CALCULUS – PARTIAL EXAM

Bachelor in Informatics Engineering

ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

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Name		GROUP	**89**

1. Consider the sequence of real numbers defined by

$$\begin{aligned} a_n &= -4 + \frac{a_{n-1}}{3}\,, \quad \text{for} \;\; n \geq 2\,, \\ a_1 &= 0\,. \end{aligned}$$

(a) Prove that $\{a_n\}$ is decreasing.

(b) Prove that $\{a_n\}$ is bounded.

(c) Calculate $\lim\limits_{n\to\infty} a_n$.

Solution. Note that

$$a_2 = -4\,, \quad a_3 = -4 - \frac{4}{3}\,, \quad a_4 = -4 - \frac{4}{3} - \frac{4}{9}\,, \quad \ldots\,,$$

thus, for each $n \geq 2$, we have

$$a_n = -4 - \frac{4}{3} - \ldots - \frac{4}{3^{n-2}}\,. \tag{1}$$

Hence

$$a_n - a_{n-1} = -\frac{4}{3^{n-2}} < 0\,,$$

which means that the sequence is (strictly) decreasing. In addition, using (1), we can identify a_n with the 'geometric' sum

$$a_n = -4 \sum_{k=0}^{n-2} \left(\frac{1}{3}\right)^k = -4\, \frac{1 - \left(\frac{1}{3}\right)^{n-1}}{1 - \frac{1}{3}} = -6 \left\{1 - \left(\frac{1}{3}\right)^{n-1}\right\}. \tag{2}$$

From (2) we can deduce that $|a_n| \leq 6$ for each $n \geq 2$, namely $\{a_n\}$ is bounded, and that

$$\lim_{n\to\infty} a_n = -6\,.$$

2. Find for which values of the parameter $\alpha \in \mathbb{R}$ the series

$$\sum_{n=0}^{\infty} \frac{(\alpha-2)^n}{n^2+1}$$

is convergent.

Solution. Note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(\alpha-2)^{n+1}}{(n+1)^2+1} \frac{n^2+1}{(\alpha-2)^n}\right| = |\alpha-2| \frac{n^2+1}{n^2+2n+2} \longrightarrow |\alpha-2| \quad \text{as } n \to \infty\,,$$

hence, according to the ratio test, the series is convergent if $|\alpha-2|<1$, namely if $1<\alpha<3$ (for $\alpha>3$ or $\alpha<1$ the series diverges). For $\alpha=3$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}\,,$$

which is convergent as $1/(n^2+1) < 1/n^2$ for $n \geq 1$ and a series with general term $1/n^2$ is a convergent p-series (this is the comparison test). Finally, for $\alpha=1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}\,,$$

which is also convergent thanks to the Leibniz test (it is an alternating series with $1/(n^2+1)$ positive, decreasing, and approaching zero as $n \to \infty$).

3. Consider the following function

$$f(x) \;=\; \begin{cases} -x^2 - 7\cos\left(\frac{\pi}{2}\,x\right) & \text{if } \; x > 2 \\ a(x+1) + b & \text{if } \; -1 < x \le 2 \\ x^3 - 12x + 5 & \text{if } \; x \le -1 \end{cases}$$

where a and b are real constants.

(i) Find the values of a and b that make $f(x)$ continuous in $\mathbb{R}$.

(ii) Calculate (if any) the local maxima and minima of $f(x)$ for $x < -1$.

Solution. The function $f(x)$ is obviously continuous for $x \in (-\infty, -1) \cup (-1, 2) \cup (2, +\infty)$ as given in terms of elementary continuous functions. Continuity at $x = -1$ is ensured by imposing that

$$f(-1) \;=\; \lim_{x\to -1^-} \; x^3 - 12x + 5 \;=\; \lim_{x\to -1^+} \; a(x+1) + b\,,$$

which yields $b = 16$. On the other hand, continuity at $x = 2$ is ensured by imposing that

$$f(2) \;=\; \lim_{x\to 2^-} \; a(x+1) + b \;=\; \lim_{x\to 2^+} \; -x^2 - 7\cos\left(\frac{\pi}{2}\,x\right)\,,$$

which yields $3a + b = 3$, namely $a = -13/3$. The calculated values for a and b make $f(x)$ continuous in $\mathbb{R}$. Now, the only critical point for $f(x)$ in the open interval $(-\infty, -1)$ is obtained as

$$f'(x) \;=\; 3x^2 - 12 \;=\; 0 \qquad \Longrightarrow \qquad x \;=\; -2\,,$$

which is a point of local maximum (because $f''(-2) = -12 < 0$) where $f(-2) = 21$. The function $f(x)$ has no local minima for $x < -1$.

4. Approximate the value of

$$\ln\left(\frac{4}{3}\right)$$

by a polynomial of degree 3 and estimate the involved error (with a suitable upper bound).

Solution. Note that

$$\ln\left(\frac{4}{3}\right) = \ln\left(1+\frac{1}{3}\right)$$

can be obtained by evaluating the function $f(x) = \ln(1+x)$ at $x = 1/3$. Such function can be expressed by using the Taylor theorem as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_3(x)\,,$$

where the remainder $R_3(x)$ is given by

$$R_3(x) = \frac{f^{(4)}(c)}{4!}\,x^4\,,$$

with $f^{(4)}(x) = -6\,(1+x)^{-4}$ and $c \in (0,x)$ if $x > 0$. Thus, we can approximate the desired value by

$$\ln\left(\frac{4}{3}\right) \approx \frac{1}{3} - \frac{1}{18} + \frac{1}{81} = \frac{47}{162} \approx 0.29$$

and estimate the involved error as

$$\left|R_3\left(\frac{1}{3}\right)\right| = \left|\frac{-6}{(1+c)^4}\,\frac{(1/3)^4}{4!}\right| = \frac{(1/3)^4}{4(1+c)^4} \leq \frac{(1/3)^4}{4} \approx 0.003\,,$$

where the inequality holds as $c \in (0,1/3)$, hence $c+1 \in (1,4/3)$ that implies $(1+c)^{-4} < 1$.