PROBLEMA 11.2

$$\boxed{\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx}$$

$$\frac{x^5-2x^3}{x^4-2x^2+1} = \frac{x^5-2x^3+x\;-x}{x^4-2x^2+1} = x - \frac{x}{x^4-2x^2+1}$$

En otras palabras:

$$\begin{array}{l|l} \cancel{x^5}+0\cdot\cancel{x^4}-2\cancel{x^3}+0\cdot\cancel{x^2}+0\cdot x+0 & x^4-2x^2+1 \\ \cline{2-2} \cancel{x^5}+\cancel{0}\cancel{x^4}-2\cancel{x^3}+\cancel{0}\cancel{x^2}+\;x & x \\ \hline \qquad\qquad\qquad\qquad\underbrace{-x} & \end{array}$$

- Por tanto:

$$\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx = \int \left(x - \frac{x}{x^4-2x^2+1} \right) dx$$

$$= \frac{x^2}{2} - \int \frac{x}{x^4-2x^2+1}\,dx$$

- $x^4-2x^2+1 = (x^2-1)^2 = (x-1)^2(x+1)^2$

$\Rightarrow$ Fracciones simples:

$$\frac{x}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$\Rightarrow \quad \frac{1}{(1+1)^2} = \frac{1}{4} = A \quad ; \quad \frac{-1}{(-1-1)^2} = -\frac{1}{4} = B$$

$$\frac{x}{(x-1)^2(x+1)^2} = \frac{1}{4}\,\frac{1}{(x-1)^2} + \frac{A}{x-1} - \frac{1}{4}\,\frac{1}{(x+1)^2} + \frac{C}{x+1}$$

$x=0$: $\quad 0 = C - A \Rightarrow A = C$

$x=2$ $\quad \frac{2}{9} = \frac{1}{4} + A - \frac{1}{4}\cdot\frac{1}{9} + \frac{A}{3} \Rightarrow A = 0$

De esta forma:

$$\frac{x}{(x-1)^2(x+1)^2} = \frac{1}{4}\,\frac{1}{(x-1)^2} - \frac{1}{4}\,\frac{1}{(x+1)^2}$$

por lo que:

$$\boxed{\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx} = \frac{x^2}{2} - \frac{1}{4}\int\frac{dx}{(x-1)^2} + \frac{1}{4}\int\frac{dx}{(x+1)^2}$$

$$\boxed{= \frac{x^2}{2} + \frac{1}{4}\,\frac{1}{x-1} - \frac{1}{4}\,\frac{1}{x+1} + C}$$

$$\boxed{\int \frac{x^2+1}{x^4-x^2}\,dx}$$

$$x^4-x^2 = x^2(x^2-1) = x^2(x-1)(x+1)$$

$\Rightarrow$ Fracciones simples:

$$\frac{x^2+1}{x^2(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x} + \frac{D}{x^2}$$

$x=1$: $\quad \dfrac{1^2+1}{1^2(1+1)} = A \Rightarrow A = 1$

$x=-1$: $\quad \dfrac{(-1)^2+1}{(-1)^2(-1-1)} = B \Rightarrow B = -1$

$x=0$: $\quad \dfrac{0^2+1}{(0-1)(0+1)} = D \Rightarrow D = -1$

$x=2$: $\quad \dfrac{2^2+1}{2^2\cdot(2-1)(2+1)} = \dfrac{1}{2-1} - \dfrac{1}{2+1} + \dfrac{C}{2} - \dfrac{1}{2^2}$

$\Rightarrow C = 0$

$$\boxed{\frac{x^2+1}{x^4-x^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{1}{x^2}}$$

Por tanto:

$$\int \frac{x^2+1}{x^4-x^2}\,dx = \int \frac{dx}{x-1} - \int \frac{dx}{x+1} - \int \frac{dx}{x^2}$$

$$= \log|x-1| - \log|x+1| + \frac{1}{x} + C$$

$$= \frac{1}{x} + \log\left|\frac{x-1}{x+1}\right| + C$$

$$\boxed{\int \frac{x^2+1}{x^4-x^2}\,dx = \frac{1}{x} + \log\left|\frac{x-1}{x+1}\right| + C}$$

$$\boxed{\int \frac{x^3+1}{x^2+4x+13}\,dx}$$

Dividiendo: $\dfrac{x^3+1}{x^2+4x+13} = x-4+\dfrac{3x+53}{x^2+4x+13}$

$$\Rightarrow \int \frac{x^3+1}{x^2+4x+13}\,dx = \frac{x^2}{2} - 4x + \int \frac{3x+53}{x^2+4x+13}\,dx$$

$$= \frac{x^2}{2} - 4x + \int \frac{3x+6+47}{x^2+4x+13}\,dx$$

$$\boxed{d(x^2+4x+13) = (2x+4)\,dx}$$

$$= \frac{x^2}{2} - 4x + \frac{3}{2}\int \frac{2x+4}{x^2+4x+13}\,dx +$$

$$+ 47 \int \frac{dx}{x^2+4x+13}\,dx$$

$$= \frac{x^2}{2} - 4x + \frac{3}{2}\log|x^2+4x+13| +$$

$$+ 47 \int \frac{dx}{x^2+4x+13}$$

Usando: $x^2+4x+13 = (x+2)^2+9$

$$\Rightarrow \int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9} =$$

$$= \frac{1}{9}\int \frac{dx}{\left(\frac{x+2}{3}\right)+1} =$$

$$\underset{\boxed{\begin{array}{l} u=\frac{x+2}{3} \\ du = dx/3 \end{array}}}{=} \frac{1}{3}\int \frac{du}{u^2+1} = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)$$

Por tanto:

$$\boxed{\int \frac{x^3+1}{x^2+4x+13}\,dx = \frac{x^2}{2} - 4x + \frac{3}{2}\log\left(x^2+4x+13\right) + \frac{47}{3}\arctan\left(\frac{x+2}{3}\right) + C}$$

$$\boxed{\int \frac{x^2+6x-1}{x^3-7x^2+15x-9}\,dx}$$

- $x^3-7x^2+15x-9 = (x-3)^2(x-1)$
- $\dfrac{x^2+6x-1}{x^3-7x^2+15x-9} = \dfrac{3}{2}\dfrac{1}{x-1} - \dfrac{1}{2}\dfrac{1}{x-3} + \dfrac{13}{(x-3)^2}$

$$\boxed{\int \frac{x^2+6x-1}{x^3-7x^2+15x-9}\,dx = \frac{3}{2}\log|x-1| - \frac{1}{2}\log|x-3| - \frac{13}{x-3} + C}$$