EXERCISE SHEET ON NUMERICAL SERIES

(1) Determine for which values of the parameter $\alpha > 0$ the following series converges:

$$\sum_{n=1}^{\infty} \left(\sin \frac{1}{n^{2\alpha}} \right) n^{\frac{5}{2}-2\alpha}.$$

(2) Determine whether the following series converges:

$$\sum_{n=1}^{+\infty} \frac{1}{n\left(\sqrt{1+\frac{5}{n^3}}-1\right)}.$$

(3) Determine whether the following series converges:

$$\sum_{n=0}^{+\infty} \frac{(\ln 2)^n}{3n+\frac{1}{2}}.$$

(4) Determine whether the series converges absolutely or conditionally

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log n}.$$

(5) Determine whether the series converges absolutely or conditionally

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{5n+3\sin n}.$$

(1) The terms of the series are all non-negative. For $\alpha > 0$, we have

$$\left(\sin\frac{1}{n^{2\alpha}}\right) n^{\frac{5}{2}-2\alpha} \sim \frac{n^{\frac{5}{2}-2\alpha}}{n^{2\alpha}} = \frac{1}{n^{4\alpha-5/2}},$$

and by the limit comparison test, the series converges if and only if $4\alpha - 5/2 > 1$, that is, if $\alpha > \frac{7}{8}$.

(2) Recall that

$$\sqrt{1+\frac{5}{n^3}} - 1 = \left(\left(1+\frac{5}{n^3}\right)^{1/2} - 1\right)\frac{\left(\sqrt{1+\frac{5}{n^3}}+1\right)}{\left(\sqrt{1+\frac{5}{n^3}}+1\right)}$$
$$= \frac{\frac{5}{2n^3}}{\left(\sqrt{1+\frac{5}{n^3}}+1\right)} \sim \frac{5}{2n^3},$$

when $n \to +\infty$, which gives

$$\frac{1}{n\left(\sqrt{1+\frac{5}{n^3}}-1\right)} \sim \frac{1}{n\left(\frac{5}{2n^3}\right)} = \frac{2}{5}n^2.$$

This is an α-series with parameter $\alpha = -2$ and diverges.

(3) We will make use of the root test. Note that

$$\lim_{n\to+\infty} \sqrt[n]{\frac{(\ln 2)^n}{3n+\frac{1}{2}}} = \frac{\ln 2}{\lim_{n\to+\infty} \sqrt[n]{3n+\frac{1}{2}}} = \ln 2 < 1.$$

By the root test we see that the series converges (because the limit is strictly less than 1).

(4) We will answer by using the Leibniz criterion for alternating series. Let us check that all the hypotheses of the Leibniz criterion are met. The sequence $\frac{1}{\log n}$ is strictly decreasing for $n \geq 2$ since $\log n$ increases with n. The limit $\lim_{n\to\infty} \frac{1}{\log n}$ is 0.

Since both conditions are satisfied, the series converges conditionally by the Leibniz criterion. However, the series of absolute values $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges by comparison with the harmonic series (as $\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 3$).

(5) We observe that $\cos(n\pi) = (-1)^n$, so the series can be rewritten as:

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{where} \quad a_n = \frac{1}{5n+3\sin n} > 0 \quad \forall n \geq 1.$$

Let us investigate the absolute convergence of the series.

Clearly, we have $a_n \geq \frac{1}{5n+3}$. Therefore, by comparison test applied with the harmonic series, the given series does not converge absolutely.

As for the conditional convergence, consider the function $f(x) = \frac{1}{5x+3\sin x}$. This function is monotonically decreasing, since its derivative:

$$f'(x) = -\frac{5+3\cos x}{(5x+3\sin x)^2} \leq 0 \quad \text{for all} \quad x \geq 1.$$

This implies that the sequence $\{a_n\}$ is also monotonically decreasing. Hence, by the Leibniz criterion, the series converges conditionally.