Introduction

1. Use Euler's formula to prove that:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{1}$$

Given that:

$$e^{ix} = \cos(x) + i\sin(x) \tag{2}$$
$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x) \tag{3}$$

Follows:

$$\begin{aligned}
\cos(x) &= \frac{e^{ix}+e^{-ix}}{2} && \text{From adding 2 and 3}\\
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} && \text{From substracting 2 and 3}
\end{aligned}$$

$$\begin{aligned}
\sin(x)\cos(x) &= \frac{e^{ix}+e^{-ix}}{2}\cdot\frac{e^{ix}-e^{-ix}}{2i} && \text{Substituting \textbf{??} and \textbf{??}}\\
&= \frac{e^{i2x}-e^{i0}+e^{i0}-e^{-i2x}}{4i} && \text{Multiplying fractions}\\
&= \frac{1}{2}\frac{e^{i2x}-e^{-i2x}}{2i} && \text{Combining exponentials}\\
&= \frac{1}{2}\sin(2x) && \text{Substituting \textbf{??}}
\end{aligned}$$

2. Show that:

$$\int_0^{T_0} \cos(k\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0, & k \neq n \\ \frac{T_0}{2}, & k = n \neq 0 \\ T_0, & k = n = 0 \end{cases} \tag{4}$$

Given that:

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + cos(x+y)\right] \tag{5}$$

$$\int_0^{T_0} \cos(k\omega_0 t)dt = \begin{cases} T_0, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{6}$$

Follows:

$$\int_0^{T_0} \cos(k\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_0^{T_0} \left[\cos\left((k-n)\omega_0 t\right) + \cos\left((k+n)\omega_0 t\right)\right]dt$$

Case $k \neq n$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(a\omega_0 t\right) + \cos\left(b\omega_0 t\right)\right]dt = \frac{1}{2}(0+0) = 0$$

Case $k = n \neq 0$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(0\right) + \cos\left(2k\omega_0 t\right)\right]dt = \frac{1}{2}T_0 + 0 = \frac{T_0}{2}$$

Case $k = n = 0$:

$$\frac{1}{2}\int_0^{T_0} \left[\cos\left(0\right) + \cos\left(0\right)\right]dt = \frac{2}{2}T_0 = T_0$$

Fourier Series

3. Show that $B_k = \frac{2}{T_0} \oint f(t)\sin(k\omega_0 t)dt$ is consistent with the Fourier series. Given that:

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\sin(k\omega_0 t)dt = \begin{cases} 0, & k \neq n \\ \frac{T_0}{2}, & k = n \neq 0 \\ T_0, & k = n = 0 \end{cases} \tag{7}$$

$$\int_{-T_0/2}^{T_0/2} \cos(n\omega_0 t)\sin(k\omega_0 t)dt = 0 \tag{8}$$

$$\oint \equiv \int_{-T_0/2}^{T_0/2} \tag{9}$$

Follows:

$$\begin{aligned} B_k &= \frac{2}{T_0} \oint \sum_{n=0}^{\infty} (A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)) \sin(k\omega_0 t)dt \\ &= \frac{2}{T_0} \sum_{n=0}^{\infty} \left\{ A_n \oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt + B_n \oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt \right\} \end{aligned}$$

Case $k = 0$:

$$B_0 = \frac{2}{T_0} \sum_{n=0}^{\infty} \{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0}\} = 0$$

Case $k > 0$:

$$B_k = \frac{2}{T_0} \sum_{n=0}^{\infty} \{A_n \underbrace{\oint \cos(n\omega_0 t)\sin(k\omega_0 t)dt}_{=0} + B_n \underbrace{\oint \sin(n\omega_0 t)\sin(k\omega_0 t)dt}_{=\frac{T_0}{2}}\} = B_k$$

4. Show that $f(t) = \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t - \phi_k)$ is equivalent to the Fourier series. Given that:

$$a = \sqrt{A^2 + B^2} \tag{10}$$

$$\phi = \begin{cases} \arctan(B/A), & A > 0 \\ \pi/2, & A = 0, B > 0 \\ -\pi/2, & A = 0, B < 0 \\ \arctan(B/A) + \pi, & A < 0, B \geq 0 \\ \arctan(B/A) - \pi, & A < 0, B < 0 \end{cases} \tag{11}$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \tag{12}$$

$$\arctan(0) = 0 \tag{13}$$

Case $A > 0,\ B = 0$:

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t - \phi_k) \\ &= \sum_{k=0}^{\infty} A_k \cos\left(k\omega_0 t - \arctan\frac{B}{A}\right) \\ &= \sum_{k=0}^{\infty} A_k \cos(k\omega_0 t) \end{aligned}$$

Case $A = 0,\ B > 0$:

$$\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} a_k \cos(k\omega_0 t - \phi_k) \\
&= \sum_{k=0}^{\infty} B_k \cos\left(k\omega_0 t - \frac{\pi}{2}\right) \\
&= \sum_{k=0}^{\infty} B_k \left[\cos(k\omega_0 t)\cos\left(\frac{\pi}{2}\right) + \sin(k\omega_0 t)\sin\left(\frac{\pi}{2}\right)\right] \\
&= \sum_{k=0}^{\infty} B_k \sin(k\omega_0 t)
\end{aligned}$$

5. Consider the triangular function:

$$f(t) = \begin{cases} 1 + \frac{2t}{T}, & \text{for} - \frac{T}{2} < t < 0 \\ 1 - \frac{2t}{T}, & \text{for} 0 \leq t \leq \frac{T}{2} \end{cases}$$

a) Derive an algebraic expression for the coefficients A_k and B_k of the Fourier series.

$$\begin{aligned}
B_0 &= 0 \\
A_0 &= \frac{1}{T}\int 1 - \frac{2t}{T} dt = \frac{1}{T}\left[t - \frac{t^2}{T}\right] \\
&= \frac{1}{T}\left[\frac{T}{2} - \frac{T^2}{4T}\right] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\
a &\equiv k\omega && \text{Shortening notation} \\
B_k &= 0 \\
A_k &= \frac{5}{T}\int_0^{T/2}\left(1 - \frac{2t}{T}\right)\cos(at)dt && \text{Using symmetry and odd property} \\
&= \frac{4}{T}\int_0^{T/2} \cos(at) - \frac{2t}{T}\cos(at)dt && \text{Multiplying cosine} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at)\right]_0^{T/2} - \int_0^{T/2}\frac{2t}{T}\cos(at)dt\right] && \text{Computing first integral} \\
&= \frac{4}{T}\left[\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at)\right]_0^{T/2} + \int_0^{T/2}\frac{2}{aT}\sin(at)dt\right] && \text{Applying product rule} \\
&= \frac{4}{T}\left[a^{-1}\sin(at) - \frac{2t}{aT}\sin(at) - \frac{2}{a^2T}\cos(at)\right]_0^{T/2} \\
&= a^{-1}\sin(aT/2) - 0 - a^{-1}\sin(aT/2) + 0 - \frac{2}{a^2T}\cos(aT/2) + \frac{2}{a^2T} && \text{Inserting boundaries} \\
&= \frac{4}{k^2\omega^2T^2}[1 - \cos(k\pi)] && \text{Replacing} a \\
&= \frac{1}{k^2\pi^2}[1 - \cos(k\pi)]
\end{aligned}$$

b) Plot A_k over k for $0 \leq k < 10$

```python
import numpy as np
import matplotlib.pyplot as plt

fn_ak = lambda k: 2/(k**2 * np.pi**2) * (1 - np.cos(k * np.pi))

ks = np.arange(0, 10)
aks = list(map(fn_ak, ks))
aks[0] = 1/2

plt.plot(ks, aks)
```

```
plt.xlabel("k")
plt.ylabel("$A_k$")
plt.show()
```