SECOND MOCK MIDTERM

Say if the following statements are true or false.

(1) A) Let a_n be a non-negative sequence and suppose that $\lim_{n\to\infty} a_n = 0$. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges as well.
B) Let a_n be a non-negative sequence. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} 1/(a_n+1)$ is convergent as well.
C) Let a_n be a non-negative sequence. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} 1/a_n$ is convergent as well.
D) Let a_n be a non-negative sequence. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} n^4 a_n$ is convergent as well.

(2) A) The series
$$\sum_{n=1}^{\infty} \frac{(n+1)\cos n}{\sqrt[3]{n^7}}$$
converges conditionally, but not absolutely.
B) The series
$$\sum_{n=1}^{\infty} (-1)^n \frac{2+n}{n^2+n+1}$$
converges.
C) The series
$$\sum_{n=1}^{\infty} (-1)^n \frac{2+n}{n^2+n+1}$$
converges absolutely.
D) The series
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(1/n)}{n^2+2}$$
converges absolutely.

(3) Consider the series
$$\sum_{n=0}^{+\infty} 3^n x^n a_n$$
A) For $a_n = 6^{-n}$, the radius of convergence of this series is 3.
B) For $a_n = 1$, the radius of convergence of this series is 0.
C) For $a_n = 2$, the radius of convergence of this series is 1.
D) For $a_n = n$, the radius of convergence of this series is $+\infty$.

(4) A) Consider the function $f(x) = e^{2x}$. The radius of convergence of its Taylor series is infinite.
B) Consider the function $f(x) = x\sin(x^2)$. The radius of convergence of its Taylor series is infinite.
C) Consider the function $f(x) = x\sin(x^2)$. The coefficient of the term of degree 1 in its Taylor series is 1.
D) Consider the function $f(x) = x\sin(x^2)$. Then $f^{(2)}(0) = 0$.

(5) A) The general solution of $x'' + x' - 6x = 0$ is of the form $x(t) = Ae^{2t} + Be^{-3t}$.
B) The general solution of $x'' - 2x' - 3x = 0$ is of the form $x(t) = Ae^{3t} + Bte^{3t}$.
C) The general solution of $x'' + 10x = 0$ is of the form $x(t) = A\cos(\sqrt{10}t) + B\sin(\sqrt{10}t)$.
D) The following Cauchy problem has solution $x(t) = e^t$

$$\begin{cases} x'(t) = x(t), \\ x(0) = 0. \end{cases}$$

(1) A) **True**.
For n big enough, say for $n > N$, we have $a_n < 1$. In particular, for $n > N$, we have $\sqrt{a_n} < 1$ and $a_n < \sqrt{a_n}$. By the comparison test we see that $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.

B) **False**.
If $\sum_{n=1}^{\infty} a_n$ is convergent, then a_n tends to 0 as n goes to ∞ (by the Cauchy criterion). As $1/(a_n + 1)$ tends to 1 as n goes to ∞, the series $\sum_{n=1}^{\infty} 1/(a_n + 1)$ cannot converge (again by the Cauchy criterion).

C) **False**.
The statement is False because $1/a_n$ does not tend to 0.

D) **False**.
In general the statement is false, consider e.g. $a_n = 1/n^4$.

(2) A) **False**.
The series

$$\sum_{n=1}^{\infty} \frac{(n+1)\cos n}{\sqrt[3]{n^7}}$$

has terms with varying signs. We now attempt to prove its absolute convergence. We have

$$\left| \frac{(n+1)\cos n}{\sqrt[3]{n^7}} \right| = \frac{(n+1)|\cos n|}{\sqrt[3]{n^7}}.$$

The terms of the series

$$\sum_{n=1}^{\infty} \frac{(n+1)|\cos n|}{\sqrt[3]{n^7}},$$

are positive. Then, we can apply the comparison test. Recall that $|\cos n| \leq 1$, and therefore

$$\sum_{n=1}^{\infty} \frac{(n+1)|\cos n|}{\sqrt[3]{n^7}} \leq \sum_{n=1}^{\infty} \frac{(n+1)}{\sqrt[3]{n^7}}.$$

By the asymptotic comparison test, the latter series converges if and only if the following series does

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}},$$

which is a generalized harmonic series with parameter $\frac{4}{3} > 1$, and it converges. Thus, the original series is absolutely convergent, and therefore it converges.

B) **True**.

First we show that the sequence $a_n = \frac{2+n}{n^2+n+1}$ is decreasing. Let us study the difference $a_{n+1} - a_n$:

$$\begin{aligned}
&a_{n+1} - a_n \\
&= \frac{n+3}{n^2+3n+3} - \frac{n+2}{n^2+n+1} \\
&= \frac{(n+3)(n^2+n+1) - (n+2)(n^2+3n+3)}{(n^2+3n+3)(n^2+n+1)} \\
&= \frac{(n^3+n^2+n+3n^2+3n+3) - (n^3+3n^2+3n+2n^2+6n+6)}{(n^2+3n+3)(n^2+n+1)} \\
&= \frac{-n^2-5n-3}{(n^2+3n+3)(n^2+n+1)} < 0 \quad \text{for all } n \geq 1
\end{aligned}$$

This proves that $a_{n+1} < a_n$, so the sequence $\{a_n\}$ is decreasing. By applying the Leibniz criterion, we get that the series is converging.

C) **False**.

We have that

$$\left((-1)^n \frac{2+n}{n^2+n+1}\right)\left(\frac{1}{n}\right)^{-1} \to 2$$

as $n \to +\infty$. By Asymptotic comparison test, the series converges absolutely if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

does. This is the harmonic series and does not converge.

D) **True**.

Let us check the absolute convergence.

We have the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin(1/n)}{n^2+2} \right|$$

Recall the approximation $\sin(1/n) \sim \frac{1}{n}$ as $n \to \infty$ and therefore the general term behaves like

$$\frac{1/n}{n^2} = \frac{1}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, by the asymptotic comparison test the original series converges absolutely.

(3) A) **False**.

The series is of the form $\sum_n (3x)^n a_n = \sum_n u_n$. We analyze its convergence based on the behavior of a_n.
For $a_n = 6^{-n}$, we have:

$$u_n = (3x)^n \cdot 6^{-n} = \left(\frac{3x}{6}\right)^n = \left(\frac{x}{2}\right)^n$$

This is a geometric series with ratio $\left|\frac{x}{2}\right|$. It converges when $\left|\frac{x}{2}\right| < 1$, i.e. $|x| < 2$.
So the radius of convergence is $R = 2$, not 3, and the statement A is false.

B) **False**.
For $a_n = 1$, we have:

$$u_n = (3x)^n$$

This is a geometric series with ratio $3x$, so it converges if $|3x| < 1$, i.e. $|x| < \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$, not 0.

C) **False**.
For $a_n = 2$, the situation is the same as $a_n = 1$, since multiplying each term by 2 doesn't affect the radius of convergence:

$$u_n = (3x)^n \cdot 2 = 2(3x)^n$$

Therefore, the radius is $R = \frac{1}{3}$, not 1.

D) **False**.
For $a_n = n$ the general term of the series is

$$u_n = 3^n x^n \cdot n = n(3x)^n$$

To find the radius of convergence, we may apply the root test

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \sqrt[n]{n|3x|^n} = |3x| \cdot \lim_{n\to\infty} \sqrt[n]{n}$$

But $\sqrt[n]{n} \to 1$, so:

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = |3x|$$

The series converges when $|3x| < 1$, that is, when $|x| < \frac{1}{3}$
Therefore, the radius of convergence is $R = \frac{1}{3}$, not $+\infty$.

(4) A) **True**.
The Taylor series of e^{2x} has the same radius of convergence as that e^x.

B) **True**.
The Taylor series of $\sin(y)$ is

$$\sum_{n=0}^{+\infty}(-1)^n\frac{y^{2n+1}}{(2n+1)!}$$

and has infinite radius of convergence. The Taylor series of $\sin(x^2)$ can be obtained, from the previous one, by the substitution $y = x^2$. By multiplying this series by x we get the one for f

$$x\left(\sum_{n=0}^{+\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}$$

Both operation do not change the radius of convergence.

C) **False**.
The Taylor series of f is

$$\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!} = x^3 + (\text{ terms of degree greater than } 7 \text{ })$$

The coefficient of x is 0.

D) **True**.
The Taylor series of f is

$$\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!} = x^3 + (\text{ terms of degree greater than } 7 \text{ })$$

The coefficient of x^k is $f^{(k)}(0)/k!$. The coefficient of x^2 is 0. Therefore, $f^{(2)}(0) = 0$.

(5) A) **True**.
The characteristic equation is $r^2 + r - 6 = 0$, with roots $r = 2$ and $r = -3$. Thus, the general solution is a linear combination of e^{2t} and e^{-3t}.

B) **False**.
The characteristic equation $r^2 - 2r - 3 = 0$ has roots $r = 3$ and $r = -1$. The general solution is $x(t) = Ae^{3t} + Be^{-t}$. The term Bte^{3t} would only appear if $r = 3$ were a repeated root (which it is not).

C) **True**.
The characteristic equation $r^2 + 10 = 0$ gives imaginary roots $r = \pm i\sqrt{10}$. The general solution for such cases is a linear combination of $\cos(\sqrt{10}t)$ and $\sin(\sqrt{10}t)$.

D) **False**.
The general solution of the differential equation $x'(t) = x(t)$ is:
$$x(t) = Ce^t,$$
where C is an arbitrary constant. Imposing the initial condition $x(0) = 0$:
$$x(0) = Ce^0 = C = 0.$$
Thus, the solution to the Cauchy problem is $x(t) \equiv 0$.