EXERCISE SHEET ON POWER SERIES

(1) Find the MacLaurin formula for $\log(1+x)^{(1+x)}$.
(2) Find the MacLaurin formula for $\cos^2 x$.
(Hint: use the double angle formula for the cosine).
(3) Determine the convergence domain of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}$$

(4) Determine the convergence domain of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{5^n (3n-1)}$$

(1) The Maclaurin series expansion for $\log(1+x)$ is given by:

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This expansion is valid for the interval $-1 < x < 1$. Therefore, we have

$$\begin{aligned} \log(1+x)^{(1+x)} &= (1+x)\log(1+x) = (1+x)\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right) \\ &= x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots = x + \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{(n-1)n} \end{aligned}$$

valid for $-1 < x < 1$.

(2) By the double angle formula for the cosine, we can write

$$\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$$

It follows that

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

valid for all $x \in \mathbb{R}$.

(3) Set

$$u_n := \frac{x^n}{n^3}.$$

By applying the ratio test to the absolute values we get

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x^{n+1}|}{(n+1)^3} \cdot \frac{n^3}{|x^n|} = |x|\left(\frac{n}{n+1}\right)^3 \to \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$$

Thus, for $|x| < 1$ the series converges absolutely. Let us study the convergence of the series at the endpoints of this interval.

- For $x = 1$: The series becomes the α-series with $\alpha = 3$ (convergent)
- For $x = -1$: The series of absolute values is the α-series with $\alpha = 3$ (absolutely convergent)

Therefore, the interval of convergence is:

$$-1 \le x \le 1$$

(4) Set

$$u_n := \frac{(x-1)^n}{5^n(3n-1)}.$$

By applying the ratio test to the absolute values, we get

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x-1|^{n+1}}{5^{n+1}(3n+2)}\cdot\frac{5^n(3n-1)}{|x-1|^n}$$
$$= \lim_{n\to\infty}\frac{|x-1|}{5}\cdot\frac{3n-1}{3n+2} = \frac{|x-1|}{5}$$

The series converges when:

$$\frac{|x-1|}{5} < 1 \Rightarrow -4 < x < 6$$

We now examine the endpoints of the interval:

- For $x=-4$: The series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{3n-1}$, which converges by the Leibniz criterion
- For $x=6$: The series becomes $\sum_{n=1}^{\infty}\frac{1}{3n-1}$, which diverges by comparison with the harmonic series

Thus, the interval of convergence is:

$$-4 \leq x < 6$$