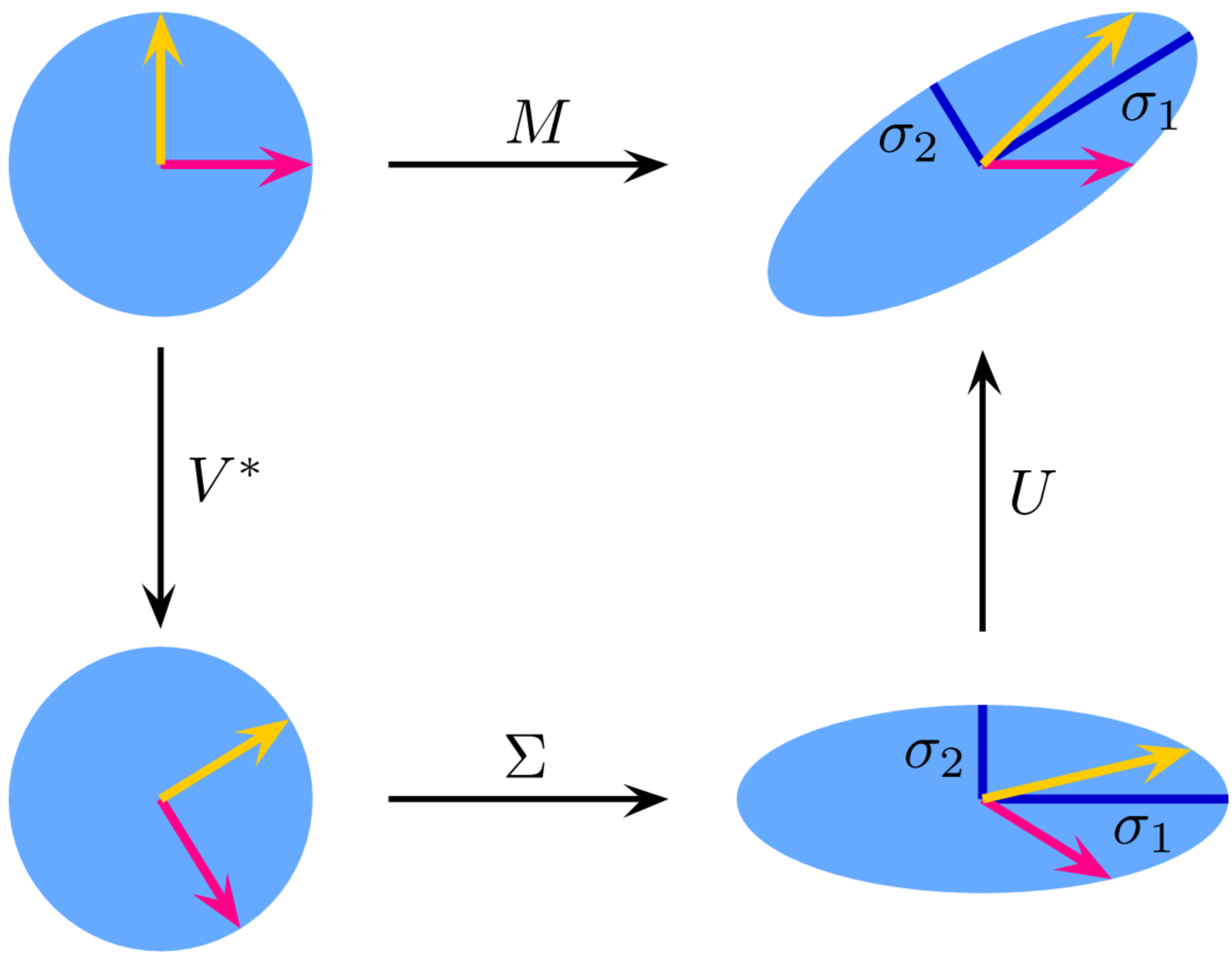
SVD

SVD stands for singular value decomposition.
Sometimes it is useful to decompose transformation matrices in multiple matrices that only do a type of transformation each.

The difference with [spectral decomposition](), is that SVD doesn't require a [symmetric matrix](), but just about any [matrix]().

SVD says that any matrix, regardless of symmetry, rank or shape, can unconditionally be decomposed into 3 special matrices.



$$M = U \cdot \Sigma \cdot V^*$$

Explanation:

$$AV = U\Sigma$$

Where:

- A is any matrix.
- V are the orthonormal vectors that, once transformed by A, remain orthonormal.
- Σ is a diagonal matrix that scales the basis vectors before they are rotated.
- U is the rotation that brings the basis vectors

What does it mean:

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$$

$$Av_1 = u_1\sigma_1$$

and

$$Av_2 = u_2\sigma_2$$

The vectors in U are the vectors of V after being transformed by A(they remain orthonormal by definition of V).

AV:
For some reason we found the pair of vectors that when transformed by A, remain orthonormal.
We now put them through A and end up with orthonormal vectors.

$U\Sigma$:
We have the scaling information in the Σ matrix, then we rotate those scaled vectors to where they are after AV(by applying U, by U definition).

Result:
If we then want A, we divide both sides by V^-1.

$$A = U\Sigma V^{-1}$$

How do we get U and V?

We make spectral decomposition of AA^T, which returns a symmetrical matrix.
We can also do A^TA, which also returns another symmetrical matrix.

Those two matrices have something in common with the original matrix apparently.
V and U's columns are the eigenvectors of those two symmetrical matrices.

$$\boxed{AA^T} = U\Sigma^2 U^T \longrightarrow Q = U,\ \Lambda = \Sigma^2$$

$$\boxed{A^T A} = V\Sigma^2 V^T \longrightarrow Q = V,\ \Lambda = \Sigma^2$$

$$U = \begin{bmatrix} | & & | \\ u_1, & \cdots, & u_m \\ | & & | \end{bmatrix}$$ $u_1, \cdots, u_m$ eigenvectors of AA^T

$$V = \begin{bmatrix} | & & | \\ v_1, & \cdots, & v_n \\ | & & | \end{bmatrix}$$ $v_1, \cdots, v_n$ eigenvectors of A^TA

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \sigma_n \end{bmatrix}$$

$\sigma_i = \sqrt{\lambda_i}$

λ_i eigenvalue