1 Kernels

1. Fix $n > 0$. For $x, y \in \{1, 2, \ldots, n\}$ define $k(x, y) = \min(x, y)$. Give an explicit feature map $\varphi : \{1, 2, \ldots, n\}$ to $\mathbb{R}^D$ (for some D) such that $k(x, y) = \varphi(x)^T \varphi(y)$.

2. Show that $k(x, y) = (x^T y)^4$ is a positive semidefinite kernel on $\mathbb{R}^d \times \mathbb{R}^d$.

3. Let $A \in \mathbb{R}^{d \times d}$ be a positive semidefinite matrix. Prove that $k(x, y) = x^T A y$ is a positive semidefinite kernel.

4. Consider the objective function
$$J(w) = \|Xw - y\|_1 + \lambda \|w\|_2^2.$$
Assume we have a positive semidefinite kernel k.
   (a) What is the kernelized version of this objective?
   (b) Given a new test point x, find the predicted value.

5. Show that the standard 2-norm on $\mathbb{R}^n$ satisfies the parallelogram law.

6. Suppose you are given an training set of distinct points $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ and labels $y_1, \ldots, y_n \in \{-1, +1\}$. Show that by properly selecting σ you can achieve perfect $0 - 1$ loss on the training data using a linear decision function and the RBF kernel.

7. Suppose you are performing standard ridge regression, which you have kernelized using the RBF kernel. Prove that any decision function $f_\alpha(x)$ learned on a training set must satisfy $f_\alpha(x) \to 0$ as $\|x\|_2 \to \infty$.

8. Consider the standard (unregularized) linear regression problem where we minimize $L(w) = \|Xw - y\|_2^2$ for some $X \in \mathbb{R}^{n \times m}$ and $y \in \mathbb{R}^n$. Assume $m > n$.
   (a) Let w^* be one minimizer of the loss function L above. Give an infinite set of minimizers of the loss function.
   (b) What property defines the minimizer given by the representer theorem (in terms of X)?