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An a priori error analysis of an HDG method for an eddy current problem

Rommel Bustinza[1] | Bibiana Lopez-Rodriguez[2] | Mauricio Osorio[2]

[1]Department of Mathematical Engineering and Mathematical Engineering Research Center (CI^{2}MA), University of Concepcion, Casilla 160-C Concepción, Chile
[2]School of Mathematics, National University of Colombia, Sede Medellín, Colombia

Correspondence
Mauricio Osorio, School of Mathematics, National University of Colombia, Sede Medellín, Colombia.
Email: maosorio@unal.edu.co

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This paper concerns itself with the development of an a priori error analysis of an eddy current problem when applying the well-known hybridizable discontinuous Galerkin (HDG) method. Up to the authors' knowledge, this kind of theoretical result has not been proved for this kind of problems. We consider nontrivial domains and heterogeneous media which contain conductor and insulating materials. When dealing with these domains, it is necessary to impose the divergence-free condition explicitly in the insulator, what is done by means of a suitable Lagrange multiplier in that material. In the end, we deduce an equivalent HDG formulation that includes as unknowns the tangential and normal trace of a vector field. This represents a reduction in the degrees of freedom when compares with the standard DG methods. For this scheme, we conduct a consistency and local conservative analysis as well as its unique solvability. After that, we introduce suitable projection operators that help us to deduce the expected a priori error estimate, which provides estimated rates of convergence when additional regularity on the exact solution is assumed.

KEYWORDS
HDG, eddy current problem

1 | INTRODUCTION

The eddy current problem is obtained from the Maxwell equations disregarding the electric displacement currents in the Ampére-Maxwell law. This simplification of the equations is reasonable when the magnitude of the displacement currents is negligible compared to other terms of the equation (see, for instance, in Bossavit[1], chapter 15 or in Alonso Rodríguez and Valli[2], chapter 1). Our goal is to study the time-harmonic eddy current problem defined in a three-dimensional domain including conducting and dielectric materials. This model arises in applications where the problem is posed in a bounded domain, and it is necessary to link the electromagnetic fields with the current source, given as a volume current, in a conducting region.

For proposing a numerical solution of the eddy current problem, two fundamental aspects should keep in mind. The first one is the need to impose a divergence-free condition to one of the electromagnetic variables in the dielectric part of the domain and the second one is that, depending on the electromagnetic properties of the coefficients in each material, discontinuities could be generated in the electromagnetic variables. The first aspect has been studied by continuous Galerkin methods by introducing a Lagrange multiplier or a scalar potential in the dielectric, see Alonso Rodríguez et al[3] with volumetric current source and other works[4-7] for different power sources. Regarding the second aspect, discontinuous Galerkin methods (DG) have been considered in recent years to deal with the discontinuity of the approximate solution, see Perugia and Schötzau.[8] However, the DG methods have a disadvantage compared to continuous methods: The number of degrees of freedom is larger.

In this paper, we propose and analyze a hybridizable DG (HDG) method for the time-harmonic eddy current problem. Hybridizable discontinuous Galerkin methods appear in 2010 as a new kind of DG method that allows to find the approximate solution by solving an equivalent system of equations associated to the skeleton of the partition of the domain. This makes the scheme to be competitive with the standard continuous Galerkin methods, for example. In Nguyen et al,[9] the authors introduce an HDG method under the assumption that the domain contains only dielectric material (conductivity is zero). They show experimentally that the method is convergent with the expected convergence rates; however, a theoretical error analysis is missing. The current paper aims to introduce a more general analysis to the model when conducting and dielectric materials are introduce in the domain. To this end, we introduce a Lagrange multiplier to impose the divergence-free condition in the dielectric domain. As far as we know, there are a few papers dealing with HDG formulations for either eddy current or Maxwell problems (see Nguyen et al[9] and Li et al[10]) and none of them present a complete error analysis.

The HDG method initially proposed here has, as unknowns, the variables related to the magnetic and electric field across the entire domain, the Lagrange multiplier in the dielectric domain, and two variables defined only on the skeleton (associated to the tangential and normal trace of the electric field). By imposing the local conservative condition, the method reduces to solving a system defined only on the skeleton, thus obtaining a complete method that preserves the discontinuities due to the different electromagnetic properties of the materials and the divergence-free condition in the dielectric, thus reducing the number of unknowns compared with standard DG methods.

Here, we propose an HDG formulation based on the normal and tangential trace of vector field. We show consistency and local conservative results, then prove that this formulation is well posed and obtain error estimates. To this end, we introduce suitable projection operators and under additional assumptions on the regularity of the exact solution we obtain theoretical rates of convergence.

The outline of the paper is as follows: In Section 2, we introduce the time-harmonic eddy current model and approximations spaces. Then, in Section 3, we propose and analyze our HDG formulation to show that it has a unique solution and the scheme is locally conservative and consistent. The hybridization of the scheme that leads to a linear system defined on the skeleton is introduced in Section 4.. Finally, in Section 5, we establish the a priori error analysis that is the main contribution of this paper.

2 | PROBLEM STATEMENT

2.1 | Eddy current problem

Maxwell's equations are used to describe electromagnetic phenomena. The eddy current model results by disregarding the effect of electric displacement in the Ampère-Maxwell law and assuming that all fields involved are sinusoidal in time. The time-harmonic eddy current model results:

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \sigma \boldsymbol{E} + \boldsymbol{J}^s \quad && \text{in} \quad \Omega, \\
i\omega\mu \boldsymbol{H} + \nabla \times \boldsymbol{E} &= \boldsymbol{0} && \text{in} \quad \Omega, \\
\nabla \cdot (\epsilon \boldsymbol{E}) &= 0 && \text{in} \quad \Omega_{\mathrm{D}},
\end{aligned}
\tag{1}
$$

where $\boldsymbol{E}$ denotes the intensity of electric field, $\boldsymbol{H}$ the intensity of magnetic field, ω the angular frequency, and $\boldsymbol{J}^s$ the density of current, which is assumed to be divergence-free. We have to take also into account the electromagnetic coefficients: magnetic permeability μ, electric permittivity ϵ, and electric conductivity σ. We remark that i denotes the imaginary unit. About the domain Ω, we assume that it is bounded, simple connected, and consists on two parts: Ω_{C} and Ω_{D}, with Ω_{C} representing the conductor domain while Ω_{D} denotes the dielectric one (cf Figure 1). The boundary of Ω, named Γ, is supposed to be connected and Lipschitz-continuous. Γ_0 denotes the interface between the conductor and dielectric domains. We point out that the electromagnetic coefficients depend on the material. In particular, we assume that μ, ϵ and σ are bounded functions satisfying

$$
\begin{aligned}
&0 < \mu_{\min} \le \mu \le \mu_{\max}, \quad \text{in}\,\Omega, \\
&0 < \epsilon_{\min} \le \epsilon \le \epsilon_{\max}, \quad \text{in}\,\Omega, \\
&0 < \sigma_{\min} \le \sigma \le \sigma_{\max}, \quad \text{in}\,\Omega_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \,\text{in}\,\Omega_{\mathrm{D}},
\end{aligned}
$$

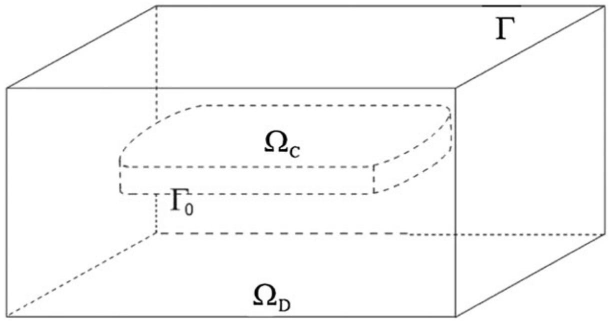

FIGURE 1 Domain Ω for problem (1)

and the following boundary condition

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{g} \quad \text{on} \quad \Gamma.$$

We introduce now the unknowns: $\boldsymbol{u} = \boldsymbol{E}$, $\boldsymbol{z} = \mu^{-1}\nabla \times \boldsymbol{u}$, and the Lagrange multiplier φ defined in Ω_D, that enforces the divergence free condition. Then, system (1) can be rewritten as the first order system

$$\begin{aligned}
\mu \boldsymbol{z} - \nabla \times \boldsymbol{u} &= 0 && \text{in} \quad \Omega, \\
\nabla \times \boldsymbol{z} + i\omega\sigma\boldsymbol{u} - \epsilon\,\nabla\varphi|_{\Omega_D} &= -i\,\omega\,\boldsymbol{J}^s && \text{in} \quad \Omega, \\
\nabla \cdot (\epsilon\,\boldsymbol{u}) &= \varphi && \text{in} \quad \Omega_D, \\
\langle \epsilon\boldsymbol{u}|_{\Omega_D} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0} &= 0, && \\
\boldsymbol{n} \times \boldsymbol{u} &= \boldsymbol{g} && \text{on} \quad \Gamma, \\
\varphi &= 0 && \text{on} \quad \Gamma.
\end{aligned} \tag{2}$$

Here, $\boldsymbol{n}$ and $\boldsymbol{n}_0$ denote the outward normal unit vector to Γ and Γ_0, disjoint components of $\partial\Omega_D$. We remark that (1) and (2) are equivalent (see Nguyen et al[9]).

Throughout this paper, when $\lesssim$ is used, it means that the inequality holds with, eventually, an omitted constant factor that does not depend either on the mesh size or the electromagnetic coefficients or stabilization parameters. Also, we will use standard notation for Sobolev spaces and norms. We will use the well-known Hilbert spaces $\mathrm{H}(\mathbf{curl};\Omega)$, $\mathrm{H}(\mathrm{div};\Omega)$, etc. (see, for instance, Amrouche et al[11]). Let us emphasize the following spaces:

$$\begin{aligned}
\mathrm{H}_\Gamma(\mathbf{curl};\Omega_D) &:= \{\boldsymbol{G} \in \mathrm{H}(\mathbf{curl};\Omega_D) \;:\; \boldsymbol{G} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma\}\,, \\
\mathrm{H}_{\Gamma_0}(\mathrm{div}^0_\epsilon;\Omega_D) &:= \{\boldsymbol{F} \in \mathrm{H}(\mathrm{div}_\epsilon;\Omega_D) \;:\; \mathrm{div}\epsilon\boldsymbol{F} = 0 \quad \text{in} \quad \Omega_D,\, \epsilon\boldsymbol{F}\cdot\boldsymbol{n} = 0 \,\text{on}\, \Gamma_0\}\,.
\end{aligned}$$

2.2 | Mesh and approximation spaces

We let now $\mathcal{T}_h$ be a shape-regular simplicial triangulation of $\bar{\Omega}$, where h also denotes the mesh size of triangulation. Next, we set $\partial\mathcal{T}_h := \{\partial K \,:\, K \in \mathcal{T}_h\}$ and define $\mathcal{E}_h^I$ and $\mathcal{E}_h^{0,I}$ as the set of interior faces F induced by $\mathcal{T}_h$, lying in Ω and Ω_D, respectively. By $\mathcal{E}_h$ we denote the set of all faces induced by $\mathcal{T}_h$, while $\mathcal{E}_h^{\Gamma_0}$ is defined as $\mathcal{E}_h \cap \Gamma_0$. In addition, we define $\mathcal{T}_h^D$ and $\mathcal{T}_h^C$ as the triangulations of $\bar{\Omega}_D$ and $\bar{\Omega}_C$, induced by $\mathcal{T}_h$, and by $\mathcal{E}_h^D$ and $\mathcal{E}_h^C$ their corresponding sets of all faces induced by $\mathcal{T}_h^D$ and $\mathcal{T}_h^C$, respectively. As usual, given $K \in \mathcal{T}_h$, $\boldsymbol{n}$ denotes the outward normal unit vector to ∂K. Let us emphasize the broken Sobolev space for any real number s

$$[\mathrm{H}^s(\mathcal{T}_h)]^3 := \left\{\boldsymbol{v} \in [\mathrm{L}^2(\mathcal{T}_h)]^3 \,: \quad \boldsymbol{v}|_K \in [\mathrm{H}^s(K)]^3 \quad \forall K \in \mathcal{T}_h\right\}\,.$$

Now, let $\boldsymbol{z}$ be a function living in $[\mathrm{H}^s(\mathcal{T}_h)]^3$ with $s > 1/2$, and let F a face in $\partial\mathcal{T}_h$, shared by 2 adjacent elements K^+ and K^- in $\mathcal{T}_h$, that is $F = \partial K^+ \cap \partial K^-$. By $\boldsymbol{z}^\pm$ we set the trace of $\boldsymbol{z}$ on F from the interior of $K^\pm$, while $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ denote the outward unit normal vectors to K^+ and K^-, respectively. Then, the jumps $[[\cdot]]$ are defined as follows. When F is an interior face, we set

$$[[\boldsymbol{n} \odot \boldsymbol{z}]] := \boldsymbol{n}^+ \odot \boldsymbol{z}^+ + \boldsymbol{n}^- \odot \boldsymbol{z}^-\,,$$

while when F is a boundary face, we define

$$[[\boldsymbol{n} \odot \boldsymbol{z}]] := \boldsymbol{n} \odot \boldsymbol{z},$$

with $\boldsymbol{n}$ being, in this case, the outward unit normal to Γ. Here, $\odot$ is either $\cdot$ or $\times$. We recall the definitions of $\boldsymbol{z}^t$, the tangential component of $\boldsymbol{z}$, as well as of $\boldsymbol{z}^n$, the normal component of $\boldsymbol{z}$:

$$\boldsymbol{z}^t := \boldsymbol{n} \times (\boldsymbol{z} \times \boldsymbol{n}), \quad \boldsymbol{z}^n := (\boldsymbol{z} \cdot \boldsymbol{n})\boldsymbol{n}.$$

We point out that the trace of $\boldsymbol{z}$ on ∂K, can be decomposed as $\boldsymbol{z} = \boldsymbol{z}^t + \boldsymbol{z}^n$.

We introduce the following approximation spaces

$$\begin{aligned}
\mathbb{P}_h &:= \{q \in \mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}) : \quad q|_K \in \mathbb{P}_k(K) \qquad \forall K \in \mathcal{T}_h^{\mathrm{D}}\},\\
\mathbb{V}_h &:= \{\boldsymbol{v} \in [\mathrm{L}^2(\mathcal{T}_h)]^3 : \quad \boldsymbol{v}|_K \in [\mathbb{P}_k(K)]^3 \qquad \forall K \in \mathcal{T}_h\},\\
\mathbb{P}_h^{\Gamma} &:= \{q \in \mathbb{P}_h : \quad q = 0 \quad \text{on} \quad \Gamma\},\\
\mathbb{M}_h^t &:= \left\{\boldsymbol{\beta} \in [\mathrm{L}^2(\mathcal{E}_h)]^3 \ : \ \boldsymbol{\beta}|_F \in [\mathbb{P}_k(F)]^3, (\boldsymbol{\beta}\cdot\boldsymbol{n})|_F = 0 \qquad \forall F \in \mathcal{E}_h\right\},\\
\mathbb{M}_h^t(\boldsymbol{g}) &:= \left\{\boldsymbol{\beta} \in \mathbb{M}_h^t \ : \ \boldsymbol{n}\times\boldsymbol{\beta} = \Pi\boldsymbol{g} \quad \text{on} \quad \Gamma\right\},\\
\mathbb{M}_h^n &:= \left\{\boldsymbol{\gamma} \in [\mathrm{L}^2(\mathcal{E}_h^{\mathrm{D}})]^3 \ : \ \boldsymbol{\gamma}|_F \in [\mathbb{P}_k(F)]^3, (\boldsymbol{\gamma}\times\boldsymbol{n})|_F = \boldsymbol{0} \qquad \forall F \in \mathcal{E}_h^{\mathrm{D}}\right\},\\
\mathbb{M}_h^n(0) &:= \left\{\boldsymbol{\gamma} \in \mathbb{M}_h^n, : \ (\boldsymbol{\gamma}\cdot\boldsymbol{n}_0)|_F = 0 \qquad \forall F \in \mathcal{E}_h^{\Gamma_0}\right\},
\end{aligned}$$

where $\boldsymbol{\Pi}\boldsymbol{g}$ represents the projection of $\boldsymbol{g}$ over $\mathbb{M}_h^t$. Hereafter, given S of positive measure, $\mathbb{P}_k(S)$ denotes the space of complex-valued polynomials of degree at most $k \geq 0$ on S. We remark that $\mathbb{M}_h^t$ ($\mathbb{M}_h^n$ resp.) consists of vector-valued functions whose normal (tangential resp.) component is zero on any face $F \in \mathcal{E}_h$ ($F \in \mathcal{E}_h^{0,I}$ resp.). The inner products to be considered here are

$$(q,p)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(q,p)_K, \quad (\boldsymbol{v},\boldsymbol{w})_{\mathcal{T}_h} := \sum_{j=1}^{d}(v_j,w_j)_{\mathcal{T}_h},$$

$$\langle q,p\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle q,p\rangle_{\partial K}, \quad \langle\boldsymbol{v},\boldsymbol{w}\rangle_{\partial\mathcal{T}_h} := \sum_{j=1}^{d}\langle v_j,w_j\rangle_{\partial\mathcal{T}_h},$$

where $(q,p)_K := \int_K q\,\bar{p}$ and $\langle q,p\rangle_{\partial K} := \int_{\partial K} q\,\bar{p}$, with $\bar{p}$ denoting the conjugate complex operator when applies to p.

3 | HDG FORMULATION

In this section, we apply the well-known HDG method to deduce a discrete variational formulation. We look for $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{P}_h^{\Gamma} \times \mathbb{M}_h^t \times \mathbb{M}_h^n(0)$, solution of the discrete scheme:

$$(\mu\boldsymbol{z}_h,\boldsymbol{r})_{\mathcal{T}_h} - (\boldsymbol{u}_h,\nabla\times\boldsymbol{r})_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{u}}_h^t,\boldsymbol{r}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{3}$$

$$(\boldsymbol{z}_h,\nabla\times\boldsymbol{v})_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{z}}_h^t,\boldsymbol{v}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + i\omega(\sigma\boldsymbol{u}_h,\boldsymbol{v})_{\mathcal{T}_h} + (\varphi_h,\nabla\cdot(\epsilon\boldsymbol{v}))_{\mathcal{T}_h^{\mathrm{D}}} - \langle\widehat{\varphi}_h,\epsilon\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = (\boldsymbol{J},\boldsymbol{v})_{\mathcal{T}_h}, \tag{4}$$

$$(\varphi_h,\rho)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{u}_h,\nabla\rho)_{\mathcal{T}_h^{\mathrm{D}}} - \langle\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n},\epsilon\rho\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0, \tag{5}$$

$$\langle\boldsymbol{n}\times\widehat{\boldsymbol{u}}_h^t,\boldsymbol{\eta}\rangle_{\Gamma} = \langle\boldsymbol{g},\boldsymbol{\eta}\rangle_{\Gamma}, \tag{6}$$

$$\langle\boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t,\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{7}$$

$$\langle\widehat{\varphi}_h,\epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0, \tag{8}$$

for all $(\boldsymbol{r},\boldsymbol{v},\rho,\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{P}_h^{\Gamma} \times \mathbb{M}_h^t \times \mathbb{M}_h^n(0)$, where $\boldsymbol{J} := -i\omega\boldsymbol{J}^s$. The numerical fluxes $\widehat{\boldsymbol{z}}_h^t$ and $\widehat{\varphi}_h$ are defined by

$$\begin{aligned}
\boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t &= \boldsymbol{n}\times\boldsymbol{z}_h^t + \tau_t(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t),\\
\widehat{\varphi}_h &= \tilde{\varphi}_h - \tau_n(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n)\cdot\boldsymbol{n},
\end{aligned} \tag{9}$$

with

$$\tilde{\varphi}_h = \begin{cases} \varphi_h & \text{on} \quad \mathcal{E}_h^{0,I}\cup\Gamma,\\ \varphi_h - \lambda\langle\epsilon\boldsymbol{u}_h|_{\Omega_D}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} & \text{on} \quad \Gamma_0. \end{cases} \tag{10}$$

Here, λ is a positive constant, and τ_t and τ_n are positive stabilization parameters defined on $\mathcal{E}_h$. Furthermore, it is straightforward to verify that $\boldsymbol{n}\times\boldsymbol{z}_h^t = \boldsymbol{n}\times\boldsymbol{z}_h$, therefore, we use these expressions indistinctly.

We say that the numerical fluxes $\hat{\boldsymbol{z}}_h^t, \hat{\varphi}_h$ are locally conservative if they are single-valued on $\mathcal{E}_h^I$ and $\mathcal{E}_h^{0,I}$, respectively. This important feature of the discrete formulation is proven next.

Lemma 3.1. *The HDG scheme defined by (3) to (8) is locally conservative and consistent.*

Proof. First, we observe that (7) implies that $[[\boldsymbol{n}\times\hat{\boldsymbol{z}}_h^t]] = 0$ on $\mathcal{E}_h^I$, by taking $\boldsymbol{\eta} := \boldsymbol{n}\times\hat{\boldsymbol{z}}_h^t$. By (9), we have

$$\begin{aligned}
0 &= [[\boldsymbol{n}\times\boldsymbol{z}_h^t + \tau_t(\boldsymbol{u}_h^t - \hat{\boldsymbol{u}}_h^t)]] \\
&= [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] + [[\tau_t(\boldsymbol{u}_h^t - \hat{\boldsymbol{u}}_h^t)]] \\
&= [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] + \tau_t^+(\boldsymbol{u}_h^t - \hat{\boldsymbol{u}}_h^t)^+ + \tau_t^-(\boldsymbol{u}_h^t - \hat{\boldsymbol{u}}_h^t)^- \\
&= [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] + \tau_t(\boldsymbol{u}_h^{t^+} - \hat{\boldsymbol{u}}_h^{t^+}) + \tau_t(\boldsymbol{u}_h^{t^-} - \hat{\boldsymbol{u}}_h^{t^-}).
\end{aligned}$$

Hence (since numerical fluxes $\hat{\boldsymbol{u}}_h^t$ and $\hat{\boldsymbol{u}}_h^n$ are single-valued),

$$\hat{\boldsymbol{u}}_h^t = \frac{1}{2\tau_t}[[\boldsymbol{n}\times\boldsymbol{z}_h^t]] + \frac{\boldsymbol{u}_h^{t^+} + \boldsymbol{u}_h^{t^-}}{2}.$$

Substituting this into (9), we obtain

$$\begin{aligned}
\hat{\boldsymbol{z}}_h^t &= \boldsymbol{z}_h^t + \tau_t\left(\boldsymbol{u}_h^t - \frac{1}{2\tau_t}[[\boldsymbol{n}\times\boldsymbol{z}_h^t]] - \frac{\boldsymbol{u}_h^{t^+} + \boldsymbol{u}_h^{t^-}}{2}\right)\times\boldsymbol{n} \\
&= \boldsymbol{z}_h^t + \frac{1}{2}\left(2\tau_t\boldsymbol{u}_h^t - [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] - \tau_t\boldsymbol{u}_h^{t^+} - \tau_t\boldsymbol{u}_h^{t^-}\right)\times\boldsymbol{n}.
\end{aligned}$$

Thus,

$$\hat{\boldsymbol{z}}_h^{t^+} = \boldsymbol{z}_h^{t^+} + \frac{1}{2}\left(\tau_t\boldsymbol{u}_h^{t^+} - [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] - \tau_t\boldsymbol{u}_h^{t^-}\right)\times\boldsymbol{n}^+ \tag{11}$$

$$\hat{\boldsymbol{z}}_h^{t^-} = \boldsymbol{z}_h^{t^-} + \frac{1}{2}\left(\tau_t\boldsymbol{u}_h^{t^-} - [[\boldsymbol{n}\times\boldsymbol{z}_h^t]] - \tau_t\boldsymbol{u}_h^{t^+}\right)\times\boldsymbol{n}^-. \tag{12}$$

Now,

$$\begin{aligned}
[[\boldsymbol{n}\times\boldsymbol{z}_h^t]]\times\boldsymbol{n}^+ &= \left(\boldsymbol{n}^+\times\boldsymbol{z}_h^{t^+} + \boldsymbol{n}^-\times\boldsymbol{z}_h^{t^-}\right)\times\boldsymbol{n}^+ \\
&= \boldsymbol{n}^+\times\left(\boldsymbol{z}_h^{t^+}\times\boldsymbol{n}^+\right) + \boldsymbol{n}^-\times\left(\boldsymbol{z}_h^{t^-}\times\boldsymbol{n}^+\right) \\
&= \boldsymbol{z}_h^{t^+}\left(\boldsymbol{n}^+\cdot\boldsymbol{n}^+\right) - \boldsymbol{n}^+\left(\boldsymbol{n}^+\cdot\boldsymbol{z}_h^{t^+}\right) + \boldsymbol{z}_h^{t^-}\left(\boldsymbol{n}^-\cdot\boldsymbol{n}^+\right) - \boldsymbol{n}^+\left(\boldsymbol{n}^-\cdot\boldsymbol{z}_h^{t^-}\right) \\
&= \boldsymbol{z}_h^{t^+} - \boldsymbol{z}_h^{t^-}, \\
[[\boldsymbol{n}\times\boldsymbol{z}_h^t]]\times\boldsymbol{n}^- &= \left(\boldsymbol{n}^+\times\boldsymbol{z}_h^{t^+} + \boldsymbol{n}^-\times\boldsymbol{z}_h^{t^-}\right)\times\boldsymbol{n}^- \\
&= \boldsymbol{n}^+\times\left(\boldsymbol{z}_h^{t^+}\times\boldsymbol{n}^-\right) + \boldsymbol{n}^-\times\left(\boldsymbol{z}_h^{t^-}\times\boldsymbol{n}^-\right) \\
&= \boldsymbol{z}_h^{t^+}\left(\boldsymbol{n}^+\cdot\boldsymbol{n}^-\right) - \boldsymbol{n}^-\left(\boldsymbol{n}^+\cdot\boldsymbol{z}_h^{t^+}\right) + \boldsymbol{z}_h^{t^-}\left(\boldsymbol{n}^-\cdot\boldsymbol{n}^-\right) - \boldsymbol{n}^-\left(\boldsymbol{n}^-\cdot\boldsymbol{z}_h^{t^-}\right) \\
&= -\boldsymbol{z}_h^{t^+} + \boldsymbol{z}_h^{t^-}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\boldsymbol{u}_h^{t^+}\times\boldsymbol{n}^+ = \left(\boldsymbol{n}^+\times\left(\boldsymbol{u}_h^+\times\boldsymbol{n}^+\right)\right)\times\boldsymbol{n}^+ = \left(\boldsymbol{u}_h^+ - \left(\boldsymbol{n}^+\cdot\boldsymbol{u}_h^+\right)\boldsymbol{n}^+\right)\times\boldsymbol{n}^+ = \boldsymbol{u}_h^+\times\boldsymbol{n}^+, \\
&\boldsymbol{u}_h^{t^-}\times\boldsymbol{n}^- = -\boldsymbol{u}_h^-\times\boldsymbol{n}^-.
\end{aligned}$$

Substituting into (11) and (12), we get

$$\begin{aligned}
\widehat{\boldsymbol{z}}_h^{t^+} &= \boldsymbol{z}_h^{t^+} + \frac{\tau_t}{2}\left(\boldsymbol{u}_h^+ \times \boldsymbol{n}^+ + \boldsymbol{u}_h^- \times \boldsymbol{n}^-\right) - \frac{1}{2}\left(\boldsymbol{z}_h^{t^+} - \boldsymbol{z}_h^{t^-}\right)\\
&= \frac{\boldsymbol{z}_h^{t^+} + \boldsymbol{z}_h^{t^-}}{2} + \frac{\tau_t}{2}\left(\boldsymbol{u}_h^+ \times \boldsymbol{n}^+ + \boldsymbol{u}_h^- \times \boldsymbol{n}^-\right)\\
&= \frac{\boldsymbol{z}_h^{t^+} + \boldsymbol{z}_h^{t^-}}{2} + \frac{\tau_t}{2}[[\boldsymbol{u}_h \times \boldsymbol{n}]]\\
\widehat{\boldsymbol{z}}_h^{t^-} &= \boldsymbol{z}_h^{t^-} + \frac{\tau_t}{2}\left(\boldsymbol{u}_h^- \times \boldsymbol{n}^- + \boldsymbol{u}_h^+ \times \boldsymbol{n}^+\right) - \frac{1}{2}\left(\boldsymbol{z}_h^{t^-} - \boldsymbol{z}_h^{t^+}\right)\\
&= \frac{\boldsymbol{z}_h^{t^-} + \boldsymbol{z}_h^{t^+}}{2} + \frac{\tau_t}{2}[[\boldsymbol{u}_h \times \boldsymbol{n}]].
\end{aligned}$$

Therefore, we have

$$\widehat{\boldsymbol{z}}_h^{t} = \frac{\boldsymbol{z}_h^{t^-} + \boldsymbol{z}_h^{t^+}}{2} + \frac{\tau_t}{2}[[\boldsymbol{u}_h \times \boldsymbol{n}]].$$

Using similar arguments, it can be proved that $\widehat{\varphi}_h$ is also single-valued across inter-element boundaries. Therefore, the numerical fluxes of the HDG scheme are locally conservative.

Now, taking $u \in H(\mathbf{curl};\Omega)$ and $\varphi \in \mathrm{H}^1_\Gamma(\Omega)$, the exact solution of (2), we have that $\boldsymbol{u}^t = \widehat{\boldsymbol{u}}^t$ and $\varphi = \widehat{\varphi}$, so from (9), we get that $\boldsymbol{z}^t = \widehat{\boldsymbol{z}}^t$ in $\mathcal{E}_h^I$ and $\boldsymbol{u}^n = \widehat{\boldsymbol{u}}^n$ in $\mathcal{E}_h^{0,I} \cup \Gamma \cup \Gamma_0$. Thus, substituting in (3) and (4) and integrating by parts, we get that

$$\begin{aligned}
(\mu \boldsymbol{z} - \nabla \times \boldsymbol{u}, \boldsymbol{r})_{\mathcal{T}_h} &= 0 && \forall \boldsymbol{r} \in \mathbb{V}_h,\\
(\nabla \times \boldsymbol{z} + i\omega\sigma \boldsymbol{u} - \epsilon \nabla\varphi|_{\Omega_{\mathrm{D}}}, \boldsymbol{v})_{\mathcal{T}_h} &= (-i\omega \boldsymbol{J}^s, \boldsymbol{v})_{\mathcal{T}_h} && \forall \boldsymbol{v} \in \mathbb{V}_h,\\
(\nabla \cdot (\epsilon \boldsymbol{u}) - \varphi, \rho)_{\mathcal{T}_h^{\mathrm{D}}} &= 0 && \forall \rho \in \mathbb{P}_h^\Gamma.
\end{aligned}$$

Thus, the exact solution satisfies the HDG formulation (3)-(8), so the scheme is also consistent. □

Next result establishes the unique solvability of discrete linear system.

Theorem 3.2. *The solution* $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{P}_h^\Gamma \times \mathbb{M}_h^t \times \mathbb{M}_h^n(0)$ *of problems (3) to (8) exists and is unique.*

Proof. Let us consider the homogeneous problem associate to problems (3) to (8) and take $\boldsymbol{r} := \boldsymbol{z}_h$, $\boldsymbol{v} := \boldsymbol{u}_h$, $\rho := \varphi_h$, $\boldsymbol{\eta} := -\widehat{\boldsymbol{u}}_h^t$, $\boldsymbol{\xi} := \widehat{\boldsymbol{u}}_h^n$ in Equations 3-5, 7, and 8, respectively. We have

$$(\mu \boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla \times \boldsymbol{z}_h)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^t, \boldsymbol{z}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{13}$$

$$(\nabla \times \boldsymbol{u}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + \langle \boldsymbol{u}_h \times \boldsymbol{n}, \widehat{\boldsymbol{z}}_h^t\rangle_{\partial\mathcal{T}_h} - i\omega(\sigma \boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h} + (\nabla \cdot (\epsilon \boldsymbol{u}_h), \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon \boldsymbol{u}_h \cdot \boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0, \tag{14}$$

$$(\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon \boldsymbol{u}_h, \nabla\varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon \widehat{\boldsymbol{u}}_h^n \cdot \boldsymbol{n}, \varphi_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0, \tag{15}$$

$$-\langle \boldsymbol{n} \times \widehat{\boldsymbol{z}}_h^t, \widehat{\boldsymbol{u}}_h^t\rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0, \tag{16}$$

$$\langle \epsilon \widehat{\boldsymbol{u}}_h^n \cdot \boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0. \tag{17}$$

Also from Equation 6, we deduce $\boldsymbol{n} \times \widehat{\boldsymbol{u}}_h^t = \boldsymbol{0}$ on Γ. This fact, together with (16), helps us to ensure that

$$\langle \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{z}}_h^t \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\overline{\langle \boldsymbol{n} \times \widehat{\boldsymbol{z}}_h^t, \widehat{\boldsymbol{u}}_h^t\rangle_{\partial\mathcal{T}_h}} = 0. \tag{18}$$

Adding (13) to, (17) we get

$$(\mu \boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + \left[(\nabla\times\boldsymbol{u}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla\times\boldsymbol{z}_h)_{\mathcal{T}_h}\right] - \langle \widehat{\boldsymbol{u}}_h^t, \boldsymbol{z}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{u}_h\times\boldsymbol{n}, \widehat{\boldsymbol{z}}_h^t\rangle_{\partial\mathcal{T}_h} - i\omega(\sigma\boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h} + (\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}}$$
$$+ \left[(\nabla\cdot(\epsilon\boldsymbol{u}_h), \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{u}_h, \nabla\varphi_h)_{\mathcal{T}_h^{\mathrm{D}}}\right] - \langle \epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n}, \varphi_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\boldsymbol{u}_h\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t, \widehat{\boldsymbol{u}}_h^t\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle \epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0. \tag{19}$$

Then, taking into account (18) and the identities

$$(\nabla\times\boldsymbol{u}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla\times\boldsymbol{z}_h)_{\mathcal{T}_h} = \langle \boldsymbol{u}_h^t, \boldsymbol{z}_h^t\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h},$$
$$(\nabla\cdot(\epsilon\boldsymbol{u}_h), \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{u}_h, \nabla\varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} = \langle \epsilon\boldsymbol{u}_h\cdot\boldsymbol{n}, \varphi_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}},$$
$$\langle \boldsymbol{u}_h\times\boldsymbol{n}, \widehat{\boldsymbol{z}}_h^t\rangle_{\partial\mathcal{T}_h} = -\langle \boldsymbol{u}_h^t, \widehat{\boldsymbol{z}}_h^t\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h},$$

(19) reduces to

$$\left[(\mu\boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + (\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - i\omega(\sigma\boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h}\right] + \langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \left(\boldsymbol{z}_h^t - \widehat{\boldsymbol{z}}_h^t\right)\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \epsilon\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}, \varphi_h - \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0.$$

Now, using the definition of the numerical traces (cf (9)-(10), we obtain

$$\left[(\mu\boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + (\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - i\omega(\sigma\boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h}\right] + \langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right)\rangle_{\partial\mathcal{T}_h} + \langle \epsilon\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}, \tau_n\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}$$
$$+ \lambda|\langle \epsilon\boldsymbol{u}_h^n|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}|^2 - \lambda\langle \epsilon\widehat{\boldsymbol{u}}_h^n|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}\overline{\langle \epsilon\boldsymbol{u}_h^n|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}} = 0,$$

and, since $\widehat{\boldsymbol{u}}_h^n \in \mathbb{M}_h^n(0)$, we deduce

$$\left[(\mu\boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + (\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - i\omega(\sigma\boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h}\right] + \langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \tau_t\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right)\rangle_{\partial\mathcal{T}_h}$$
$$+ \langle \epsilon\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}, \tau_n\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda|\langle \epsilon\boldsymbol{u}_h^n|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}|^2 = 0. \tag{20}$$

Hence, it follows from (20)

$$\boldsymbol{z}_h = 0 \quad \text{in} \quad \mathcal{T}_h, \tag{21}$$

$$\varphi_h = 0 \quad \text{in} \quad \mathcal{T}_h^{\mathrm{D}}, \tag{22}$$

$$\widehat{\boldsymbol{u}}_h^t = \boldsymbol{u}_h^t \quad \text{in} \quad \partial\mathcal{T}_h, \tag{23}$$

$$\epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n} = \epsilon\boldsymbol{u}_h^n\cdot\boldsymbol{n} \quad \text{in} \quad \partial\mathcal{T}_h^{\mathrm{D}}, \tag{24}$$

$$\langle \epsilon\boldsymbol{u}_h^n|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} = 0, \tag{25}$$

$$\boldsymbol{u}_h = 0 \quad \text{in} \quad \mathcal{T}_h^{\mathrm{C}}. \tag{26}$$

Equations 23 and 24 imply that $\boldsymbol{u}_h \in H(\mathbf{curl};\Omega)$ and $\boldsymbol{u}_h|_{\Omega_{\mathrm{D}}} \in \mathrm{H}(\mathrm{div}_\epsilon;\Omega_{\mathrm{D}})$. Equations 3 and 5 and 21 to 24 lead us to conclude that $\nabla\times\boldsymbol{u}_h = \mathbf{0}$ in Ω and $\nabla\cdot(\epsilon\boldsymbol{u}_h) = 0$ en Ω_{D}. Moreover, $\boldsymbol{u}_h|_{\Omega_{\mathrm{D}}} \in \mathrm{H}_{\partial\Omega_{\mathrm{D}}}(\mathbf{curl}^0;\Omega_{\mathrm{D}})$ because $\boldsymbol{u}_h \in H(\mathbf{curl};\Omega)$ and (26) and the fact that $\boldsymbol{n}\times\widehat{\boldsymbol{u}}_h^t = \mathbf{0}$ on Γ together with (23) . Since Γ_0 is connected, then from (25) follows that $\boldsymbol{u}_h|_{\Omega_{\mathrm{D}}}$ is also orthogonal to $\mathrm{H}_{\partial\Omega_{\mathrm{D}}}(\mathbf{curl}^0;\Omega_D)\cap\mathrm{H}(\mathrm{div}_\epsilon^0;\Omega_D)$; hence, it is zero. This let us to conclude the proof. □

4 | HYBRIDIZATION

One of the advantages of the HDG method is the fact that it is enough to solve a suitable and equivalent linear system whose degrees of freedom are defined on the skeleton of the mesh. Then, the global unknowns are recovered in a nonexpensive post process. This is not the exception here, as we describe next.

We notice that Equations 3, 4, and 5 can be written as

$$(\mu \boldsymbol{z}_h, \boldsymbol{r})_K - (\boldsymbol{u}_h, \nabla \times \boldsymbol{r}) = \langle \widehat{\boldsymbol{u}}_h^t, \boldsymbol{r} \times \boldsymbol{n} \rangle_{\partial K}, \tag{27}$$

$$\begin{aligned}(\boldsymbol{z}_h, \nabla \times \boldsymbol{v})_K - \langle \boldsymbol{z}_h^t \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial K} + \langle \tau_t \boldsymbol{u}_h^t, \boldsymbol{v} \rangle_{\partial K} + i\omega(\sigma \boldsymbol{u}_h, \boldsymbol{v})_K \\ - \langle \tilde{\varphi}_h, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K \cap \partial \mathcal{T}_h^{\mathrm{D}}} + \langle \tau_n \boldsymbol{u}_h^n \cdot \boldsymbol{n}, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K \cap \partial \mathcal{T}_h^{\mathrm{D}}} = \langle \tau_n \widehat{\boldsymbol{u}}_h^n \cdot \boldsymbol{n}, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K \cap \partial \mathcal{T}_h^{\mathrm{D}}} + \langle \tau_t \widehat{\boldsymbol{u}}_h^t, \boldsymbol{v} \rangle_{\partial K} + (\boldsymbol{J}, \boldsymbol{v})_K,\end{aligned} \tag{28}$$

$$(\varphi_h, \rho)_{K \cap \mathcal{T}_h^{\mathrm{D}}} + (\epsilon \boldsymbol{u}_h, \nabla \rho)_{K \cap \mathcal{T}_h^{\mathrm{D}}} = \langle \epsilon \widehat{\boldsymbol{u}}_h^n \cdot \boldsymbol{n}, \rho \rangle_{\partial K \cap \partial \mathcal{T}_h^{\mathrm{D}}}, \tag{29}$$

$$\varphi_h|_F = 0 \qquad \forall F = \partial K \cap \Gamma, \tag{30}$$

for all $(\boldsymbol{r}, \boldsymbol{v}, \rho) \in [\mathbb{P}_k(K)]^3 \times [\mathbb{P}_k(K)]^3 \times \mathbb{P}_k(K)$ with $\rho = 0$ on any face $F = \partial K \cap \Gamma$, for any $K \in \mathcal{T}_h$. We emphasize that (30) follows from the fact that $\varphi_h \in \mathbb{P}_h^{\Gamma}$. Therefore, if $(\widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n, \boldsymbol{J})$ is given, we can compute $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h)$ in an element-by-element fashion by solving (27) to (30) for each $K \in \mathcal{T}_h$. This allows us to introduce the local solver $\mathcal{L}$ defined by $\mathcal{L}(\widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n, \boldsymbol{J}) := (\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h)$. Then, given $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbb{M}_h^t \times \mathbb{M}_h^n$, we set

$$\begin{aligned}(\boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{u}_h^{\boldsymbol{\eta}}, \varphi_h^{\boldsymbol{\eta}}) &= \mathcal{L}(\boldsymbol{\eta}, \boldsymbol{0}, \boldsymbol{0}), \\ (\boldsymbol{z}_h^{\boldsymbol{\xi}}, \boldsymbol{u}_h^{\boldsymbol{\xi}}, \varphi_h^{\boldsymbol{\xi}}) &= \mathcal{L}(\boldsymbol{0}, \boldsymbol{\xi}, \boldsymbol{0}), \\ (\boldsymbol{z}_h^{\boldsymbol{J}}, \boldsymbol{u}_h^{\boldsymbol{J}}, \varphi_h^{\boldsymbol{J}}) &= \mathcal{L}(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{J}).\end{aligned} \tag{31}$$

Remark 1. The solvability of (27) to (30) follows similar arguments than the ones applied in the proof of Theorem 3.2.

Now, taking into account the decomposition (31) in (7) to (8), we deduce the following result.

Lemma 4.1. *Let $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n)$ be the solution of the HDG scheme (3)-(8). We have that*

$$\begin{aligned}\boldsymbol{z}_h &= \boldsymbol{z}_h^{\boldsymbol{\alpha}} + \boldsymbol{z}_h^{\boldsymbol{\delta}} + \boldsymbol{z}_h^{\boldsymbol{J}}, \\ \boldsymbol{u}_h &= \boldsymbol{u}_h^{\boldsymbol{\alpha}} + \boldsymbol{u}_h^{\boldsymbol{\delta}} + \boldsymbol{u}_h^{\boldsymbol{J}}, \\ \varphi_h &= \varphi_h^{\boldsymbol{\alpha}} + \varphi_h^{\boldsymbol{\delta}} + \varphi_h^{\boldsymbol{J}}, \\ \widehat{\boldsymbol{u}}_h^t &= \boldsymbol{\alpha}, \quad \widehat{\boldsymbol{u}}_h^n = \boldsymbol{\delta},\end{aligned}$$

where $(\boldsymbol{\alpha}, \boldsymbol{\delta}) \in \mathbb{M}_h^t(\mathbf{g}) \times \mathbb{M}_h^n(0)$ is the solution of

$$\begin{aligned}a_h(\boldsymbol{\alpha}, \boldsymbol{\beta}) + b_h(\boldsymbol{\delta}, \boldsymbol{\beta}) &= l_h(\boldsymbol{\beta}) \qquad \forall \boldsymbol{\beta} \in \mathbb{M}_h^t(\boldsymbol{0}), \\ d_h(\boldsymbol{\alpha}, \boldsymbol{\gamma}) + c_h(\boldsymbol{\delta}, \boldsymbol{\gamma}) &= f_h(\boldsymbol{\gamma}) \qquad \forall \boldsymbol{\gamma} \in \mathbb{M}_h^n(0).\end{aligned} \tag{32}$$

Here, the forms as well as the functionals in (32) are given, for any $\boldsymbol{\eta}, \boldsymbol{\beta} \in \mathbb{M}_h^t$ and $\boldsymbol{\xi}, \boldsymbol{\gamma} \in \mathbb{M}_h^n$, by

$$\begin{aligned}a_h(\boldsymbol{\eta}, \boldsymbol{\beta}) &:= \left\langle \left(\boldsymbol{z}_h^{\boldsymbol{\eta}}\right)^t \times \boldsymbol{n}, \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h} - \left\langle \tau_t \left(\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^t - \boldsymbol{\eta} \right), \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h}, \\ b_h(\boldsymbol{\xi}, \boldsymbol{\beta}) &:= \left\langle \left(\boldsymbol{z}_h^{\boldsymbol{\xi}}\right)^t \times \boldsymbol{n}, \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h} - \left\langle \tau_t \left(\boldsymbol{u}_h^{\boldsymbol{\xi}}\right)^t, \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h}, \\ c_h(\boldsymbol{\xi}, \boldsymbol{\gamma}) &:= \left\langle \varphi_h^{\boldsymbol{\xi}}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D} - \left\langle \tau_n \left(\boldsymbol{u}_h^{\boldsymbol{\xi}} - \boldsymbol{\xi}\right)^n \cdot \boldsymbol{n}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D}, \\ d_h(\boldsymbol{\eta}, \boldsymbol{\gamma}) &:= \left\langle \varphi_h^{\boldsymbol{\eta}}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D} - \left\langle \tau_n \left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^n \cdot \boldsymbol{n}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D}, \\ l_h(\boldsymbol{\beta}) &:= -\left\langle \left(\boldsymbol{z}_h^{\boldsymbol{J}}\right)^t \times \boldsymbol{n}, \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h} + \left\langle \tau_t \left(\boldsymbol{u}_h^{\boldsymbol{J}}\right)^t, \boldsymbol{\beta} \right\rangle_{\partial \mathcal{T}_h}, \\ f_h(\boldsymbol{\gamma}) &:= -\left\langle \varphi_h^{\boldsymbol{J}}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D} + \left\langle \tau_n \left(\boldsymbol{u}_h^{\boldsymbol{J}}\right)^n \cdot \boldsymbol{n}, \epsilon \boldsymbol{\gamma} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^D}.\end{aligned}$$

Proof. Substituting $\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t$ and $\widehat{\boldsymbol{u}}_h^n$ in (7) and (8), respectively, we obtain

$$\begin{aligned}
0 &= \langle \boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t, \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = -\langle \boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t, \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h}\\
&= \langle \boldsymbol{z}_h^t\times\boldsymbol{n} - \tau_t(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t), \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h}\\
&= \langle (\boldsymbol{z}_h^{\boldsymbol{\alpha}} + \boldsymbol{z}_h^{\boldsymbol{\delta}} + \boldsymbol{z}_h^{\boldsymbol{J}})^t\times\boldsymbol{n} - \tau_t((\boldsymbol{u}_h^{\boldsymbol{\alpha}} + \boldsymbol{u}_h^{\boldsymbol{\delta}} + \boldsymbol{u}_h^{\boldsymbol{J}})^t - \boldsymbol{\alpha}), \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h}\\
&= \langle (\boldsymbol{z}_h^{\boldsymbol{\alpha}})^t\times\boldsymbol{n} - \tau_t((\boldsymbol{u}_h^{\boldsymbol{\alpha}})^t - \boldsymbol{\alpha}), \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{z}_h^{\boldsymbol{\delta}})^t\times\boldsymbol{n} - \tau_t(\boldsymbol{u}_h^{\boldsymbol{\delta}})^t, \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{z}_h^{\boldsymbol{J}})^t\times\boldsymbol{n} - \tau_t(\boldsymbol{u}_h^{\boldsymbol{J}})^t, \boldsymbol{\beta}\rangle_{\partial\mathcal{T}_h}\\
0 &= \langle \widehat{\varphi}_h, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&= \langle \tilde{\varphi}_h - \tau_n(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n)\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&= \langle \varphi_h, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \tau_n(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n)\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda\langle \epsilon\boldsymbol{u}_h|_{\Omega_D}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle 1, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}_0\rangle_{\Gamma_0}\\
&= \langle \varphi_h^{\boldsymbol{\alpha}} + \varphi_h^{\boldsymbol{\delta}} + \varphi_h^{\boldsymbol{J}}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \tau_n((\boldsymbol{u}_h^{\boldsymbol{\alpha}} + \boldsymbol{u}_h^{\boldsymbol{\delta}} + \boldsymbol{u}_h^{\boldsymbol{J}})^n - \boldsymbol{\delta})\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&= \langle \varphi_h^{\boldsymbol{\alpha}} - \tau_n(\boldsymbol{u}_h^{\boldsymbol{\alpha}})^n\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle \varphi_h^{\boldsymbol{\delta}} - \tau_n(\boldsymbol{u}_h^{\boldsymbol{\delta}} - \boldsymbol{\delta})\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle \varphi_h^{\boldsymbol{J}} - \tau_n(\boldsymbol{u}_h^{\boldsymbol{J}})^n\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}.
\end{aligned}$$

Here, we have used the fact that $\boldsymbol{\beta}\in\mathbb{M}_h^t(\boldsymbol{0})$, $\boldsymbol{\gamma}\in\mathbb{M}_h^n(0)$ and $\varphi_h = 0$ on Γ_0. □

Next results give us some information on the structure of the associated matrix system to (32).

Lemma 4.2. *For all $\boldsymbol{\eta}, \boldsymbol{\beta}\in\mathbb{M}_h^t$ and $\boldsymbol{\xi}, \boldsymbol{\gamma}\in\mathbb{M}_h^n$ there hold*

$$\begin{aligned}
\left\langle \left(\boldsymbol{z}_h^{\boldsymbol{\eta}}\right)^t\times\boldsymbol{n} - \tau_t\left(\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^t - \boldsymbol{\eta}\right), \boldsymbol{\beta}\right\rangle_{\partial\mathcal{T}_h} &= \left(\mu\boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{z}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + \left(\varphi_h^{\boldsymbol{\eta}}, \varphi_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + \lambda\left\langle\epsilon\boldsymbol{u}_h^{\boldsymbol{\eta}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\overline{\left\langle\epsilon\boldsymbol{u}_h^{\boldsymbol{\beta}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}}\\
&\quad + i\omega\left(\sigma\boldsymbol{u}_h^{\boldsymbol{\eta}}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + \left\langle\tau_t\left(\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^t - \boldsymbol{\eta}\right), \left(\boldsymbol{u}_h^{\boldsymbol{\beta}}\right)^t - \boldsymbol{\beta}\right\rangle_{\partial\mathcal{T}_h}\\
&\quad + \left\langle\tau_n\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^n\cdot\boldsymbol{n}, \epsilon\left(\boldsymbol{u}_h^{\boldsymbol{\beta}}\right)^n\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D},
\end{aligned}\tag{33}$$

$$\left\langle\left(\boldsymbol{z}_h^{\boldsymbol{\xi}}\right)^t\times\boldsymbol{n} - \tau_t\left(\boldsymbol{u}_h^{\boldsymbol{\xi}}\right)^t, \boldsymbol{\beta}\right\rangle_{\partial\mathcal{T}_h} = -\left\langle\tau_n\boldsymbol{\xi}\cdot\boldsymbol{n}, \epsilon\boldsymbol{u}_h^{\boldsymbol{\beta}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D} + \left\langle\epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}, \varphi_h^{\boldsymbol{\beta}}\right\rangle_{\partial\mathcal{T}_h^D},\tag{34}$$

$$-\left\langle\left(\boldsymbol{z}_h^{\boldsymbol{J}}\right)^t\times\boldsymbol{n} - \tau_t\left(\boldsymbol{u}_h^{\boldsymbol{J}}\right)^t, \boldsymbol{\beta}\right\rangle_{\partial\mathcal{T}_h} = \left(\boldsymbol{J}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} - 2i\omega\left(\sigma\boldsymbol{u}_h^{\boldsymbol{J}}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h},\tag{35}$$

$$\begin{aligned}
\left\langle\varphi_h^{\boldsymbol{\xi}} - \tau_n\left(\boldsymbol{u}_h^{\boldsymbol{\xi}} - \boldsymbol{\xi}\right)\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D} &= \left(\mu\boldsymbol{z}_h^{\boldsymbol{\xi}}, \boldsymbol{z}_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D} + \left(\varphi_h^{\boldsymbol{\xi}}, \varphi_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D} + \lambda\left\langle\epsilon\boldsymbol{u}_h^{\boldsymbol{\xi}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\overline{\left\langle\epsilon\boldsymbol{u}_h^{\boldsymbol{\gamma}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}}\\
&\quad + \left\langle\tau_t\left(\boldsymbol{u}_h^{\boldsymbol{\xi}}\right)^t, \left(\boldsymbol{u}_h^{\boldsymbol{\gamma}}\right)^t\right\rangle_{\partial\mathcal{T}_h^D} + \left\langle\tau_n\left(\boldsymbol{u}_h^{\boldsymbol{\xi}} - \boldsymbol{\xi}\right)\cdot n, \epsilon\left(\boldsymbol{u}_h^{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\right)\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D},\\
-\left\langle\varphi_h^{\boldsymbol{J}} - \tau_n\left(\boldsymbol{u}_h^{\boldsymbol{J}}\right)^n\cdot\boldsymbol{n}, \epsilon\boldsymbol{\gamma}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D} &= \left(\boldsymbol{J}, \boldsymbol{u}_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D}.
\end{aligned}\tag{36}$$

Proof. The strategy here is very similar to the one applied in Nguyen et al.[9] To make the paper self-contained, we start proving (33). First, we write down the local system that defines $\mathcal{L}(\boldsymbol{\eta}, \boldsymbol{0}, \boldsymbol{0})$:

$$(\mu\boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{r})_K - (\boldsymbol{u}_h^{\boldsymbol{\eta}}, \nabla\times\boldsymbol{r})_K = \langle\boldsymbol{\eta}, \boldsymbol{r}\times\boldsymbol{n}\rangle_{\partial K},\tag{37}$$

$$\begin{aligned}
&(\boldsymbol{z}_h^{\boldsymbol{\eta}}, \nabla\times\boldsymbol{v})_K - \left\langle\left(\boldsymbol{z}_h^{\boldsymbol{\eta}}\right)^t\times\boldsymbol{n}, \boldsymbol{v}\right\rangle_{\partial K} + \left\langle\tau_t\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^t, \boldsymbol{v}\right\rangle_{\partial K} + i\omega(\sigma\boldsymbol{u}_h^{\boldsymbol{\eta}}, \boldsymbol{v})_K\\
&+ (\varphi_h^{\boldsymbol{\eta}}, \nabla\cdot(\epsilon\boldsymbol{v}))_{K\cap\mathcal{T}_h^{\mathrm{D}}} - \left\langle\tilde{\varphi}_h^{\boldsymbol{\eta}} - \tau_n\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^n\cdot\boldsymbol{n}, \epsilon\boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial K\cap\partial\mathcal{T}_h^{\mathrm{D}}} = \langle\tau_t\boldsymbol{\eta}, \boldsymbol{v}\rangle_{\partial K},
\end{aligned}\tag{38}$$

$$(\varphi_h^{\boldsymbol{\eta}}, \rho)_{K\cap\mathcal{T}_h^{\mathrm{D}}} + (\epsilon \boldsymbol{u}_h^{\boldsymbol{\eta}}, \nabla\rho)_{K\cap\mathcal{T}_h^{\mathrm{D}}} = 0, \tag{39}$$

$$\varphi_h^{\boldsymbol{\eta}}|_F = 0 \qquad \forall F = \partial K \cap \Gamma. \tag{40}$$

Now, considering $\boldsymbol{\beta}$ instead of $\boldsymbol{\eta}$ in (37), then testing it with $\boldsymbol{r} := \boldsymbol{z}_h^{\boldsymbol{\eta}}$, we find that its conjugate is written as

$$\left(\mu \boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{z}_h^{\boldsymbol{\beta}}\right)_K - \left(\nabla\times\boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_K = \left\langle \boldsymbol{z}_h^{\boldsymbol{\eta}}\times\boldsymbol{n}, \boldsymbol{\beta}\right\rangle_{\partial K}. \tag{41}$$

Similarly, after replacing $\boldsymbol{\beta}$ instead of $\boldsymbol{\eta}$ in (39), taking $\rho := \varphi_h^{\boldsymbol{\eta}}$ and conjugating the resulting expression, we have

$$\left(\varphi_h^{\boldsymbol{\eta}}, \varphi_h^{\boldsymbol{\beta}}\right)_{K\cap\mathcal{T}_h^{\mathrm{D}}} + \left(\nabla\varphi_h^{\boldsymbol{\eta}}, \epsilon \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{K\cap\mathcal{T}_h^{\mathrm{D}}} = 0. \tag{42}$$

Next, we evaluate (38) for $\boldsymbol{v} := \boldsymbol{u}_h^{\boldsymbol{\beta}}$, and add Equations 41 and 42. After replacing $\tilde{\varphi}_h^{\boldsymbol{\eta}}$ (defined as in (10) but with the superscript $\boldsymbol{\eta}$) and suitable simplifications, we establish (33). The rest of the identities (34)-(**??**) are proved in analogous way. □

As a consequence of Lemmata 4.1 and 4.2, we conclude the following result that gives some knowledge on the structure of system (32).

Lemma 4.3. *Let $(\boldsymbol{\alpha}, \boldsymbol{\delta}) \in \mathbb{M}_h^t(\mathbf{g}) \times \mathbb{M}_h^n(0)$, be the solution of (32). It satisfies the following system:*

$$\begin{aligned} a_h(\boldsymbol{\alpha}, \boldsymbol{\beta}) + b_h(\boldsymbol{\delta}, \boldsymbol{\beta}) &= l_h(\boldsymbol{\beta}) && \forall \boldsymbol{\beta} \in \mathbb{M}_h^t(\mathbf{0}), \\ \overline{b_h(\boldsymbol{\gamma}, \boldsymbol{\alpha})} + c_h(\boldsymbol{\delta}, \boldsymbol{\gamma}) &= f_h(\boldsymbol{\gamma}) && \forall \boldsymbol{\gamma} \in \mathbb{M}_h^n(0), \end{aligned} \tag{43}$$

where the forms and functionals are given, for all $\boldsymbol{\eta}$, $\boldsymbol{\beta} \in \mathbb{M}_h^t$ and $\boldsymbol{\xi}$, $\boldsymbol{\gamma} \in \mathbb{M}_h^n$, by

$$\begin{aligned} a_h(\boldsymbol{\eta}, \boldsymbol{\beta}) &:= \left(\mu \boldsymbol{z}_h^{\boldsymbol{\eta}}, \boldsymbol{z}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + \left(\varphi_h^{\boldsymbol{\eta}}, \varphi_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + i\omega\left(\sigma \boldsymbol{u}_h^{\boldsymbol{\eta}}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} + \left\langle \tau_n \left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^n \cdot \boldsymbol{n}, \epsilon\left(\boldsymbol{u}_h^{\boldsymbol{\beta}}\right)^n \cdot \boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D} \\ &\quad + \lambda\left\langle \epsilon \boldsymbol{u}_h^{\boldsymbol{\eta}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\right\rangle_{\Gamma_0} \overline{\left\langle \epsilon \boldsymbol{u}_h^{\boldsymbol{\beta}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}} + \left\langle \tau_t\left(\left(\boldsymbol{u}_h^{\boldsymbol{\eta}}\right)^t - \boldsymbol{\eta}\right), \left(\boldsymbol{u}_h^{\boldsymbol{\beta}}\right)^t - \boldsymbol{\beta}\right\rangle_{\partial\mathcal{T}_h}, \\ b_h(\boldsymbol{\xi}, \boldsymbol{\beta}) &:= -\left\langle \tau_n \boldsymbol{\xi}\cdot\boldsymbol{n}, \epsilon \boldsymbol{u}_h^{\boldsymbol{\beta}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D} + \left\langle \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}, \varphi_h^{\boldsymbol{\beta}}\right\rangle_{\partial\mathcal{T}_h^D}, \\ c_h(\boldsymbol{\xi}, \boldsymbol{\gamma}) &:= \left(\mu \boldsymbol{z}_h^{\boldsymbol{\xi}}, \boldsymbol{z}_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D} + \left(\varphi_h^{\boldsymbol{\xi}}, \varphi_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D} + \left\langle \tau_t\left(\boldsymbol{u}_h^{\boldsymbol{\xi}}\right)^t, \left(\boldsymbol{u}_h^{\boldsymbol{\gamma}}\right)^t\right\rangle_{\partial\mathcal{T}_h^D}, \\ l_h(\boldsymbol{\beta}) &:= \left(\boldsymbol{J}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h} - 2i\omega\left(\sigma \boldsymbol{u}_h^{\boldsymbol{J}}, \boldsymbol{u}_h^{\boldsymbol{\beta}}\right)_{\mathcal{T}_h}, \\ f_h(\boldsymbol{\gamma}) &:= \left(\boldsymbol{J}, \boldsymbol{u}_h^{\boldsymbol{\gamma}}\right)_{\mathcal{T}_h^D}. \end{aligned}$$

Remark 2. We notice that the complex-valued matrix of discrete system (43) is symmetric but not hermitian. This is due to the presence of a term that contains the imaginary unit i as a factor in the sesquilinear form a_h. We point out that the unknowns in this system are defined only on the skeleton of $\mathcal{T}_h$, and then it should be less expensive to solve than the original HDG scheme (3)-(8). Once (43) is solved, we recover the global unknowns by solving (31), which again represents an inexpensive post-process.

5 | A PRIORI ERROR ANALYSIS

5.1 | Projection operators

We begin this section by introducing the projector operators for $\boldsymbol{z}$, $\boldsymbol{u}$ and φ, that help us to obtain a priori error estimates of the method. This definition will depend on whether the element is taken from $\mathcal{T}_h^{\mathrm{C}}$ or $\mathcal{T}_h^{\mathrm{D}}$. Moreover, when we are on $\mathcal{T}_h^{\mathrm{D}}$, we distinguish the operators defined on $K \in \mathcal{T}_h^{\mathrm{D}}$ that have at least one face lying on Γ and that have none. It is important to remark that the operators we consider here have been presented and studied in Cockburn et al[13] and Cockburn and Sayas.[14]

We start by introducing $P_{\mathbb{V}}\boldsymbol{z} \in \mathbb{V}_h$ as the standard piecewise orthogonal L^2-projection of $\boldsymbol{z}$ onto $\mathbb{V}_h$. This means that on each $K \in \mathcal{T}_h$:

$$(P_{\mathbb{V}}\boldsymbol{z}, \boldsymbol{r})_K = (\boldsymbol{z}, \boldsymbol{r})_K \quad \forall \boldsymbol{r} \in [\mathbb{P}_k(K)]^3. \tag{44}$$

Now, since φ is defined only on the dielectric domain, we set $\Pi_{\mathbb{V}}\boldsymbol{u} \in \mathbb{V}_h$ in the conductor domain as in Cockburn and Sayas[14], section 6:

$$\forall K \in \mathcal{T}_h^{\mathrm{C}} \ : \begin{cases} (\Pi_{\mathbb{V}}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K & \forall \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^3, \\ \langle \Pi_{\mathbb{V}}\boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = \langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} & \forall \boldsymbol{v} \in [\mathbb{P}_k^{\perp}(K)]^3, \end{cases} \tag{45}$$

where $\mathbb{P}_k^{\perp}(K) := \{p \in \mathbb{P}_k(K) \ : \ (p,q)_K = 0 \quad \forall q \in \mathbb{P}_{k-1}(K)\}$.

Concerning the dielectric region, we first propose a suitable decomposition of $\mathcal{T}_h^{\mathrm{D}}$. To this aim, we set $A^{\mathrm{D}} := \left\{K \in \mathcal{T}_h^{\mathrm{D}} \ : \ |\partial K \cap \Gamma|_{\mathbb{R}^{d-1}} > 0\right\}$, and $B^{\mathrm{D}} := \mathcal{T}_h^{\mathrm{D}} \backslash A^{\mathrm{D}}$. Then, we define $(\Pi_{\mathbb{V}}\boldsymbol{u}, \Pi_{\mathbb{P}}\varphi)$ as follows:

$$\forall K \in B^{\mathrm{D}} \ : \begin{cases} (\Pi_{\mathbb{V}}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K & \forall \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^3, \\ (\Pi_{\mathbb{P}}\varphi, \psi)_K = (\varphi, \psi)_K & \forall \psi \in \mathbb{P}_{k-1}(K), \\ \langle \Pi_{\mathbb{P}}\varphi - \tau_n \Pi_{\mathbb{V}}\boldsymbol{u} \cdot \boldsymbol{n}, \eta \rangle_F = \langle \varphi - \tau_n \boldsymbol{u} \cdot \boldsymbol{n}, \eta \rangle_F & \forall \eta \in \mathbb{P}_k(F) \quad \forall F \in \partial K, \end{cases} \tag{46}$$

and

$$\forall K \in A^{\mathrm{D}} \ : \begin{cases} (\Pi_{\mathbb{V}}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K & \forall \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^3, \\ \langle \tau_n \Pi_{\mathbb{V}}\boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = \langle \tau_n \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} & \forall \boldsymbol{v} \in [\mathbb{P}_k^{\perp}(K)]^3, \\ (\Pi_{\mathbb{P}}\varphi, \psi)_K = (\varphi, \psi)_K & \forall \psi \in \mathbb{P}_{k-1}(K), \\ \langle \Pi_{\mathbb{P}}\varphi, \eta \rangle_{F_0} = \langle \varphi, \eta \rangle_{F_0} & \forall \eta \in \mathbb{P}_k(F_0), \, F_0 \subseteq \partial K \cap \Gamma. \end{cases} \tag{47}$$

We point out that since $\varphi = 0$ on Γ, its projection $\Pi_{\mathbb{P}}\varphi$ also vanishes on Γ. We notice that in B^{D}, we define $(\Pi_{\mathbb{V}}\boldsymbol{u}, \Pi_{\mathbb{P}}\varphi)$ as in Cockburn et al,[13] while in A^{D}, we follow the description given in Cockburn and Sayas[14], section 6 to define $(\Pi_{\mathbb{V}}\boldsymbol{u}, \Pi_{\mathbb{P}}\varphi)$. In what follows, we resume the approximation properties of the introduced projector operators, according to the region where they are defined.

Lemma 5.1. *Considering $\Pi_{\mathbb{V}}\boldsymbol{u}$ given by (45), and $l_{\boldsymbol{u}} \in [0,k]$, then for any $K \in \mathcal{T}_h^{\mathrm{C}}$ there hold*

(a) $\|(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,
(b) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_K \lesssim h_K^{l_{\boldsymbol{u}}+1} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,
(c) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$.

Proof. They are consequences in Cockburn and Sayas.[14], propositions 6.2 and 6.3 □

Lemma 5.2. *Assume that $\tau_n|_{\partial K} > 0$, for any $K \in A^D$. Then, for $l_{\boldsymbol{u}}, l_\varphi \in [0,k]$, the operators $\Pi_{\mathbb{V}}$ and $\Pi_{\mathbb{P}}$ defined in (47), satisfy*

(a) $\|(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,
(b) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,
(c) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_K \lesssim h_K^{l_{\boldsymbol{u}}+1} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,
(d) $\|\Pi_{\mathbb{P}}\varphi - \varphi\|_{\partial K} \lesssim h_K^{l_\varphi+1/2} |\varphi|_{l_\varphi+1,K} \quad \forall \varphi \in \mathrm{H}^{l_\varphi+1}(K)$,
(e) $\|\Pi_{\mathbb{P}}\varphi - \varphi\|_K \lesssim h_K^{l_\varphi+1} |\varphi|_{l_\varphi+1,K} \quad \forall \varphi \in \mathrm{H}^{l_\varphi+1}(K)$.

Proof. (a) to (c) are deduced in Cockburn and Sayas,[14], propositions 6.2 and 6.3 while for (d) to (e), we refer to the proof of Cockburn et al.[15], proposition 2.1-(vii) □

Lemma 5.3. *Suppose that $\tau_n|_{\partial K} > 0$, for any $K \in B^D$. Then, for $l_{\boldsymbol{u}}, l_\varphi \in [0, k]$, the operators $\Pi_{\mathbb{V}}$ and $\Pi_{\mathbb{P}}$ defined in (46), satisfy*

(a) $\|\Pi_{\mathbb{P}}\varphi - \varphi\|_K \lesssim h_K^{l_\varphi+1} |\varphi|_{l_\varphi+1,K} \quad \forall \varphi \in \mathrm{H}^{l_\varphi+1}(K)$,

(b) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_K \lesssim h_K^{l_{\boldsymbol{u}}+1} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} + \frac{h_K^{l_\varphi+1}}{(\tau_n)_K^*} |\varphi|_{l_\varphi+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$,

(c) $\|\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} |\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,K} \quad \forall \boldsymbol{u} \in [\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3$.

Here, $(\tau_n)_K^* := \min \tau_n|_{\partial K}$.

Proof. These results correspond to Cockburn and Cui[16], theorem 2.1 (see also Cockburn et al[13], theorem 2.1). □

Lemma 5.4. *Given $l_{\boldsymbol{z}} \in [0, k]$, there hold on any $K \in \mathcal{T}_h$*

(a) $\|P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\|_K \lesssim h_K^{l_{\boldsymbol{z}}+1} |\boldsymbol{z}|_{l_{\boldsymbol{z}}+1,K} \quad \forall \boldsymbol{z} \in [\mathrm{H}^{l_{\boldsymbol{z}}}(\mathcal{T}_h)]^3$,

(b) $\|P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\|_{\partial K} \lesssim h_K^{l_{\boldsymbol{z}}+1/2} |\boldsymbol{z}|_{l_{\boldsymbol{z}}+1,K} \quad \forall \boldsymbol{z} \in [\mathrm{H}^{l_{\boldsymbol{z}}}(\mathcal{T}_h)]^3$.

Proof. We refer to Ciarlet.[17] □

5.2 | Error estimates

The idea here is to bound the errors involving the discrete solution and its corresponding projection. Then, the a priori error estimate is derived by applying triangle inequality. To this end, we first introduce the following notations, which we call the *projection errors*:

$$
\begin{aligned}
\boldsymbol{e}_h^{\varphi} &:= \Pi_{\mathbb{P}}\varphi - \varphi_h, \qquad & \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} &:= P_{\mathbb{M}_h^t}\boldsymbol{u}^t - \widehat{\boldsymbol{u}}_h^t, \qquad & \boldsymbol{e}_h^{\boldsymbol{z}} &:= P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}_h, \\
\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} &:= P_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n - \widehat{\boldsymbol{u}}_h^n, \qquad & \boldsymbol{e}_h^{\boldsymbol{u}} &:= \Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}_h,
\end{aligned}
$$

where $P_{\mathbb{M}_h^t}$ and $P_{\mathbb{M}_h^n(0)}$ denote the L^2-projection onto $\mathbb{M}_h^t$ and $\mathbb{M}_h^n(0)$, respectively. Now, we relate these projection errors in the next result.

Lemma 5.5. *Let $(\boldsymbol{z}, \boldsymbol{u}, \varphi)$ and $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n)$ be the solutions of (2) and (3) to (8), respectively. Then, the projection errors $\boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{e}_h^{\boldsymbol{u}}, \boldsymbol{e}_h^{\varphi}, \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}, \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}$ satisfy*

$$
(\mu \boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{r})_{\mathcal{T}_h} - (\boldsymbol{e}_h^{\boldsymbol{u}}, \nabla \times \boldsymbol{r})_{\mathcal{T}_h} - \left\langle \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{r} \times \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h} = 0, \tag{48}
$$

$$
\begin{aligned}
&\left(\boldsymbol{e}_h^{\boldsymbol{z}}, \nabla \times \boldsymbol{v}\right)_{\mathcal{T}_h} + \left\langle \boldsymbol{n} \times \left(\boldsymbol{e}_h^{\boldsymbol{z}}\right)^t + \tau_t \left(\left(\boldsymbol{e}_h^{\boldsymbol{u}}\right)^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} \right), \boldsymbol{v} \right\rangle_{\partial \mathcal{T}_h} + (\boldsymbol{e}_h^{\varphi}, \nabla \cdot (\epsilon \boldsymbol{v}))_{\mathcal{T}_h^{\mathrm{D}}} - \left\langle \boldsymbol{e}_h^{\varphi} + \tau_n \left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \left(\boldsymbol{e}_h^{\boldsymbol{u}}\right)^n \right) \cdot \boldsymbol{n}, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&+ i\omega(\sigma \boldsymbol{e}_h^{\boldsymbol{u}}, \boldsymbol{v})_{\mathcal{T}_h} + \lambda \left\langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \right\rangle_{\Gamma_0} \overline{\left\langle \epsilon \boldsymbol{v}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \right\rangle_{\Gamma_0}} = \left\langle \tau_t \left((\Pi_{\mathbb{V}}\boldsymbol{u})^t - P_{\mathbb{M}_h^t}\boldsymbol{u}^t \right), \boldsymbol{v} \right\rangle_{\partial \mathcal{T}_h} + i\omega(\sigma (\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v})_{\mathcal{T}_h} \\
&- \sum_{K \in A^{\mathrm{D}}} \langle \Pi_{\mathbb{P}}\varphi - \varphi, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} - \sum_{K \in B^{\mathrm{D}}} \left\langle \tau_n \left((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \boldsymbol{u}^n \right) \cdot \boldsymbol{n}, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial K} + \left\langle \tau_n \left((\Pi_{\mathbb{V}}\boldsymbol{u})^n - P_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n \right) \cdot \boldsymbol{n}, \epsilon \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&+ \lambda \left\langle \epsilon (\Pi_{\mathbb{V}}\boldsymbol{u})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \right\rangle_{\Gamma_0} \overline{\left\langle \epsilon \boldsymbol{v}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \right\rangle_{\Gamma_0}} + \langle \boldsymbol{n} \times (P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}), \boldsymbol{v} \rangle_{\partial \mathcal{T}_h}
\end{aligned} \tag{49}
$$

$$
\left(\boldsymbol{e}_h^{\varphi}, \rho\right)_{\mathcal{T}_h^D} + \left(\epsilon \boldsymbol{e}_h^{\boldsymbol{u}}, \nabla \rho\right)_{\mathcal{T}_h^D} - \left\langle \epsilon \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}, \rho \right\rangle_{\partial \mathcal{T}_h^D} = (\Pi_{\mathbb{P}}\varphi - \varphi, \rho)_{\mathcal{T}_h^D}, \tag{50}
$$

$$\left\langle \boldsymbol{n}\times \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{\eta}\right\rangle_{\Gamma} = 0, \tag{51}$$

$$\left\langle -\boldsymbol{n}\times\left(\boldsymbol{e}_h^{\boldsymbol{z}}\right)^t + \tau_t\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - \left(\boldsymbol{e}_h^{\boldsymbol{u}}\right)^t\right), \boldsymbol{\eta}\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = \left\langle -\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z}) + \tau_t(\boldsymbol{u}-\Pi_{\mathbb{V}}\boldsymbol{u})^t, \boldsymbol{\eta}\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma}, \tag{52}$$

$$\begin{aligned}
&\left\langle e_h^{\varphi} + \tau_n\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \boldsymbol{e}_h^{\boldsymbol{u}}\right)\cdot\boldsymbol{n}, \epsilon\xi\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\overline{\langle \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}}\\
&= -\lambda\langle \epsilon\left(\Pi_{\mathbb{V}}\boldsymbol{u}\right)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}\overline{\langle \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}} - \left\langle \tau_n\left(\Pi_{\mathbb{V}}\boldsymbol{u} - P_{\mathbb{M}_h^n(0)}\boldsymbol{u}\right)\cdot\boldsymbol{n}, \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^D}\\
&\quad + \langle \Pi_{\mathbb{P}}\varphi - \varphi, \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^D} + \langle \varphi, \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^D},
\end{aligned} \tag{53}$$

for all $(\boldsymbol{r}, \boldsymbol{v}, \rho, \boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbb{V}_h\times\mathbb{V}_h\times\mathbb{P}_h^{\Gamma}\times\mathbb{M}_h^t\times\mathbb{M}_h^n(0)$.

Proof. First, we derive from (3):

$$\left(\mu\left(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{e}_h^{\boldsymbol{z}}\right), \boldsymbol{r}\right)_{\mathcal{T}_h} - (\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{e}_h^{\boldsymbol{u}}, \nabla\times\boldsymbol{r})_{\mathcal{T}_h} - \left\langle P_{\mathbb{M}_h^t}\boldsymbol{u}_h^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{r}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} = 0.$$

Then, we have

$$\begin{aligned}
(\mu\boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{r})_{\mathcal{T}_h} - (\boldsymbol{e}_h^{\boldsymbol{u}}, \nabla\times\boldsymbol{r})_{\mathcal{T}_h} - \left\langle \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{r}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h} &= (\mu P_{\mathbb{V}}\boldsymbol{z}, \boldsymbol{r})_{\mathcal{T}_h} - (\Pi_{\mathbb{V}}\boldsymbol{u}, \nabla\times\boldsymbol{r})_{\mathcal{T}_h} - \left\langle P_{\mathbb{M}_h^t}\boldsymbol{u}_h^t, \boldsymbol{r}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h}\\
&= \left(\mu\left(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\right), \boldsymbol{r}\right)_{\mathcal{T}_h} + \left[(\mu\boldsymbol{z}, \boldsymbol{r})_{\mathcal{T}_h} - (\boldsymbol{u}, \nabla\times\boldsymbol{r})_{\mathcal{T}_h} - \left\langle \boldsymbol{u}^t, \boldsymbol{r}\times\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h}\right],
\end{aligned}$$

and (48) is deduced, after applying the consistency of the scheme and the fact that $P_{\mathbb{V}}\boldsymbol{z}$ is the L^2-projection of $\boldsymbol{z}$ onto $\mathbb{V}_h$. (49) to (53) are established by similar arguments. We omit further details. □

Next result establishes an important inequality that helps us to conclude on the convergence of the method.

Theorem 5.6. *The projection errors* $\boldsymbol{e}_h^{\boldsymbol{z}}$, $\boldsymbol{e}_h^{\boldsymbol{u}}$, e_h^{φ}, $\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}$, *and* $\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}$ *satisfy*

$$\begin{aligned}
&\left\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\right\|_{\mathcal{T}_h}^2 + \omega\left\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\right\|_{\mathcal{T}_h}^2 + \left\|e_h^{\varphi}\right\|_{\mathcal{T}_h^D}^2 + \lambda\left|\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\right|^2 + \left\|(\tau_n\epsilon)^{1/2}\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\right)\cdot\boldsymbol{n}\right\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 + \left\|\tau_t^{1/2}\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right)^t\right\|_{\partial\mathcal{T}_h}^2\\
&\lesssim \omega\left\|\sigma^{1/2}\left(\Pi_{\boldsymbol{V}}\boldsymbol{u} - \boldsymbol{u}\right)\right\|_{\mathcal{T}_h}^2 + \|\Pi_{\mathbb{P}}\varphi - \varphi\|_{\mathcal{T}_h^D}^2 + \left\|\tau_t^{1/2}\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right)\right\|_{\partial\mathcal{T}_h}^2 + \left\|(\epsilon\tau_n)^{1/2}\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right)\cdot\boldsymbol{n}\right\|_{\partial A^{\mathrm{D}}}^2\\
&\quad + \left\|\left(\epsilon\tau_n^{-1}\right)^{1/2}\left(\Pi_{\mathbb{P}}\varphi - \varphi\right)\right\|_{\partial A^D}^2 + \lambda\left|\left\langle \epsilon\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}|_{\Omega_{\mathrm{D}}}\right)\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\right|^2 + \left\|\tau_t^{-1/2}\boldsymbol{n}\times\left(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\right)\right\|_{\partial\mathcal{T}_h}^2.
\end{aligned}$$

Proof. After reducing $\overline{(49)}+(50)+\overline{(51)}$ (where the bar denotes conjugate), for $\boldsymbol{r} := \boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{v} := \boldsymbol{e}_h^{\boldsymbol{u}}$, and $\rho := e_h^{\varphi}$, we obtain

$$\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|_{\mathcal{T}_h}^2 + \left\langle \boldsymbol{n}\times\boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right)^t, \boldsymbol{e}_h^{\boldsymbol{u}}\right\rangle_{\partial\mathcal{T}_h} + i\omega\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}^2 + \|e_h^{\varphi}\|_{\mathcal{T}_h^{\mathrm{D}}}^2 - \left\langle e_h^{\varphi}, \epsilon\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad - \left\langle \tau_n\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \boldsymbol{e}_h^{\boldsymbol{u}}\right)\cdot\boldsymbol{n}, \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\left|\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\right|^2\\
&= i\omega\left(\sigma\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right), \boldsymbol{e}_h^{\boldsymbol{u}}\right)_{\mathcal{T}_h} + \left(e_h^{\varphi}, \Pi_{\mathbb{P}}\varphi - \varphi\right)_{\mathcal{T}_h^{\mathrm{D}}} + \left\langle \boldsymbol{n}\times\left(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\right), \boldsymbol{e}_h^{\boldsymbol{u}}\right\rangle_{\partial\mathcal{T}_h} + \left\langle \tau_t\left((\Pi_{\mathbb{V}}\boldsymbol{u})^t - P_{\mathbb{M}_h^t}\boldsymbol{u}^t\right), \boldsymbol{e}_h^{\boldsymbol{u}}\right\rangle_{\partial\mathcal{T}_h}\\
&\quad - \sum_{K\in A^{\mathrm{D}}}\left\langle \Pi_{\mathbb{P}}\varphi - \varphi, \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n}\right\rangle_{\partial K} - \sum_{K\in B^{\mathrm{D}}}\left\langle \tau_n\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right)\cdot\boldsymbol{n}, \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n}\right\rangle_{\partial K} + \left\langle \tau_n\left(\Pi_{\mathbb{V}}\boldsymbol{u} - P_{\mathbb{M}_h^n(0)}\boldsymbol{u}\right)\cdot\boldsymbol{n}, \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad + \lambda\left\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\overline{\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}}.
\end{aligned} \tag{54}$$

We notice that (51) implies $\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} = \boldsymbol{0}$ on Γ(take for example $\boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}$). Then, taking $\boldsymbol{\eta} := \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}$ and $\xi := \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}$ in (52) and (53) respectively, and summing the resulting relations with (54), we have (after some simplifications)

$$\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|_{\mathcal{T}_h}^2 + \left\langle \tau_t\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right)^t, \left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right)^t \right\rangle_{\partial\mathcal{T}_h} + i\omega\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\|_{\mathcal{T}_h}^2 + \|\boldsymbol{e}_h^{\varphi}\|_{\mathcal{T}_h^{\mathrm{D}}}^2 + \left\langle \tau_n\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \boldsymbol{e}_h^{\boldsymbol{u}}\right)\cdot\boldsymbol{n}, \epsilon\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \boldsymbol{e}_h^{\boldsymbol{u}}\right)\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&+ \lambda\left|\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\right|^2 = i\omega\left(\sigma\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right), \boldsymbol{e}_h^{\boldsymbol{u}}\right)_{\mathcal{T}_h} + \left(\boldsymbol{e}_h^{\varphi}, \Pi_{\mathbb{P}}\varphi - \varphi\right)_{\mathcal{T}_h^{\mathrm{D}}} + \left\langle \tau_t\left(\Pi_{\mathbb{V}}\boldsymbol{u} - P_{\mathbb{M}_h^t}\boldsymbol{u}\right)^t, \boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} \right\rangle_{\partial\mathcal{T}_h} \\
&\qquad + \left\langle \tau_n\left(\Pi_{\mathbb{V}}\boldsymbol{u} - P_{\mathbb{M}_h^n(0)}\boldsymbol{u}\right)\cdot\boldsymbol{n}, \epsilon(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n})\cdot\boldsymbol{n} \right\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \left\langle \Pi_{\mathbb{P}}\varphi - \varphi, \epsilon\left(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - \boldsymbol{e}_h^{\boldsymbol{u}}\right)\cdot\boldsymbol{n} \right\rangle_{\partial A^{\mathrm{D}}} \\
&\qquad - \left\langle \tau_n\left(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}\right)\cdot\boldsymbol{n}, \epsilon(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n})\cdot\boldsymbol{n} \right\rangle_{\partial B^{\mathrm{D}}} + \left\langle \boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}), \boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} \right\rangle_{\partial\mathcal{T}_h} \\
&\qquad + \lambda\left\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\overline{\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}}.
\end{aligned}$$

Next, taking into account the property $\frac{1}{2}(|\mathrm{Re}(w)| + |\mathrm{Im}(w)|) \leq |w|, \forall w \in \mathbb{C}$, and a discrete version of Cauchy-Schwarz inequality, the estimate of the theorem is obtained. We omit further details. □

As by product, we derive the following result.

Theorem 5.7. *Assuming that the exact solution of (2), $(\boldsymbol{z}, \boldsymbol{u}, \varphi) \in [H^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3 \times [H^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3 \times H^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}})$, $l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi \in [0, k]$, there holds*

$$\begin{aligned}
&\left\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\right\|_{\mathcal{T}_h}^2 + \omega\left\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\right\|_{\mathcal{T}_h}^2 + \left\|\boldsymbol{e}_h^{\varphi}\right\|_{\mathcal{T}_h^{D}}^2 + \lambda\left|\left\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}^2\cdot\boldsymbol{n}_0, 1\right\rangle_{\Gamma_0}\right|^2 + \left\|\tau_t^{1/2}\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\right)^t\right\|_{\partial\mathcal{T}_h}^2 + \left\|(\tau_n\epsilon)^{1/2}\left(\boldsymbol{e}_h^{\boldsymbol{u}} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\right)\cdot\boldsymbol{n}\right\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 \\
&\quad\lesssim C_1 h^{2(l_{\boldsymbol{u}}+1/2)}|\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,\mathcal{T}_h^{\mathrm{D}}}^2 + C_2 h^{2(l_{\boldsymbol{u}}+1/2)}|\boldsymbol{u}|_{l_{\boldsymbol{u}}+1,\mathcal{T}_h^{\mathrm{C}}}^2 + C_3 h^{2(l_\varphi+1/2)}|\varphi|_{l_\varphi+1,\mathcal{T}_h^{\mathrm{D}}}^2 + C_4 h^{2(l_{\boldsymbol{z}}+1/2)}|\boldsymbol{z}|_{l_{\boldsymbol{z}}+1,\mathcal{T}_h}^2,
\end{aligned}$$

where

$$C_1 := \max\left\{\max_{K\in A^{\mathrm{D}}}\left(\epsilon(\tau_n)_K^{\mathrm{m}}\right), \max_{K\in\mathcal{T}_h^{\mathrm{D}}}(\tau_t)_K^{\mathrm{m}}, \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\left(\frac{\epsilon}{(\tau_t)_K^*}\right), \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\lambda\epsilon\right\},$$

$$C_2 := \max\left\{\omega\max_{K\in\mathcal{T}_h^{\mathrm{C}}}(\sigma h_K), \max_{K\in\mathcal{T}_h^{\mathrm{C}}}(\tau_t)_K^{\mathrm{m}}\right\},$$

$$C_3 := \max\left\{\max_{K\in\mathcal{T}_h^{\mathrm{D}}} h_K, \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\left(\frac{\epsilon}{(\tau_n)_K^*}\right)\right\}, \quad \textit{and}$$

$$C_4 := \max_{K\in\mathcal{T}_h}\left(\frac{1}{(\tau_t)_K^*}\right).$$

Here, given $K \in \mathcal{T}_h$, we have $(\tau_n)_K^{\mathrm{m}} := \max\tau_n|_{\partial K}$, $(\tau_t)_K^{\mathrm{m}} := \max\tau_t|_{\partial K}$, $(\tau_n)_K^* := \min\tau_n|_{\partial K}$, and $(\tau_t)_K^* := \min\tau_t|_{\partial K}$.

Proof. It follows straightforwardly by using triangle inequalities, applying Theorem 5.6 and some approximation properties (cf Lemmata 5.1-5.4). We omit further details. □

Now, to deduce an error estimate for the numerical fluxes, we introduce the norm $\|\cdot\|_{h,\mathcal{T}_h^{\mathrm{D}}}$ given by $\|\theta\|_{h,\mathcal{T}_h^{\mathrm{D}}}^2 := \sum_{K\in\mathcal{T}_h^{\mathrm{D}}} h_K\|\theta\|_{\partial K}^2$ for any function $\theta \in \mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}) := \prod_{K\in\mathcal{T}_h^{\mathrm{D}}}\mathrm{L}^2(\partial K)$. The definition of $\|\boldsymbol{\theta}\|_{h,\mathcal{T}_h^{\mathrm{C}}}$, when $\boldsymbol{\theta} \in [\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{C}})]^3$, is defined in analogous way. So, we derive the following results.

Theorem 5.8. *Under the same assumptions in Theorem 5.7, there hold*

a) $\left\| \boldsymbol{e}_h^{\hat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n} \right\|_{h,\mathcal{T}_h^{\mathrm{D}}}^2 \lesssim \sum_{K\in\mathcal{T}_h^{\mathrm{D}}} \frac{h_K}{\epsilon\,(\tau_n)_K^*} \left\| (\epsilon\,\tau_n)^{1/2} \left(\boldsymbol{e}_h^{\hat{\boldsymbol{u}}_h^n} - \left(\boldsymbol{e}_h^{\boldsymbol{u}}\right)^n \right) \cdot \boldsymbol{n} \right\|_{\partial K}^2 + \left\| \boldsymbol{e}_h^{\boldsymbol{u}} \right\|_K^2$,

b) $\left\| \boldsymbol{e}_h^{\hat{\boldsymbol{u}}_h^t} \right\|_{h,\mathcal{T}_h^{\mathrm{C}}}^2 \lesssim \sum_{K\in\mathcal{T}_h^{\mathrm{C}}} \frac{h_K}{(\tau_t)_K^*} \left\| (\tau_t)^{1/2} \left(\boldsymbol{e}_h^{\hat{\boldsymbol{u}}_h^t} - \left(\boldsymbol{e}_h^{\boldsymbol{u}}\right)^t \right) \right\|_{\partial K}^2 + \left\| \boldsymbol{e}_h^{\boldsymbol{u}} \right\|_K^2$.

Proof. Just apply triangle inequality and Theorem 5.7. We omit further details. □

Finally, we conclude the main result of this paper.

Theorem 5.9. *Under the same assumptions in Theorem 5.7, we have that*

$$\|\boldsymbol{z} - \boldsymbol{z}_h\|_{\mathcal{T}_h} = \mathcal{O}(h^{\ell+1/2}), \qquad \left\|\boldsymbol{u}^n - \hat{\boldsymbol{u}}_h^n\right\|_{h,\mathcal{T}_h^{\mathrm{D}}} = \mathcal{O}(h^{\ell+1/2}),$$

$$\left\|\sigma^{1/2}(\boldsymbol{u} - \boldsymbol{u}_h)\right\|_{\mathcal{T}_h} = \mathcal{O}(h^{\ell+1/2}), \qquad \left\|\boldsymbol{u}^t - \hat{\boldsymbol{u}}_h^t\right\|_{h,\mathcal{T}_h^{\mathrm{C}}} = \mathcal{O}(h^{\ell+1/2}),$$

$$\|\varphi - \varphi_h\|_{\mathcal{T}_h^{\mathrm{D}}} = \mathcal{O}(h^{\ell+1/2}),$$

where $\ell = \min\{l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi\}$, *provided* τ_n *and* τ_n^{-1}, *as well as* τ_t *and* τ_t^{-1} *remain of order one on* $\partial\mathcal{T}_h^{\mathrm{D}}$ *and* $\partial\mathcal{T}_h$, *respectively.*

These results show that we achieve rates of convergence for the method, but without applying any duality argument. In addition, since σ vanishes on Ω_{D}, the current analysis is not able to find an a priori error estimate for the error of $\boldsymbol{u}_h$ in the dielectric domain. Nevertheless, in practice, this knowledge is not so useful.

6 | CONCLUDING REMARKS

Using a simple strategy, we have been able to develop an a priori error analysis of an HDG method for an eddy current problem. We point out that, as far as we know, this kind of analysis has not been done before. We observe that under enough regularity of the exact solution, the error of the method behaves as $\mathcal{O}(h^{k+1/2})$, where $k \geq 0$. Moreover, the same analysis does not let us prove super-convergence for the error of involved numerical fluxes. The introduction of adjoint problem and use of dual arguments do not help us in this matter. It remains open, then, if this rate of convergence is indeed the best it could be, or it can be improved, for example, by defining another suitable projection operators. Some numerical examples could be helpful also in this direction. These would be the subject of future work.

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ORCID

Mauricio Osorio http://orcid.org/0000-0001-5359-4816

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