

# Math Lecture Notes Template

Angad Kapoor

Separate colored box environments for each section (with proofs), and some extras, made using the `tcolorbox` package and more.

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## 1 Definitions

### Definition 1.1 Cauchy Sequences

A sequence  $\{a_n\}$  of real numbers is a **Cauchy sequence** if  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $|a_m - a_n| < \epsilon \forall m, n \geq N$ .

## 2 Theorems

### Theorem 2.1 Differentiability implies Continuity

If a function  $f$  is differentiable at a point  $a$ , then it is continuous at  $a$ .

*Proof:* We want to show that  $\lim_{x \rightarrow a} f(x) = f(a)$ . First, notice that  $f(x)$  can be rewritten as  $f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$ . Since  $f$  is differentiable at  $a$ , we know that  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  exists and is finite.  $x$  is continuous, so  $\lim_{x \rightarrow a} (x - a) = 0$ , so  $f(x) \xrightarrow{x \rightarrow a} f(a) + f'(a) \cdot 0 = f(a)$ . ■

### 3 Propositions

#### Proposition 3.1

A Cauchy sequence with a convergent subsequence converges.

*Proof:* Let  $\{a_n\}$  be a Cauchy sequence, and let  $\{a_{k_n}\}$  be a convergent subsequence of  $\{a_n\}$ . Let  $\lim_{n \rightarrow \infty} a_{k_n} = a$ . Let  $\epsilon > 0$ . Since  $a$  is the limit of the subsequence, we know  $\exists N_1 \in \mathbb{N}$  such that  $|a - a_{k_n}| < \frac{\epsilon}{2} \forall n \geq N_1$ . Similarly, since  $\{a_n\}$  is Cauchy, we know  $\exists N_2 \in \mathbb{N}$  such that  $|a_m - a_n| < \frac{\epsilon}{2} \forall m, n \geq N_2$ . Choose  $N = \max\{N_1, N_2\}$ . Then, for  $n \geq N$ , we know  $k_n \geq n \geq N$  (by the definition of a subsequence). So,  $|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . Then, by the definition of the limit,  $\{a_n\}$  converges to  $a$ . ■

### 4 Corollaries

#### Corollary 4.1

A Cauchy sequence converges.

*Proof:* Every Cauchy sequence has a convergent subsequence, and we just saw that every Cauchy sequence with a convergent subsequence converges. Hence, every Cauchy sequence converges.

### 5 Lemmas

#### Lemma 5.1

This is a lemma.

*Proof:* This is the proof.

## 6 Examples

### Example 6.1 Using Cesaro-Stolz

Show that the sequence defined by  $x_n = n^{\frac{1}{n}}$  converges, and find its limit.

We can define  $x_n$  as  $x_n = a_n^{\frac{1}{n}}$ , where  $a_n$  is another sequence  $a_n = n$ . Thus,  $\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$ , which converges to 1 as  $n \rightarrow \infty$ . Then, by the Cesaro-Stolz theorem,  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ .

## 7 Miscellaneous

Environments can be created inside of one another, if required.

### Theorem 7.1 Theorem A

We will prove Theorem A.

*Proof:* To prove Theorem A, we will appeal to Lemma A.1

### Lemma 7.1 Lemma A.1

Statement of lemma

*Proof:* Proof of lemma

From this, it follows that Theorem A is true.

The Remark environment can be used as alternative to the built-in quote environment.

This is a remark that contains some relevant information.