EXERCISE SHEET ON ORDINARY DIFFERENTIAL EQUATIONS

(1) Solve the Cauchy problem:

$$\begin{cases} y'(t) + 5y(t) = 2e^{-5t} \\ y(0) = 1 \end{cases}$$

(2) Solve the Cauchy problem:

$$\begin{cases} y'(t) + \frac{y(t)}{t} = e^{t} \\ y(1) = 2 \end{cases}$$

(3) Solve the Cauchy problem for the homogeneous differential equation:

$$y''(t) - 4y'(t) + 5y(t) = 0,$$

with initial conditions $y(0) = 0$ and $y'(0) = 2$.

(4) Solve the Cauchy problem for the homogeneous differential equation:

$$y''(t) - 6y'(t) + 9y(t) = 0,$$

with initial conditions $y(0) = 1$ and $y'(0) = 1$.

(5) Solve the following Cauchy problem

$$\begin{cases} y'(t) = \dfrac{5t}{y^4(t)} \\ y(0) = 1 \end{cases}$$

(1) First we find a solution to the associated homogeneous equation, namely $y'(t) = -5y(t)$. This is an equation of the form $y'(t) = a(t)y(t)$. We have

$$A(t) = \int a(t)\,dt = \int -5\,dt = -5t + h, \qquad h \in \mathbb{R}$$

and thus a solution of the homogeneous equation is of the form

$$y(t) = Ce^{-5t} \qquad C \in \mathbb{R}$$

By the variation of constant method we look for a solution of the original equation of the form

$$y(t) = C(t)e^{-5t} \qquad C \in \mathbb{R}$$

The function $y(t)$ must satisfy the equation

$$y'(t) = C'(t)e^{-5t} - 5C(t)e^{-5t} \stackrel{!}{=} -5C(t)e^{-5t} + 2e^{-5t}$$

which leads to

$$C'(t) = 2$$

that is $C(t) = 2t + K$, for some $K \in \mathbb{R}$. Therefore, we have

$$y(t) = (2t + K)e^{-5t}$$

By setting $y(0) = 1$, we see that $K = 1$.

(2) First we find a solution to the associated homogeneous equation, namely $y'(t) = -y(t)/t$. The solution will be defined either in $(-\infty, 0)$, or $(0, +\infty)$. As we require that $y(1) = 2$, the function must be defined in $t = 1$ and we limit ourselves to the interval $(0, +\infty)$. The equation is of the form $y'(t) = a(t)y(t)$. We have

$$A(t) = \int a(t)\,dt = \int -\frac{1}{t}\,dt = -\ln(t) + h, \qquad h \in \mathbb{R}$$

and thus a solution of the homogeneous equation is of the form

$$y(t) = C \cdot \frac{1}{e^{\ln t}} = \frac{C}{t} \qquad C \in \mathbb{R}$$

By the variation of constant method we look for a solution of the original equation of the form

$$y(t) = \frac{C(t)}{t} \qquad C \in \mathbb{R}$$

The function $y(t)$ must satisfy the equation

$$y'(t) = \frac{C'(t)}{t} - \frac{C(t)}{t^2} \stackrel{!}{=} -\frac{C(t)}{t^2} + e^t$$

which leads to

$$C'(t) = te^t$$

that is $C(t) = (t-1)e^t + K$, for some $K \in \mathbb{R}$. Therefore, we have

$$y(t) = [(t-1)e^t + K] \cdot \frac{1}{t}$$

By setting $y(1) = 2$, we see that $K = 2$.

(3) The characteristic equation is:

$$z^2 - 4z + 5 = 0,$$

which has two complex conjugate roots: $2+i$ and $2-i$, each with multiplicity 1. The general homogeneous solution is:

$$y(t) = e^{2t}(c_1 \cos t + c_2 \sin t).$$

Imposing the initial conditions:

$$y(0) = c_1 = 0,$$

so $y(t) = c_2 e^{2t} \sin t$. Since:

$$y'(t) = c_2 e^{2t}(2 \sin t + \cos t),$$

we have:

$$y'(0) = c_2 = 2.$$

The solution is:

$$y(t) = 2e^{2t} \sin t.$$

(4) The characteristic equation is:

$$z^2 - 6z + 9 = 0,$$

which has a single root $z = 3$ with multiplicity 2. The general homogeneous solution is:

$$y(t) = (c_1 t + c_2)e^{3t}.$$

Imposing the initial conditions:

$$y(0) = c_2 = 1,$$

and since:

$$y'(t) = c_1 e^{3t} + 3(c_1 t + c_2)e^{3t},$$

we have:

$$y'(0) = c_1 + 3c_2 = 1.$$

Solving the system:

$$\begin{cases} c_2 = 1 \\ c_1 + 3c_2 = 1 \end{cases}$$

yields $c_1 = -2$ and $c_2 = 1$. The solution is:

$$y(t) = (-2t + 1)e^{3t}.$$

(5) We separate the variables

$$\frac{y^4}{5} y' = t$$

and integrate w.r.t. t

$$\int \frac{y^4}{5} dy = \int t dt \quad \Rightarrow \quad \frac{y^5}{25} = \frac{t^2}{2} + C$$

By imposing the initial condition $y(0) = 1$, we get

$$\frac{1^5}{25} = \frac{0^2}{2} + C \quad \Rightarrow \quad C = 1/25$$

The solution satisfies the following relation

$$\frac{y^5}{25} = \frac{t^2}{2} + \frac{1}{25}$$

Therefore, we have

$$y(t) = \left(1 + \frac{25}{2} t^2\right)^{1/5}$$

The solution is defined for all $t \in \mathbb{R}$.