P2

1) $a_{n+1} - a_n = 3^n$, $n \geq 0$, $a_0 = 1$

$$a_{n+1} = a_n + 3^n \Rightarrow \left(\text{Si } f(x) = \sum_{n=0}^{\infty} a_n x^n\right) \Rightarrow$$

$$\sum_{n=0}^{\infty} a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} 3^n x^n \Rightarrow$$

$$\frac{f(x) - a_0}{x} = f(x) + \sum_{n=0}^{\infty} (3x)^n \Rightarrow$$

$$\frac{f(x) - 1}{x} = f(x) + \frac{1}{1-3x} \Rightarrow$$

$$f(x) - 1 = x f(x) + \frac{x}{1-3x} \Rightarrow$$

$$f(x) = \frac{1-2x}{(1-x)(1-3x)} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1-3x}\right] \Rightarrow$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n\right] \Rightarrow$$

$$\boxed{a_n = \frac{1+3^n}{2}}$$

2) $a_{n+1} - a_n = n^2 \Rightarrow \left(\text{Si } f(x) = \sum_{n=0}^{\infty} a_n x^n \right)$

$$\sum_{n=0}^{\infty} a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n^2 x^n \Rightarrow$$

$$\frac{f(x)-a_0}{x} - f(x) = x\left[x\left(\frac{1}{1-x}\right)'\right]' \Rightarrow$$

$$\frac{f(x)-1}{x} - f(x) = \frac{x+x^2}{(1-x)^3} \Rightarrow f(x) = \frac{4x^2-3x+1}{(1-x)^4}$$

y por otro lado $f(x) = \frac{4}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4} \Rightarrow$
(fracciones simples)

$$\boxed{a_n = 4\binom{n+1}{1} - 5\binom{n+2}{2} + 2\binom{n+3}{3}}$$

4) $a_{n+2} - 3a_{n+1} + 2a_n = 0$; $a_0 = 1$; $a_1 = 6$

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 3\sum_{n=0}^{\infty} a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\frac{f(x)-a_0-a_1 x}{x^2} - 3\,\frac{f(x)-a_0}{x} + 2f(x) = 0 \Rightarrow f(x) = \frac{3x+1}{2x^2-3x+1} = \frac{3x+1}{(1-x)(1-2x)}$$

$$\Rightarrow f(x) = \frac{-4}{1-x} + \frac{5}{1-2x} \Rightarrow \boxed{a_n = -4 + 5\cdot 2^n}$$

P4

$$n a_n = 2\left(a_{n-1} + a_{n-2}\right) \; ; \; n \geq 2 \; ; \; a_0 = e \; ; \; a_1 = 2e$$

equivalente a:

$$(n+2) a_{n+2} = 2\left(a_{n+1} + a_n\right) \; ; \; n \geq 0 \; ; \; a_0 = e \; ; \; a_1 = 2e$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2) a_{n+2} x^n = 2\left(\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n\right) \Rightarrow$$

$$\frac{\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+2}}{x^2} = 2\left(\frac{\sum_{n=0}^{\infty} a_{n+1} x^{n+1}}{x} + f(x)\right) \Rightarrow$$

$$\frac{\sum_{n=0}^{\infty} n a_n x^n - a_1 x}{x^2} = 2\left(\frac{\sum_{n=0}^{\infty} a_n x^n - a_0}{x} + f(x)\right) \Rightarrow$$

$$\frac{x f'(x) - a_1 x}{x^2} = 2\left(\frac{f(x) - a_0}{x} + f(x)\right) \Rightarrow$$

$$f'(x) - a_1 = 2\left(f(x) - a_0 + x f(x)\right) \Rightarrow$$

$$f'(x) - 2e = 2\left[f(x) - e + x f(x)\right] \quad \text{ó bien}$$

$$\boxed{f'(x) - (2x+2) f(x) = 0}$$

P1)

$$x_1 + x_2 + x_3 = 17$$

a) $x_i \in \{0, 1, 2, \dots, 6\}$

La función generatriz asociada será:

$$f(x) = \left(1 + x + x^2 + \dots + x^6\right)^3 = \left(\frac{1 - x^7}{1 - x}\right)^3 =$$

$$= \left(1 - x^7\right)^3 \left(1 - x\right)^{-3}$$

La solución será el coeficientes de grado 17 de $f(x)$ (a_{17})

Grados en $(1-x^7)^3$	Coeficiente	Grados en $(1-x)^{-3}$	Coeficiente
0	1	17	$-\binom{-3}{17} = \binom{19}{17}$
7	$-\binom{3}{1}$	10	$\binom{-3}{10} = \binom{12}{10}$
14	$\binom{3}{2}$	3	$-\binom{-3}{3} = \binom{5}{3}$

Habrá que multiplicar por filas los coeficientes, y después, sumar:

$$\boxed{a_{17} = \binom{19}{17} - \binom{3}{1}\binom{12}{10} + \binom{3}{2}\binom{5}{3} = 3}$$

b) $x_1, x_2 \in 2\mathbb{N}$ pares y $x_3 \geq 0$ impar

Equivalente a $2y_1 + 2y_2 + 2y_3 - 1 = 17$ con $y_i \geq 1$

$\Rightarrow$ $y_1 + y_2 + y_3 = 9$ que tendrá como función generatriz asociada $f(y) = (y + y^2 + y^3 + \dots)^3 = y^3(1 + y + y^2 + \dots)^3 = y^3(1-y)^{-3}$

La solución será el coeficiente de grado 9 $\Rightarrow$ $Coef_6\left[(1-y)^{-3}\right] =$

$$= \binom{-3}{6} = \binom{8}{6} = \boxed{28}$$

c) Equivalente a:

$$2y_1+1+2y_2+1+2y_3+1=17 \quad ; \quad y_i \geq 0$$

$$\Rightarrow \quad y_1+y_2+y_3=7 \quad ; \quad y_i \geq 0$$

Función generatriz $f(y)=(1+y+y^2+\cdots)^3=\left(\frac{1}{1-y}\right)^3=(1-y)^{-3}$

para hallar el coeficiente de grado 7

$$a_7=-\binom{-3}{7}=\binom{9}{7}=\boxed{36}$$

Código Maple

```
s:=0
for i from 1 by 2 to 17 do
  for j from 1 by 2 to 17 do
    for k from 1 by 2 to 17 do
      if (i+j+k=17) then s:=s+1
        print(i,j,k)
      fi
    od
  od
od
print(s)
```

P5] Explicado ya

P3] $x_1 + x_2 + \cdots = N$

a) Enteros distintos. Función generatriz asociada:

$$f(x) = (1+x)(1+x^2)(1+x^3)\cdots = \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\ldots =$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\ldots$$

b) Enteros impares. Función generatriz asociada:

$$g(x) = (1+x+x^2+\ldots)(1+x^3+x^6+x^9+\ldots)(1+x^5+x^{10}+\ldots)\cdots =$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\ldots$$

Puesto que las funciones generatrices coinciden, queda demostrado el aserto.

Problema 9.2.6

$$a_{n+2} - 2a_{n+1} - a_n = 2^n \; ; \; n \geq 1 \, , \; a_0 = 1 \, , \; a_1 = 2$$

$$a_2 - 2a_1 - a_0 = 2^0 \; ; \; \boxed{a_2 = 2a_1 + a_0 + 1 = 6}$$

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\sum_{n=1}^{\infty} a_{n+2} x^n - 2 \sum_{n=1}^{\infty} a_{n+1} x^n - \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2^n x^n$$

$$\frac{F(x) - a_0 - a_1 x - a_2 x^2}{x^2} - 2\,\frac{F(x) - a_0 - a_1 x}{x} - \left(F(x) - a_0\right) = \frac{1}{1-2x} - 1$$

$$F(x) - 1 - 2x - 6x^2 - 2x\left(F(x) - 1 - 2x\right) - x^2\left(F(x) - 1\right) = \frac{1}{1-2x} - 1$$

$$F(x)\left(1 - 2x - x^2\right) \cancel{-1} \cancel{-2x} - 6x^2 \cancel{+2x} + 4x^2 + x^2 = \frac{1}{1-2x} \cancel{-1}$$

$$F(x)\left(1 - 2x - x^2\right) = \frac{1}{1-2x} + x^2$$

$$F(x) = \frac{1}{(1-2x)(1-2x-x^2)} + \frac{x^2}{1-2x-x^2} = \frac{-1}{(1-2x)(x^2+2x-1)} + \frac{-x^2}{x^2+2x-1}$$

$$F(x) = \frac{A}{1-2x} + \frac{B}{(\alpha - x)} + \frac{C}{(\beta - x)} + \frac{D}{(\alpha - x)} + \frac{E}{(\beta - x)}$$

donde $\alpha = -1 + \sqrt{2}$ y $\beta = -1 - \sqrt{2}$

(1)

$$A(\alpha-x)(\beta-x)+B(1-2x)(\beta-x)+C(1-2x)(\alpha-x)=-1$$

Si $x=\alpha \Rightarrow B(1-2\alpha)(\beta-\alpha)=-1$

$$1-2\alpha = 1-2(-1+\sqrt{2}) = 1+2-2\sqrt{2} = 3-2\sqrt{2}$$

$$\beta-\alpha = -1-\sqrt{2}+1-\sqrt{2} = -2\sqrt{2}$$

$$B(8-6\sqrt{2}) = -1 \Rightarrow \boxed{B = \frac{1}{6\sqrt{2}-8}}$$

Si $x=\beta \Rightarrow C(1-2\beta)(\alpha-\beta)=-1$

$$1-2\beta = 1-2(-1-\sqrt{2}) = 1+2+2\sqrt{2} = 3+2\sqrt{2}$$

$$\alpha-\beta = 2\sqrt{2}$$

$$C(8+6\sqrt{2}) = -1 \Rightarrow \boxed{C = \frac{-1}{6\sqrt{2}+8}}$$

Si $x=\frac{1}{2} \Rightarrow A\left(\alpha-\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)=-1 \Leftrightarrow A(2\alpha-1)(2\beta-1)=-4$

$$2\alpha-1 = 2(-1+\sqrt{2})-1 = -2+2\sqrt{2}-1 = -3+2\sqrt{2}$$

$$2\beta-1 = 2(-1-\sqrt{2})-1 = -2-2\sqrt{2}-1 = -3-2\sqrt{2}$$

$$A(9-8) = -4 \Rightarrow \boxed{A=-4}$$

---

(2) $D(\beta-x)+E(\alpha-x) = -x^2$

$\beta-\alpha = -2\sqrt{2}$

Si $x=\alpha \Rightarrow D(\beta-\alpha) = -\alpha^2$

$-\alpha^2 = -(-1+\sqrt{2})^2 = -3+2\sqrt{2}$

$$\boxed{D = \frac{2\sqrt{2}-3}{-2\sqrt{2}} = -1+\frac{3}{2\sqrt{2}} = -1+\frac{3\sqrt{2}}{4}}$$

Si $x=\beta \Rightarrow E(\alpha-\beta) = -\beta^2 \Rightarrow E = -1-\frac{3\sqrt{2}}{4}$

$$a_n = A \cdot 2^n + (B+D)\left(\frac{1}{\alpha}\right)^{n+1} + (C+E)\left(\frac{1}{\beta}\right)^{n+1}$$

$$A = -4$$

$$B+D = \frac{1}{6\sqrt{2}-8} - 1 + \frac{3}{4}\sqrt{2} = \frac{6\sqrt{2}+8}{8} - 1 + \frac{3}{4}\sqrt{2} = \boxed{\frac{3}{2}\sqrt{2}}$$

$$C+E = \frac{-1}{6\sqrt{2}+8} - 1 - \frac{3}{4}\sqrt{2} = -\cancel{\frac{6\sqrt{2}}{8} - 1 - 1 - \frac{3}{4}\sqrt{2}} =$$

$$= \frac{-6\sqrt{2}+8}{8} - 1 - \frac{3}{4}\sqrt{2} = \boxed{-\frac{3}{2}\sqrt{2}}$$

$$a_n = -2^{n+2} + \frac{3}{2}\sqrt{2}\left(\frac{1}{\alpha}\right)^{n+1} - \frac{3}{2}\sqrt{2}\left(\frac{1}{\beta}\right)^{n+1}$$