POPRAWKA TOPOGRAFICZNA - TERENOWA rozdział 8.7

Chapter 8

Spectral techniques, FFT

8.1 The Stokes equation as a convolution

We start from the Stokes equation

$$T(\phi,\lambda)=\frac{R}{4\pi}\iint_{\sigma}S(\psi)\,\Delta g(\phi',\lambda')\,d\sigma',$$

where (ϕ',λ') is the location of the moving point (integration or data point) on the Earth surface, and (ϕ,λ) the location of the evaluation point, it too on the Earth surface. Generally the locations of both points are given in spherical co-ordinates (ϕ,λ), and correspondingly the integration is executed over the surface of the unit sphere σ: a surface element is $d\sigma=\cos\phi\,d\phi d\lambda$, where the factor $\cos\phi$ represents the *determinant of Jacobi,* or *jacobian*, for these spherical co-ordinates (ϕ,λ).

However *locally*, in a sufficiently small area, one may write the point co-ordinates also in rectangular form, and then also the integral in rectangular co-ordinates. Suitable rectangular co-ordinates are, e.g., map projection co-ordinates, see figure 8.1.

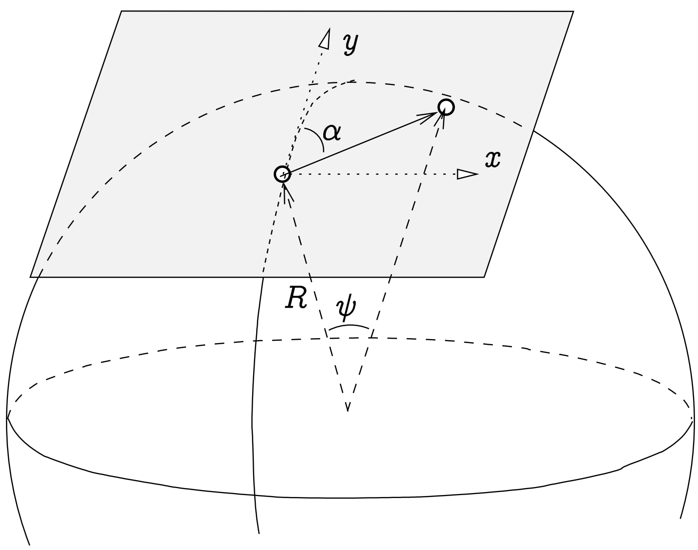

Figure 8.1 – Map projection co-ordinates x, y in the local tangent plane.

A simple example of rectangular co-ordinates in the tangent plane would be

$$\begin{aligned} x &= \psi R \sin\alpha, \\ y &= \psi R \cos\alpha, \end{aligned} \tag{8.1}$$

where α is the azimuth between evaluation and moving point. The centre of this projection is the point where the tangent plane touches the sphere. The location of other points is measured by the angle at the Earth's centre, ψ, the spherical distance, and by the direction angle in the tangent plane or *azimuth* α.

A more realistic example uses a popular conformal map projection, the *stereographic projection*:

$$\begin{aligned} x &= 2\sin\left(\psi/2\right)\ R\sin\alpha, \\ y &= 2\sin\left(\psi/2\right)\ R\cos\alpha. \end{aligned}$$

In the limit for small values of ψ this is the same as equations 8.1.

Taking the squares of equations 8.1, summing them, and dividing them by R^2 yields

$$\psi^2 \approx \frac{x^2+y^2}{R^2}.$$

More generally ψ is the angular distance between the two points (x, y) (evaluation point) and (x', y') (integration or moving point) seen from the Earth's centre, approximately

$$\psi^2 \approx \left(\frac{x-x'}{R}\right)^2 + \left(\frac{y-y'}{R}\right)^2.$$

Furthermore we must account for the Jacobi determinant of the projection:

$$d\sigma = R^{-2} dx dy$$

and the Stokes equation becomes now

$$T(x,y) \approx \frac{1}{4\pi R} \iint_{\infty}^{\infty} S(x-x', y-y')\, \Delta g(x', y')\, dx' dy', \tag{8.2}$$

a two-dimensional *convolution integral*.

Convolutions have nice properties in Fourier theory. If we designate the Fourier transform with the symbol $\mathcal{F}$, and convolution with the symbol $\circ$, we may abbreviate the above equation as follows:

$$T = \frac{1}{4\pi R} S \circ \Delta g,$$

and according to the *convolution theorem* ("Fourier transforms a convolution into a multiplication"):

$$\mathcal{F}\{T\} = \frac{1}{4\pi R} \mathcal{F}\{S\}\, \mathcal{F}\{\Delta g\}.$$

This (x, y) plane approximation works only, if the *needed integration can be restricted to a local area where the curvature of the Earth surface may be neglected.* This is possible thanks to the use of global spherical harmonic expansions, because these describe the global part of the variability of the Earth's gravity field. After we have removed from the observed gravity anomalies Δg the effect of the global spherical harmonic model (the "Remove" step) we may safely forget the effect of areas far removed from the evaluation point: after this removal, the anomaly field will contain only the remaining short wavelength parts, the effect of which cancels out at greater distances. Of course, once the integral has been computed and the local disturbing potential T_{loc} has been obtained, we must remember to add to it again the T_{glob} effect of the global spherical harmonic expansion to be calculated separately (the "Restore" step).

8.2 Integration by FFT

The Fourier transform needed for applying the convolution theorem is calculated as a *discrete Fourier transform.* For this purpose exists the highly efficient Fast Fourier Transform, FFT (e.g., Kakkuri, 1981 pp. 183-200). There are several formulations of the discrete Fourier transform to be found in the literature; it doesn't really matter which one is chosen, as long as it is a compatible pair of a forward Fourier transform $\mathcal{F}$ and an inverse Fourier transform $\mathcal{F}^{-1}$.

In preparation for this we first compute a discrete *grid representation* of the function $\Delta g(x, y)$, a rectangular table of Δg values on an equidistant (x, y) grid of points. The values may be, e.g., the function values themselves at the grid points[1]:

$$\Delta g_{ij} = \Delta g(x_i, y_j),$$

where the co-ordinates of the grid points are:

$$\begin{aligned} x_i &= i\delta x, \quad i = -\frac{n}{2}, \ldots, \frac{n}{2} - 1, \\ y_j &= j\delta y, \quad j = -\frac{n}{2}, \ldots, \frac{n}{2} - 1, \end{aligned}$$

for suitably chosen grid spacings $(\delta x, \delta y)$. The sequence of values of the subscripts i and j has been chosen so, that the centre of the area $(x = y = 0)$ is also in the centre of the constructed data table Δg_{ij}, $(i = j = 0)$. The integer n is here the grid size, assumed the same in both directions (this is not a necessary assumption).

Next, we do the same for the kernel function

$$S(\psi) = S(x - x', y - y') = S(\Delta x, \Delta y),$$

[1] There exist alternatives to this. E.g., one could calculate for every grid point the average over a square surrounding the point.

i.e., we write

$$S_{ij} = S(\Delta x_i, \Delta y_j),$$

where again (n being the grid size):

$$\Delta x_i = i\delta x, \quad i = -\frac{n}{2}, \ldots, \frac{n}{2} - 1,$$
$$\Delta y_j = j\delta y, \quad j = -\frac{n}{2}, \ldots, \frac{n}{2} - 1.$$

Also here, the choice of the value sequences for the i and j subscripts is based in the wish to get the central peak at the origin of the S function – $S(\psi) \to \infty$ when $(\Delta x, \Delta y) \to (0, 0)$ – placed at the centre of the grid of values S_{ij}[2].

Next:

1. the grid representations Δg_{ij} and S_{ij} thus obtained of the functions S and Δg are transformed to the *frequency domain* – they become functions $\mathcal{S}_{uv}$ and $\mathcal{G}_{uv}$ of the two "frequencies", the wave numbers u and v in the x and y directions. The spatial frequencies are $\omega_u = {}^u\!/\!_L, \omega_v = {}^v\!/\!_L$, where L is the size of the area (assumed square).

2. They are multiplied with each other "one frequency pair at a time", i.e., we calculate

$$\mathcal{T}_{uv} = \sum_{u=0}^{n-1} \sum_{v=0}^{n-1} \mathcal{S}_{uv} \mathcal{G}_{uv};$$

3. and we transform the result, $\mathcal{T} = \mathcal{F}\{T\}$, back to the space domain, i.e., to a grid $T_{ij} = T(x_i, y_j)$ describing the disturbing potential T. The disturbing potential of an arbitrary point can be obtained from this grid by interpolation. The co-ordinates x_i, y_j run as functions of i, j in the same way as described above for Δg[3] .

This method is good for computing the disturbing potential T – and similarly the geoid height $N = {}^T\!/\!_\gamma$ – from gravity anomalies using the Stokes equation. It is just as good for evaluating other quantities, like, e.g., the vertical gradient of gravity by the Poisson equation. The only requirement is, that the equation can be expressed as a *convolution*.

Also the *reverse computation* is easy: in the Fourier or *spectral domain* it is just a simple division.

Using the discrete FFT transform requires that the input data in a field to be integrated – in the example gravity anomalies – is given on a regular grid covering the area of computation, or can be converted to one. The result – e.g., the disturbing potential – is obtained on a regular grid in the same geometry. Values can then be *interpolated* to chosen locations.

The FFT method may again be depicted as a *commutative diagram*:

[2] Without this measure the result of the calculation would be correct, but in the wrong place...

[3] In fact, for both Δg and T we could choose the simpler grid geometry where the subscript sequences are $i, j = 0, \cdots, n-1$; however, for S it is mandatory to have the origin in the middle of the grid.

Free observation point selection	$\Rightarrow$ Interpolation to grid points $\Rightarrow$	Point grid
$\downarrow$ (Direct solution) $\downarrow$		$\Downarrow$ FFT $\Downarrow$
Solution points in their own places	$\Leftarrow$ Interpolation to sol. points $\Leftarrow$	Point grid

Appendix C offers a short explanation of why FFT works and what makes it as efficient as it is.

8.3 Solution in rectangular co-ordinates

In the above equation 8.2, the co-ordinates x and y are rectangular. In practice, often latitude and longitude (ϕ, λ) are taken, causing additional errors due to meridian convergence – as latitude and longitude aren't actually rectangular. Slightly more suitable would be the pair $(\phi, \lambda \cos \phi)$.

The problem has also been addressed on a more conceptual level.

8.3.1 The Strang van Hees method

The Stokes kernel function $S(\psi)$ depends only on the angular distance ψ between evaluation point (ϕ, λ) and the data point (ϕ', λ'). The angular distance may be written as follows (spherical approximation):

$$\cos\psi = \sin\phi' \sin\phi + \cos\phi' \cos\phi \cos(\lambda' - \lambda).$$

Substitute

$$\begin{aligned}
\cos(\lambda' - \lambda) &= 1 - 2\sin^2\frac{(\lambda' - \lambda)}{2} \\
\cos\psi &= 1 - 2\sin^2\frac{\psi}{2}, \\
\cos(\phi' - \phi) &= 1 - 2\sin^2\frac{\phi' - \phi}{2},
\end{aligned}$$

and obtain

$$\begin{aligned}
\cos\psi &= \cos(\phi' - \phi) - 2\cos\phi'\cos\phi\sin^2\frac{\lambda' - \lambda}{2} \Rightarrow \\
\sin^2\frac{\psi}{2} &= \sin^2\frac{\phi' - \phi}{2} + \cos\phi'\cos\phi\sin^2\frac{\lambda' - \lambda}{2}.
\end{aligned}$$

Here we may use the following approximation:

$$\cos\phi' \approx \cos\phi \equiv \cos\phi_0,$$

where ϕ_0 is a reference value in the middle of the calculation area. Now the above formula becomes:

$$\sin^2 \frac{\psi}{2} \approx \sin^2 \frac{\phi' - \phi}{2} + \cos^2 \phi_0 \sin^2 \frac{\lambda' - \lambda}{2}, \tag{8.3}$$

which depends *only* on the differences $\Delta\phi \equiv \phi' - \phi$ and $\Delta\lambda \equiv \lambda' - \lambda$; a requirement for convolution.

After this, the FFT method may be applied by using co-ordinates ϕ, λ[4] and the modified Stokes function

$$S^*(\Delta\phi, \Delta\lambda) \equiv S\left(2\sqrt{\arcsin\left(\sin^2 \frac{\Delta\phi}{2} + \cos^2 \phi_0 \sin^2 \frac{\Delta\lambda}{2}\right)}\right).$$

This clever way of using FFT in geographical co-ordinates was invented by the Dutchman G. Strang van Hees[5] v. 1990.

8.3.2 "Spherical FFT", multi-band model

We divide the area in several narrow latitude bands. In each band we apply the Strang van Hees method using its own optimal central latitude.

Write the Stokes equation as follows:

$$N(\phi, \lambda) = \frac{R}{4\pi\gamma} \iint S(\phi - \phi', \lambda - \lambda'; \phi) \left[\Delta g(\phi'\lambda') \cos\phi'\right] d\phi' d\lambda', \tag{8.4}$$

where we have expressed $S(\cdot)$ as a function of latitude difference, longitude difference and *evaluation latitude*. Now, choose two support latitudes: ϕ_i ja ϕ_{i+1}. Assume furthermore that S is between these a linear function of ϕ. In that case we may write:

$$S(\Delta\phi, \Delta\lambda, \phi) = \frac{(\phi - \phi_i) S_{i+1}(\Delta\phi, \Delta\lambda) + (\phi_{i+1} - \phi) S_i(\Delta\phi, \Delta\lambda)}{\phi_{i+1} - \phi_i},$$

where $\Delta\phi = \phi - \phi'$, $\Delta\lambda = \lambda - \lambda'$ and

$$\begin{aligned} S_i(\Delta\phi, \Delta\lambda) &= S(\phi - \phi', \lambda - \lambda'; \phi_i), \\ S_{i+1}(\Delta\phi, \Delta\lambda) &= S(\phi - \phi', \lambda - \lambda'; \phi_{i+1}). \end{aligned}$$

We obtain by substitution into integral equation 8.4:

$$\begin{aligned} N(\phi, \lambda) &= \frac{R}{4\pi\gamma} \left\{ \left[\frac{\phi - \phi_i}{\phi_{i+1} - \phi_i}\right] \iint S_{i+1}(\Delta\phi, \Delta\lambda) \left[\Delta g(\phi', \lambda') \cos\phi'\right] d\phi' d\lambda' + \right. \\ &\quad \left. + \left[\frac{\phi_{i+1} - \phi}{\phi_{i+1} - \phi_i}\right] \iint S_i(\Delta\phi, \Delta\lambda) \left[\Delta g(\phi', \lambda') \cos\phi'\right] d\phi' d\lambda' \right\}. \end{aligned} \tag{8.5}$$

[4] In practice one uses the geodetic/geographic latitude φ instead of ϕ without significant error.
[5] Govert L. Strang van Hees (1932-2012) was a Dutch gravimetric geodesist.

This equation is the *sum of two convolutions*. Both are evaluated by FFT and from the solutions obtained one forms the weighted mean according to equation 8.5.

In this method we may use, instead of the approximative equation 8.3, an exact equation, in which ϕ' is expressed into ϕ and $\Delta\phi$:

$$\sin^2\frac{\psi}{2} = \sin^2\frac{\phi'-\phi}{2} + \cos\phi'\cos\phi\sin^2\frac{\lambda'-\lambda}{2} =$$
$$= \sin^2\frac{\Delta\phi}{2} + \cos\left(\phi-\Delta\phi\right)\cos\phi\sin^2\frac{\Delta\lambda}{2}.$$

Here again, we calculate S_i and S_{i+1} for the values $\phi = \phi_i$ and $\phi = \phi_{i+1}$, we evaluate the integrals with the aid of the convolution theorem, and interpolate $N\left(\phi,\lambda\right)$ according to equation 8.5 when $\phi_i \leq \phi < \phi_{i+1}$. After this, the solution isn't entirely exact, because inside every band we still use linear interpolation. However by making the bands narrower, we can keep the error arbitrarily small.

8.3.3 "Spherical FFT", Taylor expansion model

This somewhat more complicated but also more versatile approach expands the Stokes kernel into a Taylor expansion with respect to latitude about a *reference latitude* located in the middle of the computation area[6]. Each term in the expansion depends only on the *difference* in latitude. The integral to be calculated similarly expands into terms, of which each contains a pure convolution.

Let us write the *general* problem as follows:

$$\ell\left(\varphi,\lambda\right) = \int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C\left(\varphi,\varphi',\Delta\lambda\right)\left[m\left(\varphi',\lambda'\right)\cos\varphi'\right]d\varphi' d\lambda',$$

where ℓ contains values to be computed, m contains values given, and C is the coefficient or kernel function. Here is assumed only *rotational symmetry* for the geometry, i.e., the kernel function depends only on the difference between longitudes $\Delta\lambda$ rather than the absolute longitudes λ, λ'.

In a concrete case m contains for example Δg values in various points (φ', λ'), ℓ contains geoid heights N in various points (φ, λ), and C contains coefficients calculated using the Stokes function.

We first change the dependence upon φ, φ' into a dependence upon $\varphi, \Delta\varphi$:

$$C = C\left(\varphi,\varphi',\Delta\lambda\right) = C\left(\Delta\varphi,\Delta\lambda,\varphi\right).$$

Linearize:

$$C = C_0\left(\Delta\varphi,\Delta\lambda\right) + \left(\varphi-\varphi_0\right)C_\varphi\left(\Delta\varphi,\Delta\lambda\right) + \ldots$$

[6] In the literature the method has been generalized by expanding the kernel also with respect to height.

where we define for a suitable *reference latitude* φ_0:

$$C_0\left(\Delta\varphi,\Delta\lambda\right) \equiv C\left(\Delta\varphi,\Delta\lambda,\varphi_0\right),$$
$$C_\varphi\left(\Delta\varphi,\Delta\lambda\right) \equiv \frac{\partial}{\partial\varphi} C\left(\Delta\varphi,\Delta\lambda,\varphi\right)\Big|_{\varphi=\varphi_0}.$$

Substitution yields

$$\begin{aligned}
\ell &= \int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} Cm'\cos\phi\varphi' d\varphi' d\lambda' = \\
&= \int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} \left[C_0 + \left(\varphi-\varphi'\right)C_\varphi\right] m'\cos\varphi' d\varphi' d\lambda' = \\
&= \int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C_0 m'\cos\varphi' d\varphi' d\lambda' + \\
&= + \left(\varphi-\varphi_0\right)\int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C_\varphi m'\cos\varphi' d\varphi' d\lambda'.
\end{aligned} \tag{8.6}$$

Important here is now, that the integrals in the first and second terms, i.e.,

$$\int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C_0 m'\cos\varphi' d\varphi' d\lambda' =$$
$$= \int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C_0\left(\Delta\varphi,\Delta\lambda\right)\left[m'\cos\varphi'\right] d\varphi' d\lambda' \equiv C_0 \circ \left[m\cos\varphi\right], \quad \text{ja}$$
$$\int_0^{2\pi}\int_{-\pi/2}^{+\pi/2} C_\varphi\left[m'\cos\varphi'\right] d\varphi' d\lambda' =$$
$$= \int_0^{\pi}\int_{-\pi/2}^{+\pi/2} C_\varphi\left(\Delta\varphi,\Delta\lambda\right)\left[m'\cos\varphi'\right] d\varphi' d\lambda' \equiv C_\varphi \circ \left[m\cos\varphi\right],$$

are both convolutions: both C functions depend only on $\Delta\varphi$ and $\Delta\lambda$. Both integrals can be calculated only if the corresponding $\Delta\varphi = \varphi - \varphi'$ and $\Delta\lambda = \lambda - \lambda'$, and the corresponding coefficient grids C_0, C_φ, are calculated first. After this (in principle expensive, but, thanks to FFT and the convolution theorem, a lot cheaper) integration, computing the compound 8.6 is cheap: one multiplication and one addition for each evaluation point (φ,λ).

Example: let the evaluation area at latitude 60° be $10^\circ \times 20^\circ$ in size. If the grid mesh size is $5' \times 10'$, the number of cells is 120×120. Let us choose, e.g., a 256×256 grid (i.e., $n = 256$) and fill the missing values with extrapolated values.

Also the values of the kernel functions C_0 and C_φ are calculated on a 256×256 size $(\Delta\varphi,\Delta\lambda)$ grid. The number of these is thus also 65536. Calculating the convolutions $C_0 \circ [m\cos\varphi]$ and $C_\varphi \circ [m\cos\varphi]$ by means of FFT – i.e.,

$$\iint C_0\, m'\cos\varphi' d\varphi' d\lambda' = C_0 \circ \left[m\cos\varphi\right] = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{C_0\right\}\mathcal{F}\left\{m\cos\varphi\right\}\right\} \quad \text{ja}$$
$$\iint C_\varphi\, m'\cos\varphi' d\varphi' d\lambda' = C_\phi \circ \left[m\cos\varphi\right] = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{C_\varphi\right\}\mathcal{F}\left\{m\cos\varphi\right\}\right\}$$

requires $(n^2) \times^2 \log(n^2) = 65536 \times 16 =$ more than a million operations, multiplication with $(\varphi - \varphi_0)$ and adding together, each again 65536 operations.

The grid matrices corresponding to functions C_0 and C_φ are obtained as follows: for three reference latitudes $\varphi_{-1}, \varphi_0, \varphi_{+1}$ we compute numerically the grids

$$\begin{aligned} C_{-1} &= C(\Delta\varphi, \Delta\lambda, \varphi_{-1}), \\ C_0 &= C(\Delta\varphi, \Delta\lambda, \varphi_0), \\ C_{+1} &= C(\Delta\varphi, \Delta\lambda, \varphi_{+1}), \end{aligned}$$

where C_0 is directly available, and

$$C_\varphi \approx \frac{C_{+1} - C_{-1}}{\varphi_{+1} - \varphi_{-1}}.$$

Also *inversion* is thus directly feasible. Let ℓ be given in suitable point grid form. We compute the first approximation to m as follows:

$$\mathcal{F}\{C_0\}\,\mathcal{F}\{m\cos\varphi\} = \mathcal{F}\{\ell\} \Rightarrow [m\cos\varphi]^{(0)} = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{\ell\}}{\mathcal{F}\{C_0\}}\right\}.$$

The second approximation is obtained by first calculating

$$\ell^{(0)} = C_0 \circ [m\cos\phi]^{(0)} + (\phi - \phi_0) \cdot C_\phi \circ [m\cos\phi]^{(0)},$$

after which we make the improvement:

$$[m\cos\varphi]^{(1)} = [m\cos\varphi]^{(0)} + \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\left\{\ell - \ell^{(0)}\right\}}{\mathcal{F}\{C_0\}}\right\},$$

and so on, iteratively. Two, three interations are usually enough. This method has been used to compute underground mass points to represent gravity anomalies in the exterior gravity field of the Earth. More is explained in Forsberg and Vermeer (1992).

8.3.4 "1D-FFT"

This is a limiting case of the previous ones, where FFT is used only in the longitude direction. In other words, a zones method where the zones are only one point narrow. This method is exact if all longitudes $(0^\circ - 360^\circ)$ are along in the calculation. It requires a bit more computing time compared to the previous methods. In fact, it is identical to a Fourier transform in variable λ, longitude. Details are found in Haagmans et al. (1993).

8.4 Complications; bordering, tapering

The discrete Fourier transform presupposes the data to be *periodically continuous*. In practice it is not. Therefore, always when using FFT with the convolution theorem 8.2, we

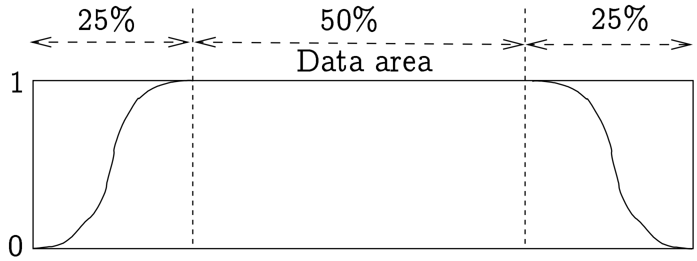

Figure 8.2 – "Tapering" 25%.

continue the data by adding a border to the data area, so-called *bordering*. Often the border is 25% of the size of the data area; then, the size of the whole calculation area will become four times that of the data area itself. The border is often filled with zeroes, although predicted values – or even measured values, if those exist – are a better choice.

Also the calculation area for the *kernel function* is made similarly four times larger; in this case, as the function is *symmetric*, the border is filled with real (computable) values, miking it automatically periodically continuous.

Because the discrete Fourier transform assumes periodicity, one should make sure that the data really is periodic. If the values at the borders are not zero, they may be forced to zero by multiplying the whole data area by a so-called *tapering* function, which goes smoothly to zero towards the edges. Such a function can easily be built, e.g., a third-degree spline polynomial or cosine. See figure 8.2, showing a 25% tapering function, as well as example images 8.3, where one sees how non-periodicity (differing left and right, and upper and lower, edges) causes horizontal and vertical artefacts.

Many journal articles have appeared on these technicalities. Groups that have been involved in early development of FFT geoid computation (in the 1980s) are Forsberg's group in Copenhagen, the group of Schwarz and Sideris in Calgary, Canada, the Delft group (Strang van Hees, Haagmans, De Min, Van Gelderen), the Milanese (Sansò, Barzaghi, Brovelli), the Hannover "Institut für Erdmessung", and many others.

8.5 Computing a geoid model with FFT

Nowadays computing a geoid or quasi-geoid model is easy thanks to increased computing power, especially using FFT. On the other hand the spread of geodetic GPS has made the availability of precise geoid models an important issue, so that one can use GPS for rapid and inexpensive height determination.

8.5.1 The GRAVSOFT software

The GRAVSOFT geoid computation software has been mainly produced in Denmark. Authors include Carl Christian Tscherning (1942–2014), René Forsberg, Per Knudsen, and the Greek Dimitris Arabelos. http://www.academia.edu/9206363/An_overview_manual_for_the_GRAVSOFT_Geodetic_Gravity_Field_Modelling_Programs.

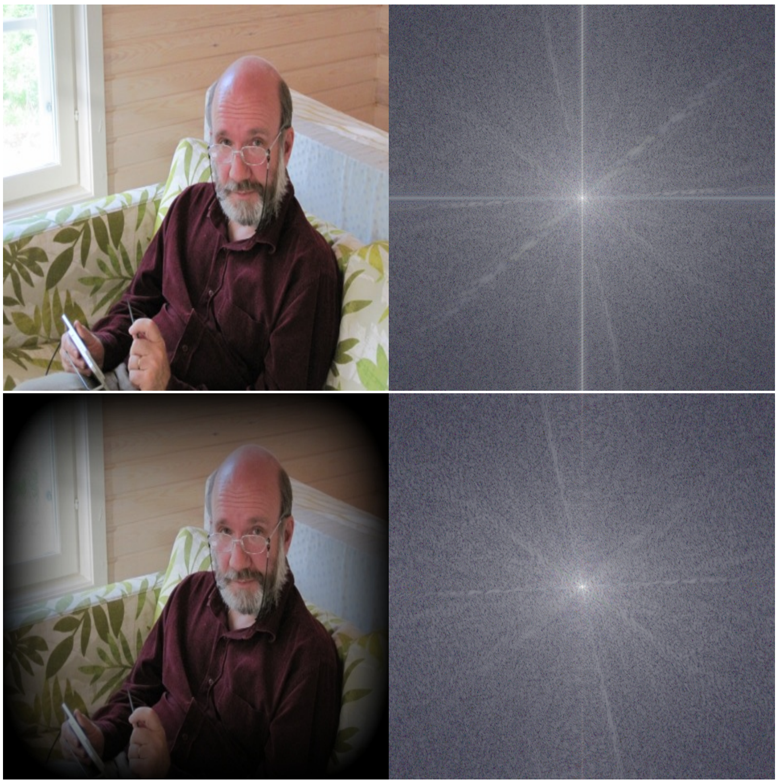

Figure 8.3 – Example images for FFT transform without (above) and with (below) tapering. Used on-line FFT http://www.ejectamenta.com/Imaging-Experiments/fourierimagefiltering.html.

This package is in widespread use and offers, in addition to variants of FFT geoid computation, also, e.g., least-squares collocation, routines for evaluation of various terrain effects, etc. Its spread can be partly explained by it being free for scientific use, and being distributed as source code. It is also well documented. Therefore it has also found commercial use, e.g., in the petroleum extraction industry.

GRAVSOFT has been used a lot also for teaching, e.g., at many research schools organized by the IAG (International Association of Geodesy) in various countries. http://www.isgeoid.polimi.it/Schools/schools.html.

8.5.2 The Finnish FIN2000 geoid

Currently two geoid models are in use in Finland: FIN2000 (figure 8.4) and FIN2005N00 (Bilker-Koivula and Ollikainen, 2009). The first model is a reference surface for the N60 vertical reference system: using it together with GNSS positioning allows determination of the N60 heights of points. The geoid heights given by the model are above the GRS80 reference ellipsoid. The second model is similarly a reference surface for the new N2000 height system. It too gives heights from the GRS80 reference ellipsoid.

The precisions (mean errors) of FIN2000 and FIN2005N00 are on the level of $\pm 2 - 3$ cm.

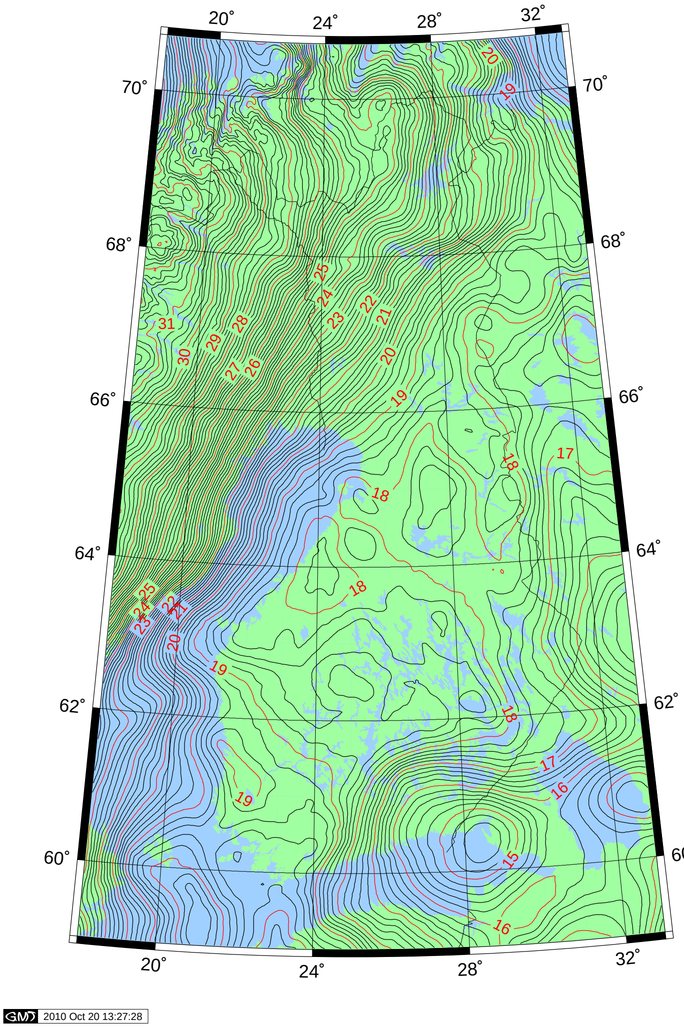

Figure 8.4 – The Finnish FIN2000-geoid. Data source: Finnish Geodetic Institute.

8.6 Use of FFT computation in other contexts

8.6.1 Altimetry

Per Knudsen and Ole Balthasar Andersen have computed a gravity map of the world ocean by starting from satellite altimetry derived "geoid heights" and inverting them to gravity anomalies (Andersen et al., 2010). A pioneer of the method has been David Sandwell.

8.6.2 Satellite gravity missions; airborne gravimetry

Also the data from satellite gravity missions (like CHAMP, GRACE and GOCE) can be regionally processed using the FFT method: in the case of GOCE, the inversion of gradiometric measurements, i.e., calculating geoid heights on the Earth surface from measurements at satellite level. Also airborne gravity measurements are processed in this way. The problem is called "downward continuation" and is in principle unstable.

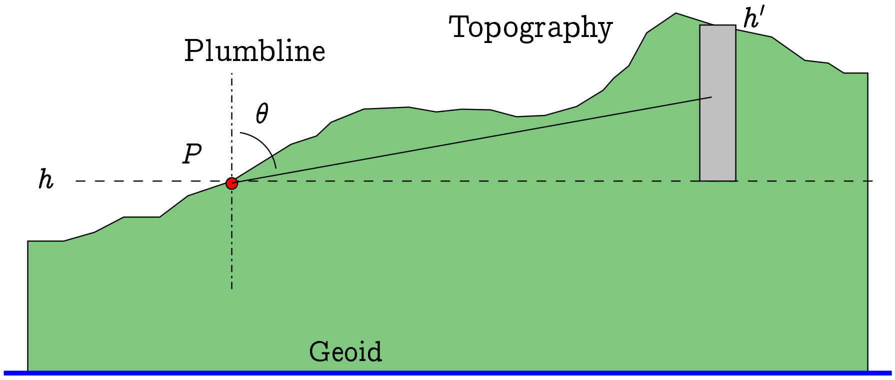

Figure 8.5 – Calculating the terrain correction with the FFT method.

Airborne gravimetry is a practical method for gravimetric mapping of large areas; in the pioneering days, the gravity field over Greenland was mapped, as well as many areas around the Arctic and Antarctic. Later, areas were measured like the Brazilian Amazonas, Mongolia and Ethiopia, where no full-coverage terrestrial gravimetric data existed. The advantage of the method is that one measures rapidly large areas in a homogeneous way. Also for the processing of airborne gravimetry data, FFT is suitable.

8.7 Computing terrain corrections with FFT

The *terrain correction* is a very localized phenomenon, the calculation of which requires high-resolution terrain data from a relatively small area around the computation point. Thus, calculating the terrain correction is ideally suited to applying the FFT method.

We show how, with FFT, we can simply and efficiently evaluate the terrain correction. We make the following simplifying *assumptions*:

1. terrain slopes are relatively small;
2. the density ρ of the Earth's crust is constant;
3. the Earth is flat.

These assumptions are not mandatory. The general case however leads us into a jungle of formulas without aiding the conceptual picture.

The *terrain correction*, the joint effect of all the topographic masses, or lacking topographic masses, above and below the height level h of the evaluation point, can be calculated under these assumptions using the following rectangular equation, which describes the attraction of rock columns projected vertically (figure 8.5):

$$\begin{aligned} TC\,(x,y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{G\rho\,(h'-h)}{\ell^2}\cos\theta\,dx'dy' \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{G\rho\,(h'-h)}{\ell^2}\cdot\frac{1}{2}\frac{h'-h}{\ell}dx'dy' \\ &= \frac{1}{2}G\rho\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{(h'-h)^2}{\ell^3}dx'dy'. \end{aligned} \tag{8.7}$$

Here $G\rho\left(h'-h\right)\ell^{-2}$ is the attraction of the column and $\frac{1}{2}\left(h'-h\right)\ell^{-1}$ is the cosine of the angle θ between the force vector and the vertical direction.

This is a linear approximation, wherein ℓ, the slant distance between the evaluation point (x, y) and the moving data point (x', y'), is also the *horizontal distance*:

$$\ell^2 = \left(x'-x\right)^2 + \left(y'-y\right)^2.$$

Equation 8.7 is easy to check straight from Newton's law of gravitation. When it is assumed that the terrain is relatively free of steep slopes, then ℓ is large compared to $h'-h$.

From the above we obtain by development into terms:

$$\begin{aligned} TC\left(x,y\right) &= \frac{1}{2}G\rho h^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\ell^3} dx'dy' - G\rho h \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{h'}{\ell^3} dx'dy' + \\ &\quad + \frac{1}{2}G\rho \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\left(h'\right)^2}{\ell^3} dx'dy', \end{aligned} \tag{8.8}$$

where every integral is a *convolution* with kernel ℓ^{-3}, and functions to be integrated 1, h' and $\left(h'\right)^2$. Unfortunately the function ℓ^{-3} has no Fourier transform, wherefore we change the above definition a tiny bit by adding a small term:

$$\ell^2 = \left(x'-x\right)^2 + \left(y'-y\right)^2 + \delta^2. \tag{8.9}$$

Then, the terms in the above sum are large numbers that almost cancel each other, giving a nearly correct result. However, numerically this is an unpleasant situation.

If ℓ is defined according to equation 8.9, then the Fourier transform of kernel ℓ^{-3} is (Harrison and Dickinson, 1989):

$$\mathcal{F}\{\ell^{-3}\} = \frac{2\pi}{\delta}\exp\left(-2\pi\delta q\right) = \frac{2\pi}{\delta}\left[1 - 2\pi\delta q + \frac{4\pi^2 q^2\delta^2}{1\cdot 2} - \ldots\right],$$

where $q \equiv \sqrt{u^2+v^2}$, and u, v are wave numbers (i.e., "frequencies") in the x and y directions in the (x, y) plane. If we substitute this into equation 8.8, we notice that the terms containing δ^{-1} sum to zero, and of course also the terms containing positive powers of δ vanish when $\delta \to 0$. Next (Harrison and Dickinson, 1989):

$$\begin{aligned} \mathcal{F}\left\{TC\right\} &\approx \frac{1}{2}G\rho h^2 \mathcal{F}\left\{1\right\}\left[\frac{2\pi}{\delta}\left(1-2\pi\delta q\right)\right] - \\ &- G\rho h \mathcal{F}\left\{h'\right\}\left[\frac{2\pi}{\delta}\left(1-2\pi\delta q\right)\right] + \\ &+ \frac{1}{2}G\rho \mathcal{F}\left\{\left(h'\right)^2\right\}\left[\frac{2\pi}{\delta}\left(1-2\pi\delta q\right)\right] \end{aligned}$$

where we leave off all terms in higher powers of δ.

Re-order the terms:

$$\mathcal{F}\{TC\} = \frac{\pi}{\delta} G\rho \left[h^2 \mathcal{F}\{1\} - 2h\mathcal{F}\{h'\} + \mathcal{F}\left\{(h')^2\right\}\right] +$$
$$4\pi^2 q \left[-\frac{1}{2} G\rho h^2 \mathcal{F}\{1\} + G\rho h \mathcal{F}\{h'\} - \frac{1}{2} G\rho \mathcal{F}\left\{(h')^2\right\}\right].$$

Because $\mathcal{F}\{1\} = 0$ if $q \neq 0$, the first term inside the second term will always vanish. We obtain (remember that h is a constant, the height of the evaluation point):

$$\mathcal{F}\{TC\} = \frac{\pi}{\delta} G\rho \left[\mathcal{F}\left\{ h^2 - 2hh' + (h')^2 \right\}\right] +$$
$$4\pi^2 q \left[G\rho h \mathcal{F}\{h'\} - \frac{1}{2} G\rho \mathcal{F}\left\{(h')^2\right\}\right]$$

and the inverse Fourier transform yields:

$$\begin{aligned} TC &= \frac{2\pi G\rho}{\delta} \left[\frac{1}{2} h^2 - h'h + \frac{1}{2}(h')^2 \right] + \\ &+ G\rho h \mathcal{F}^{-1} \left\{ \mathcal{F}\{h'\} \cdot 4\pi^2 q \right\} - \\ &- \frac{1}{2} G\rho \mathcal{F}^{-1} \left\{ \mathcal{F}\left\{(h')^2\right\} \cdot 4\pi^2 q \right\}. \end{aligned}$$

In the first term

$$\frac{1}{2} h^2 - h'h + \frac{1}{2}(h')^2 = \frac{1}{2}(h' - h)^2 = 0$$

in point (x, y) where $h' = h$, and we obtain:

$$TC_P = 4\pi^2 G\rho \mathcal{F}^{-1} \left\{ q \cdot \left[h\mathcal{F}\{h'\} - \frac{1}{2}\mathcal{F}\left\{(h')^2\right\}\right]\right\},$$

from which now the troublesome δ^{-1} has vanished.

A condition for this “regularization” or “renormalization” is, that at point (x, y) $h' = h$, i.e., the evaluation happens at the Earth surface. The convolutions above are evaluated by the FFT technique; a more detailed account is found, e.g., from the article Vermeer (1992).

For calculating the terrain effect TC in the *exterior space* – airborne gravimetry, but also the effect of the sea floor at the sea surface, or the effect of the Mohorovičić discontinuity at the Earth's surface – there are techniques that express TC as a sum of convolutions (as a Taylor expansion). An early paper on this is Parker (1972).