A&A-II 404.1 - Assignment-II

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(Dated: March 31, 2024)

I. QUESTION 1

the functions given, being just a function of t and the unit vectors not being a function of t, we can directly do this

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2}{dt^2}\left(r_x\hat{\mathbf{x}} + r_y\hat{\mathbf{y}} + r_z\hat{\mathbf{z}}\right) \\
\ddot{\mathbf{r}} &= \frac{d^2 r_x}{dt^2}\hat{\mathbf{x}} + \frac{d^2 r_y}{dt^2}\hat{\mathbf{y}} + \frac{d^2 r_z}{dt^2}\hat{\mathbf{z}} \\
\ddot{\mathbf{r}} &= \dot{\mathbf{v}} = \ddot{r_x}\hat{\mathbf{x}} + \ddot{r_y}\hat{\mathbf{y}} + \ddot{r_z}\hat{\mathbf{z}} \\
|\ddot{\mathbf{r}}|^2 &= |\dot{\mathbf{v}}|^2 = \ddot{r_x}^2 + \ddot{r_y}^2 + \ddot{r_z}^2
\end{aligned}$$

Now looking at the Larmor's Formula, we directly get the answer.

$$P = \frac{2}{3}\frac{q'^2}{4\pi\epsilon_0 c^3}|\dot{\mathbf{v}}|^2$$

Note: We are also given that $q' = 4q$

(a) Now substitute the function given in the question

$$\begin{aligned}
x(t) &= R_0\cos(\omega t) \\
y(t) &= R\sin(\omega t) \\
z(t) &= R
\end{aligned}$$

Therefore,

$$\begin{aligned}
\ddot{x}(t) &= -\omega^2 R_0\cos(\omega t) \\
\ddot{y}(t) &= -\omega^2 R\sin(\omega t) \\
\ddot{z}(t) &= 0
\end{aligned}$$

$$\begin{aligned}
P &= \frac{2}{3}\frac{q'^2}{4\pi\epsilon_0 c^3}|\dot{\mathbf{v}}|^2 \\
&= \frac{32}{3}\frac{q^2}{4\pi\epsilon_0 c^3}[\omega^4(R_0^2\cos^2(\omega t) + R^2\sin^2(\omega t))]
\end{aligned}$$

(b) Now substitute the function given in the part b of the question

$$\begin{aligned}
x(t) &= x_0 + v_0\tanh(t-10) \\
y(t) &= y_0 + v_0\tanh(t-10)
\end{aligned}$$

where t is in seconds. Then we get

$$\begin{aligned}
\ddot{x}(t) &= -2v_0\,\mathrm{sech}^2(t-10)\,\tanh(t-10) \\
\ddot{y}(t) &= -2v_0\,\mathrm{sech}^2(t-10)\,\tanh(t-10) \\
\implies |\dot{\mathbf{v}}|^2 &= \ddot{x}(t)^2 + \ddot{y}(t)^2 \\
&= 8v_0\,\mathrm{sech}^2(t-10)\,\tanh(t-10)
\end{aligned}$$

Therefore Power is,

$$\begin{aligned}
P &= \frac{2}{3}\frac{q'^2}{4\pi\epsilon_0 c^3}|\dot{\mathbf{v}}|^2 \\
&= \frac{32}{3}\frac{q^2}{4\pi\epsilon_0 c^3}[8v_0^2\,\mathrm{sech}^4(t-10)\,\tanh^2(t-10)]
\end{aligned}$$

II. QUESTION 2

We know that the optical depth is $\tau = \int n_i\sigma(\omega)cdt$. And we are given $n_i = 50\,per\,cm^{-3}$. And since $\mathrm{T} = \frac{1}{100\omega_0} << \frac{1}{\omega_0}$, we can use the bound system analysis. Now using the equation

$$\sigma(\omega) = \sigma_T\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega_0^3\mathrm{T})^2}$$

where $\sigma_T = 6.652 * 10^{-29}m^2$.

Now plotting Optical depth vs ω, we will get these graphs. We have two plots to show the nature of the plot.

In the starting, we can see its a straight line with slope $10^4\,over\,10^1$ in log log plot. And in log log plot, we can also clearly see optical depth asymptotically approaches a value and that is because σ asymptotically approaches σ_T.

Point to keep in mind, the shape of the curve is the exact same for both $\sigma\,vs\,\omega$ and $Optical\,Depth\,vs\,\omega$.

Note: here we are considering everything in SI units, optical depth is dimensionless and ω is in Hz.

III. QUESTION 3

For Thermal distribution, we have power equation for Bremsstralung emission.

$$P = \frac{32\pi e^6 g_{\text{eff}}}{3mc^3}\sqrt{\frac{2\pi}{3km}}\exp\left(-\frac{h\nu}{kT}\right)\frac{Z^2 n_e n_i}{\sqrt{T}}$$

The graph is given in here.

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IV. QUESTION 4

We know,

$$\alpha_\nu = \frac{4e^6}{3mhc} g_{eff} \left(\frac{2\pi}{3km}\right)^{0.5} \frac{1}{\sqrt{T}} \frac{Z^2 n_e n_i}{\nu^3} \left(1 - e^{-\frac{h\nu}{kT}}\right)$$

Using this we get this plot. Note: we can clearly see that, as L increases, the whole graph is scaled up by the same factor and for same temperature, same shape is preserved.

V. QUESTION 5

We know the for Synchroton emission, the power emitted is

$$P_{sync} = \frac{1}{6\pi} \sigma_T c \beta^2 \gamma^2 B^2 \tag{1}$$

And the cooling timescale is given by

$$t_{cool} = \frac{\gamma mc^2}{P_{sync}} \tag{2}$$

(a) Now substitute the given conditions

$$\begin{aligned} B &= 10^{-5}\ Gauss \\ &= 10^{-9}\ T \\ v &= 0.999c \end{aligned}$$

Therefore,

$$\begin{aligned} P_{sync} &= 5.29 \times 10^{-37}\ W \\ t_{cool} &= 3.47 \times 10^{24}\ s \end{aligned}$$

(b) Now part b

$$\begin{aligned} B &= 10^{6}\ Gauss \\ &= 10^{2}\ T \\ v &= 0.999c \end{aligned}$$

Therefore,

$$\begin{aligned} P_{sync} &= 5.29 \times 10^{-15}\ W \\ t_{cool} &= 3.47 \times 10^{2}\ s \end{aligned}$$

(c) Now part c

$$\begin{aligned} B &= 10^{6}\ Gauss \\ &= 10^{2}\ T \\ v &= 0.1c \end{aligned}$$

Therefore,

$$\begin{aligned} P_{sync} &= 1.07 \times 10^{-19}\ W \\ t_{cool} &= 7.70 \times 10^{5}\ s \end{aligned}$$

You can find a code in the `GitHub link` which has a python program to solve for this.

VI. QUESTION 6

For power law distribution, the electron follows this

$$N(E) = CE^{-p}$$

Meanwhile the polarization facotr is given by

$$\Pi = \frac{p+1}{p+\frac{7}{3}}$$

And the total power emitted by the synchroton is

$$P_{tot}(\omega) = \frac{\sqrt{3}e^3 CB\sin\alpha}{2\pi mc^2(p+1)} \Gamma(\frac{p}{4}+\frac{19}{12})\Gamma(\frac{p}{4}-\frac{1}{12})(\frac{mc\omega}{3eB\sin\alpha})^{-(p-1)/2}$$

(a) Given:

$$N(E) = E^{-5/2}$$

Thenrefore polarization fraction is:

$$\begin{aligned} \Pi &= \tfrac{\frac{5}{2}+1}{\frac{5}{2}+\frac{7}{3}} \\ &= 0.72 \end{aligned}$$

On solving and plotting this, we get these graphs.

(b) Part b of the question

$$N(E) = E^{-4}$$

Thenrefore polarization fraction is:

$$\begin{aligned} \Pi &= \tfrac{4+1}{4+\frac{7}{3}} \\ &= 0.79 \end{aligned}$$

On solving and plotting these we get these graphs.

Note: We are just taking Q5 Part a and c into consideration, because part b is redundant, because we dont use beta anywhere.

VII. QUESTION 7

We the absorption coefficient for this case is given by,

$$\alpha_\nu = \frac{\sqrt{3}e^3}{8\pi m}(\frac{3e}{2\pi m^3 c^5})^{p/2} C(B\sin\alpha)^{(p+2)/2}\Gamma(\frac{3p+2}{12})\Gamma(\frac{3p+22}{12})\nu^{-(p+4)/2}$$

And we know the optical depth is $\tau = \int n_i \sigma(\omega) c dt$.

(a) Given:

$$N(E) = E^{-5/2}$$

ie. $p = 5/2$

(i) for $L = 10^2$ Km. On solving and plotting this, we get these graphs.

(i) for $L = 10^{10}$ Km. On solving and plotting this, we get these graphs.

(b) Part b of the question

$$N(E) = E^{-4}$$

ie. $p = 4$

(i) for $L = 10^2$ Km. On solving and plotting this, we get these graphs.

(i) for $L = 10^{10}$ Km. On solving and plotting this, we get these graphs.

Note: Again Wwe are just taking Q5 Part a and c into consideration, because part b is redundant, because we dont use beta anywhere.

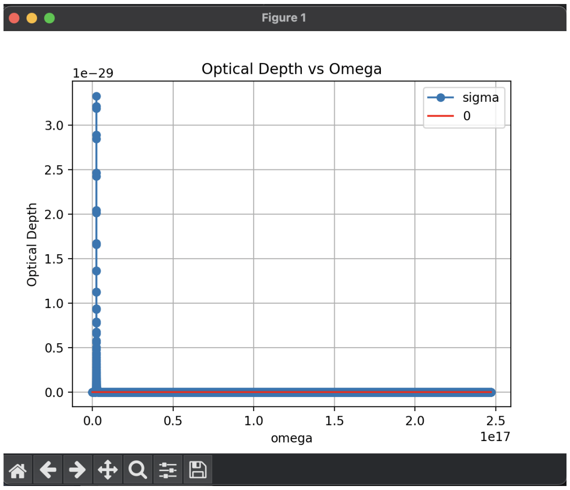
Using a python code, we are getting these graphs.

ACKNOWLEDGEMENT

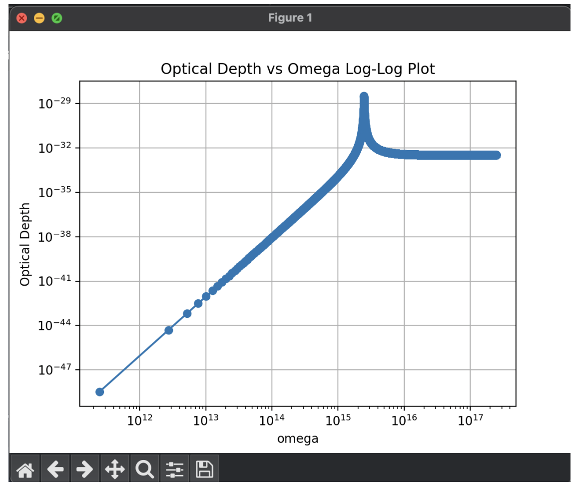
Discussed with Vidit for the Question 5 and 7. Code is in this `GitHub link`. And I have used numpy, and matplotlib and scipy(for gamma functions) in my codes.

GRAPHS

Graphs are on the next page



(a) Optical Depth vs ω



(b) Log Log plot of Optical Depth vs ω

FIG. 1: Optical Depth of a Harmonically Bound System

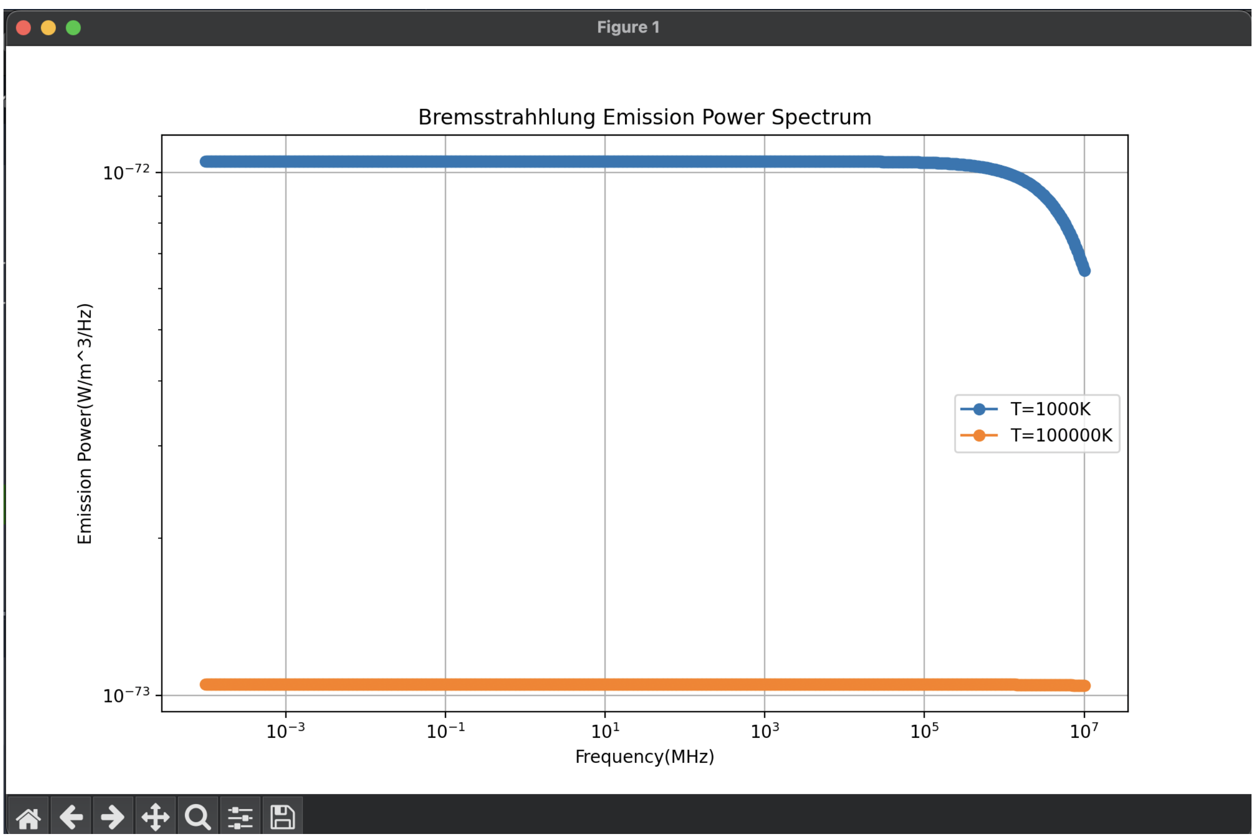


FIG. 2: Power Spectrum of Thermal Distribution at different Temperatures

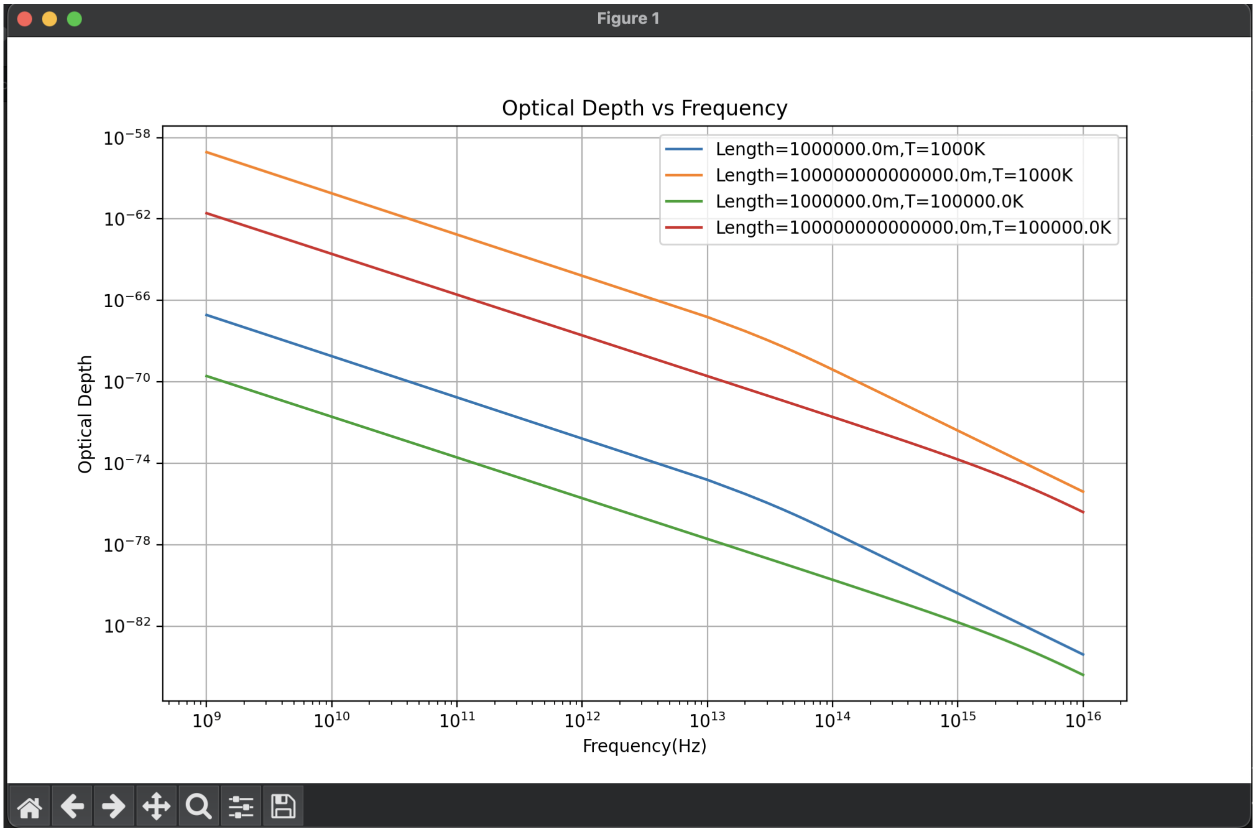
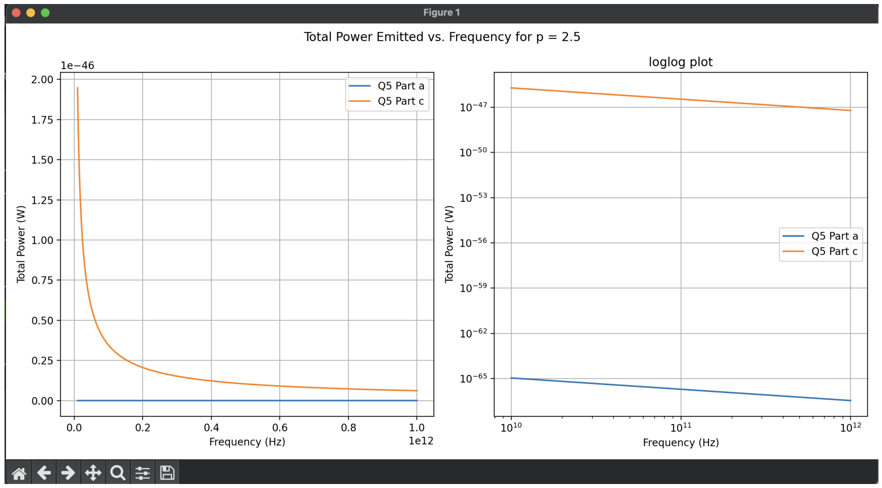
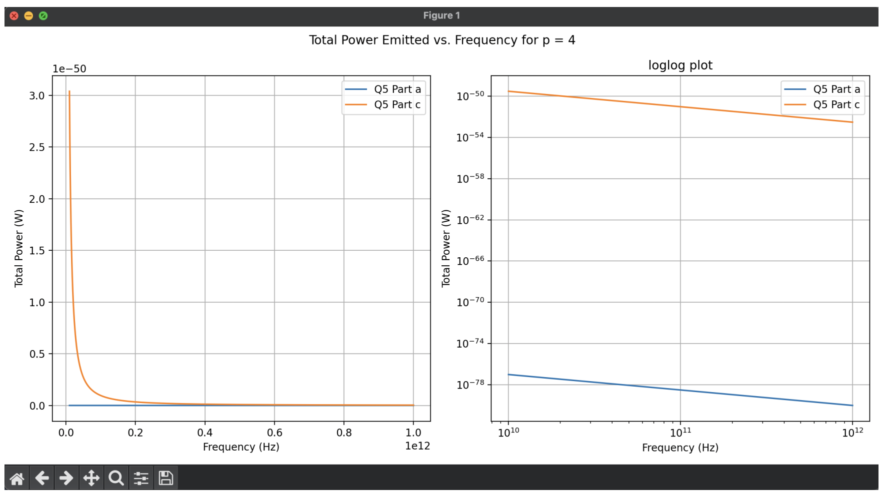


FIG. 3: Log Log Plot of Optical Depth at different Temperatures and Lengths

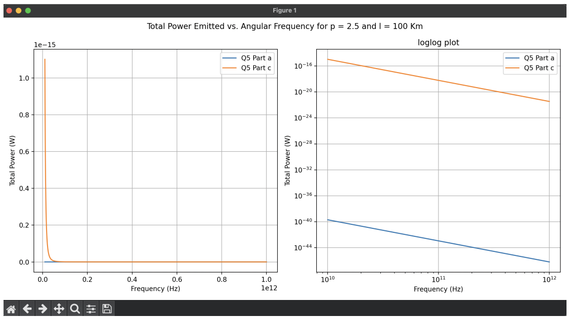


(a) Total Power Emitted vs. Angular Frequency for p = 2.5

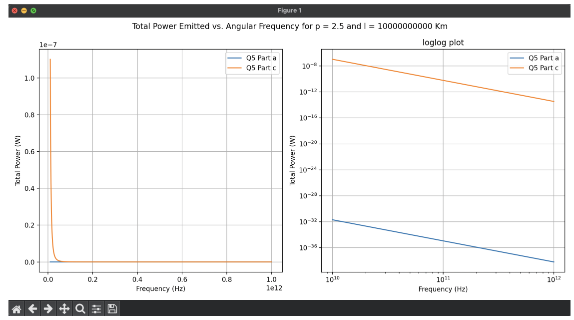


(b) Total Power Emitted vs. Angular Frequency for p = 4

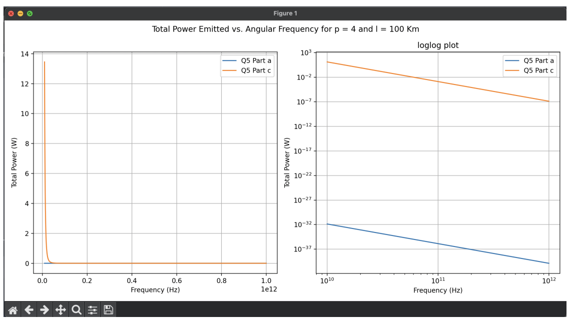
FIG. 4: Total Power Emitted vs. Angular Frequency



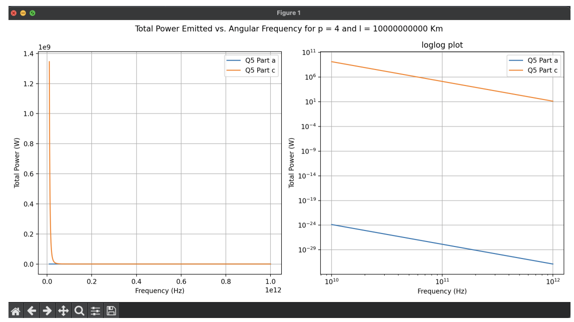
(a) Total Power Emitted vs. Angular Frequency for p = 2.5 and l = 10^2 Km



(b) Total Power Emitted vs. Angular Frequency for p = 2.5 and l = 10^{10} Km



(c) Total Power Emitted vs. Angular Frequency for p = 4 and l = 10^2 Km



(d) Total Power Emitted vs. Angular Frequency for p = 4 and l = 10^{10} Km

FIG. 5: Total Power Emitted vs. Angular Frequency