Math Lecture Notes Template

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Separate colored box environments for each section (with proofs), made using the `tcolorbox` package.

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1 Definitions

Definition 1.1 Cauchy Sequences

A sequence $\{a_n\}$ of real numbers is a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon \; \forall m, n \geq N$.

2 Theorems

Theorem 2.1 Differentiability implies Continuity

If a function f is differentiable at a point a, then it is continuous at a.

Proof: We want to show that $\lim_{x \to a} f(x) = f(a)$. First, notice that $f(x)$ can be rewritten as $f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$. Since f is differentiable at a, we know that $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. x is continuous, so $\lim_{x \to a}(x - a) = 0$, so $f(x) \xrightarrow{x \to a} f(a) + f'(a) \cdot 0 = f(a)$. ■

3 Propositions

Proposition 3.1

A Cauchy sequence with a convergent subsequence converges.

Proof: Let $\{a_n\}$ be a Cauchy sequence, and let $\{a_{k_n}\}$ be a convergent subsequence of $\{a_n\}$. Let $\lim_{n \to \infty} x_{a_{k_n}} = a$. Let $\epsilon > 0$. Since a is the limit of the subsequence, we know $\exists N_1 \in \mathbb{N}$ such that $|a - a_{k_n}| < \frac{\epsilon}{2}\ \forall n \geq N_1$. Similarly, since $\{a_n\}$ is Cauchy, we know $\exists N_2 \in \mathbb{N}$ such that $|a_m - a_n| < \frac{\epsilon}{2}\ \forall m, n \geq N_2$. Choose $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we know $k_n \geq n \geq N$ (by the definition of a subsequence). So, $|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Then, by the definition of the limit, $\{a_n\}$ converges to a. ■

4 Corollaries

Corollary 4.1

A Cauchy sequence converges.

Proof: Every Cauchy sequence has a convergent subsequence, and we just saw that every Cauchy sequence with a convergent subsequence converges. Hence, every Cauchy sequence converges.

5 Examples

Example 5.1 Using Cesaro-Stolz

Show that the sequence defined by $x_n = n^{\frac{1}{n}}$ converges, and find its limit.

We can define x_n as $x_n = a_n^{\frac{1}{n}}$, where a_n is another sequence $a_n = n$. Thus, $\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$, which converges to 1 as $n \to \infty$. Then, by the Cesaro-Stolz theorem, $\lim_{n \to \infty} x_n = \lim_{n \to \infty} a_n^{\frac{1}{n}} = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$.