1 Optimization and linear algebra

1.1 Optimization Prerequisites for Lasso

1. Given $a \in \mathbb{R}$ we define a^+, a^- as follows:

$$a^+ = \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \quad a^- = \begin{cases} -a & \text{if } a < 0, \\ 0 & \text{otherwise.} \end{cases}$$

   We call a^+ the *positive part* of a and a^- the *negative part* of a. Note that $a^+, a^- \geq 0$.

   (a) Give an expression for a in terms of a^+, a^-.
   (b) Give an expression for $|a|$ in terms of a^+, a^-.
   For $x \in \mathbb{R}^d$ define $x^+ = (x_1^+, \ldots, x_d^+)$ and $x^- = (x_1^-, \ldots, x_d^-)$.
   (c) Give an expression for x in terms of x^+, x^-.
   (d) Give an expression for $\|x\|_1$ without using any summations or absolute values. [Hint: Use x^+, x^- and the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$.]

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $S \subseteq \mathbb{R}$. Consider the two optimization problems

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}} & |x| \\ \text{subject to} & f(x) \in S \end{array} \qquad \text{and} \qquad \begin{array}{ll} \text{minimize}_{a,b \in \mathbb{R}} & a + b \\ \text{subject to} & f(a-b) \in S \\ & a, b \geq 0. \end{array}$$

   Solve the following questions.

   (a) If x in the first problem satisfies $f(x) \in S$ show how to quickly compute (a, b) for the second problem with $a + b = |x|$ and $f(a - b) \in S$.
   (b) If a, b in the second problem satisfy $f(a - b) \in S$, show how to quickly compute an x for the first problem with $|x| \leq a + b$ and $f(x) \in S$.
   (c) Assume x is a minimizer for the first problem with minimum value p_1^* and (a, b) is a minimizer for the second problem with minimum p_2^*. Using the previous two parts, conclude that $p_1^* = p_2^*$.

3. Let $f : \mathbb{R}^d \to \mathbb{R}$, $S \subseteq \mathbb{R}$ and consider the following optimization problem:

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^d} & \|x\|_1 \\ \text{subject to} & f(x) \in S, \end{array}$$

   where $\|x\|_1 = \sum_{i=1}^d |x_i|$. Give a new optimization problem with a linear objective function and the same minimum value. Show how to convert a solution to your new problem into a solution to the given problem. [Hint: Use the previous two problems.]

1.2 Ellipsoids

1. ($\star$) Describe the following set geometrically:

$$\left\{ v \in \mathbb{R}^2 \;\middle|\; v^T \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} v = 4 \right\}.$$

1.3 ($\star$) Linear Algebra Prerequisites for Linear Regressions

1. When performing linear regression we obtain the *normal equations* $A^T A x = A^T y$ where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$.

   (a) If $\mathbf{rank}(A) = n$ then solve the normal equations for x.
   (b) ($\star$) What if $\mathbf{rank}(A) \neq n$?

2. Prove that $A^T A + \lambda \mathbf{I}_{n \times n}$ is invertible if $\lambda > 0$ and $A \in \mathbb{R}^{n \times n}$.