# Home Assignments
## Digital Signal Processing

Marco Schwarz (m01602279)

# Introduction

## 1. Use Euler's formula to prove that:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

**Given that:**

$$\begin{align}
e^{ix} &= \cos(x) + i\sin(x) \tag{1}\\
e^{-ix} &= \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x) \tag{2}
\end{align}$$

**Follows:**

$$\begin{align}
\cos(x) &= \frac{e^{ix}+e^{-ix}}{2} && \text{Added (1) and (2)} \tag{3}\\
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} && \text{Subtracted (1) from (2)} \tag{4}
\end{align}$$

$$\begin{aligned}
\sin(x)\cos(x) &= \frac{e^{ix}+e^{-ix}}{2}\cdot\frac{e^{ix}-e^{-ix}}{2i} && \text{Substituting (3) and (4)}\\
&= \frac{e^{i2x}-e^{i0}+e^{i0}-e^{-i2x}}{4i} && \text{Multiplied fractions}\\
&= \frac{1}{2}\frac{e^{i2x}-e^{-i2x}}{2i} && \text{Combined exponentials}\\
&= \frac{1}{2}\sin(2x) && \text{Substituted (4)}
\end{aligned}$$

## 2. Show that:

$$\int_0^{T_0}\cos(k\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0, & k\neq n\\ \frac{T_0}{2}, & k=n\neq 0\\ T_0, & k=n=0\end{cases}$$

**Given that:**

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y)+cos(x+y)\right] \tag{5}$$

$$\int_0^{T_0}\cos(k\omega_0 t)dt = \begin{cases} T_0, & k=0\\ 0, & k\neq 0\end{cases} \tag{6}$$

**Follows:**

$$\int_0^{T_0}\cos(k\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_0^{T_0}\left[\cos\left((k-n)\omega_0 t\right)+\cos\left((k+n)\omega_0 t\right)\right]dt$$

Case $k\neq n$:

$$\frac{1}{2}\int_0^{T_0}\left[\cos\left(a\omega_0 t\right)+\cos\left(b\omega_0 t\right)\right]dt = \frac{1}{2}(0+0) = 0$$

Case $k=n\neq 0$:

$$\frac{1}{2}\int_0^{T_0}\left[\cos\left(0\right)+\cos\left(2k\omega_0 t\right)\right]dt = \frac{1}{2}T_0 + 0 = \frac{T_0}{2}$$

Case $k=n=0$:

$$\frac{1}{2}\int_0^{T_0}\left[\cos\left(0\right)+\cos\left(0\right)\right]dt = \frac{2}{2}T_0 = T_0$$

## Fourier Series

### 3. Show that $B_k = \frac{2}{T_0}\oint f(t)\sin(k\omega_0 t)dt$ is consistent with the Fourier series.

**Given that:**

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\sin(k\omega_0 t)dt = \begin{cases} 0, & k \neq n \\ \frac{T_0}{2}, & k = n \neq 0 \\ T_0, & k = n = 0 \end{cases} \tag{7}$$

$$\int_{-T_0/2}^{T_0/2} \cos(n\omega_0 t)\sin(k\omega_0 t)dt = 0 \tag{8}$$

$$\oint \equiv \int_{-T_0/2}^{T_0/2} \tag{9}$$

**Follows:**

$$\begin{aligned} B_k &= \frac{2}{T_0}\oint \sum_{n=0}^{\infty} (A_n\cos(n\omega_0 t) + B_n\sin(n\omega_0 t))\sin(k\omega_0 t)dt \\ &= \frac{2}{T_0}\sum_{n=0}^{\infty}\left\{A_n \oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt + B_n \oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\right\} \end{aligned}$$

Case $k = 0$:

$$B_0 = \frac{2}{T_0}\sum_{n=0}^{\infty}\{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\}}_{=0} = 0$$

Case $k > 0$:

$$B_k = \frac{2}{T_0}\sum_{n=0}^{\infty}\{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt\}}_{=\frac{T_0}{2}} = B_k$$

### 4. Show that $f(t) = \sum_{k=0}^{\infty} a_k\cos(k\omega_0 t - \phi_k)$ is equivalent to the Fourier series.

**Given that:**

$$a = \sqrt{A^2 + B^2} \tag{10}$$

$$\phi = \begin{cases} \arctan(B/A), & A > 0 \\ \pi/2, & A = 0, B > 0 \\ -\pi/2, & A = 0, B < 0 \\ \arctan(B/A) + \pi, & A < 0, B \geq 0 \\ \arctan(B/A) - \pi, & A < 0, B < 0 \end{cases} \tag{11}$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \tag{12}$$

$$\arctan(0) = 0 \tag{13}$$

**Follows:**

Case $A > 0,\ B = 0$:

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k\cos(k\omega_0 t - \phi_k) \\ &= \sum_{k=0}^{\infty} A_k\cos\left(k\omega_0 t - \arctan\frac{B}{A}\right) \\ &= \sum_{k=0}^{\infty} A_k\cos(k\omega_0 t) \end{aligned}$$

Case $A = 0,\ B > 0$:

$$\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t - \phi_k) \\
&= \sum_{k=0}^{\infty} B_k \cos\left(k\omega_0 t - \frac{\pi}{2}\right) \\
&= \sum_{k=0}^{\infty} B_k \left[\cos(k\omega_0 t)\cos\left(\frac{\pi}{2}\right) + \sin(k\omega_0 t)\sin\left(\frac{\pi}{2}\right)\right] \\
&= \sum_{k=0}^{\infty} B_k \sin(k\omega_0 t)
\end{aligned}$$

## 5. Consider the triangular function:

$$f(t) = \begin{cases} 1 + \frac{2t}{T}, & \text{for} - \frac{T}{2} < t < 0 \\ 1 - \frac{2t}{T}, & \text{for} 0 \leq t \leq \frac{T}{2} \end{cases}$$

**a) Derive an algebraic expression for the coefficients $A_k$ and $B_k$ of the Fourier series.**

$$\begin{aligned}
B_0 &= 0 \\
A_0 &= \frac{1}{T}\int 1 - \frac{2t}{T} dt = \frac{1}{T}\left[t - \frac{t^2}{T}\right] \\
&= \frac{1}{T}\left[\frac{T}{2} - \frac{T^2}{4T}\right] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
a &\equiv k\omega && \text{Shortening notation} \\
B_k &= 0 \\
A_k &= \frac{5}{T}\int_0^{T/2}\left(1 - \frac{2t}{T}\right)\cos(at)dt && \text{Used symmetry and odd property} \\
&= \frac{4}{T}\int_0^{T/2} \cos(at) - \frac{2t}{T}\cos(at)dt && \text{Multiplied cosine} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at)\right]_0^{T/2} - \int_0^{T/2}\frac{2t}{T}\cos(at)dt\right] && \text{Computed first integral} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at)\right]_0^{T/2} + \int_0^{T/2}\frac{2}{aT}\sin(at)dt\right] && \text{Applied product rule} \\
&= \frac{4}{T}\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at) - \frac{2}{a^2T}\cos(at)\right]_0^{T/2} && \text{Computed second integral} \\
&= a^{-1}\sin(aT/2) - 0 - a^{-1}\sin(aT/2) + 0 - \frac{2}{a^2T}\cos(aT/2) + \frac{2}{a^2T} && \text{Inserted boundaries} \\
&= \frac{4}{k^2\omega^2T^2}[1 - \cos(k\pi)] && \text{Replaced } a \\
&= \frac{1}{k^2\pi^2}[1 - \cos(k\pi)] && \text{Replaced } T
\end{aligned}$$

**b) Plot $A_k$ over $k$ for $0 \leq k < 10$**

**c) Plot the Fourier series from [-T0;T0] for kmax = 1 and kmax = 9.**

Code 1: Python code for 5b

```
import numpy as np
import matplotlib.pyplot as plt

fn_ak = lambda k: 2/(k**2 * np.pi**2) * (1 - np.cos(k * np.pi))

ks = np.arange(0, 10)
aks = list(map(fn_ak, ks))
aks[0] = 1/2

plt.plot(ks, aks)
plt.xlabel("k")
plt.ylabel("$A_k$")
plt.show()
```

Code 2: Python code for 5c

```
fn_term = lambda t, T, k: fn_ak(k) * np.cos(k * 2 * np.pi * t / T)

def fn_fourier(ts, k_max, T):
    y = np.ones_like(ts) * 1/2
    for i in range(1, k_max+1):
        y += np.array([fn_term(t, T, i) for t in ts])
    return y

T = 5
ts = np.linspace(-T, T, 1000)

plt.plot(ts, fn_fourier(ts, 1, T), label="$k_{max}=1$")
plt.plot(ts, fn_fourier(ts, 9, T), label="$k_{max}=9$")
plt.xticks([-T, 0, T], ["$-T_0$",0 , "$T_0$"])
plt.ylabel("$f(x)$")
plt.legend()
plt.show()
```

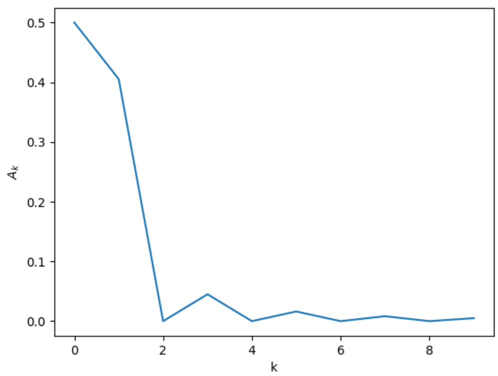


Figure 1: Plot for 5b. The value of $A_k$ for different values of $k$

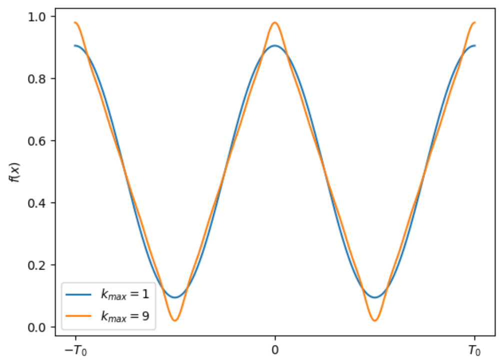


Figure 2: Plot for 5c. The Fourier series with two different values for $k_{max}$

## 6. Consider the Fourier series of the rectangular function (see course notes).

Code 3: Python code for 6a

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize

def fn_term(t, T, k):
    b = 0 if k % 2 == 0 else 2/(np.pi*k)
    return b * np.sin(k * 2 * np.pi * t / T)

def fn_fourier(ts, T, k_max):
    y = np.zeros_like(ts)
    for i in range(1, k_max):
        y += np.array([fn_term(t, T, i) for t in ts])
    return y

T = 5
ks = np.arange(2, 100)
xs1, heights = np.zeros(len(ks)), np.zeros(len(ks))
for i, k_max in enumerate(ks):
    sol = minimize(
        lambda t, k_max: 0.5 - fn_fourier(t, T, k_max), (0,),
        method="Nelder-Mead", bounds=((0, T/4),), args=(k_max,)
    )
    xs1[i], heights[i] = sol.x, -sol.fun

plt.plot(ks, heights)
plt.xlabel("$k_{max}$")
plt.ylabel("Height")
plt.show()
```

Code 4: Python code for 6c

```
widths = np.zeros(len(ks))
for i, k_max in enumerate(ks):
    root_first = minimize(
        lambda t, k_max: np.abs(0.5 - fn_fourier(t, T, k_max)), (0,),
        method="Nelder-Mead", bounds=((0, xs1[i]),), args=(k_max,)
    ).x
    x2 = minimize(
        lambda t, k_max: fn_fourier(t, T, k_max), (xs1[i],),
        method="Nelder-Mead", bounds=((xs1[i], T/4),), args=(k_max,)
    ).x
    root_second = minimize(
        lambda t, k_max: np.abs(0.5 - fn_fourier(t, T, k_max)), (xs1[i],),
        method="Nelder-Mead", bounds=((xs1[i], x2),), args=(k_max,)
    ).x
    widths[i] = root_second - root_first

plt.plot(ks, widths)
plt.xlabel("$k_{max}$")
plt.ylabel("Width")
plt.show()
```

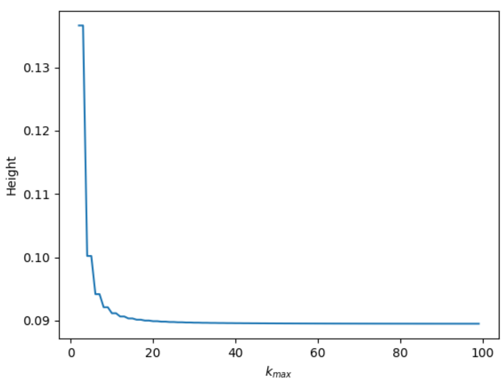


Figure 3: Plot for 6a. The overshoot height for different values of $k_{max}$

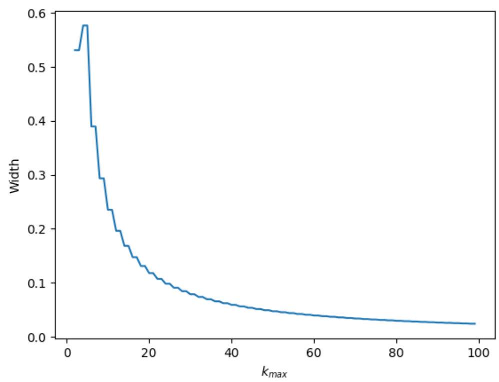


Figure 4: Plot for 6c. The overshoot width for different values of $k_{max}$

## 7. Proof the modulation theorem

Given That:

$$F(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{14}$$

$$\cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \tag{15}$$

Follows:

$$\begin{aligned}
F(x(t)\cos(\omega_0 t)) &= \int_{-\infty}^{\infty} x(t)\cos(\omega_0 t)e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} x(t)\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}e^{-i\omega t}dt \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(t)e^{i\omega_0 t}e^{-i\omega t}dt + \int_{-\infty}^{\infty} x(t)e^{-i\omega_0 t}e^{-i\omega t}dt\right] \\
&= \frac{1}{2}\left[F\left[x(t)e^{i\omega_0 t}\right] + F\left[x(t)e^{-i\omega_0 t}\right]\right] \\
&= \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]
\end{aligned}$$

## 8. sinc-function

### a) Show that the spectral density of the sinc-function is a boxcar.

Given that:

$$\sin(x) = \frac{1}{2i}\left(e^{x} - e^{-x}\right) \tag{16}$$

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} \tag{17}$$

Follows:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{i\omega t}d\omega = \frac{1}{2\pi i t}\left(e^{iWt} - e^{-iWt}\right) \\
&= \frac{W}{\pi W t}\sin(Wt) = \frac{W}{\pi}\operatorname{sinc}(Wt)
\end{aligned}$$

### b) What is the relation of the distance between the zero crossings of the sinc to the width of the boxcar? Specifically, what happens for very dense/very sparse zero crossings?

Given that $2W$ represents the width of the boxcar function follows:

$$\begin{aligned}
x(t) &= \frac{W}{\pi}\operatorname{sinc}(Wt) = \frac{\sin(Wt)}{\pi t} \\
&\Rightarrow x(t) = 0 \ \forall\ t \in \left\{n\frac{\pi}{W} \mid n \in \mathbb{Z} \setminus \{0\}\right\} \\
&\Longrightarrow l_0 = \frac{\pi}{W}
\end{aligned}$$

The distance between zero crossings $l_0$ is inverse proporitional to the width of the boxcar $2W$.