Problema 5.5

1) Consideremos la función:

$$f: (-\infty,0)\cup(0,\infty) \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = \arctan(x) + \arctan(1/x)$$

f es continua y derivable en $(-\infty,0)\cup(0,\infty)$.

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) =$$
$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Aunque $f'(x) = 0 \ \forall x \in (-\infty,0)\cup(0,\infty)$, no podemos garantizar que f sea constante ya que el dominio no es un intervalo. En este punto sólo podemos garantizar que:

$$f(x) = \arctan(x) + \arctan(1/x) = \begin{cases} c_1 & \text{si } x \in (-\infty,0) \\ c_2 & \text{si } x \in (0,\infty) \end{cases}$$

- Para calcular c_1 basta evaluar f en un punto de $(-\infty,0)$:
$$c_1 = f(-1) = 2\arctan(-1) = -\frac{\pi}{2}$$

- Para calcular c_2 basta evaluar f en un punto de $(0,\infty)$:
$$c_2 = f(1) = 2\arctan(1) = \pi/2$$

$$\Rightarrow \begin{cases} \arctan(x) + \arctan(1/x) = \frac{\pi}{2} & \text{si } x>0 \\ \arctan(x) + \arctan(1/x) = -\frac{\pi}{2} & \text{si } x<0 \end{cases}$$

2) Consideremos la función:

$$f : (-\infty, 1) \cup (1, \infty) \longrightarrow \mathbb{R}$$
$$x \mapsto f(x) = \arctan\left(\frac{1+x}{1-x}\right) - \arctan(x)$$

f es continua y derivable en $(-\infty, 1) \cup (1, \infty)$

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \frac{1-x+1+x}{(1-x)^2} - \frac{1}{1+x^2} =$$

$$= \frac{2}{(1-x)^2+(1+x)^2} - \frac{1}{1+x^2}$$

$$= \frac{2}{2+2x^2} - \frac{1}{1+x^2} = 0$$

Por tanto:

$$\arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = \begin{cases} c_1 & \text{si } x \in (-\infty, 1) \\ c_2 & \text{si } x \in (1, \infty) \end{cases}$$

donde $c_1 = \arctan\left(\frac{1+0}{1-0}\right) - \arctan(0) = \frac{\pi}{4}$

$$c_2 = \lim_{x \to \infty} \left(\arctan\left(\frac{1+x}{1-x}\right) - \arctan(x)\right) =$$

$$= \arctan(-1) - \frac{\pi}{2} = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$$

$$\Rightarrow \begin{cases} \arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = \frac{\pi}{4} & \text{si } x < 1 \\ \arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = -\frac{3\pi}{4} & \text{si } x > 1 \end{cases}$$

3) Consideremos la función:

$$f(x) = 2\arctan(x) + \operatorname{arcsen}\left(\frac{2x}{1+x^2}\right)$$

Nótese que $\operatorname{Dom} f = \left\{x : -1 \leq \frac{2x}{1+x^2} \leq 1\right\} =$

$$= \{x : \underbrace{-1-x^2 \leq 2x \leq 1+x^2}\} = \mathbb{R}$$

$$-1-x^2-2x \leq 0 \leq 1+x^2-2x$$

$$-(1+x)^2 \leq 0 \leq (1-x)^2$$

Además, f es derivable si $\left\{x : -1 < \frac{2x}{1+x^2} < 1\right\} =$

$$= \mathbb{R} \setminus \{\pm 1\}$$

Si $x \neq \pm 1$ se tiene que:

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$= \frac{2}{1+x^2} + \frac{1+x^2}{\sqrt{1+x^4-2x^2}} \quad \frac{2-2x^2}{(1+x^2)^2}$$

$$= \frac{2}{1+x^2} + \frac{2}{\sqrt{(1-x^2)^2}} \quad \frac{1-x^2}{1+x^2}$$

$\sqrt{z^2} = |z|$

$$= \frac{2}{1+x^2} + \frac{1-x^2}{|1-x^2|} \cdot \frac{2}{1+x^2} =$$

$$= \frac{2}{1+x^2}\left(1 + \frac{1-x^2}{|1-x^2|}\right)$$

$z \neq 0$: $\frac{z}{|z|} = \operatorname{sig}(z)$

$$= \frac{2}{1+x^2}\left(1 + \operatorname{sig}(1-x^2)\right)$$

$$= \begin{cases} \frac{4}{1+x^2} & \text{si } 1-x^2 > 0 \\ 0 & \text{si } 1-x^2 < 0 : \end{cases}$$

Por tanto: $f'(x) = 0 \iff 1 - x^2 < 0$

$$\iff 1 < x^2$$

$$\iff 1 < |x|$$

$$\iff x \in (-\infty, -1) \cup (1, \infty)$$

$\Rightarrow$ Si $x > 1 \Rightarrow f(x) = c_1$

Si $x < -1 \Rightarrow f(x) = c_2$

En particular, como f es continua:

$$c_1 = \lim_{x \to 1} f(x) = f(1) = 2\arctan(1) + \arcsen(1)$$

$$= 2 \cdot \frac{\pi}{4} + \frac{\pi}{2} = \pi$$

$$c_2 = \lim_{x \to -1} f(x) = f(-1) = 2\arctan(-1) + \arcsen(-1)$$

$$= -2 \cdot \frac{\pi}{4} - \frac{\pi}{2} = -\pi$$

$\Rightarrow$

- $2\arctan(x) + \arcsen\left(\frac{2x}{1+x^2}\right) = \pi$ si $x \geq 1$
- $2\arctan(x) + \arcsen\left(\frac{2x}{1+x^2}\right) = -\pi$ si $x \leq -1$
- $f'(x) \neq 0$ si $x \in (-1, 1) \Rightarrow$ La función <u>no</u> es constante en el intervalo $-1 \leq x \leq 1$