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Master theorem Examples - part 2

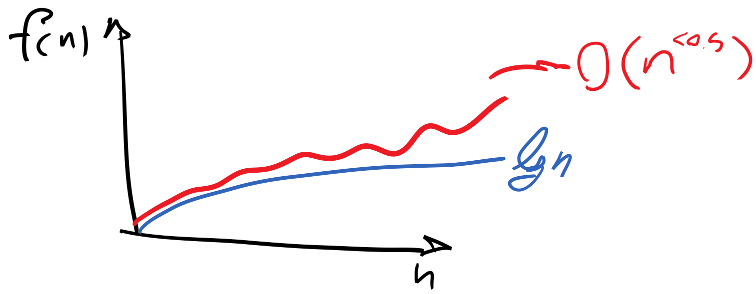
(11) $T(n) = \sqrt{2}T(n/2) + \log n$:

$a = \sqrt{2}$

$b = 2$

$f(n) = \lg n$

It is case 1 because: $O(n^{\lg_b a - \varepsilon}) = O(n^{\lg \sqrt{2} - \varepsilon})$ → < 0.5



Then: $T(n) = \Theta(n^{\lg_b a})$

$T(n) = \Theta(n^{\lg \sqrt{2}})$

$= \Theta(\sqrt{n})$

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(13) $T(n) = 3T(n/3) + \sqrt{n}$

$a = 3$

$b = 3$

$f(n) = \sqrt{n}$

It is case 1: $O(n^{\lg_3 3 - \varepsilon})$ → for small $\varepsilon > 0$ → $\sqrt{n} \leq O(n^{\lg_3 3 - \varepsilon})$

Then: $T(n) = \Theta(n^{\lg_3 3}) = \Theta(n)$

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(14) $T(n) = 4T(n/2) + cn$

$a = 4$

$b = 2$

$f(n) = cn$

It is case 1 because: $O(n^{\lg_2 4 - \varepsilon})$

for all $0 < \varepsilon < 2$, $n \leq O(n^{\lg 3 \ldots})$

Then $T(n) = \Theta(n^{\lg_2 4}) = \Theta(n^2)$

15 $T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$

$a = 3$
$b = 4$

$f(n) = n\lg n$

It is not case 1 because: $O\left(n^{\lg_4^{3-\varepsilon}}\right)$ <1

Then $n\lg n > n^{\lg <1}$

It is not case 2: $\Theta\left(n^{\lg_4^3} \lg^k n\right) \Rightarrow$

we cannot find a range for k to sandwitch $f(n)$

for $k=0 \longrightarrow n^{\lg_4^3} \cancel{(\lg n)^0} = n^\alpha < n$

$\alpha < 1$

$k=1 \longrightarrow n^\alpha \lg n$

It might be case 3: $\beta < 1$

$f(n) \geqslant \Omega\left(n^{\lg_4^{3+\varepsilon}}\right) \longrightarrow n\lg n > \Omega(n^\beta)$

But we have to examine regulation condition befor applying the rule in case 3.

$3\left(\frac{n}{4}\lg\frac{n}{4}\right) \leqslant c\, n\lg n$

$\frac{3\cancel{n}}{4}\lg\left(\frac{n}{4}\right) \leqslant c\cancel{n}\lg n$

$\frac{3}{4}\left(\lg n - \cancel{\lg 4}\right) \leqslant c\lg n$ $\quad 2$

for large n $\longrightarrow \frac{3}{4}\cancel{\lg n} \leqslant c\cancel{\lg n} \rightarrow \frac{3}{4} \leqslant c < 1$

Then: $T(n) = \Theta(n\lg n)$

(16) $T(n) = 3T(n/3) + \frac{n}{2}$

$a = 3$

$b = 3$

$f(n) = \frac{n}{2}$

It is not case 1 because: $\frac{n}{2} > O(n^{\lg_3^{3-\varepsilon}})$ <1

$\frac{n}{2} \to$ linear

It is case 2: $\theta(n^{\lg_3^3} \lg_n^k) = \theta(n \lg n^k)$

for $\boxed{k=0} \to f(n) \geq \theta(n)$

for $k=1 \to f(n) < \theta(n \lg n)$

Then: $T(n) = \theta(n^{\lg_3^3} \lg n^{k+1}) = \theta(n \lg n)$

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(17) $T(n) = 6T(\frac{n}{3}) + n^2 \lg n$

$a = 6$

$b = 3$

$f(n) = n^2 \lg n$

It is case 3 because: $n^2 \lg n \geq \Omega(n^{\lg_3^{6+\varepsilon}})$ $\to$ for $0<\varepsilon<3$

We have to examine regulation condition:

$6(\frac{n}{3})^2 \lg \frac{n}{3} \leq c n^2 \lg n$

$\frac{6}{9} \cancel{n^2} \cancel{\lg \frac{n}{3}} \leq c \cancel{n^2} \lg n$

$\hookrightarrow$ for large $n \to \lg \frac{n}{3} \approx \lg n$

$\frac{6}{9} \leq c < 1$

Then: $\boxed{T(n) = \theta(n^2 \lg n)}$

(19) $T(n) = 64\,T(n/8) - n^2 \lg n$

$a = 64$

$b = 8$

$f(n) = \underline{-n^2 \lg n}$ → $f(n)$ is negative. Master theorem cannot be applied. $f(n)$ must be positive.

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(20) $T(n) = 7\,T(n/3) + n^2$

$a = 7$

$b = 3$

$f(n) = n^2$

It is case 3, because $n^2 > \Omega(n^{\lg_3^{7-\varepsilon}})$ → <2 for all $0<\varepsilon<2$

But we have to examine regulation condition:

$$7\left(\frac{n}{3}\right)^2 \leqslant c n^2 \rightarrow \frac{7}{9}\cancel{n^2} \leqslant c\cancel{n^2} \rightarrow \frac{7}{9} \leqslant c < 1$$

Then : $\boxed{T(n) = \Theta(n^2)}$

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(21) $T(n) = 4\,T(n/2) + \lg n$

$a = 4$

$b = 2$

$f(n) = \lg n$

It is case 1, $\lg n \leqslant O(n^{\lg_2^{4-\varepsilon}})$ → for $0<\varepsilon<2$

Then $T(n) = \Theta(n^{\lg_2^4}) = \Theta(n^2)$