a} \times \vec{b}$ then

$$||c|| = ||a||\,||b||$$

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knowing

$$\begin{aligned} \vec{a} \times \vec{b} &= < (a_y b_z - a_z b_y), (a_x b_z - a_z b_x), (a_x b_y - a_y b_y) > \\ &= ||\vec{a}||\,||\vec{b}|| sin(\theta) \end{aligned}$$

where θ is the angle between the vectors. For the vectors to be orthogonal, the angle between them must be 90^0 (i.e., $\frac{\pi}{2}$). That means the sin of the angle will be 1, so the cross product will be the magnitude of the two vectors $||\vec{a}||\,||\vec{b}||$.

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3. Find the cross product of the vectors $\vec{a} =< 1, 0, 3 >$ and $\vec{b} =< 9, -3, 1 >$. If $\vec{c} = \vec{a} \times \vec{b}$, verify that $\vec{c}^T \vec{a} = \vec{c}^T \vec{b} = 0$

$$\begin{aligned} \vec{a} \times \vec{b} &= < 1, 0, 3 > \times < 9, -3, 1 > \\ &= < (0)(1) - (3)(-3), (3)(9) - (1)(1), (1)(-3) - (0)(9) > \\ &= < 9, 26, -3 > \end{aligned}$$

second part is simple matrix multiplication:

$$\begin{aligned} \vec{c}^T \vec{a} &= \begin{bmatrix} 9 \\ 26 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \\ &= (9)(1) + (26)(0) + (-3)(3) = 0 \end{aligned}$$

$$\begin{aligned} \vec{c}^T \vec{b} &= \begin{bmatrix} 9 \\ 26 \\ -3 \end{bmatrix} \begin{bmatrix} 9 & -3 & 1 \end{bmatrix} \\ &= (9)(9) + (26)(-3) + (-3)(1) = 0 \end{aligned}$$

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6 Question

Differentiate:

1. $\frac{d}{dx}(3x^2)$

   $6x$

2. $\frac{d}{dx}(4x^4 - 2)$

   $16x^3$

3. $\frac{d}{dx}(x + \frac{1}{x})$

   $1 - \frac{1}{x^2}$

4. $\frac{d}{dx}(\sqrt[4]{x})$

   $\frac{1}{4x^{\frac{4}{4}}}$

5. $\frac{d}{dx}(\sqrt[5]{x} + \frac{5}{\sqrt{x}})$

   $\frac{1}{5x^{\frac{4}{5}}} - \frac{5}{2x^{\frac{3}{2}}}$

7 Question

Integrate:

1. $\int 4x^3 dx$

   x^4

2. $\int 2x^4 dx$

   $\frac{2}{5}x^5$

3. $\int x^{-4} dx$

$\frac{-1}{3} x^{-3}$

4. $\int 5x^{-3} dx$

$\frac{-5}{2} x^{-2}$

5. $\int \frac{6}{x^2} dx$

$\frac{-6}{x}$

8 Question

Compute:

1. $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 16 & -6 \\ -2 & -5 \end{bmatrix}$$

2. $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} -40 \\ -41 \end{bmatrix}$$

3. $\begin{bmatrix} 1 \\ -6 \end{bmatrix} \times \begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}$

Not defined

4. $\begin{bmatrix}1\\6\end{bmatrix} \times \begin{bmatrix}3 & 2\end{bmatrix}$

$$\begin{bmatrix}3 & 2\\18 & 12\end{bmatrix}$$

5. $\begin{bmatrix}2 & -1\end{bmatrix} \times \begin{bmatrix}1\\-6\end{bmatrix}$

$$\begin{bmatrix}8\end{bmatrix}$$

9 Question

Generate the 4×4 transformation matrix for the following transformations:

1. A translation of $< 4, 6, 7 >$

$$\begin{bmatrix}1 & 0 & 0 & 4\\0 & 1 & 0 & 6\\0 & 0 & 1 & 7\\0 & 0 & 0 & 1\end{bmatrix}$$

2. A scaling of $< 10, 5, 2 >$

$$\begin{bmatrix}10 & 0 & 0 & 0\\0 & 5 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

3. A z-axis rotation of $\frac{\pi}{2}$ radians

$$\begin{bmatrix}0 & -1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

4. A x-axis rotation of π radians

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. A y-axis rotation of $\frac{\pi}{4}$ radians

$$\begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10 Question

Transform the following vectors using the transformation matrix:

$$\begin{bmatrix} 2.121 & -2.121 & 0 & 10 \\ 2.121 & 2.121 & 0 & 15 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. $< 4, 5, 10 >$

$< 7.879, 34.089, 42 >$

2. $< 12, 8, 16 >$

$< 18.484, 57.420, 60 >$

3. $< 1, 4, 0 >$

$< 3.637, 25.605, 12 >$

4. $< -10, 12, -4 >$

$< -36.662, 19.242, 0 >$

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5. $< 0, 5, -11 >$

$< -0.605, 25.605, -21 >$

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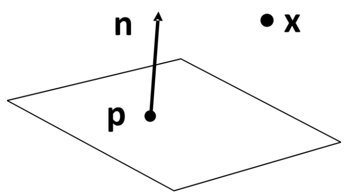
11 Question

A plane is described by a point $p < 1, 2, 1 >$ on the plane and a unit normal $n < 0, 1, 0 >$. Find the distance from point $x < 2, 2, 0 >$ to the plane



The distance is the length of the projection of $\vec{x} - \vec{p}$ onto $\hat{n}$:

$$\begin{aligned} dist &= (\vec{x} \bullet \hat{n}) - (\vec{p} \bullet \hat{n}) \\ &= (\vec{x} - \vec{p}) \bullet \hat{n} \\ &= (< 2, 2, 0 > - < 1, 2, 1 >) \bullet < 0, 1, 0 > \\ &= < 1, 0, -1 > \bullet < 0, 1, 0 > \\ &= < 0, 0, 0 > \end{aligned} \tag{1}$$

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12 Question

Find the solution of the following system of algebraic equations:

1. $$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 - x_2 &= 1.5 \\ -x_2 + x_3 &= 5 \end{aligned}$$

The strategy is to reduce this to two equations in two unknowns. Do that by eliminating one of the unknowns from two pairs of equations: either from equations a) and b), or a) and c), or b) and c).

$$\begin{aligned} a)&\quad -x_1+2x_2-x_3 &= 2\\ b)&\quad 2x_1-x_2 &= 1.5\\ c)&\quad -x_2+x_3 &= 5 \end{aligned}$$

a + c:

$$\begin{aligned} d)&\quad -x_1+x_2 &= 7\\ b)&\quad 2x_1-x_2 &= 1.5 \end{aligned}$$

d + b:

$$e)\quad x_1 = 8.5$$

sub into b:

$$b)\quad 2(8.5)-x_2 = 1.5$$

$$\therefore$$

$$x_2 = 17-1.5 = 15.5$$

sub into c:

$$c)\quad -(15.5)+x_3 = 5 \therefore$$

$$x_3 = 15.5+5 = 20.5$$

check in a, b and c:

$$\begin{aligned} a)&\quad -(8.5)+2(15.5)-(20.5) &= 2\\ b)&\quad 2(9)-(15.5) &= 1.5\\ c)&\quad -(15.5)+(20.5) &= 5 \end{aligned}$$

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2. $$\begin{aligned} -3x_2+5x_3 &= 0\\ -2x_1+2x_2-3x_3 &= 0\\ 6x_1-2x_2 &= 5.5 \end{aligned}$$

The strategy is to reduce this to two equations in two unknowns. Do that by eliminating one of the unknowns from two pairs of equations: either from equations a) and b), or a) and c), or b) and c).

$$\begin{aligned} a)&\quad -3x_2+5x_3 &= 0\\ b)&\quad -2x_1+2x_2-3x_3 &= 0\\ c)&\quad 6x_1-2x_2 &= 5.5 \end{aligned}$$

3b +c:

$$\begin{aligned} &a) \quad -3x_2 + 5x_3 &= 0 \\ &d) \quad 4x_2 - 9x_3 &= 5.5 \end{aligned}$$

4a+3d:

$20x_3 - 27x_3 = 16.5$ $x_3 = -16.5/7 = -2.36$

sub into a:

$a) \quad -3x_2 + 5(-2.36) = 0$

$\therefore \quad x_2 = -3.933$

sub into c:

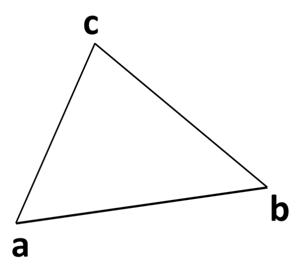
$6x_1 - 2(-3.933) = 5.5$ $x_1 = -0.39$

check with b:

$-2(-0.39) + 2(-3.933) - 3(-2.36) = 0$

13 Question

A triangle is defined by 3D points $\vec{a}$, $\vec{b}$, and $\vec{c}$
Find the area of the triangle given $\vec{a} < 1, 1, 0 >$, $\vec{b} < 4, 5, 1 >$, and $\vec{c} < 0, 2, 0 >$.



The area of the triangle is given by:

$$
\begin{aligned}
area &= \frac{1}{2}|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})| \\
&= \frac{1}{2}|(< 4, 5, 1 > - < 1, 1, 0 >) \times (< 0, 2, 0 > - < 1, 1, 0 >)| \\
&= \frac{1}{2}|(< 3, 4, 1 >) \times (< -1, 1, 0 >)| \\
&= \frac{1}{2}| < (4)(0) - (1)(1), (3)(0) - (1)(-1), (3)(1) - (4)(-1) > | \\
&= \frac{1}{2}| < -1, 1, 7 > | \\
&= \frac{1}{2}\sqrt{51}
\end{aligned} \tag{2}
$$