BONUS EXERCISE SHEET

1) Determine whether the series converges or not

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{5n+1}$$

2) Determine for which real values of x the series converges

$$\sum_{n=1}^{\infty} \frac{|x-7|^{n+1}}{2n^2 3^{-n}}$$

3) Compute the definite integral:

$$\int_0^1 \frac{x+11}{x^2+5}\,dx$$

4) Find the general solution of:

$$2y''(x) + 5y(x) = \sin x$$

5) Solve the Cauchy problem:

$$\begin{cases} y' = 2y + e^{7x} \\ y(0) = 0 \end{cases}$$

6) Consider the Cauchy problem for each n:

$$\begin{cases} y_n' = -ny_n^4 \\ y_n(0) = 1 \end{cases}$$

Compute $\lim_{n\to\infty} y_n(1)^2$.

1) The given series diverges, since:

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{5n+1} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{5n+1}}_{\text{Divergent (compare to harmonic series)}} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}}_{\text{Convergent (Leibniz test)}}$$

5) This is a first-order linear ODE equation. The integrating factor is $\mu(x) = e^{\int -2\,dx} = e^{-2x}$. If we multiply the equation by it, we get

$$\frac{d}{dx}\left(e^{-2x}y\right) = e^{-2x}y' - 2e^{-2x}y = e^{5x}$$

After Integrating both sides we obtain

$$e^{-2x}y = \frac{1}{5}e^{5x} + C$$

The initial condition $y(0) = 0$ allows us to determine C

$$1 \cdot 0 = \frac{1}{5} + C \Rightarrow C = -\frac{1}{5}$$

Therefore, the final solution is

$$y(x) = \frac{1}{5}e^{7x} - \frac{1}{5}e^{2x} = \frac{1}{5}(e^{7x} - e^{2x})$$