The Prime–Factor Exponent Vector System (PFS)

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Abstract

We introduce the Prime–Factor Exponent Vector System (PFS) as an infinite-dimensional Hilbert space encoding natural numbers by their prime exponents. We prove foundational lemmas on orthogonality and a Pythagorean identity for gcd and lcm.

Contents

1 The Prime–Factor Exponent Vector System (PFS)

Definition 1.1 (Prime–Factor Exponent Vector)**.** *Let* $(p_1, p_2, p_3, \dots)$ *be the sequence of all primes in increasing order. For each positive integer* n*, define its PFS vector*

$$\Phi(n) = \mathbf{n} = (\nu_{p_1}(n), \nu_{p_2}(n), \nu_{p_3}(n), \dots) \in \mathbb{N}_0^{(\infty)},$$

where $\nu_{p_i}(n)$ *is the exponent of* p_i *in the prime factorization of* n*. The inverse map*

$$\pi : \mathbb{N}_0^{(\infty)} \to \mathbb{N}_{>0}, \quad \pi((x_i)) = \prod_{i=1}^{\infty} p_i^{x_i}$$

satisfies $\pi(\Phi(n)) = n$*.*

2 The PFS Hilbert Space

Definition 2.1 (Hilbert Space Structure)**.** *Define*

$$\mathcal{H}_{\mathrm{PFS}} = \Big\{ \mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\infty : \sum_{i=1}^{\infty} x_i^2 < \infty \Big\},$$

which is a real Hilbert space under the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} x_i \, y_i, \quad \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

The subspace of finitely supported integer vectors

$$\mathcal{PFS} = \mathbb{Z}^{(\infty)} = \{ \mathbf{x} \in \mathbb{Z}^\infty : \textit{only finitely many } x_i \neq 0 \}$$

contains all PFS images $\Phi(n)$ *for* $n \in \mathbb{N}_{>0}$*.*

2.1 Arithmetic Operations in PFS

For $\mathbf{a}, \mathbf{b} \in \mathcal{PFS}$, componentwise:

$$\begin{aligned} \Phi(a \cdot b) &= \mathbf{a} + \mathbf{b}, \\ \Phi\big(a/b\big) &= \mathbf{a} - \mathbf{b} \quad (b \mid a), \\ \gcd(\mathbf{a}, \mathbf{b}) &= \min(\mathbf{a}, \mathbf{b}), \\ \operatorname{lcm}(\mathbf{a}, \mathbf{b}) &= \max(\mathbf{a}, \mathbf{b}). \end{aligned}$$

3 Foundational Lemmas and the Pythagorean Identity

Let $a, b \in \mathbb{N}_{>0}$ with

$$\mathbf{a} = (a_i), \quad \mathbf{b} = (b_i), \quad \mathbf{g} = \min(\mathbf{a}, \mathbf{b}), \quad \mathbf{r} = \mathbf{a} - \mathbf{g}, \quad \mathbf{s} = \mathbf{b} - \mathbf{g}.$$

Lemma 3.1 (Orthogonal Remainders)**.** $\langle \mathbf{r}, \mathbf{s} \rangle = 0$.

Proof. For each index i, either $a_i \leq b_i$ (so $a_i - g_i = 0$) or $b_i \leq a_i$ (so $b_i - g_i = 0$). Hence $(a_i - g_i)(b_i - g_i) = 0$ for all i, implying orthogonality. □

Lemma 3.2 (Decomposition)**.** *The vectors decompose as*

$$\mathbf{a} = \mathbf{g} + \mathbf{r}, \quad \mathbf{b} = \mathbf{g} + \mathbf{s}, \quad \ell = \max(\mathbf{a}, \mathbf{b}) = \mathbf{g} + \mathbf{r} + \mathbf{s}.$$

Lemma 3.3 (Dot-Product Identities)**.**

$$\begin{aligned} \langle \mathbf{a}, \mathbf{b} \rangle &= |\mathbf{g}|^2, \\ \langle \mathbf{a}, \mathbf{r} \rangle &= |\mathbf{r}|^2, \quad \langle \mathbf{b}, \mathbf{s} \rangle = |\mathbf{s}|^2. \end{aligned}$$

Lemma 3.4 (Pythagorean Identity)**.**

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{g}\|^2 + \|\ell\|^2.$$

4 Generalized Orthogonal Decomposition

For n numbers $a_1, \ldots, a_n$ with PFS vectors $\mathbf{v}_i$, and each nonempty subset $S \subseteq \{1, \ldots, n\}$ define

$$\mathbf{g}_S = (\min_{i \in S} (\mathbf{v}_i)_k)_k, \quad \mathbf{r}_S = \mathbf{g}_S - \sum_{T \supsetneq S} \mathbf{r}_T.$$

Theorem 4.1 (Inclusion–Exclusion Decomposition)**.** *The vectors* $\{\mathbf{r}_S\}$ *are pairwise orthogonal, and each*

$$\mathbf{v}_i = \sum_{S \ni i} \mathbf{r}_S, \quad \sum_{i=1}^{n} \|\mathbf{v}_i\|^2 = \sum_{\emptyset \neq S \subseteq \{1, \ldots, n\}} |S| \, \|\mathbf{r}_S\|^2.$$

5 Conclusion

This short paper is just a summary of the Prime-Factor System (PFS) shortened down to the most important details, the purpose of this is to support the understanding (the mathematical background) behind my project on https://summer.hackclub.com/projects/2895 .