DMA1 Proofs

Closure Lemma

Lemma

A functional dependency $X \to Y$ is in the armstrong-extended F if and only if Y is determined by X.

$$X \to Y \in F^A \quad <=> \quad Y \subseteq X^+$$

Hint
Obviously, $X \to Y$ needs to belong to F before belonging to F^A.
$Y \subseteq X^+$ is equal to saying that the dependency is in F.

Proof

"$=>$" - If $X \to Y$ is in F^A then $Y \subseteq X^+$

Warning
The premise here is that $X \to Y$ is in F^A.

1. By decomposition rule, we obtain:

$$X \to K \in F^A \quad \forall K \in Y$$

2. Knowing the definition of X^+, we also obtain that:

$$K \in X^+ \quad \forall K \in Y$$

Hint
I guess that the definition of X^+ implies this equivalence:

$$X \to K \in F^A \quad == \quad K \in X^+$$

3. Which means that :

$$Y \subseteq X^+$$

Example

$$X \to BC \in F^A$$

$$X \to B \in F^A \quad \text{and} \quad X \to C \in F^A$$
$$B \in X^+ \quad \text{and} \quad C \in X^+$$
$$BC \subseteq X^+$$

"$<=$" - If $Y \subseteq X^+$ then $X \to Y$ is in F^A

> ⚠ **Warning**
> The premise here is that Y is contained in X^+

1. Since $Y \subseteq X^+$, it holds that:

$$X \to K \in F^A \quad \forall K \in Y$$

2. Wich, by union rule, implies that:

$$X \to Y \in F^A$$

> **Example**
>
> $$BC \subseteq X^+$$
> $$X \to B \in F^A \quad \text{and} \quad X \to C \in F^A$$
> $$X \to BC \in F^A$$

Closure inclusion Lemma

Lemma

If F is dontained in G^+, then F^+ is also contained in G^+.

$$\text{If} \quad F \subseteq G^+ \quad \text{then} \quad F^+ \subseteq G^+$$

Proof

> ⚠ **Warning**
> The premise is that we know that $F \subseteq G^+$ is True.

Let F' be a set of functional dependencies derived by F through the axioms. We indicate this process of deriving by:

$$F \to^A F'$$

1. It holds that F' can be derived through the axioms from F if and only if F' is in F^+.

$$F \to^A F' \quad \text{iff} \quad F \subseteq F^+$$

> **Hint**
> Since F^+ contains all possible derivate dependencies of F, if F' was derived from F, it must be a subset of F^+.

2. This implies that F^+ can be derived from F through the axioms.

$$F \to^A F^+$$

3. Now we have demonstrated that we can get F^+ from F. If we can get F from G, then we know that we can get F^+ from G.

$$F \subseteq G^+ \quad \text{iff} \quad G \to^A F \to^A F^+$$

> **Hint**
> The premise here is that $F \subseteq G^+$ is True, so $G \to^A F \to^A F^+$ is also True.

4. $G \to^A F \to^A F^+$ being true means that we can get F^+ from G, and so G^+ also contains F^+.

$$F^+ \subseteq G^+$$

$F^A = F^+$

Theorem

F^+ is equal to F^A.

Proof (By double inclusion)

$F^A \subseteq F^+$

Base case(n=0):

We don't use any axiom, $X \to Y$ already belongs to F and also to F^+.

> **Hint**
> Are there dependencies in F^A that we can obtain with 0 applications of the axioms? YES! The ones in F.

Induction case:

Here we are assuming that the dependencies obtained with n applications of the axioms are already in both F^A and F^+.

What we are trying to do is apply the axioms another time(n+1) and prove that the resulting dependencies are in F^+.

Reflexivity

We are obtaining $X \to Y \in F^A$ because Y is part of X.

> **Hint**
> We know for sure that the dependency is in F^A, because we obtained i through axioms, now we gotta demonstrate that it's also in F^+.

We know that the dependencies in F^+ need to satisfy every legal instance of R, so we need to ensure that by using a random dependency.

$X \to Y$ is legal in R if:

$$t_1[x] = t_2[x] \to t_1[y] = t_2[y]$$

In the case of reflexivity, this is always true, because Y is part of X.