Hybridizable discontinuous Galerkin method for the transient eddy current problem

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Abstract

In this paper we develop the first (up to authors' knowledge) a priori error analysis for a transient eddy current problem when applying the Hybridizable discontinuous Galerkin (HDG) method. For this purpose we introduce a curl-curl formulation based in the one developed for the same authors in a previous work, for the same authors, and analyze it for both the semidiscrete and the fully discrete scheme. We consider non trivial domains and heterogeneous media which contain conductor and insulating materials and impose the divergence free condition explicitly in the insulator, by means of a suitable Lagrange multiplier in that material. The HDG formulation includes as unknowns the tangential and normal trace of a vector field and therefore, represents a reduction in the degrees of freedom when compare other DG methods. For this scheme we conduct a complete error analysis that includes unique solvability and estimated convergence rates, for different regularity conditions on the exact solution.

Keywords: HDG, transient eddy current problem.

Mathematics Subject Classification: 65N30; 65M60; 65N12; 35L65

1 Introduction

The electromagnetic phenomena is based on the eddy current model that is obtained from the the Maxwell's equations disregarding displacement currents in the Ampére-Maxwell law, which is reasonable when the magnitude of the displacement currents is negligible compared to the other terms of the equation (see, for instance Chapter 15 in [7] or Chapter 1 in [4]). It is important to point out that the analysis of three dimensional eddy current problems is strongly related with the chosen formulation (either in terms of fields or potentials) and the way of imposing the sources. In [4] a very complete numerical and mathematical analysis can be found, for the time harmonic case. However, since the electromagnetic forming process begins with a pulse of transient electric current, the model that best fits this situation is the transient eddy current model. Our goal is to study the transient eddy current problem defined in a three-dimensional domain including conducting and dielectric materials.

The finite element method (FEM) have been extensively used for the numerical analysis of the time harmonic and the transient eddy current problem (see [1, 2, 8, 17, 18]), however two fundamental aspects should keep in mind when dealing with the numerical solution of an eddy current problem, namely, the need to impose a divergence free condition in one of the variables in the dielectric domain (see [3, 5, 6, 16], for instance, for FEM) and the possibility of discontinuities depending on the electromagnetic properties of each material. The last aspect is of fundamental importance in real life situations and leads to the necessity of using discontinuous Galerkin (DG) methods (see [20]). However must of the DG methods exhibit a great disadvantage when compared with continuous methods: the larger number of degrees of freedom.

In 2010 a new kind of DG methods appeared in order to alleviate the increasing in degrees of freedom. It was called de hybridizable discontinuous Galerkin method (HDG) and allows to find the approximate solution by solving an equivalent system of equations defined on the skeleton of the mesh. This makes the HDG very competitive when compared with standard continuous Galerkin methods as the FEM. In this paper, we propose and analyse an HDG method in space and a backward Euler squeme in time, for the transient eddy current problem, following the ideas introduced in [9] for the time harmonic case, and in [15] for the interior penalty DG method for the Maxwell equation. We introduce a Lagrange multiplier to impose the divergence-free condition in the dielectric domain. The model has as unknows the variables related to the electric and magnetic field, the Lagrange multiplier and two variables defined on the skeleton. These last two variables are the only ones we have to solve to, once we impose the local conservative condition typical of HDG methods; so the number of unknowns (degrees of freedom) is reduced compared with standard DG methods.

As point it out by Griesmaier and Monk in [14], discontinuos Galerkin methods are not usually convenient when using implicit time schemes, due to the larger number of spatial degrees of freedom; however, when using the HDG method this difficulty is overcome since at each time step the degrees of freedom are reduced to cheap degrees of freedom associate with numerical traces on the skeleton of the mesh. This has been done for the wave equation (see [14]), acoustic and elastodymanics problems (see [19]), heat equation (see [10]) and others; but not for the transient eddy current problem (up to authors knowledge). Here, we propose an HDG formulation in space based on the normal and tangential trace of vector field. We prove that this formulation is well-posed and obtain error estimates. To this end, we introduce suitable projection operators and under additional assumptions on the regularity of the exact solution we obtain theoretical rates of convergence.

The outline of the paper is as follows: in Section 2 we introduce the transient eddy current model. Then in Section 3, we introduce approximations spaces, propose an HDG formulation for the semi-discrete case and show it has a unique solution. In Section 4 we recalled convenient operators in order to establish the a priori error analysis for different regularities of the exact solution. Finally, in Section 5 we introduce a fully discrete formulation based on a backward Euler scheme in time, and show the a priori error analysis for it.

2 Eddy current problem

Maxwell's equations are used to describe electromagnetic phenomena. The eddy current model results by disregarding the effect of electric displacement in the Ampère-Maxwell law. This yields the model problem:

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \sigma\, \boldsymbol{E} + \boldsymbol{J} \quad \text{in } \Omega\,, \\ \partial_t(\mu \boldsymbol{H}) + \nabla \times \boldsymbol{E} &= \mathbf{0} \quad \text{in } \Omega\,, \\ \nabla \cdot (\epsilon\, \boldsymbol{E}) &= 0 \quad \text{in } \Omega_{\mathrm{D}}\,, \end{aligned} \tag{1}$$

where $\boldsymbol{E}(t,\boldsymbol{x})$ denotes the intensity of electric field, $\boldsymbol{H}(t,\boldsymbol{x})$ the intensity of magnetic field and $\boldsymbol{J}(t,\boldsymbol{x})$ the density of source current, which is assumed to be divergence-free. We have to take also into account the electromagnetic coefficients are time-independent: magnetic permeability μ, electric permittivity ϵ and electric conductivity σ. About the domain Ω, we assume that it is bounded, simple connected, and consists on two parts: Ω_{C} and Ω_{D}, with Ω_{C} representing the conductor domain while Ω_{D} denotes the dielectric one (cf. Figure 1). The boundary of Ω, named Γ, is supposed to be connected and Lipschitz-continuous. Γ_0 denotes the interface between the conductor and dielectric domains. We point out that the electromagnetic coefficients depend on the material. In particular, we assume that μ, ϵ and σ are bounded functions satisfying:

$$
\begin{aligned}
&0 < \mu_{\min} \leq \mu \leq \mu_{\max}, \quad \text{in } \Omega\,,\\
&0 < \epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}, \quad \text{in } \Omega\,,\\
&0 < \sigma_{\min} \leq \sigma \leq \sigma_{\max}, \quad \text{in } \Omega_{\mathrm{C}} \quad \text{and} \quad \sigma \equiv 0 \text{ in } \Omega_{\mathrm{D}}\,.
\end{aligned}
$$

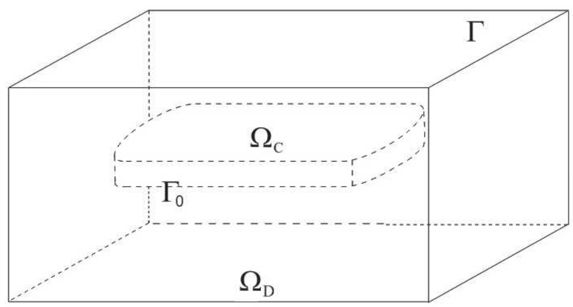

Figure 1: Domain Ω for problem (1)

The equations above must be completed with suitable boundary and initial conditions guaranteeing the well-posedness of the problem. Namely

$$
\begin{aligned}
\langle \epsilon \boldsymbol{E}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0} &= 0\,,\\
\boldsymbol{n} \times \boldsymbol{E} &= \boldsymbol{g} \quad \text{on} \quad \Gamma\,,\\
\boldsymbol{E}(0,\cdot) &= \boldsymbol{u}_0(\cdot) \quad \text{in } \Omega_{\mathrm{C}},
\end{aligned}
\tag{2}
$$

here, $\boldsymbol{n}$ and $\boldsymbol{n}_0$ denote the outward normal unit vector to Γ and Γ_0, disjoint components of $\partial\Omega_D$; $\boldsymbol{u}_0$ is the electric field a the initial time and $\boldsymbol{g}$ is given.
Let us now to propose a formulation base on the electric field introducing a Lagrange multiplier to impose the divergence free condition on the dielectric.
We introduce now the unknowns: $\boldsymbol{u} = \boldsymbol{E}$ and $\boldsymbol{z} = \mu^{-1}\nabla \times \boldsymbol{u}$, and the Lagrange multiplier φ. Then,

system (1)–(2) can be rewritten as the first order system (assuming $\boldsymbol{J}$ smooth enough)

$$
\begin{aligned}
\mu \boldsymbol{z} - \nabla \times \boldsymbol{u} &= \boldsymbol{0} \quad \text{in } \Omega \,, \\
\sigma \partial_t \boldsymbol{u} + \nabla \times \boldsymbol{z} - \epsilon \nabla \varphi|_{\Omega_{\mathrm{D}}} &= -\partial_t \boldsymbol{J} \quad \text{in } \Omega \,, \\
\nabla \cdot (\epsilon\, \boldsymbol{u}) &= \varphi \quad \text{in } \Omega_{\mathrm{D}} \,, \\
\langle \epsilon \boldsymbol{u}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} &= 0 \,, \\
\boldsymbol{n} \times \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma \,, \\
\varphi &= 0 \quad \text{on } \Gamma \,, \\
\boldsymbol{u}(0, \cdot) &= \boldsymbol{u}_0(\cdot) \quad \text{in } \Omega_{\mathrm{C}} \,.
\end{aligned} \tag{3}
$$

Let us now to propose a numerical solution base on a HDG formulation.

3 The HDG method

We introduce the following approximation spaces

$$
\begin{aligned}
&\mathbb{P}_h := \{\rho_h \in \mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}) : \ \rho_h|_K \in \mathbb{P}_k(K) \ \ \forall K \in \mathcal{T}_h^{\mathrm{D}}\} \,, \\
&\mathbb{P}_h^{\Gamma} := \{\rho_h \in \mathbb{P}_h \ : \ \ \rho_h = 0 \text{ on } \Gamma\} \,, \\
&\mathbb{V}_h := \{\boldsymbol{v}_h \in \left[\mathrm{L}^2(\mathcal{T}_h)\right]^3 : \ \boldsymbol{v}_h|_K \in [\mathbb{P}_k(K)]^3 \ \ \forall K \in \mathcal{T}_h\} \,, \\
&\mathbb{M}_h^t := \left\{\boldsymbol{\beta} \in \left[\mathrm{L}^2(\mathcal{E}_h)\right]^3 \ : \ \boldsymbol{\beta} \in [\mathbb{P}_k(F)]^3,\, (\boldsymbol{\beta} \cdot \boldsymbol{n})|_F = 0 \ \ \forall F \in \mathcal{E}_h \right\} \,, \\
&\mathbb{M}_h^n := \left\{\boldsymbol{\gamma} \in \left[\mathrm{L}^2\left(\mathcal{E}_h^{\mathrm{D}}\right)\right]^3 \ : \ \boldsymbol{\gamma} \in [\mathbb{P}_k(F)]^3,\, (\boldsymbol{\gamma} \times \boldsymbol{n})|_F = \boldsymbol{0} \ \ \forall F \in \mathcal{E}_h^{\mathrm{D}} \right\} \,, \\
&\mathbb{M}_h^n(0) := \left\{\boldsymbol{\gamma} \in \mathbb{M}_h^n, : \ (\boldsymbol{\gamma} \cdot \boldsymbol{n}_0)|_F = 0 \ \ \forall F \in \mathcal{E}_h^{\Gamma_0} \right\} \,.
\end{aligned}
$$

Hereafter, given S of positive measure, $\mathbb{P}_k(S)$ denotes the space of polynomials of degree at most $k \geq 0$ on S. We remark that $\mathbb{M}_h^t$ ($\mathbb{M}_h^n$ resp.) consists of vector-valued functions whose normal (tangential resp.) component is zero on any face $F \in \mathcal{E}_h$ ($F \in \mathcal{E}_h^{\mathrm{D}}$ resp.).
Then, multiplying the first three equations of (3) by test functions $\boldsymbol{r}, \boldsymbol{v}$ and ρ, for all $K \in \mathcal{T}_h$ and $K \in \mathcal{T}_h^{\mathrm{D}}$, applying the Green identities we obtain

$$
\begin{aligned}
&(\mu \boldsymbol{z}, \boldsymbol{r})_K - (\boldsymbol{u}, \nabla \times \boldsymbol{r})_K - \langle \boldsymbol{u}^t, \boldsymbol{r} \times \boldsymbol{n} \rangle_{\partial K} = 0 \\
&(\sigma \partial_t \boldsymbol{u}, \boldsymbol{v})_K + (\boldsymbol{z}, \nabla \times \boldsymbol{v})_K + \langle \boldsymbol{z}^t, \boldsymbol{v} \times \boldsymbol{n} \rangle_{\partial K} + (\varphi, \nabla \cdot (\epsilon \boldsymbol{v}))_K - \langle \epsilon \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{\partial K} = -(\partial_t \boldsymbol{J}, \boldsymbol{v})_K \\
&(\varphi, \rho)_K + (\epsilon \boldsymbol{u}, \nabla \rho)_K - \langle \epsilon \boldsymbol{u} \cdot \boldsymbol{n}, \rho \rangle_{\partial K} = 0
\end{aligned}
$$

where $(q,p)_D := \displaystyle\int_D q\, p$, with D a domain in $\mathbb{R}^n$, and $\langle q,p \rangle_D := \displaystyle\int_D q\, p$, whenever D is a domain in $\mathbb{R}^{n-1}$. For vector functions the notation is similar, with the integrand being a dot product.
Adding over all $K \in \mathcal{T}_h$ and denoting $(u,v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u,v)_K$, (likewise for $\langle u, v \rangle_{\partial \mathcal{T}_h}$), we obtain the next HDG formulation:

Problem 1 *Find* $(\boldsymbol{z}_h, \boldsymbol{u}_h, \varphi_h, \widehat{\boldsymbol{u}}_h^t, \widehat{\boldsymbol{u}}_h^n) \in \mathrm{L}^2(0,T;\mathbb{V}_h)\times\mathrm{L}^2(0,T;\mathbb{V}_h)\cap\mathrm{H}^1(0,T;\mathbb{V}_h|_{\mathcal{T}_h^{\mathrm{C}}})\times\mathrm{L}^2(0,T;\mathbb{P}_h^\Gamma)\times \mathrm{L}^2(0,T;\mathbb{M}_h^t)\times\mathrm{L}^2(0,T;\mathbb{M}_h^n(0))$ *such that*

$$
\begin{aligned}
&(\mu\boldsymbol{z}_h, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla\times\boldsymbol{r}_h)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^t, \boldsymbol{r}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0,\\
&(\sigma\partial_t\boldsymbol{u}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\boldsymbol{z}_h, \nabla\times\boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{z}}_h^t, \boldsymbol{v}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + (\varphi_h, \nabla\cdot(\epsilon\boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -(\partial_t\boldsymbol{J}, \boldsymbol{v}_h)_{\mathcal{T}_h},\\
&(\varphi_h, \rho_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{u}_h, \nabla\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n}, \rho_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0,\\
&\langle \boldsymbol{n}\times\widehat{\boldsymbol{u}}_h^t, \boldsymbol{\eta}\rangle_\Gamma = \langle \boldsymbol{g}, \boldsymbol{\eta}\rangle_\Gamma,\\
&\langle \boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t, \boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0,\\
&\langle \epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0,\\
&\boldsymbol{u}_h|_{t=0} = \Pi_{\mathbb{V}}\boldsymbol{u}_0,
\end{aligned}
\tag{4}
$$

for all $(\boldsymbol{r}_h, \boldsymbol{v}_h, \rho_h, \boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbb{V}_h\times\mathbb{V}_h\times\mathbb{P}_h^\Gamma\times\mathbb{M}_h^t\times\mathbb{M}_h^n(0)$*. The numerical fluxes* $\widehat{\boldsymbol{z}}_h^t$ *and* $\widehat{\varphi}_h$ *are defined by*

$$
\begin{aligned}
\boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t &= \boldsymbol{n}\times\boldsymbol{z}_h^t + \tau_t(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t)\,,\\
\widehat{\varphi}_h &= \widetilde{\varphi}_h - \tau_n(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n)\cdot\boldsymbol{n}\,,
\end{aligned}
\tag{5}
$$

with

$$
\widetilde{\varphi}_h = \begin{cases} \varphi_h & \text{on } \mathcal{E}_h^{\mathrm{D},I}\cup\Gamma\,,\\ \varphi_h - \lambda\langle\epsilon\boldsymbol{u}_h|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} & \text{on } \Gamma_0\,. \end{cases}
\tag{6}
$$

Here $\Pi_{\mathbb{V}}$ is a projector to be defined later, λ is a positive constant, and τ_t and τ_n are positive stabilization parameters defined on $\mathcal{E}_h$. Furthermore, it is straightforward to verify that $\boldsymbol{n}\times\boldsymbol{z}_h^t = \boldsymbol{n}\times\boldsymbol{z}_h$, therefore, we use these expressions indistinctly.
Notice that the fourth equation in (3) is imposed in (6), the fifth is weakly imposed, the sixth is incorporated into the space and the fifth equation in (4) guarantees the well definition of $\widehat{\boldsymbol{z}}_h^t$.

Proposition 2 *(Uniqueness) Assume* $\boldsymbol{J}\in\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h)]^3)$*, if a solution of the HDG problem 1 exists it is unique.*

Proof. Let us consider the homogeneous problem associate to problem (3) ($\boldsymbol{J}=\boldsymbol{0}$, $\boldsymbol{g}=\boldsymbol{0}$ and $\boldsymbol{u}_0=\boldsymbol{0}$) and take $\boldsymbol{r}_h := \boldsymbol{z}_h, \boldsymbol{v} := \boldsymbol{u}_h, \rho := \varphi_h, \boldsymbol{\eta} := -\widehat{\boldsymbol{u}}_h^t, \boldsymbol{\xi} := \widehat{\boldsymbol{u}}_h^n$ in (4), we have

$$
\begin{aligned}
&(\mu\boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} - (\boldsymbol{u}_h, \nabla\times\boldsymbol{z}_h)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^t, \boldsymbol{z}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0\\
&(\sigma\partial_t\boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h} + (\boldsymbol{z}_h, \nabla\times\boldsymbol{u}_h)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{z}}_h^t, \boldsymbol{u}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + (\varphi_h, \nabla\cdot(\epsilon\boldsymbol{u}_h))_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\boldsymbol{u}_h\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0\\
&(\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{u}_h, \nabla\varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n}, \varphi_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0\\
&-\langle \boldsymbol{n}\times\widehat{\boldsymbol{z}}_h^t, \widehat{\boldsymbol{u}}_h^t\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0\\
&\langle \epsilon\widehat{\boldsymbol{u}}_h^n\cdot\boldsymbol{n}, \widehat{\varphi}_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0.
\end{aligned}
$$

(Also by the fourth equation $\boldsymbol{n}\times\widehat{\boldsymbol{u}}_h^t = 0$ in Γ a.e. t). Adding the resulting equations all together, we get

$$(\mu \boldsymbol{z}_h, \boldsymbol{z}_h)_{\mathcal{T}_h} + (\varphi_h, \varphi_h)_{\mathcal{T}_h^{\mathrm{D}}} + \frac{1}{2}\frac{d}{dt}(\sigma \boldsymbol{u}_h, \boldsymbol{u}_h)_{\mathcal{T}_h^{\mathrm{C}}} + \langle \boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t, \tau_t \left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right)\rangle_{\partial\mathcal{T}_h}$$
$$+ \langle \epsilon \left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}, \tau_n \left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle \epsilon \boldsymbol{u}_h^n\cdot\boldsymbol{n}, 1\rangle_{\Gamma_0}^2 = 0,$$

thus

$$\|\mu^{1/2}\boldsymbol{z}_h\|_{\mathcal{T}_h} + \|\varphi_h\|_{\mathcal{T}_h^{\mathrm{D}}} + \frac{1}{2}\frac{d}{dt}\|\sigma^{1/2}\boldsymbol{u}_h\|_{\mathcal{T}_h^{\mathrm{C}}} + \|\tau_t^{1/2}\left(\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t\right)\|_{\partial\mathcal{T}_h}$$
$$+ \|(\epsilon\tau_n)^{1/2}\left(\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n\right)\cdot\boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle \epsilon \boldsymbol{u}_h^n\cdot\boldsymbol{n}, 1\rangle_{\Gamma_0}^2 = 0.$$

Integrating in $(0,t)$ $(0 < t \leq T)$, we conclude $\frac{1}{2}\|\sigma^{1/2}\boldsymbol{u}_h(t)\|_{\mathcal{T}_h^{\mathrm{C}}} \leq 0$, therefore $\boldsymbol{u}_h(t) = \boldsymbol{0}$ a.e. $t\in(0,T)$ in $\mathcal{T}_h^{\mathrm{C}}$, hence

$$\begin{aligned}
&\boldsymbol{z}_h(t) = 0 \qquad \text{a.e. } t\in(0,T) \qquad \text{in } \mathcal{T}_h\\
&\varphi_h(t) = 0 \qquad \text{a.e. } t\in(0,T) \qquad \text{in } \mathcal{T}_h^{\mathrm{D}}\\
&\widehat{\boldsymbol{u}}_h^t(t) = \boldsymbol{u}_h^t(t) \qquad \text{a.e. } t\in(0,T) \qquad \text{in } \partial\mathcal{T}_h\\
&\epsilon\widehat{\boldsymbol{u}}_h^n(t)\cdot\boldsymbol{n} = \epsilon\boldsymbol{u}_h^n(t)\cdot\boldsymbol{n} \qquad \text{a.e. } t\in(0,T) \qquad \text{in } \partial\mathcal{T}_h^{\mathrm{D}}\\
&\langle\epsilon\boldsymbol{u}_h^n(t)\cdot\boldsymbol{n}, 1\rangle_{\Gamma_0} = 0 \qquad \text{a.e. } t\in(0,T)
\end{aligned}$$

Thus, is clear that $u_h(t)\in \mathrm{H}(\mathbf{curl};\Omega)$, $u_h(t)|_{\Omega_D}\in\mathrm{H}(\mathrm{div}_\epsilon;\Omega)$, $u_h(t)\in\mathrm{H}_\Gamma(\mathbf{curl}^0;\Omega)\cap\mathrm{H}_{\Gamma_0}(\mathrm{div}_\epsilon^0;\Omega)$ a. e. $t\in(0,T)$, and this implies that $u_h(t)=0$ also in $\mathcal{T}_h^{\mathrm{D}}$ a. e. $t\in(0,T)$. This concludes the proof. ■

4 Estimates of the a priori errors for the semi-discrete case

We begin this section by introducing the same projector operators defined in [9] (that were introduced and studied in [11], [12]), and omit the time dependence by readability.
$P_{\mathbb{V}}\boldsymbol{z}\in\mathbb{V}_h$ is the standard piecewise orthogonal L^2-projection of $\boldsymbol{z}$ onto $\mathbb{V}_h$. This means that on each $K\in\mathcal{T}_h$:

$$(P_{\mathbb{V}}\boldsymbol{z},\boldsymbol{r})_K = (\boldsymbol{z},\boldsymbol{r})_K \quad \forall\,\boldsymbol{r}\in[\mathbb{P}_k(K)]^3\,. \tag{7}$$

Since φ is defined only on the dielectric domain, we set $\Pi_{\mathrm{v}}\boldsymbol{u}\in\mathbb{V}_h$ in the conductor domain as:

$$\forall\,K\in\mathcal{T}_h^{\mathrm{C}}\;:\;\begin{cases}(\Pi_{\mathrm{v}}\boldsymbol{u},\boldsymbol{v})_K = (\boldsymbol{u},\boldsymbol{v})_K & \forall\,\boldsymbol{v}\in[\mathbb{P}_{k-1}(K)]^3\,,\\[2mm] \langle\Pi_{\mathrm{v}}\boldsymbol{u}\cdot\boldsymbol{n},\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = \langle\boldsymbol{u}\cdot\boldsymbol{n},\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} & \forall\,\boldsymbol{v}\in[\mathbb{P}_k^{\perp}(K)]^3\,,\end{cases} \tag{8}$$

where $\mathbb{P}_k^{\perp}(K) := \{p\in\mathbb{P}_k(K)\,:\,(p,q)_K = 0 \quad \forall\,q\in\mathbb{P}_{k-1}(K)\}$.
Now we set $A^{\mathrm{D}} := \left\{K\in\mathcal{T}_h^{\mathrm{D}}\,:\,|\partial K\cap\Gamma|_{\mathbb{R}^{d-1}}>0\right\}$, and $B^{\mathrm{D}} := \mathcal{T}_h^{\mathrm{D}}\backslash A^{\mathrm{D}}$. Then, we define $(\Pi_{\mathrm{v}}\boldsymbol{u},\Pi_{\mathbb{P}}\varphi)$ as follows:

$$\forall\,K\in B^{\mathrm{D}}\;:\;\begin{cases}(\Pi_{\mathrm{v}}\boldsymbol{u},\boldsymbol{v})_K = (\boldsymbol{u},\boldsymbol{v})_K & \forall\,\boldsymbol{v}\in[\mathbb{P}_{k-1}(K)]^3\,,\\[2mm] (\Pi_{\mathrm{P}}\varphi,\psi)_K = (\varphi,\psi)_K & \forall\,\psi\in\mathbb{P}_{k-1}(K)\,,\\[2mm] \langle\Pi_{\mathrm{P}}\varphi-\tau_n\,\Pi_{\mathrm{v}}\boldsymbol{u}\cdot\boldsymbol{n},\eta\rangle_F = \langle\varphi-\tau_n\,\boldsymbol{u}\cdot\boldsymbol{n},\eta\rangle_F & \forall\,\eta\in\mathbb{P}_k(F)\quad\forall\,F\in\partial K\,,\end{cases} \tag{9}$$

and

$$\forall\, K \in A^{\mathrm{D}} \, : \, \begin{cases} (\Pi_{\mathbb{V}} \boldsymbol{u}, \boldsymbol{v})_K \,=\, (\boldsymbol{u}, \boldsymbol{v})_K & \forall\, \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^3 \,, \\ \langle \tau_n \, \Pi_{\mathbb{V}} \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} \,=\, \langle \tau_n \, \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} & \forall\, \boldsymbol{v} \in [\mathbb{P}_k^{\perp}(K)]^3 \,, \\ (\Pi_{\mathbb{P}} \varphi, \psi)_K \,=\, (\varphi, \psi)_K & \forall\, \psi \in \mathbb{P}_{k-1}(K) \,, \\ \langle \Pi_{\mathbb{P}} \varphi, \eta \rangle_{F_0} \,=\, \langle \varphi, \eta \rangle_{F_0} & \forall\, \eta \in \mathbb{P}_k(F_0) \,, \, F_0 \,\subseteq\, \partial K \cap \Gamma \,. \end{cases} \tag{10}$$

We point out that since $\varphi = 0$ on Γ, its projection $\Pi_{\mathbb{P}} \varphi$ also vanishes on Γ.

These projector operators satisfy the properties summarized by the next lemmas.

Lemma 3 *Considering* $\Pi_{\mathbb{V}} \boldsymbol{u}$ *given by* (8)*, and* $l_{\boldsymbol{u}} \in [0, k]$*, then for any* $K \in \mathcal{T}_h^{\mathrm{C}}$ *there hold*

(a) $\|(\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(b) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(c) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,.$

Lemma 4 *Assume that* $\tau_n|_{\partial K} > 0$*, for any* $K \in A^{\mathrm{D}}$*. Then, for* $l_{\boldsymbol{u}}\,,\, l_{\varphi} \in [0, k]$*, the operators* $\Pi_{\mathbb{V}}$ *and* $\Pi_{\mathbb{P}}$ *defined in* (10)*, satisfy*

(a) $\|(\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(b) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(c) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(d) $\|\Pi_{\mathbb{P}} \varphi - \varphi\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial K))} \lesssim h_K^{l_{\varphi}+1/2} \, |\varphi|_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))} \quad \forall\, \varphi \in \mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))\,,$

(e) $\|\Pi_{\mathbb{P}} \varphi - \varphi\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(K))} \lesssim h_K^{l_{\varphi}+1} \, |\varphi|_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))} \quad \forall\, \varphi \in \mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))\,.$

Lemma 5 *Suppose that* $\tau_n|_{\partial K} > 0$*, for any* $K \in B^{\mathrm{D}}$*. Then, for* $l_{\boldsymbol{u}}\,,\, l_{\varphi} \in [0, k]$*, the operators* $\Pi_{\mathbb{V}}$ *and* $\Pi_{\mathbb{P}}$ *defined in* (9)*, satisfy*

(a) $\|\Pi_{\mathbb{P}} \varphi - \varphi\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(K))} \lesssim h_K^{l_{\varphi}+1} \, |\varphi|_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))} \quad \forall\, \varphi \in \mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))\,,$

(b) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \, + \, \dfrac{h_K^{l_{\varphi}+1}}{(\tau_n)_K^*} \, |\varphi|_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\varphi}+1}(K))}$
$\forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,,$

(c) $\|\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial K)]^3)} \lesssim h_K^{l_{\boldsymbol{u}}+1/2} \, |\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)} \quad \forall\, \boldsymbol{u} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)]^3)\,.$

Here $(\tau_n)_K^* := \min \tau_n|_{\partial K}$.

We also have the known result for standard orthogonal L^2-projections given by the next lemma.

Lemma 6 *Given $l_{\boldsymbol{z}} \in [0,k]$, there hold on any $K \in \mathcal{T}_h$*

(a) $\|P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)} \lesssim h_K^{l_{\boldsymbol{z}}+1}\, |\boldsymbol{z}(t)|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(K)]^3)} \quad \forall\, \boldsymbol{z} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(K)]^3)\,,$

(b) $\|P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial K)]^3)} \lesssim h_K^{l_{\boldsymbol{z}}+1/2}\, |\boldsymbol{z}(t)|_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(K)]^3)} \quad \forall\, \boldsymbol{z} \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(K)]^3)\,.$

Remark 7 *Notice that, for $l_{\boldsymbol{u}}$, l_φ , $l_{\boldsymbol{z}} \in [0,k]$ if $\boldsymbol{u} \in \mathrm{L}^\infty(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)$, $\varphi \in \mathrm{L}^\infty(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h))$ and $\boldsymbol{z} \in \mathrm{L}^\infty(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)$, we can get similar bounds in the corresponding spaces.*
Likewise we can get

$$\|\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi)\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)} \lesssim h^{\ell_\varphi+1/2}|\partial_t\varphi|_{\mathrm{L}^2(0,T;\mathrm{H}^{\ell_\varphi+1}(A^{\mathrm{D}}))} \lesssim h^{\ell_\varphi+1/2}\|\varphi\|_{\mathrm{H}^1(0,T;\mathrm{H}^{\ell_\varphi+1}(A^{\mathrm{D}}))}\,,$$
$$\|\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} \lesssim h^{\ell_{\boldsymbol{u}}+1/2}|\partial_t\boldsymbol{u}|_{\mathrm{L}^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)} \lesssim h^{\ell_{\boldsymbol{u}}+1/2}\|\boldsymbol{u}\|_{\mathrm{H}^1(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)}$$

provided that $\varphi \in \mathrm{H}^1(0,T;\mathrm{H}^{\ell_\varphi+1}(A^{\mathrm{D}}))$ and $\boldsymbol{u} \in \mathrm{H}^1(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)$.

The idea now is to bound the errors involving the discrete solution and their corresponding projection. Then, the a priori error estimate is derived by applying the triangle inequality. To this end, we first introduce the following notations for a.e. $t \in (0,T)$, which we call the *projection errors*:

$$\begin{aligned} &\boldsymbol{e}_h^{\varphi} := \Pi_{\mathbb{P}}\varphi - \varphi_h, \qquad &&\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} := \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t - \widehat{\boldsymbol{u}}_h^t, \qquad \boldsymbol{e}_h^{\boldsymbol{z}} := P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z}_h,\\ &\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} := \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n - \widehat{\boldsymbol{u}}_h^n, \qquad &&\boldsymbol{e}_h^{\boldsymbol{u}} := \Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}_h, \end{aligned}$$

where $\mathrm{P}_{\mathbb{M}_h^t}$ and $\mathrm{P}_{\mathbb{M}_h^n(0)}$ denote the L^2-projection onto $\mathbb{M}_h^t$ and $\mathbb{M}_h^n(0)$, respectively. The projection errors satisfy the following equations.

Lemma 8 *Let $(\boldsymbol{z},\boldsymbol{u},\varphi)$ and $(\boldsymbol{z}_h,\boldsymbol{u}_h,\varphi_h,\widehat{\boldsymbol{u}}_h^t,\widehat{\boldsymbol{u}}_h^n)$ be the solutions of (3) and Problem 1, respectively. Then, the projection errors $\boldsymbol{e}_h^{\boldsymbol{u}},\boldsymbol{e}_h^{\boldsymbol{z}},\boldsymbol{e}_h^{\varphi},\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}$ satisfy*

$$\begin{aligned}
&(\mu\boldsymbol{e}_h^{\boldsymbol{z}},\boldsymbol{r}_h)_{\mathcal{T}_h} - (\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\times\boldsymbol{r}_h)_{\mathcal{T}_h} - \langle\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{r}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0\\
&(\sigma\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} + (\boldsymbol{e}_h^{\boldsymbol{z}},\nabla\times\boldsymbol{v}_h)_{\mathcal{T}_h} + \langle\boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z}})^t,\boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} - \langle\tau_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t),\boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h}\\
&- \langle\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n},\boldsymbol{e}_h^{\varphi} + \tau_n(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n))\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\langle\epsilon\boldsymbol{v}_h|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\\
&+ (\boldsymbol{e}_h^{\varphi},\nabla\cdot\epsilon\boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{D}}}\\
&\quad = (\sigma\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}),\boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} + \langle\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})^t,\boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} + \langle\tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t),\boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad - \langle\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n},\Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial A^{\mathrm{D}}} + \langle\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n},\tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u})^n\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad - \langle\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n},\tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} + \lambda\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\langle\epsilon\boldsymbol{v}_h|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\\
&(\boldsymbol{e}_h^{\varphi},\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle\epsilon\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n},\rho_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = (\Pi_{\mathbb{P}}\varphi - \varphi,\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} \qquad (11)\\
&\langle\boldsymbol{n}\times\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{\eta}\rangle_{\Gamma} = 0\\
&\langle\boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z}})^t,\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} - \langle\tau_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t),\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = \langle\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})^t,\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad + \langle \tau_t((\Pi_{\mathbb{V}} \boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^t), \boldsymbol{\eta} \rangle_{\partial \mathcal{T}_h \setminus \Gamma} \\
&\langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \boldsymbol{e}_h^{\varphi} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} - \lambda \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} + \langle \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n}, \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\quad = \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \Pi_{\mathbb{P}} \varphi - \varphi \rangle_{\partial A^{\mathrm{D}}} + \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \tau_n (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n} \rangle_{\partial B^{\mathrm{D}}} + \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \varphi \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\quad - \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} - \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \tau_n ((\Pi_{\mathbb{V}} \boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^n) \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\boldsymbol{e}_h^{\boldsymbol{u}}(t=0) = \mathbf{0}
\end{aligned}
$$

for all $(\boldsymbol{r}_h, \boldsymbol{v}_h, \rho_h, \boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{P}_h^{\Gamma} \times \mathbb{M}_h^t \times \mathbb{M}_h^n(0)$.

Proof. The first, third, fourth and fifth equations are an easy consequence of the consistency and the properties of projectors. The second and sixth equations are similar so we just describe how to obtain the second one. Subtracting the second equations in (3) and Problem 1, we have:

$$
\begin{aligned}
(\sigma \partial_t (\boldsymbol{u}_h - \boldsymbol{u}), \boldsymbol{v}_h)_{\mathcal{T}_h} + (\boldsymbol{z}_h - \boldsymbol{z}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{z}}_h^t - \boldsymbol{z}^t, \boldsymbol{v}_h \times \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\
+ (\varphi_h - \varphi, \nabla \cdot (\epsilon \boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \widehat{\varphi}_h - \varphi \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} = 0.
\end{aligned}
$$

We know that $\langle (\widehat{\boldsymbol{z}}_h^t - \boldsymbol{z}^t), \boldsymbol{v}_h \times \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = \langle \boldsymbol{n} \times (\widehat{\boldsymbol{z}}_h^t - \boldsymbol{z}^t), \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h}$, using (5) we get

$$
\begin{aligned}
\boldsymbol{n} \times (\widehat{\boldsymbol{z}}_h^t - \boldsymbol{z}^t) &= \boldsymbol{n} \times \boldsymbol{z}_h^t + \tau_t (\boldsymbol{u}_h^t - \widehat{\boldsymbol{u}}_h^t) - \boldsymbol{n} \times \boldsymbol{z}_h^t \\
&= -\boldsymbol{n} \times (\boldsymbol{e}_h^{\boldsymbol{z}})^t + \boldsymbol{n} \times (P_{\mathbb{V}} \boldsymbol{z} - \boldsymbol{z})^t + \tau_t (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t + (\Pi_{\mathbb{V}} \boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^t),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\varphi}_h - \varphi &= \begin{cases} \varphi_h - \varphi - \tau_n (\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n) \cdot \boldsymbol{n} & \text{on } \mathcal{E}_h^{\mathrm{D},I} \cup \Gamma\,, \\ \varphi_h - \varphi - \tau_n (\boldsymbol{u}_h^n - \widehat{\boldsymbol{u}}_h^n) \cdot \boldsymbol{n} - \lambda \langle \epsilon \boldsymbol{u}_h|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} & \text{on } \Gamma_0\,. \end{cases} \\
&= \begin{cases} -\boldsymbol{e}_h^{\varphi} + (\Pi_{\mathbb{P}} \varphi - \varphi) - \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n} - \tau_n ((\Pi_{\mathbb{V}} \boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^n) \cdot \boldsymbol{n} & \text{on } \mathcal{E}_h^{\mathrm{D},I} \cup \Gamma\,, \\ -\boldsymbol{e}_h^{\varphi} + (\Pi_{\mathbb{P}} \varphi - \varphi) - \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n} - \tau_n ((\Pi_{\mathbb{V}} \boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^n) \cdot \boldsymbol{n} & \\ \qquad\qquad - \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{e}_h^{\boldsymbol{u}})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} & \text{on } \Gamma_0\,. \end{cases}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\sigma \partial_t (\boldsymbol{u}_h - \boldsymbol{u}), \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} &= -(\sigma \partial_t (\boldsymbol{e}_h^{\boldsymbol{u}}), \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} + (\sigma \partial_t (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} \\
(\boldsymbol{z}_h - \boldsymbol{z}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} &= (\boldsymbol{z}_h - P_{\mathbb{V}} \boldsymbol{z} + P_{\mathbb{V}} \boldsymbol{z} - \boldsymbol{z}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} = -(\boldsymbol{e}_h^{\boldsymbol{z}}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} \\
(\varphi_h - \varphi, \nabla \cdot (\epsilon \boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} &= (\varphi_h - \Pi_{\mathbb{P}} \varphi + \Pi_{\mathbb{P}} \varphi - \varphi, \nabla \cdot (\epsilon \boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} = -(\boldsymbol{e}_h^{\varphi}, \nabla \cdot (\epsilon \boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} \\
\langle \boldsymbol{n} \times (\widehat{\boldsymbol{z}}_h^t - \boldsymbol{z}^t), \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} &= -\langle \boldsymbol{n} \times (\boldsymbol{e}_h^{\boldsymbol{z}})^t, \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \boldsymbol{n} \times (P_{\mathbb{V}} \boldsymbol{z} - \boldsymbol{z})^t, \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} \\
&\quad + \langle \tau_t (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t), \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau_t ((\Pi_{\mathbb{V}} \boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^t), \boldsymbol{v}_h \rangle_{\partial \mathcal{T}_h} \\
\langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \widehat{\varphi}_h - \varphi \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} &= -\langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \boldsymbol{e}_h^{\varphi} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} + \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \Pi_{\mathbb{P}} \varphi - \varphi \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\quad - \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\quad - \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \tau_n ((\Pi_{\mathbb{V}} \boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^n) \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h^{\mathrm{D}}} \\
&\quad + \lambda \langle \epsilon \boldsymbol{v}_h|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} \\
&\quad - \lambda \langle \epsilon \boldsymbol{v}_h|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0} \langle \epsilon \Pi_{\mathbb{V}} \boldsymbol{u}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1 \rangle_{\Gamma_0},
\end{aligned}
$$

and substituting these back into the second equation, we have

$$
\begin{aligned}
&(\sigma\partial_t(\boldsymbol{e}_h^{\boldsymbol{u}}), \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} + (\boldsymbol{e}_h^{\boldsymbol{z}}, \nabla\times\boldsymbol{v}_h)_{\mathcal{T}_h} + \langle \boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z}})^t, \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} - \langle \tau_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t), \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} \\
&\quad + (\boldsymbol{e}_h^{\varphi}, \nabla\cdot(\epsilon\boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \boldsymbol{e}_h^{\varphi}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \tau_n(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\quad + \lambda\langle \epsilon\boldsymbol{v}_h|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} \\
&\qquad = (\sigma\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{C}}} + \langle \boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})^t, \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t), \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} \\
&\qquad\quad + \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \tau_n((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle \epsilon\boldsymbol{v}_h|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle \epsilon\Pi_{\mathbb{V}}\boldsymbol{u}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} \\
&\qquad\quad - \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}.
\end{aligned}
$$

Since $\boldsymbol{u}$ satisfies $\langle \epsilon\boldsymbol{u}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0} = 0$ we write $\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}$ instead of $\langle \epsilon\Pi_{\mathbb{V}}\boldsymbol{u}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0, 1\rangle_{\Gamma_0}$. Moreover,

$$
\begin{aligned}
\langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} &= \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial A^{\mathrm{D}}} + \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial B^{\mathrm{D}}} \\
&= \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial A^{\mathrm{D}}} + \langle \epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}, \tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}}
\end{aligned}
$$

and we get the second equation in (11). ■

As a consequence of previous Lemma we can get bounds for the projection errors, as follows.

Theorem 9 *For $T > 0$, any $k \geq 0$ and $l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi \in [0, k]$, assuming that the exact solution of* (3), $(\boldsymbol{z}, \boldsymbol{u}, \varphi) \in \mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)\times\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)\cap\mathrm{H}^1(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{C}})]^3)\times\mathrm{L}^2(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))$, *there holds*

$$
\begin{aligned}
&\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\|^2_{\mathrm{L}^\infty(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \|\boldsymbol{e}_h^{\boldsymbol{z}}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h)]^3)} + \|(\tau_n\epsilon)^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^n - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n})\cdot\boldsymbol{n}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))} \\
&+ \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \lambda\int_0^T |\langle \epsilon\boldsymbol{e}_h^{\boldsymbol{u}}(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_0}|^2 ds + \|\boldsymbol{e}_h^{\varphi}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))} \\
&\lesssim C_1\ h^{2(l_{\boldsymbol{u}}+1/2)}(|\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)} + |\partial_t\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{C}})]^3)}) + C_2\ h^{2(l_\varphi+1)}|\varphi|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))} \\
&\quad + C_3\ h^{2(l_{\boldsymbol{z}}+1/2)}|\boldsymbol{z}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \max\left\{|\Omega|\ \max_{K\in\mathcal{T}_h^{\mathrm{C}}}\sigma_K, \max_{K\in\mathcal{T}_h}(\tau_t)_K^{\mathrm{m}}, \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\epsilon(\tau_n)_K^{\mathrm{m}}, \lambda|\Gamma_0|\ \max_{K\in B^{\mathrm{D}}}\epsilon_K\right\} \\
C_2 &= \max\left\{1, \max_{K\in A^{\mathrm{D}}}\frac{\epsilon}{(\tau_n)_K^*}\right\}\ \textit{and} \\
C_3 &= \max_{K\in\mathcal{T}_h}\frac{1}{(\tau_t)_K^*}.
\end{aligned}
$$

Here, given $K \in \mathcal{T}_h$, we have $(\tau_n)_K^{\mathtt{m}} := \max\tau_n|_{\partial K}$, $(\tau_t)_K^{\mathtt{m}} := \max\tau_t|_{\partial K}$, $(\tau_n)_K^* := \min\tau_n|_{\partial K}$, and $(\tau_t)_K^* := \min\tau_t|_{\partial K}$.

Proof. First notice that from the fourth equation in (11), we can conclude that $\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} = \boldsymbol{0}$ on Γ. Also, by definition, $\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} = \boldsymbol{0}$ on Γ_0. Now, taking $\boldsymbol{r}_h := \boldsymbol{e}_h^{\boldsymbol{z}}, \boldsymbol{v}_h := \boldsymbol{e}_h^{\boldsymbol{u}}, \rho_h := \boldsymbol{e}_h^{\varphi}, \boldsymbol{\eta} := -\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{\xi} := \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}$

in (11), and adding the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|^2_{\mathcal{T}_h} + \|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)\|^2_{\partial\mathcal{T}_h} + \lambda|\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}|^2 + \|\boldsymbol{e}_h^{\varphi}\|^2_{\mathcal{T}_h^{\mathrm{D}}}\\
&+\|(\tau_n\epsilon)^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^n - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n})\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} = (\sigma\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u}),\boldsymbol{e}_h^{\boldsymbol{u}})_{\mathcal{T}_h^{\mathrm{C}}} + (\Pi_{\mathbb{P}}\varphi-\varphi,\boldsymbol{e}_h^{\varphi})_{\mathcal{T}_h^{\mathrm{D}}}\\
&-\langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)\cdot\boldsymbol{n},\tau_n((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z})^t,\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t\rangle_{\partial\mathcal{T}_h}\\
&+\langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)^n\cdot\boldsymbol{n},\Pi_{\mathbb{P}}\varphi-\varphi\rangle_{\partial A^{\mathrm{D}}} + \langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)^n\cdot\boldsymbol{n},\tau_n(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}}\\
&+\lambda\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0} - \langle\tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t),\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
\tag{12}
$$

Integrating in $(0,t)$ $(0<t\le T)$ and using Young's inequality, we get

$$
\begin{aligned}
&\frac{1}{2}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}(t)\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \int_0^t\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(s)\|^2_{\mathcal{T}_h}ds + \frac{1}{2}\int_0^t\|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)(s)\|^2_{\partial\mathcal{T}_h}ds + \frac{1}{2}\int_0^t\|\boldsymbol{e}_h^{\varphi}(s)\|^2_{\mathcal{T}_h^{\mathrm{D}}}ds\\
&+\frac{\lambda}{2}\int_0^t|\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}|^2ds + \frac{1}{2}\int_0^t\|(\tau_n\epsilon)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(s)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}ds\\
&\le\frac{1}{2}\int_0^t\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}(s)\|^2_{\mathcal{T}_h^{\mathrm{C}}}ds + \frac{1}{2}\|\sigma^{1/2}\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \frac{1}{2}\|(\Pi_{\mathbb{P}}\varphi-\varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))}\\
&\quad+\|\tau_t^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \|(\epsilon\tau_n)^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)}\\
&\quad+\|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial B^{\mathrm{D}})]^3)} + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi-\varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial A^{\mathrm{D}}))}\\
&\quad+\|\tau_t^{-1/2}\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \frac{\lambda}{2}|\Gamma_0|\,\|\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})|_{\Omega_{\mathrm{D}}}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\Gamma_0)]^3)}.
\end{aligned}
$$

Gronwall's inequality allows us to show that

$$
\begin{aligned}
&\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}(t)\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \int_0^t\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(s)\|^2_{\mathcal{T}_h}ds + \int_0^t\|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)(s)\|^2_{\partial\mathcal{T}_h}ds + \int_0^t\|\boldsymbol{e}_h^{\varphi}(s)\|^2_{\mathcal{T}_h^{\mathrm{D}}}ds\\
&+\lambda\int_0^t|\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}|^2ds + \int_0^t\|(\tau_n\epsilon)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(s)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}ds\\
&\lesssim\|\sigma^{1/2}\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi-\varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial A^{\mathrm{D}}))}\\
&\quad+\|(\epsilon\tau_n)^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)} + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial B^{\mathrm{D}})]^3)}\\
&\quad+\|\tau_t^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \|\tau_t^{-1/2}\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)}\\
&\quad+\|(\Pi_{\mathbb{P}}\varphi-\varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))} + \lambda|\Gamma_0|\,\|\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})|_{\Omega_{\mathrm{D}}}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\Gamma_0)]^3)}.
\end{aligned}
$$

We know that, for all K a.e. t

$$
\begin{aligned}
\|(\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n\|_{[\mathrm{L}^2(\partial K)]^3} &\le \|(\Pi_{\mathbb{V}}\boldsymbol{u})^n - \boldsymbol{u}^n\|_{[\mathrm{L}^2(\partial K)]^3} + \|\boldsymbol{u}^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n\|_{[\mathrm{L}^2(\partial K)]^3}\\
&\lesssim \|\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u}\|_{[\mathrm{L}^2(\partial K)]^3} + h_K^{l_{\boldsymbol{u}}+1/2}|\boldsymbol{u}|_{\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)},\\
\|(\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t\|_{[\mathrm{L}^2(\partial K)]^3} &\le \|(\Pi_{\mathbb{V}}\boldsymbol{u})^t - \boldsymbol{u}^t\|_{[\mathrm{L}^2(\partial K)]^3} + \|\boldsymbol{u}^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t\|_{[\mathrm{L}^2(\partial K)]^3}\\
&\lesssim \|\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u}\|_{[\mathrm{L}^2(\partial K)]^3} + h_K^{l_{\boldsymbol{u}}+1/2}|\boldsymbol{u}|_{\mathrm{H}^{l_{\boldsymbol{u}}+1}(K)}.
\end{aligned}
$$

Now, using Lemmas 3, 4, 5 and 6 and the previous inequalities, we obtain

$$
\begin{aligned}
&\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u}}(t)\|^2_{\mathcal{T}_h^{\mathrm{C}}} + 2\int_0^t \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(s)\|^2_{\mathcal{T}_h} ds + \int_0^t \|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)(s)\|^2_{\partial\mathcal{T}_h} ds + \int_0^t \|\boldsymbol{e}_h^{\varphi}(s)\|^2_{\mathcal{T}_h^{\mathrm{D}}} ds \\
&+ \int_0^t \|(\tau_n\epsilon)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(s)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} ds + \lambda\int_0^t |\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_0}|^2 ds \\
&\lesssim \left(\max_{K\in\mathcal{T}_h^{\mathrm{C}}} \sigma_K\right) h^{2(l_{\boldsymbol{u}}+1)}|\partial_t\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{C}}))} + h^{2(l_\varphi+1)}|\varphi|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))} \\
&\quad + \left(\max_{K\in\mathcal{T}_h^{\mathrm{D}}} \epsilon(\tau_n)_K^{\mathrm{m}}\right) h^{2(l_{\boldsymbol{u}}+1/2)}|\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{D}}))} + \left(\max_{K\in A^{\mathrm{D}}} \frac{\epsilon}{(\tau_n)_K^*}\right) h^{2(l_\varphi+1/2)}|\varphi|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_\varphi+1}(A^{\mathrm{D}}))} \\
&\quad + \left(\max_{K\in\mathcal{T}_h} \frac{1}{(\tau_t)_K^*}\right) h^{2(l_{\boldsymbol{z}}+1/2)}|\boldsymbol{z}|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h))} + \lambda|\Gamma_0|\left(\max_{K\in B^{\mathrm{D}}} \epsilon_K\right) h^{2(l_{\boldsymbol{u}}+1/2)}|\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(B^{\mathrm{D}})]^3)} \\
&\quad + \left(\max_{K\in\mathcal{T}_h} (\tau_t)_K^{\mathrm{m}}\right) h^{2(l_{\boldsymbol{u}}+1/2)}|\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h))}.
\end{aligned}
$$

And this concludes the proof. ■

In order to deduce an error bound for the numerical fluxes, we introduce the norm

$$\|\theta\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{D}}))} := \sum_{K\in\mathcal{T}_h^{\mathrm{D}}} h_k\|\theta\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))},$$

for any $\theta \in \mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})$. Analogously, for $\boldsymbol{\theta} \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)$ we define $\|\boldsymbol{\theta}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)}$.

Corollary 10 *Under the same assumptions in Theorem 9, we have*

- $\|\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{D}}))} \lesssim \sum_{K\in\mathcal{T}_h^{\mathrm{D}}} \frac{h_K}{\epsilon(\tau_n)_K^*}\|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)\cdot\boldsymbol{n}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial K))} + \|\boldsymbol{e}_h^{\boldsymbol{u}}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)}$.
- $\|\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{C}}))} \lesssim \sum_{K\in\mathcal{T}_h^{\mathrm{C}}} \frac{h_K}{(\tau_t)_K^*}\|(\tau_t)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial K))} + \|\boldsymbol{e}_h^{\boldsymbol{u}}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(K)]^3)}$.

By Theorem 9, previous Corollary and triangle inequality, we can prove the convergence order given by the next result.

Theorem 11 *Under the same assumptions in Theorem 9, we have that*

$$
\begin{aligned}
&\left\|\sigma^{1/2}(\boldsymbol{u}-\boldsymbol{u}_h)\right\|_{\mathrm{L}^\infty(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} = \mathcal{O}(h^{\ell+1/2}), \quad && \|(\boldsymbol{u}^n - \widehat{\boldsymbol{u}}_h^n)\cdot\boldsymbol{n}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{D}}))} = \mathcal{O}(h^{\ell+1/2}), \\
&\|\boldsymbol{z}-\boldsymbol{z}_h\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h)]^3)} = \mathcal{O}(h^{\ell+1/2}), && \|\boldsymbol{u}^t - \widehat{\boldsymbol{u}}_h^t\|^2_{\mathrm{L}^2(0,T;\mathrm{L}_h^2(\partial\mathcal{T}_h^{\mathrm{C}}))} = \mathcal{O}(h^{\ell+1/2}), \\
&\|\varphi-\varphi_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))} = \mathcal{O}(h^{\ell+1/2}).
\end{aligned}
$$

where $\ell = \min\{l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi\}$, *provided* τ_n *and* τ_n^{-1}, *as well as* τ_t *and* τ_t^{-1} *remain of order one on* $\partial\mathcal{T}_h^{\mathrm{D}}$ *and* $\partial\mathcal{T}_h$, *respectively.*

If we assume additional regularity over the solution of (3), we get bounds in L^∞ for the rest of the variables.

Theorem 12 *For $T>0$, any $k\geq 0$ and $l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi \in [0,k]$, assuming that the exact solution of (3), $(\boldsymbol{z},\boldsymbol{u},\varphi)\in \mathrm{H}^1(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)\times \mathrm{H}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)\times \mathrm{H}^1(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))$, there holds*

$$
\begin{aligned}
&\|\partial_t \boldsymbol{e}_h^{\boldsymbol{u}}\|_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|_{\mathrm{L}^\infty(0,T;[\mathrm{L}^2(\mathcal{T}_h)]^3)} + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)\cdot\boldsymbol{n}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))}\\
&+\|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)\|_{\mathrm{L}^\infty(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \|\boldsymbol{e}_h^{\varphi}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))} + \lambda \sup_{0\leq s\leq T}\langle \epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}^2\\
&\lesssim C_1\ h^{2(l_{\boldsymbol{u}}+1/2)}\big(|\boldsymbol{u}|^2_{\mathrm{L}^\infty(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)} + |\partial_t\boldsymbol{u}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{C}})]^3)}\big)\\
&\quad + C_2\ h^{2(l_\varphi+1/2)}\big(|\varphi|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))} + |\partial_t\varphi|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{l_\varphi+1}(A^{\mathrm{D}}))}\big)\\
&\quad + C_3\ h^{2(l_{\boldsymbol{z}}+1/2)}\big(|\boldsymbol{z}|^2_{\mathrm{L}^\infty(0,T;[\mathrm{H}^{\boldsymbol{z}+1}(\mathcal{T}_h)]^3)} + |\partial_t\boldsymbol{z}|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)}\big)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \max\left\{\max_{K\in\mathcal{T}_h^{\mathrm{C}}}\sigma_K,\ \max_{K\in\mathcal{T}_h}(\tau_t)_K^{\mathrm{m}},\ \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\epsilon(\tau_n)_K^{\mathrm{m}},\ \lambda|\Gamma_0|\max_{K\in B^{\mathrm{D}}}\epsilon_K\right\}\\
C_2 &= \max\left\{|\Omega|,\ \max_{K\in A^{\mathrm{D}}}\frac{\epsilon}{(\tau_n)_K^*}\right\}\ \textit{and}\\
C_3 &= \max_{K\in\mathcal{T}_h}\frac{1}{(\tau_t)_K^*}.
\end{aligned}
$$

Proof. First we differentiate (with respect to t) the first, third and fourth equation of (11) to get,

$$
\begin{aligned}
&(\mu\partial_t\boldsymbol{e}_h^{\boldsymbol{z}},\boldsymbol{r}_h)_{\mathcal{T}_h} - (\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\times\boldsymbol{r}_h)_{\mathcal{T}_h} - \langle\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{r}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0\\
&(\partial_t\boldsymbol{e}_h^{\varphi},\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\rho_h)_{\mathcal{T}_h^{\mathrm{D}}} - \langle\epsilon\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n},\rho_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = (\partial_t(\Pi_{\mathbb{P}}\varphi-\varphi),\rho_h)_{\mathcal{T}_h^{\mathrm{D}}}\\
&\langle\boldsymbol{n}\times\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{\eta}\rangle_\Gamma = 0.
\end{aligned}
\tag{13}
$$

Taking $\boldsymbol{r}_h := \boldsymbol{e}_h^{\boldsymbol{z}}, \rho_h := \boldsymbol{e}_h^{\varphi}$ and $\boldsymbol{\eta} := \boldsymbol{e}_h^{\boldsymbol{z}}$ in (13) and $\boldsymbol{v}_h =: \partial_t\boldsymbol{e}_h^{\boldsymbol{u}}, \boldsymbol{\eta} := -\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}, \boldsymbol{\xi} := \partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}$ in (11), and adding the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|^2_{\mathcal{T}_h} - (\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\times\boldsymbol{e}_h^{\boldsymbol{z}})_{\mathcal{T}_h} - \langle\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{e}_h^{\boldsymbol{z}}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \|\sigma^{1/2}\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\|^2_{\mathcal{T}_h^{\mathrm{C}}} + (\boldsymbol{e}_h^{\boldsymbol{z}},\nabla\times\partial_t\boldsymbol{e}_h^{\boldsymbol{u}})_{\mathcal{T}_h}\\
&+\langle\boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z}})^t,\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\rangle_{\partial\mathcal{T}_h} - \langle\tau_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t),\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{e}_h^{\varphi},\nabla\cdot\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}})_{\mathcal{T}_h^{\mathrm{D}}} - \langle\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n},\boldsymbol{e}_h^{\varphi}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&-\langle\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\cdot\boldsymbol{n},\tau_n(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n))\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\langle\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0} + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h^{\varphi}\|^2_{\mathcal{T}_h^{\mathrm{D}}}\\
&+(\epsilon\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\nabla\boldsymbol{e}_h^{\varphi})_{\mathcal{T}_h^{\mathrm{D}}} - \langle\epsilon\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n},\boldsymbol{e}_h^{\varphi}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle\boldsymbol{n}\times\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t},\boldsymbol{e}_h^{\boldsymbol{z}}\rangle_\Gamma - \langle\boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z}})^t,\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\rangle_{\partial\mathcal{T}_h\setminus\Gamma}\\
&+\langle\tau_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t),\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle\boldsymbol{e}_h^{\varphi},\epsilon\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\langle\epsilon\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}_0,1\rangle_{\Gamma_0}\\
&+\langle\tau_n(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)\cdot\boldsymbol{n},\epsilon\partial_t\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&= (\sigma\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u}),\partial_t\boldsymbol{e}_h^{\boldsymbol{u}})_{\mathcal{T}_h^{\mathrm{C}}} + \langle\boldsymbol{n}\times(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z})^t,\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}\rangle_{\partial\mathcal{T}_h} + \langle\partial_t\boldsymbol{e}_h^{\boldsymbol{u}},\tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)\rangle_{\partial\mathcal{T}_h}
\end{aligned}
$$

$$
\begin{aligned}
&- \langle \Pi_{\mathbb{P}}\varphi - \varphi, \epsilon\partial_t \boldsymbol{e}_h^{\boldsymbol{u}} \cdot \boldsymbol{n}\rangle_{\partial A^{\mathrm{D}}} - \langle \tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}, \epsilon\partial_t \boldsymbol{e}_h^{\boldsymbol{u}} \cdot \boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} \\
&+ \langle \tau_n((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n) \cdot \boldsymbol{n}, \epsilon\partial_t \boldsymbol{e}_h^{\boldsymbol{u}} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle \epsilon\partial_t \boldsymbol{e}_h^{\boldsymbol{u}}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0} \\
&+ (\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi), \boldsymbol{e}_h^{\varphi})_{\mathcal{T}_h^{\mathrm{D}}} - \langle \boldsymbol{n} \times (P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})^t, \partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} - \langle \tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t), \partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} \\
&+ \langle \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}, \varphi\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial A^{\mathrm{D}}} + \langle \tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}, \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} \\
&- \lambda\langle (\epsilon\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0} - \langle \tau_n((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n) \cdot \boldsymbol{n}, \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}.
\end{aligned}
$$

For the left hand side, using Green's identities and the fact that $\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}|_\Gamma = 0$ *a.e.* t (by the third equation of (11)) implies $\langle \tau_t((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}), \partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t}\rangle_\Gamma = 0$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}\|_{\mathcal{T}_h}^2 + \|\boldsymbol{e}_h^{\varphi}\|_{\mathcal{T}_h^{\mathrm{D}}}^2 + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})\|_{\partial\mathcal{T}_h}^2 + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 \\
&\qquad + \lambda\langle \epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0}^2\Big] + \|\sigma^{1/2}\partial_t \boldsymbol{e}_h^{\boldsymbol{u}}\|_{\mathcal{T}_h^{\mathrm{C}}}^2.
\end{aligned}
$$

For the right hand side, given that $\langle \epsilon\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n}, \varphi\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = 0$ (since φ is single valued, $\varphi|_\Gamma = 0$ and $\partial_t \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} \cdot \boldsymbol{n} = 0$ on Γ_0), then

$$
\begin{aligned}
&(\sigma\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}), \partial_t \boldsymbol{e}_h^{\boldsymbol{u}})_{\mathcal{T}_h^{\mathrm{C}}} + \langle \tau_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t), \partial_t((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})\rangle_{\partial\mathcal{T}_h} \\
&+ \langle \boldsymbol{n} \times (P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})^t, \partial_t((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})\rangle_{\partial\mathcal{T}_h} + \langle \tau_n((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n) \cdot \boldsymbol{n}, \epsilon\partial_t((\boldsymbol{e}_h^{\boldsymbol{u}})^n - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n}) \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&+ \lambda\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0}\langle \epsilon\partial_t((\boldsymbol{e}_h^{\boldsymbol{u}})^n - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_0, 1\rangle_{\Gamma_0} + (\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi), \boldsymbol{e}_h^{\varphi})_{\mathcal{T}_h^{\mathrm{D}}} \\
&+ \langle \epsilon\partial_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n}, \Pi_{\mathbb{P}}\varphi - \varphi\rangle_{\partial A^{\mathrm{D}}} + \langle \tau_n(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}, \epsilon\partial_t(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n) \cdot \boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}}.
\end{aligned}
$$

Now, integrating in $(0,t)$ $(0 < t \le T)$ the whole expression, using integrating by parts and Young's inequality, we have

$$
\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(t)\|_{\mathcal{T}_h}^2 + \|\boldsymbol{e}_h^{\varphi}(t)\|_{\mathcal{T}_h^{\mathrm{D}}}^2 + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})(t)\|_{\partial\mathcal{T}_h}^2 + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t) \cdot \boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 \\
&+ \lambda\langle \epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0}^2 + \int_0^t \|\sigma^{1/2}\partial_t \boldsymbol{e}_h^{\boldsymbol{u}}(s)\|_{\mathcal{T}_h^{\mathrm{C}}}^2\, ds \\
&\lesssim \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(0)\|_{\mathcal{T}_h}^2 + \|\boldsymbol{e}_h^{\varphi}(0)\|_{\mathcal{T}_h^{\mathrm{D}}}^2 + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})(0)\|_{\partial\mathcal{T}_h} + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(0) \cdot \boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&+ \|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})(t)\|_{\partial\mathcal{T}_h}^2 + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi - \varphi)(t)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(t)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 \\
&+ \|\tau_t^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)(t)\|_{\partial\mathcal{T}_h}^2 + \lambda\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(t)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0}^2 + \|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})(0)\|_{\partial\mathcal{T}_h}^2 \\
&+ \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi - \varphi)(0)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(0)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2 + \|\tau_t^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})^t(0)\|_{\partial\mathcal{T}_h}^2 \\
&+ \lambda\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(0)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0}^2 + \int_0^t \|\sigma^{1/2}\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(s)\|_{\mathcal{T}_h^{\mathrm{C}}}^2\, ds + \int_0^t \|\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi)(s)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2\, ds \\
&+ \int_0^t \|\tau_t^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)(s)\|_{\partial\mathcal{T}_h}^2\, ds + \int_0^t \lambda\langle \epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(s)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0}^2\, ds \\
&+ \int_0^t \|\epsilon^{1/2}\tau_n^{-1/2}\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi)(s)\|_{\partial\mathcal{T}_h^{\mathrm{D}}}^2\, ds + \int_0^t \|\tau_t^{-1/2}\boldsymbol{n} \times \partial_t(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})(s)\|_{\partial\mathcal{T}_h}^2\, ds
\end{aligned}
$$

$$+\int_0^t \|(\epsilon\tau_n)^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)(s)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\,ds + \int_0^t \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})^t(s)\|^2_{\partial\mathcal{T}_h}\,ds$$
$$+\int_0^t \lambda\langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}\,ds + \int_0^t \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})^t(s)\|^2_{\partial\mathcal{T}_h}\,ds$$
$$+\int_0^t \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(s)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\,ds + \int_0^t \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(s)\|^2_{\mathcal{T}_h}\,ds + \int_0^t \|\boldsymbol{e}_h^{\varphi}(s)\|^2_{\mathcal{T}_h^{\mathrm{D}}}\,ds.$$

To bound the terms at $t=0$, we evaluate (12) at $t=0$ and use Young's inequality, so

$$\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(0)\|^2_{\mathcal{T}_h} + \|\tau_t^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t} - (\boldsymbol{e}_h^{\boldsymbol{u}})^t)^t(0)\|^2_{\partial\mathcal{T}_h} + \|(\tau_n\epsilon)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(0)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} + \|\boldsymbol{e}_h^{\varphi}(0)\|^2_{\mathcal{T}_h^{\mathrm{D}}}\\
&\lesssim \|\tau_t^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)(0)\|^2_{\partial\mathcal{T}_h} + \|(\tau_n\epsilon)^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)(0)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi - \varphi)(0)\|^2_{\partial A^{\mathrm{D}}} + \|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})(0)\|^2_{\partial\mathcal{T}_h} + \|(\Pi_{\mathbb{P}}\varphi - \varphi)(0)\|^2_{\mathcal{T}_h^{\mathrm{D}}}.
\end{aligned}$$

Using the previous expression, together with Gronwall's inequality and straightforward calculation, we get

$$\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(t)\|^2_{\mathcal{T}_h} + \|\boldsymbol{e}_h^{\varphi}(t)\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})(t)\|_{\partial\mathcal{T}_h} + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)\cdot\boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&+ \lambda\langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0} + \int_0^t \|\sigma^{1/2}\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}(s)\|^2_{\mathcal{T}_h^{\mathrm{C}}}\,ds\\
&\lesssim \max_{0\le s\le T}\Big(\|(\Pi_{\mathbb{P}}\varphi - \varphi)(s)\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \|\tau_t^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})^t(s)\|^2_{\partial\mathcal{T}_h} + \|(\tau_n\epsilon)^{1/2}((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)^n(s)\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&+\|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi - \varphi)(s)\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} + \|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})(s)\|^2_{\partial\mathcal{T}_h} + \lambda\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})(s)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}\Big)\\
&+ \|\sigma^{1/2}\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \|\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))}\\
&+ \|(\epsilon\tau_n)^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^n - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\cdot\boldsymbol{n}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))} + \|\tau_t^{-1/2}\partial_t(P_{\mathbb{V}}\boldsymbol{z} - \boldsymbol{z})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)}\\
&+ \|\epsilon^{1/2}\tau_n^{-1/2}\partial_t(\Pi_{\mathbb{P}}\varphi - \varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))} + \|\tau_t^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)}\\
&+ \lambda|\Gamma_0|^{1/2}\|\epsilon\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n}\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\Gamma_0)]^3)}.
\end{aligned}$$

So, using Lemmas 3, 4, 5 and 6 we obtain,

$$\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z}}(t)\|^2_{\mathcal{T}_h} + \|\boldsymbol{e}_h^{\varphi}(t)\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u}})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t})(t)\|_{\partial\mathcal{T}_h} + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)\cdot\boldsymbol{n}\|_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&+ \lambda\langle\epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n} - (\boldsymbol{e}_h^{\boldsymbol{u}})^n)(t)|_{\Omega_{\mathrm{D}}}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0} + \int_0^t \|\sigma^{1/2}\partial_t\boldsymbol{e}_h^{\boldsymbol{u}}(s)\|^2_{\mathcal{T}_h^{\mathrm{C}}}\,ds\\
&\lesssim \left(\max_{K\in\mathcal{T}_h}(\tau_t)^{\mathrm{m}}_K\right) h^{2(\ell_{\boldsymbol{u}}+1/2)}\|\boldsymbol{u}\|^2_{\mathrm{L}^\infty(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3)} + h^{2(\ell_\varphi+1/2)}\|\varphi\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))}\\
&+ \left(\max_{K\in\mathcal{T}_h^{\mathrm{D}}}\epsilon(\tau_n)^{\mathrm{m}}_K\right)\ h^{2(\ell_{\boldsymbol{u}}+1/2)}\ \|\boldsymbol{u}\|^2_{\mathrm{L}^\infty(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{D}})]^3)} + h^{2(\ell_\varphi+1/2)}\|\partial_t\varphi\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{\ell_\varphi+1}(A^{\mathrm{D}}))}\\
&+ \lambda|\Gamma_0|\left(\max_{K\in B^{\mathrm{D}}}\epsilon_K\right) h^{2(\ell_{\boldsymbol{u}}+1/2)}\ \|\boldsymbol{u}\|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(B^{\mathrm{D}})]^3)} + \|\sigma^{1/2}\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u} - \boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)}\\
&+ \left(\max_{K\in A^{\mathrm{D}}}\frac{\epsilon}{(\tau_n)^*_K}\right) h^{2(\ell_\varphi+1/2)}\|\varphi\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell_\varphi+1}(A^{\mathrm{D}}))} + \|\tau_t^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^t - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^t)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\max_{K\in\mathcal{T}_h}\frac{1}{(\tau_t)_K^*}\right)h^{2(\ell_{\boldsymbol{z}}+1/2)}\;\|\boldsymbol{z}\|^2_{\mathrm{L}^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)}+\|\tau_t^{-1/2}\partial_t(P_{\mathbb{V}}\boldsymbol{z}-\boldsymbol{z})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)}\\
&+\|\epsilon^{1/2}\tau_n^{-1/2}\partial_t(\Pi_{\mathbb{P}}\varphi-\varphi)\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}}))}+\|(\epsilon\tau_n)^{1/2}\partial_t((\Pi_{\mathbb{V}}\boldsymbol{u})^n-\mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^n)\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)}\\
&+\lambda|\Gamma_0|^{1/2}\|\epsilon\partial_t(\Pi_{\mathbb{V}}\boldsymbol{u}-\boldsymbol{u})|_{\Omega_{\mathrm{D}}}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma_0)]^3)}.
\end{aligned}
$$

Finally, using Remark 7 and Theorem 9 we conclude the proof. ■

5 A fully discrete HDG scheme

To define a fully discrete scheme, we divide the time interval $(0,T)$ into M uniform subintervals $I^k=[t_{k-1},t_k]$, with size $\Delta t=t_k-t_{k-1}=T/M, 1\le k\le M$, such that $0=t_0<t_1<t_2<\cdots<t_M=T$. Let us denote (for example) $\boldsymbol{u}_h^k=\boldsymbol{u}_h(\cdot,t_k), 0\le k\le M$, so if we use the backward Euler finite difference scheme to discretize the time derivative, we get the fully discrete variational formulation: For each $0\le k\le M$, find $(\boldsymbol{z}_h^k,\boldsymbol{u}_h^k,\varphi_h^k,\widehat{\boldsymbol{u}}_h^{t,k},\widehat{\boldsymbol{u}}_h^{n,k})\in\mathbb{V}_h\times\mathbb{V}_h\times\mathbb{P}_h^\Gamma\times\mathbb{M}_h^t\times\mathbb{M}_h^n(0)$ such that

$$
\begin{aligned}
&(\mu\boldsymbol{z}_h^k,\boldsymbol{r}_h)_{\mathcal{T}_h}-(\boldsymbol{u}_h^k,\nabla\times\boldsymbol{r}_h)_{\mathcal{T}_h}-\langle\widehat{\boldsymbol{u}}_h^{t,k},\boldsymbol{r}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}=0\\
&\left(\sigma\frac{\boldsymbol{u}_h^k-\boldsymbol{u}_h^{k-1}}{\Delta t},\boldsymbol{v}_h\right)_{\mathcal{T}_h}+(\boldsymbol{z}_h^k,\nabla\times\boldsymbol{v}_h)_{\mathcal{T}_h}+\langle\boldsymbol{z}_h^{t,k},\boldsymbol{v}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}+(\varphi_h^k,\nabla\cdot(\epsilon\boldsymbol{v}_h))_{\mathcal{T}_h^{\mathrm{D}}}\\
&\qquad+\langle\tau_t(\boldsymbol{u}_h^{t,k}-\widehat{\boldsymbol{u}}_h^{t,k})\times\boldsymbol{n},\boldsymbol{v}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}+\langle\tau_n(\boldsymbol{u}_h^{n,k}-\widehat{\boldsymbol{u}}_h^{n,k}),\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^0}\\
&\qquad-\langle\widetilde{\varphi}_h^k,\epsilon\boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^0}=-(\partial_t\boldsymbol{J},\boldsymbol{v}_h)_{\mathcal{T}_h}\\
&(\varphi_h^k,\rho_h)_{\mathcal{T}_h^{\mathrm{D}}}+(\epsilon\boldsymbol{u}_h^k,\nabla\rho_h)_{\mathcal{T}_h^{\mathrm{D}}}-\langle\epsilon\widehat{\boldsymbol{u}}_h^{n,k}\cdot\boldsymbol{n},\rho_h\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}=0\\
&\langle\boldsymbol{n}\times\widehat{\boldsymbol{u}}_h^{t,k},\boldsymbol{\eta}\rangle_\Gamma=\langle\boldsymbol{g},\boldsymbol{\eta}\rangle_\Gamma\\
&\langle\boldsymbol{n}\times\boldsymbol{z}_h^{t,k},\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma}+\langle\tau_t(\boldsymbol{u}_h^{t,k}-\widehat{\boldsymbol{u}}_h^{t,k}),\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h\setminus\Gamma}=0\\
&\langle\widetilde{\varphi}_h^k,\epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}-\langle\tau_n(\boldsymbol{u}_h^{n,k}-\widehat{\boldsymbol{u}}_h^{n,k})\cdot\boldsymbol{n},\epsilon\boldsymbol{\xi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}=0\\
&\boldsymbol{u}_h^0=\Pi_v\boldsymbol{u}_0.
\end{aligned}
\tag{14}
$$

As we did for the semidiscrete case, let us introduce the *projection errors* defined as

$$
\begin{aligned}
&\boldsymbol{e}_h^{\varphi,k}:=\Pi_{\mathbb{P}}\varphi^k-\varphi_h^k, &&\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k}:=\mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^{t,k}-\widehat{\boldsymbol{u}}_h^{t,k}, &&\boldsymbol{e}_h^{\boldsymbol{z},k}:=P_{\mathbb{V}}\boldsymbol{z}^k-\boldsymbol{z}_h^k,\\
&\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}:=\mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^{n,k}-\widehat{\boldsymbol{u}}_h^{n,k}, &&\boldsymbol{e}_h^{\boldsymbol{u},k}:=\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}_h^k.
\end{aligned}
$$

The projection errors satisfy the following equations.

Lemma 13 *Let $(\boldsymbol{z},\boldsymbol{u},\varphi)$ and $(\boldsymbol{z}_h^k,\boldsymbol{u}_h^k,\varphi_h^k,\widehat{\boldsymbol{u}}_h^{t,k},\widehat{\boldsymbol{u}}_h^{n,k})$ be the solutions of* (3) *and* (14) *at the time t and t_k, respectively. Then, the projection errors $\boldsymbol{e}_h^{\boldsymbol{u},k},\boldsymbol{e}_h^{\boldsymbol{z},k},\boldsymbol{e}_h^{\varphi,k},\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k},\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}$ satisfy*

$$
\begin{aligned}
&(\mu\boldsymbol{e}_h^{\boldsymbol{z},k},\boldsymbol{r}_h)_{\mathcal{T}_h}-(\nabla\times\boldsymbol{r}_h,\boldsymbol{e}_h^{\boldsymbol{u},k})_{\mathcal{T}_h}-\langle\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k},\boldsymbol{r}_h\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}=0\\
&\left(\sigma\frac{\boldsymbol{e}_h^{\boldsymbol{u},k}-\boldsymbol{e}_h^{\boldsymbol{u},k-1}}{\Delta t},\boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}}+(\nabla\times\boldsymbol{v}_h,\boldsymbol{e}_h^{\boldsymbol{z},k})_{\mathcal{T}_h}+\langle\boldsymbol{n}\times(\boldsymbol{e}_h^{\boldsymbol{z},k})^t,\boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h}+(\nabla\cdot\epsilon\boldsymbol{v}_h,\boldsymbol{e}_h^{\varphi,k})_{\mathcal{T}_h^{\mathrm{D}}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \langle \tau_t (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k} - \boldsymbol{e}_h^{\boldsymbol{u},k})^t \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \boldsymbol{e}_h^{\varphi,k} + \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - (\boldsymbol{e}_h^{\boldsymbol{u},k})^n)) \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&+ \lambda \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u},k}|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \\
&= \left(\sigma \frac{(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}} \boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t}, \boldsymbol{v}_h \right)_{\mathcal{T}_h^{\mathrm{C}}} + \langle (P_{\mathbb{V}} \boldsymbol{z}^k - \boldsymbol{z}^k)^t, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle \tau_t (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^k) \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \Pi_{\mathbb{P}} \varphi^k - \varphi^k, \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial A^{\mathrm{D}}} \\
&\quad - \langle \tau_n (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} + \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, \tau_n (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^k) \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\quad + \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k)|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \left(\sigma \left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} - \partial_t \boldsymbol{u}^k \right), \boldsymbol{v}_h \right)_{\mathcal{T}_h^{\mathrm{C}}} \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{e}_h^{\varphi,k}, \rho_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\nabla \rho_h, \epsilon \boldsymbol{e}_h^{\boldsymbol{u},k})_{\mathcal{T}_h^{\mathrm{D}}} - \langle \rho_h, \epsilon \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = (\Pi_{\mathbb{P}} \varphi^k - \varphi^k, \rho_h)_{\mathcal{T}_h^{\mathrm{D}}} \\
&\langle \boldsymbol{n} \times \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k}, \boldsymbol{\eta}\rangle_{\Gamma} = 0 \\
&\langle \boldsymbol{n} \times (\boldsymbol{e}_h^{\boldsymbol{z},k})^t, \boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h \setminus \Gamma} - \langle \tau_t (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k} - (\boldsymbol{e}_h^{\boldsymbol{u},k})^t), \boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h \setminus \Gamma} = \langle \boldsymbol{n} \times (P_{\mathbb{V}} \boldsymbol{z}^k - \boldsymbol{z}^k), \boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h \setminus \Gamma} \\
&\qquad\qquad - \langle \tau_t (\mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^k - \Pi_{\mathbb{V}} \boldsymbol{u}^k)^t), \boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h \setminus \Gamma} \\
&\langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \boldsymbol{e}_h^{\varphi,k}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u},k} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \tau_n (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - (\boldsymbol{e}_h^{\boldsymbol{u},k})^n) \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\quad = \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \Pi_{\mathbb{P}} \varphi^k - \varphi^k\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \\
&\quad\quad - \langle \epsilon \boldsymbol{\xi} \cdot \boldsymbol{n}, \tau_n (\mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^k - \Pi_{\mathbb{V}} \boldsymbol{u}^k)^n \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\boldsymbol{e}_h^{\boldsymbol{u},0} = \boldsymbol{0}
\end{aligned}
$$

for all $(\boldsymbol{r}_h, \boldsymbol{v}_h, \rho_h, \boldsymbol{\eta}, \boldsymbol{\xi}) \in \mathbb{V}_h \times \mathbb{V}_h \times \mathbb{P}_h^{\Gamma} \times \mathbb{M}_h^t \times \mathbb{M}_h^n(0)$.

Proof. The proof is similar to the one of Lemma 8. So we just sketch how to obtain the second equation on (15). Notice that, using consistency together with the properties of the projectors operators we have

$$
\begin{aligned}
&(\boldsymbol{z}_h^k, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} = (P_{\mathbb{V}} \boldsymbol{z}^k - \boldsymbol{e}_h^{\boldsymbol{z},k}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} = (\boldsymbol{z}^k, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} - (\boldsymbol{e}_h^{\boldsymbol{z},k}, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_h} \\
&\langle \boldsymbol{z}_h^{t,k}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle (P_{\mathbb{V}} \boldsymbol{z}^k - \boldsymbol{z}^k)^t, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{z}^{t,k}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle (\boldsymbol{e}_h^{\boldsymbol{z},k})^t, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\langle \tau_t (\boldsymbol{u}_h^{t,k} - \widehat{\boldsymbol{u}}_h^{t,k}) \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = \langle \tau_t ((\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{e}_h^{\boldsymbol{u},k})^t - (\mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^k - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})) \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\qquad = \langle \tau_t (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^k) \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \tau_t (\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k} - (\boldsymbol{e}_h^{\boldsymbol{u},k})^t) \times \boldsymbol{n}, \boldsymbol{v}_h \times \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&(\epsilon \varphi_h^k, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{D}}} = (\epsilon (\Pi_{\mathbb{P}} \varphi^k - \boldsymbol{e}_h^{\varphi,k}), \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{D}}} = -(\epsilon \boldsymbol{e}_h^{\varphi,k}, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{D}}} + (\epsilon \varphi^k, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h^{\mathrm{D}}} \\
&\langle \epsilon \widetilde{\varphi}_h^k, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = \langle \epsilon \varphi_h^k, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda \langle \epsilon \boldsymbol{u}_h^k \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \\
&\qquad = \langle \epsilon (\Pi_{\mathbb{P}} \varphi^k - \varphi^k), \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle \epsilon \varphi^k, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \langle \epsilon \boldsymbol{e}_h^{\varphi,k}, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\qquad\quad - \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \lambda \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u},k} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \\
&\qquad = -\langle \Pi_{\mathbb{P}} \varphi^k - \varphi^k, \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial A^{\mathrm{D}}} - \langle \tau_n (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} + \langle \epsilon \varphi^k, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
&\qquad\quad - \langle \epsilon \boldsymbol{e}_h^{\varphi,k}, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} - \lambda \langle \epsilon (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} + \lambda \langle \epsilon \boldsymbol{e}_h^{\boldsymbol{u},k} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \langle \epsilon \boldsymbol{v}_h \cdot \boldsymbol{n}, 1\rangle_{\Gamma_0} \\
&\langle \epsilon \tau_n (\boldsymbol{u}_h^{n,k} - \widehat{\boldsymbol{u}}_h^{n,k}) \cdot \boldsymbol{n}, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} = \langle \epsilon \tau_n (\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{e}_h^{\boldsymbol{u},k} - (\mathrm{P}_{\mathbb{M}_h^n(0)} \boldsymbol{u}^{n,k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})) \cdot \boldsymbol{n}, \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}
\end{aligned}
$$

$$= \langle \epsilon\tau_n(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^k)\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \langle \epsilon\tau_n(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - \boldsymbol{e}_h^{\boldsymbol{u},k})\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}$$

$$\left(\sigma\frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}, \boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}} = \left(\sigma\frac{\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{e}_h^{\boldsymbol{u},k} - \Pi_{\mathbb{V}}\boldsymbol{u}^{k-1} + \boldsymbol{e}_h^{\boldsymbol{u},k-1}}{\Delta t}, \boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}}$$

$$= -\left(\sigma\frac{\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\boldsymbol{u},k-1}}{\Delta t}, \boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}} + \left(\sigma\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t}, \boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}}$$

$$+ \left(\sigma\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t}, \boldsymbol{v}_h\right)_{\mathcal{T}_h^{\mathrm{C}}}.$$

So adding these equations all together, we get the second equation of (15). ■
Now, if we call

$$\|\Phi_{\Delta t}\|^2_{\ell^2(t^n,t^m;E)} := \Delta t\sum_{k=n}^{m}\|\Phi^k\|_E^2 \qquad \text{and} \qquad \|\Phi_{\Delta t}\|^2_{\ell^\infty(t^n,t^m;E)} := \max_{n\le k\le m}\|\Phi^k\|_E^2,$$

then, as a consequence of this Lemma we get,

Theorem 14 *For $T > 0$, any $s \geq 0$ and $l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi \in [0, s]$, assuming that the exact solution of* (3), $(\boldsymbol{z}, \boldsymbol{u}, \varphi) \in \mathrm{H}^1(0, T; [\mathrm{H}^{l_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3) \times \mathrm{H}^2(0, T; [\mathrm{H}^{l_{\boldsymbol{u}}+1}(\mathcal{T}_h)]^3) \times \mathrm{H}^1(0, T; \mathrm{H}^{l_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))$*, there holds*

$$\begin{aligned}
&\|\sigma^{1/2}\boldsymbol{e}^{\boldsymbol{u}}_{h,\Delta t}\|^2_{\ell^\infty(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + \|\mu^{1/2}\boldsymbol{e}^{\boldsymbol{z}}_{h,\Delta t}\|^2_{\ell^2(0,T;[\mathrm{L}^2(\mathcal{T}_h)]^3)} + \|\boldsymbol{e}^{\varphi}_{h,\Delta t}\|^2_{\ell^2(0,T;\mathrm{L}^2(\mathcal{T}_h^{\mathrm{D}}))}\\
&+ \|\tau_t^{1/2}((\boldsymbol{e}^{\boldsymbol{u}}_{h,\Delta t})^t - \boldsymbol{e}^{\widehat{\boldsymbol{u}}_h^t}_{h,\Delta t})\times\boldsymbol{n}\|^2_{\ell^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h)]^3)} + \|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}^{\boldsymbol{u}}_{h,\Delta t} - \boldsymbol{e}^{\widehat{\boldsymbol{u}}_h^n}_{h,\Delta t})\cdot\boldsymbol{n}\|^2_{\ell^2(0,T;[\mathrm{L}^2(\partial\mathcal{T}_h^{\mathrm{D}})]^3)}\\
&+ \lambda\langle\epsilon\boldsymbol{e}^{\boldsymbol{u}}_{h,\Delta t}\cdot\boldsymbol{n}, 1\rangle^2_{\ell^2(0,T;[\mathrm{L}^2(\Gamma_0)]^3)}\\
&\lesssim \Delta t^2\|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)} + C_1 h^{2(\ell_{\boldsymbol{z}}+1/2)}\ \|\boldsymbol{z}_{\Delta t}\|^2_{\ell^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{z}}+1}(\mathcal{T}_h)]^3)} + C_2 h^{2(\ell_{\boldsymbol{u}}+1/2)}\|\boldsymbol{u}_{\Delta t}\|^2_{\ell^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{D}})]^3)}\\
&\quad + C_3 h^{2(\ell_\varphi+1/2)}\|\varphi_{\Delta t}\|^2_{\ell^2(0,T;\mathrm{H}^{\ell_\varphi+1}(\mathcal{T}_h^{\mathrm{D}}))} + C_4 h^{2(\ell_{\boldsymbol{u}}+1/2)}\|\boldsymbol{u}_{\Delta t}\|^2_{\ell^2(0,T;[\mathrm{H}^{\ell_{\boldsymbol{u}}+1}(\mathcal{T}_h^{\mathrm{C}})]^3)}.
\end{aligned}$$

where

$$C_1 := \max\left\{\max_{K\in A^{\mathrm{D}}}\left(\epsilon\,(\tau_n)_K^{\mathtt{m}}\right),\ \max_{K\in\mathcal{T}_h^{\mathrm{D}}}(\tau_t)_K^{\mathtt{m}},\ \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\left(\frac{\epsilon}{(\tau_t)_K^*}\right)^{1/2},\ \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\lambda\epsilon\right\},$$

$$C_2 := \max\left\{|\Omega|\max_{K\in\mathcal{T}_h^{\mathrm{C}}}(\sigma h_K),\ \max_{K\in\mathcal{T}_h^{\mathrm{C}}}(\tau_t)_K^{\mathtt{m}}\right\},$$

$$C_3 := \max\left\{\max_{K\in\mathcal{T}_h^{\mathrm{D}}}h_K,\ \max_{K\in\mathcal{T}_h^{\mathrm{D}}}\left(\frac{\epsilon}{(\tau_n)_K^*}\right)\right\},\ \textit{and}$$

$$C_4 := \max_{K\in\mathcal{T}_h}\left(\frac{1}{(\tau_t)_K^*}\right).$$

Here, given $K \in \mathcal{T}_h$, we have $(\tau_n)_K^{\mathtt{m}} := \max\tau_n|_{\partial K}$, $(\tau_t)_K^{\mathtt{m}} := \max\tau_t|_{\partial K}$, $(\tau_n)_K^* := \min\tau_n|_{\partial K}$, and $(\tau_t)_K^* := \min\tau_t|_{\partial K}$.

Proof. We proceed as in Theorem 12, taking $\boldsymbol{r}_h := \boldsymbol{e}_h^{\boldsymbol{z},k}, \boldsymbol{v}_h := \boldsymbol{e}_h^{\boldsymbol{u},k}, \rho_h := \boldsymbol{e}_h^{\varphi,k}, \boldsymbol{\eta} := -\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k}, \boldsymbol{\xi} := \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}$ in (15), adding the resulting equations all together and using Green's identities, to obtain: for the left hand side:

$$
\begin{aligned}
&\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z},k}\|^2_{\mathcal{T}_h} + \left(\sigma\frac{\boldsymbol{e}_h^{\boldsymbol{u},k}-\boldsymbol{e}_h^{\boldsymbol{u},k-1}}{\Delta t}, \boldsymbol{e}_h^{\boldsymbol{u},k}\right)_{\mathcal{T}_h^{\mathrm{C}}} + \langle \tau_t((\boldsymbol{e}_h^{\boldsymbol{u},k})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}, \boldsymbol{e}_h^{\boldsymbol{u},k}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \|\boldsymbol{e}_h^{\varphi,k}\|^2_{\mathcal{T}_h^{\mathrm{D}}}\\
&+ \langle \epsilon(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}, \boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0} - \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}\langle\epsilon\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}\\
&- \langle \tau_t((\boldsymbol{e}_h^{\boldsymbol{u},k})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}\, \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} - \langle \tau_n((\boldsymbol{e}_h^{\boldsymbol{u},k})^n - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}, \epsilon\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\geq \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z},k}\|^2_{\mathcal{T}_h} + \|\boldsymbol{e}_h^{\varphi,k}\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u},k})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})^t\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h} + \|(\tau_n\epsilon)^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad + \lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0} + \frac{1}{2\Delta t}\left(\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}} - \|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k-1}\|^2_{\mathcal{T}_h^{\mathrm{C}}}\right),
\end{aligned}
$$

for the right hand side:

$$
\begin{aligned}
&\left(\sigma\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t}, \boldsymbol{e}_h^{\boldsymbol{u},k}\right)_{\mathcal{T}_h^{\mathrm{C}}} + \langle (P_{\mathbb{V}}\boldsymbol{z}^k - \boldsymbol{z}^k)^t, (\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle \tau_t(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^k)\times\boldsymbol{n}, (\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\times\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \Pi_{\mathbb{P}}\varphi^k - \varphi^k, \epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - \boldsymbol{e}_h^{\boldsymbol{u},k})\cdot\boldsymbol{n}\rangle_{\partial A^{\mathrm{D}}}\\
&\quad + \langle \tau_n(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k)\cdot\boldsymbol{n}, \epsilon(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - \boldsymbol{e}_h^{\boldsymbol{u},k})\cdot\boldsymbol{n}\rangle_{\partial B^{\mathrm{D}}} + \lambda\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k)\cdot\boldsymbol{n},1\rangle_{\Gamma_0}\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}\\
&\quad + \langle \epsilon\tau_n(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^k)\cdot\boldsymbol{n}, (\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h^{\mathrm{D}}} + \left(\sigma\left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} - \partial_t\boldsymbol{u}^k\right), \boldsymbol{e}_h^{\boldsymbol{u},k}\right)_{\mathcal{T}_h^{\mathrm{C}}}\\
&\quad + (\Pi_{\mathbb{P}}\varphi^k - \varphi^k, \boldsymbol{e}_h^{\varphi,k})_{\mathcal{T}_h^{\mathrm{D}}}\\
&\leq \frac{T}{8}\left\|\sigma^{1/2}\left[\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t}\right]\right\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \frac{1}{8T}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}\\
&\quad + \|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z}^k - \boldsymbol{z}^k)^t\|^2_{\partial\mathcal{T}_h} + \frac{1}{4}\|\tau_t^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h} + \|\tau_t^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^k)\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h}\\
&\quad + \frac{1}{4}\|\tau_t^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h} + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi^k - \varphi^k)\|^2_{\partial A^{\mathrm{D}}} + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k)\|^2_{\partial B^{\mathrm{D}}}\\
&\quad + \frac{1}{4}\|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k} - \boldsymbol{e}_h^{\boldsymbol{u},k})\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^k)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad + \frac{1}{4}\|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} + \frac{\lambda}{2}\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}^k - \boldsymbol{u}^k)\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0} + \frac{\lambda}{2}\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}\\
&\quad + \frac{1}{2}\|\boldsymbol{e}_h^{\varphi,k}\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \frac{1}{2}\|\Pi_{\mathbb{P}}\varphi^k - \varphi^k\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \frac{T}{8}\left\|\sigma^{1/2}\left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} - \partial_t\boldsymbol{u}^k\right)\right\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \frac{1}{8T}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}.
\end{aligned}
$$

Now, putting both sides together we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}} - \frac{1}{2\Delta t}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k-1}\|^2_{\mathcal{T}_h^{\mathrm{C}}} + \|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z},k}\|^2_{\mathcal{T}_h} + \frac{1}{2}\|\boldsymbol{e}_h^{\varphi,k}\|^2_{\mathcal{T}_h^{\mathrm{D}}}\\
&+ \frac{1}{2}\|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u},k})^t - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h} + \frac{1}{2}\|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k} - \boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}\|^2_{\mathcal{T}_h^{\mathrm{D}}} + \frac{\lambda}{2}\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{T}{8}\left\|\sigma^{1/2}\left[\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)-(\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1}-\boldsymbol{u}^{k-1})}{\Delta t}\right]\right\|^2_{\mathcal{T}_h^{\mathrm{C}}}+\frac{T}{8}\left\|\sigma^{1/2}\left(\frac{\boldsymbol{u}^k-\boldsymbol{u}^{k-1}}{\Delta t}-\partial_t\boldsymbol{u}^k\right)\right\|^2_{\mathcal{T}_h^{\mathrm{C}}} \qquad (16)\\
&\quad+\|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z}^k-\boldsymbol{z}^k)^t\|^2_{\partial\mathcal{T}_h}+\|\tau_t^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^k)\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h}\\
&\quad+\|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi^k-\varphi^k)\|^2_{\partial A^{\mathrm{D}}}+\|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^k)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}\\
&\quad+\frac{\lambda}{2}\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}+\|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)\|^2_{\partial B^{\mathrm{D}}}+\frac{1}{4T}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}.
\end{aligned}
$$

Let us call,

$$
\begin{aligned}
f^k:=&\ \frac{T}{8}\left\|\sigma^{1/2}\left[\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)-(\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1}-\boldsymbol{u}^{k-1})}{\Delta t}\right]\right\|^2_{\mathcal{T}_h^{\mathrm{C}}}+\|\tau_t^{-1/2}(P_{\mathbb{V}}\boldsymbol{z}^k-\boldsymbol{z}^k)^t\|^2_{\partial\mathcal{T}_h}\\
&+\|\tau_t^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\mathrm{P}_{\mathbb{M}_h^t}\boldsymbol{u}^k)\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h}+\|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}}\varphi^k-\varphi^k)\|^2_{\partial A^{\mathrm{D}}}\\
&+\|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\mathrm{P}_{\mathbb{M}_h^n(0)}\boldsymbol{u}^k)\cdot\boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}}+\frac{\lambda}{2}\langle\epsilon(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}\\
&+\|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)\|^2_{\partial B^{\mathrm{D}}}+\frac{T}{8}\left\|\sigma^{1/2}\left(\frac{\boldsymbol{u}^k-\boldsymbol{u}^{k-1}}{\Delta t}-\partial_t\boldsymbol{u}^k\right)\right\|^2_{\mathcal{T}_h^{\mathrm{C}}},
\end{aligned}
$$

hence,

$$
\left(1-\frac{1}{2N}\right)\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}\leq\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k-1}\|^2_{\mathcal{T}_h^{\mathrm{C}}}+2\Delta t f^k,
$$

and using the discrete Gronwall's inequality (see Lemma 6.26 from [13]), we conclude that there exists a constant C such that

$$
\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}\leq C\Delta t\sum_{m=1}^{k-1}f^m\leq C\Delta t\sum_{m=1}^{M}f^m,
$$

thus

$$
\sum_{k=1}^{m}\|\sigma^{1/2}\boldsymbol{e}_h^{\boldsymbol{u},k}\|^2_{\mathcal{T}_h^{\mathrm{C}}}\leq C\sum_{k=1}^{m}\left(\Delta t\sum_{m=1}^{M}f^m\right)\leq MC\frac{T}{M}\sum_{k=1}^{M}f^k=C\sum_{k=1}^{M}f^k. \qquad (17)
$$

Multiplying (16) by $2\Delta t$, summing from 1 to m ($m\leq M$), taking maximum and using (17), we have

$$
\begin{aligned}
&\max_{1\leq m\leq M}\|\sigma^{1/2}\boldsymbol{e}_h^{u,m}\|^2_{\mathcal{T}_h^{\mathrm{C}}}+\Delta t\sum_{k=1}^{M}\Big[\|\mu^{1/2}\boldsymbol{e}_h^{\boldsymbol{z},k}\|^2_{\mathcal{T}_h}+\|\boldsymbol{e}_h^{\varphi,k}\|^2_{\mathcal{T}_h^{\mathrm{D}}}+\|\tau_t^{1/2}((\boldsymbol{e}_h^{\boldsymbol{u},k})^t-\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^t,k})\times\boldsymbol{n}\|^2_{\partial\mathcal{T}_h}\\
&\quad+\|(\epsilon\tau_n)^{1/2}(\boldsymbol{e}_h^{\boldsymbol{u},k}-\boldsymbol{e}_h^{\widehat{\boldsymbol{u}}_h^n,k})\cdot\boldsymbol{n}\|^2_{\mathcal{T}_h^{\mathrm{D}}}+\lambda\langle\epsilon\boldsymbol{e}_h^{\boldsymbol{u},k}\cdot\boldsymbol{n},1\rangle^2_{\Gamma_0}\Big]\\
&\quad\lesssim\Delta t\sum_{k=1}^{M}\left\|\sigma^{1/2}\left[\frac{(\Pi_{\mathbb{V}}\boldsymbol{u}^k-\boldsymbol{u}^k)-(\Pi_{\mathbb{V}}\boldsymbol{u}^{k-1}-\boldsymbol{u}^{k-1})}{\Delta t}\right]\right\|^2_{\mathcal{T}_h^{\mathrm{C}}}\\
&\qquad+\Delta t\sum_{k=1}^{M}\left\|\sigma^{1/2}\left(\frac{\boldsymbol{u}^k-\boldsymbol{u}^{k-1}}{\Delta t}-\partial_t\boldsymbol{u}^k\right)\right\|^2_{\mathcal{T}_h^{\mathrm{C}}} \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
+ \Delta t \sum_{k=1}^{M} \Big[& \|\tau_t^{-1/2}(P_{\mathbb{V}} \boldsymbol{z}^k - \boldsymbol{z}^k)^t\|^2_{\partial\mathcal{T}_h} + \|\tau_t^{1/2}(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_h^t} \boldsymbol{u}^k) \times \boldsymbol{n}\|^2_{\partial\mathcal{T}_h} \\
& + \|\epsilon^{1/2}\tau_n^{-1/2}(\Pi_{\mathbb{P}} \varphi^k - \varphi^k)\|^2_{\partial A^{\mathrm{D}}} + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \mathrm{P}_{\mathbb{M}_{h(0)}^n} \boldsymbol{u}^k) \cdot \boldsymbol{n}\|^2_{\partial\mathcal{T}_h^{\mathrm{D}}} \\
& + \frac{\lambda}{2}|\Gamma_0|\langle \epsilon(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) \cdot \boldsymbol{n}, 1\rangle^2_{\Gamma_0} + \|(\epsilon\tau_n)^{1/2}(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k)\|^2_{\partial B^{\mathrm{D}}} \Big].
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left| \partial_t \boldsymbol{u}^k - \frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} \right| &= \left| \frac{1}{2\Delta t} \int_{t_{k-1}}^{t_k} \partial_{tt} \boldsymbol{u}(t)(t - t_{k-1})^2 \, dt \right| \le \frac{1}{2\Delta t} \int_{t_{k-1}}^{t_k} |\partial_{tt}\boldsymbol{u}(t)||t - t_{k-1}|^2 \, dt \\
&\le \frac{\Delta t}{2} \int_0^T |\partial_{tt}\boldsymbol{u}(t)| dt \le \frac{\Delta t}{2} T^{1/2} \left(\int_0^T |\partial_{tt}\boldsymbol{u}(t)|^2 dt \right)^{1/2},
\end{aligned}
$$

therefore

$$
\begin{aligned}
\left\| \sigma^{1/2} \left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} - \partial_t \boldsymbol{u}^k \right) \right\|^2_{\mathcal{T}_h^{\mathrm{C}}} &= \sum_{K \in \mathcal{T}_h^{\mathrm{C}}} \int_K \left| \sigma^{1/2} \left(\partial_t \boldsymbol{u}^k - \frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} \right) \right|^2 d\mathbf{x} \\
&\le \sum_{K \in \mathcal{T}_h^{\mathrm{C}}} \int_K \left(\sigma \frac{T}{4} (\Delta t)^2 \int_0^T |\partial_{tt}\boldsymbol{u}(t)|^2 dt \right) d\mathbf{x} \\
&\le \frac{\sigma T}{4} (\Delta t)^2 \int_0^T \int_{\mathcal{T}_h^{\mathrm{C}}} |\partial_{tt}\boldsymbol{u}(t)|^2 d\mathbf{x} \, dt \\
&\le \frac{\sigma T}{4} (\Delta t)^2 \|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)},
\end{aligned}
$$

whence,

$$
\Delta t \sum_{k=1}^{M} \left\| \sigma^{1/2} \left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\Delta t} - \partial_t \boldsymbol{u}^k \right) \right\|^2_{\mathcal{T}_h^{\mathrm{C}}} \le C \Delta t^2 \|\boldsymbol{u}\|^2_{\mathrm{H}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)}.
$$

Likewise,

$$
\left| \sigma^{1/2} \frac{(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}} \boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t} \right| \le \frac{\sigma^{1/2}}{\Delta t^{1/2}} \left(\int_0^T |\partial_t (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u})(t)|^2 \, dt \right)^{1/2},
$$

thus

$$
\begin{aligned}
\left\| \sigma^{1/2} \left[\frac{(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}} \boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t} \right] \right\|^2_{\mathcal{T}_h^{\mathrm{C}}} &\le \frac{\sigma}{\Delta t} \int_0^T \left(\int_{\mathcal{T}_h^{\mathrm{C}}} |\partial_t (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u})(t)|^2 \, d\mathbf{x} \right) dt \\
&= \frac{\sigma}{\Delta t} \|\partial_t (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)},
\end{aligned}
$$

and

$$
\Delta t \sum_{k=1}^{M} \left\| \sigma^{1/2} \left[\frac{(\Pi_{\mathbb{V}} \boldsymbol{u}^k - \boldsymbol{u}^k) - (\Pi_{\mathbb{V}} \boldsymbol{u}^{k-1} - \boldsymbol{u}^{k-1})}{\Delta t} \right] \right\|^2_{\mathcal{T}_h^{\mathrm{C}}} \le M\sigma \|\partial_t (\Pi_{\mathbb{V}} \boldsymbol{u} - \boldsymbol{u})\|^2_{\mathrm{L}^2(0,T;[\mathrm{L}^2(\mathcal{T}_h^{\mathrm{C}})]^3)}.
$$

Finally, substituting these results in (18) and proceeding as in Theorem 12 we get what we want to prove. ■

Now, for the fully discrete case, we can obtain a result as the one in Corollary 10, and then by Theorem 14 and triangle inequality we can prove the convergence order given by the next theorem.

Theorem 15 *Under the same assumptions in Theorem 14, we have that*

$$\|\boldsymbol{z}-\boldsymbol{z}_{h,\Delta t}\|_{\ell^\infty(0,T;[\mathrm{L}^2(\Omega)]^3)} = \mathcal{O}(\Delta t + h^{\ell+1/2}), \qquad \left\|(\boldsymbol{u}^n-\widehat{\boldsymbol{u}}^n_{h,\Delta t})\cdot\boldsymbol{n}\right\|_{\ell^2(0,T;[\mathrm{L}^2_h(\mathcal{T}_h^{\mathrm{D}})]^3)} = \mathcal{O}(\Delta t + h^{\ell+1/2}),$$

$$\left\|\sigma^{1/2}(\boldsymbol{u}-\boldsymbol{u}^k_{h,\Delta t})\right\|_{\ell^2(0,T;[\mathrm{L}^2(\Omega)]^3)} = \mathcal{O}(\Delta t + h^{\ell+1/2}), \quad \left\|\boldsymbol{u}^t-\widehat{\boldsymbol{u}}^t_{h,\Delta t}\right\|_{\ell^2(0,T;[\mathrm{L}^2_h(\mathcal{T}_h^{\mathrm{C}})]^3)} = \mathcal{O}(\Delta t + h^{\ell+1/2}),$$

$$\|\varphi-\varphi_{h,\Delta t}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} = \mathcal{O}(\Delta t + h^{\ell+1/2}).$$

where $\ell = \min\{l_{\boldsymbol{z}}, l_{\boldsymbol{u}}, l_\varphi\}$*, provided* τ_n *and* τ_n^{-1}*, as well as* τ_t *and* τ_t^{-1} *remain of order one on* $\partial\mathcal{T}_h^{\mathrm{D}}$ *and* $\partial\mathcal{T}_h$*, respectively.*

6 Concluding remarks

We have been able to develop a complete error analysis for an HDG method applied to a transient eddy current problem in both the semi and fully discrete case. For it we were able to show that the errors behave as $\mathcal{O}(h^{\ell+1/2})$ in space (where $\ell \geq 0$) and $\mathcal{O}(\Delta t)$ in time (for the fully discrete case using a backward Euler scheme in time), when enough regularity of the exact solution is assumed. For the semi-discrete case, these bounds are valid in both $L^\infty(0,T;\cdot)$ and $\mathrm{L}^2(0,T;\cdot)$, according with the assumed regularity of the exact solution. It remains an open question, to investigate whether or not this rate of convergence (in space) is indeed the best it could be, or it can be improved. Some numerical examples could be helpful also in this direction and are the subject of future work.

Also, the current analysis does not allow us to find an a priori error estimate for the error of $\boldsymbol{u}_h$ in the dielectric domain (since σ vanishes on Ω_{D}). Nevertheless, in practice, this knowledge is not so useful.

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