PROBLEMA 7.7

① $f(x) = e^{ax^2}$

$$f(x) = e^{ax^2}$$
$$= 1 + ax^2 + \frac{(ax^2)^2}{2!} + \frac{(ax^2)^3}{3!} + \dots + \frac{(ax^2)^n}{n!} + o(x^{2n})$$
$$= 1 + ax^2 + \frac{a^2x^4}{2!} + \frac{a^3x^6}{3!} + \dots + \frac{a^n x^{2n}}{n!} + o(x^{2n})$$

② $f(x) = \cos(ax)$

$$f(x) = \cos(ax)$$
$$= 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} + \dots + (-1)^n \frac{(ax)^{2n}}{(2n)!} + o(x^{2n+1})$$
$$= 1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} + \dots + (-1)^n \frac{a^{2n}x^{2n}}{(2n)!} + o(x^{2n+1})$$

③ $f(x) = \frac{1+x}{1-x}$

$$f(x) = \frac{1+x}{1-x} = \frac{1}{1-x} + x \cdot \frac{1}{1-x} =$$
$$= \left(1 + x + x^2 + \dots + x^n + o(x^n)\right) +$$
$$+ x\left(1 + x + x^2 + \dots + x^n + o(x^n)\right)$$
$$= 1 + 2x + 2x^2 + \dots + 2x^n + o(x^n)$$

④ $f(x) = xe^{-x^2}$

$$f(x) = x\,e^{-x^2} = x\left(1 - x^2 + \frac{(-x^2)^2}{2!} + \dots + \frac{(-x^2)^n}{n!} + o(x^{2n})\right)$$

$$= x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{n!} + o(x^{2n})$$

⑤ $f(x) = \dfrac{e^{ax} - e^{-ax}}{2}$

$$f(x) = \frac{1}{2}e^{ax} - \frac{1}{2}e^{-ax} =$$

$$= \frac{1}{2}\left(\cancel{1} + ax + \cancel{\frac{(ax)^2}{2}} + \frac{(ax)^3}{3!} + \dots + \frac{(ax)^n}{n!} + o(x^n)\right)$$

$$- \frac{1}{2}\left(\cancel{1} - ax + \cancel{\frac{(ax)^2}{2}} - \frac{(ax)^3}{3!} + \dots + (-1)^n \frac{(ax)^n}{n!} + o(x^n)\right)$$

$$= ax + \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + \dots + \frac{(ax)^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$= ax + \frac{a^3}{3!}x^3 + \frac{a^5}{5!}x^5 + \dots + \frac{a^{2n+1}}{(2n+1)!}x^{2n+1} + o(x^{2n+2})$$

⑥ $f(x) = \dfrac{e^{ax} + e^{-ax}}{2}$

$$f(x) = \frac{1}{2}e^{ax} + \frac{1}{2}e^{-ax} =$$

$$= \frac{1}{2}\left(1 + \cancel{ax} + \frac{(ax)^2}{2!} + \cancel{\frac{(ax)^3}{3!}} + \dots + \frac{(ax)^n}{n!} + o(x^n)\right)$$

$$+ \frac{1}{2}\left(1 - \cancel{ax} + \frac{(ax)^2}{2!} - \cancel{\frac{(ax)^3}{3!}} + \dots + (-1)^n \frac{(ax)^n}{n!} + o(x^n)\right)$$

$$= 1 + \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} + \dots + \frac{(ax)^{2n}}{(2n)!} + o(x^{2n+1})$$

$$= 1 + \frac{a^2}{2!}x^2 + \frac{a^4}{4!}x^4 + \dots + \frac{a^{2n}}{(2n)!}x^{2n} + o(x^{2n+1})$$

Obs 1 $\left.\begin{array}{l} f \in C^{\infty}(\mathbb{R}) \\ f \text{ PAR} \end{array}\right\} \Rightarrow f(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2n} x^{2n} + o(x^{2n+1})$

Dem:

$$f(x) = f(-x)$$
$$f'(x) = -f'(-x) \Rightarrow f'(0) = -f'(0) = 0$$
$$f''(x) = f''(-x)$$
$$f'''(x) = -f'''(-x) \Rightarrow f'''(0) = -f'''(0) = 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$f^{(2n+1)}(0) = -f^{(2n+1)}(0) = 0$$

Obs 2: $\left.\begin{array}{l} f \in C^{\infty}(\mathbb{R}) \\ f \text{ IMPAR} \end{array}\right\} \Rightarrow f(x) = a_1 x + a_3 x^3 + \dots + a_{2n+1} x^{2n+1} + o(x^{2n+2})$

Dem:

$$f(x) = -f(-x) \Rightarrow f(0) = -f(0) = 0$$
$$f'(x) = f'(-x)$$
$$f''(x) = -f''(-x) \Rightarrow f''(0) = -f''(0) = 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$f^{(2n)}(0) = -f^{(2n)}(0) = 0$$