Lezione 16 Analisi Reale

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0.1 altri teoremi

Teorema 1 (Lisin)
$f : \mathbb{R} \to \mathbb{R}$ *misurabile* $\Rightarrow \forall \delta > 0 \exists g_\delta \in C(\mathbb{R}) \mid m(\{f\}) < \delta \quad \sup|g_\delta| \leq \sup|f|$

Dimostrazione
Procediamo per passi:

1. $s : [a, b] \to \mathbb{R}$ *semplice.*
   Tesi: $\forall \delta \exists F \subseteq [a, b] \mid m([a, b] \setminus F) < \delta, s|_F$ *continua*
   $\exists n \in \mathbb{N}, c \in \mathbb{R}^n, E \subseteq M^n \mid s(x) = \sum_{k=1}^n c_k \chi_{E_n}(x)$ *SPDG* $E_k \cap E_h = \emptyset \ \ \forall h, k$
   $\forall \delta > 0, k \in [n] \exists F_k \subseteq E_k$ *chiuso* $\mid m(E_k \setminus F_k) < \frac{\delta}{n}$
   $\Rightarrow F := \bigcup_{k=1}^n F_k, \quad m([a, b] \setminus F) = m(\bigcup_{k=1}^n E_k \bigcup_{k=1}^n F_k) \leq \sum_{k=1}^n m(E_k \setminus \bigcup_{k=1}^n F_k) = \sum_{k=1}^n m(E_k \setminus F_k) < \delta$
   $s|_F$ *continua pK*

2. $f : [a, b] \to \mathbb{R}$ *misurabile*
   Obiettivo: $\forall \delta \in (0, +\infty) \exists F \subseteq [a, b] \mid m([a, b] \setminus F) < \delta, f)_F \in C(F)$
   La dimostrazione è poco chiara. Guarda da Spadaro.

□

0.2 Spazi L^p

(X, μ) spazio di misura
$L^p(X) = L^p(X, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ misurabile} \mid \int_X |f|^p d\mu < +\infty\}$
$f(x) = \frac{\chi_{|x| \geq 1}(x)}{|x|} \in L^p(\mathbb{R})$?
$\int_{\mathbb{R}} |f(x)|^p d\mu = 2 \int_1^+ \infty \frac{1}{x^p} dm = 2 \frac{x^{1-p}}{1-p}|_{p=1}^{+\infty} = \begin{cases} +\infty & p \leq 1 \\ \frac{2}{1-p} & p > 1 \end{cases} \Rightarrow f \in L^p \ \ \forall p > 1$
$f(x) = x^{-\frac{1}{2}} \in L^p((0, 1)) \Leftrightarrow \int_0^1 x^{-\frac{p}{2}} dx < +\infty \quad \Leftrightarrow \quad \frac{2x^{\frac{2-p}{2}}}{2-p}|_{x=0}^1 < +\infty \quad \Leftrightarrow \frac{2-p}{2} > 0 \Leftrightarrow p < 2$

Lemma 1 (Disugualgianza di convessità)
$\forall a, b \in [0, +\infty)$

1. *(Disuguaglianza di Young)*
   $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \quad \forall p, p' \in \mathbb{R} \mid \frac{1}{p} + \frac{1}{p'} = 1 \quad (p' = \frac{p}{p-1})$

2. $a^p + b^p \leq (a+p)^p \leq 2^{p-1}(a^p + b^p) \forall p \in [1, +\infty)$

3. $a^p + b^p \geq (a+b)^p \geq 2^{p-1}(a^p + b^p) \quad \forall p \in (0, 1)$

Dimostrazione *1.* $\ln(x)$ *concava* $\forall x \in (0, +\infty)$

$\ln(\frac{1}{p}a^p + \frac{}{1}p' b^{p'}) \geq \frac{1}{p}\ln(a^p) + \frac{1}{p'}\ln(b^p) = \ln(a) + \ln(b) = \ln(ab)$

$\Rightarrow \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \geq ab$

2. $p \geq 1 \Rightarrow f(x) = x^p, x \in [0, +\infty)$ *convessa*

$$\frac{1}{2^p}(a+b)^p(\frac{1}{2}a + \frac{1}{2}b)^p \leq \frac{1}{2}a^p + \frac{1}{2}b^p = \frac{1}{2}(a^p + b^p) \quad (ottimale, a = b \Rightarrow =).$$

$b = 0 \Rightarrow a^p + b^p \leq (a+b)^p \quad \forall p \geq 1$

$b > 0 \Rightarrow f(x) = (x+1)^p - x^p - 1$

$\Rightarrow f'(x) = p(x+1)^{p-1} - px^{p-1} = (x+1)^{p-1} - x^{p-1} \geq 0 \ \ \forall x \geq 0, p \geq 1$

$\Rightarrow f \nearrow, f(0) = 0 \Rightarrow f \geq 0 \ \ \forall x \geq 0$

$\Rightarrow (\frac{a}{b} + 1)^p - (\frac{a}{b})^p - 1 = f(\frac{a}{b}) \geq 0$

$\Rightarrow (\frac{a}{b} + 1)^p \geq (\frac{a}{b})^p + 1$

$\square$