P 14.1.a

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\boxed{\begin{aligned}&p_{A^TA}(\lambda)=\lambda^2-6\lambda+8=(\lambda-4)(\lambda-2)\\&\sigma\left(A^TA\right)=\{2,4\}\end{aligned}}$$

Valores singulares: $\sigma_1=2 \quad \sigma_2=\sqrt{2}$

$$\boxed{\Sigma=\begin{pmatrix}2 & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{pmatrix}}$$

Vectores singulares derechos (autovectores de $A^T A$)

$$V_{A^T A}(4) = Nul\left(A^T A - 4I\right) = Nul\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = Gen\left\{(1,1)\right\}$$

$$V_{A^T A}(2) = Nul\left(A^T A - 2I\right) = Nul\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = Gen\left\{(1,-1)\right\}$$

Normalizando:

$$v_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^T$$

$$\boxed{V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}^T \qquad u_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_3 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}^T$$

$$\boxed{U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \quad \textit{ortogonal}}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & -1/2 \end{pmatrix}$$

P 14.1.b

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$p_{A^T A}(\lambda) = \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$
$$\sigma\left(A^T A\right) = \{0, 10\}$$

Valores singulares: $\sigma_1 = \sqrt{10} \quad \sigma_2 = 0$

$$\Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}$$

Vectores singulares derechos (autovectores de $A^T A$)

$$V_{A^T A}(10) = Nul\left(A^T A - 10I\right) = Nul\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} = Gen\left\{(1,2)\right\}$$

$$V_{A^T A}(0) = Nul\left(A^T A\right) = Nul\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = Gen\left\{(2,-1)\right\}$$

Normalizando: $v_1 = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}^T$

$$\boxed{V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U):
Como A es de rango 1, una imagen será nula.

$$Av_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^T$$

$$\boxed{U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad ortogonal}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A^+ = \frac{1}{10}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

P 14.1.c

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

Rango completo (invertible)

$$A^T A = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$$

$$\boxed{\begin{aligned} &p_{A^TA}(\lambda) = \lambda^2 - 17\lambda + 16 = (\lambda - 16)(\lambda - 1) \\ &\sigma\left(A^TA\right) = \{1, 16\} \end{aligned}}$$

Valores singulares: $\sigma_1 = 4 \quad \sigma_2 = 1$

$$\boxed{\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}}$$

Vectores singulares derechos (autovectores de A^TA)

$$V_{A^TA}(16) = Nul\left(A^TA - 16I\right) = Nul\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} = Gen\left\{(1,2)\right\}$$

$$V_{A^TA}(1) = Nul\left(A^TA - I\right) = Nul\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} = Gen\left\{(2,-1)\right\}$$

Normalizando: $v_1 = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}^T$

$$\boxed{V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U):
Como A es de rango 1, una imagen será nula.

$$Av_1 = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad Av_2 = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}^T \quad u_2 = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}^T$$

$$\boxed{U = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad \textit{ortogonal}}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$A^+ = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} = A^{-1}$$

P 14.1.d

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$$

$$p_{A^T A}(\lambda) = \lambda^2 - 8\lambda = \lambda(\lambda - 8)$$

$$\sigma\left(A^T A\right) = \{0, 8\}$$

Valores singulares: $\sigma_1 = \sqrt{8} \quad \sigma_2 = 0$

$$\Sigma = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Vectores singulares derechos (autovectores de $A^T A$)

$$V_{A^T A}(8) = Nul\left(A^T A - 8I\right) = Nul\begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix} = Gen\left\{(0,1)\right\}$$

$$V_{A^T A}(0) = Nul\left(A^T A\right) = Nul\begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix} = Gen\left\{(1,0)\right\}$$

$$v_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$$

$$\boxed{V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \qquad u_3 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}^T$$

$$\boxed{U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \quad \mathit{ortogonal}}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{8} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 0 \end{pmatrix}$$

P 14.1.e

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} p_{A^T A}(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \\ \sigma\left(A^T A\right) = \{1, 2\} \end{array}}$$

Valores singulares: $\sigma_1 = \sqrt{2} \quad \sigma_2 = 1$

$$\boxed{\Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

Vectores singulares derechos (autovectores de $A^T A$)

$$V_{A^T A}(2) = Nul\left(A^T A - 2I\right) = Nul\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = Gen\left\{(1,0)\right\}$$

$$V_{A^T A}(1) = Nul\left(A^T A - I\right) = Nul\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Gen\left\{(0,1)\right\}$$

$$v_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

$$\boxed{V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}^T \qquad u_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_3 = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}^T$$

$$\boxed{U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \quad ortogonal}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{8} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

P 14.1.f

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\boxed{\begin{aligned} &p_{A^T A}(\lambda) = \lambda^2 - 9\lambda = \lambda(\lambda - 9) \\ &\sigma\left(A^T A\right) = \{0,9\} \end{aligned}}$$

Valores singulares: $\sigma_1 = 3 \quad \sigma_2 = 0$

$$\boxed{\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

Vectores singulares derechos (autovectores de A^TA)

$$V_{A^TA}(9) = Nul\left(A^TA - 9I\right) = Nul\begin{pmatrix} -6 & 3 & 3 \\ 3 & -6 & 3 \\ 3 & 3 & -6 \end{pmatrix} = Gen\left\{(1,1,1)\right\}$$

$$V_{A^TA}(0) = Nul\left(A^TA\right) = Nul\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = Gen\left\{(1,-1,0);(1,0,-1)\right\}$$

Como la base conseguida no es ortogonal, se ortogonaliza y después se normaliza:

$$v_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 1/\sqrt{5} & 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}^T$$

$$\boxed{V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & 0 & -2/\sqrt{5} \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \quad \textit{las demás imágenes serán nulas.}$$

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}^T \quad u_3 = \begin{pmatrix} 1/\sqrt{5} & 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}^T$$

$$\boxed{U = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & 0 & -2/\sqrt{5} \end{pmatrix} \quad \textit{ortogonal}}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & 0 & -2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{5} & 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

$$A^+ = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

P 14.4.a

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

Como los espectros de $A^T A$ y AA^T coinciden

$$AA^T = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$\boxed{\begin{aligned} & p_{A^T A}(\lambda) = \lambda^2 - 6\lambda = \lambda(\lambda - 6) \\ & \sigma\left(A^T A\right) = \{0, 6\} \end{aligned}}$$

Valores singulares: $\sigma_1 = \sqrt{6} \quad \sigma_2 = 0$

$$\boxed{\Sigma = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

Vectores singulares derechos (autovectores de A^TA)

$$V_{A^TA}(6) = Nul\left(A^TA - 6I\right) = Nul\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} = Gen\left\{(1,1,1)\right\}$$

$$V_{A^TA}(0) = Nul\left(A^TA\right) = Nul\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = Gen\left\{(1,-1,0);(1,0,-1)\right\}$$

La base obtenida no es ortogonal, por lo que se ortogonaliza y normaliza:

$$v_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}^T \quad v_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}^T \quad v_3 = \begin{pmatrix} 1/\sqrt{5} & 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}^T$$

$$\boxed{V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & 0 & -2/\sqrt{5} \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

La restante imagen será nula

Normalizando:

$$u_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^T$$

$$\boxed{U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad ortogonal}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{3} & 0 & -2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A^+ = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

La solución de mínimos cuadrados (de norma mínima) será:

$$x = A^{+}b = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Pero hay que tener en cuenta que como la matriz A no es de rango completo, habrá infinitas soluciones de mínimos cuadrados.
La solución encontrada será la de norma mínima.

P 14.4.a

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\boxed{\begin{array}{l} p_{A^T A}(\lambda) = \lambda^2 - 12\lambda + 35 = (\lambda - 7)(\lambda - 5) \\ \sigma\left(A^T A\right) = \{5, 7\} \end{array}}$$

Valores singulares: $\sigma_1 = \sqrt{7} \quad \sigma_2 = \sqrt{5}$

$$\boxed{\Sigma = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix}}$$

Vectores singulares derechos (autovectores de A^TA)

$$V_{A^TA}(7) = Nul\left(A^TA - 7I\right) = Nul\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = Gen\left\{(1,1)\right\}$$

$$V_{A^TA}(5) = Nul\left(A^TA - 5I\right) = Nul\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = Gen\left\{(1,-1)\right\}$$

Normalizando:

$$v_1 = \left(\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}\right)^T \qquad v_2 = \left(\tfrac{1}{\sqrt{2}} \quad -\tfrac{1}{\sqrt{2}}\right)^T$$

$$\boxed{V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad ortogonal}$$

Vectores singulares izquierdos (matriz U)

$$Av_1 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

Normalizando:

$$u_1 = \begin{pmatrix} 3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{pmatrix}^T \qquad u_1 = \begin{pmatrix} -1/\sqrt{10} & 3/\sqrt{10} & 0 \end{pmatrix}^T$$

Completando la base ortonormal:

$$u_3 = \begin{pmatrix} 3/\sqrt{35} & 1/\sqrt{35} & -5/\sqrt{35} \end{pmatrix}^T$$

$$\boxed{U = \begin{pmatrix} 3/\sqrt{14} & -1/\sqrt{10} & 3/\sqrt{35} \\ 1/\sqrt{14} & 3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \end{pmatrix} \quad ortogonal}$$

Se verifica (DVS):

$$A = U\Sigma V^T$$

Pseudoinversa:

$$A^+ = V\Sigma^+ U^T$$

$$A^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{7} & 0 & 0 \\ 0 & 1/\sqrt{5} & 0 \end{pmatrix}\begin{pmatrix} 3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \\ -1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{35} & 1/\sqrt{35} & -5/\sqrt{35} \end{pmatrix}$$

$$A^+ = \frac{1}{35}\begin{pmatrix} 4 & 13 & 5 \\ 11 & -8 & 5 \end{pmatrix}$$

La solución (única) de mínimos cuadrados será:

$$x = A^{+}b = \frac{1}{35}\begin{pmatrix} 4 & 13 & 5 \\ 11 & -8 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{35}\begin{pmatrix} 43 \\ 22 \end{pmatrix}$$

P 14.5.e

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix} \qquad R = \begin{pmatrix} 3/\sqrt{3} & 1/\sqrt{3} \\ 0 & 4/\sqrt{6} \end{pmatrix}$$

$$\Pr oy_{Col(A)}(\mathrm{e}_1) = AA^{+}e_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Pr oy_{Col(A)}(\mathrm{e}_1) = QQ^{T}e_1 =$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

P 14.5.c

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

$$\Pr oy_{Col(A)}(\mathrm{e}_1) = AA^{+}e_1 = Ie_1 = e_1$$

$$\Pr oy_{Col(A)}(\mathrm{e}_1) = QQ^{T}e_1 = Ie_1 = e_1$$

P 14.5.e

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \quad R = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\operatorname{Pr}oy_{Col(A)}(\mathrm{e}_1) = AA^{+}e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\operatorname{Pr}oy_{Col(A)}(\mathrm{e}_1) = QQ^{T}e_1 =$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$