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CRITICAL BEHAVIOR AND THRESHOLD OF COEXISTENCE OF A PREDATOR–PREY STOCHASTIC MODEL IN A 2D LATTICE

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We investigate the critical behavior of a stochastic lattice model describing a predator–prey system. By means of Monte Carlo procedure we simulate the model defined on a regular square lattice and determine the threshold of species coexistence, that is, the critical phase boundaries related to the transition between an active state, where both species coexist and an absorbing state where one of the species is extinct. A finite size scaling analysis is employed to determine the order parameter, order parameter fluctuations, correlation length and the critical exponents. Our numerical results for the critical exponents agree with those of the directed percolation universality class. We also check the validity of the hyperscaling relation and present the data collapse curves.

Keywords: Criticality; directed percolation; Monte Carlo.

1. Introduction

The analysis of phase transitions and critical phenomena of nonequilibrium systems is of fundamental importance in the study of several features of complex phenomena in physics. Nonequilibrium phase transitions from an active state into an absorbing state are commonplace; the contact process (CP) being one of the simplest models presenting such a dynamic transition [Harris, 1974; Marro & Dickman, 1999; Dickman, 1996]. Its universality class is the directed percolation (DP) class [Marro & Dickman, 1999], which includes several systems presenting a dynamic transition into a single absorbing state. The CP is the prototype model for this class. It consists of a D-dimensional lattice where each site can be in either an active or inactive state. Upon contact with an active neighbor, the inactive sites become active according to a given probability. The active state is transient, having a finite lifetime. A short lifetime of active sites results in the whole systems being driven to the absorbing state with only inactive sites. Above a critical lifetime, the system reaches a stationary active state with a fluctuating finite fraction of active sites. In previous works [Fulco *et al.*, 2001; da Costa *et al.*, 2007; Tomé & de Oliveira, 2005] some of the authors analyzed the critical behavior of one-dimensional models. In [Fulco *et al.*, 2001] a model of an epidemic process in a population of diffusive individuals, through a contact-like reaction–diffusion decay process of two species was considered; a process mediated by a density of diffusive individuals which can infect a static population was the subject of [Macnadbay *et al.*, 2005] and a diffusion-limited reaction model consisting of two interacting species that simulates the spreading of an epidemiological process in a diffusive population mediated by a static vector was studied in [da Costa *et al.*, 2007]. The dynamic transition of the model considered in [Macnadbay *et al.*, 2004] does not belong to the usual DP class, in agreement with the fact that diffusion is an important mechanism that can influence the critical behavior of absorbing states. In [da Costa *et al.*, 2007] it was found that the critical behavior has a dynamical phase transition from an absorbing to a stationary state with part of the population in the active state, with large deviations from the DP class. It was found that the critical exponents are very distinct from the expected values of directed percolation. In particular, the value for $1/\nu z$ is less than half the expected value of DP. This is evidence that higher order terms in the action functional of the field-theoretical analysis of the contact process with diffusive and conserved fields are necessary to correctly predict the critical behavior in 1D. In this paper, we study an interacting population biological model whose critical behavior also places the model in the DP universality class. In particular, we consider a predator–prey system and investigate the threshold of species coexistence. Our approach is the one based on stochastic spatial structured models. In the last years a great amount of works have shown the relevance of this kind of approach to describe biological population problems [Tainaka, 1989; Satulovsky & Tomé, 1994; Durrett & Levin, 1994; Boccara *et al.*, 1994; Provata *et al.*, 1999; Antal & Droz, 2001; Antal *et al.*, 2001; Ovaskainen *et al.*, 2002; Aguiar *et al.*, 2003; Carvalho & Tomé, 2004; Szabó & Sznaider, 2004; Mobilia *et al.*, 2006; Arashiro & Tomé, 2007; Arashiro *et al.*, 2008; Rodrigues & Tomé, 2008]. We focus on the stochastic lattice model for a predator–prey system introduced by Satulovsky and Tomé [1994]. This model exhibits a phase diagram with active states where both species coexist and an absorbing state where one of the species has been extinct. Here, we calculate the static critical exponents associated to the nonequilibrium phase transition from the active state into the absorbing state. It is worth mentioning that the dynamic critical exponents associated to the synchronous version of this model have been obtained by some of the present authors [Arashiro & Tomé, 2007]. In order to find the critical properties we perform Monte Carlo simulations of the model and we use finite size scaling analysis. In that way we determine a set of critical exponents which allow the classification of the model in a universality class. We have found that the present system exhibit a continuous phase transition which belongs to the DP universality class. The motivation of the present work is to study the critical behavior of this model to find the threshold of species coexistence of a predator–prey system. We show that the continuous second order phase transition exist and locate it by means of the relative fluctuation (cumulant) technique. We also calculate the critical exponents and compare them with the DP universality class.

2. Model and Simulations

Let us denote by X, Y and Z respectively prey, predators and empty sites (which plays the role of food for prey). The habitat where the individuals survive, interact and proliferate is represented here by a regular square lattice. Each site of the lattice can be empty or occupied by just one individual of each species. Therefore, a site in the lattice can be in one of the three states: occupied by a prey individual; occupied by a predator individual or can be empty. The model comprehends the following set of reactions:

$$Z + X \xrightarrow{a} 2X$$

$$X + Y \xrightarrow{b} 2Y$$

and

$$Y \xrightarrow{c} Z$$

which describe the cyclic process $X \to Y \to Z \to X$. Those are basic and relevant reactions that characterize a simple predator–prey system. They are taken into account here by considering the stochastic lattice model defined by an asynchronous global dynamics composed by a set of local Markovian rules:

(a) *Prey proliferation*. In an empty site, a prey individual can be born with a probability $a/4$ times the number of sites occupied by prey in the neighborhood. a is a parameter related to prey

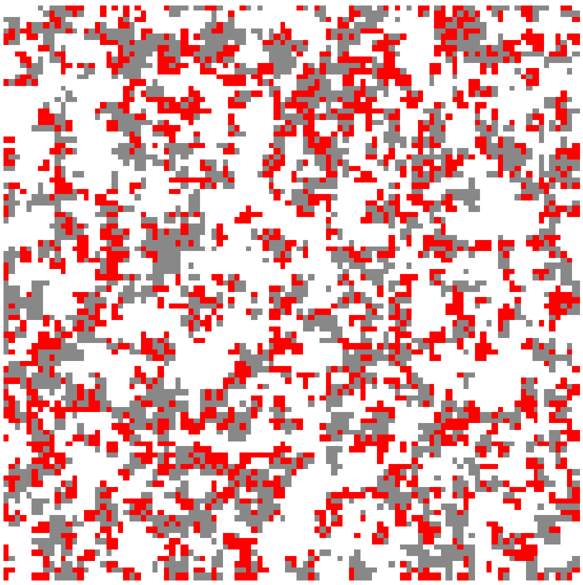

Fig. 1. Configuration at $t = 470$ MCS for the predator–prey model on a square lattice with $N = 100^2$ sites and at $c = 0.15$ ($a = 0.4$ and $b = 0.45$). Predators are in red, preys are in white and empty sites are in grey.

proliferation and it is divided by $z = 4$, where z is the coordination number of the lattice;

(b) *Predation*. Death of prey and birth of predators. In a site occupied by a prey, a predator individual can be born with probability $b/4$ times the number of sites occupied by predators in the neighborhood. In this process there occurs an instantaneous prey death;

(c) *Death of predators*. Predators die spontaneously with probability c and leave an empty site. This process reintegrates to the system the resources for prey proliferation.

The model has three parameters a, b and c which assume values in the interval $(0, 1)$. Without loss of generality, we consider $a + b + c = 1$. Therefore, we are left with only two independent parameters.

The critical behavior will be characterized by measuring a set of relevant static critical exponents obtained by the use of a finite size scaling analysis of the critical order parameter and its relative fluctuation.

In what follows, we show results from simulations on finite lattices with $N = L^2$ sites (L is the linear size). Each lattice sweep is considered as the time unit or one Monte Carlo step (MCS). The process is updated sequentially. We start from an initial condition with a single predator at the center of the square lattice covered by prey. Once the system is placed in the initial condition we apply the local rules (a)–(c). An example of a configuration obtained by simulations is shown in Fig. 1. The system evolves in time and eventually reaches stationary states. To avoid the system to become trapped in an absorbing state, we activate a predator individual in a random chosen site whenever the system becomes trapped in it. We measure the order parameter and the density of predators $\rho = \langle N_Y \rangle / N$, where $\langle N_Y \rangle$ is the number of predators in the stationary regime as a function of the parameter c (probability of predators death).

3. Results

In Fig. 2 we show the density of predators ρ as a function of the spontaneous death probability c, as obtained from simulations on lattices of distinct sizes L. As $L \to \infty$ a transition from a state with nonzero density of predators to the prey absorbing state takes place by increasing the values of c. The values used in our simulations were: $a = 0.4$ and c ranging from 0.18 up to 0.2 with step 0.01 (b is calculated by the relation $a + b + c = 1$). To precisely locate the critical spontaneous death probability c_c, we measured the relative fluctuation, that is, the ratio between the second moment and squared first moment of predators' number, defined as

$$U_L(c) = \frac{\langle N_Y^2 \rangle}{\langle N_Y \rangle^2} \tag{1}$$

which is independent of the system size at the critical point. In Fig. 3, we plot $U_L(c)$ obtained from simulations performed in distinct lattice sizes,

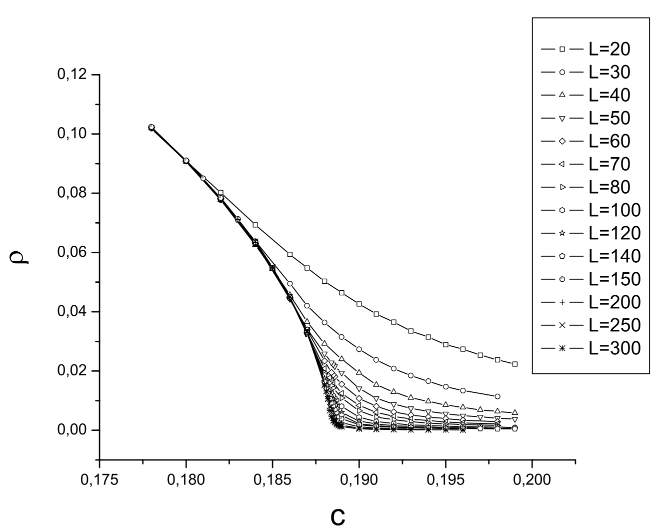

Fig. 2. Density of predators ρ in the active state versus c for distinct linear lattice sizes L.

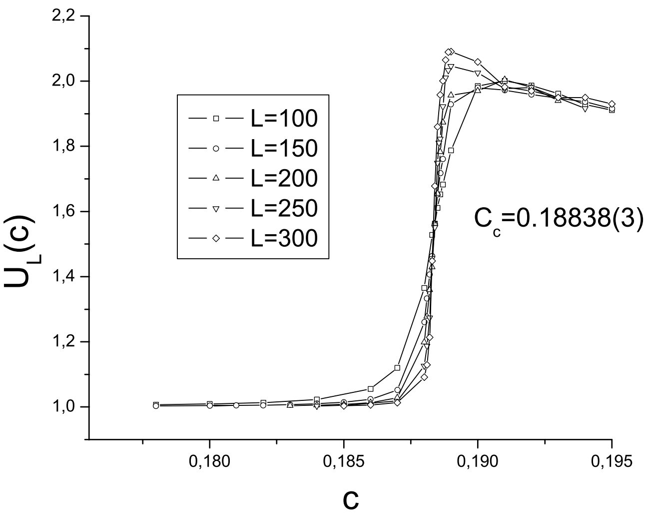

Fig. 3. The moment ratio $U_L(c)$ as a function of the predator death probability c for distinct lattice sizes. The scale invariance at the critical point allowed us to precisely estimate the critical predator spontaneous death rate $c_c = 0.18838(3)$.

which allow us to estimate the critical probability as $c_c = 0.18838(3)$. At this critical value, we found that $U_L(c_c) \approx 1.5$, similar to the value reported in [Dickman & Kamphorst, 1998].

Once having located the critical point, finite size scaling relations were used to compute the critical exponents characterizing such nonequilibrium phase transition. In particular, the order parameter obeys the power law $\rho \propto L^{-\beta/\nu}$ and its logarithmic derivative $(d \ln \rho / dc) \propto L^{1/\nu}$. These scaling

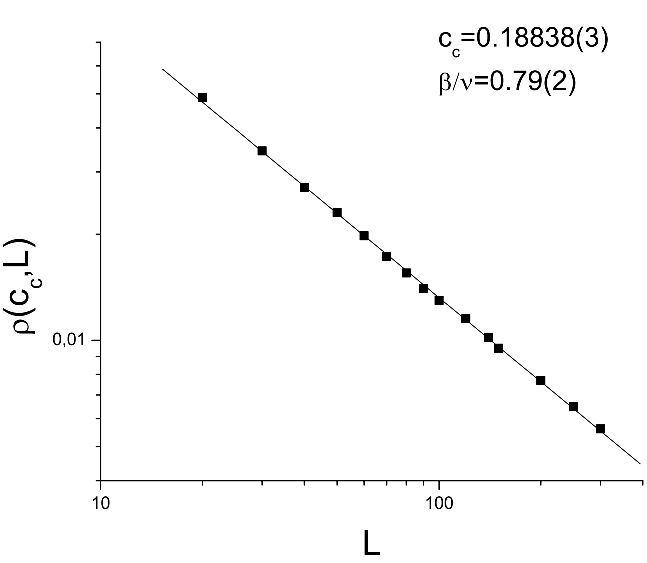

Fig. 4. Log-log plot of the order parameter versus the linear size L. From the best fit to a power-law, we estimate the critical exponent ratio $\beta/\nu = 0.79(2)$ for the square lattice.

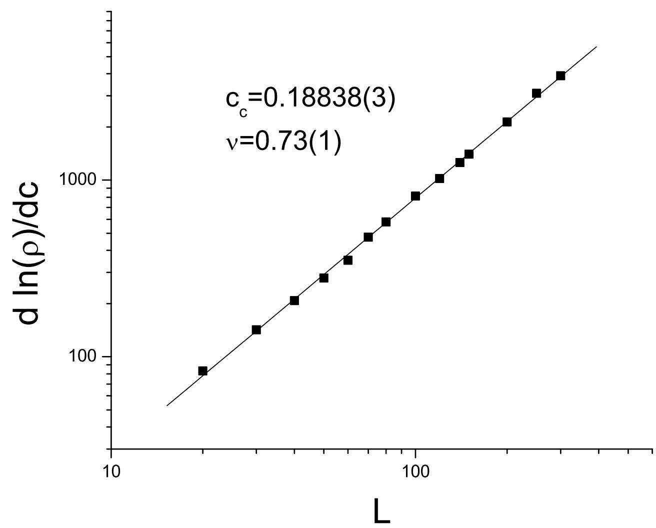

Fig. 5. The logarithmic derivative of the critical order parameter versus L. From the best fit to a power-law, we estimate the critical exponent $\nu = 0.73(1)$ for the square lattice.

laws are depicted in Figs. 4 and 5 from which we estimate $\beta/\nu = 0.79(2)$ and $\nu = 0.73(1)$ for the square lattice.

In Fig. 6, we calculate the order parameter fluctuations:

$$\Delta\rho = \frac{\langle N_Y^2 \rangle - \langle N_Y \rangle^2}{N} \tag{2}$$

for the square lattice versus c for several lattices. The increasing peaks signals the diverging fluctuations at the critical point as the thermodynamical

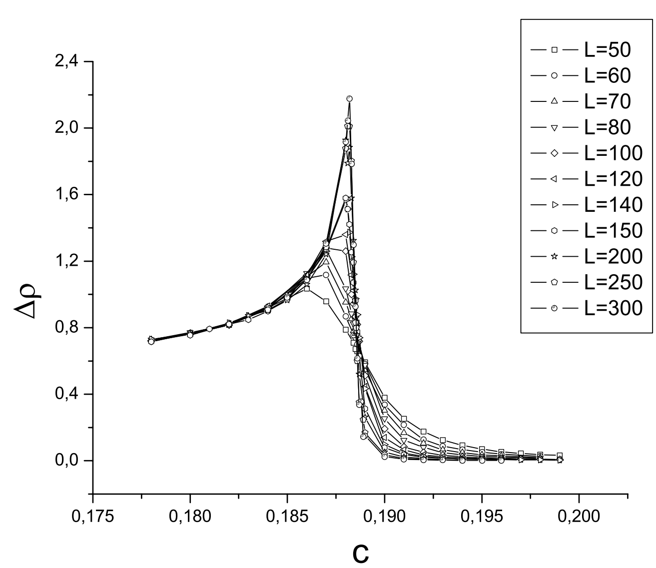

Fig. 6. Order parameter fluctuations $\Delta\rho$ versus c for distinct linear lattice sizes L. The peak in the vicinity of the critical density signalizes the enhancement of the order parameter fluctuations near the transition.

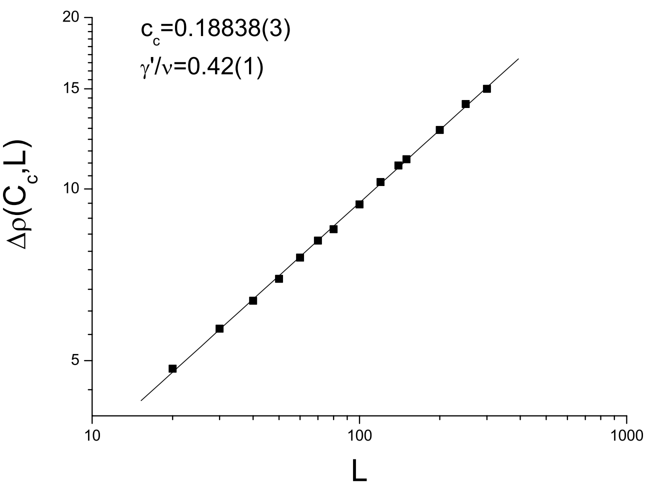

Fig. 7. Log-log plot of the order parameter fluctuations $\Delta\rho$ versus L at the critical point. The exponent ratio γ'/ν is estimated from the slope of the fitted straight line from which we obtained $\gamma'/\nu = 0.42(1)$.

limit is approached. The data for the order parameter fluctuations at the critical point are used in Fig. 7 to obtain critical exponents ratio γ'/ν since $\Delta\rho \propto L^{\gamma'/\nu}$.

In Fig. 8, we present data collapse of the order parameter density computed from different lattice sizes. Using $c_c = 0.18838(3)$, the ratio of critical exponents $\beta/\nu = 0.79(2)$ and the critical exponent $\nu = 0.73(1)$ is estimated. We also present in Fig. 9, the data collapse of the order parameter density fluctuation computed from different lattice sizes. These are the results for the square lattice and using

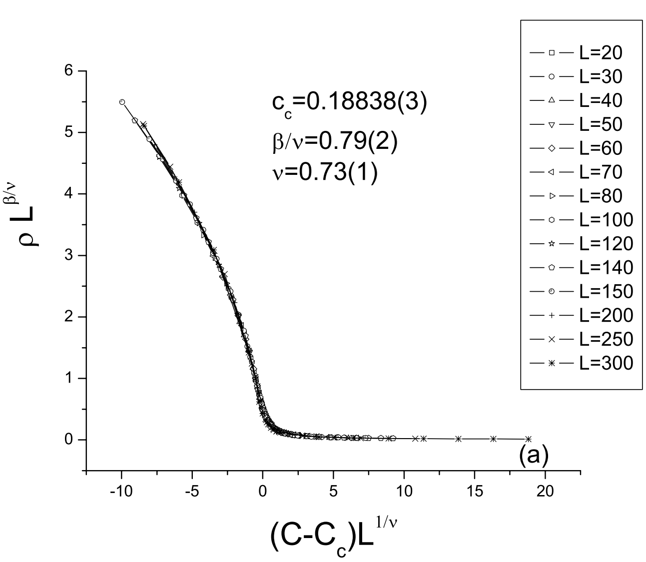

Fig. 8. Data collapse of the order parameter density computed from different linear lattice sizes L and using $c_c = 0.18838(3)$, $\beta/\nu = 0.79(2)$ and $\nu = 0.73(1)$.

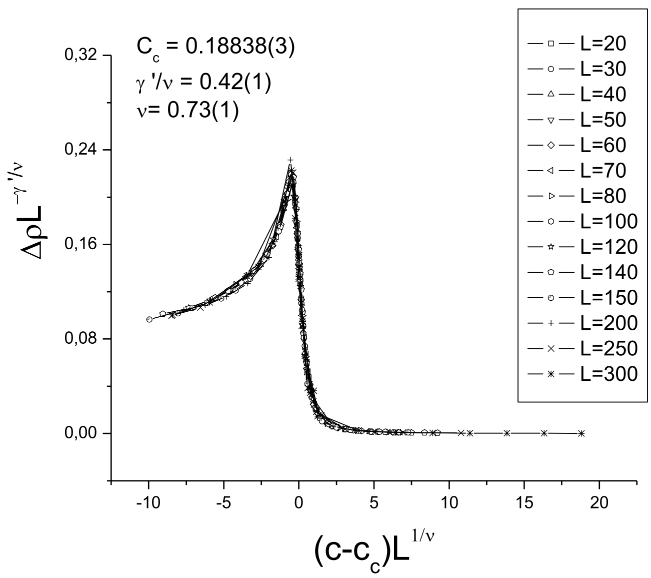

Fig. 9. Data collapse of the order parameter density fluctuations computed from different linear lattice sizes L and using $c_c = 0.18838(3)$, $\gamma'/\nu = 0.42(1)$ and $\nu = 0.73(1)$.

Table 1. Values of critical exponents β/ν, ν and γ'/ν for the square lattice. For comparison, we show in the last rows the corresponding values for the contact process [Lubeck & Willmann, 2005].

—	β/ν	ν	γ'/ν
This model	0.79(2)	0.73(1)	0.42(1)
CP-2D	0.796(2)	0.733(7)	0.409(1)

$c_c = 0.18838(3)$, the exponents $\gamma'/\nu = 0.42(1)$ and $\nu = 0.73(1)$ are estimated considering this critical density. At this point, we emphasize that the uncertainty in the location of the critical point was taken into account in the estimation of the error bars in the critical exponents.

Finally, in Table 1, we present the obtained values of the critical exponents for the square lattice. The values of the CP in 2D are shown at the end of the table for comparison. The results are similar to those of DP. Our estimated value for γ'/ν is consistent with the hyperscaling relation $2\beta/\nu + \gamma'/\nu = 2$.

4. Conclusions

We have investigated the critical behavior of a stochastic spatial structured model in which prey and predator individuals reside on the sites of a square lattice and are described by discrete stochastic variables. We showed that the proposed model presents a transition from an active state to an absorbing one at a critical predator death probability c_c. From numerical simulations of this irreversible model defined in two-dimensions and using finite size scaling analysis, we computed relevant critical exponents governing this nonequilibrium phase transition. The results are consistent with the ones of the directed percolation universality class and agree with the dynamical results previously presented by some of the present authors [Arashiro & Tomé, 2007].

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