1 Defining the AIMD Recurrence for 3 Users

We represent the congestion window as a vector composed of the **three users**, U_1, U_2, U_3, at time step k:

$$X_k = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The **AIMD Algorithm** has the following phases:

- **Additive Increase:** $x_i \to x_i + \alpha$ when there is no congestion.
- **Multiplicative Decrease:** $x_i \to \beta x_i$ when congestion occurs.

For our algorithm, the values used are:

$$\alpha = 1, \quad \beta = 0.5$$

2 Representing AIMD as a Matrix System

Over multiple AIMD cycles, the system alternates between the two phases:

2.1 Phase 1: Additive Increase

During the additive increase phase, each user increases their congestion window by α, so:

$$X_{k+1} = X_k + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2.2 Phase 2: Multiplicative Decrease

When congestion occurs, all users reduce their congestion window by a factor of β, leading to:

$$X_{k+1} = \beta X_k$$

2.3 Combining Both Phases into a Single Recurrence

The combined recurrence can be expressed as:

$$X_{k+1} = M X_k + B$$

where:

- **Multiplicative Decrease Matrix** M:

$$M = \beta I = 0.5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Additive Increase Vector** B:

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, our **matrix recurrence equation** is:

$$X_{k+1} = 0.5X_k + B$$

3 Convergence Point

To find the convergence point, we first expand the recurrence iteratively:

$$X_k = 0.5^k X_0 + \sum_{i=0}^{k-1} 0.5^i B$$

This equation is evaluated using the **geometric series formula**:

$$\sum_{i=0}^{k-1} 0.5^i = \frac{1 - 0.5^k}{1 - 0.5} = 2(1 - 0.5^k)$$

This leads to the following equation :

$$X_k = 0.5^k X_0 + 2(1 - 0.5^k)B$$

Taking the **limit as** $k \to \infty$:

- The term $0.5^k X_0$ **vanishes** as $0.5^k \to 0$.
- The summation term **converges** to $2B$, giving:

$$X_\infty = 2B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

This shows that AIMD will lead to the window sizes of the users **converging to become equal**.

4 Scaling to Capacity C

Since the total bandwidth is C, the steady-state must satisfy:

$$\sum_{i=1}^{N=3} x_i = C. \tag{1}$$

Since all users share the bandwidth equally, we get:

$$x_1 = x_2 = x_3 \to X_\infty, \tag{2}$$

which is equivalent to the previous X_∞ we obtained. This equation gives:

$$3X_\infty = C. \tag{3}$$

Solving for X_∞:

$$X_\infty = \frac{C}{3}. \tag{4}$$

Thus, the final congestion window for each user is:

$$X_\infty = \begin{bmatrix} \frac{C}{3} \\ \frac{C}{3} \\ \frac{C}{3} \end{bmatrix} \tag{5}$$

This confirms that all 3 users, U_1, U_2, U_3 will **converge to an equal share of the Total Capacity,** C.

5 Additional Note : Generalization to N Users

The proof also works with N users, where the recurrence equation generalizes to:

$$X_k = 0.5^k X_0 + 2(1 - 0.5^k)B$$

which **always converges** to:

$$X_\infty = 2B = \begin{bmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{bmatrix}$$

In the same manner as the previous proof, all N Users will converge to an **equal share of the Total Capacity**, C, leading to the final congestion window:

$$X_\infty = \begin{bmatrix} \frac{C}{N} \\ \frac{C}{N} \\ \vdots \\ \frac{C}{N} \end{bmatrix}. \tag{6}$$