PROBLEMA 7.18 Error menor que 10^{-3}

$\boxed{\cos 1}$ $$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$|R_{2n+1}(x)| = \frac{|\cos c|}{(2n+2)!} x^{2n+2} \quad ; \quad c \in (0,x)$$

$$\Rightarrow |R_{2n+1}(x)| \leq \frac{x^{2n+2}}{(2n+2)!}$$

$$|R_{2n+1}(1)| \leq \frac{1}{(2n+2)!}$$

Imponemos $\frac{1}{(2n+2)!} < 10^{-3} \Rightarrow n \geq 3$
$[2n \geq 6]$

$$\boxed{\cos 1 \simeq 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \text{ con un error } < 10^{-3}}$$

$\boxed{\text{sen } 3}$ $\text{sen}\,\pi = 0$; $\pi = 3.14\cdots$

Consideramos $f(x) = \text{sen}(\pi + x) = -\text{sen}\,x$

queremos aproximar el valor de $f(x)$ para $x \simeq 0$:

$$f(\underbrace{3-\pi}_{\text{proximo a } 0}) = -\text{sen}(\underbrace{3-\pi}_{\text{proximo a } 0})$$

Usando:

$$f(x) = -\text{sen}\,x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)} + o(x^{2n+2})$$

$$|R_{2n+2}(x)| = \frac{|\cos(c)|}{(2n+3)!}|x|^{2n+3} \leq \frac{|x|^{2n+3}}{(2n+3)!}$$

se tiene que:

$$|R_{2n+2}(3-\pi)| \leq \frac{|3-\pi|^{2n+3}}{(2n+3)!} < \frac{(0.2)^{2n+3}}{(2n+3)!}$$

Si imponemos:

$$\frac{(0.2)^{2n+3}}{(2n+3)!} < 10^{-3} \Rightarrow n \geq 1$$

$$[2n+1 \geq 3]$$

se tiene que

$$\boxed{\begin{array}{l} \operatorname{sen} 3 = -\operatorname{sen}(3-\pi) \simeq (\pi-3) - \dfrac{(\pi-3)^3}{3!} \\ \text{aproxima sen 3 con un error menor que } 10^{-3} \end{array}}$$

$\boxed{e}$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

donde $R_n(x) = \dfrac{e^c}{(n+1)!} x^n$ con $c \in (0,x)$

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$$

donde $|R_n(1)| = \dfrac{e}{(n+1)!} < \dfrac{3}{(n+1)!}$

Imponiendo $\dfrac{3}{(n+1)!} < 10^{-3} \Rightarrow n \geq 6$

$$\boxed{\begin{array}{l} e \simeq 1 + 1 + \dfrac{1}{2} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!} \\ \text{con un error menor que } 10^{-3} \end{array}}$$

$\boxed{e^{-2}}$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x)$$

donde $R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$ con $c \in (x,0)$

$$|R_n(-2)| = \frac{e^c}{(n+1)!} 2^{n+1} \underset{\substack{\uparrow \\ c\in(-2,0)}}{\leq} \frac{2^{n+1}}{(n+1)!}$$

Imponiendo:

$$\frac{2^{n+1}}{(n+1)!} < 10^{-3} \Rightarrow n \geq 9$$

se tiene que:

$$e^{-2} \simeq 1 - 2 + \frac{2^2}{2} - \frac{2^3}{3!} + \frac{2^4}{4!} - \frac{2^5}{5!} + \frac{2^6}{6!} - \frac{2^7}{7!} + \frac{2^8}{8!} - \frac{2^9}{9!}$$

con un error $< 10^{-3}$

$\boxed{\log(3/2)}$

Consideremos $f(x) = \log(1+x)$

$$f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k} \; ; \; k \geq 1$$

$$\log(1+x) = x - \frac{x^2}{2} + \dots + (-1)^{n+1} \frac{x^n}{n} + R_n(x)$$

donde $|R_n(x)| = \frac{|x|^{n+1}}{(n+1)(1+c)^{n+1}}$ con $c \in (0,x)$

En particular:

$$|R_n(1/2)| = \frac{1}{(n+1)\, 2^{n+1} (1+c)^{n+1}} \underset{\substack{\uparrow \\ c\in(0,1/2)}}{<} \frac{1}{2^{n+1}(n+1)}$$

Imponiendo

$$\frac{1}{2^{n+1}(n+1)} < 10^{-3} \Rightarrow n \geq 7$$

se tiene qe:

$$\log(1+1/2) = \log(3/2)$$

$$\simeq 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} + \frac{(0.5)^5}{5} - \frac{(0.5)^6}{6} + \frac{(0.5)^7}{7}$$

con un error menor qe 10^{-3}

$\boxed{\log(4/3)}$ $$|R_n(x)| = \frac{|x|^{n+1}}{(1+n)\cdot(1+c)^{n+1}} \quad \text{con } c \in (0,x)$$

$$|R_n(1/3)| = \frac{1}{(1+n)\,3^{n+1}\,(1+c)^{n+1}} \underset{c\in(0,1/3)}{<} \frac{1}{(n+1)\,3^{n+1}}$$

Imponiendo $\frac{1}{(n+1)\,3^{n+1}} < 10^{-3} \Rightarrow n \geq 4$

$$\log(1+1/3) = \log(4/3)$$

$$\simeq \frac{1}{3} - \frac{(1/3)^2}{2} + \frac{(1/3)^3}{3} - \frac{(1/3)^4}{4}$$

con un error menor qe 10^{-3}

$\boxed{\log 2}$ $\qquad R_n(x) = \dfrac{x^{n+1}}{(n+1)\,(1+c)^{n+1}}$ con $c \in (0,x)$

$$|R_n(1)| = \frac{1}{(n+1)(1+c)^{n+1}} \underset{c\in(0,1)}{<} \frac{1}{n+1}$$

Imponiendo:

$$\frac{1}{n+1} < 10^{-3} \Rightarrow n+1 > 1000 \Rightarrow n \geq 1000$$

$\log(1+1) = \log 2$

$$\simeq 1 - 1/2 + 1/3 - 1/4 + \cdots + \frac{1}{999} - \frac{1}{1000}$$

con un error menor qe 10^{-3}

$\boxed{\log(1/2)}$

$\log(1/2) = -\log(2)$

$$\simeq -1 + 1/2 - 1/3 + \cdots + \frac{1}{1000}$$

con un error menor qe 10^{-3}