PROBLEMA 7.1

1) $f(x) = \frac{1}{x}$; $x_0 = -1$

$$f(x) = \frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \frac{1}{1 + \left(\frac{x-x_0}{x_0}\right)}$$

$$= \frac{1}{x_0}\left(1 - \left(\frac{x-x_0}{x_0}\right) + \left(\frac{x-x_0}{x_0}\right)^2 + \cdots + (-1)^n \left(\frac{x-x_0}{x_0}\right)^n + o\left((x-x_0)^n\right)\right)$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + o(z^n) \quad \text{GEOMÉTRICA}$$

$$\Rightarrow f(x) = \frac{1}{x} = \frac{1}{x_0} - \frac{1}{x_0^2}(x-x_0) + \frac{1}{x_0^3}(x-x_0)^2 + \cdots + \frac{(-1)^n}{x_0^{n+1}}(x-x_0)^n + o\left((x-x_0)^n\right)$$

Si $x_0 = -1$:

$$\boxed{f(x) = -1 - (x+1) - (x+1)^2 - \cdots - (x+1)^n + o\left((x+1)^n\right)}$$

2) $f(x) = x e^x$; $x_0 = 0$

$$f(x) = x e^x = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)\right)$$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + o(z^n)$$

$$\Rightarrow \boxed{f(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n-1)!} + o(x^n)}$$

3) $f(x) = (1+e^x)^2$; $x_0 = 0$.

$$f(x) = (1+e^x)^2 = 1 + 2e^x + e^{2x}$$

$$= 1 + 2\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o(x^n)\right) +$$

$$+ 1 + 2x + \frac{(2x)^2}{2} + \cdots + \frac{(2x)^n}{n!} + o(x^n)$$

$e^z = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + o(z^n)$

$$\Rightarrow \boxed{f(x) = 4 + \frac{2+2}{1!}x + \frac{2+2^2}{2!}x^2 + \frac{2+2^3}{3!}x^3 + \cdots + \frac{2+2^n}{n!}x^n + o(x^n)}$$