CALCULUS FINAL EXAM

January 14th, 2014 **Bachelor in Informatics Engineering**

SURNAME			
NAME		GROUP	

Problem 1. Consider the monotonically *increasing* sequence defined by the following recursive formula

$$a_1 = 0\,,$$
$$a_{n+1} = \sqrt{4a_n + 5}\,, \quad n \geq 1\,.$$

Prove that $(a_n)_{n\in\mathbb{N}}$ is bounded *[0.5 points]*. Calculate $\lim\limits_{n\to\infty} a_n$ *[0.5 points]*.

SOLUTION

Let us suppose that the given sequence has a (finite) limit, namely $a = \lim\limits_{n\to\infty} a_n$. Then, letting $n \to \infty$ in both sides of the recursive formula for the sequence, we have

$$a = \sqrt{4a+5} \quad \Rightarrow \quad a^2 = 4a + 5 \quad \Rightarrow \quad a = -1, 5\,,$$

where the value $a = -1$ has to be excluded as the sequence is increasing with nonnegative terms. Hence, $a = 5$ is the only possible *candidate* to be the limit value.

Now, let us prove by *induction* that $(a_n)_{n\in\mathbb{N}}$ is bounded above by 5, namely $0 \leq a_n \leq 5$ with $n \in \mathbb{N}$. First, such property holds for $n = 1$, that is $0 \leq a_1 = 0 \leq 5$. Then, supposing that $0 \leq a_n \leq 5$, we get

$$0 \leq a_{n+1} = \sqrt{4a_n + 5} \leq \sqrt{4\cdot 5 + 5} = 5$$

for each $n \in \mathbb{N}$. Thus, the sequence is bounded, hence it has a (finite) limit thanks to its increasing behavior. As a consequence, the desired value for such limit is $a = 5$ as calculated above.

Problem 2. Find *all* values of the parameter $x \in \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} \frac{3^{2k}\, x^{3k}}{(2k+1)\, 5^k}$$

is convergent *[1 point]*.

SOLUTION

Let $a_k = \dfrac{3^{2k}\, x^{3k}}{(2k+1)\, 5^k}$. Then, we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{9}{5}\,|x|^3\, \frac{2k+1}{2k+3} \quad \longrightarrow \quad \frac{9}{5}\,|x|^3 \quad \text{as } k \to \infty\,.$$

Thus, according to the *ratio test*, the series converges if

$$\frac{9}{5}\,|x|^3 < 1 \quad \Longleftrightarrow \quad |x|^3 < \frac{5}{9} \quad \Longleftrightarrow \quad x \in \left(-\frac{5^{1/3}}{9^{1/3}}\,,\ \frac{5^{1/3}}{9^{1/3}}\right)\,,$$

while the series diverges if

$$\frac{9}{5}\,|x|^3 > 1 \quad \Longleftrightarrow \quad x \in \left(-\infty\,,\ -\frac{5^{1/3}}{9^{1/3}}\right) \cup \left(\frac{5^{1/3}}{9^{1/3}}\,,\ +\infty\right)\,.$$

On the other hand, if

$$\frac{9}{5}\,|x|^3 = 1 \quad \Longleftrightarrow \quad |x|^3 = \frac{5}{9} \quad \Longleftrightarrow \quad x^3 = \pm\frac{5}{9}\,,$$

the series becomes in the first case

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(+1)^k}{2k+1} = \sum_{k=1}^{\infty} \frac{1}{2k+1}\,,$$

which diverges, and it becomes in the second case

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}\,,$$

which converges by the Leibniz test. Hence, we can finally conclude that the given series converges if and only if

$$x \in \left[-\frac{5^{1/3}}{9^{1/3}}\,,\ \frac{5^{1/3}}{9^{1/3}}\right)\,.$$

Problem 3. Consider the function

$$F(x) = \int_0^{5x} e^{-7t^4}\, dt\,, \quad x \in \mathbb{R}\,.$$

- Prove that F is an *odd* function *[0.5 points]*.
- Prove the *existence* of the limit $\ell = \lim_{x\to\infty} F(x)$ *[1 point]*.
- Prove that the function $F : \mathbb{R} \to (-\ell, \ell)$ is monotonically *increasing* *[0.25 points]*.
- Calculate $(F^{-1})'(0)$ *[0.25 points]*.
- Calculate the limit $\lim_{x\to 0} \dfrac{5x - F(x)}{x^5}$ *[1 point]*.

SOLUTION

- The function F is *odd* as

$$F(-x) \;=\; \int_0^{-5x} e^{-7t^4}\, dt \;=\; -\int_0^{5x} e^{-7u^4}\, du \;=\; -F(x)\,,$$

where the second equality is obtained by means of the change of variable $u = -t$.

- First, note that

$$\ell \;=\; \lim_{x\to\infty} F(x) \;=\; \lim_{x\to\infty} \int_0^{5x} e^{-7t^4}\, dt \;=\; \int_0^{\infty} e^{-7t^4}\, dt\,,$$

which is an *improper* integral (of a positive function). Then, we have

$$\lim_{t\to\infty} \frac{e^{-7t^4}}{e^{-t}} \;=\; \lim_{t\to\infty} e^{-7t^4+t} \;=\; 0\,.$$

Thus, as $\int_0^\infty e^{-t}\, dt$ converges, by the limit comparison test we can conclude that $\int_0^\infty e^{-7t^4}\, dt$ also converges, that is the desired limit ℓ exists.

- The function F is monotonically increasing as

$$F'(x) \;=\; 5\, e^{-7(5x)^4} \;>\; 0$$

for each $x \in \mathbb{R}$. Note that the derivative F' above is deduced by the fundamental theorem of calculus.

• Thanks to the result in the previous item, F^{-1} exists. In addition, we have

$$(F^{-1})'(x) = \frac{1}{F'\big(F^{-1}(x)\big)} \quad \Longrightarrow \quad (F^{-1})'(0) = \frac{1}{F'\big(F^{-1}(0)\big)} = \frac{1}{F'(0)} = \frac{1}{5},$$

where the last but one equality holds as $F(0) = 0$, while the last one comes from the expression for F' calculated in the previous item.

• The given limit provides an indeterminate form of the type $0/0$. Thus, using the l'Hôpital rule two times we get

$$\lim_{x\to 0} \frac{5x - F(x)}{x^5} = \lim_{x\to 0} \frac{5 - F'(x)}{5x^4} = \lim_{x\to 0} \frac{5 - 5\, e^{-7(5x)^4}}{5x^4} = \lim_{x\to 0} 7 \cdot 5^4\, e^{-7(5x)^4} = 7 \cdot 5^4 .$$

Problem 4. Calculate

$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}} \qquad \textit{[1 point]}\,.$$

(HINT: find a suitable change of variable)

SOLUTION

Let us use the change of variable

$$u = (x+1)^{1/3}, \quad du = \frac{1}{3}(x+1)^{-2/3}dx \,\Big(\Rightarrow \; dx = 3u^2 du\Big)\,.$$

Then, the given integral becomes

$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}} = 3\int \frac{u^2}{u^4-u^2}du = 3\int \frac{du}{u^2-1}\,.$$

Now, using the partial fractions method, we can write

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{1/2}{u-1} - \frac{1/2}{u+1}\,,$$

which yields

$$\int \frac{du}{u^2-1} = \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c\,,$$

where c is an arbitrary constant. Thus, we can finally write

$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}} = \frac{3}{2}\ln\left|\frac{(x+1)^{1/3}-1}{(x+1)^{1/3}+1}\right| + c\,.$$