Introduction

1. Use Euler's formula to prove that:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

Given that:

$$e^{ix} = \cos(x) + i\sin(x) \tag{1}$$
$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x) \tag{2}$$

Follows:

$$\begin{aligned}
\cos(x) &= \frac{e^{ix} + e^{-ix}}{2} && \text{From adding 1 and 2} \\
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} && \text{From substracting 1 and 2}
\end{aligned}$$

$$\begin{aligned}
\sin(x)\cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{ix} - e^{-ix}}{2i} && \text{Substituting \textbf{??} and \textbf{??}} \\
&= \frac{e^{i2x} - e^{i0} + e^{i0} - e^{-i2x}}{4i} && \text{Multiplying fractions} \\
&= \frac{1}{2}\frac{e^{i2x} - e^{-i2x}}{2i} && \text{Combining exponentials} \\
&= \frac{1}{2}\sin(2x) && \text{Substituting \textbf{??}}
\end{aligned}$$

2. Show that:

$$\int_0^{T_0} \cos(k\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0, & k \neq n \\ \frac{T_0}{2}, & k = n \neq 0 \\ T_0, & k = n = 0 \end{cases}$$

Given that:

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + cos(x+y)\right] \tag{3}$$

$$\int_0^{T_0} \cos(k\omega_0 t)dt = \begin{cases} T_0, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{4}$$

Follows:

$$\int_0^{T_0} \cos(k\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_0^{T_0} \left[\cos\left((k-n)\omega_0 t\right) + \cos\left((k+n)\omega_0 t\right)\right] dt$$

Case $k \neq n$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(a\omega_0 t\right) + \cos\left(b\omega_0 t\right)\right] dt = \frac{1}{2}(0+0) = 0$$

Case $k = n \neq 0$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(0\right) + \cos\left(2k\omega_0 t\right)\right] dt = \frac{1}{2}T_0 + 0 = \frac{T_0}{2}$$

Case $k = n = 0$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(0\right) + \cos\left(0\right)\right] dt = \frac{2}{2}T_0 = T_0$$

Fourier Series

3. Show that $B_k = \frac{2}{T_0}\oint f(t)\sin(k\omega_0 t)dt$ is consistent with the Fourier series.

Given that:

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\sin(k\omega_0 t)dt = \begin{cases} 0, & k \neq n \\ \frac{T_0}{2}, & k = n \neq 0 \\ T_0, & k = n = 0 \end{cases} \tag{5}$$

$$\int_{-T_0/2}^{T_0/2} \cos(n\omega_0 t)\sin(k\omega_0 t)dt = 0 \tag{6}$$

$$\oint \equiv \int_{-T_0/2}^{T_0/2} \tag{7}$$

Follows:

$$\begin{aligned} B_k &= \frac{2}{T_0}\oint \sum_{n=0}^{\infty} (A_n\cos(n\omega_0 t) + B_n\sin(n\omega_0 t))\sin(k\omega_0 t)dt \\ &= \frac{2}{T_0}\sum_{n=0}^{\infty}\left\{A_n \oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt + B_n \oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\right\} \end{aligned}$$

Case $k = 0$:

$$B_0 = \frac{2}{T_0}\sum_{n=0}^{\infty}\{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\}}_{=0} = 0$$

Case $k > 0$:

$$B_k = \frac{2}{T_0}\sum_{n=0}^{\infty}\{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\}}_{=\frac{T_0}{2}} = B_k$$

4. Show that $f(t) = \sum_{k=0}^{\infty} a_k\cos(k\omega_0 t - \phi_k)$ is equivalent to the Fourier series.

Given that:

$$a = \sqrt{A^2 + B^2} \tag{8}$$

$$\phi = \begin{cases} \arctan(B/A), & A > 0 \\ \pi/2, & A = 0, B > 0 \\ -\pi/2, & A = 0, B < 0 \\ \arctan(B/A) + \pi, & A < 0, B \geq 0 \\ \arctan(B/A) - \pi, & A < 0, B < 0 \end{cases} \tag{9}$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \tag{10}$$

$$\arctan(0) = 0 \tag{11}$$

Follows:

Case $A > 0,\ B = 0$:

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k\cos(k\omega_0 t - \phi_k) \\ &= \sum_{k=0}^{\infty} A_k\cos\left(k\omega_0 t - \arctan\frac{B}{A}\right) \\ &= \sum_{k=0}^{\infty} A_k\cos(k\omega_0 t) \end{aligned}$$

Case $A = 0,\ B > 0$:

$$
\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t - \phi_k) \\
&= \sum_{k=0}^{\infty} B_k \cos\left(k\omega_0 t - \frac{\pi}{2}\right) \\
&= \sum_{k=0}^{\infty} B_k \left[\cos(k\omega_0 t)\cos\left(\frac{\pi}{2}\right) + \sin(k\omega_0 t)\sin\left(\frac{\pi}{2}\right)\right] \\
&= \sum_{k=0}^{\infty} B_k \sin(k\omega_0 t)
\end{aligned}
$$

5. Consider the triangular function:

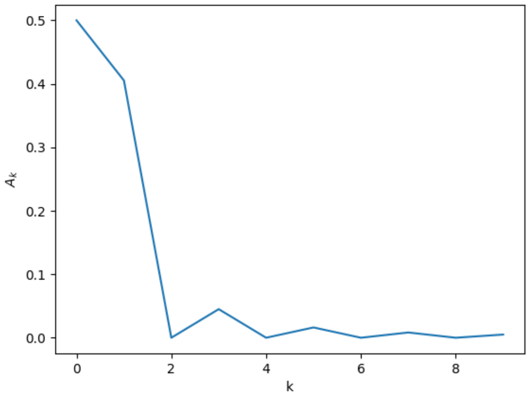
$$
f(t) = \begin{cases} 1 + \frac{2t}{T}, & \text{for} - \frac{T}{2} < t < 0 \\ 1 - \frac{2t}{T}, & \text{for} 0 \leq t \leq \frac{T}{2} \end{cases}
$$

a) Derive an algebraic expression for the coefficients A_k and B_k of the Fourier series.

$$
\begin{aligned}
B_0 &= 0 \\
A_0 &= \frac{1}{T}\int 1 - \frac{2t}{T} dt = \frac{1}{T}\left[t - \frac{t^2}{T}\right] \\
&= \frac{1}{T}\left[\frac{T}{2} - \frac{T^2}{4T}\right] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
a &\equiv k\omega && \text{Shortening notation} \\
B_k &= 0 \\
A_k &= \frac{5}{T}\int_0^{T/2} \left(1 - \frac{2t}{T}\right)\cos(at)dt && \text{Using symmetry and odd property} \\
&= \frac{4}{T}\int_0^{T/2} \cos(at) - \frac{2t}{T}\cos(at)dt && \text{Multiplying cosine} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at)\right]_0^{T/2} - \int_0^{T/2} \frac{2t}{T}\cos(at)dt\right] && \text{Computing first integral} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at)\right]_0^{T/2} + \int_0^{T/2} \frac{2}{aT}\sin(at)dt\right] && \text{Applying product rule} \\
&= \frac{4}{T}\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at) - \frac{2}{a^2T}\cos(at)\right]_0^{T/2} \\
&= a^{-1}\sin(aT/2) - 0 - a^{-1}\sin(aT/2) + 0 - \frac{2}{a^2T}\cos(aT/2) + \frac{2}{a^2T} && \text{Inserting boundaries} \\
&= \frac{4}{k^2\omega^2T^2}[1 - \cos(k\pi)] && \text{Replacing} a \\
&= \frac{1}{k^2\pi^2}[1 - \cos(k\pi)]
\end{aligned}
$$

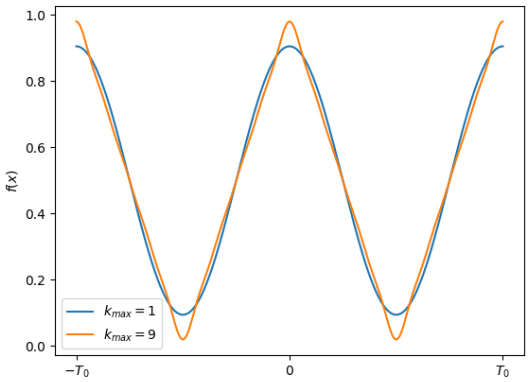
b) Plot A_k over k for $0 \leq k < 10$

```python
import numpy as np
import matplotlib.pyplot as plt

fn_ak = lambda k: 2/(k**2 * np.pi**2) * (1 - np.cos(k * np.pi))

ks = np.arange(0, 10)
aks = list(map(fn_ak, ks))
aks[0] = 1/2

plt.plot(ks, aks)
plt.xlabel("k")
plt.ylabel("$A_k$")
plt.show()
```



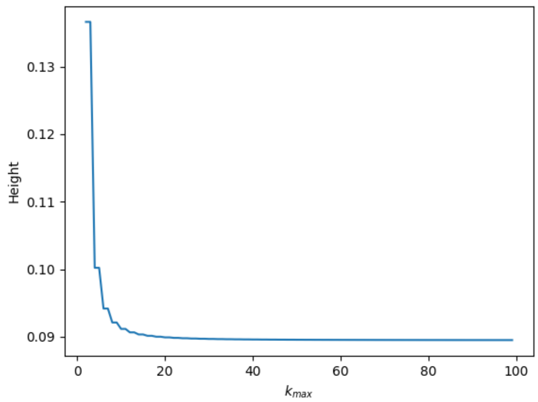
c) Plot the Fourier series from [-T0;T0] for kmax = 1 and kmax = 9.

```python
fn_term = lambda t, T, k: fn_ak(k) * np.cos(k * 2 * np.pi * t / T)

def fn_fourier(ts, k_max, T):
    y = np.ones_like(ts) * 1/2
    for i in range(1, k_max+1):
        y += np.array([fn_term(t, T, i) for t in ts])
    return y

T = 5
ts = np.linspace(-T, T, 1000)

plt.plot(ts, fn_fourier(ts, 1, T), label="$k_{max}=1$")
plt.plot(ts, fn_fourier(ts, 9, T), label="$k_{max}=9$")
plt.xticks([-T, 0, T], ["$-T_0$",0 , "$T_0$"])
plt.ylabel("$f(x)$")
plt.legend()
plt.show()
```

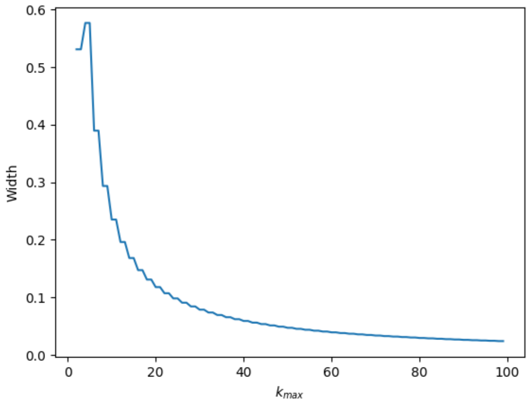


6. Consider the Fourier series of the rectangular function (see course notes).
a)

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize

def fn_term(t, T, k):
    b = 0 if k % 2 == 0 else 2/(np.pi*k)
    return b * np.sin(k * 2 * np.pi * t / T)

def fn_fourier(ts, T, k_max):
    y = np.zeros_like(ts)
    for i in range(1, k_max):
        y += np.array([fn_term(t, T, i) for t in ts])
    return y

T = 5
ks = np.arange(2, 100)
xs1, heights = np.zeros(len(ks)), np.zeros(len(ks))
for i, k_max in enumerate(ks):
    sol = minimize(
        lambda t, k_max: 0.5 - fn_fourier(t, T, k_max), (0,),
        method="Nelder-Mead", bounds=((0, T/4),), args=(k_max,)
    )
    xs1[i], heights[i] = sol.x, -sol.fun

plt.plot(ks, heights)
plt.xlabel("$k_{max}$")
plt.ylabel("Height")
plt.show()
```




```
widths = np.zeros(len(ks))
for i, k_max in enumerate(ks):
    root_first = minimize(
        lambda t, k_max: np.abs(0.5 - fn_fourier(t, T, k_max)), (0,),
        method="Nelder-Mead", bounds=((0, xs1[i]),), args=(k_max,)
    ).x
    x2 = minimize(
        lambda t, k_max: fn_fourier(t, T, k_max), (xs1[i],),
        method="Nelder-Mead", bounds=((xs1[i], T/4),), args=(k_max,)
    ).x
    root_second = minimize(
        lambda t, k_max: np.abs(0.5 - fn_fourier(t, T, k_max)), (xs1[i],),
        method="Nelder-Mead", bounds=((xs1[i], x2),), args=(k_max,)
    ).x
    widths[i] = root_second - root_first

plt.plot(ks, widths)
plt.xlabel("$k_{max}$")
plt.ylabel("Width")
plt.show()
```


0.6
0.5
0.4
0.3
0.2
0.1
0.0
Width
0
20
40
60
80
100
k_max