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BCS303 : Discrete Structures & Theory of Logic

1
UNIT

Sets, Relations, POSET and Lattices

CONTENTS

PART-1

Set Theory : Introduction, Combination of Sets.

Que 1.1. **What do you understand by set ? Explain different types of set.**

Answer

1. A set is a collection of well defined objects, called elements or members of the set.
2. These elements may be anything like numbers, letters of alphabets, points etc.
3. Sets are denoted by capital letters and their elements by lower case letters.
4. If an object x is an element of set A, we write it as $x \in A$ and read it as 'x belongs to A' otherwise $x \notin A$ (x does not belong to A).

Types of set :

1. **Finite set :** A set with finite or countable number of elements is called finite set.
2. **Infinite set :** A set with infinite number of elements is called infinite set.
3. **Null set :** A set which contains no element at all is called a null set. It is denoted by ϕ or { }. It is also called empty or void set.
4. **Singleton set :** A set which has only one element is called singleton set.
5. **Subset :** Let A and B be two sets, if every elements of A also belongs to B *i.e.*, if every element of set A is also an element of set B, then A is called subset of B and it is denoted by $A \subseteq B$. Symbolically, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.
6. **Superset :** If A is subset of a set B, then B is called superset of A.
7. **Proper subset :** Any subset A is said to be proper subset of another set B, if there is at least one element of B which does not belong to A, *i.e.*, if $A \subseteq B$ but $A \neq B$. It is denoted by $A \subset B$.
8. **Universal set :** In many applications of sets, all the sets under consideration are considered as subsets of one particular set. This set is called universal set and is denoted by U.
9. **Equal set :** Two set A and B are said to be equal if every element of A belong to set B and every element of B belong to set A. It is written as $A = B$.
   Symbolically, $A = B$ if $x \in A$ and $x \in B$.
10. **Disjoint set :** Let A and B be two sets, if there is no common element between A and B, then they are said to be disjoint.

Que 1.2. **Describe the different types of operation on sets.**

Answer

1. **Union :** Let A and B be two sets, then the union of sets A and B is a set that contain those elements that are either in A or B or in both. It is denoted by $A \cup B$ and is read as 'A union B'.
   Symbolically, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
   For example : $A = \{1, 2, 3, 4\}$
   $B = \{3, 4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}$
2. **Intersection :** Let A and B be two sets, then intersection of A and B is a set that contain those elements which are common to both A and B. It is denoted by $A \cap B$ and is read as 'A intersection B'.
   Symbolically, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
   For example : $A = \{1, 2, 3, 4\}$
   $B = \{2, 4, 6, 7\}$
   then $A \cap B = \{2, 4\}$
3. **Complement :** Let U be the universal set and A be any subset of U, then complement of A is a set containing elements of U which do not belong to A. It is denoted by A^c or A' or $\bar{A}$.
   Symbolically, $A^c = \{x \mid x \in U \text{ and } x \notin A\}$
   For example : $U = \{1, 2, 3, 4, 5, 6\}$
   and $A = \{2, 3, 5\}$
   then $A^c = \{1, 4, 6\}$
4. **Difference of sets :** Let A and B be two sets. Then difference of A and B is a set of all those elements which belong to A but do not belong to B and is denoted by $A - B$.
   Symbolically, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
   For example : Let $A = \{2, 3, 4, 5, 6, 7\}$
   and $B = \{4, 5, 7\}$
   then $A - B = \{2, 3, 6\}$
5. **Symmetric difference of set :** Let A and B be two sets. The symmetric difference of A and B is a set containing all the elements that belong to A or B but not both. It is denoted by $A \oplus B$ or $A \Delta B$.
   Also $A \oplus B = (A \cup B) - (A \cap B)$
   For example : Let $A = \{2, 3, 4, 6\}$
   $B = \{1, 2, 5, 6\}$
   then $A \oplus B = \{1, 3, 4, 5\}$

PART-2

Relations : Definition, Operations on Relations.

Que 1.3. **Describe the term relation along with its types.**

Let A and B be two non-empty sets, then R is relation from A to B if R is subset of $A \times B$ and is set of ordered pair (a, b) where $a \in A$ and $b \in B$. It is

denoted by aRb and read as "a is related to b by R".
Symbolically, $R = \{(a, b) : a \in A, b \in B, a R b\}$
If $(a, b) \notin R$ then $a \not R b$ and read as "a is not related to b by R".

For example :
Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$ and aRb iff $a \times b$ = even number
Then $R = \{(1, 2), (2, 1), (2, 2), (3, 2), (4, 1), (4, 2)\}$

Types of relation :

1. **Universal relation :** A relation R is called universal relation on A if $R = A \times A$. In case where R is defined from A to B, then R is universal relation if $R = A \times B$.
   For example :
   If $A = \{1, 2, 3\}$
   then $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   is universal relation over A.
2. **Identity relation :** A relation R is called identity relation on A if $R = \{(a, a) \mid a \in A\}$. It is denoted by I_A or Δ_A or Δ. It is also called diagonal relation.
   For example :
   If $A = \{1, 2, 3\}$, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$
   is identity relation on A.
3. **Void relation :** A relation R is called a void relation on A if $R = \phi$. It is also called null relation.
   For example :
   If $A = \{1, 2, 3\}$ and R is defined as $R = \{(a + b) \mid a + b > 5\}$, $a, b \in A$ then $R = \phi$.
4. **Inverse relation :** A relation R defined from B to A is called inverse relation of R defined from A to B if
   $$R^{-1} = \{(b, a) : b \in B \text{ and } a \in A \text{ and } (a, b) \in R\}.$$
   For example : Consider relation
   $R = \{(1, 1), (1, 2), (1, 3), (3, 2)\}$
   then $R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 3)\}$
5. **Complement of a relation :** Let relation R is defined from A to B, then complement R is set of ordered pairs $\{(a, b) : (a, b) \notin R\}$. It is also called complementary relation.
   For example :
   Let $A = \{1, 2, 3\}$ $B = \{4, 5\}$
   Then $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$
   Let R be defined as $R = \{(1, 4), (3, 4), (3, 5)\}$
   Then $R^C = \bar{R} = \{(1, 5), (2, 4), (2, 5)\}$

Que 1.4. **Explain operation on relation with example.**

Answer

1. Relations are sets of ordered pairs so all set operations can be done on relations.

2. The resulting sets contain ordered pairs and are, therefore, relations.
3. If R and S denote two relations, then $R \cap S$, known as intersection of R and S, defines a relation such that
   $x(R \cap S)\, y = x R y \wedge x S y$
4. Similarly, $R \cup S$, known as union of R and S, such that
   $x(R \cap S)\, y = x R y \vee x S y$
   Also, $x(R - S)\, y = x R y \wedge x \$ y$ where $R - S$ is known as different of R and S
   and $x(R')\, y = x R y$ where R' is the complement of R

For example : $A = \{x, y, z\}$, $B = \{x, y, z\}$, $C = \{x, y, z\}$
$D = \{Y, z\}$, $R = \{(x, X), (x, Y), (y, z)\}$
$S = \{(x, Y), (y, z)\}$

The complement of R consists of all pairs of the Cartesian product $A \times B$ that are not R. Thus $A \times B = \{(x, X), (x, Y), (x, Z), (y, X), (y, Y), (y, Z), (z, X), (z, Y), (z, Z)\}$

Hence
$R' = \{(x, Z), (y, X), (y, Y), (z, X), (z, Y), (z, Z)\}$
$R \cup S = \{(x, X), (x, Y), (y, Z)\}$
$R \cap S = \{(x, Y), (y, Z)\}$
$R - S = \{(x, Y)\}$

PART-3

Properties of Relations.

Que 1.5. **Give properties of relation.**

Answer

Properties of relation are :

1. **Reflexive relation :** A binary relation R on set A is said to be reflexive if every element of set A is related to itself.
   i.e., $\forall\, a \in A, (a, a) \in R$ or aRa.
   For example : Let $R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$ be a relation defined on set $A = \{1, 2, 3\}$. As $(1, 1) \in R$, $(2, 2) \in R$ and $(3, 3) \in R$. Therefore, R is reflexive relation.
2. **Irreflexive relation :** A binary relation R defined on set A is said to irreflexive if there is no element in A which is related to itself *i.e.*, $\forall\, a \in A$ such that $(a, a) \notin R$.
   For example : Let $R = \{(1, 2), (2, 1), (3, 1)\}$ be a relation defined on set $A = \{1, 2, 3\}$. As $(1, 1) \notin R$, $(2, 2) \notin R$ and $(3, 3) \notin R$. Therefore, R is irreflexive relation.
3. **Non-reflexive relation :** A relation R defined on set A is said to be non-reflexive if it is neither reflexive nor irreflexive *i.e.*, some elements are related to itself but there exist at least one element not related to itself.
4. **Symmetric relation :** A binary relation on a set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$.

5. **Asymmetric relation :** A binary relation on a set A is said to be asymmetric if $(a, b) \in R \Rightarrow (b, a) \notin R$.
6. **Antisymmetric relation :** A binary relation R defined on a set A is said to antisymmetric relation if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ i.e., aRb and $bRa \Rightarrow a = b$ for $a, b \in R$.
7. **Transitive relation :** A binary relation R on a set A is transitive whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ i.e., aRb and $bRc \Rightarrow aRc$.

Que 1.6. **Identify whether the each of the following relations defined on the set $X = \{1, 2, 3, 4\}$ are reflexive, symmetric, transitive and/or antisymmetric ?**
i. $R_1 = \{(1, 1), (1, 2), (2, 1)\}$
ii. $R_2 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
iii. $R_3 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

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Answer

i. $R_1 = \{(1, 1), (1, 2), (2, 1)\}$
Set $A = \{1, 2, 3, 4\}$
Reflexive : $(1, 1) \in R_1$ $(2, 2) \notin R_1$
Therefore, R_1 is not reflexive relation.
Symmetric : $(1, 2) \in R_1$, $(2, 1) \in R_1$
R_1 is symmetric relation.
Transitive : $(1, 2) \in R_1$ and $(2, 1) \in R_1$ then $(1, 1) \in R_1$
R_1 is transitive relation.
Antisymmetric : $(1, 2) \in R_1$ but $(2, 1) \in R_1$ and $2 \neq 1$
Therefore R_1 is not antisymmetric relation.

ii. $R_2 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
Reflexive : $(1, 1) \in R_2$, $(2, 2) \in R_2$, $(3, 3) \in R_2$ and $(4, 4) \in R_2$
Therefore R_2 is reflexive relation.
Symmetric : $(1, 2) \in R_2$, then $(2, 1) \in R_2$
$(1, 4) \in R_2$ then $(4, 1) \in R_2$
Therefore R_2 is symmetric relation.
Transitive : $(1, 4) \in R_2$ and $(4, 1) \in R_2$ then $(1, 1) \in R_2$
Therefore R_2 is transitive relation.
Antisymmetric : $(1, 4) \in R_2$ and $(4, 1) \in R_2$ only if $4 \neq 1$
Therefore R_2 is not antisymmetric relation.

iii. $R_3 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
Reflexive : $(1, 1) \notin R_3$. Therefore R_3 is not reflexive relation.
Symmetric : $(2, 1) \in R_3$ but $(1, 2) \notin R_3$. Therefore R_3 is not symmetric relation.
Transitive : $(3, 1) \in R_3$ but $(1, 3) \notin R_3$. Therefore R_3 is not transitive relation.
Antisymmetric : $(3, 2) \in R_3$ but $(2, 3) \notin R_3$, only if $2 \neq 3$. Therefore R_3 is not antisymmetric relation.

PART-4
Composite Relations, Equality of Relations.

Que 1.7. **Write short notes on :**
a. Equivalence relation
b. Composition of relation
OR
Write a short note on equality of relation.

Answer

a. **Equivalence relation :**
1. A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.
2. The two elements a and b related by an equivalence relation are called equivalent.
3. So, a relation R is called equivalence relation on set A if it satisfies following three properties :
   i. $(a, a) \in R \ \forall\, a \in A$ (Reflexive)
   ii. $(a, b) \in R \Rightarrow (b, a) \in R$ (Symmetric)
   iii. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ (Transitive)

b. **Composition of relation :**
1. Let R be a relation from a set A to B and S be a relation from set B to C then composition of R and S is a relation consisting or ordered pair (a, c) where $a \in A$ and $c \in C$ provided that there exist $b \in B$ such that $(a, b) \in R \subseteq A \times B$ and $(b, c) \in S \subseteq B \times C$. It is denoted by RoS.
2. Symbolically, $R \circ S = \{(a, c) \mid \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$

PART-5
Recursive Definition of Relation.

Que 1.8. **Describe recursive definition of relation.**

Answer

1. The characteristic function C_R of a relation $R \subseteq N^k$ is defined as follows :
   $- C_S(X_1, ..., X_k) = 1$ if $<X_1, ..., X_k> \in S$
   $- C_S(X_1, ..., X_k) = 0$ if $<X_1, ..., X_k> \notin S$
2. A relation R is a recursive set iff its characteristic function C_R is a recursive function.
3. Examples of recursive relations : $<, >, \leq, =$
   $c_<(x, y) = sg(y + x)$

$c_>(x, y) = sg(x + y)$

$c_\leq(x, y) = \overline{sg}\,(x + y)$

$c_=(x, y) = \overline{sg}\,(x + y) \times c_\leq(x, y) = \overline{sg}\,(y + x)$

4. Consider relation $R(x, y, z)$ defined as follows :
   $- R(x, y, z)$ iff $y \times z \leq x$
5. We see that R is the result of substituting the recursive function × into recursive relation ≤.
6. Thus, R is recursive.
7. (Technically, R is the result of substituting the functions $f_1(x, y, z) = y \times z$ and $f_2(x, y, z) = x$ into ≤, and we need to show that $f_1(x, y, z) = y \times z$ and $f_2(x, y, z) = x$ are recursive ... but that's trivial using the identity functions).

PART-6

Order of Relations.

Que 1.9. Define the term partial order relation or partial ordering relation.

Answer

A binary relation R defined on set A is called Partial Order Relation (POR) if R satisfies following properties :

i. $(a, a) \in R \ \forall\, a \in A$ (Reflexive)

ii. If $(a, b) \in R$ and $(b, a) \in R$, the $a = b$ (Antisymmetric)

iii. If $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ where $a, b, c \in A$ (Transitive)

A set A together with a partial order relation R is called partial order set or poset.

Que 1.10. Write short notes on :

a. **Closure of relations**
b. **Total order**
c. **Compatibility relation**

Answer

a. **Closure of relations :**

   i. **Reflexive closure :** Let R be a relation defined on set A. The $R \cup I_A$ is called reflexive closure of R, where $I_A = \{(a, a) \mid a \in A\}$ is diagonal or identity relation.

   ii. **Symmetric closure :** Let R be a relation defined on set A. Then $R \cup R^{-1}$ is called symmetric closure of R, where R^{-1} is inverse of R on A.

b. **Total order :** A binary relation R on a set A is said to be total order iff it is

   i. Partial order

   ii. $(a, b) \in R$ or $(b, a) \in R \ \forall\, a, b \in A$

   It is also called linear order.

c. **Compatibility relation :** A binary relation R defined on set A is said to be compatible relation if it is reflexive and symmetric. It is denoted by '≈'.

Que 1.11. Is the "divides" relation on the set of positive integers transitive ? What is the reflexive and symmetric closure of the relation ?

$R = \{(a, b) \mid a > b\}$ on the set of positive integers.

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Answer

Yes, divides relation on the set of positive integers is transitive.

Numerical :

Reflexive : $a = a \Rightarrow a \geq a$

$\therefore$ $(a, a) \in R \ \forall\, a$ is real number.

$\therefore$ R is reflexive

Symmetric : Let $(a, b) \in R$

$\Rightarrow$ $a \geq b \not\Rightarrow b \geq a \Rightarrow (b, a) \notin R$

$\therefore$ R is not symmetric

$\therefore$ R is not an equivalence relation.

Que 1.12. Show that $R = \{(a, b) \mid a \equiv b \pmod m\}$ is an equivalence relation on Z. Show that if $x_1 \equiv y_1$ and $x_2 \equiv y_2$ then $(x_1 + x_2) \equiv (y_1 + y_2)$.

Answer

$R = \{(a, b) \mid a \equiv b \pmod m\}$

For an equivalence relation it has to be reflexive, symmetric and transitive.

Reflexive : For reflexive $\forall\, a \in Z$ we have $(a, a) \in R$ *i.e.*,

$a \equiv a \pmod m$

$\Rightarrow$ $a - a$ is divisible by m *i.e.*, 0 is divisible by m

Therefore aRa, $\forall\, a \in Z$, it is reflexive.

Symmetric : Let $(a, b) \in Z$ and we have

$(a, b) \in R$ *i.e.*, $a \equiv b \pmod m$

$\Rightarrow$ $a - b$ is divisible by m

$\Rightarrow$ $a - b = km$, k is an integer

$\Rightarrow$ $(b - a) = (-k)\, m$

$\Rightarrow$ $(b - a) = p\, m$, p is also an integer

$\Rightarrow$ $b - a$ is also divisible by m

$\Rightarrow$ $b \equiv a \pmod m \Rightarrow (b, a) \in R$

It is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then

$(a, b) \in R \Rightarrow a - b$ is divisible by m

$\Rightarrow$ $a - b = t\,m$, t is an integer ...(1.12.1)

$(b, c) \in R \Rightarrow b - c$ is divisible by m

$\Rightarrow$ $b - c = s\,m$, s is an integer ...(1.12.2)

From eq. (1.12.1) and (1.12.2)

$$a - b + b - c = (t + s)\,m$$

$a - c = lm$, l is also an integer

$a - c$ is divisible by m

$a \equiv c \pmod{m}$, yes it is transitive.

R is an equivalence relation.

To show : $(x_1 + x_2) \equiv (y_1 + y_2)$:

It is given $x_1 \equiv y_1$ and $x_2 \equiv y_2$

i.e., $x_1 - y_1$ divisible by m

$x_2 - y_2$ divisible by m

Adding above equation :

$(x_1 - y_1) + (x_2 - y_2)$ is divisible by m

$\Rightarrow$ $(x_1 + x_2) - (y_1 + y_2)$ is divisible by m

i.e., $(x_1 + x_2) \equiv (y_1 + y_2)$

Que 1.13. **State principle of duality. Let A denote the set of real numbers and a relation R is defined on A such that $(a, b)R\,(c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$. Justify that R is an equivalence relation.**

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Answer

Principle of duality : The principle of duality is a type of pervasive property of algebraic structure in which two concepts are interchangeable only if all results held in one formulation also hold in another. This concept is known as dual formulation.

Numerical : $a^2 + b^2 = c^2 + d^2$

Let $(a, c) \in N * N$

then $a^2 + c^2 = a^2 + c^2$

$(a, c)\,R\,(a, c)$

Hence, R is reflexive.

Let $(a, c)\,(b, d) \in N * N$ such that $(a, c)\,R\,(b, d)$

$\Rightarrow$ $a^2 + b^2 = c^2 + d^2$

$\Rightarrow$ $c^2 + d^2 = a^2 + b^2$

$\Rightarrow$ $(c, d)\,R(a, b)$

Hence, R is symmetric.

Let $(a, b), (c, d), (e, f) \in N * N$ such that

$(a, b)\,R\,(c, d)$ and $(c, d)\,R\,(e, f)$

$a^2 + b^2 = c^2 + d^2$ ...(1.13.1)

$c^2 + f^2 = d^2 + e^2$ ...(1.13.2)

Adding (1.13.1) and (1.13.2)

$$a^2 + b^2 + c^2 + f^2 = c^2 + d^2 + f^2 + e^2$$

$$a^2 + b^2 = d^2 + e^2$$

$$(a, b)\,R\,(d, e)$$

Hence, R is transitive.

Hence, R is an equivalence relation.

Que 1.14. **Let R be binary relation on the set of all strings of 0's and 1's such that $R = \{(a, b) \mid a$ and b are strings that have the same number of 0's$\}$. Is R is an equivalence relation and a partial ordering relation ?**

Answer

For equivalence relation :

Reflexive : $a\,R\,a \Rightarrow (a, a) \in R\ \forall a \in R$

where a is a string of 0's and 1's.

Always a is related to a because both a has same number of 0's.

It is reflexive.

Symmetric : Let $(a, b) \in R$

then a and b both have same number of 0's which indicates that again both b and a will also have same number of zeros. Hence $(b, a) \in R$. It is symmetric.

Transitive : Let $(a, b) \in R$, $(b, c) \in R$

$(a, b) \in R \Rightarrow a$ and b have same number of zeros.

$(b, c) \in R \Rightarrow b$ and c have same number of zeros.

Therefore a and c also have same number of zeros, hence $(a, c) \in R$.

It is transitive.

$\therefore$ R is an equivalence relation.

For partial order, it has to be reflexive, antisymmetric and transitive. Since, symmetricity and antisymmetricity cannot hold together. Therefore, it is not partial order relation.

Que 1.15. **Let A {1, 2, 3,.........., 13}. Consider the equivalence relation on $A \times A$ defined by $(a, b)\,R\,(c, d)$ if $a + d = b + c$. Find equivalence classes of (5, 8).**

Answer

$$A = \{1, 2, 3, \ldots, 13\}$$

$$[(5, 8)] = \{(a, b) : (a, b)\ R\ (5, 8), (a, b) \in A \times A\}$$
$$= \{(a, b) : a + 8 = b + 5\}$$
$$= \{(a, b) : a + 3 = b\}$$
$$[5, 8] = \{(1, 4), (2, 5), (3, 6), (4, 7)\ (5, 8), (6, 9), (7, 10), (8, 11)\ (9, 12), (10, 13)\}$$

Que 1.16. **The following relation on $A = \{1, 2, 3, 4\}$. Determine whether the following :**

a. $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$

b. $R = A \times A$

Is an equivalence relation or not ?

Answer

a. $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$

Reflexive : $(a, a) \in R\ \forall\ a \in A$

$\because$ $(1, 1) \in R, (2, 2) \notin R$

$\therefore$ R is not reflexive.

Symmetric : Let $(a, b) \in R$ then $(b, a) \in R$.

$\because$ $(1, 3) \in R$ so $(3, 1) \in R$

$\because$ $(1, 2) \in R$ but $(2, 1) \notin R$

$\therefore$ R is not symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$\because$ $(1, 3) \in R$ and $(3, 1) \in R$ so $(1, 1) \in R$

$\because$ $(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$

$\therefore$ R is not transitive.

Since, R is not reflexive, not symmetric, and not transitive so R is not an equivalence relation.

b. $R = A \times A$

Since, $A \times A$ contains all possible elements of set A. So, R is reflexive, symmetric and transitive. Hence R is an equivalence relation.

Que 1.17. **Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t beings with the n characters at the start of s.**

Answer

We have to show that the relation R_n is reflexive, symmetric, and transitive.

1. **Reflexive :** The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S.
2. **Symmetric :** If $sR_n t$, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that $tR_n s$. We conclude that R_n is symmetric.
3. **Transitive :** Now suppose that $sR_n t$ and $tR_n u$. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters, *i.e.*, $sR_n u$. Consequently, R_n is transitive.

Que 1.18. **Let $X = \{1, 2, 3, \ldots, 7\}$ and $R = \{(x, y) \mid (x - y)$ is divisible by $3\}$. Is R equivalence relation ? Draw the digraph of R.**

Answer

Given that $X = \{1, 2, 3, 4, 5, 6, 7\}$

and $R = \{(x, y) : (x - y)$ is divisible by $3\}$

Then R is an equivalence relation if

i. **Reflexive :** $\forall\ x \in X \Rightarrow (x - x)$ is divisible by 3

So, $(x, x) \in X\ \forall\ x \in X$ or, R is reflexive.

ii. **Symmetric :** Let $x, y \in X$ and $(x, y) \in R$

$\Rightarrow (x - y)$ is divisible by 3 $\Rightarrow$ $(x - y) = 3n_1$, (n_1 being an integer)

$\Rightarrow (y - x) = -3n_2 = 3n_2$, n_2 is also an integer

So, $y - x$ is divisible by 3 or R is symmetric.

iii. **Transitive :** Let $x, y, z \in X$ and $(x, y) \in R$, $(y, z) \in R$

Then $x - y = 3n_1$, $y - z = 3n_2$, n_1, n_2 being integers

$\Rightarrow x - z = 3(n_1 + n_2)$, $n_1 + n_2 = n_3$ be any integer

So, $(x - z)$ is also divisible by 3 or $(x, z) \in R$

So, R is transitive.

Hence, R is an equivalence relation.

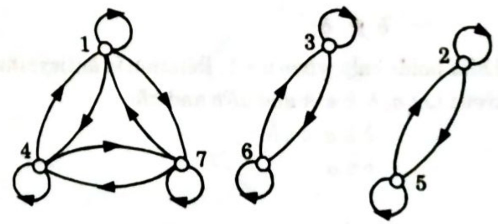


Fig. 1.18.1. Diagraph of R.

PART-7
POSET.

Que 1.19. **Define the term partially ordered set (POSET) ?**

Answer

Let R be a relation on a set A satisfying following properties :

i. For any $a \in A$ $(a, a) \in R$ *i.e.,* aRa (Reflexive property)

ii. For $a, b \in A$ if aRb and bRa then $a = b$ (Antisymmetric property)

iii. For $a, b, c \in A$ if aRb and bRc then aRc (Transitive property)

Then R is called partial order relation or simply order relation or R is said to define a partial ordering of A. A set A together with relation R (partial order relation) is called partially ordered set or poset denoted by (A, R). A partially ordered relation is denoted by $\leq$. $a \leq b$ is read as "a precedes b".

Que 1.20. **Let $(A, \leq)$ be a partially ordered set. Let $\leq$ be a binary relation A such that for a and b in A, a is related to b iff $b \leq a$.**
i. Show that $\leq$ is partially ordered relation.
ii. Show that $(A, \leq)$ is lattice or not.

Answer

i. $(A, \leq)$ is a partially ordered relation if it is reflexive, antisymmetric and transitive.

Reflexive : Let $a \in A$ then by definition of relation, $aRa \Rightarrow a \leq a$ which is true.

Hence, the relation R *i.e.,* $\leq$ is reflexive.

Antisymmetric : Let $a, b \in A$ and

$$aRb \Rightarrow b \leq a$$

$$\Rightarrow \quad a \not\leq b$$

$$\Rightarrow \quad b \not R\ a$$

aRb and bRa holds only when $a = b$. Relation is antisymmetric.

Transitive : Let $a, b, c \in A$ and aRb and bRc

$$\Rightarrow \quad b \leq a, c \leq b$$

$$\Rightarrow \quad c \leq a$$

$$\Rightarrow \quad aRc$$

Hence, relation is transitive.

Therefore, $\leq$ is a partial order relation.

ii. Since all the elements of the given set A are comparable to each other, we always have a least upper bound and greatest lower bound for each pair of elements of A and A is also a partial order relation. Hence, $(A, \leq)$ is a lattice.

PART-8
Hasse Diagram.

Que 1.21. **What do you mean by Hasse Diagram ?**

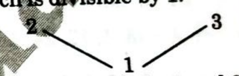
Answer

Let A be a poset and $a, b \in A$. Then a is immediate predecessor of b or b is immediate successor of a if $a < b$, but no element of A lies between a and b denoted by $a << b$. We can also say that b is cover of a.

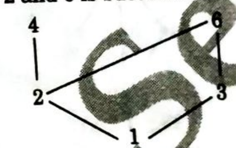
Hasse diagram of a poset A is a directed graph whose vertices are elements of A and there is a directed edge from a to b whenever $a << b$. In Hasse diagram we will place b higher than a and draw a line between them to indicate succession instead of drawing an arrow.

For example : Let A be set of factors of 12 and $\leq$ be the relation defined on A such that $x \leq y$ means x divides y. Now $A = \{1, 2, 3, 4, 6, 12\}$.

Step I : 1 divides 2 and 3 i.e., 2 and 3 are successors of 1. There is no element between 1, 2 and 1, 3 which is divisible by 1.



Step II : Now both 4 and 6 are divisible by 2 and 6 is also divisible by 3. Then 4 and 6 are successors of 2 and 6 is successor of 3.



Step III : Now 12 is divisible by both 4 and 6.

Therefore, Hasse diagram is shown below :

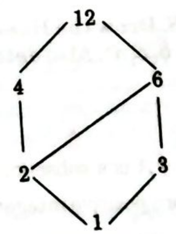


Fig. 1.21.1.

Que 1.22. **Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered set with relation "x divides y". Draw its Hasse diagram.**

Answer

The Hasse diagram of poset is shown below :

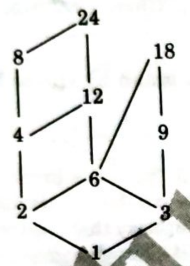


Fig. 1.22.1.

Que 1.23. **Draw the Hasse diagram of $(A, \leq)$, where $A = \{3, 4, 12, 24, 48, 72\}$ and relation $\leq$ be such that $a \leq b$ if a divides b.**

Answer

Hasse diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$

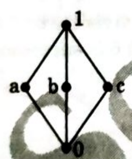


Fig. 1.23.1.

Que 1.24. **Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$. Also determine whether $(P(S), \subseteq)$ is a lattice.**

Answer

Reflexivity : $A \subseteq A$ whenever A is a subset of S.

Antisymmetry : If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Transitivity : If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Hasse diagram :

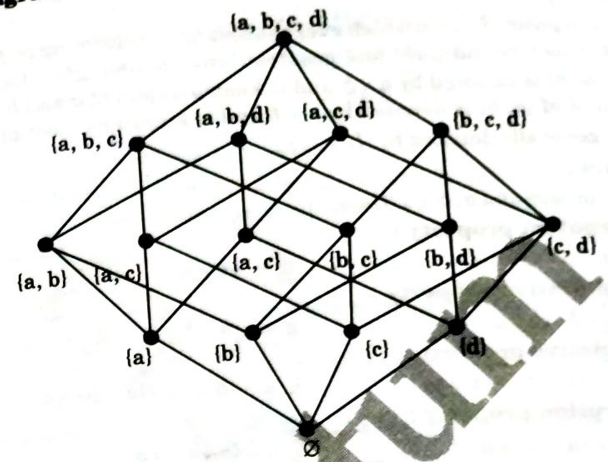


Fig. 1.24.1.

$(P(S), \subseteq)$ is not a lattice because $(\{a, b\}, \{b, d\})$ has no *lub* and *glb*.

Que 1.25. **Let $A = \{2, 3, 6, 12, 24, 36\}$ and relation '$\leq$' be such that '$x \leq y$' if x divides y. Draw the Hasse diagram of $(A, \leq)$.**

Answer

The Hasse diagram is shown below :

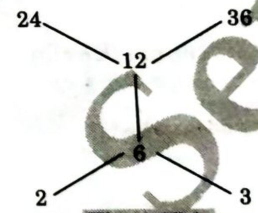


Fig. 1.25.1.

PART-9

Lattices : Definition and Properties of Lattices-Bounded, Complemented, Distributed, Modular and Complete Lattice.

Que 1.26. **Define lattice. Give its properties.**

Answer

A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of 2 elements has least upper bound (*lub*) and greatest lower bound (*glb*). Least upper bound of $\{a, b\}$ is denoted by $a \vee b$ and is known as join of a and b. Greatest lower bound of $\{a, b\}$ is denoted by $a \wedge b$ and is known as meet of a and b. Lattice is generally denoted by $(L, \wedge, \vee)$.

Properties :

Let L be a lattice and $a, b \in L$ then

1. **Idempotent property :**
   i. $a \vee a = a$ ii. $a \wedge a = a$
2. **Commutative property :**
   i. $a \vee b = b \vee a$ ii. $a \wedge b = b \wedge a$
3. **Associative property :**
   i. $a \vee (b \vee c) = (a \vee b) \vee c$ ii. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
4. **Absorption property :**
   i. $a \vee (a \wedge b) = a$ ii. $a \wedge (a \vee b) = a$
5. i. $a \vee b = b$ iff $a \preccurlyeq b$ ii. $a \wedge b = a$ iff $a \preccurlyeq b$
   iii. $a \wedge b = a$ iff $a \vee b = b$
6. **Distributive Inequality :**
   i. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ ii. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
7. **Isotonicity :**
   Let $a, b, c, \in L$ and $a \preccurlyeq b$, then
   i. $a \vee c \preccurlyeq b \vee c$ ii. $a \wedge c \preccurlyeq b \wedge c$
8. **Stability :**
   Let $a, b, c, \in L$, then
   i. $a \preccurlyeq b, a \preccurlyeq c \Rightarrow a \preccurlyeq (b \vee c), a \preccurlyeq (b \wedge c)$
   ii. $a \succcurlyeq b, a \succcurlyeq c \Rightarrow a \succcurlyeq (b \wedge c), a \succcurlyeq (b \wedge c)$

Que 1.27. **Explain types of lattice.**

Answer

Types of lattice :

1. **Bounded lattice :** A lattice L is said to be bounded if it has a greatest element 1 and a least element 0. In such lattice we have

$$a \vee 1 = 1, a \wedge 1 = a$$
$$a \vee 0 = a, a \wedge 0 = 0$$
$$\forall a \in L \text{ and } 0 \leq a \leq I$$

2. **Complemented lattice :** Let L be a bounded lattice with greatest element 1 and least element 0. Let $a \in L$ then an element $a' \in L$ is complement of a if,

$$a \vee a' = 1 \text{ and } a \wedge a' = 0$$

A lattice L is called complemented if is bounded and if every element in L has a complement.

3. **Distributive lattice :** A lattice L is said to be distributive if for any element a, b and c of L following properties are satisfied :
   i. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   ii. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
   otherwise L is non-distributive lattice.
4. **Complete lattice :** A lattice L is called complete if each of its non-empty subsets has a least upper bound and greatest lower bound.
   For example :
   i. $(Z, \leq)$ is not a complete lattice.
   ii. Let S be the class of all subsets of some universal set A and a relation $\leq$ is defined as $X \leq Y \Rightarrow X$ is a subset of Y such that $X \wedge Y = X \cap Y$ and $X \cup Y$. Every subset of S has *glb* and *lub*. So, S is complete.
5. **Modular lattice :** A lattice $(L, \leq)$ is called modular lattice if, $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$.

Que 1.28. **If the lattice is represented by the Hasse diagram given below :**

i. **Find all the complements of 'e'.**

ii. **Prove that the given lattice is bounded complemented lattice.**

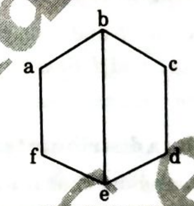


Fig. 1.28.1.

Answer

i. Complements of e are c and d which are as follows :
   $c \vee e = b$, $c \wedge e = f$
   $d \vee e = b$, $d \wedge e = f$
ii. A lattice is bounded if it has greatest and least elements. Here b is greatest and f is least element.

Que 1.29. **Define modular lattice. Justify that if 'a' and 'b' are the elements in a bounded distributive lattice and if 'a' has complement a', then**

i. $a \vee (a' \wedge b) = a \vee b$

ii. $a \wedge (a' \vee b) = a \wedge b$ **AKTU 2021-22, Marks 10**

Answer

Modular lattice : Refer Q. 1.27, Page 1–18F, Unit-1.

i. $a \vee (a' \wedge b) = a \vee b$

$\Rightarrow a \vee (a' \vee b) = (a \vee a') \wedge (a \vee b)$ (Distributive)

$\Rightarrow 1 \vee (a \vee b) = a \vee b$

ii. $a \wedge (a' \vee b) = a \wedge b$

$\Rightarrow (a \wedge a') \vee (a \wedge b)$

$\Rightarrow 0 \vee (a \wedge b)$

$= a \wedge b$

Que 1.30. **Define a lattice. For any a, b, c, d in a lattice $(A, \leq)$ if $a \leq b$ and $c \leq d$ then show that $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.**

Answer

Lattice : Refer Q. 1.26, Page 1–17F, Unit-1.

Numerical :

As $a \leq b$ and $c \leq d$, $a \leq b \leq b \vee d$ and $c \leq d \leq b \vee d$.

By transitivity of $\leq$, $a \leq b \vee d$ and $c \leq b \vee d$.

So $b \vee d$ is an upper bound of a and c.

So $a \vee c \leq b \vee d$.

As $a \wedge c \leq a$ and $a \wedge c \leq c$, $a \wedge c \leq a \leq b$ and $a \wedge c \leq c \leq d$.

Hence $a \wedge c$ is a lower bound of b and d. So $a \wedge c \leq b \wedge d$.

So $a \wedge c \leq b \wedge d$.

Que 1.31. **Let $(L, \wedge, \vee, \leq)$ be a distribute lattice and $a, b \in L$. If $a \wedge b = a \vee c$ and $a \wedge b = a \wedge c$ then show that $b = c$.**

AKTU 2019-20, Marks 10

Answer

$b = b \wedge (a \vee b)$ (Absorption)

$= b \wedge (a \vee c)$ (hypothesis)

$= (b \wedge a) \vee (b \wedge c)$ (distributive law)

$= (a \wedge c) \vee (b \wedge c)$ (hypothesis)

$= (a \vee b) \wedge c$ (distributive law)

$= (a \vee c) \wedge c$ (hypothesis)

$= c$ (Absorption)

Que 1.32. **Define complemented lattice and then show that in a distributive lattice, if an element has a complement then this complement is unique.**

AKTU 2022-23, Marks 10

Answer

Complemented lattice : A lattice L is called complete if each of its non-empty subsets has a least upper bound and greatest lower bound.

For example :

i. $(Z, \leq)$ is not a complete lattice.

ii. Let S be the class of all subsets of some universal set A and a relation $\leq$ is defined as $X \leq Y \Rightarrow X$ is a subset of Y such that $X \wedge Y = X \cap Y$ and $X \cup Y$. Every subset of S has glb and lub. So, S is complete.

Proof : Let $(L, \leq)$ be a bounded distributive lattice. Let $a \in L$ having two complement b and c then show $b = c$

Since b and c be complement of a then

$a \vee b = 1$ $\quad a \wedge b = 0$

$a \vee c = 1$ $\quad a \wedge c = 0$

$b = b \wedge 1$

Now $= b \wedge (a \vee c)$ [by distributive law]

$= (b \wedge a) \vee (b \wedge c)$ $[a \wedge b = b \wedge a]$

$= (a \wedge b) \vee (b \wedge c)$ $[a \wedge b = 0]$

$= 0 \vee (b \wedge c)$ $[0 = a \wedge c]$

$= (a \wedge c) \vee (b \wedge c)$ $[a \vee b = 1]$

$= (a \vee b) \wedge c$

$= 1 \wedge c = c$

Que 1.33. **Let L be a bounded distributed lattice, prove if a complement exists, it is unique. Is D_{12} a complemented lattice ? Draw the Hasse diagram of $[P(a, b, c), \leq]$, (Note : '$\leq$' stands for subset). Find greatest element, least element, minimal element and maximal element.**

OR

Draw the Hasse diagram of $[P(a, b, c), \subseteq]$ (Note : '$\subseteq$' stands for subset). Find greatest element, least element, minimal element and maximal element.

Answer

Let a_1 and a_2 be two complements of an element $a \in L$.

Then by definition of complement

$$\left.\begin{array}{l} a \vee a_1 = I \\ a \wedge a_1 = 0 \end{array}\right\} \quad \text{...(1.33.1)}$$

$$\left.\begin{array}{l} a \vee a_2 = I \\ a \wedge a_2 = 0 \end{array}\right\} \quad \text{...(1.33.2)}$$

Consider

$a_1 = a_1 \vee 0$ [from (1.33.2)]

$= a_1 \vee (a \wedge a_2)$ [Distributive property]

$= (a_1 \vee a) \wedge (a_1 \vee a_2)$

$$= (a \vee a_1) \wedge (a_1 \vee a_2) \quad \text{[Commutative property]}$$
$$= I \wedge (a_1 \vee a_2) \quad \text{[from (1.33.1)]}$$
$$= a_1 \vee a_2 \quad \text{...(1.33.3)}$$

Now Consider

$$a_2 = a_2 \vee 0$$
$$= a_2 \vee (a \wedge a_1) \quad \text{[from (1.33.2)]}$$
$$= (a_2 \vee a) \wedge (a_2 \vee a_1) \quad \text{[Distributive property]}$$
$$= (a \vee a_2) \wedge (a_1 \vee a_2) \quad \text{[Commutative property]}$$
$$= I \wedge (a_1 \vee a_2) \quad \text{[from (1.33.1)]}$$
$$= a_1 \vee a_2 \quad \text{...(1.33.4)}$$

Hence, from (1.33.3) and (1.33.4),

$$a_1 = a_2$$

So, for bounded distributive lattice complement is unique.

Hasse diagram of $[P(a, b, c), \subseteq]$ or $[P(a, b, c), \leq]$ is shown in Fig. 1.33.1.

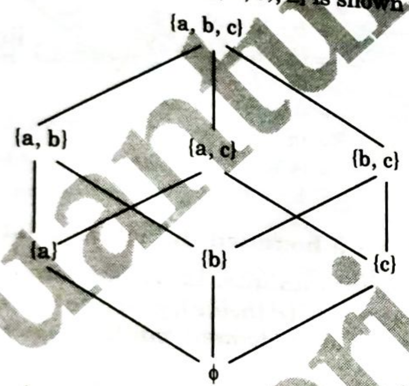


Fig. 1.33.1.

Greatest element is $\{a, b, c\}$ and maximal element is $\{a, b, c\}$..
The least element is ϕ and minimal element is ϕ.

Que 1.34. **Prove that in any lattice the following distributive inequalities hold**

i. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

ii. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ **AKTU 2020-21, Marks 10**

Answer

Proof : Since $a \leq a \vee b$ and $a \leq a \vee c$, $a \leq (a \vee b) \wedge (a \wedge c)$.
Similarly, $b \wedge c \leq b \leq a \vee b$ and $b \wedge c \leq c \leq a \vee c$ imply $b \wedge c \leq (a \vee b) \wedge (a \vee c)$.
Together we have $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.
The second inequality is the dual of the first one.
The two inequalities above are called the distributive inequalities.

Que 1.35. **The directed graph G for a relation R on set $A = \{1, 2, 3, 4\}$ is shown below :**

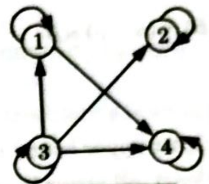


Fig. 1.35.1.

i. Verify that (A, R) is a poset and find its Hasse diagram.
ii. Is this a lattice ?
iii. How many more edges are needed in the Fig. 1.35.1 to extend (A, R) to a total order ?
iv. What are the maximal and minimal elements ?

Answer

i. The relation R corresponding to the given directed graph is,
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 1), (3, 4), (1, 4), (3, 2)\}$
R is a partial order relation if it is reflexive, antisymmetric and transitive.
Reflexive : Since $aRa, \forall a \in A$, hence, it is reflexive.
Antisymmetric : Since aRb and bRa then we get $a = b$ otherwise aRb or bRa.
Hence, it is antisymmetric.
Transitive: For every aRb and bRc we get aRc. Hence, it is transitive.
Therefore, we can say that (A, R) is poset. Its Hasse diagram is :

Fig. 1.35.2.

ii. Since there is no *lub* of 1 and 2 and same for 2 and 4. The given poset is not a lattice.

∨	1	2	3	4
1	1	–	1	1
2	–	2	2	–
3	1	2	3	1
4	1	–	1	4

iii. Only one edge (4, 2) is included to make it total order.

iv. Maximals are {1, 2} and minimals are {3, 4}.

Que 1.36. **In a lattice if $a \le b \le c$, then show that**

a. $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$

Answer

a. **Given :** $a \le b \le c$

Now
$$\begin{aligned} a \vee b &= \text{least upper bound of } a, b \\ &= \text{least \{all upper bounds of } a, b\} \\ &= \text{least } \{b, c, ...\} \quad [\text{using } a \le b \le c] \\ &= b \quad ...(1.36.1) \end{aligned}$$

and
$$\begin{aligned} b \wedge c &= \text{greatest lower bound of } b, c \\ &= \text{maximum \{all lower bounds of } b, c\} \\ &= \text{maximum } \{b, a, ...\} \quad [\text{using } a \le b \le c] \\ &= b \quad ...(1.36.2) \end{aligned}$$

Eq. (1.36.1) and (1.36.2) gives, $a \vee b = b \wedge c$

b. $(a \vee b) \vee (b \wedge c) \Rightarrow (a \vee b) \wedge (a \vee c) = b$

Consider, $(a \vee b) \vee (b \wedge c)$

$$= b \vee b \; [\text{using } a \le b \le c \text{ and definition of } \vee \text{ and } \wedge]$$
$$= b \quad ...(1.36.3)$$

and $(a \vee b) \wedge (a \vee c) = b \wedge c$
$$= b \quad ...(1.36.4)$$

From eq. (1.36.3) and (1.36.4), $(a \vee b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$.

Que 1.37.

a. Prove that every finite subset of a lattice has an LUB and a GLB.

b. Give an example of a lattice which is a modular but not a distributive.

Answer

a.

1. The theorem is true if the subset has 1 element, the element being its own *glb* and *lub*.
2. It is also true if the subset has 2 elements.
3. Suppose the theorem holds for all subsets containing 1, 2, ..., k elements, so that a subset $a_1, a_2, ..., a_k$ of L has a *glb* and a *lub*.
4. If L contains more than k elements, consider the subset $\{a_1, a_2, ..., a_{k+1}\}$ of L.
5. Let $w = lub\ (a_1, a_2, ..., a_k)$.
6. Let $t = lub\ (w, a_{k+1})$.
7. If s is any upper bound of $a_1, a_2, ..., a_{k+1}$, then s is $\ge$ each of $a_1, a_2, ..., a_k$ and therefore $s \ge w$.
8. Also, $s \ge a_{k+1}$ and therefore s is an upper bound of w and a_{k+1}.
9. Hence $s \ge t$.
10. That is, since $t \ge$ each a_1, t is the *lub* of $a_1, a_2, ..., a_{k+1}$.
11. The theorem follows for the *lub* by finite induction.
12. If L is finite and contains m elements, the induction process stops when $k + 1 = m$.

b.

1. The diamond is modular, but not distributive.
2. Obviously the pentagon cannot be embedded in it.
3. The diamond is not distributive :
$$y \vee (x \wedge z) = y\ (y \vee x) \wedge (y \vee z) = 1$$
4. The distributive lattices are closed under sublattices and every sublattice of a distributive lattice is itself a distributive lattice.
5. If the diamond can be embedded in a lattice, then that lattice has a non-distributive sublattice, hence it is not distributive.

Que 1.38. **Explain modular lattice, distributive lattice and bounded lattice with example and diagram.**

Answer

Modular, distributive and bounded lattice : Refer Q. 1.27, Page 1–18F, Unit-1.

Example :

Let consider a Hasse diagram :

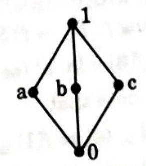


Fig. 1.38.1.

Modular lattice :

$0 \le a$ *i.e.*, taking $b = 0$

$b \vee (a \wedge c) = 0 \vee 0 = 0, a \wedge (b \vee c) = a \wedge c = 0$

Distributive lattice :

For a set S, the lattice $P(S)$ is distributive, since union and intersection each satisfy the distributive property.

Bounded lattice : Since, the given lattice has 1 as greatest and 0 as least element so it is bounded lattice.

Que 1.39.

i. **Justify that $(D_{36}, \backslash)$ is lattice.**

ii. **Let L_1 be the lattice defined as D_6 and L_2 be the lattice $(P(S), \subseteq)$, where $P(S)$ be the power set defined on set $S = \{a, b\}$. Justify that the two lattices are isomorphic.** **AKTU 2021-22, Marks 10**

OR

For any positive integer D36, then find whether (D36, '|') is lattice or not ?

Answer

i. D36 = Divisor of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}

Hasse diagram :

$$(1 \vee 3) = \{3, 6\}, (1 \vee 2) = \{2, 4\}, (2 \vee 6) = \{6, 18\}, (9 \vee 4) = \{\phi\}$$

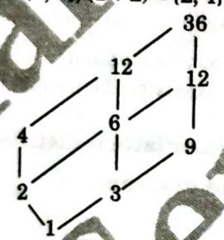


Fig. 1.39.1.

Since, $9 \vee 4 = \{\phi\}$

So, D36 is not a lattice.

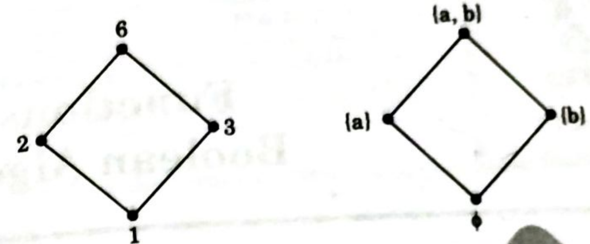
ii. Let $A = \{a, b\}$ and $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Then the lattice $(P(S), \subseteq)$ is isomorphic to lattice $(D_6, 1)$ with divisibility as the partial order relation. In fact, we define a mapping $f : D_6 \rightarrow P(S)$ by

$f(1) = \phi, f(2)\ \{a\}, f(3) = \{b\}, f(6) = \{a, b\}$ (see Fig. 1.39.1)

Then, f is bijective and we note that

$$1 \mid 2 \Leftrightarrow \{\phi\} \subseteq \{a\} \Leftrightarrow f(1) \subseteq f(2),$$
$$2 \mid 6 \Leftrightarrow \{a\} \subseteq \{a, b\} \Leftrightarrow f(2) \subseteq f(6)$$

and so on. Hence f is isomorphism. Thus, the two lattices are isomorphic.



D_6 **Fig. 1.39.2.** P({a, b})

☺☺☺

2 UNIT

Functions and Boolean Algebra

CONTENTS

PART-1

Functions : Definition, Classification of Functions, Operations on Functions.

Que 2.1. **Define the term function. Also, give classification of it.**

Answer

1. Let X and Y be any two non-empty sets. A function from X to Y is a rule that assigns to each element $x \in X$ a unique element $y \in Y$.
2. If f is a function from X to Y we write $f : X \to Y$.
3. Functions are denoted by f, g, h, i etc.
4. It is also called mapping or transformation or correspondence.

Domain and co-domain of a function : Let f be a function from X to Y. Then set X is called domain of function f and Y is called co-domain of function f.

Range of function : The range of f is set of all images of elements of X.

i.e., Range $(f) = \{y : y \in Y \text{ and } y = f(x)\ \forall x \in X\}$

Also Range $(f) \subseteq Y$

Classification of functions :

1. **Algebraic functions :** Algebraic functions are those functions which consist of a finite number of terms involving powers and roots of the independent variable x.

   Three particular cases of algebraic functions are :

   i. **Polynomial functions :** A function of the form $a_0x^n + a_1x^{n-1} + ... + a_n$ where n is a positive integer and $a_0, a_1, ..., a_n$ are real constants and $a_0 \neq 0$ is called a polynomial of x in degree n, for example $f(x) = 2x^3 + 5x^2 + 7x - 3$ is a polynomial of degree 3.

   ii. **Rational functions :** A function of the form $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

   iii. **Irrational functions :** Functions involving radicals are called irrational functions. $f(x) = \sqrt[3]{x} + 5$ is an example of irrational function.

2. **Transcendental functions :** A function which is not algebraic is called transcendental function.

   i. **Trigonometric functions :** Six functions $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$ where the angle x is measured in radian are called trigonometric functions.

ii. **Inverse trigonometric functions :** Six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$, are called inverse trigonometric functions.

iii. **Exponential functions :** A function $f(x) = a^x$ $(a > 0)$ satisfying the law $a' = a$ and $a^x a^y = a^{x+y}$ is called the exponential function.

iv. **Logarithm functions :** The inverse of the exponential function is called logarithm function.

Que 2.2. **Give the types / operations on functions.**

Answer

1. **One-to-one function (Injective function or injection) :** Let $f : X \to Y$ then f is called one-to-one function if for distinct elements of X there are distinct image in Y *i.e.*, f is one-to-one iff

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \ \forall \ x_1, x_2, \in X$$

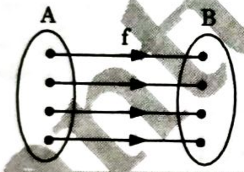


Fig. 2.2.1. One-to-one.

2. **Onto function (Surjection or surjective function) :** Let $f : X \to Y$ then f is called onto function iff for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$ or f is onto if Range $(f) = Y$.

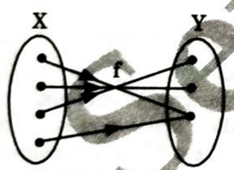


Fig. 2.2.2. Onto.

3. **One-to-one onto function (Bijective function or bijection) :** A function which is both one-to-one and onto is called one-to-one onto function or bijective function.

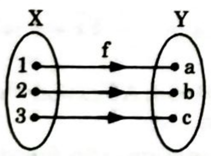


Fig. 2.2.3. One-to-one onto.

4. **Many one function :** A function which is not one-to-one is called many one function *i.e.*, two or more elements in domain have same image in co-domain *i.e.*,

If $f : X \to Y$ then $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$.

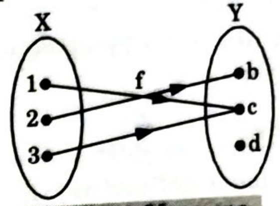


Fig. 2.2.4. Many one.

5. **Identity function :** Let $f : X \to X$ then f is called identity function if $f(a) = a \ \forall \ a \in X$ *i.e.*, every element of X is image of itself. It is denoted by I.

6. **Inverse function (Invertible function) :** Let f be a bijective function from X to Y. The inverse function of f is the function that assigns an element $y \in Y$, a unique element $x \in X$ such that $f(x) = y$ and inverse of f denoted by f^{-1}. Therefore if $f(x) = y$ implies $f^{-1}(y) = x$.

Que 2.3. **Determine whether each of these functions is a bijective from R to R.**

a. $f(x) = x^2 + 1$
b. $f(x) = x^3$
c. $f(x) = (x^2 + 1)/(x^2 + 2)$

Answer

a. $f(x) = x^2 + 1$

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$
$$x_1^2 + 1 = x_2^2 + 1$$
$$x_1^2 = x_2^2$$
$$x_1 = \pm x_2$$

Therefore, if $x_2 = 1$ then $x_1 = \pm 1$

So, f is not one-to-one.

Hence, f is not bijective.

b. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$x_1^3 = x_2^3$$
$$x_1 = x_2$$

$\therefore$ f is one-to-one.

Let $y \in R$ such that

$$y = x^3$$
$$x = (y)^{1/3}$$

For $\forall \ y \in R \ \exists$ a unique $x \in R$ such that $y = f(x)$

$\therefore$ f is onto.

Hence, f is bijective.

c. Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1^2+1}{x_1^2+2} = \frac{x_2^2+1}{x_2^2+2}$$

If $\quad x_1 = 1, x_2 = -1$ then $f(x_1) = f(x_2)$

but $\quad x_1 \neq x_2$

$\therefore$ f is not one-to-one.

Hence, f is not bijective.

Que 2.4. **Let a function is defined as $f: R - \{3\} \to R - \{1\}, f(x) = (x-1)/(x-3)$, then show that f is a bijective function and also compute the inverse of f. Where R is a set of real numbers.**

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Answer

$$f(x) = \frac{(x-1)}{(x-3)}$$

Let $x_1, x_2 \in R - \{3, 1\}$ such that

$$f(x_1) = f(x_2)$$

$$\frac{x_1 - 1}{x_1 - 3} = \frac{x_2 - 1}{x_2 - 3}$$

$$x_1x_2 - 3x_1 - x_2 + 3 = x_1x_2 - x_1 - 3x_2 + 3$$

$$x_1x_2 - x_2 - 3x_1 = x_1x_2 - x_1 - 3x_2$$

$$x_2 + 3x_1 = x_1 + 3x_2$$

$$3x_1 - x_1 = 3x_2 - x_2$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

$\therefore$ f is one-to-one.

Let $y \in R - \{1, 3\}$ such that

$$y = \frac{x-1}{x-3}$$

$$yx - 3y = x - 1$$

$$-3y + 1 = x - yx$$

$$1 - 3y = x(y-1)$$

$$x = \frac{1-3y}{y-1}$$

$\because$ f is not onto

Hence, f is not bijective

Inverse of $\quad f(x) = \dfrac{1-3y}{y-1}$

Que 2.5.

i. Let $R = \{(1, 2) (2, 3) (3, 1)\}$ defined on $A = \{1, 2, 3\}$. Find the transitive closure of R using Warshall's algorithm.

ii. Justify that "If $f: A \to B$ and $g: B \to C$ be one-to-one onto functions, then gof is also one to one onto and $(gof)^{-1} = f^{-1} o\, g^{-1}$".

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OR

If $f: A \to B, g: B \to C$ are invertible functions, then show that $g \circ f: A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Answer

i.
$$R = \{(1, 2) (2, 3) (3, 1)\}$$

Step 1 : Representing the relation R in matrix form

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}_{3\times3}$$

Step 2 : Now, consider Column 1 and Row 1 of the above matrix

So, $\quad C_1 = \{3\}$ and $R_1 = \{2\}$

$$C_1 \times R_1 = \{3, 2\}$$

So,
$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Step 3 : Now consider Column 2 and Row 2 of above matrix

$$C_2 = \{1\} \text{ and } R_2 = \{3\}$$

$$C_2 \times R_2 = \{1, 3\}$$

So,
$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Step 4 : Now consider C_3 and R_3

i.e., $\quad C_3 = \{2\}$ and $R_3 = \{1\}$

$$C_3 \times R_3 = \{2, 1\}$$

So,
$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence, $\quad R_t = \{(1, 2) (2, 1) (2, 3) (3, 1)\}$

ii. If $f: A \to B$ and $g: B \to C$ be one-to-one onto functions, then $g \circ f$ is also one-onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof : Since f is one-to-one, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in R$
Again since g is one-to-one, $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ for $y_1, y_2 \in R$
Now $g \circ f$ is one-to-one, since $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow \quad f(x_1) = f(x_2)$ [g is one-to-one]

$\Rightarrow \quad x_1 = x_2$ [f is one-to-one]

Since g is onto, for $z \in C$, there exists $y \in B$ such that $g(y) = z$. Also f being onto there exists $x \in A$ such that $f(x) = y$. Hence $z = g(y) = g[f(x)] = (g \circ f)(x)$

This shows that every element $z \in C$ has pre-image under gof. So, $g \circ f$ is onto.

Thus, $g \circ f$ is one-to-one onto function and hence $(g \circ f)^{-1}$ exists.

By the definition of the composite functions, $g \circ f : A \to C$. So, $(g \circ f)^{-1} : C \to A$.

Also $g^{-1} : C \to B$ and $f^{-1} : B \to A$.

Then by the definition of composite functions, $f^{-1} \circ g^{-1} : C \to A$.

Therefore, the domain of $(g \circ f)^{-1}$ = the domain of $f^{-1} \circ g^{-1}$.

Now $\quad (g \circ f)^{-1}(z) = x \Leftrightarrow (g \circ f)(x) = z$

$\Leftrightarrow g(f(x)) = z$

$\Leftrightarrow g(y) = z$ where $y = f(x)$

$\Leftrightarrow y = g^{-1}(z)$

$\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1} \circ g^{-1})(z)$

$\Leftrightarrow x = (f^{-1} \circ g^{-1})(z) \; [f^{-1}(y) = x]$

Thus, $\quad (g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$.

So, $\quad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

PART-2

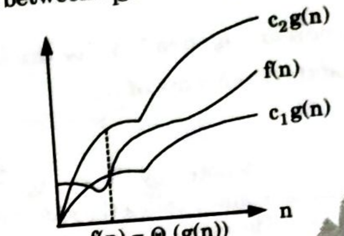
Growth of Functions.

Que 2.6. **Write short note on growth of functions.**

Answer

1. We need to approximate the number of steps required to execute any algorithm because of the difficulty involved in expression or difficulty in evaluating an expression. We compare one function with another function to know their rate of growths.
2. If f and g are two functions we can give the statements like 'f has same growth rate as g' or 'f has higher growth rate than g'.

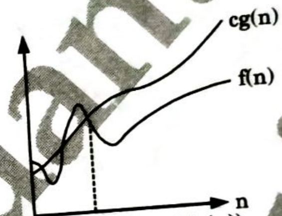
3. Growth rate of function can be defined through notation.
   a. **θ-Notation (Same order) :** This notation bounds a function to within constant factors. We say $f(n) = \theta g(n)$ if there exist positive constants n_0, c_1 and c_2 such that to the right of n_0 the value of $f(n)$ always lies between $c_1g(n)$ and $c_2g(n)$ inclusive.



f(n) = Θ (g(n))

Fig. 2.6.1.

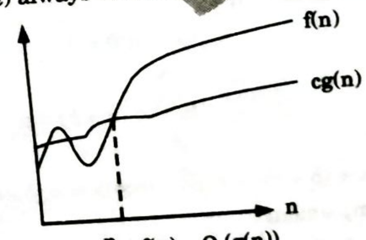
   b. **O-Notation (Upper bound) :** This notation gives an upper bound for a function to within a constant factor. We write $f(n) = O(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or below $cg(n)$.



f(n) = O(g(n))

Fig. 2.6.2.

   c. **Ω-Notation (Lower bound) :** This notation gives a lower bound for a function to within a constant factor. We write $f(n) = \Omega\, g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or above $cg(n)$.



f(n) = Ω (g(n))

Fig. 2.6.3.

PART-3

Boolean Algebra : Introduction, Axioms and Theorems of Boolean Algebra.

Que 2.7. What is Boolean algebra ? Write the axioms of Boolean algebra. Also, describe the theorems of it.

Answer

A Boolean algebra is generally denoted by $(B, +, ., 0, 1)$ where $(B, +, .)$ is a lattice with binary operations '+' and '.' called the join and meet respectively and (') is unary operation in B. The elements 0 and 1 are zero (least) and unit (greatest) elements of lattice $(B, +, .)$. B is called a Boolean algebra if the following axioms are satisfied for all a, b, c in B.

Axioms of Boolean algebra :

If $a, b, c \in B$, then

1. Commutative laws :
   a. $a + b = b + a$ b. $a.b = b.a$
2. Distributive laws :
   a. $a + (b.c) = (a + b).(a + c)$ b. $a.(b + c) = (a.b) + (a.c)$
3. Identity laws :
   a. $a + 0 = a$ b. $a.1 = a$
4. Complement laws :
   a. $a + a' = 1$ b. $a.a' = 0$

Basic theorems :

Let $a, b, c \in B$, then

1. Idempotent laws :
   a. $a + a = a$ b. $a.a = a$
2. Boundedness (Dominance) laws :
   a. $a + 1 = 1$ b. $a.0 = 0$
3. Absorption laws :
   a. $a + (a.b) = a$ b. $a.(a + b) = a$
4. Associative laws :
   a. $(a + b) + c = a + (b + c)$ b. $(a.b).c = a.(b.c)$
5. Uniqueness of complement :
   $a + x = 1$ and $a.x = 0$, then $x = a$
6. Involution law : $(a')' = a$

7. a. $0' = 1$ b. $1' = 0$
8. De-Morgan's laws :
   a. $(a + b)' = a'.b'$ b. $(a.b)' = a' + b'$

Que 2.8. Prove the following theorems :

a. Absorption law : Prove that $\forall a, b, \in B$
   i. $a.(a + b) = a$ ii. $a + a.b = a$

b. Idempotent law : Prove that $\forall a \in B, a + a = a$ and $a.a = a$.

c. De Morgan's law : Prove that $\forall a, b, \in B$
   i. $(a + b)' = a'.b'$ ii. $(a. b)' = a' + b'$

d. Prove that $0' = 1$ and $1' = 0$.

Answer

a. **Absorption law :**

i. **To prove :** $a.(a + b) = a$

Let
$$\begin{aligned} a.(a+b) &= (a+0).(a+b) && \text{by Identity law} \\ &= a + 0.b && \text{by Distributive law} \\ &= a + b.0 && \text{by Commutative law} \\ &= a + 0 && \text{by Dominance law} \\ &= a && \text{by Identity law} \end{aligned}$$

ii. **To prove :** $a + a.b = a$

Let
$$\begin{aligned} a + a.b &= a.1 + a.b && \text{by Identity law} \\ &= a.(1 + b) && \text{by Distributive law} \\ &= a.(b + 1) && \text{by Commutative law} \\ &= a.1 && \text{by Dominance law} \\ &= a && \text{by Identity law} \end{aligned}$$

b. **Idempotent law :**

To prove : $a + a = a$ and $a.a = a$

Let
$$\begin{aligned} a &= a + 0 && \text{by Identity law} \\ &= a + a.a' && \text{by Complement law} \\ &= (a + a).(a + a') && \text{by Distributive law} \\ &= (a + a).1 && \text{by Complement law} \\ &= a + a && \text{by Identity law} \end{aligned}$$

Now let
$$\begin{aligned} a &= a.1 && \text{by Identity law} \\ &= a.(a + a') && \text{by Complement law} \\ &= a.a + a.a' && \text{by Distributive law} \\ &= a.a + 0 && \text{by Complement law} \\ &= a.a && \text{by Identity law} \end{aligned}$$

c. **De Morgan's law :**

i. **To prove :** $(a + b)' = a'.b'$

To prove the theorem we will show that
$$(a + b) + a'.b' = 1$$

Consider
$$\begin{aligned} (a + b) + a'.b' &= \{(a + b) + a'\}.\{(a + b) + b'\} && \text{by Distributive law} \\ &= \{(b + a) + a'\}.\{(a + b) + b'\} && \text{by Commutative law} \end{aligned}$$

$= \{b + (a + a')\}.\{a + (b + b')\}$

$= (b + 1).(a + 1)$ by Associative law

$= 1.1$ by Complement law

$= 1$ by Dominance law ...(3.14.1)

Also consider $(a + b).a'b' = a'b'.(a + b)$ by Commutative law

$= a'b'.a + a'b'.b$ by Distributive law

$= a.(a'b') + a'.(b'b)$ by Commutative law

$= (a.a').b' + a'.(b.b')$ by Associative law

$= 0.b' + a'.0$ by Complement law

$= b'.0 + a'.0$ by Commutative law

$= 0 + 0$ by Dominance law

$= 0$ ...(3.14.2)

From eq. (3.14.1) and (3.14.2), we get

$a'b'$ is complement of $(a + b)$ *i.e.*, $(a + b)' = a'b'$.

ii. **To prove :** $(a.b)' = a' + b'$

Follows from principle of duality, that is, interchange operations + and • and interchange the elements 0 and 1.

d. **To prove :** $0' = 1$ and $1' = 0$.

$0' = (a a')'$

$= a' + (a')'$ by Complement law

$= a' + a$ by De Morgan's law

$= a + a'$ by Involution law

$= 1$ by Commutative law

Now $(0')' = 1'$ by Complement law

$\Rightarrow$ $0 = 1'$

$\Rightarrow$ $1' = 0$.

Que 2.9. **Justify that for any sets A, B, and C :**

i. $(A - (A \cap B)) = A - B$

ii. $(A - (B \cap C)) = (A - B) \cup (A - C)$

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Answer

i.

L.H.S $= A - (A \cap B)$

$= A \cap (A \cap B)'$ $(\because A - B = A \cap B')$

$= A \cap (A' \cup B')$ (De Morgan law)

$= (A \cap A') \cup (A \cap B')$ (Distribution law)

$= \phi \cup (A \cap B')$

$= A \cap B'$

$= A - B$

$=$ R.H.S

ii. $(A - (B \cap C)) = (A - B) \cup (A - C)$

L.H.S $= (A - (B \cap C))$

$= A \cap (B \cap C)'$

$= A \cap (B' \cup C')$

$= (A \cap B') \cap (A \cap C')$

$= (A - B) \cup (A - C) =$ R.H.S

Que 2.10. **Define Boolean algebra. Show that in a Boolean algebra meet and join operations are distributive to each other.**

Answer

Boolean algebra : Refer Q. 2.7, Page 2–9F, Unit-2.

Meet and join operations are distributive :

1. Let L be a poset under an ordering $\leq$. Let $a, b \in L$.
2. We define :

   $a \vee b$ (read "a join b") as the least upper bound of a and b, and

   $a \wedge b$(read "a meet b") as greatest lower bound of a and b.
3. Since the join and meet operation produce a unique result in all cases where they exist, we can consider them as binary operations on a set if they always exist.
4. A lattice is a poset L (under $\leq$) in which every pair of elements has a *lub* and a *glb*.
5. Since a lattice L is an algebraic system with binary operations $\wedge$ and $\vee$, it is denoted by $[L, \vee, \wedge]$.
6. Let us consider,

   a. $[P(A), \vee, \wedge]$ is a lattice for any set A and

   b. The join operation is the set operation of union and the meet operation is the operation of intersection; that is, $\vee = \cup$ and $\wedge = \cap$.
7. It can be shown that the commutative laws, associative laws, idempotent laws, and absorption laws are all true for any lattice.
8. An example of this is clearly $[P(A); \cup, \cap]$, since these laws hold in the algebra of sets.
9. This lattice is also distributive such that join is distributive over meet and meet is distributive over join.

Que 2.11. **Define Boolean algebra. Show that $a'.[(b' + c)' + b.c] + [(a + b')'.c] = a'.b$ using rules of Boolean Algebra. Where a' is the complement of an element a.**

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Answer

Boolean algebra : Refer Q. 2.7, Page 2–9F, Unit-2.

Numerical :

LHS $= a' . [(b' + c)' + b.c] + [(a + b')' . c]$

$= a' . [b . c' + b . c] + [a' . b . c]$ [de Morgan's Law]

$= a' . [b(c' + c)] + a'.b.c$ $(a + b)' = a' . b'$

$= a' . [b . 1] + a.'b.c$ [Complement Law]

$= a'b + a'b.c$ $[\because a + \bar{a} = 1]$

$= a'b\,(1 + c)$ [Identity Law $\because 1 + a = 1$]

$= a'b = \text{RHS}$

Que 2.12. **Show that the following are equivalent in a Boolean algebra.**

$$a \le b \Leftrightarrow a * b' = 0 \Leftrightarrow b' \le a' \Leftrightarrow a' \oplus b = 1$$

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Answer

First prove the LHS, for this we use $a * b' = 0$ or $a \wedge b' = 0$ (say)

Now, $a = a \wedge I$ [By identity of bounded lattice]

$= a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$ [By law of distribution]

$= (a \wedge b) \vee 0$ $[a \wedge b' = 0]$

$= a \wedge b \le b$ [Partial order relation]

Hence it is clear that if $a * b' = 0$ then relation $a \le b$ is a partial order relation.

Since, it is evident in Boolean algebra for any a, b,

$a \le b$ iff $b' \le a'$

[By law of duality]

Note : In the given question, the equivalency cannot be solved if we use $a' \oplus b = 1$. So, assuming $a' + b = 1$.

Since, $a \le b$ so we can say that $a \wedge b = a \equiv a \vee b = b$

If $a' + b = 1$ then by de Morgan's law and involution,

$1 = 0' = (a * b')' = a' + b'' = a' + b$

$0 = 1' = (a' + b)' = a'' * b' = a * b'$

So, $a * b' = 0$ is equivalent to $a' + b = 1$

PART-4

Algebraic Manipulation of Boolean Expressions.

Que 2.13. **Express each Boolean expression as SOP and then in its complete sum-of-products form.**

a. $E = x(xy' + x'y + y'z)$

b. $E = z(x' + y) + y'$.

Answer

a. $E = x.x.y' + x.x'.y + x.y'.z = x.y' + x.y'.z = x.y'$

$(\because\ x.x'.y = 0$ and $x.y'$ is contained in $x.y'.z)$

Complete SOP form $= x.y'\,(z + z')$ $(\because\ z$ is missing$)$

$= x.y'.z + x.y'.z'$

b. $E = z.(x' + y) + y' = x'.z + y.z + y'$

Then $E = x'.z(y + y') + y.z.(x + x') + y'.(x + x')(z + z')$

(by Complement law)

$= x'.y.z + x'.y'.z + x.y.z + x'.y.z + x.y'.z + x.z'.y' + x'.y'.z + x'.y'.z'$

$= x'.y.z + x'.y'.z + x.y.z + x.y'.z + x.y'.z' + x'.y'.z'$

PART-5

Simplification of Boolean Functions, Karnaugh Maps.

Que 2.14. **Define a Boolean function of degree *n*. Simplify the following Boolean expression using Karnaugh maps**

$xyz + xy'z + x'y'z + x'yz + x'y'z'$

Answer

Boolean function of degree *n* :

1. Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i\, B$ for $1 \le i \le n\}$ is the set of all possible n-tuples of 0s and 1s.
2. The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1.
3. A function from B^n to B is called a Boolean function of degree n.
4. For example, the function $F(x, y) = xy$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$, is a Boolean function of degree z with $F(1, 1) = 1$, $F(1, 0) = 0$, $F(0, 1) = 0$ and $F(0, 0) = 0$.

Numerical : The Karnaugh map for the given function is :

$xyz + xy'z + x'y'z + x'yz + x'y'z'$

x \ yz	y'z'	y'z	yz	yz'
x'	1	1	1	
x		1	1	

Fig. 2.14.1.

Then the simplified expression is : $z + x'y'$.

Que 2.15. **Simplify the following Boolean functions using three variable maps :**

a. $F(x, y, z) = \Sigma(0, 1, 5, 7)$ **b.** $F(x, y, z) = \Sigma(1, 2, 3, 6, 7)$

Answer

a. $f(x, y, z) = \Sigma(0, 1, 5, 7)$

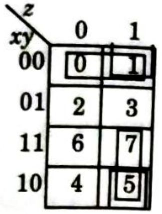

Fig. 2.15.1.

$f = \bar{x}\bar{y} + xz$

b. $f(x, y, z) = \Sigma(1, 2, 3, 6, 7)$

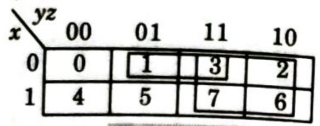

Fig. 2.15.2.

$f = \bar{x}z + y$

Que 2.16. **Simplify the following Boolean function using K-map :**

$$F(x, y, z) = \Sigma(0, 2, 3, 7)$$

Answer

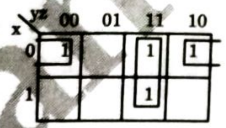

Fig. 2.16.1.

$F = \bar{x}\bar{z} + yz$

Que 2.17. **Solve $E(x, y, z, t) = \Sigma(0, 2, 6, 8, 10, 12, 14, 15)$ using K-map.**

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Answer

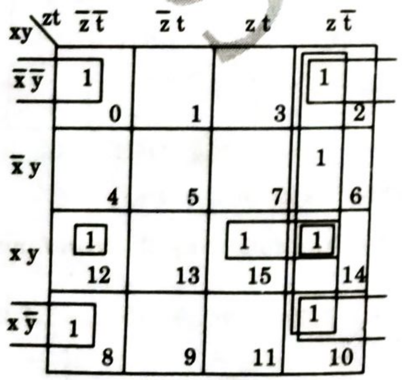

Fig. 2.17.1.

$\Rightarrow z\bar{t} + xyz + xy\bar{z} + \bar{y}\bar{t}$

Que 2.18. **Simplify the following Boolean expressions using K-map :**

a. $Y = ((AB)' + A' + AB)'$

b. $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

Answer

a. $Y = ((AB)' + A' + AB)'$

$$\begin{aligned} &= ((AB)')' (A' + (AB))' \\ &= (AB)((A')'(AB)') \\ &= AB(A(A' + B')) \\ &= AB(AA' + AB') \\ &= AB(0 + AB') = AB\,AB' \\ &= ABB' \\ &= 0 \end{aligned}$$

Here, we find that the expression is not in minterm. For getting minterm, we simplify and find that its value is already zero. Hence, no need to use K-map for further simplification.

b. $A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD' = A'B'$

$= A'B'C'D' + A'B'C'D + A'B'CD + A'B'CD'$

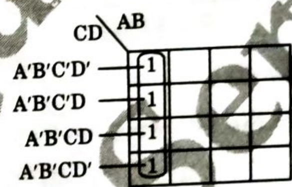

Fig. 2.18.1.

On simplification by K-map, we get $A'B'$ corresponding to all the four one's.

Que 2.19. **Solve the following Boolean functions using K-map:**

i. $F(A, B, C, D) = \Sigma\{m_0, m_1, m_2, m_4, m_5, m_6, m_8, m_9, m_{12}, m_{13}, m_{14}\}$

ii. $F(A, B, C, D) = \Sigma\{0, 2, 5, 7, 8, 10, 13, 15\}$

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Answer

i. $F(A, B, C, D) = \Sigma\{m_0, m_1, m_2, m_4, m_5, m_6, m_8, m_9, m_{12}, m_{13}, m_{14}\}$

AB \ CD	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	1 (0)	1 (1)	(3)	1 (2)
$\bar{A}B$	1 (4)	1 (5)	(7)	1 (6)
AB	1 (12)	1 (13)	(15)	1 (14)
$A\bar{B}$	1 (8)	1 (9)	(11)	(10)

$\bar{A}\bar{D}$, $AB\bar{D}$, $\bar{C}\bar{D}$

Fig. 2.19.1.

$F(A, B, C, D) = \bar{A}\,\bar{D} + \bar{C}\,\bar{D} + AB\bar{D}$

ii. $F(A, B, C, D) = \Sigma\{0, 2, 5, 7, 8, 10, 13, 15\}$

AB \ CD	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	1 (0)	(1)	(3)	1 (2)
$\bar{A}B$	(4)	1 (5)	1 (7)	(6)
AB	(12)	1 (13)	1 (15)	(14)
$A\bar{B}$	1 (8)	(9)	(11)	1 (10)

BD, $\bar{B}\bar{D}$

Fig. 2.20.1.

$F(A, B, C, D) = \bar{B}\,\bar{D} + BD$

Que 2.20. Simplify the following boolean expressions using Boolean algebra

i. $xy + x'z + yz$

ii. $C(B + C)(A + B + C)$

iii. $A + B(A + B) + A(A' + B)$

iv. $XY + (XZ)' + XY'Z(XY + Z)$

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Answer

i. $xy + x'z + yz$

$= xy + z(x' + y) = (xy + z)\,(xy + x' + y)$

$= (xy + z)\,(y(x + 1) + x') = (xy + z)\,(y + x')$

ii. $= C(B + C)(A + B + C) = (BC + CC)\,(A + B + C)$

$= (BC + C)\,(A + B + C)$ by Idempotent law

$= C(A + B + C)$ by Absorption law

$= AC + BC + CC$

$= AC + BC + C$ by Idempotent law

$= AC + C$ by Absorption law

$= C$ by Absorption law

iii. $= A + B(A + B) + A(A' + B)$

$= A + AB + BB + AA' + AB$

$= A + AB + B + AA'$ by Idempotent law

$= A + AB + B$ by Complement law

$= A + B$ by Absorption law

iv. $= xy + (xz)' + xy'z(xy + z)$

$= xy + x' + z' + xy'z(xy + z)$ by de Morgan's law

$= xy + x' + z' + xy'zxy + xy'zz$

$= xy + x' + z' + xy'zy + xy'z$ by Idempotent law

$= xy + x' + z' + xy'z$ by Complement law

$= x(y + y'z) + x' + z'$

$= x(y + z) + x' + z'$ $[x + x'y = x + y]$

$= xy + xz + x' + z'$

$= x' + y + z' + x$ $[x + x'y = x + y]$

$= 1 + y + z'$ by Complement law

$= 1$ by Boundedness law

Que 2.21. Find the Boolean algebra expression for the following system.

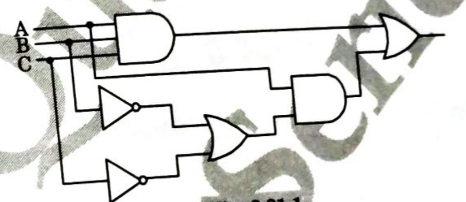

Fig. 2.21.1.

Answer

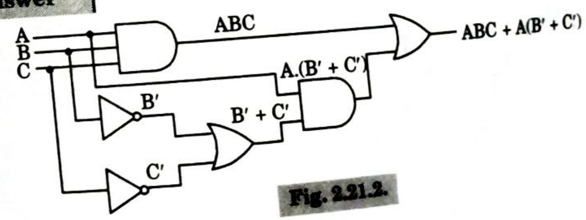

Fig. 2.21.2.

Que 2.22. **Find the Sum-Of-Products and Product-Of-sum expansion of the Boolean function $F(x, y, z) = (x + y) z'$.**

Answer

$$F(x, y, z) = (x + y)z'$$

x	y	z	$x + y$	z'	$(x + y)z'$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Sum-Of-Product :

$$F(x, y, z) = xyz' + xy'z' + x'yz'$$

Product-Of-Sum :

$$F(x, y, z) = (x + y + z)(x + y' + z)(x' + y + z)(x' + y' + z)(x' + y' + z')$$

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3 UNIT

Theory of Logics

CONTENTS

PART-1

Proposition, Truth Tables.

Que 3.1. **Define the term proposition. Also, explain compound proposition with example.**

Answer

Proposition : Proposition is a statement which is either true or false but not both. It is a declarative statement. It is usually denoted by lower case letters p, q, r, s, t etc. They are called Boolean variable or logic variable.

For example :

1. Dr. A.P.J. Abdul Kalam was Prime Minister of India.
2. Roses are red.
3. Delhi is in India

(1) proposition is false whereas (2) and (3) are true.

Compound proposition : A compound proposition is formed by composition of two or more propositions called components or sub-propositions.

For example :

1. Risabh is intelligent and he studies hard.
2. Sky is blue and clouds are white.

Here first statement contain two propositions "Risabh is intelligent" and "he studies hard" whereas second statement contain propositions "sky is blue" and "clouds are white". As both statements are formed using two propositions. So they are compound propositions.

Que 3.2. **Discuss connectives in detail with truth tables.**

Answer

1. The words or phrases used to form compound proposition are called connectives.
2. There are five basic connectives as shown in the Table 3.2.1.

Table. 3.2.1.

S. No.	Connective words	Name of connective	Symbol
1.	Not	Negation	~ or ¬ or –
2.	And	Conjunction	$\wedge$
3.	Or	Disjunction	$\vee$
4.	If-then	Implication	$\rightarrow$
5.	If and only if	Biconditional	$\leftrightarrow$

i. **Negation :** If P is a proposition then negation of P is a proposition which is true when p is false and false when p is true. It is denoted by $\sim p$ or $\neg$ or p' or $\overline{p}$.

Truth table :

p	$\sim p$
T	F
F	T

ii. **Conjunction :** If p and q are two propositions then conjunction of p and q is a proposition which is true when both p and q are true otherwise false. It is denoted by $p \wedge q$.

Truth table :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

iii. **Disjunction :** If p and q be two propositions, then disjunction of p and q is a proposition which is true when either one of p or q or both are true and is false when both p and q are false and it is denoted by $p \vee q$.

Truth table :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Que 3.3. **Write short notes on :**

i. **Truth table** **ii.** **Logical equivalence**

Answer

i. **Truth table :** A truth table is a table that shows the truth value of a compound proposition for all possible cases.

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$(p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\sim p$
T	F
F	T

ii. **Logical equivalence :** If two propositions $P(p, q, ...)$ and $Q(p, q, ...)$ where $p, q, ...$ are propositional variables, have the same truth values in every possible case, the propositions are called logically equivalent or simply equivalent, and denoted by

$$P(p, q, ...) \equiv Q(p, q, ...)$$

Que 3.4. **Explain the following terms with suitable example :**

i. **Conjunction**
ii. **Disjunction**
iii. **Conditional**
iv. **Converse**
v. **Contrapositive**

OR

Define inverse.

Answer

i. **Conjunction :** If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". Its truth table is,

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example :

p **:** Ram is healthy.

q **:** He has blue eyes.

$p \wedge q$ **:** Ram is healthy and he has blue eyes.

ii. **Disjunction :** If p and q are two statements, the disjunction of p and q is the compound statement denoted by $p \vee q$ and it is read as "p or q". Its truth table is,

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example :

p **:** Ram will go to Delhi.

q **:** Ram will go to Calcutta.

$p \vee q$ **:** Ram will go to Delhi or Calcutta.

iii. **Conditional :** If p and q are propositions. The compound proposition if p then q denoted by $p \Rightarrow q$ or $p \rightarrow q$ and is called conditional proposition or implication. It is read as "If p then q" and its truth table is,

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example :

p **:** Ram works hard.

q **:** He will get good marks.

$p \rightarrow q$ **:** If Ram works hard then he will get good marks.

For converse and contrapositive :

Let p : It rains.

q : The crops will grow.

iv. **Converse :** If $p \Rightarrow q$ is an implication then its converse is given by $q \Rightarrow p$ states that S : If the crops grow, then there has been rain.

v. **Contrapositive :** If $p \Rightarrow q$ is an implication then its contrapositive is given by $\sim q \Rightarrow \sim p$ states that,

t : If the crops do not grow then there has been no rain.

Inverse :

If $p \Rightarrow q$ is implication the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

Consider the statement

p : It rains.

q : The crops will grow

The implication $p \Rightarrow q$ states that,

r : If it rains then the crops will grow.

The inverse of the implication $p \Rightarrow q$, namely $\sim p \Rightarrow \sim q$ states that.

u : If it does not rain then the crops will not grow.

Que 3.5.

i. Express Converse, Inverse and Contrapositive of the following statement "If $x + 5 = 8$ then $x = 3$"

ii. Show that the statements $P \leftrightarrow Q$ and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ are equivalent.

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Answer

i. Consider the statements

$p : x + 5 = 8$

$q : x = 3$

Converse : If $p \Rightarrow q$ then its converse is $q \Rightarrow p$

if $x = 3$ then $x + 5 = 8$

Inverse : If $p \Rightarrow q$ then inverse is $\sim p \Rightarrow \sim q$

If $x + 5 \neq 8$ then $x \neq 3$

Contrapositive : If $p \Rightarrow q$ then its contrapositive is $\sim q \Rightarrow \sim p$

If $x \neq 3$ then $x + 5 \neq 8$

ii.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim P \wedge \sim Q$	$P \leftrightarrow Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
T	T	F	F	T	F	T	T
F	F	T	T	F	T	T	T

Since all rows of $(P \rightarrow Q)$ and $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ are identical.
Therefore $(P \rightarrow Q) \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$

Que 3.6. Construct the truth table for the following statements :

i. $(P \rightarrow Q') \rightarrow P'$

ii. $P \leftrightarrow (P' \vee Q')$

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Answer

i. $(P \rightarrow Q') \rightarrow P'$

P	Q	Q'	$P \rightarrow Q'$	P'	$(P \rightarrow Q') \rightarrow P'$
T	T	F	F	F	T
T	F	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	T

Hence, $(P \rightarrow Q') \rightarrow P'$ is a tautology

ii. $P \leftrightarrow (P' \vee Q')$

P	Q	P'	Q'	$P' \vee Q'$	$P \leftrightarrow (P' \vee Q')$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

Hence, $P \leftrightarrow (P' \vee Q')$ is not a tautology

PART-2

Tautology, Satisfiability, Contradiction, Algebra of Proposition, Theory of Inference.

Que 3.7. Explain tautologies, contradictions, satisfiability and contingency.

Answer

1. **Tautology :** Tautology is defined as a compound proposition that is always true for all possible truth values of its propositional variables and it contains T in last column of its truth table.

   Propositions like,

   i. The doctor is either male or female.

   ii. Either it is raining or not.

are always true and are tautologies.

2. **Contradiction :** Contradiction is defined as a compound proposition that is always false for all possible truth values of its propositional variables and it contains F in last column of its truth table.

   Propositions like,

   i. x is even and x is odd number.

   ii. Tom is good boy and Tom is bad boy.

   are always false and are contradiction.

3. **Contingency :** A proposition which is neither tautology nor contradiction is called contingency.

   Here the last column of truth table contains both T and F.

4. **Satisfiability :**

   A compound statement formula $A(P_1, P_2, \ldots P_n)$ is said to be satisfiable, if it has the truth value T for at least one combination of truth value of $P_1, P_2, \ldots P_n$.

Que 3.8. **Write short note on algebra of propositions.**

Answer

Proposition satisfies various laws which are useful in simplifying complex expressions. These laws are listed as :

1. Idempotent laws :
   a. $p \vee p \equiv p$
   b. $p \wedge p \equiv p$
2. Associative laws :
   a. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   b. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3. Commutative laws :
   a. $p \vee q \equiv q \vee p$
   b. $p \wedge q \equiv q \wedge p$
4. Distributive laws :
   a. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   b. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. Identity laws :
   a. $p \vee F \equiv p$
   b. $p \vee T \equiv T$
   c. $p \wedge F \equiv F$
   d. $p \wedge T \equiv p$
6. Complement laws :
   a. $p \vee \sim P \equiv T$
   b. $p \wedge \sim p \equiv F$

   c. $\sim T \equiv F$
   d. $\sim F \equiv T$
7. Involution law :
   a. $\sim(\sim p) \equiv p$
8. de Morgan's laws :
   a. $\sim (p \vee q) \equiv \sim p \wedge \sim q$
   b. $\sim (p \wedge q) \equiv \sim p \vee \sim q$
9. Absorption laws :
   a. $p \vee (p \wedge q) \equiv p$
   b $p \wedge (p \vee q) \equiv p$

These laws can easily be verified using truth table.

Que 3.9. **Explain various rules of inference for propositional logic.**

OR

Discuss theory of inference in propositional logic.

AKTU 2019-20, Marks 10

Answer

Rules of inference are the laws of logic which are used to reach the given conclusion without using truth table. Any conclusion which can be derived using these laws is called valid conclusion and hence the given argument is valid argument.

1. **Modus ponens (Law of detachment) :** By this rule if an implication $p \to q$ is true and the premise p is true then we can always conclude that q is also true.

   The argument is of the form :

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore \; q \end{array}$$

2. **Modus tollens (Law of contraposition) :** By this rule if an implication $p \to q$ is true and conclusion q is false then the premise p must be false.

   The argument is of the form :

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

3. **Hypothetical syllogism :** By this rule whenever the two implications $p \to q$ and $q \to r$ are true then the implication $p \to r$ is also true.

   The argument is of the form :

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

4. **Disjunctive syllogism :** By this rule if the premises $p \vee q$ and $\sim q$ are true then p is true.
   The argument is of the form :

$$\begin{array}{c} p \vee q \\ \sim q \\ \hline \therefore p \end{array}$$

5. **Addition :** By this rule if p is true then $p \vee q$ is true regardless the truth value of q.
   The argument is of the form :

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

6. **Simplification :** By this rule if $p \wedge q$ is true then p is true.
   The argument is of form :

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array} \text{ or } \begin{array}{c} p \wedge q \\ \hline \therefore q \end{array}$$

7. **Conjunction :** By this rule if p and q are true then $p \wedge q$ is true.
   The argument is of the form :

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

8. **Constructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $p \vee r$ are true then $q \vee s$ is true.
   The argument is of form :

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

9. **Destructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $\sim q \wedge s$ are true.
   The argument is of the form :

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ \sim q \wedge s \\ \hline \therefore \sim p \wedge r \end{array}$$

10. **Absorption :** By this rule if $p \to q$ is true then $p \to (p \wedge q)$ is true.
    The argument is of the form :

$$\begin{array}{c} p \to q \\ \hline \therefore p \to (p \wedge q) \end{array}$$

Que 3.10. What do you mean by valid argument ? Are the following arguments valid ? If valid, construct a formal proof ; if not, explain why.
For students to do well in discrete structure course, it is necessary that they study hard. Students who do well in courses do not skip classes. Student who study hard do well in courses. Therefore students who do well in discrete structure course do not skip class.

Answer

Valid arguments :

1. An argument $P_1, P_2, ..., P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, ..., P_n$ are true.
2. For example : Consider the argument : $p \to q, q \vdash p$.

C	P	P
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

where P denotes the premise and C denotes the conclusion.

3. From the truth table we can see in first and third rows both the premises q and $p \to q$ are true, but the conclusion p is false in third row. Therefore, this is not a valid argument.
4. First and third rows are called critical rows.
5. This method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called truth table technique.
6. Also, we can say the argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if the proposition $P_1 \wedge P_2 \wedge ... \wedge P_n$ is true or we can say if $P_1 \wedge P_2 \wedge ... \wedge P_n \to Q$ is a tautology.
   For example : Consider the argument $p \to q, p \vdash q$.

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

4. **Disjunctive syllogism :** By this rule if the premises $p \vee q$ and $\sim q$ are true then p is true.
The argument is of the form :

$$\begin{array}{c} p \vee q \\ \sim q \\ \hline \therefore p \end{array}$$

5. **Addition :** By this rule if p is true then $p \vee q$ is true regardless the truth value of q.
The argument is of the form :

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

6. **Simplification :** By this rule if $p \wedge q$ is true then p is true.
The argument is of form :

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array} \text{ or } \begin{array}{c} p \wedge q \\ \hline \therefore q \end{array}$$

7. **Conjunction :** By this rule if p and q are true then $p \wedge q$ is true.
The argument is of the form :

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

8. **Constructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $p \vee r$ are true then $q \vee s$ is true.
The argument is of form :

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

9. **Destructive dilemma :** By this rule if $(p \to q) \wedge (r \to s)$ and $\sim q \wedge s$ are true.
The argument is of the form :

$$\begin{array}{c} (p \to q) \wedge (r \to s) \\ \sim q \wedge s \\ \hline \therefore \sim p \wedge r \end{array}$$

10. **Absorption :** By this rule if $p \to q$ is true then $p \to (p \wedge q)$ is true.
The argument is of the form :

$$\begin{array}{c} p \to q \\ \hline \therefore p \to (p \wedge q) \end{array}$$

Que 3.10. **What do you mean by valid argument ? Are the following arguments valid ? If valid, construct a formal proof ; if not, explain why.**
For students to do well in discrete structure course, it is necessary that they study hard. Students who do well in courses do not skip classes. Student who study hard do well in courses. Therefore students who do well in discrete structure course do not skip class.

Answer

Valid arguments :

1. An argument $P_1, P_2, ..., P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, ..., P_n$ are true.
2. For example : Consider the argument : $p \to q, q \vdash p$.

C	P	P
p	q	$p \to q$
$\boxed{T}$	T	$\boxed{T}$
T	F	F
$\boxed{F}$	T	$\boxed{T}$
F	F	T

where P denotes the premise and C denotes the conclusion.

3. From the truth table we can see in first and third rows both the premises q and $p \to q$ are true, but the conclusion p is false in third row. Therefore, this is not a valid argument.
4. First and third rows are called critical rows.
5. This method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called truth table technique.
6. Also, we can say the argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if the proposition $P_1 \wedge P_2 \wedge ... \wedge P_n$ is true or we can say if $P_1 \wedge P_2 \wedge ... \wedge P_n \to Q$ is a tautology.
For example : Consider the argument $p \to q, p \vdash q$.

Then from the truth table :

p	q	$p \to q$	$p \wedge p \to q$	$p \wedge (p \to q) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$p \wedge (p \to q) \to q$ is a tautology since the last column contains T only.

$\therefore$ $p \to q, p \vdash q$ is a valid argument.

Numerical :

Let the propositional variables be :

$p \to$ Do well in the course.

$q \to$ They study hard.

$r \to$ Do not skip classes.

1. For students to do well in discrete structure course, it is necessary that they study hard : $p \to q$
2. Students who do well in the courses do not skip classes : $p \to r$
3. Students who study hard do well in courses : $q \to p$
4. Therefore, students who do well in discrete structure course do not skip classes : $p \to r$

Therefore, we have,

Given : $p \to q$ (I) $\quad p \to r$ (II) $\quad q \to p$ (III) $\quad$ **Conclusion :** $p \to r$ (IV)

Proof : Taking III and II together we get

$q \to p, p \to r$ gives $q \to r$ (V) (Using hypothetical syllogism)

Now taking I and V

$p \to q$ and $q \to r$ we get $p \to r$ (Using hypothetical syllogism)

Hence, $p \to r$ is conclusion, so it is valid.

Yes, the statement is valid.

Que 3.11. **Use rules of inference to justify that the three hypotheses**

i. "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on."

ii. "If the sailing race is held, then the trophy will be awarded."

iii. "The trophy was not awarded." imply the conclusion

iv. "It rained."

AKTU 2021-22, Marks 10

Answer

Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \to (s \wedge l)$, $s \to t$, and $\neg t$. We want to conclude r.

	Step	**Reason**
1.	$\neg t$	Hypothesis
2.	$s \to t$	Hypothesis
3.	$\neg s$	Modus tollens using (1) and (2)
4.	$(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5.	$(\neg (s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6.	$(\neg s \vee \neg l) \to (r \wedge f)$	de Morgan's law and double negative
7.	$\neg s \vee \neg l$	Addition, using (3)
8.	$r \wedge f$	Modus ponens using (6) and (7)
9.	r	Simplification using (8)

Que 3.12.

i. Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

ii. Rewrite the following arguments using quantifiers, variables and predicate symbols :

a. All birds can fly.

b. Some men are genius.

c. Some numbers are not rational.

d. There is a student who likes mathematics but not geography.

OR

Show that $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$ is a tautology without using truth table.

Answer

i. We have

$((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \vee r)$

$\equiv ((p \vee q) \wedge \sim(\sim p \wedge \sim(q \wedge r))) \vee (\sim(p \vee q) \vee \sim(p \vee r))$

(Using de Morgan's Law)

$\equiv [(p \vee q)] \wedge (p \vee (q \wedge r)) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv [(p \vee q) \wedge (p \vee q) \wedge (p \wedge r)] \vee \sim((p \vee q) \wedge (p \vee r))$

(Using Distributive Law)

$\equiv [((p \vee q) \wedge (p \vee q)] \wedge (p \vee r) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv ((p \vee q) \wedge (p \vee r)) \vee \sim((p \vee q) \wedge (p \vee r))$

$\equiv x \vee \sim x$ where $x \equiv (p \vee q) \wedge (p \wedge r)$

$\equiv T$

ii. a. $\forall x\, [B(x) \Rightarrow F(x)]$ b. $\exists x\, [M(x) \wedge G(x)]$

c. $\sim [\, \exists(x)\, (N(x) \wedge R(x)]$ d. $\exists x\, [S(x) \wedge M(x) \wedge \sim G(x)]$

Que 3.13. **Show that $((P \vee Q) \wedge \neg(\neg Q \vee \neg R)) \vee (\neg P \vee \neg Q) \wedge (\neg P \wedge \neg R)$ is tautology by using equivalences.** **AKTU 2019-20, Marks 10**

Answer

$= ((P \vee Q) \wedge \neg(\neg Q \vee \neg R)) \vee (\neg P \vee \neg Q) \wedge (\neg P \wedge \neg R)$

$= ((P \vee Q) \wedge (Q \vee R)) \vee ((\neg P \vee \neg Q) \wedge (\neg P \wedge \neg R))$

$= ((P \wedge Q \wedge R) \vee (Q \wedge Q \wedge R)) \vee ((\neg P \wedge \neg P \wedge \neg R) \vee (\neg Q \wedge \neg P \wedge \neg R))$

$= (P \wedge Q \wedge R) \vee (Q \wedge R) \vee (\neg P \wedge \neg R) \vee (\neg Q \wedge \neg P \wedge \neg R)$

$= (P \vee T)\, (Q \wedge R) \vee (T \vee \neg Q)\, (\neg P \wedge \neg R)$ $[\because\ T \vee P = T]$

$= (Q \wedge R) \vee (\neg P \wedge \neg R)$

So, the given expression is not a tautology.

Que 3.14. **"If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument using truth table.**

Answer

Let p_1 : The labour market is perfect.

p_2 : Wages of all persons in a particular employment will be equal.

$\sim p_2$: Wages for such persons are not equal.

$\sim p_1$: The labour market is not perfect.

The premises are $p_1 \Rightarrow p_2$, $\sim p_2$ and the conclusion is $\sim p_1$. The argument $p_1 \Rightarrow p_2, \sim p_2 \Rightarrow \sim p_1$ is valid if $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology.

Its truth table is,

p_1	p_2	$\sim p_1$	$\sim p_2$	$p_1 \Rightarrow p_2$	$(p_1 \Rightarrow p_2) \wedge \sim p_2$	$(p_1 \Rightarrow p_2 \wedge \sim p_2) \Rightarrow \sim p_1$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since $((p_1 \Rightarrow p_2) \wedge \sim p_2) \Rightarrow \sim p_1$ is a tautology. Hence, this is valid argument.

Que 3.15. **Prove the validity of the following argument. If Mary runs for office, she will be elected. If Mary attends the meeting, she will run for office. Either Mary will attend the meeting or she will go to India. But Mary cannot go to India. "Thus Mary will be elected".** **AKTU 2022-23, Marks 10**

Answer

Let p_1 : Mary run for office

p_2 : She will be elected

p_3 : Mary attends the meeting

p_4 : She will go to India

Conclusion : p_2

Hypothesis becomes

$R_1 : p_1 \to p_2$

$R_2 : p_3 \to p_1$

$R_3 : p_3 \vee p_4$

$R_4 : \sim p_4$

S. No.	Step	Reason
1.	$p_3 \to p_1$	Rule 2
2.	$p_1 \to p_2$	Rule 1
3.	$p_3 \to p_2$	Hypothetical syllogism using Rule 1 and Rule 2
4.	$p_3 \vee p_4$	Rule 3
5.	$\sim p_4$	Rule 4
6.	p_3	Disjunctive syllogism using Rule 3 and Rule 4
7.	p_2	Modus ponens using step 3 and step 6

Que 3.16. **Prove the validity of the following argument "if the races are fixed so the casinos are cooked, then the tourist trade will decline. If the tourist trade decreases, then the police will be happy. The police force is never happy. Therefore, the races are not fixed."** **AKTU 2019-20, Marks 10**

Answer

Let

p : Race are fixed.

q : Casinos are cooked.

r : Tourist trade will decline.

s : Police will be happy.

The above argument can be written in symbolic form as

$$(p \wedge q) \rightarrow r$$
$$r \rightarrow s \sim s$$
$$\therefore \sim p$$

So,

1. $(p \vee q) \rightarrow r$ (Given Premise)
2. $r \rightarrow s$ (Given Premise)
3. $\sim s$ (Given Premise)
4. $\sim r$ Modus tollens using 2 and 3
5. $\sim (p \vee q)$ Modus tollen using 1 and 4
6. $\sim p \wedge \sim q$

Que 3.17. **Prove the validity of the following argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job, or I will not work hard."**

AKTU 2020-21, Marks 10

Answer

Let p : I get the job

q : I work hard

r : I get promoted

s : I will be happy

Then the given arguments can be written in symbolic from as

$$(p \wedge q) \rightarrow r$$
$$r \rightarrow s$$
$$\sim s$$

So

1.	$(p \wedge q) \rightarrow r$	Premise (Given)
2.	$r \rightarrow s$	Premise (Given)
3.	$(p \wedge q) \rightarrow s$	Hypothetical syllogism
4.	$\sim s$	Premise (Given)
5.	$\sim(p \wedge q)$	Modus tollens
6.	$\sim p \vee - q$	Conclusion

Hence the argument is valid.

Que 3.18. **Define tautology, contradiction and contingency ? Check whether $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ is a tautology, contradiction or contingency.**

AKTU 2020-21, Marks 10

Answer

1. **Tautology and contradiction, contingency : Refer Q. 3.7, Page 3–7F, Unit-3.**

Proof : $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$

p	q	r	$\sim P$	$(p \vee q) = A$	$(\sim p \vee r) = B$	$(A \wedge B) = C$	$(q \vee r) = D$	$C \rightarrow D$
F	F	F	T	F	T	F	F	T
F	F	T	T	F	T	F	T	T
F	T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	F	F	F
T	F	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T	T
T	T	T	F	T	T	T	T	T

So, $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ is contingency.

Que 3.19.

i. **Find a compound proposition involving the propositional variables p, q, r and s that is true when exactly three propositional variables are true and is false otherwise.**

ii. **Show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday", "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a canoe trip." and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."**

OR

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip." and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Answer

i. The compound proposition will be : $(p \wedge q \wedge r) \Leftrightarrow s$

ii. Let p be the proposition "It is sunny this afternoon", q be the proposition "It is colder than yesterday", r be the proposition "We will go swimming", s be the proposition "We will take a canoe trip", and t be the proposition "We will be home by sunset".

Then the hypothesis becomes $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$. The conclusion is simply t.

We construct an argument to show that our hypothesis lead to the conclusion as follows :

S. No.	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using step 1
3.	$r \to p$	Hypothesis
4.	$\neg r$	Modus tollens using steps 2 and 3
5.	$\neg r \to s$	Hypothesis
6.	s	Modus ponens using steps 4 and 5
7.	$s \to t$	Hypothesis
8.	t	Modus ponens using steps 6 and 7

Que 3.20. **Obtain the principle disjunction and conjunctive normal forms of the formula $(p \to r) \wedge (q \leftrightarrow p)$.**

AKTU 2019-20, Marks 10

Answer

Principle disjunction normal form :

$(p \to r) \wedge (p \leftrightarrow q)$

$(\sim p \vee r) \wedge ((p \to q) \wedge (p \to q))$ [By using distributive property]

$\downarrow \quad \downarrow \qquad\qquad \downarrow$

$B \quad C \qquad\qquad A$

$(B \wedge A) \vee (C \wedge A)$

$A = (q \to p) \wedge (p \to q) = (\sim q \vee p) \wedge (\sim p \vee q)$

$= (\sim q \vee p) \wedge \sim p \vee (\sim q \vee p) \vee q$

$= (\sim q \wedge \sim p) \vee (p \wedge p) \vee (\sim q \wedge \sim q) \vee (p \wedge q)$

[By using distributive property]

$= (\sim q \wedge \sim p) \vee F \vee F \vee (p \wedge q) = (\sim q \wedge \sim p) \vee (p \wedge q)$

Now, $(B \wedge A) \vee (C \wedge A)$

$= \sim p \wedge ((\sim q \wedge \sim p) \vee (p \wedge q)) \vee (r \wedge (\sim q \wedge \sim p) \vee (p \wedge q))$

$= (\sim p \wedge \sim q \wedge \sim p) \vee (\sim p \wedge p \wedge q)) \vee (r \wedge \sim q \wedge \sim p) \vee (r \wedge p \wedge q))$

$= (\sim q \wedge \sim p) \vee (r \wedge \sim q \wedge \sim p) \vee (r \wedge p \wedge q)$

$= (T \wedge r)(\sim q \wedge \sim p) \vee (r \wedge p \wedge q) = (\sim q \wedge \sim p) \vee (r \wedge p \wedge q)$

Principle conjunctive normal form :

$(p \to r) \wedge (p \leftrightarrow q)$

$(p \to r) \wedge (q \to p) \wedge (p \to q)$

$(\sim p \to r) \wedge (\sim q \to p) \wedge (\sim p \to q)$

Que 3.21. **Show that : $(r \to \sim q, r \vee S, S \to \sim q, p \to q) \leftrightarrow \sim p$ are inconsistent.**

Answer

Following the indirect method, we introduce p as an additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $p \to q$	Rule P
{2}	(2) p	Rule P (assumed)
{1, 2}	(3) q	Rule T, (1), (2) and modus ponens
{4}	(4) $s \to \bar{q}$	Rule P
{1, 2, 4}	(5) $\bar{s}$	Rule T, (3), (4) and modus tollens
{6}	(6) $r \vee s$	Rule P
{1, 2, 4, 6}	(7) r	Rule T, (5), (6) disjunctive syllogism
{8}	(8) $r \to \bar{q}$	Rule P
{8}	(9) $\bar{r} \vee \bar{q}$	Rule T, (8) and EQ_{16} $(p \to q \equiv \bar{p} \vee q)$
{8}	(10) $\overline{r \wedge q}$	Rule T, (8) and De Morgan's law
{1, 2, 4, 6}	(11) $r \wedge q$	Rule T, (7), (3) and conjunction
{1, 2, 4, 6, 8}	(12) $r \wedge q \wedge \overline{r \wedge q}$	Rule T, (10), (11) and conjunction.

Since, we know that set of formula is inconsistent if their conjunction implies contradiction. Hence it leads to a contradiction. So, it is inconsistent.

Que 3.22. **Justify that the following premises are inconsistent :**

i. **If Nirmala misses many classes through illness then she fails high school.**
ii. **If Nirmala fails high school, then she is uneducated.**
iii. **If Nirmala reads a lot of books then she is not uneducated.**
iv. **Nirmala misses many classes through illness and reads a lot of books.**

AKTU 2021-22, Marks 10

Answer

Let,

C = Nirmala misses many classes through illness
F = Nirmala fails high school
ε = Nirmala is uneducated
B = Nirmala reads lot of books

Symbolic representation is

$C \to F, F \to \varepsilon, B \to \neg\,\varepsilon, C \wedge B$ are inconsistent

1.	$C \wedge B$	Rule P
2.	C	Rule T, 1 and Simplification
3.	B	Rule T, 1 and Simplification
4.	$C \to F$	Rule P
5.	$F \to \varepsilon$	Rule P
6.	$C \to \varepsilon$	Rule T, 4, 5 and hypothetical syllogism
7.	$B \to \neg\,\varepsilon$	Rule P
8.	$\neg\,\varepsilon$	Rule T, 3, 7 and modus ponens
9.	ε	Rule T, 2, 6 and modus ponens
10.	$\varepsilon \wedge \neg\,\varepsilon$	Rule T, 8, 9 and conjunction
11.	F	Rule T, 10 and negation law

The set of given premises are inconsistent.

PART-3

Predicate Logic : First Order Predicate, Well Formed Formula of Predicate, Quantifiers, Inference Theory of Predicate Logic.

Que 3.23. Write a short note on

1. First order logic
2. Quantifiers

Answer

1. **First order logic :**
   i. First order logic is the extension of propositional logic by generalizing and quantifying the propositions over given universe of discourse.
   ii. In first order logic every individual has the property p (say).
   iii. It is also called first order predicate calculus.
   iv. Predicate calculus is generalization of propositional calculus. Predicate calculus allows us to manipulate statements about all or something.
   v. Universe of Discourse (UD) : It is the set of all possible values that can be substituted in place of predicate variable.
2. **Quantifiers :** There are following two types of quantifiers :
   i. **Universal quantifier :** Let $p(x)$ be a propositional function defined on set A. Consider the expression.

$$(\forall\, x \in A)\ P(x) \text{ or } \forall\, x\ P(x) \qquad ...(3.23.1)$$

   Here the symbol "$\forall$" is read as "for all" or "for every" and is called universal quantifier. Then the statement (3.23.1) is read as "For every $x \in A$, $P(x)$ is true."
   ii. **Existential quantifier :** Let $Q(x)$ be a propositional function defined on set B. Consider the expression

$$(\exists x \in B)\ Q(x) \text{ or } \exists x\ Q(x) \qquad ...(3.23.2)$$

   Here the symbol "$\exists$" is read as "for some" or "for at least one" or "there exists" and is called existential quantifier.
   Then the statement (3.23.2) is read as "For some $x \in B$, $Q(x)$ is true".

Que 3.24. Explain rules of inference in predicate logic.

Answer

Rule of inference is a logical form consisting of function which takes premises, analyzes their syntax and returns a conclusion.

Rules of inference :

i. **Universal specification :** By this rule if the premise $(\forall\, x)\, P(x)$ is true then $P(c)$ is true where c is particular member of UD.

$$\frac{(\forall x)P(x)}{\therefore P(c)}$$

ii. **Universal generalization :** By this rule if $P(c)$ is true for all c in UD then $(\forall x)\ P(x)$ is true.

$$\frac{P(c)}{\therefore (\forall x)P(x)}$$

x is not free in any of given premises.

iii. Existential specification : By this rule if $(\exists x)\, P(x)$ is true then $P(x)$ is true for some particular member of UD.

$$\frac{(\exists x)P(x)}{\therefore P(c)}$$

c is some member of UD.

iv. Existential generalization : By this rule if $P(c)$ is true for some particular member c in UD, then $(\exists x)\, P(x)$ is true

$$\frac{P(c)}{\therefore (\exists x)P(x)}$$

c is some member of UD.

v. Universal modus ponens : By this rule if $P(x) \to Q(x)$ is true for every x and $P(c)$ is true for some particular member c in UD then $Q(c)$ is true.

$$\frac{\begin{array}{r}(\forall x)\, P(x) \to Q(x)\\ P(c)\end{array}}{\therefore Q(c)}$$

vi. Universal modus tollens : By this rule if $P(x) \to Q(x)$ is true for every x and $\sim Q(c)$ is true for some particular c in UD then $\sim Q(c)$ is true.

$$\frac{\begin{array}{r}(\forall x)\, P(x) \to Q(x)\\ \sim Q(c)\end{array}}{\therefore \ \sim P(c)}$$

Que 3.25. **Write the symbolic form and negate the following statements :**

a. Everyone who is healthy can do all kinds of work.
b. Some people are not admired by everyone.
c. Everyone should help his neighbours, or his neighbours will not help him.

Answer

a. Symbolic form :

Let $P(x)$: x is healthy and $Q(x)$: x do all work

$\forall x(P(x) \to Q(x))$

Negation : $\neg (\forall x\, (P(x) \to Q(x))$

b. Symbolic form :

Let $P(x)$: x is a person

$A(x, y)$: x admires y

The given statement can be written as "There is a person who is not admired by some person" and it is $(\exists x)\,(\exists y)[P(x) \wedge P(y) \wedge \neg A(x, y)]$

Negation : $(\exists x)\,(\exists y)\,[P(x) \wedge P(y) \wedge A(x, y)]$

c. Symbolic form :

Let $N(x, y)$: x and y are neighbours

$H(x, y)$: x should help y

$P(x, y)$: x will help y

The statement can be written as "For every person x and every person y, if x and y are neighbours, then either x should help y or y will not help x" and it is $(\forall x)\,(\forall y)[N(x, y) \to (H(x, y) \ \neg P(y, x))]$

Negation : $(\forall x)\,(\forall y)\,[N(x, y) \to \neg (H(x, y)\, P(y, x))]$

Que 3.26. **Translate the following statements in symbolic form**

i. The sum of two positive integers is always positive.
ii. Everyone is loved by someone.
iii. Some people are not admired by everyone.
iv. If a person is female and is a parent, then this person is someone's mother.

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Answer

i. For two positive integer $P(x)$ and $P(y)$

$P(x)$: x is positive integer

$P(y)$: y is positive integer

$S(x, y)$: $x + y$ is positive integer

$\forall x \forall y\, (P(x) \wedge P(y) \to S(x, y))$

ii. Let x and y are persons

Loves (x, y) : x is loved by y

$\forall x \exists y$ Loves (y, x)

iii. Refer Q. 3.25, Page 3–22F, Unit-3.

iv. $P(x)$: x is parent

$F(x)$: x is female

$M(x, y)$: x is the mother of y

$\forall x((F(x) \wedge P(x) \to \exists y\, M(x, y))$

Que 3.27. **Express the following statements using quantifiers and logical connectives.**

a. Mathematics book that is published in India has a blue cover.

b. All animals are mortal. All human being are animal. Therefore, all human being are mortal.
c. There exists a mathematics book with a cover that is not blue.
d. He eats crackers only if he drinks milk.
e. There are mathematics books that are published outside India.
f. Not all books have bibliographies.

Answer

a. $P(x)$: x is a mathematic book published in India

$Q(x)$: x is a mathematic book of blue cover

$\forall x\, P(x) \rightarrow Q(x)$.

b. $P(x)$: x is an animal

$Q(x)$: x is mortal

$\forall x\ P(x) \rightarrow Q(x)$

$R(x)$: x is a human being

$\therefore\ \forall x\ R(x) \rightarrow P(x)$.

c. $P(x)$: x is a mathematics book

$Q(x)$: x is not a blue color

$\exists x, P(x) \wedge Q(x)$.

d. $P(x)$: x drinks milk

$Q(x)$: x eats crackers

for x, if $P(x)$ then $Q(x)$.

or x, $P(x) \Rightarrow Q(x)$.

e. $P(x)$: x is a mathematics book

$Q(x)$: x is published outside India

$\exists x\ P(x) \wedge Q(x)$.

f. $P(x)$: x is a book having bibliography $\sim \forall x, P(x)$.

Que 3.28.

i. Express this statement using quantifiers : "Every student in this class has taken some course in every department in the school of mathematical sciences".

ii. If $\forall x \exists y P(x, y)$ is true, does it necessarily follow that $\forall y P(x, y)$ is true ? Justify your answer.

Answer

i. $\forall x\ p(x) \Rightarrow \exists\ \forall z\ Q(y, z)$

where $P(x)$ is student of class.

$Q(y, z)$ is the course from department.

ii. Let $P(x, y)$ be $x + y = 3$ and x, y belong to some set of integers $\forall x \exists y P(x, y)$ is true means for all x there exists some y for which $x + y = 3$ is true but for all y we conclude that $P(x, y)$ will not be true.

Que 3.29. **Translate the following sentences in quantified expressions of predicate logic.**
i. All students need financial aid.
ii. Some cows are not white.
iii. Suresh will get if division if and only if he gets first div.
iv. If water is hot, then Shyam will swim in pool.
v. All integers are either even or odd integer.

Answer

i. $\forall x\ [S(x) \Rightarrow F(x)]$

ii. $\sim [\exists(x)\ (C(x) \wedge W(x))]$

iii. Sentence is incorrect so cannot be translated into quantified expression.

iv. $W(x)$: x is water

$H(x)$: x is hot

$S(x)$: x is Shyam

$P(x)$: x will swim in pool

$\forall x\ [((W(x) \wedge H(x)) \Rightarrow (S(x) \wedge P(x))]$

v. $E(x)$: x is even

$O(x)$: x is odd

$\forall x\ (E(x) \vee O(x))$

Que 3.30. **Convert the following two statements in quantified expressions of predicate logic**
i. For every number there is a number greater than that number.
ii. Sum of every two integer is an integer.
iii. Not Every man is perfect.
iv. There is no student in the class who knows Spanish and German.

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Answer

i. $P(x)$: x is a number

$Q(y)$: y is a number greater than x

$\forall x\,(P(x) \Rightarrow Q(y))$

ii. $P(x)$: x is a integer

$S(x)$: x is sum of integer

$\forall x\,(S(x) \Rightarrow P(x))$

iii. $P(x)$: x is perfect man

$\sim \forall x\,(P(x))$

iv. $P(x)$: x is a student

$L(x)$: x knows Spanish and German

$\exists x\,(P(x) \vee L(x))$

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4 UNIT

Algebraic Structures

CONTENTS

PART - 1

Algebraic Structures : Definition, Groups, Subgroups and Order.

Que 4.1. **What is algebraic structure ? List properties of algebraic system.**

Answer

Algebraic structure : An algebraic structure is a non-empty set G equipped with one or more binary operations. Suppose * is a binary operation on G. Then $(G, *)$ is an algebraic structure.

Properties of algebraic system : Let $\{S, *, +\}$ be an algebraic structure where * and + binary operation on S :

1. **Closure property :** $(a * b) \in S \quad \forall a, b, \in S$
2. **Associative property :** $(a * b) * c = a * (b * c) \; \forall a, b, c \in S$
3. **Commutative property :** $(a * b) = (b * a) \quad \forall a, b \in S$
4. **Identity element :** $\exists e \in S$ such that $a * e = a$ (right identity) $\forall a \in S$, e is called identity element of S with respect to operation *.
5. **Inverse element :** For every $a \in S$, $\exists a^{-1} \in S$ such that
$$a * a^{-1} = e = a^{-1} * a$$
here a^{-1} is called inverse of 'a' under operation *.
6. **Cancellation property :**
$a * b = a * c \Rightarrow b = c$ and $b * a = c * a \Rightarrow b = c \; \forall a, b, c \in S$ and $a \neq 0$
7. **Distributive property :** $\forall a, b, c \in S$
$a * (b + c) = (a * b) + (a * c)$ (right distributive)
$(b + c) * a = (b * a) + (c * a)$ (left distributive)
8. **Idempotent property :** An element $a \in S$ is called idempotent element with respect to operation * if $a * a = a$.

Que 4.2. **Write short notes on :**

i. **Group**
ii. **Abelian group**
iii. **Finite and infinite group**
iv. **Order of group**
v. **Groupoid**

Answer

i. **Group :** Let $(G, *)$ be an algebraic structure where * is binary operation then $(G, *)$ is called a group if following properties are satisfied :
   1. $a * b \in G \; \forall \; a, b \in G$ [closure property]

   2. $a * (b * c) = (a * b) * c \quad \forall a, b, c \in G$ [associative property]
   3. There exist an element $e \in G$ such that for any $a \in G$
   $a * e = e * a = e$ [existence of identity]
   4. For every $a \in G$, $\exists$ element $a^{-1} \in G$
   such that $a * a^{-1} = a^{-1} * a = e$

   For example : $(Z, +)$, $(R, +)$, and $(Q, +)$ are all groups.

ii. **Abelian group :** A group $(G, *)$ is called abelian group or commutative group if binary operation * is commutative *i.e.*, $a * b = b * a \; \forall \; a, b \in G$

   For example : $(Z, +)$ is an abelian group.

iii. **Finite group :** A group $\{G, *\}$ is called a finite group if number of elements in G are finite. For example $G = \{0, 1, 2, 3, 4, 5\}$ under $\otimes_6$ is a finite group.

   Infinite group : A group $\{G, *\}$ is called infinite group if number of element in G are infinite.

   For example, $(Z, +)$ is infinite group.

iv. **Order of group :** Order of group G is the number of elements in group G. It is denoted by o (G) or $|G|$. A group of order 1 has only the identity element.

v. **Groupoid :** Let $(S, *)$ be an algebraic structure in which S is a non-empty set and * is a binary operation on S. Thus, S is closed with the operation *. Such a structure consisting of a non-empty set S and a binary operation defined in S is called a groupoid.

Que 4.3. **Describe subgroup with example.**

Answer

If $(G, *)$ is a group and $H \subseteq G$. The $(H, *)$ is said to subgroup of G if $(H, *)$ is also a group by itself. *i.e.*,

(1) $a * b \in H \quad \forall a, b \in H$ (Closure property)

(2) $\exists e \in H$ such that $a * e = a = e * a \; \forall a \in H$

Where e is called identity of G.

(3) $\exists a^{-1} \in H$ such that $a * a^{-1} = e = a^{-1} * a \; \forall a \in H$

For example : The set Q^+ of all non-zero +ve rational number is subgroup of $Q - \{0\}$.

Que 4.4. **Show that the set $G = \{x + y\sqrt{2} \mid x, y \in Q\}$ is a group with respect to addition.**

Answer

i. **Closure :**

Let $X = x_1 + \sqrt{2}\, y_1$

$Y = x_2 + \sqrt{2}\, y_2$

where $x_1, x_2, y_1, y_2 \in Q$ and $X, Y \in G$

Then $X + Y = (x_1 + \sqrt{2}\, y_1) + (x_2 + \sqrt{2}\, y_2)$

$= (x_1 + x_2) + (y_1 + y_2)\sqrt{2}$

$= X_1 + \sqrt{2} Y_1 \in G$ where $X_1, Y_1 \in Q$

Therefore, G is closed under addition [$\because$ Sum of two rational numbers is rational].

ii. **Associativity :**

Let X, Y and $Z \in G$

$\therefore$ $X = x_1 + \sqrt{2}\, y_1\ Y = x_2 + \sqrt{2}\, y_2\ Z = x_3 + \sqrt{2}\, y_3$

where $x_1, x_2, x_3, y_1, y_2, y_3 \in Q$

Consider $(X + Y) + Z = (x_1 + \sqrt{2}\, y_1 + x_2 + \sqrt{2}\, y_2) + (x_3 + \sqrt{2}\, y_3)$

$= ((x_1 + x_2) + (y_1 + y_2)\sqrt{2}) + (x_3 + \sqrt{2}\, y_1)$

$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)\sqrt{2}$ ...(4.4.1)

Also $X + (Y + Z) = (x_1 + \sqrt{2}\, y_1) + ((x_2 + \sqrt{2}\, y_2) + (x_3 + \sqrt{2}\, y_3))$

$= (x_1 + \sqrt{2}\, y_1) + ((x_2 + x_3) + (y_2 + y_3)\sqrt{2})$

$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)\sqrt{2}$ ...(4.4.2)

From eq. (4.4.1) and (4.4.2)

$(X + Y) + Z = X + (Y + Z)$

Therefore, G is associative under addition.

iii. **Identity element :**

Let $e \in G$ be identity elements of G under addition then

$(x + y\sqrt{2}) + (e_1 + e_2\sqrt{2}) = x + y\sqrt{2}$

where $e = e_1 + e_2\sqrt{2}$ and $e_1, e_2, x, y \in Q$

$e_1 + e_2\sqrt{2} = 0 + 0\sqrt{2}$

$e_1 = 0$ and $e_2 = 0$

Therefore, $0 \in G$ is identity element.

iv. **Inverse element :**

$-x - y\sqrt{2} \in G$ is inverse of $x + y\sqrt{2} \in G$.

Therefore, inverse exist for every element $x + y\sqrt{2} \in G$ such that, $y \in Q$.

Hence, G is a group under addition.

Que 4.5. **Let H be a subgroup of a finite group G. Prove that order of H is a divisor of order of G.**

Answer

1. Let H be any sub-group of order m of a finite group G of order n. Let us consider the left coset decomposition of G relative to H.
2. We will show that each coset aH consists of m different elements.
   Let $H = \{h_1, h_2, \ldots, h_m\}$
3. Then $ah_1, ah_2, \ldots, ah_m$, are the members of aH, all distinct.
   For, we have
   $$ah_i = ah_j \Rightarrow h_i = h_j$$
   by cancellation law in G.
4. Since G is a finite group, the number of distinct left cosets will also be finite, say k. Hence the total number of elements of all cosets is k_m which is equal to the total number of elements of G.
   Hence
   $$n = mk$$
   This show that m, the order of H, is a divisor of n, the order of the group G.
   We also find that the index k is also a divisor of the order of the group.

Que 4.6. **Define identity and zero elements of a set under a binary operation *. What do you mean by an inverse element ?**

Answer

Identity element : An element e in a set S is called an identity element with respect to the binary operation * if, for any element a in S

$$a * e = e * a = a$$

If $a * e = a$, then e is called the right identity element for the operation * and if $e * a = a$, then e is called the left identity element for the operation *.

Zero element : Let R be an abelian group with respect to addition. The element $0 \in R$ will be the additive identity. It is called the zero element of R.

Inverse element : Consider a set S having the identity element e with respects to the binary operation *. If corresponding to each element $a \in S$ there exists an element $b \in S$ such that

$$a * b = b * a = e$$

Then b is said to be the inverse of a and is usually denoted by a^{-1}. We say a is invertible.

Que 4.7. **Define the binary operation * on Z by $x * y = x + y + 1$ for all x, y belongs to set of integers. Verify that (Z, *) is abelian group ? Discuss the properties of abelian group.**

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Answer

i. **Closure property :** Let $x, y \in Z$

$x * y = x + y + 1 \in Z$

as $xy \neq 0$

$\therefore$ * is closure in Z

ii. **Associativity :** Let $x, y, c, \in Z$

Consider $x * (y + z)$

$\Rightarrow \quad x * (y + z + 1)$

$\Rightarrow x + y + z + 1 + 1 \Rightarrow x + y + z + 2$

$(x + y) * z = (x + y + 1) + z = z + y + 1 + z + 1$

$= x + y + z + 2$

$\Rightarrow$ * is associative in Z.

iii. **Existence of the identity :** Let $x \in Z$ and $\exists c$ such that

$x + c = x + e + 1 = x + 1$

$\therefore$ 1 is the identity element.

iv. **Existence of the inverse :** Let $x \in Z$ and $y \in Z$

Such that $x + y = e = 2$

$x + y + 1 = 2$

$x + y = 1$

$\therefore$ The inverse of $x = 1 - y, \ \forall x \in Z$

v. **Commutative :** Let $x, y \in Z$

$x * y = x + y + 1$

$y * x = y + x + 1$

$\therefore$ * is commutative.

Thus, $(Z, *)$ is an abelian group.

Properties of abelian group : An abelian group G is a group for which the element pair $(a, b) \in G$ always holds commutative law. An abelian group satisfies five properties : Closure Property, Associative Property, Identity Property, Inverse Property, and Commutative Property.

Que 4.8. **Prove that $(Z_6, (+_6))$ is an abelian group of order 6, where $Z_6 = \{0, 1, 2, 3, 4, 5\}$.**

Answer

The composition table is :

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Since $\quad 2 +_6 1 = 3$

$4 +_6 5 = 3$

From the table we get the following observations :

Closure : Since all the entries in the table belong to the given set Z_6. Therefore, Z_6 is closed with respect to addition modulo 6.

Associativity : The composition '$+_6$' is associative. If a, b, c are any three elements of Z_6,

$a +_6 (b +_6 c) = a +_6 (b + c) \ [\because b +_6 c = b + c \ (\text{mod } 6)]$

= least non-negative remainder when $a + (b + c)$ is divided by 6.

= least non-negative remainder when $(a + b) + c$ is divided by 6.

$= (a + b) +_6 c = (a +_6 b) +_6 c.$

Identity : We have $0 \in Z_6$. If a is any element of Z_6, then from the composition table we see that

$0 +_6 a = a = a +_6 0$

Therefore, 0 is the identity element.

Inverse : From the table we see that the inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively. For example $4 +_6 2 = 0 = 2 +_6 4$ implies 4 is the inverse of 2.

Commutative : The composition is commutative as the elements are symmetrically arranged about the main diagonal. The number of elements in the set Z_6 is 6.

$\therefore \quad (Z_6, +_6)$ is a finite abelian group of order 6.

Que 4.9. **Let $G = \{1, -1, i, -i\}$ with the operation of ordinary multiplication on G be a algebraic structure, where $i = \sqrt{-1}$.**

i. Determine whether G is abelian.

ii. Determine the order of each element in G.

iii. Determine whether G is a cyclic group, if G is a cyclic group, then determine the generator/generators of the group G.

iv. Determine a subgroup of the group G.

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OR

Let $G = \{1, -1, i, -i\}$ with the binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$. Determine whether G is an abelian or not.

Answer

i. The composition table of G is

*	1	−1	i	$-i$
1	1	−1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

1. **Closure property :** Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.
2. **Associativity :** The elements of G are complex numbers, and we know that multiplication of complex numbers is associative.
3. **Identity :** Here, 1 is the identity element.
4. **Inverse :** From the composition table, we see that the inverse elements of $1, -1, i, -i$ are $1, -1, -i, i$ respectively.
5. **Commutativity :** The corresponding rows and columns of the table are identical. Therefore the binary operation is commutative. Hence, $(G, *)$ is an abelian group.

ii. $G = \{1, -1, i, -i\}$

Order of 1 : $(1)^1 = 1$, $O(1) = 1$

Order of − 1 : $(-1)^1 = -1$

$(-1)^2 = -1 \times -1 = 1$, $O(-1) = 2$

Order of i :

$(i)^1 = i$

$(i)^2 = \sqrt{-1} \times \sqrt{-1} = -1$

$(i)^3 = i^2 \times i = -i$

$(i)^4 = i^2 \times i^2$

$= (-i)(-i) = 1$

$O(i) = 4$

Order of $-i$:

$(-i)^1 = -i$

$(-i)^2 = -1$

$(-i)^3 = -1 \times -i = i$

$(-i)^4 = -1 \times -1 = 1$

$O(-i) = 4$

iii. As N and M are subgroups of G then $N \cap M$ is a subgroup of G. Let $g \in G$ and $a \in N \cap M$

$a \in N$ and $a \in M$.

Since N is normal subgroup of G, $gag^{-1} \in N$

Since M is normal subgroup of G, $gag^{-1} \in M$

$\therefore$ $gag^{-1} \in N \cap M$ is a normal subgroup of G.

Hence $N \cap M$ is a normal subgroup of G.

iv. $H = \{1, -1\}$ is the subgroup of group G.

Que 4.10. Write the properties of group. Show that the set {1, 2, 3, 4, 5} is not group under addition and multiplication modulo 6.

Answer

Properties of group : Refer Q. 4.2(i), Page 4–2F, Unit-4.

Numerical :

Addition modulo 6 ($+_6$) : Composition table of $S = \{1, 2, 3, 4, 5\}$ under operation $+_6$ is given as :

$+_6$	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Since, $1 +_6 5 = 0$ but $0 \notin S$ *i.e.*, S is not closed under addition modulo 6.

So, S is not a group.

Multiplication modulo 6 ($*_6$) :

Composition table of $S = \{1, 2, 3, 4, 5\}$ under operation $*_6$ is given as

$*_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Since, $2 *_6 3 = 0$ but $0 \notin S$ i.e., S is not closed under multiplication modulo 6. So, S is not a group.

Que 4.11. **Let $G = \{a, a^2, a^3, a^4, a^5, a^6 = e\}$. Find the order of every element.**

Answer

$$o(a) : a^6 = e \to o(a) = 6$$
$$o(a^2) : (a^2)^3 = a^6 = e \to o(a^2) = 3$$
$$o(a^3) : (a^3)^2 = a^6 = e \to o(a^3) = 2$$
$$o(a^4) : (a^4)^3 = a^{12} = (a^6)^2 = e^2 = e \to o(a^4) = 3$$
$$o(a^5) : (a^5)^6 = a^{30} = (a^6)^5 = e^5 = e \to o(a^5) = 6$$
$$o(a^6) : (a^6)^1 = a^6 = e \to o(a^6) = 1$$

Que 4.12. **Let G be a group and let $a, b \in G$ be any elements. Then**

i. $(a^{-1})^{-1} = a$

ii. $(a * b)^{-1} = b^{-1} * a^{-1}$.

OR

In a group (G, *) prove that

i. $(a^{-1})^{-1} = a$

ii. $(ab)^{-1} = b^{-1}a^{-1}$.

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Answer

i. Let e be the identity element for * in G.

Then we have $a * a^{-1} = e$, where $a^{-1} \in G$.

Also $(a^{-1})^{-1} * a^{-1} = e$

Therefore, $(a^{-1})^{-1} * a^{-1} = a * a^{-1}$.

Thus, by right cancellation law, we have $(a^{-1})^{-1} = a$.

ii. Let a and $b \in G$ and G is a group for *, then $a * b \in G$ (closure)

Therefore, $(a * b)^{-1} * (a * b) = e$. ....(4.12.1)

Let a^{-1} and b^{-1} be the inverses of a and b respectively, then a^{-1}, $b^{-1} \in G$.

Therefore, $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$ (associativity)

$= b^{-1} * e * b = b^{-1} * b = e$ ...(4.12.2)

From (4.12.1) and (4.12.2), we have

$$(c * b)^{-1} * (a * b) = (b^{-1} * a^{-1}) * (a * b)$$
$$(a * b)^{-1} = b^{-1} * a^{-1} \quad \text{(by right cancellation law)}$$

Que 4.13.

i. Justify that "The intersection of any two subgroup of a group (G, *) is again a subgroup of (G, *)".

ii. Justify that "If a, b are the arbitrary elements of a group G then $(ab)^2 = a^2b^2$ if and only if G is abelian.

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OR

Prove that the intersection of two subgroups of a group is also subgroup.

Answer

i. Let H_1 and H_2 be any two subgroups of G. Since at least the identity element e is common to both H_1 and H_2.

$$\therefore \quad H_1 \cap H_2 \neq \phi$$

In order to prove that $H_1 \cap H_2$ is a subgroup, it is sufficient to prove that $a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1, H_2 are subgroups. Therefore,

$a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally, $ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Thus, we have shown that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Hence, $H_1 \cap H_2$ is a subgroup of G.

ii. Let G is an abelian group

$$ab = ba \quad \forall\, a, b \in G$$

Consider
$$(ab)^2 = (ab)(ab)$$
$$= a(ba)b \quad \text{[By associativity law]}$$
$$= a(ab)b \quad [\because G \text{ is an abelian } ba = ab]$$
$$= (aa)(bb)$$
$$(ab)^2 = a^2b^2$$

Conversely, let $(ab)^2 = a^2b^2 \;\; \forall\, a, b \in G$

$$(ab)(ab) = (aa)(bb)$$
$$a(ba)b = a(ab)b$$
$$ba = ab \quad \text{[Cancellation law]}$$
$$ba = ab \;\; \forall\, a, b \in G$$

Que 4.14. **Let G be the set of all non-zero real number and let $a * b = ab/2$. Show that $(G\, *)$ be an abelian group.**

Answer

i. **Closure property :** Let $a, b \in G$.

$$a * b = \frac{ab}{2} \in G \text{ as } ab \neq 0$$

$\Rightarrow$ * is closure in G.

ii. **Associativity :** Let $a, b, c \in G$

Consider $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{abc}{4}$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4} = \frac{abc}{4}$$

$\Rightarrow$ * is associative in G.

iii. **Existence of the identity :** Let $a \in G$ and $\exists\, e$ such that

$$a * e = \frac{ae}{2} = a$$

$\Rightarrow$ $ae = 2a$

$\Rightarrow$ $e = 2$

$\therefore$ 2 is the identity element in G.

iv. **Existence of the inverse :** Let $a \in G$ and $b \in G$ such that $a * b = e = 2$

$\Rightarrow$ $\frac{ab}{2} = 2$

$\Rightarrow$ $ab = 4$

$\Rightarrow$ $b = \frac{4}{a}$

$\therefore$ The inverse of a is $\frac{4}{a}, \forall\, a \in G$.

v. **Commutative :** Let $a, b \in G$

$$a * b = \frac{ab}{2}$$

and $b * a = \frac{ba}{2} = \frac{ab}{2}$

$\Rightarrow$ * is commutative.

Thus, $(G, *)$ is an abelian group.

Que 4.15. **Prove that inverse of each element in a group is unique.**

Answer

Let (if possible) b and c be two inverses of element $a \in G$.

Then by definition of group :

$$b * a = a * b = e$$

and $a * c = c * a = e$

where e is the identity element of G

Now
$$\begin{aligned} b = e * b &= (c * a) * b \\ &= c * (a * b) \\ &= c * c \\ &= c \\ b &= c \end{aligned}$$

Therefore, inverse of an element is unique in $(G, *)$.

PART-2
Cyclic Group.

Que 4.16. **Define cyclic group with suitable example.**

Answer

Cyclic group : A group G is called a cyclic group if $\exists$ at least one element a in G such that every element $x \in G$ is of the form a^n, where n is some integer. The element $a \in G$ is called the generator of G.

For example :

Show that the multiplicative group $G = \{1, -1, i, -i\}$ is cyclic. Also find its generators.

We have, $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$

and $(-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i, (-i)^4 = 1$

Thus, every element in G be expressed as i^n or $(-i)^n$

$\therefore$ G is cyclic group and its generators are i and $-i$.

Que 4.17. **Prove that every group of prime order is cyclic.**

Answer

1. Let G be a group whose order is a prime p.
2. Since $P > 1$, there is an element $a \in G$ such that $a \neq e$.
3. The group $<a>$ generated by 'a' is a subgroup of G.
4. By Lagrange's theorem, the order of 'a' divides $|G|$.
5. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$ we have $|<a>| >$ 1, so $|<a>| = p$.
6. Hence, $<a> = G$ and G is cyclic.

Que 4.18. **Show that every group of order 3 is cyclic.**

Answer

1. Suppose G is a finite group whose order is a prime number p, then to prove that G is a cyclic group.
2. An integer p is said to be a prime number if $p \neq 0, p \neq \pm 1$, and if the only divisors of p are $\pm 1, \pm p$.
3. Some G is a group of prime order; therefore G must contain at least 2 elements. Note that 2 is the least positive prime integer.
4. Therefore, there must exist an element $a \in G$ such that $a \neq$ the identity element e.
5. Since a is not the identity element, therefore $o(a)$ is definitely ≥ 2. Let $o(a) = m$. If H is the cyclic subgroup of G generated by a then $o\,(H = o\,(a) = m)$.
6. By Lagrange's theorem m must be a divisor of p. But p is prime and $m \geq 2$. Hence, $m = p$.
7. $\therefore H = G$. Since H is cyclic therefore G is cyclic and a is a generator of G.

Que 4.19. **Show that group $(G, +_5)$ is a cyclic group where $G = \{0, 1, 2, 3, 4\}$. What are its generators ?**

Answer

Composition table of G under operation $+_5$ is shown in Table 4.19.1.

Table 4.19.1.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

0 is the identity element of G under $+_5$.

$$1^1 = 1 \equiv 1 \bmod 5$$
$$1^2 = 1 + 1 \equiv 2 \bmod 5$$
$$1^3 = 1 + 1 + 1 \equiv 3 \bmod 5$$
$$1^4 = 1 + 1 + 1 + 1 \equiv 4 \bmod 5$$
$$1^5 = 1 = 1 + 1 + 1 + 1 + 1 \equiv 0 \bmod 5$$

where 1^n means 1 is added n times

Therefore, 1 is generator of G.

Similarly, 2, 3, 4 are also generators of G.

Therefore, G is a cyclic group with generator 1, 2, 3, 4.

Que 4.20. **Show that $G = [(1, 2, 4, 5, 7, 8), X_9]$ is cyclic. How many generators are there ? What are they ?**

Answer

Composition table for X_9 is

X_9	1	2	4	5	7	8
1	2	3	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

1 is identity element of group G

$$2^1 = 2 \equiv 2 \bmod 9$$
$$2^2 = 4 \equiv 4 \bmod 9$$
$$2^3 = 8 \equiv 8 \bmod 9$$
$$2^4 = 16 \equiv 7 \bmod 9$$
$$2^5 = 32 \equiv 5 \bmod 9$$
$$2^6 = 64 \equiv 1 \bmod 9$$

Therefore, 2 is generator of G. Hence G is cyclic.

Similarly, 5 is also generator of G.

Hence there are two generators 2 and 5.

PART-3

Cosets, Lagrange's Theorem.

Que 4.21. **Define cosets. Write and prove properties of cosets.**

Answer

Let H be a subgroup of group G and let $a \in G$ then the set $Ha = \{ha : h \in H\}$ is called right coset generated by H and a.

Also the set $aH = \{ah : h \in H\}$ is called left coset generated by a and H.

Properties of cosets : Let H be a subgroup of G and let a and b belong to G. Then

1. $a \in aH$

   Proof : $a = ae \in aH$

   Since e is identity element of G.

2. $aH = H$ iff $a \in H$.

   Proof : Let $aH = H$.

   Then $a = ae \in aH = H$ (e is identity in G and so is in H)

   $\Rightarrow a \in H$

3. $aH = bH$ or $aH \cap bH = \phi$

   Proof : Let $aH = bH$ or $aH \cap bH = \phi$ and to prove that $aH = bH$.

   Let $aH \cap bH$

   Then there exists $h_1, h_2 \in H$ such that

   $x = ah_1$ and $x = bh_2$

   $a = xh_1^{-1} = bh_2 h_1^{-1}$

   Since H is a subgroup, we have $h_2 h_1^{-1} \in H$

   let $h_2 h_1^{-1} = h \in H$

   Now, $aH = bh_2 h_1^{-1} H = (bh) H = b\,(hH) = bh$ ($\because hH = H$ by property 2)

   $\therefore \quad aH = bH$ if $aH \cap bH \neq \phi$

   Thus, either $aH \cap bH = \phi$ or $aH = bH$.

4. $aH = bH$ iff $a^{-1} b \in H$.

   Proof : Let

   $$aH = bH.$$
   $$a^{-1} aH = a^{-1} bH$$
   $$eH = a^{-1} bH$$
   $$H = (a^{-1} b) H \quad (e \text{ is identity in } G)$$

   Therefore by property (2); $a^{-1} b \in H$.

   Conversely, now if $a^{-1} b \in H$.

   Then consider $bH = e\,(bH) = (a\,a^{-1})\,(bH)$

   $$= a\,(1^{-1} b)\,H$$
   $$= aH$$

   Thus $\quad aH = bH$ iff $a^{-1} b \in H$.

5. aH is a subgroup of G iff $a \in H$.

   Proof : Let aH is a subgroup of G then it contains the identity e of G.

   Thus, $aH \cap eH \neq \phi$

   then by property (3) ; $aH = eH = H$

$aH = H \Rightarrow a \in H$

Conversely, if $a \in H$ then by property (2); $aH = H$.

Que 4.22. **State and prove Lagrange's theorem for group. Is the converse true ?**

OR

State and prove Lagrange theorem for group.

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Answer

Lagrange's theorem :

Statement : The order of each subgroup of a finite group is a divisor of the order of the group.

Proof : Let G be a group of finite order n. Let H be a subgroup of G and let $O(H) = m$. Suppose $h_1, h_2, \ldots, h_m$ are the m members of H.

Let $a \in G$, then Ha is the right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, \ldots h_m a\}$$

Ha has m distinct members, since $h_i a = h_j a \Rightarrow h_i = h_j$

Therefore, each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint *i.e.*, they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say, equal to k. The union of these k distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, \ldots, Ha_k$ are the k distinct right cosets of H in G. Then $G = Ha_1 \cup Ha_2 \cup Ha_3 \cup \ldots \cup Ha_k$

$\Rightarrow$ the number of elements in G = the number of elements in Ha_1 + ... + the number of elements in Ha_2 + ...+ the number of elements in Ha_k

$$\Rightarrow \quad O(G) = km$$
$$\Rightarrow \quad n = km$$
$$\Rightarrow \quad k = \frac{n}{m}$$

$\Rightarrow$ m is a divisor of n.

$\Rightarrow$ $O(H)$ is a divisor of $O(G)$.

Proof of converse : If G be a finite group of order n and $n \in G$, then

$$a^n = e$$

Let o $(a) = m$ which implies $a^m = e$.

Now, the subset H of G consisting of all the integral power of a is a subgroup of G and the order of H is m.

Then, by the Lagrange's theorem, m is divisor of n.

Let $n = mk$, then

$$a^n = a^{mk} = (a^m)^k = e^k = e$$

$\therefore$ Yes, the converse is true.

Que 4.23. **State and explain Lagrange's theorem.**

Answer

Lagrange's theorem :

If G is a finite group and H is a subgroup of G then $o(H)$ divides $o(G)$. Moreover, the number of distinct left (right) cosets of H in G is $o(G)/o(H)$.

Proof : Let H be subgroup of order m of a finite group G of order n.

Let $H \{h_1, h_2, ..., h_m\}$

Let $a \in G$. Then aH is a left coset of H in G and $aH = \{ah_1, ah_2, ..., ah_m\}$ has m distinct elements as $ah_i = ah_j \Rightarrow h_i = h_j$ by cancellation law in G.

Thus, every left coset of H in G has m distinct elements.

Since G is a finite group, the number of distinct left cosets will also be finite. Let it be k. Then the union of these k-left cosets of H in G is equal to G.

i.e., if $a_1H, a_2H, ..., a_kH$ are right cosets of H in G then

$$G = a_1H \cup a_2H \cup ... \cup a_kH.$$

$\therefore$ $o(G) = o(a_1H) + o(a_2H) + ... + o(a_kH)$

(Since two distinct left cosets are mutually disjoint.)

$\Rightarrow$ $n = m + m + ... + m$ (k times)

$\Rightarrow$ $n = mk \;\Rightarrow\; k = \dfrac{n}{m}$

$\therefore$ $k = \dfrac{o(G)}{o(H)}$.

Thus order of each subgroup of a finite group G is a divisor of the order of the group.

Cor 1 : If H has m different cosets in G then by Lagrange's theorem :

$$o(G) = m\, o(H)$$

$\Rightarrow$ $m = \dfrac{o(G)}{o(H)}$

$\therefore$ $[G : H] = \dfrac{o(G)}{o(H)}$

Cor 2 : If $|G| = n$ and $a \in G$ then $a^n = e$

Let $|a| = m \;\Rightarrow\; a^m = e$

Now, the subset H of G consisting of all integral powers of a is a subgroup of G and the order of H is m.

Then by Lagrange's theorem, m is divisor of n.

Let $n = mk$, then

$$a^n = a^{mk} = (a^m)^k = e^k = e$$

Que 4.24. **Explain cyclic group. Let H be a subgroup of a finite group G. Justify the statement "the order of H is a divisor of the order of G".**

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Answer

Cyclic group : Refer Q. 4.16, Page 4–13F, Unit-4.

Proof : Let G be a group of finite order n. Let H be a subgroup of G and let $O(H) = m$. Suppose $h_1, h_2, ..., h_m$ are the m members of H.

Let $a \in G$, then Ha is the right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, ... h_m a\}$$

Ha has m distinct members, since $= h_i a = h_j a \Rightarrow h_i = h_j$

Therefore, each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint *i.e.*, they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say, equal to k. The union of these k distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, ..., Ha_k$ are the k distinct right cosets of H in G. Then $G = Ha_1 \cup Ha_2 \cup Ha_3 \cup .... \cup Ha_k$

$\Rightarrow$ The number of elements in G = the number of elements in Ha_1 + ... + the number of elements in Ha_2 + ... + the number of elements in Ha_k

$\Rightarrow$ $O(G) = km$

$\Rightarrow$ $n = km$

$\Rightarrow$ $k = \dfrac{n}{m}$

$\Rightarrow$ m is a divisor of n.

$\Rightarrow$ $O(H)$ is a divisor of $O(G)$.

Que 4.25.

a. **Prove that every cyclic group is an abelian group.**
b. **Obtain all distinct left cosets of {(0), (3)} in the group $(Z_6, +_6)$ and find their union.**
c. **Find the left cosets of {[0], [3]} in the group $(Z_6, +_6)$.**

Answer

a. Let G be a cyclic group and let a be a generator of G so that

$$G = \langle a \rangle = \{a^n : n \in Z\}$$

If g_1 and g_2 are any two elements of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$. Then

$$g_1 g_2 = a^r\, a^s = a^{r+s} = a^{s+r} = a^s \,.\, a^r = g_2 g_1$$

So, G is abelian.

b. $\therefore$ $[0] + H = [3] + H$, $[1] + [4] + H$ and $[2] + H = [5] + H$ are the three distinct left cosets of H in $(Z_6, +_6)$.

We would have the following left cosets :

$$g_1 H = \{g_1\, h, h \in H\}$$
$$g_2 H = \{g_2\, h, h \in H\}$$
$$g_n H = \{g_n\, h, h \in H\}$$

The union of all these sets will include all the g's, since for each set

$$g_k = \{g_k\, h, h \in H\}$$

we have $g_e \in g_k = \{g_k\, h, h \in H\}$

where e is the identity.

Then if we make the union of all these sets we will have at least all the elements of g. The other elements are merely g_h for some h. But since $g_h \in G$ they would be repeated elements in the union. So, the union of all left cosets of H in G is G, *i.e.*,

$$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$$

c. Let $Z_6 = \{[0], [1], [2], [3], [4], [5]\}$ be a group.

$H = \{[0], [3]\}$ be a subgroup of $(Z_6, +_6)$.

The left cosets of H are,

$$[0] + H = \{[0], [3]\}$$
$$[1] + H = \{[1], [4]\}$$
$$[2] + H = \{[2], [5]\}$$
$$[3] + H = \{[3], [0]\}$$
$$[4] + H = \{[4], [1]\}$$
$$[5] + H = \{[5], [2]\}$$

Que 4.26. **Write and prove the Lagrange's theorem. If a group $G = \{....,-3, 2, -1, 0, 1, 2, 3,.....\}$ having the addition as binary operation. If H is a subgroup of group G where $x^2 \in H$ such that $x \in G$. What is H and its left coset w.r.t 1 ?**

Answer

Lagrange's theorem : Refer Q. 4.23, Page 4–18F, Unit-4.

Numerical :

$$H = \{x^2 : x \in G\} = \{0, 1, 4, 9, 16, 25 ....\}$$

Left coset of H will be $1 + H = \{1, 2, 5, 10, 17, 26, ...\}$

Que 4.27. **What do you mean by cosets of subgroup ? Consider the group Z of integers under addition and the subgroup $H = \{..., -10, -5, 0, 5, 10, ...\}$ considering the multiple of 5.**
i. Find the cosets of H in Z.
ii. What is index of H in Z ?

Answer

Coset : Refer Q. 4.21, Page 4–15F, Unit-4.

Numerical :

i. We have $Z = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ...\}$

and $H = \{...., -10, -5, 0, 5, 10, ...\}$

Let $0 \in Z$ and the right cosets are given as

$$H = H + 0 = \{...., -10, -5, 0, 5, 10, ...\}$$
$$H + 1 = \{...., -9, -4, 1, 6, 11, ...\}$$
$$H + 2 = \{...., -8, -3, 2, 7, 12, ...\}$$
$$H + 3 = \{...., -7, -2, 3, 8, 13, ...\}$$
$$H + 4 = \{....., -6, -1, 4, 9, 14, ...\}$$
$$H + 5 = \{...., -10, -5, 0, 5, 10, ...\} = H$$

Now, its repetition starts. Now, we see that the right cosets, $H, H + 1, H + 2, H + 3, H + 4$ are all distinct and more over they are disjoint. Similarly the left cosets will be same as right cosets.

ii. Index of H in G is the number of distinct right/left cosets.

Therefore, index is 5.

Que 4.28. **What do you mean by cosets of a subgroup ? Consider the group Z of integers under addition and the subgroup $H = \{....., -12, -6, 0, 6, 12, ...\}$ considering of multiple of 6**
i. Find the cosets of H in Z.
ii. What is the index of H in Z.

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Answer

Cosets of a subgroup : Refer Q. 4.21, Page 4–15F, Unit-4.

Numerical :

i. We have $Z = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ...\}$

and $H = \{...., -12, -6, 0, 6, 12, ...\}$

Let $0 \in Z$ and the right cosets are given as

$$H = H + 0 = \{...., -12, -6, 0, 6, 12, ...\}$$
$$H + 1 = \{...., -11, -5, 1, 7, 13, ...\}$$

$$H + 2 = \{\ldots, -10, -4, 2, 8, 14, \ldots\}$$
$$H + 3 = \{\ldots, -9, -3, 3, 9, 15, \ldots\}$$
$$H + 4 = \{\ldots, -8, -2, 4, 10, 16, \ldots\}$$
$$H + 5 = \{\ldots, -7, -1, 5, 11, 17, \ldots\}$$
$$H = H + 6 = \{\ldots, -6, 0, 6, 12, 18, \ldots\}$$

Now, its repetition starts. Now, we see that the right cosets, $H, H+1, H+2, H+3, H+4, H+5$ are all distinct and more over they are disjoint. Similarly the left cosets will be same as right cosets.

ii. Index of H in Z is the number of distinct right/left cosets. Therefore, index is 6.

PART-4

Normal Subgroups, Permutation and Symmetric Groups.

Que 4.29. **Write short notes on :**

a. Normal subgroup
b. Permutation group

Answer

a. **Normal subgroup :** A subgroup H of G is said to be normal subgroup of G if $Ha = aH \ \forall \ a \in G$ *i.e.*, the right coset and left coset of H is G generated by a are the same.

i. Clearly, every subgroup H of an abelian group G is a normal subgroup of G. For $a \in G$ and $h \in H$, $ah = ha$.

ii. Since a cyclic group is abelian, every subgroup of a cyclic group is normal.

b. **Permutation group :** A set G of all permutation on a non-empty set A under the binary operation * is called permutation group.

If $A = \{1, 2, 3, \ldots, n\}$, the given permutation group formed by A is also called symmetric group of degree n denoted by S_n. Number of elements of S_n will be $n!$.

Cyclic permutation : Let $A = \{x_1, x_2, \ldots, x_n\}$. Then let $t_1, t_2, \ldots, t_k$ be elements of set A and permutation $P : A \rightarrow$ defined by

$$P(t_1) = t_2$$
$$P(t_2) = t_3$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$P(t_{k-1}) = t_k$$
$$P(t_k) = t_1$$

is called a cyclic permutation of length k.

For example : Consider $A = \{a, b, c, d, e\}$. Then let $P = \begin{pmatrix} a & b & c & d & e \\ c & b & d & a & e \end{pmatrix}$.

Then P has a cycle of length 3 given by (a, c, d).

Que 4.30. **Define the subgroup of a group. Let (G, o) be a group. Let $H = \{a \mid a \in G$ and $a \ o \ b = b \ o \ a$ for all $b \in G\}$. Show that H is a normal subgroup.**

Answer

Subgroup : If $(G, *)$ is a group and $H \subseteq G$. Then $(H, *)$ is said to subgroup of G if $(H, *)$ is also a group by itself.

i.e.,

1. $a * b \in H \ \forall \ a, b \in H$ (Closure property)
2. $\exists \ e \in H$ such that $a * e = a = e * a \quad \forall \ a \in H$ where e is called identity of G.
3. $\exists \ a^{-1} \in H$ such that $a * a^{-1} = e = a^{-1} * a \ \forall \ a \in H$

Numerical : Let (G, x) be a group. A non-empty subset H of a group G is said to be a subgroup of G if $(H, *)$ itself is a group.

Given that, $\quad H = \{a \mid a \in G$ and $a \ o \ b = b \ o \ a; \ \forall \ b \in G\}$

Let $\quad a, b \in H \Rightarrow a \ o \ x = x \ o \ a$ and $b \ o \ x = x \ o \ b, \ \forall \ x \in G$

$\Rightarrow \quad (b \ o \ x)^{-1} = (x \ o \ b)^{-1}$

$\Rightarrow \quad x^{-1} o \ b^{-1} = b^{-1} o \ x^1$

$\Rightarrow \quad b^{-1} \in H.$

Now, $\quad (a \ o \ b^{-1}) \ o \ x = a \ o \ (b^{-1} o \ x) \quad$ [$\because$ o is associative]

$= a \ o \ (x \ o \ b^{-1}) \quad$ [$\because$ use $b^{-1} \in H$]

$= (a \ o \ x) \ o \ b^{-1} \quad$ [$\because$ $a \in H$]

$= (x \ o \ a) \ o \ b^{-1}$

$= x \ o \ (a \ o \ b^{-1})$

$\Rightarrow \quad a \ o \ b^{-1} \in H$

Therefore, H is a subgroup of group G.

Let $h \in H$ and $g \in G$ and any x in G.

Consider

$(g \ o \ h \ o \ g^{-1}) \ o \ x = (g \ o \ g^{-1} o \ h) \ o \ x \quad$ [$\because$ $h \in H$]

$= (e \ o \ h) \ o \ x = h \ o \ x$

$= x \ o \ h \quad$ [$\because$ $h \in H$]

$= x \ o \ (h \ o \ g \ o \ g^{-1})$

$$= x \circ (g \circ h \circ g^{-1})$$

$\Rightarrow$ $g \circ h \circ g^{-1} \in H$ for any $g \in G$ [$\because$ $h \in H$]

$\therefore$ H is a normal subgroup of G.

Que 4.31. **A subgroup H of a group G is a normal subgroup if and only if $g^{-1}hg \in H$ for every $h \in H$ and $g \in G$.**

AKTU 2020-21, Marks 10

Answer

Let G be a group and let $H \leq G$ be a subgroup. We say that the subgroup H is normal in G, denoted $H \triangleleft G$, if for every $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$, that is, if for every $g \in H$ we have $gHg^{-1} \subseteq H$.

Proof : Let $g \in G$ be arbitrary. We know $Hg = gH$. Therefore for every $h \in H$ $gh \in Hg$, that is, there exists $h' \in H$ such that $gh = h'g$. Hence $ghg^{-1} = h' \in H$. Since $h \in H$ was arbitrary, this implies that $gHg^{-1} \subseteq H$. Therefore $H \triangleleft G$.

Que 4.32. **If N and M are normal subgroup of G then $N \cap M$ is a normal subgroup of G.**

Answer

As N and M are subgroups of G then $N \cap M$ is a subgroup of G. Let $g \in G$ and $a \in N \cap M$

$a \in N$ and $a \in M$.

Since N is normal subgroup of G, $gag^{-1} \in N$

Since M is normal subgroup of G, $gag^{-1} \in M$

$\therefore$ $gag^{-1} \in N \cap M$ is a normal subgroup of G.

Hence $N \cap M$ is *a* normal subgroup of G.

Que 4.33. **Prove or disprove that intersection of two normal subgroups of a group G is again a normal subgroup of G.**

AKTU 2019-20, Marks 10

Answer

Let H_1 and H_2 be any two subgroups of G. Since at least the identity element e is common to both H_1 and H_2.

$$\therefore \quad H_1 \cap H_2 \neq \phi$$

In order to prove that $H_1 \cap H_2$ is a subgroup, it is sufficient to prove that $a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1, H_2 are subgroups. Therefore,

$a \in H_1, b \in H_1 \Rightarrow ab^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally, $ab^{-1} \in H_1, ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Thus, we have shown that

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Hence, $H_1 \cap H_2$ is a subgroup of G.

PART-5

Groups Homomorphism, Definition and Elementary Properties of Ring and Fields.

Que 4.34. **Discuss homomorphism and isomorphic group.**

Answer

Homomorphism : Let $(G_1, \bullet)$ and $(G_2, *)$ be two groups then a mapping $f: G_1 \to G_2$ is called a homomorphism if $f(a \bullet b) = f(a) * f(b)$ for all $a, b \in G_1$. Thus f is homomorphism from G_1 to G_2 then f preserves the composition in G_1 and G_2 *i.e.*, image of composition is equal to composition of images.

The group G_2 is said to be homomorphic image of group G_1 if there exist a homomorphism of G_1 onto G_2.

Isomorphism : Let $(G_1, \bullet)$ and $(G_2, *)$ be two groups then a mapping $f: G_1 \to G_2$ is an isomorphism if

i. f is homomorphism.

ii. f is one to one *i.e.*, $f(x) = f(y) \Rightarrow x = y \ \forall \ x, y \in G_1$.

iii. f is onto.

Que 4.35. **Let (G,*) and G',*) be any two groups and let e and e' be their respective identities. If f is a homomorphism of G into G', the prove that**

i. $f(e) = e'$

ii. $f(x^{-1}) = [f(x)]^{-1}, \forall \ x \in G$

AKTU 2022-23, Marks 10

Answer

i. **Proof :** Let e and e' be identity of G and G' respectively.

Let $\quad x \in G \Rightarrow f(x) \in G'$ [$f: G \to G'$]

Now $\quad e' f(x) = f(x) = f(xe)$ $\quad$ [$xe = x$]

$= f(e) f(x)$ [f is homomorphism of isomorphism]

$\Rightarrow \quad e' = f(e)$ $\quad$ [right cancellation law]

$\therefore \quad f(e)$ is identity of G'

ii. **If e is the identity of $G \Rightarrow f(e)$ is identity of G'. If x^{-1} is the inverse of x in G, then**

$x^{-1}x = e = xx^{-1}$

Now, $\quad x^{-1}x = e \Rightarrow f(x^{-1}x) = f(e)$

$\Rightarrow \quad f(x^{-1}) f(x) = f(e)$

Again $\quad xx^{-1} = e \Rightarrow f(xx^{-1}) = f(e)$ $\quad$ [f is homo or isomorphism]

$\Rightarrow \quad f(x) f(x^{-1}) = f(e)$

$\therefore \quad f(x^{-1}) f(x) = f(e) = f(x) f(x^{-1})$

$\Rightarrow \quad f(x^{-1}) = [f(x)]^{-1}$

Que 4.36. **Give the definitions of rings, integral domains and fields.**

Answer

Ring : A ring is an algebraic system $(R, +, \bullet)$ where R is a non-empty set and $+$ and $\bullet$ are two binary operations (which can be different from addition and multiplication) and if the following conditions are satisfied :

1. $(R, +)$ is an abelian group.
2. $(R, \bullet)$ is semigroup *i.e.*, $(a \bullet b) \bullet c = a \bullet (b \bullet c) \ \forall \ a, b, c \in R$.
3. The operation $\bullet$ is distributive over $+$.

i.e., for any $a, b, c \in R$

$$a \bullet (b + c) = (a \bullet b) + (a \bullet c) \text{ or } (b + c) \bullet a = (b \bullet a) + (c \bullet a)$$

Integral domain : A ring is called an integral domain if :

i. It is commutative
ii. It has unit element
iii. It is without zero divisors

Field : A ring R with at least two elements is called a field if it has following properties :

i. R is commutative
ii. R has unity
iii. R is such that each non-zero element possesses multiplicative inverse.

For example, the rings of real numbers and complex numbers are also fields.

Que 4.37. **Consider a ring $(R, +, \bullet)$ defined by $a \bullet a = a$, determine whether the ring is commutative or not.**

Answer

Let $\ a, b \in R \ (a + b)^2 = (a + b)$

$\Rightarrow \quad (a + b)(a + b) = (a + b)$

$(a + b)a + (a + b)b = (a + b)$

$(a^2 + ba) + (ab + b^2) = (a + b)$

$(a + ba) + (ab + b) = (a + b)$ $\quad$ ($\because a^2 = a$ and $b^2 = b$)

$(a + b) + (ba + ab) = (a + b) + 0$

$\Rightarrow \quad ba + ab = 0$

$a + b = 0 \Rightarrow a + b = a + a$ [being every element of its own additive inverse]

$\Rightarrow \quad b = a$

$\Rightarrow \quad ab = ba$

$\therefore \quad R$ is commutative ring.

Que 4.38. **Write out the operation table for $[Z_2, +_2, *_2]$. Is Z_2 a ring ? Is an integral domain ? Is a field ? Explain.**

Answer

The operation tables are as follows :

we have $Z_2 = \{0, 1\}$

$+_2$	0	1
0	0	1
1	1	0

$*_2$	0	1
0	0	0
1	0	1

Since $(Z_2, +_2, *_2)$ satisfies the following properties :

i. **Closure axiom :** All the entries in both the tables belong to Z_2. Hence, closure is satisfied.

ii. **Commutative :** In both the tables all the entries about the main diagonal are same therefore commutativity is satisfied.

iii. **Associative law :** The associative law for addition and multiplication are also satisfied.

iv. Here 0 is the additive identity and 1 is the multiplicative identity. Identity property is satisfied.

v. Inverse exists in both the tables. The additive inverse of 0, 1 are 1, 0 respectively and the multiplicative inverse of non-zero element of Z_2 is 1.

vi. Multiplication is distributive over addition.

Hence $(Z_2, +_2, *_2)$ is a ring as well as field. Since, we know that every field is an integral domain therefore it is also an integral domain.

Que 4.39. **Prove that the set of residues $F = \{0, 1, 2, 3, 4\}$ modulo 5 is a field w.r.t. addition and multiplication of residue classes modulo 5. *i.e.*, $(F, +_5, X_5)$ is a field.**

AKTU 2022-23, Marks 10

Answer

The composition tables for addition and multiplication modulo 5 are shown below :

+5	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	4	3

×5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

From composition tables, we see that F is closed w.r.t. both addition and multiplication modulo 5. F is associative by the property of integers under both addition and multiplication modulo 5.

$0 \in F$ and $1 \in F$ are the additive and multiplicative identity elements.

The additive inverse of 0, 1, 2, 3, 4 are respectively 0, 4, 3, 2, 1.

The multiplicative inverses of 1, 2, 3, 4 are respectively 1, 3, 2, 4.

From the symmetry of the tables, it follows that F is commutative.

The distributive laws hold by the property of integers. Hence, F is a field.

Que 4.40. **If the permutation of the elements of {1, 2, 3, 4, 5} are given by $a = (1\,2\,3)\,(4\,5)$, $b = (1)\,(2)\,(3)\,(4\,5)$, $c = (1\,5\,2\,4)\,(3)$. Find the value of x, if $ax = b$. And also prove that the set $Z_4 = (0, 1, 2, 3)$ is a commutative ring with respect to the binary modulo operation $+_4$ and $*_4$.**

Answer

$ax = b \;\Rightarrow\; (123)\,(45)\,x = (1)\,(2)\,(3)\,(4, 5)$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3 & 4 & 5\\2 & 3 & 1 & 4 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix} x = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\2 & 3 & 1 & 5 & 4\end{pmatrix} x = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix}$$

$$x = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\2 & 3 & 1 & 5 & 4\end{pmatrix}^{-1}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix}$$

$$= \begin{pmatrix}2 & 3 & 1 & 5 & 4\\1 & 2 & 3 & 4 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 3 & 4 & 5\\3 & 1 & 2 & 5 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 5 & 4\end{pmatrix}$$

$$x = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\3 & 1 & 2 & 4 & 5\end{pmatrix}$$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

We find from these tables :

i. All the entries in both the tables belong to Z_4. Hence, Z_4 is closed with respect to both operations.

ii. **Commutative law :** The entries of 1st, 2nd, 3rd, 4th rows are identical with the corresponding elements of the 1st, 2nd, 3rd, 4th columns respectively in both the tables. Hence, Z_4 is commutative with respect to both operations.

iii. **Associative law :** The associative law for addition and multiplication
$a +_4 (b +_4 c) = (a +_4 b) +_4 c$ for all $a, b, c \in Z_4$
$a \times_4 (b \times_4 c) = (a \times_4 b) \times_4 c$, for all $a, b, c \in Z_4$
can easily be verified.

iv. **Existence of identity :** 0 is the additive identity and 1 is multiplicative identity for Z_4.

v. **Existence of inverse :** The additive inverses of 0, 1, 2, 3 are 0, 3, 2, 1 respectively.

Multiplicative inverse of non-zero element 1, 2, 3 are 1, 2, 3 respectively.

vi. **Distributive law :** Multiplication is distributive over addition *i.e.*,

$$a \times_4 (b +_4 c) = a \times_4 b + a \times_4 c$$

$$(b +_4 c) \times_4 a = b \times_4 a + c \times_4 a$$

For, $a \times_4 (b +_4 c) = a \times_4 (b + c)$ for $b +_4 c = b + c \pmod 4$

= least positive remainder when $a \times (b + c)$ is divided by 4

= least positive remainder when $ab + ac$ is divided by 4

$= ab +_4 ac$

$= a \times_4 b +_4 a \times_4 c$

For $a \times_4 b = a \times b \pmod 4$

Since $(Z_4, +_4)$ is an abelian group, $(Z_4, \times_4)$ is a semigroup and the operation is distributive over addition. The $(Z_4, +_4, \times_4)$ is a ring. Now $(Z_4, \times_4)$ is commutative with respect to $\times_4$. Therefore, it is a commutative ring.

Que 4.41. **What is meant by ring ? Give examples of both commutative and non-commutative rings.**

Answer

Ring : Refer Q. 4.36, Page 4–26F, Unit-4.

Example of commutative ring : Refer Q. 4.37, Page 4–26F, Unit-4.

Example of non-commutative ring : Consider the set R of 2×2 matrix with real element. For $A, B, C \in R$

$$A * (B + C) = (A * B) + (A * C)$$

also, $$(A + B) * C = (A * C) + (B * C)$$

$\therefore$ * is distributive over +.

$\therefore$ $(R, +, *)$ is a ring.

We know that $AB \neq BA$, Hence $(R, +, *)$ is non-commutative ring.

Que 4.42. **What is ring ? Define elementary properties of ring with example.** **AKTU 2019-20, Marks 10**

Answer

Ring : Refer Q. 4.36, Page 4–26F, Unit-4.

Elementary properties of a ring :

Let a, b and c belongs to a ring R. Then

1. $a.0 = 0.a = 0$
2. $a.(-b) = (-a).b = -(a.b)$
3. $(-a).(-b) = a.b$
4. $a(b - c) = a.b - a.c$ and $(b - c).a = b.a - c.a$

For example : If $a, b \in R$ then $(a + b)^2 = a^2 + ab + ba + b^2$

We have $(a + b)^2 = (a + b)(a + b)$

$= a(a + b) + b(a + b)$ [by right distributive law]

$= (aa + ab) + (ba + bb)$ [by right distributive law]

$= a^2 + ab + ba + b^2$

☺☺☺

5 UNIT

Graphs and Combinatorics

CONTENTS

PART-1

Graphs : Definition and Terminology, Representation of Graphs.

Que 5.1. What do you mean by graph ? Also, explain directed and undirected graph.

Answer

A graph is a non-linear data structure consisting of nodes and edges. A graph consists of two sets as follows :

1. Set V of nodes or point or vertices of graph G.
2. Set E of ordered or unordered pairs of distinct edges of G.

We denote such a graph by $G(V, E)$ and set of vertices as $V(G)$ and set of edges as $E(G)$.

For example :

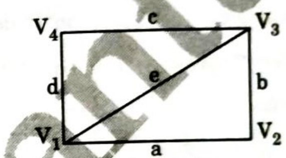


Fig. 5.1.1.

Order : If G is finite then number of vertices in G denoted by $|V(G)|$ is called order of G.

Size : The number of edges denoted by $|E(G)|$ in a finite graph G is called size of G.

Directed graph : A graph $G(V, E)$ is said to be directed graph or digraph if each edge $e \in E$ is associated with an ordered pair of vertices as shown below :

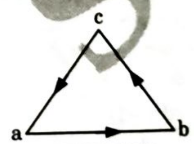


Fig. 5.1.2.

Undirected graph : A graph $G(V, E)$ is said to be undirected if each edge $e \in E$ is associated with an unordered pair of vertices as shown below :

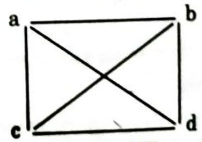


Fig. 5.1.3.

Que 5.2.

i. **Justify that "In a undirected graph the total number of odd degree vertices is even".**

ii. **Justify that "The maximum number of edges in a simple graph is $n(n-1)/2$".**

AKTU 2021-22, Marks 10

Answer

i. Let $G = (V, E)$ a undirected graph,
Let U denote the set of even degree vertices in G and W denote the set of odd degree vertices.
Then,

$$\sum_{v_i \in V} \deg_G(v_i) = \sum_{v_i \in U} \deg_G(v_i) + \sum_{v_i \in W} \deg_G(v_i)$$

$$2e - \sum_{v_i \in U} \deg_G(v_i) = \sum_{v_i \in W} \deg_G(v_i) \quad \text{...(5.2.1)}$$

Now, $\sum_{v_i \in W} \deg_G(v_i)$ is also even as the sum of degrees of even degree vertices is always even. Therefore, from eq. (5.2.1),

$$\sum_{v_i \in W} \deg_G(v_i) \text{ is even}$$

$$v_i \in W$$

$\therefore$ Since for each $v_i \in W$, $\deg_G(v_i)$ is odd, the number of odd vertices in G must be even.

ii. By handshaking theorem, $\sum_{i=1}^{n} d(v_i) = 2e$

Where e is the number of edges with n vertices in graph G.

$$\Rightarrow d(v_1) + d(v_2) + ... + d(v_n) = 2e \quad \text{...(5.2.2)}$$

We know that,
Maximum number degree of each vertices in the graph can be $(n-1)$.
Therefore, eq. (5.2.2) becomes

$$(n-1) + (n-1) + ... \text{ to } n \text{ terms} = 2e$$

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2}$$

Que 5.3. **Discuss representation of graph.**

Answer

Graph can be represented in following two ways :

1. **Matrix representation :**
Matrices are commonly used to represent graphs for computer processing. Advantages of representing the graph in matrix lies in the

fact that many results of matrix algebra can be readily applied to study the structural properties of graph from an algebraic point of view.

a. **Adjacency matrix :**

i. **Representation of undirected graph :**
The adjacency matrix of a graph G with n vertices and no parallel edges is a $n \times n$ matrix $A = [a_{ij}]$ whose elements are given by
$a_{ij} = 1$, if there is an edge between i^{th} and j^{th} vertices
$= 0$, if there is no edge between them

ii. **Representation of directed graph :**
The adjacency matrix of a digraph D, with n vertices is the matrix

$$A = [a_{ij}]_{n \times n} \text{ in which}$$

$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in } D$$

$$= 0 \text{ otherwise}$$

For example :

Fig. 5.3.1.

$$A = \begin{array}{c|c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b. **Incidence matrix :**

i. **Representation of undirected graph :**
Consider an undirected graph $G = (V, E)$ which has n vertices and m edges all labelled. The incidence matrix $I(G) = [b_{ij}]$, is then $n \times m$ matrix, where

$$b_{ij} = 1 \quad \text{when edge } e_j \text{ is incident with } v_i$$

$$= 0 \quad \text{otherwise}$$

ii. **Representation of directed graph :**
The incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which.

$$b_{ij} = 1 \text{ if arc } j \text{ is directed away from a vertex } v_i$$

$$= -1 \text{ if arc } j \text{ is directed towards vertex } v_i$$

$$= 0 \text{ otherwise.}$$

Find the incidence matrix to represent the graph shown in Fig. 5.3.2.

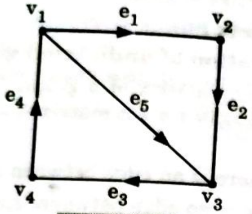


Fig. 5.3.2.

The incidence matrix of the digraph of Fig. 5.3.2 is

$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

2. **Linked representation :** In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a care has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex.

PART-2

Multigraphs, Biparite Graphs, Planar Graph, Isomorphism and Homomorphism of Graphs, Euler and Hamiltonian Paths.

Que 5.4. **Write short notes on :**

a. **Simple and multigraph**
b. **Complete graph and regular graph**
c. **Bipartite graph**
d. **Planar graph**

Answer

a. **Simple and multigraph :**

i. **Simple graph :** A graph in which there is only one edge between a pair of vertices is called a simple graph.

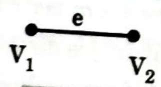


Fig. 5.4.1.

ii. **Multigraph :** Any graph which contains some parallel edges is called a multigraph.

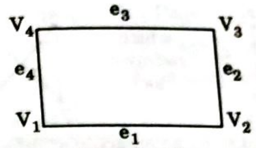


Fig. 5.4.2.

b. **Complete graph and regular graph :**

i. **Complete graph :** A simple graph, in which there is exactly one edge between each pair of distinct vertices is called a complete graph. The complete graph of n vertices is denoted by K_n. The graphs K_1 to K_5 are shown below in Fig. 5.4.3.

Fig. 5.4.3.

K_n has exactly $\dfrac{n(n-1)}{2} = {}^nC_2$ edges

ii. **Regular graph (n-regular graph) :** If every vertex of a simple graph has equal edges then it is called regular graph.
If the degree of each vertex is n then the graph is called n-regular graph.

Fig. 5.4.4.

The graphs shown in Fig. 5.4.4 are 2-regular graphs.

The graph shown in Fig. 5.4.5 is 3-regular graph.

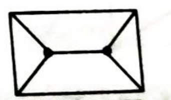


Fig. 5.4.5.

c. **Bipartite graph :**

i. **Bipartite graph :** A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2. (V_1, V_2) is called a bipartition of G.

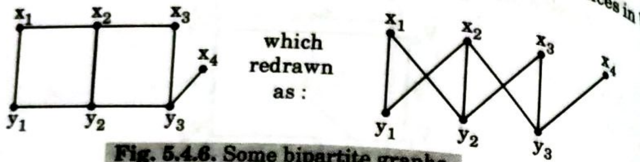


Fig. 5.4.6. Some bipartite graphs.

ii. **Complete bipartite graph :** The complete bipartite graph on m and n vertices, denoted $K_{m,n}$ is the graph, whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices v_1 and v_2 where v_1 is in V_1 and v_2 is in V_2. The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$

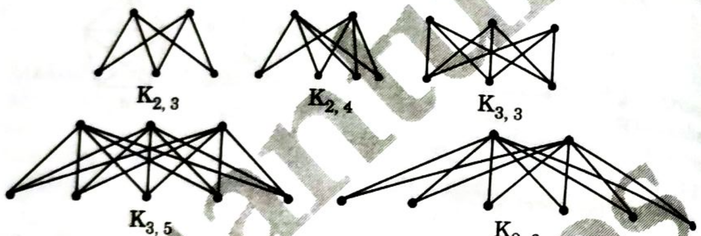


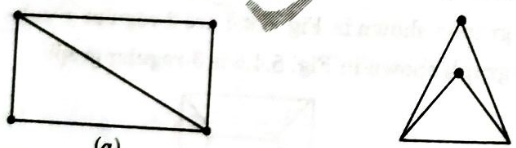
Fig. 5.4.7. Some complete bipartite graphs.

d. **Planar graph :**

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex.

i. A graph is said a planar graph, if it cannot be drawn on a plane without a crossover between its edges crossing.

ii. The graphs shown in Fig. 5.4.8(a) and (b) are planar graphs.



(a) (b)

Fig. 5.4.8. Some planar graph.

Que 5.5. **Define the following with one example :**

i. **Bipartite graph**
ii. **Complete graph**
iii. **How many edges in K_7 and $K_{3,6}$**
iv. **Planar graph**

Answer

i. **Bipartite graph :** Refer Q. 5.4(c), Page 5–5F, Unit-5.

ii. **Complete graph :** Refer Q. 5.4(b), Page 5–5F, Unit-5.

iii. **Number of edge in K_7 :** Since, K_n is complete graph with n vertices.

Number of edge in $K_7 = \dfrac{7(7-1)}{2} = \dfrac{7 \times 6}{2} = 21$

Number of edge in $K_{3,6}$:

Since, $K_{n,m}$ is a complete bipartite graph with $n \in V_1$ and $m \in V_2$

Number of edge in $K_{3,6} = 3 \times 6 = 18$

iv. **Planar graph :** Refer Q. 5.4(d), Page 5–5F, Unit-5.

Que 5.6. **Explain isomorphism and homomorphism of graph.**

Answer

Isomorphism of graph : Two graphs are isomorphic to each other if :

i. Both have same number of vertices and edges.

ii. Degree sequence of both graphs are same (degree sequence is the sequence of degrees of the vertices of a graph arranged in non-increasing order).

Example :

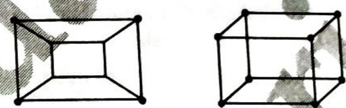


Fig. 5.6.1.

Homomorphism of graph : Two graphs are said to be homomorphic if one graph can be obtained from the other by the creation of edges in series (*i.e.*, by insertion of vertices of degree two) or by the merger of edges in series.

(a) (b) (c)

Fig. 5.6.2.

Que 5.7. **Prove that K_3 and K_4 are planar graphs. Prove that K_5 is non-planar.**

Answer

The complete K_3 graph has 3 edges and 3 vertices.

For a graph to be planar $3v - e \geq 6$

$$3v - e = 3 \times 3 - 3 = 9 - 3 = 6 \geq 6$$

$\therefore$ K_3 is planar graph

Similarly complete K_4 graph has 4 vertices and 6 edges.

$$3v - e = 3 \times 4 - 6 = 12 - 6 = 6 \geq 6$$

$\therefore$ K_4 is planar graph

The complete K_5 graph contains 5 vertices and 10 edges.

Now $3v - e = 3 \times 5 - 10 = 15 - 10 = 5 \geq 6$

Hence K_5 is non planar since for a graph to be planar $3v - e \geq 6$.

Que 5.8. **Define planar graph. Prove that for any connected planar graph, $v - e + r = 2$ where v, e, r is the number of vertices, edges, and regions of the graph respectively.**

AKTU 2019-20, Marks 10

OR

If a connected planar graph G has n vertices, e edges and r region, then $n - e + r = 2$.

AKTU 2020-21, Marks 10

Answer

Planar graph :

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex.

i. A graph is said a planar graph, if it cannot be drawn on a plane without a crossover between its edges crossing.

ii. The graphs shown in Fig. 5.8.1(a) and (b) are planar graphs.

(a) (b)

Fig. 5.8.1. Some planar graph.

Proof : We will use induction to prove this theorem.

Step I : Inductive base :

Assume that $e = 1$ Then we have two cases given in figure below :

(a) (b)

Fig. 5.8.2.

In Fig. 5.8.2 (*a*) we have $v = 2$ and $r = 1 \Rightarrow v + r - e = 2 + 1 - 1 = 2$

In Fig. 5.8.2 (*b*) we have $v = 1$ and $r = 2 \Rightarrow v + r - e = 1 + 2 - 1 = 2$

Hence verified

Step II : Inductive hypothesis :

Let us assume that given theorem is true for $e = k$

i.e., for k edges

Step III : Inductive step :

We have to show that theorem is true for $k + 1$ edges.

Let graph G has $k + 1$ edges.

Case I : We suppose that g contain no circuits. Now take a vertex v and find a path starting at v. Since g has no circuit so whenever we find an edge we have a new vertex at least we will reach a vertex with degree one as shown in Fig. 5.8.3.

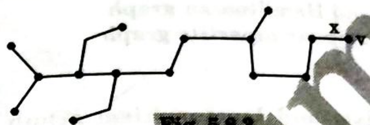


Fig. 5.8.3.

Now remove vertex x and edge incident on v.
Then we will left with graph G^* given as

Fig. 5.8.4.

Therefore Euler's formula holds for graphs in Fig. 5.8.4, since it has k edges [By inductive hypothesis]

Since G has one more edge than G^* and one more vertices than G^*.

So, let $v = v_1 + 1$ and $e = e_1 + 1$ where $G^* = (v_1, e_1)$

$$v + r - e = v_1 + 1 + r - e_1 - 1$$
$$= v_1 + r - e_1 = 2 \quad \text{[By inductive hypothesis]}$$

Hence Euler's formula holds true.

Case II : We assume that G has a circuit and e is edge in circuit. Let G be given in Fig. 5.8.5.

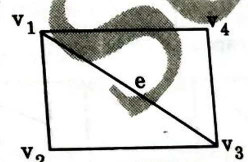


Fig. 5.8.5.

Now e is the part of boundary for 2 region so after removing edge we are left with graph G^* as shown in Fig. 5.8.6.

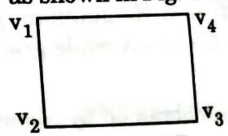


Fig. 5.8.6.

Now number of edges in G^* are k so by inductive hypothesis, Euler formula holds for G^*.

Now since G has one more edges and region than G^* with same vertices
So $v + r - e = v + r_1 + 1 - e_1 - 1 = v + r_1 - e_1 = 2$
Hence Euler's formula also holds for G.
Hence by Principle of mathematical induction Euler's Theorem holds true.

Que 5.9. **What are Euler and Hamiltonian graph ?**

OR

Explain the following terms with example :
i. Homomorphism and Isomorphism graph
ii. Euler graph and Hamiltonian graph
iii. Planar and Complete bipartite graph

Answer

i. **Homomorphism and Isomorphism graph :** Refer Q. 5.6, Page 5–8F, Unit-5.

ii. **Eulerian path :** A path of graph G which includes each edge of G exactly once is called Eulerian path.

Eulerian circuit : A circuit of graph G which include each edge of G exactly once.

Eulerain graph : A graph containing an Eulerian circuit is called Eulerian graph.

For example : Graphs given below are Eulerian graphs.

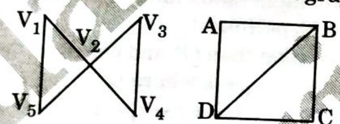
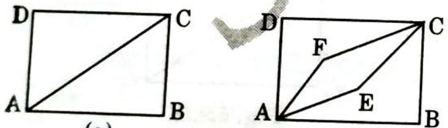


Fig. 5.9.1.

Hamiltonian graph : A Hamiltonian circuit in a graph G is a closed path that visit every vertex in G exactly once except the end vertices. A graph G is called Hamiltonian graph if it contains a Hamiltonian circuit.

For example : Consider graphs given below :



(a) (b)

Fig. 5.9.2.

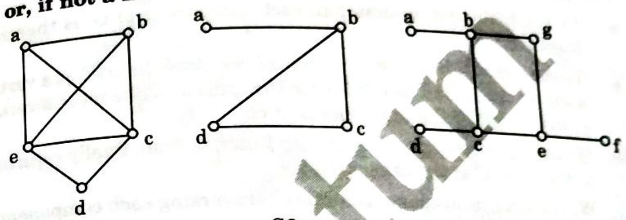
...oh **given is Fig.** 5.9.2(a) is a Hamiltonian graph since it contains a ...miltonian circuit $A - B - C - D - A$ while graph in Fig 5.15.2(b) is not a ...amiltonian graph.

...miltonian path : The path obtained by removing any one edge from a Hamiltonian circuit is called Hamiltonian path. Hamiltonian path is subgraph of Hamiltonian circuit. But converse is not true.

The length of Hamiltonian path in a connected graph of n vertices is $n - 1$ if it exists.

iii. **Planar and Complete bipartite graph :** Refer Q.5.4, Page 5–5F, Unit-5.

Que 5.10.
a. Prove that a connected graph G is Euler graph if and only if every vertex of G is of even degree.
b. Which of the following simple graph have a Hamiltonian circuit or, if not a Hamiltonian path ?



G1 G2 G3

Fig. 5.10.1.

Answer

a.
1. First of all we shall prove that if a non-empty connected graph is Eulerian then it has no vertices of odd degree.
2. Let G be Eulerian.
3. Then G has an Eulerian trail which begins and ends at u.
4. If we travel along the trail then each time we visit a vertex. We use two edges, one in and one out.
5. This is also true for the start vertex because we also end there.
6. Since an Eulerian trail uses every edge once, the degree of each vertex must be a multiple of two and hence there are no vertices of odd degree.
7. Now we shall prove that if a non-empty connected graph has no vertices of odd degree then it is Eulerian.
8. Let every vertex of G have even degree.
9. We will now use a proof by mathematical induction on $|E(G)|$, the number of edges of G.

Basis of induction :

Let $|E(G)| = 0$, then G is the graph K_1, and G is Eulerian.

Inductive step :

1. Let $P(n)$ be the statement that all connected graphs on n edges of even degree are Eulerian.

2. Assume $P(n)$ is true for all $n < |E(G)|$.
3. Since each vertex has degree at least two, G contains a cycle C.
4. Delete the edges of the cycle C from G.
5. The resulting graph, G' say, may not be connected.
6. However, each of its components will be connected, and will have fewer than $|E(G)|$ edges.
7. Also, all vertices of each component will be of even degree, because the removal of the cycle either leaves the degree of a vertex unchanged, or reduces it by two.
8. By the induction assumption, each component of G' is therefore Eulerian.
9. To show that G has an Eulerian trail, we start the trail at a vertex u say, of the cycle C and traverse the cycle until we meet a vertex c_1 say, of one of the components of G'.
10. We then traverse that component's Eulerian trail, finally returning to the cycle C at the same vertex, c_1.
11. We then continue along the cycle C, traversing each component of G' as it meets the cycle.
12. Eventually, this process traverses all the edges of G and arrives back at u, thus producing an Eulerian trail for G.
13. Thus, G is Eulerian by the principle of mathematical induction.

b. **G1 :** The graph G1 shown in Fig. 5.10.1 contains Hamiltonian circuit, *i.e.*, $a-b-c-d-e-a$ and also a Hamiltonian path, *i.e.*, *abcde*.

G2 : The graph G2 shown in Fig. 5.10.1 does not contain Hamiltonian circuit since every cycle containing every vertex must contain the edge e twice. But the graph does have a Hamiltonian path $a-b-c-d$.

G3 : The graph G3 shown in Fig. 5.17.1 neither have Hamiltonian circuit nor have Hamiltonian path because any traversal does not cover all the vertices.

Que 5.11. **Prove that a simple graph with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.**

Answer

Let the number of vertices in each of the k-components of a graph G be n_1, n_2, ..., n_k, then we get

$n_1 + n_2 + ... + n_k = n$ where $n_i \geq 1$ $(i = 1, 2, ..., k)$

Now,
$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

$$\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

$\therefore$ or $\displaystyle\sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$

or $\displaystyle\sum_{i=1}^{k}(n_i - 1)^2 + 2(\text{non-negative terms}) = n^2 + k^2 - 2nk$

$[\because n_i - 1 \geq 0, n_j - 1 \geq 0]$

or
$$\sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

or
$$\sum_{i=1}^{k} n_i^2 - n \leq n^2 + k^2 - 2nk - k + n$$
$$= n(n-k+1) - k(n-k+1)$$
$$= (n-k)(n-k+1)$$

We know that the maximum number of edges in the i^{th} component of $G = {}^{n_i}C_2 = \dfrac{n_i(n_i - 1)}{2}$

Therefore, the maximum number of edges in G is :
$$\frac{1}{2}\sum n_i(n_i - 1) = \frac{1}{2}\left(\sum n_i^2 - \sum n_i\right) = \frac{1}{2}\left(\sum n_i^2 - n\right)$$
$$\leq \frac{1}{2}(n-k)(n-k+1) \text{ by (1)}$$

Que 5.12. **What are different ways to represent a graph ? Define Euler circuit and Euler graph. Give necessary and sufficient conditions for Euler circuits and paths.**

Answer

Representation of graph : Refer Q. 5.3, Page 5–3F, Unit-5.

Euler circuit and Euler graph : Refer Q. 5.9, Page 5–11F, Unit-5.

Necessary and sufficient condition for Euler circuits and paths :

1. A graph has an Euler circuit if and only if the degree of every vertex is even.
2. A graph has an Euler path if and only if there are at most two vertices with odd degree.

Que 5.13. **Define and explain any two the following :**

1. **BFS and DFS in trees**
2. **Euler graph**
3. **Adjacency matrix of a graph**

Answer

1. **Breadth First Search (BFS) :** Breadth First Search (BFS) is an algorithm for traversing or searching tree or graph data structures. It starts at the tree root and explores the neighbour nodes first, before moving to the next level neighbours.

Algorithmic steps :

Step 1 : Push the root node in the queue.

Step 2 : Loop until the queue is empty.

Step 3 : Remove the node from the queue.

Step 4 : If the removed node has unvisited child nodes, mark them as visited and insert the unvisited children in the queue.

Depth First Search (DFS) :

Depth First Search (DFS) is an algorithm for traversing or searching tree or graph data structures. One starts at the root (selecting some arbitrary node as the root in the case of a graph) and explores as far as possible along each branch before backtracking.

Algorithmic steps :

Step 1 : Push the root node in the stack.

Step 2 : Loop until stack is empty.

Step 3 : Pick the node of the stack.

Step 4 : If the node has unvisited child nodes, get the unvisited child node, mark it as traversed and push it on stack.

Step 5 : If the node does not have any unvisited child nodes, pop the node from the stack.

2. **Euler graph :** Refer Q. 5.9, Page 5–11F, Unit-5.
3. **Adjacency matrix of a graph :** Refer Q. 5.3, Page 5–3F, Unit-5.

Que 5.14. **Express the following :**

i. **Euler graph and Hamiltonian graph**
ii. **Chromatic number of a graph**
iii. **Walk and path**
iv. **Bipartite graph**

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Answer

i. **Euler graph and Hamiltonian graph :** Refer Q. 5.9(ii), Page 5–11F, Unit-5.

ii. **Chromatic number of a graph :** The chromatic number can be described as the minimum number of colors required to properly color any graph, in such a way that no two adjacent vertices of a graph will be assigned the same color.

iii. **Walk :**

1. A walk can be defined as a sequence of edges and vertices of a graph. When we have a graph and traverse it, then that traverse will be known as a walk.
2. In a walk, there can be repeated edges and vertices.
3. The number of edges which is covered in a walk is known as the length of the walk. In a graph, there can be more than one walk.
4. For a walk, the following two points are important, which are described as follows :
   a. Edges can be repeated
   b. Vertex can be repeated

Path :

1. A path is a type of open walk where neither edges nor vertices are allowed to repeat.
2. For a path, the following two points are important, which are described as follows :
   a. Edges cannot be repeated
   b. Vertex cannot be repeated

iv.

i. **Bipartite graph :** Refer Q. 5.4(c), Page 5–5F, Unit-5.

PART-3

Graph Coloring.

Que 5.15. **Write a short note on graph coloring.**

Answer

1. Suppose that $G(V, E)$ is a graph with no multiple edges, a vertex colouring of G is an assignment of colours.
2. A graph G is m-colourable if there exists a colouring of G which uses m colours.
3. Colouring the vertices such a way such that no two adjacent vertices have same colour is called proper colouring otherwise it is called improper colouring.

Que 5.16. **Explain the following terms with example :**

i. Graph coloring and chromatic number.
ii. How many edges in K_7 and K_{33}.
iii. Isomorphic graph and Hamiltonian graph.
iv. Bipartite graph.
v. Handshaking theorem.

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Answer

i. **Chromatic number :** Refer Q. 5.14(ii), Page 5–15F, Unit-5.

Graph coloring : Refer Q. 5.15, Page 5–16F, Unit-5.

ii. **Number of edge in K_7 :** Since, K_n is complete graph with n vertices.

Number of edge in $K_7 = \frac{7(7-1)}{2} = \frac{7 \times 6}{2} = 21$

Number of edge in $K_{3,3}$:

Since, $K_{n,m}$ is a complete bipartite graph with $n \in V_1$ and $m \in V_2$

Number of edge in $K_{3,3} = 3 \times 3 = 9$

iii. **Isomorphic graph :** Refer Q. 5.6, Page 5–8F, Unit-5.

Hamiltonian graph : Refer Q. 5.9(ii), Page 5–11F, Unit-5.

iv.

i. **Bipartite graph :** Refer Q. 5.4(c), Page 5–5F, Unit-5.

v. **Handshaking theorem :**

1. Handshaking theorem states that the sum of degrees of the vertices of a graph is twice the number of edges.
2. If $G = (V, E)$ be a graph with E edges, then $\Sigma \deg_G(V) = 2E$.
3. The following conclusions may be drawn from the Handshaking Theorem :

In any graph,

i. The sum of degree of all the vertices is always even.
ii. The sum of degree of all the vertices with odd degree is always even.
iii. The number of vertices with odd degree is always even.

PART-4

Combinatorics : Introduction, Counting Techniques.

Que 5.17. **Define permutation and combination. Also, write difference between them.**

Answer

1. Permutation refers to different ways of arranging a set of object in a sequential order.
2. The number of permutations of n different things taken r ($\leq n$) at a time is denoted by $p(n, r)$ or ${}^nP_r$.
3. Combination refers to several ways of choosing items from a large set of object.
4. The number of combinations of n different thing taken r ($\leq n$) at a time is denoted by $C(n, r)$ or ${}^nC_r$.

The selection of two letters from three letters a, b, c are ab bc ca and thus, the number of combinations of 3 letters taken 2 at a time is $C(3, 2) = 3$

Difference between a permutation and combination :

S. No.	Permutation	Combination
1.	Both selection and arrangement are made.	Only selection is made.
2.	Ordering of the selected object is essential.	Ordering of the selected object is not essential.
3.	Multiple permutations can be derive from combination.	Single combination is derive from single permutation.
4.	${}^nP_r = \frac{n!}{(n-r)!}$	${}^nC_r = \frac{n!}{r!(n-r)!}$

Que 5.18. **Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen ?**

Answer

As the order in which each cookie is chosen does not matter and each kind of cookies can be chosen as many as 6 times, the number of ways these cookies can be chosen is the number of 6-combination with repetition allowed from a set with 4 distinct elements.

The number of ways to choose six cookies in the bakery shop is the number of 6 combinations of a set with four elements.

$$C(4 + 6 - 1.6) = C(9.6)$$

Since $C(9.6) = C(9.3) = 9.8.7/1.2.3 = 84$

Therefore, there are 84 different ways to choose the six cookies.

Que 5.19. **A collection of 10 electric bulbs contain 3 defective ones.**
i. In how many ways can a sample of four bulbs be selected ?
ii. In how many ways can a sample of 4 bulbs be selected which contain 2 good bulbs and 2 defective ones ?
iii. In how many ways can a sample of 4 bulbs be selected so that either the sample contain 3 good ones and 1 defectives ones or 1 good and 3 defectives ones ?

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Answer

i. Four bulbs can be selected out of 10 bulbs in

$$^{10}C_4 = \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210 \text{ ways}$$

ii. Two bulbs can be selected out of 7 good bulbs in $^7C_2$ ways and 2 defective bulbs can be selected out of 3 defective bulbs in $^3C_2$ ways. Thus, the number of ways in which a sample of 4 bulbs containing 2 good bulbs and 2 defective bulbs can be selected as

$$^7C_2 \times {}^3C_2 = \frac{7!}{2!\,5!} \times \frac{3!}{2!\,1!} = \frac{7 \times 6}{2} \times 3 = 63$$

iii. Three good bulbs can be selected from 7 good bulbs in $^7C_3$ ways and 1 defective bulb can be selected out of 3 defective ones in $^3C_1$ ways.

Similarly, one good bulb can be selected from 7 good bulb in $^7C_1$ ways and 3 defective ones in $^3C_3$ ways.

So, the number of ways of selecting a sample of 4 bulbs containing 3 good ones and 1 defective or 1 good and 3 defective ones are

$$^7C_3 \times {}^3C_1 + {}^7C_1 \times {}^3C_3 = \frac{7!}{3!\,4!} \times \frac{3!}{1!\,2!} + \frac{7!}{1!\,6!} \times \frac{3!}{3!\,0!}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2} \times 3 + 7 = 35 \times 3 + 7 = 112$$

PART-5

Pigeonhole Principle.

Que 5.20. **Write short notes on pigeonhole principle**

Answer

Pigeonhole principle :

The pigeonhole principle is sometime useful in counting methods.

If n pigeons are assigned to m pigeonholes then at least one pigeonhole contains two or more pigeons ($m < n$).

Proof :

1. Let m pigeonholes be numbered with the numbers 1 through m.
2. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number.
3. Since $m < n$ *i.e.*, the number of pigeonhole is less than the number of pigeons, n-m pigeons are left without having assigned a pigeonhole.
4. Thus, at least one pigeonhole will be assigned to a more than one pigeon.
5. We note that the pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons.
6. It only asserts the existence of a pigeon hole containing two or more pigeons.
7. To apply the principle one has to decide which objects will play the role of pigeon and which objects will play the role of pigeonholes.

Que 5.21. **Find the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5, and 7.**

Answer

1, 2, 3, -------------- 250

Number of integers between 1 and 250 that are divisible by 2 :

Quotient of last number ÷ 2 – quotient of first number ÷ 2

$$(250 \div 2) - (1 \div 2) = 125 - 0 = 125$$

Number of integers between 1 and 250 that are divisible by 3

$$(250 \div 3) - (1 \div 3) = 83 - 0 = 83$$

Number of integers between 1 and 250 that are divisible by 5

$$(250 \div 5) - (1 \div 5) = 50 - 0 = 50$$

Number of integers between 1 and 250 that are divisible by 7

$$(250 \div 7) - (1 \div 7) = 35 - 0 = 5$$

$\therefore$ Total number of integers divisible by only 2, 3, 5, 7, individually are $125 + 83 + 50 + 35 = 293$

Number of integers divisible by (2 and 3) = 41
Number of integers divisible by (2 and 5) = 25
Number of integers divisible by (2 and 7) = 17
Number of integers divisible by (3 and 5) = 16
Number of integers divisible by (3 and 7) = 11
Number of integers divisible by (5 and 7) = 7
Number of integers divisible by (2, 3 and 5) = 8
Number of integers divisible by (2, 3, and 7) = 5
Number of integers divisible by (2, 5, and 7) = 3
Number of integers divisible by (3, 5 and 7) = 2

135

$\therefore$ Number of integers between 1 and 250 that are divisible by any of the integer (2, 3, 5 and 7) = 293 – 135 = 158.

Que 5.22. **Find the numbers between the 100 to 1000 that are divisible by 3 or 5 or 7.** **AKTU 2020-21, Marks 10**

Answer

Let A, B, and C be the sets of integer, between 100 and 1000 (inclusive) which are divisible by 3, 5 and 7 respectively. Total integers = 1000 – 100 + 1 = 901

$$|A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$$|A| = \frac{901}{3} = 300 \; |B| = \frac{901}{5} = 180 \; |C| = \frac{901}{7} = 128$$

$$|A \cap B| = \frac{901}{15} = 60 \; |B \cap C| = \frac{901}{35} = 25 \; |C \cap A| = \frac{901}{21} = 42$$

$$|A \cap B \cap C| = \frac{901}{106} = 8$$

So, $|A \cup B \cup C| = 300 + 180 + 128 - 60 - 25 - 42 + 8 = 489$

Hence, the number between the 100 to 1000 that are divisble by 3 or 5 or 7 is 489.

Que 5.23. **Find the number between 1 to 500 that are not divisible by any of the integers 2 or 3 or 5 or 7.** **AKTU 2019-20, Marks 10**

Answer

Let A, B, C and D be the sets of integers between 1 and 500(inclusive) which are divisible by 2, 3, 5, and 7, respectively. We want $|(A \cup B \cup C \cup D)^c|$.

Now

$|(A \cup B \cup C \cup D)| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

We have

$$|A| = \left\lfloor \frac{500}{2} \right\rfloor = 250 \; |B| = \left\lfloor \frac{500}{3} \right\rfloor = 166 \; |C| = \left\lfloor \frac{500}{5} \right\rfloor = 100 \; |D| = \left\lfloor \frac{500}{7} \right\rfloor = 71$$

$$|A \cap B| = \left\lfloor \frac{500}{6} \right\rfloor = 83 \; |A \cap C| = \left\lfloor \frac{500}{10} \right\rfloor = 50 \; |A \cap D| = \left\lfloor \frac{500}{14} \right\rfloor = 35$$

$$|B \cap C| = \left\lfloor \frac{500}{15} \right\rfloor = 33 \; |B \cap D| = \left\lfloor \frac{500}{21} \right\rfloor = 23 \; |C \cap D| = \left\lfloor \frac{500}{35} \right\rfloor = 14$$

$$|A \cap B \cap C| = \left\lfloor \frac{500}{30} \right\rfloor = 16 \; |A \cap B \cap D| = \left\lfloor \frac{500}{42} \right\rfloor = 11$$

$$|A \cap C \cap D| = \left\lfloor \frac{500}{70} \right\rfloor = 7 \; |B \cap C \cap D| = \left\lfloor \frac{500}{105} \right\rfloor = 4$$

$$|A \cap B \cap C \cap D| = \left\lfloor \frac{500}{210} \right\rfloor = 2$$

So $|A \cup B \cup C \cup D| = 385$.

Hence, the number between 1 to 500 that are not divisible by 2 or 3 or 5 or 7 is 500 – 385 = 115.

Que 5.24. **Using Pigeonhole principle find the minimum number n of integers to be selected from S = {1, 2, 3, 4, 5, 6, 7, 8, 9} so that**

i. **the sum of two of the integers is even**
ii. **the difference of two of the n integers is 5**

Answer **AKTU 2022-23, Marks 10**

i. The sum of two even integers or of two odd integers is even. Consider the subsets {1, 3, 5, 7, 9} and {2, 4, 6, 8} of S as pigeonholes. Hence, $n = 3$.

ii. Consider the five subsets {1, 6}, {2, 7}, {3, 8}, {4, 9}, {5} of S as pigeonholes. Then, $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

Que 5.25. **How many different rooms are needed to assign 500 classes, if there are 45 different time periods during in the university time table that are available ?**

Answer

Using pigeonhole principle :

Here

$$n = 500, m = 45 = \left[\frac{n-1}{m}\right]+1 = \left[\frac{500-1}{45}\right]+1$$

At least 12 rooms are needed.

Que 5.26. **A total of 1232 student have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further 103 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French and Russian, how many students have taken a course in all three languages ?**

Answer

Let S be the set of students who have taken a course in Spanish, F be the set of students who have taken a course in French, and R be the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|S \cap R| = 14$, and $|S \cup F \cup R| = 23$.

Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |S \cap R| + |S \cup F \cup R|$,

$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$,

$|S \cap F \cap R| = 7$.

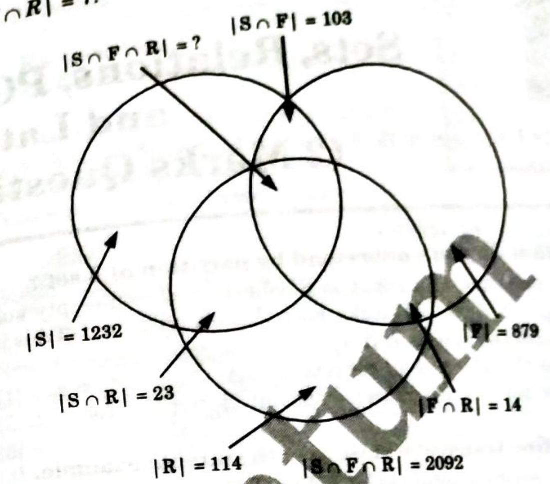

Fig. 5.26.1.

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Quantum Series

UNIT 1

Sets, Relations, POSET and Lattices (2 Marks Questions)

1.1. What do you understand by partition of a set ?

Ans. A partition of a set A is a collection of non-empty subsets A_1, A_2, ..., A_n called blocks, such that each element of A is in exactly one of the blocks, *i.e.*,

i. A is the union of all subsets $A_1 \cup A_2 \cup ..... \cup A_n = A$.

ii. The subsets are pairwise disjoint, $A_i \cap A_j = \varnothing$ for $i \neq j$.

1.2. Define transitive closure with suitable example.

Ans. The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the transitive closure of the relation. The transitive closure of R is denoted by R^+.

1.3. Let R be a relation on the set of natural numbers N, as $R = \{(x, y) : x, y \in N, 3x + y = 19\}$. Find the domain and range of R. Verify whether R is reflexive.

Ans. By definition of relation,

$R = \{(1, 16), (2, 13), (3, 10), (4, 7), (5, 4), (6, 1)\}$

$\therefore$ Domain = $\{1, 2, 3, 4, 5, 6\}$

$\therefore$ Range = $\{16, 13, 10, 7, 4, 1\}$

R is not reflexive since $(1, 1) \notin R$.

1.4. Show that the relation R on the set Z of integers given by $R = \{(a, b) : 3$ divides $a - b\}$, is an equivalence relation.

Ans. **Reflexive :** $a - a = 0$ is divisible by 3

$\therefore$ $(a, a) \in R \ \forall a \in Z$

$\therefore$ R is reflexive.

Symmetric : Let $(a, b) \in R \Rightarrow a - b$ is divisible by 3

$\Rightarrow$ $-(a - b)$ is divisible by 3

$\Rightarrow$ $b - a$ is divisible by 3

$\Rightarrow$ $(b, a) \in R$

$\therefore$ R is symmetric.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$a - b$ is divisible by 3 and $b - c$ is divisible by 3

Then $a - b + b - c$ is divisible by 3

$a - c$ is divisible by 3

$\therefore$ $(a, c) \in R$

$\therefore$ R is transitive.

Hence, R is equivalence relation.

1.5. Let $A = \{1, 2, 3, 4, 5, 6\}$ be the set and $R = \{(1, 1) (1, 5) (2, 2) (2, 3) (2, 6) (3, 2) (3, 3) (3, 6) (4, 4) (5, 1) (5, 5) (6, 2) (6, 3) (6, 6)\}$ be the relation defined on set A. Find equivalence classes induced by R.

AKTU 2021-22, Marks 02

Ans. Given : $A = \{1, 2, 3, 4, 5, 6\}$ and

$R = \{(1, 1) (1, 5) (2, 2) (2, 3) (2, 6) (3, 2) (3, 3) (3, 6) (4, 4) (5, 1) (5, 5) (6, 2) (6, 3) (6, 6)\}$

Now, finding equivalence class as

$[1] = \{1, 5\}$

$[2] = \{2, 3, 6\}$

$[3] = \{3, 2, 6\} \Rightarrow \{2, 3, 6\}$

$[4] = \{4\}$

$[5] = \{5, 1\} \Rightarrow \{1, 5\}$

$[6] = \{6, 2, 3\} \Rightarrow \{2, 3, 6\}$

1.6. Mention some properties of cartesian product.

Ans. For the four sets A, B, C and D

i. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

ii. $(A - B) \times C = (A \times C) - (B \times C)$

iii. $(A \cup B) \times C = (A \times C) \cup (B \times C)$

iv. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

1.7. Discuss countable and uncountable sets.

Ans. A set which is either empty, finite or countably infinite is called countable otherwise uncountable. Thus, A is countable if there exists a one-to-one function F from N onto A.

1.8. Write a short note on composition of relation.

Ans. The composite of, say R and S denoted by RoS, is the relation consisting of ordered pairs (a, c) when $a \in A$ and $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

1.9. How many symmetric and reflexive relations are possible from a set A containing 'n' elements ?

AKTU 2019-20, Marks 02

Ans. There are $2^{n(n+1)/2}$ symmetric binary relations and $2^{n(n-1)}$ reflexive binary relations are possible on a set S with cardinality.

1.10. Show that if set A has 3 elements, then we can have 2^6 symmetric relations on A.

Ans. Number of elements in set = 3
Number of symmetric relations if number of elements is $n = 2^{n(n+1)/2}$

Here, $n = 3$

$\therefore$ Number of symmetric relations

$$= 2^{3(3+1)/2}$$
$$= 2^{3(4)/2}$$
$$= 2^6$$

Hence proved.

1.11. Define cardinality.

Ans. Cardinality of a set is defined as the total number of elements in a finite set.

1.12. Let R be a relation on set A with cardinality n. Write down the number of reflexive and symmetric relation on set A.

AKTU 2020-21, Marks 02

Ans. Total number of reflexive relations $= 2^{n(n-1)}$
Total number of symmetric relation $= 2^{n(n-1)/2}$

1.13. Let $A = (2, 4, 5, 7, 8) = B$, aRb if and only if $a + b <= 12$. Find relation matrix.

Ans. $R = \{(2, 4), (2, 5), (2, 7), (2, 8)$ $(4, 2), (4, 5), (4, 7), (4, 8)$ $(5, 2), (5, 4), (5, 7), (7, 2), (7, 4), (7, 5), (8, 2), (8, 4), (2, 2), (4, 4), (5, 5)\}$

$$m_{ij} = \begin{array}{c|c} & \begin{array}{ccccc} 2 & 4 & 5 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 4 \\ 5 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

1.14. Explain maximal and minimal element.

Ans. **Maximal element :** An element 'a' in the poset is called a maximal element of P if $a \le x$ for no 'x' in P, that is, if no element of P strictly succeeds 'a'.

Minimal element : An element 'b' in P is called a minimal element of P if $x \le b$ for no 'x' in P.

1.15. What do you mean by sublattice ?

Ans. A non-empty subset L' of a lattice L is called a sublattice of L if $a, b \in L'$ so that $a \vee b$, $a \wedge b \in L'$ *i.e.*, the algebra $(L', \vee, \wedge)$ is a sublattice of $(L, \wedge, \vee)$ iff L' is closed under both operations $\wedge$ and $\vee$.

1.16. Explain lattice homomorphism and lattice isomorphism.

Ans. **Lattice homomorphism :** Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. A mapping $g : L \to S$ is called a lattice homomorphism from the lattice $(L, *, \oplus)$ to $(S, \wedge, \vee)$ if for any $a, b \in L$

$$g(a * b) = g(a) \wedge g(b) \text{ and } (g \oplus b) = g(a) \vee g(b).$$

Lattice isomorphism : If a homomorphism $g : L \to S$ of two lattices $(L, *, \oplus)$ and $(S, \wedge, \vee)$ is bijective *i.e.*, one-to-one onto, then g is called an isomorphism.

1.17. Show that the relation $\ge$ is a partial ordering on the set of integers, Z.

Ans. Since :
1. $a \ge a$ for every a, $\ge$ is reflexive.
2. $a \ge b$ and $b \ge a$ imply $a = b$, $\ge$ is antisymmetric.
3. $a \ge b$ and $b \ge c$ imply $a \ge c$, $\ge$ is transitive.

It follows that $\ge$ is a partial ordering on the set of integers and $(Z, \ge)$ is a poset.

1.18. Consider $A = \{x \in R : 1 < x < 2\}$ with $\le$ as the partial order. Find
i. All the upper and lower bounds of A.
ii. Greatest lower bound and least upper bound of A.

Ans.
i. Every real number ≥ 2 is an upper bound of A and every real number ≤ 1 is a lower bound of A.
ii. 1 is a greatest lower bound and 2 is the least upper bound of A.

1.19. Determine
i. All maximal and minimal elements
ii. Greatest and least element
iii. Upper and lower bounds of 'a' and 'b', 'c' and 'd'

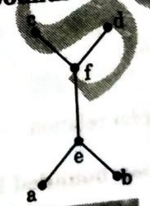


Fig. 1.19.1.

Ans.
i. Maximal elements = c, d, Minimal element = a, b
ii. Greatest and least elements do not exist.
iii. Upper bound for a, b are e, f, c, d.
Upper bound for c, d are does not exist.
Lower bound for a, b are does not exist.
Lower bound for c, d are f, e, a, b.

1.20. Let (A, ≤) be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a then $x = y$.

Ans. We have
$$x = x \vee (x \wedge a) = x \vee (y \wedge a) \quad (\because \text{ Given condition})$$
$$= (x \vee y) \wedge (x \vee a)$$
$$= (x \vee y) \wedge (y \vee a) \quad (\because \text{ Distributive property})$$
$$= y \vee (x \wedge a)$$
$$x = x \vee (y \wedge a)$$
$$x = y$$

1.21. If L be a lattice, then for every a and b in L prove that $a \wedge b = a$ if and only if $a \leq b$. **AKTU 2020-21, Marks 02**

Ans. Let $a \wedge b = a$. Since $a \wedge b \leq b$, we have $a \leq b$. Conversely, if $a \leq b$ and since $a \leq a$, a is a lower bound of a and b and so, by the definition of greatest lower bound, we have $a \leq a \wedge b$
Since $a \wedge b$ is lower bound, $a \wedge b \leq a$
Hence $a \wedge b = a$

1.22. Show that the "greater than or equal" relation (>=) is a partial ordering on the set of integers.

Ans. **Reflexive :**
$a \geq a \ \forall a \in Z$ (set of integer)
$(a, a) \in A$
$\therefore$ R is reflexive.
Antisymmetric : Let $(a, b) \in R$ and $(b, a) \in R$
$\Rightarrow$ $a \geq b$ and $b \geq a$
$\Rightarrow$ $a = b$
$\therefore$ R is antisymmetric.
Transitive : Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $a \geq b$ and $b \geq c$
$\Rightarrow$ $a \geq c \Rightarrow (a, c) \in R$
$\therefore$ R is transitive.
Hence, R is partial order relation.

1.23. Distinguish between bounded lattice and complemented lattice.

Ans. **Bounded lattice :** A lattice which has both elements 0 and 1 is called a bounded lattice.
Complemented lattice : A lattice L is called complemented lattice if it is bounded and if every element in L has complement.

1.24. Differentiate complemented lattice and distributed lattice. **AKTU 2021-22, Marks 02**

Ans. **Complemented lattice :** Let L be a bounded lattice with greatest element 1 and least element 0. Let $a \in L$ then an element $a' \in L$ is complement of a if,
$$a \vee a' = 1 \text{ and } a \wedge a' = 0$$
A lattice L is called complemented if is bounded and if every element in L has a complement.
Distributive lattice : A lattice L is said to be distributive if for any element a, b and c of L following properties are satisfied :
i. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
ii. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
otherwise L is non-distributive lattice.

1.25. Draw the Hasse diagram of D_{30}.

Ans.

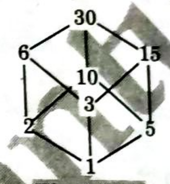


Fig. 1.25.1.

1.26. Find the Maximal elements and minimal elements from the following Hasse's diagram.

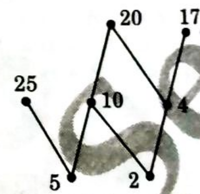
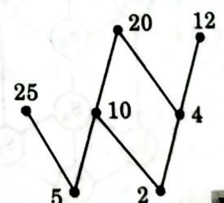


Fig. 1.26.1.

AKTU 2022-23, Marks 02

Ans.



Maximal elements = {20, 12}
Maximal elements = {2, 5}

Fig. 1.26.2.

1.27. Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation 'a divides b'. Find the Hasse diagram.

AKTU 2020-21, Marks 02

Ans. $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

Find poset for the divisibility

$A = \{(1, 2), (1, 3), (1, 4), (1, 6), (1, 8), (1, 9), (1, 12), (1, 18), (1, 24), (2, 4), (2, 6), (2, 8), (2, 12), (2, 18), (2, 24), (3, 6), (3, 9), (3, 12), (3, 18), (3, 24), (4, 8), (4, 12), (4, 24), (6, 12), (6, 18), (6, 24), (8, 24), (9, 18), (12, 24)\}$

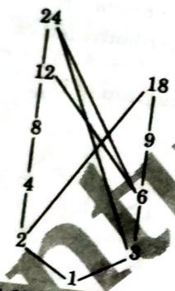


Fig. 1.27.1. Hasse diagram.

1.28. Draw the Hasse's diagram of the POSET $(L, \subseteq)$ where $L = \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$, where the sets are given by

$S_0 = \{a, b, c, d, e, f\}$
$S_1 = \{a, b, c, d, e\}$
$S_2 = \{a, b, c, e, f\}$
$S_3 = \{a, b, c, e\}$
$S_4 = \{a, b, c\}$
$S_5 = \{a, b\}$
$S_6 = \{a, c\}$
$S_7 = \{a\}$

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Ans.

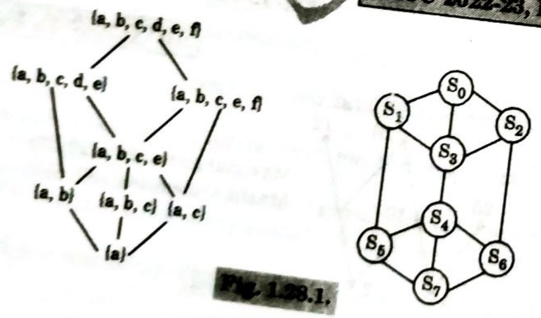

Fig. 1.28.1.

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2 UNIT Functions and Boolean Algebra (2 Marks Questions)

2.1. Define various types of functions.

AKTU 2019-20, Marks 02

Ans. **Various types of functions :**

1. One-to-one function (Injective function).
2. Onto function (Surjective function).
3. One-to-one onto function (Bijective function).
4. Many one function.
5. Identity function.
6. Inverse function (Invertible function).

2.2. If the function $f : R \to R$ defined by $f(x) = x^2$, find $f^{-1}(4)$ and $f^{-1}(-4)$.

Ans.

$$f^{-1}(4) = \{x \in R : f(x) = 4\} = \{x \in R : x^2 = 4\} = \{x \in R : x = \pm 2\} = \{-2, 2\}$$

$$f^{-1}(-4) = \{x \in R : f(x) = -4\} = \{x \in R : x^2 = -4\} = \{x \in R : x = \pm 2\sqrt{-1}\} = \phi \text{ since } \pm 2\sqrt{-1}$$

are imaginary numbers

2.3. Identify whether $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$, $\forall x, y \in R$, where $\lceil x \rceil$ is a ceiling function.

AKTU 2022-23, Marks 02

Ans. $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$, $\forall x, y \in R$,

Let consider $x = 4.5, y = 5.5$

therefore $\lceil x \rceil = 5 = \lceil y \rceil = 6$

$\lceil x + y \rceil = \lceil 10 \rceil = 10$

$\lceil x \rceil + \lceil y \rceil = 5 + 6 = 11$

So, $\lceil x + y \rceil \neq \lceil x \rceil + \lceil y \rceil$

2.4. If $f : A \to B$ is one-to-one onto mapping, then prove that $f^{-1} : B \to A$ will be one-to-one onto mapping.

Ans. **Proof :** Here $f : A \to B$ is one-to-one and onto.
$a_1, a_2 \in A$ and $b_1, b_2 \in B$ so that
$b_1 = f(a_1)$, $b_2 = f(a_2)$ and $a_1 = f^{-1}(b_1)$, $a_2 = f^{-1}(b_2)$

As f is one-to-one

$$f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$$
$$b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$$
i.e., $$f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$$

$\therefore$ f^{-1} is one-to-one function.

As f is onto.

Every element of B is associated with a unique element of A i.e., for any $a \in A$ is pre-image of some $b \in B$ where $b = f(a) \Rightarrow a = f^{-1}(b)$ i.e., for $b \in B$, there exists f^{-1} image $a \in A$.

Hence, f^{-1} is onto.

2.5. Check whether the function $f(x) = x^2 - 1$ is injective or not for $f : R \to R$.

AKTU 2020-21, Marks 02

Ans.
$$f(x) = x^2 - 1$$
Assume $f(a) = f(b)$ for all $a, b, \varepsilon R$

This follows that $a^2 - 1 = b^2 - 1 \Rightarrow a^2 = b^2$

$$\sqrt{a^2} = \sqrt{b^2}$$
$$\pm a = \pm b$$

However, this does not show that $a = b$ as $a = + b$ or $a = - b$

Therefore $f(x)$ is not an injective function.

2.6. Find the composite mapping gof if $f : R \to R$ is given by $f(x) = e^x$ and $g : R \to Z$ is given by $g(x) = \sin x$

AKTU 2022-23, Marks 02

Ans.
$$f(x) = e^x, g(x) = \sin x$$
$$\text{gof} = g(f(x)) = g(e^x) = \sin e^x$$

2.7. Define what is Big-O notation with respect of growth of functions ?

AKTU 2022-23, Marks 02

Ans.
1. We need to approximate the number of steps required to execute any algorithm because of the difficulty involved in expression or difficulty in evaluating an expression. We compare one function with another function to know their rate of growths.
2. If f and g are two functions we can give the statements like 'f has same growth rate as g' or 'f has higher growth rate than g'.
3. Growth rate of function can be defined through notation.

2.8. Define injective, surjective and bijective function.

Ans.
1. **One-to-one function (Injective function or injection) :** Let $f : X \to Y$ then f is called one-to-one function if for distinct elements of X there are distinct image in Y i.e., f is one-to-one iff
$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2 \ \forall \ x_1, x_2 \in X$$

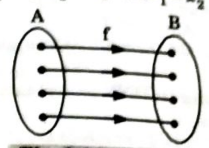

Fig. 2.8.1. One-to-one.

2. **Onto function (Surjection or surjective function) :** Let $f : X \to Y$ then f is called onto function iff for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$ or f is onto if Range $(f) = Y$.

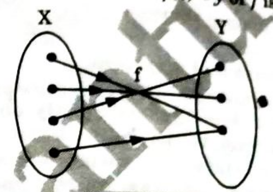

Fig. 2.8.2. Onto.

3. **One-to-one onto function (Bijective function or bijection) :** A function which is both one-to-one and onto is called one-to-one onto function or bijective function.

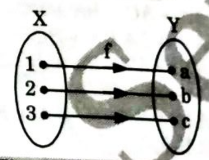

Fig. 2.8.3. One-to-one onto.

2.9. Solve Ackermann function A (2, 1).

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Ans. **Given :** $A(2, 1)$

Here $m = 2$ and $n = 1$

$$\begin{aligned} A(2, 1) &= A(1, A(2, 0)) \\ &= A(1, A(1, 1)) \\ &= A(1, A(0, A(1, 0))) \\ &= A(1, A(0, A(0, 1))) \\ &= A(1, A(0, 2)) \end{aligned}$$

$= A(1, 3)$
$= A(0, A(1, 2))$
$= A(0, A(0, A(1, 1)))$
$= A(0, A(0, A(0, A(1, 0))))$
$= A(0, A(0, A(0, A(0, 1))))$
$= A(0, A(0, A(0, 2)))$
$= A(0, A(0, 3))$
$= A(0, 4) = 5$

2.10. Write the following in DNF $(x + y)(x' + y')$.

Ans. **Given :** $(x + y)(x' + y')$

The complete CNF in two variables (x, y)

$$= (x + y)(x' + y')(x + y')(x' + y)$$

Hence, $f'(x, y) = (x' + y)(x + y')$

$\therefore$ $[f'(x, y)]' = [(x' + y)(x + y')]'$

$$= xy' + x'y$$

which is the required DNF.

2.11. Define minterm and maxterm.

Ans. **Minterm :** A minterm of 'n' variables is a product of 'n' literals in which each variable appears exactly once in either true or complemented form, but not both.

Maxterm : A maxterm of 'n' variables is a sum of 'n' literals in which each variable appears exactly once in either true or complemented form, but not both.

2.12. Prove that $(A + B)(A + C) = A + BC$

Ans.

$$\begin{aligned}
\text{L.H.S} &= (A + B)(A + C) \\
&= AA + AC + BA + BC \\
&= A + AC + BA + BC && (\because\ AA = 1) \\
&= A(1 + C) + BA + BC \\
&= A + AB + BC && (\because\ 1 + C = 1) \\
&= A(1 + B) + BC = A + BC && (\because\ 1 + B = 1) \\
&= \text{R.H.S.}
\end{aligned}$$

2.13. For a given function, $F = x\bar{y} + x\bar{y}$, find complement of F.

Ans.

$F = x\bar{y} + x\bar{y}$

$F = x\bar{y}$ $\qquad (A + A = A)$

Take the complement of both sides,

$\bar{F} = \overline{\bar{x}\,\bar{y}}$

Using de Morgan's first law, we get

$\bar{F} = \bar{x} + \bar{\bar{y}}$

$\bar{F} = \bar{x} + y$

2.14. Obtain an equivalent expression for $[(x.y)(z' + xy')]$.

Ans. Applying general form of De-Morgan's theorem, we get

$$[(x.y)(z' + xy')] = (x.y)' + (z' + xy')' = x' + y' [z + (x' + y)]$$

$$= x' + y' + zx' + zy = x' + x'.z + y' + yz = x' + y' + z'$$

[Applying $x + xy = x$ and $x + x'y = x + y$]

2.15. Define the terms : DNF and CNF.

Ans. **Disjunction Normal Form (DNF) :** A logical expression is said to be in Disjunction Normal Form if it is the sum of elementary product, *i.e.*, join of elementary product.

Conjunctive Normal Form (CNF) : A logical expression is said to be in Conjunctive Normal Form if it is the product of elementary sums.

Example : $p \wedge q$, $(p \vee q) \wedge (\sim p \vee q)$

2.16. Prove that a lattice with 5 elements is not a boolean algebra. **AKTU 2019-20, Marks 02**

Ans. To be a boolean algebra the number of elements in the lattice should be of the form 2^m. Since the number of elements is 5 which is not of the form 2^m. So, it is not a boolean algebra.

2.17. State de Morgan's law and Absorption law. **AKTU 2021-22, Marks 02**

Ans. **de Morgan's law :** De Morgan's laws states that the complement of the union of two sets is the intersection of their complements, and also, the complement of intersection of two sets is the union of their complements.

Absorption law : In algebra, the absorption law or absorption identity is an identity linking a pair of binary operations.

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3 UNIT

Theory of Logics (2 Marks Questions)

3.1. What is compound proposition ?

Ans. A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition is referred to as composite or compound proposition.

3.2. Show the implications without constructing the truth table $(P \to Q) \to Q \Rightarrow P \vee Q$.

Ans. $(P \to Q) \to Q \Rightarrow P \vee Q$

Take L.H.S

$$(P \to Q) \to Q = (\sim P \vee Q) \to Q$$
$$= (\sim(\sim P \vee Q)) \vee Q$$
$$= (P \vee \sim Q) \vee Q$$
$$= (P \vee Q) \vee (\sim Q \vee Q)$$
$$= (P \vee Q) \wedge T = P \vee Q$$

It is equivalent.

3.3. Prove that $(P \vee Q) \to (P \wedge Q)$ is logically equivalent to $P \leftrightarrow Q$.

Ans. $(P \vee Q) \to (P \wedge Q) = P \leftrightarrow Q$

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \leftrightarrow (P \wedge Q)$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

3.4. Show that the propositions $p \to q$ and $\neg p \wedge q$ are logically equivalent.

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Ans. Truth table :

$p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg p \wedge q$:

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

No, both are not equivalent.

3.5. The converse of a statement is : If a steel rod is stretched, then it has been heated. Write the inverse of the statement.

Ans. The statement corresponding to the given converse is "If a steel rod is stretched, then it has been heated". Now the inverse of this statement is "If a steel rod is not stretched then it has not been heated".

3.6. Give truth table for converse, contrapositive and inverse.

Ans. The truth table of the four propositions are as follows :

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

3.7. Show that contrapositive are logically equivalent; that is $\sim q \Rightarrow \sim p \equiv p \Rightarrow q$

Ans. The truth table of $\sim q \Rightarrow \sim p$ and $p \Rightarrow q$ are shown below and the logical equivalence is established by the last two columns of the table, which are identical.

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

3.8. Give truth table for NOR and XOR.

Ans.

Truth table for NOR

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Truth table for XOR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

3.9. Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.

Ans.

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

3.10. Show that $[((p \vee q) \rightarrow r) \wedge (\sim p)] \rightarrow (q \wedge r)$ is tautology or contradiction.

Ans.

p	q	r	$p \vee q$	$\sim p$	$q \wedge r$	$(p \vee q) \rightarrow r$	$((p \vee q) \rightarrow r) \wedge (\sim p)$	$[((p \vee q) \rightarrow r) \wedge (\sim p)] \rightarrow (q \wedge r)$
T	T	T	T	F	T	T	F	T
T	T	F	T	F	F	F	F	T
T	F	T	T	F	F	T	F	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T	F
F	F	F	F	T	F	T	T	F

Question is incorrect. Since the result of the question is contingency.

3.11. Identify whether $(p \wedge q) \rightarrow (p \vee q)$ is tautology or contradiction with using Truth table.

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Ans.

P	q	$P \wedge q$	$P \vee q$	$(P \wedge q) \rightarrow (P \vee q)$
T	F	F	T	T
F	T	F	T	T
T	T	T	T	T
F	F	F	F	T

Since all the rows in last column is true. So the given expression is tautology.

3.12. What are the contrapositive, converse, and the inverse of the conditional statement : "The home team wins whenever it is raining" ?

Ans. **Given :** The home team wins whenever it is raining.

q**(conclusion) :** The home team wins.

p**(hypothesis) :** It is raining.

Contrapositive : $\sim q \rightarrow \sim p$ is "if the home team does not win then it is not raining".

Converse : $q \rightarrow p$ is "if the home team wins then it is raining".

Inverse : $\sim p \rightarrow \sim q$ is "if it is not raining then the home team does not win".

3.13. Translate the conditional statement "If it rains, then I will stay at home" into contrapositive, converse and inverse statement.

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Ans. Let, p : It rains

q : I will stay at home

Symbolic form of given statement is

$p \rightarrow q$

Converse : $q \rightarrow p$

i.e., If I will stay at home, then it rains.

Inverse : $\sim p \to \sim q$

i.e., If it does not rain, then I will not stay at home.

Contrapositive : $\sim q \to \sim p$

i.e., If I will not stay at home, then it does not rain.

3.14. Write the contrapositive of the implication: "if it is Sunday then it is a holiday".

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Ans. Consider the statements :

p : It is Sunday

q : It is a holiday

Contrapositive : $\sim q \;\Rightarrow\; \sim p$

3.15. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Ans. **To prove :** $(p + q)' = p'.q'$

To prove the theorem we will show that

$$(p+q)+p'.q' = 1$$

Consider $(p+q)+p'.q' = \{(p+q)+p'\}.\{(p+q)+q'\}$ by Distributive law

$= \{(q+p)+p'\}.\{(p+q)+q'\}$ by Commutative law

$= \{q+(p+p')\}.\{p+(q+q')\}$ by Associative law

$= (q+1).(p+1)$ by Complement law

$= 1.1$ by Dominance law

$= 1$ ...(3.15.1)

Also consider

$(p+q).p'q' = p'q'.(p+q)$ by Commutative law

$= p'q'.p+p'q'.q$ by Distributive law

$= p.(p'q')+p'.(q'q)$ by Commutative law

$= (p.p').q'+p'.(q.q')$ by Associative law

$= 0.q'+p'.0$ by Complement law

$= q'.0+p'.0$ by Commutative law

$= 0+0$ by Dominance law

$= 0$ ...(3.15.2)

From (3.15.1) and (3.15.2), we get

$p'q'$ is complement of $(p+q)$ *i.e.*, $(p+q)' = p'q'$.

3.16. Find the contrapositive of "If he has courage, then he will win".

Ans. If he will not win then he does not have courage.

3.17. State Universal Modus Ponens and Universal Modus Tollens laws.

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Ans. **Universal modus ponens :** By this rule if $P(x) \to Q(x)$ is true for every x and $P(c)$ is true for some particular member c in UD then $Q(c)$ is true.

$$\begin{array}{r} (\forall\, x)\, P(x) \to Q(x) \\ P(c) \\ \hline \therefore\; Q(c) \end{array}$$

Universal modus tollens : By this rule if $P(x) \to Q(x)$ is true for every x and $\sim Q(c)$ is true for some particular c in UD then $\sim Q(c)$ is true.

$$\begin{array}{r} (\forall\, x)\, P(x) \to Q(x) \\ \sim Q(c) \\ \hline \therefore\; \sim P(c) \end{array}$$

3.18. Write the negation of the following statement : "If I wake up early in the morning, then I will be healthy."

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Ans. **Given :** If I wake up early in the morning, then I will be healthy :

Let p : I wake up early in the morning

q : I will be healthy

Negation : $p \to q$

$$\sim(p \to q) = \sim(\sim p \vee q) = \sim(\sim p) \wedge \sim q = p \wedge \sim q$$

I wake up early in the morning and I will not be healthy.

3.19. Express the following statement in symbolic form : "All flowers are beautiful."

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Ans. $F(x)$: x is a flower

$B(x)$: x is beautiful

$\forall x[F(x) \Rightarrow B(x)]$

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UNIT 4

Algebraic Structures (2 Marks Questions)

4.1. List the properties of cosets.

Ans. Let H be a subgroup of G and 'a' and 'b' belong to G. Then,

i. $a \in aH$
ii. $aH = H$ iff $a \in H$
iii. $aH = bH$
iv. $aH = bH$ iff $a^{-1} b \in H$

4.2. List types of permutation group.

Ans. **Types of permutation group are :**

i. Identity permutation
ii. Inverse permutation
iii. Cyclic permutation
iv. Even and odd permutation

4.3. Define group. **AKTU 2020-21, Marks 02**

Ans. **Group :** Let $(G, *)$ be an algebraic structure where $*$ is binary operation then $(G, *)$ is called a group if following properties are satisfied :

1. $a * b \in G \;\forall\; a, b \in G$ [closure property]
2. $a * (b * c) = (a * b) * c \quad \forall\, a, b, c \in G$ [associative property]
3. There exist an element $e \in G$ such that for any $a \in G$
$a * e = e * a = e$ [existence of identity]
4. For every $a \in G$, $\exists$ element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

For example : $(Z, +)$, $(R, +)$, and $(Q, +)$ are all groups.

4.4. If a and b are any two elements of group G then prove $(a * b)^{-1} = (b^{-1} * a^{-1})$.

Ans. Consider $(a * b) * (b^{-1} * a^{-1})$

$$= a * (b * b^{-1}) * a^{-1}$$
$$= a * e * a^{-1}$$
$$= a * a^{-1} = e$$

Also $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$

$$= b^{-1} * e * b$$

$$= b^{-1} * b = e$$

Therefore $(a * b)^{-1} = b^{-1} * a^{-1}$ for any $a, b \in G$

4.5. What do you mean by integral domain ?

Ans. A ring $(R, +, \cdot)$ is called an integral domain if,

i. It is commutative.
ii. It has multiplicative identity element.
iii. It is without zero divisors.

4.6. Let Z be the group of integers with binary operation $*$ defined by $a * b = a + b - 2$, for all $a, b \in Z$. Find the identity element of the group $(Z, *)$. **AKTU 2019-20, Marks 02**

Ans. Since Z is closed for addition, as we have

$$a + b \in Z \text{ for all } a, b \in Z$$
$$\Rightarrow \quad a + b - 2 \in Z$$
$$\Rightarrow \quad a * b \in Z$$

So $*$ is a binary operation on Z.

Again, $a * b = a + b - 2 = b + a - 2$ (by commutative law of addition on Z)

$$= b * a \text{ for all } a, b \in Z$$

Hence $*$ is commutative.

Again, $(a * b) * c = (a + b - 2) * c$

$$= (a + b - 2) + c - 2 = (a + b + c) - 4$$

and $a * (b * c) = a * (b + c - 2)$

$$= a + (b + c - 2) - 2 = (a + b + c) - 4$$

Thus, $(a * b) * c = a * (b * c)$ for all $a, b, c \in Z$

Hence, $*$ is associative.

Now, if e is the identity element in Z for $*$, then for all $a \in Z$

$$a * e = a \Rightarrow a + e - 2 = a \Rightarrow e = 2 \in Z$$

So, 2 is the identity element for $*$ in Z.

Let the integer a have its inverse b. Then,

$$a * b = -1 \Rightarrow a + b - 2 = 1 \Rightarrow b = 3 - a$$

$\Rightarrow$ So, the inverse of a is $(3 - a)$

4.7. Define ring and give an example of a ring with zero divisors.

OR

Define ring and field.

OR

Define rings and write its properties.

OR

State Ring and Field wtih example. **AKTU 2021-22, Marks 02**

OR

Define ring. **AKTU 2020-21, Marks 02**

Ans. **Ring :** A non-empty set R is a ring if it is equipped with two binary operations called addition and multiplication and denoted by '+' and '.' respectively *i.e.*, for all $a, b \in R$ we have $a + b \in R$ and $a.b \in R$ and it satisfies the following properties :

i. Addition is associative, *i.e.*,
$(a + b) + c = a + (b + c) \; \forall \, a, b, c \in R$

ii. Addition is commutative, *i.e.*,
$a + b = b + a \; \forall \, a, b \in R$

iii. There exists an element $0 \in R$ such that
$0 + a = a = a + 0, \; \forall \, a \in R$

iv. To each element a in R there exists an element $-a$ in R such that
$a + (-a) = 0$

v. Multiplication is associative, *i.e.*,
$a.(b.c) = (a.b).c, \; \forall \, a\, b, c \in R$

vi. Multiplication is distributive with respect to addition *i.e.*, for all $a, b, c \in R$,

Example of ring with zero divisors : R = {a set of 2×2 matrices}.

Field : A ring R with at least two elements is called a field if it has following properties :

i. R is commutative
ii. R has unity
iii. R is such that each non-zero element possesses multiplicative inverse.

For example : The rings of real numbers and complex numbers are also fields.

4.8. Prove that if $a^2 = a$, then $a = e$, a being an element of a group.

Ans. Let a be an element of a group G such that $a^2 = a$.
To prove that $a = e$.

$$a^2 = a \Rightarrow a.a = a \Rightarrow (aa)\, a^{-1} = aa^{-1}$$
$$\Rightarrow a(aa^{-1}) = e \qquad (\because aa^{-1} = e)$$
$$\Rightarrow ae = e \Rightarrow a = e \qquad (\because ae = a)$$

4.9. Define normal subgroup.

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Ans. **Normal subgroup :** A subgroup H of G is said to be normal subgroup of G if $Ha = aH \; \forall \; a \in G$ *i.e.*, the right coset and left coset of H is G generated by a are the same.

i. Clearly, every subgroup H of an abelian group G is a normal subgroup of G. For $a \in G$ and $h \in H$, $ah = ha$.

ii. Since a cyclic group is abelian, every subgroup of a cyclic group is normal.

4.10. If H is a subgroup of G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G.

Ans. For any $g \in G$, $h \in H$; $(gh)^2 \in H$ and $g^{-2} \in H$.
Since H is a subgroup, $h^{-1} g^{-2} \in H$ and so $(gh)^2 h^{-1} g^{-2} \in H$. This gives that $gh\, gh\, h^{-1} g^{-2} \in H$, *i.e.*, $ghg^{-1} \in H$. Hence, H is a normal subgroup of G.

4.11. State and justify "Every cyclic group is an abelian group".

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Ans. **Abelian group :** A group $(G, *)$ is called abelian group or commutative group if binary operation * is commutative *i.e.*, $a*b = b*a \; \forall \; a, b \in G$.
Let G be a cyclic group and let a be a generator of G so that
$$G = \langle a \rangle = \{a^n : n \in Z\}$$
If g_1 and g_2 are any two elements of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$. Then
$$g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s \cdot a^r = g_2 g_1$$
So, G is abelian.

4.12. Prove that if $a, b \in R$ then $(a + b)^2 = a^2 + ab + ba + b^2$.

Ans. We have
$$(a + b)^2 = (a + b)\,(a + b)$$
$$= a\,(a + b) + b\,(a + b) \quad \text{[by right distributive law]}$$
$$= (aa + ab) + (ba + bb) \quad \text{[by left distributive law]}$$
$$= a^2 + ab + ba + b^2.$$

4.13. In an integral domain D, show that if $ab = ac$ with $a \neq 0$ then $b = c$.

Ans. Since $ab = ac$ we have
$ab - ac = 0$ and so $a\,(b - c) = 0$
Since $a \neq 0$, we must have $b - c = 0$, since D has no zero divisors.
Hence $b = c$.

4.14. Prove that left inverse of an element is also its right inverse *i.e.*, $a^{-1} * a = e = a * a^{-1}$

Ans. Now $a^{-1} * (a * a^{-1}) = (a^{-1} * a) * a^{-1}$ (Associativity)
$$= e * a^{-1}$$
$$= a^{-1} * e$$
Thus, $a^{-1} * (a * a^{-1}) = a^{-1} * e$
$$a * a^{-1} = e$$
Thus, the left inverse of an element in a group is also its right inverse.

4.15. Define Lagrange's theorem. What is the use of the theorem ?

Ans. **Lagrange's theorem :**
Statement : The order of each subgroup of a finite group is a divisor of the order of the group.

Use of theorem :

i. It can be used to find the subgroup of any order for the symmetric group.

ii. It tells that the number of subgroups of the cyclic group of order p, when p is prime then there is only one subgroup and that is (ϕ).

4.16. Show that every cyclic group is abelian.

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Ans. Let G be a cyclic group and let a be a generator of G so that

$$G = \langle a \rangle = \{a^n : n \in Z\}$$

If g_1 and g_2 are any two elements of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$. Then

$$g_1 g_2 = a^r a^s = a^{r+s} = a^{s+r} = a^s \cdot a^r = g_2 g_1$$

So, G is abelian.

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5 UNIT Graphs and Combinatorics (2 Marks Questions)

5.1. Define multigraph. Explain with example in brief.

Ans. A multigraphs $G(V, E)$ consists of a set of vertices V and a set of edges E such that edge set E may contain multiple edges and self loops.

Example :

a. **Undirected multigraph :**

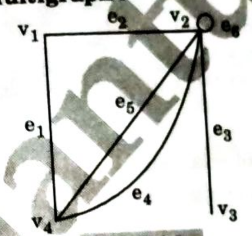

Fig. 5.1.1.

b. **Directed multigraph :**

Fig. 5.1.2.

5.2. Define complete and regular graph.

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Ans. **Complete graph :** A simple graph, in which there is exactly one edge between each pair of distinct vertices is called a complete graph.

Regular graph (n-regular graph) : If every vertex of a simple graph has equal edges then it is called regular graph.

5.3. Let G be a graph with ten vertices. If four vertices has degree four and six vertices has degree five, then find the number of edges of G.

Ans. We know that

$$\sum_i \deg(v_i) = 2e$$

$$4+4+4+4+4+5+5+5+5+5+5 = 2e$$
$$16 + 30 = 2e$$
$$2e = 46$$
$$e = 23$$

5.4. Find the minimum number of students in a class to show that five of them are born in same month.

Ans. **Using pigeonhole principle :**

Five of them are born in same month, so $n = ?, m = 12$.

$$5 = \left[\frac{n-1}{m}\right] + 1$$

$$4 = \frac{n-1}{12}$$

$$48 = n - 1$$

$$n = 49$$

$\therefore$ 49 students are there to show that at least 5 of them are born in same month.

5.5. Explain edge coloring and *k*-egde coloring.

Ans. **Edge coloring :** An edge coloring of a graph *G* may also be thought of as equivalent to a vertex coloring of the line graph *L*(*G*), the graph that has a vertex for every edge of *G* and an edge for every pair of adjacent edges in *G*.

***k*-edge coloring :** A proper edge coloring with *k* different colors is called a (proper) *k*-edge coloring.

5.6. State and prove pigeonhole principle.

Ans. **Pigeonhole principle :** If *n* pigeons are assigned to *m* pigeonholes then at least one pigeon hole contains two or more pigeons ($m < n$).

Proof :

1. Let *m* pigeonholes be numbered with the numbers 1 through *m*.
2. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number.
3. Since $m < n$ *i.e.*, the number of pigeonhole is less than the number of pigeons, $n - m$ pigeons are left without having assigned a pigeon hole.
4. Thus, at least one pigeonhole will be assigned to a more than one pigeon.

5.7. Draw the digraph *G* corresponding to adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Ans. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required d_i graph is shown in Fig. 5.7.1.

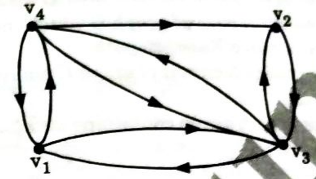


Fig. 5.7.1.

5.8. A connected plane graph has 10 vertices each of degree 3. Into how many regions, does a representation of this planar graph split the plane ?

Ans. Here $n = 10$ and degree of each vertex is 3.

$\Sigma \deg(v) = 3 \times 10 = 30$

But $\Sigma \deg(v) = 2e \Rightarrow 30 = 2e \Rightarrow e \Rightarrow 15$

By Euler's formula, we have $n - e + r = 2$

$10 - 15 + r = 2 \Rightarrow r = 7$.

5.9. How many permutations of the letters of the word BANANA ?

Ans. There are 6 letters in the word BANANA of which three are alike of one kind (3A's), two are alike of second kind (2N's) and the rest one letter is different.

Hence, the required number of permutations $= \frac{6!}{3!2!} = 60$.

5.10. How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed ?

Ans. When repetition is allowed :

The thousands place can be filled by 4 ways.

The hundreds place can be filled by 4 ways.

The tens place can be filled by 4 ways.

The units place can be filled by 4 ways.

$\therefore$ Total number of 4 digit number $= 4 \times 4 \times 4 \times 4 = 256$

5.11. How many bit strings of length eight either start with a '1' bit or end with the two bit '00' ?

Ans.
1. Number of bit strings of length eight that start with a 1 bit : $2^7 = 128$.
2. Number of bit strings of length eight that end with bits 00 : $2^6 = 64$.
3. Number of bit strings of length eight $2^5 = 32$ that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

5.12. Define Eulerian path, circuit and graph.

Ans. **Eulerian path :** A path of graph G which includes each edge of G exactly once is called Eulerian path.

Eulerian circuit : A circuit of graph G which include each edge of G exactly once.

Eulerain graph : A graph containing an Eulerian circuit is called Eulerian graph.

5.13. Define chromatic number and isomorphic graph.

Ans. **Chromatic number :** The minimum number of colours required for the proper colouring of a graph so that no two adjacent vertices have the same colour, is called chromatic number of a graph.

Isomorphic graph : If two graphs are isomorphic to each other then :

i. Both have same number of vertices and edges.
ii. Degree sequence of both graphs are same (degree sequence is the sequence of degrees of the vertices of a graph arranged in non-increasing order).

5.14. Define walk.

Ans. In a graph G, a finite alternating sequence of vertices and edges v_1, e_1, v_2, e_2 ... starting and ending with vertices such that each edge in sequence is incident with the vertices following and preceding, it is called walk.

5.15. Define non-planar graph.

Ans. A graph G is said to be non-planar graph if it cannot be drawn in a plane so that no edges cross.

5.16. Explain Euler's formula. Determine number of regions if a planar graph has 30 vertices of degree 3 each.

AKTU 2021-22, Marks 02

Ans. Euler's formula :

$$F + V = E + 2$$

where, F = Number of faces
V = Number of vertices
ε = Number of edges

Numerical :

Given : $V = 30, \varepsilon = 3$
To Find : $F = ?$

$$F = |\varepsilon + 2 - V|$$
$$= |5 - 30| = 25$$

5.17. Explain pigeonhole principle with example.

AKTU 2021-22, Marks 02

OR

Define Pigeonhole principle. **AKTU 2019-20, Marks 02**

Ans. **Pigeonhole principle :**
The pigeonhole principle is sometime useful in counting methods. If n pigeons are assigned to m pigeonholes then at least one pigeonhole contains two or more pigeons ($m < n$).

Example : Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution : Here n = 12 months are the pigeonholes

And $k + 1 = 3$
$k = 2$

5.18. Draw an adjacency matrix for the following graph.

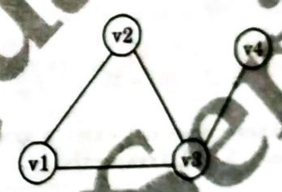


Fig. 5.18.1.

AKTU 2022-23, Marks 02

Ans. Adjacency matrix is given is :

$$\begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 1 & 0 \\ V_2 & 1 & 0 & 1 & 0 \\ V_3 & 1 & 1 & 0 & 1 \\ V_4 & 0 & 0 & 1 & 0 \end{array}$$

Fig. 5.18.2.

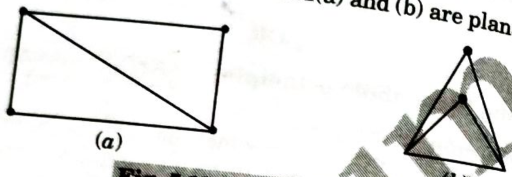
5.19. Describe planar graph and express Euler's formula for planar graph.

AKTU 2022-23, Marks 02

Ans. **Planar graph :** A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex :

i. A graph is said a planar graph, if it cannot be drawn on a plane without a crossover between its edges crossing.

ii. The graphs shown in Fig. 5.19.1(a) and (b) are planar graphs.



(a) (b)

Fig. 5.19.1. Some planar graph.

Euler's formula :

$$F + V = E + 2$$

where,
F = Number of faces
V = Number of vertices
ε = Number of edges

Numerical :

Given : $V = 30$, $\varepsilon = 3$
To Find : $F = ?$

$$F = |\varepsilon + 2 - V|$$
$$= |5 - 30| = 25$$

5.20. Show that there does not exist a graph with 5 vertices with degrees 1, 3, 4, 2, 3 respectively.

AKTU 2019-20, Marks 02

Ans. According to hand shaking theorem, sum of degree of all the vertices is even. But in the given degree sequence

$$1 + 3 + 4 + 2 + 3 = 13$$

Sum of degrees of all vertices is 13 which is an odd number. Hence, no such graph exists with the given degree sequence.

☺☺☺

B.Tech.

(SEM. III) ODD SEMESTER THEORY EXAMINATION, 2019-20

DISCRETE STRUCTURES & THEORY OF LOGIC

Time : 3 Hours **Max. Marks : 100**

Note : 1. Attempt **all** Section.

Section-A

1. Answer **all** questions in brief. **(2 × 10 = 20)**

a. **Define various types of functions.**

Ans. Refer Q. 2.1, Page SQ–8F, Unit-2, Two Marks Questions.

b. **How many symmetric and reflexive relations are possible from a set A containing 'n' elements ?**

Ans. Refer Q. 1.9, Page SQ–2F, Unit-1, Two Marks Questions.

c. **Let Z be the group of integers with binary operation * defined by $a * b = a + b - 2$, for all $a, b \in Z$. Find the identity element of the group (Z, *).**

Ans. Refer Q. 4.6, Page SQ–20F, Unit-4, Two Marks Questions.

d. **Show that every cyclic group is abelian.**

Ans. Refer Q. 4.16, Page SQ–23F, Unit-4, Two Marks Questions.

e. **Prove that a lattice with 5 elements is not a boolean algebra.**

Ans. Refer Q. 2.16, Page SQ–12F, Unit-2, Two Marks Questions.

f. **Write the contrapositive of the implication: "if it is Sunday then it is a holiday".**

Ans. Refer Q. 3.14, Page SQ–17F, Unit-3, Two Marks Questions.

g. **Show that the propositions $p \to q$ and $\neg p \wedge q$ are logically equivalent.**

Ans. Refer Q. 3.4, Page SQ–13F, Unit-3, Two Marks Questions.

h. **Show that there does not exist a graph with 5 vertices with degrees 1, 3, 4, 2, 3 respectively.**

Ans. Refer Q. 5.20, Page SQ–29F, Unit-5, Two Marks Questions.