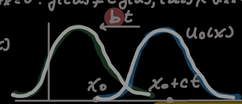
$u=u(x,t)$, $D>0$

(i) Heat Eq. (Diffusing Particles) $u_t=Du_{xx}$; (ii) Wave eq (wave prop) $u_{tt}=c^2u_{xx}$ (iii) Transport eq. (passive trans.) $u_t+cu_x=0$; $u=u(x,y)$ $\forall c$. $u_x=0 \implies u(x,y)=f(y)$; $y''+y=0 \implies y=A\cos(x)+B\sin(x)$ $\forall A,B\in\mathbb{R}$

∴ $u_{xx}+u=0 \implies u(x,y)=A(y)\cos(x)+B(y)\sin(x)$. Order: Highest derivative, Transformation $\mathcal{J}$ linear if (i) $\mathcal{J}(u+v)=\mathcal{J}(u)+\mathcal{J}(v)$ (ii) $\mathcal{J}(cu)=c\mathcal{J}(u)$; non-lin. PDE Linear if $\mathcal{J}(u)=f$, $\mathcal{J}$: Lin. tran. f ind. of u. 2° PDE:

$u_t-u_{xx}=0$: $\mathcal{J}(u)=u_t-u_{xx}$, $f=0$. $\mathcal{J}(u+v)=\mathcal{J}(u+v)$, $\mathcal{J}(cu)=c\mathcal{J}(u)$. (ii) $u_t+uu_x+u_{xxx}=0$: $\mathcal{J}(cu)\neq c\mathcal{J}(u)$, (iii) $x^3u_{xx}+y^2u_y+5u=0$: Linear; $au_{xx}+bu_{xy}+cu_{yy}+du_x+eu_y+fu=g$. $D=b^2-4ac$

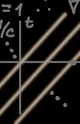
(i) $D<0$: Elliptic (ii) $D>0$: Hyperb. (iii) $D=0$: parabolic. $u_t=u_x$: $u(x,0)=u_0(x)$



$u(x_0,t)=u_0(x_0+bt)$. LHS: $\frac{\partial u(x,t)}{\partial t}=\frac{\partial u_0(x+bt)}{\partial t}=u_0'(x+bt)\cdot\frac{\partial(x+bt)}{\partial t}$

LHS $=bu_0'(x+bt)$; RHS: $\frac{\partial u(x,t)}{\partial x}=u_0'(x+bt)\frac{\partial(x+bt)}{\partial x}=u_0'(x+bt)$ ∴ LHS = RHS $\iff b=1$ ∴ $\forall u_0(x)$, $u(x,t)=u_0(x+t)$ solution to $u_t=u_x$

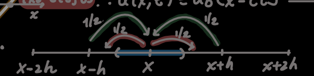
(Gradient): (Directional D. of u in direction $\vec{v}$) (Characteristic f) $m=1/c$



Sol. Transport Eq speed $=c$

$u=u(x,y)$, $\nabla u=(u_x,u_y)$, if $\|\vec{v}\|=1 \implies \nabla u\cdot\vec{v}$ ∴ $cu_x+u_t=\nabla u\cdot\langle c,1\rangle=0 \implies u$ constant in direction $\vec{v}=\langle c,1\rangle$ ∴ $u(x,t)=u_0(x-ct)$ ⟶

Heat: $u_t=Du_{xx}$. Let $u(x,t)$:= unit length #particles (pos: x, time: t). For $0<h<<1$, $hu(x,t)\approx\#p\in[x-\frac{h}{2},x+\frac{h}{2}]$. $\forall p$, $\mathbb{P}(p\to)=\mathbb{P}(\leftarrow)=1/2$



$t\longrightarrow t+\tau$

$\#p|_{t+\tau}$ $\#p|_t$ ⟵Inflow# ⟵Inflow# #leaving or

$hu(x,t+\tau)=hu(x,t)+\frac{h}{2}u(x-h,t)+\frac{h}{2}u(x+h,t)-hu(x,t)$ ∴ § $\frac{u(x,t+\tau)-u(x,t)}{\tau}=\frac{h^2}{2\tau}\frac{u(x-h,t)+u(x+h,t)-2u(x,t)}{h^2}$. Taylor Exp: $u(x,t+\tau)=u(x,t)+u_t(x,t)\tau+\dots$

$u(x+h,t)=u(x,h)+u_x(x,t)h+u_{xx}(x,t)\frac{h^2}{2}+\dots$; $u(x-h,t)=u(x,h)-u_x(x,h)h+u_{xx}(x,t)\frac{h^2}{2}+\dots$ ∴ $u(x,t+\tau)-u(x,t)=u_t(x,t)+O(\tau^2)$; $u(x-h,t)+u(x+h,t)-2u(x,t)=u_{xx}(x,t)h^2+O(h^3)$

$\frac{h^2}{2\tau}$, choose $\tau=\frac{h^2}{2D}$ ∴ $\frac{h^2}{2\tau}=D$. Take $h\to0$ ∴ § $u_t=Du_{xx}$

Euler's F: $e^{i\varphi}=\cos\varphi+i\sin\varphi$ ∴ $e^{ikx}=\cos(kx)+i\sin(kx)$. Fourier Transform: $\hat{u}(k)$ of $u(x)$ $\forall k,x\in\mathbb{R}$

$(\mathcal{F}u)(k)=\hat{u}(k)=\int_{\mathbb{R}}e^{ikx}u(x)dx$. Let $u(x)=e^{-ax^2}$ ($a>0$) ∴ $\hat{u}(k)=\int_{\mathbb{R}}e^{ikx}e^{-ax^2}dx \implies \frac{d\hat{u}(k)}{dk}=\int_{\mathbb{R}}\frac{\partial}{\partial k}(e^{ikx})e^{-ax^2}dx=\int_{\mathbb{R}}ie^{ikx}xe^{-ax^2}dx=\int_{\mathbb{R}}ie^{ikx}\left(\frac{-1}{2a}\frac{d}{dx}(e^{-ax^2})\right)dx=\frac{-i}{2a}\int_{\mathbb{R}}e^{ikx}\frac{d}{dx}(e^{-ax^2})dx\stackrel{IBP}{=}\frac{i}{2a}\int_{\mathbb{R}}\frac{d}{dx}(e^{ikx})e^{-ax^2}dx$

$=\frac{-k}{2a}\int_{\mathbb{R}}e^{ikx}e^{-ax^2}dx=\frac{-k}{2a}\hat{u}(k)$ ∴ $\frac{d\hat{u}(k)}{dk}=\frac{-k}{2a}\hat{u}(k)$; $(\ln\hat{u}(k))'=\frac{-k}{2a}\implies\hat{u}(k)=\hat{u}(0)e^{-k^2/4a}$; $\hat{u}(0)=\int_{\mathbb{R}}e^{ikx}e^{-ax^2}dx\Big|_{k=0}=\int_{\mathbb{R}}e^{-ax^2}dx=\sqrt{\frac{\pi}{a}}$ ∴ $\hat{u}(k)=\sqrt{\frac{\pi}{a}}e^{-\frac{k^2}{4a}}$ ($\mathcal{S}$) Schwartz Class

$u(x)\in\mathcal{S}$ if $\forall n,m\in\mathbb{N}\ \exists u^{(m)}(x)$, $\lim_{|x|\to\infty}|x|^n\left|\frac{d^mu(x)}{dx^m}\right|=0$. I.F.T: $u(x)\in\mathcal{S}\Leftrightarrow\hat{u}(k)\in\mathcal{S}$ and $u(x)=(\mathcal{F}^{-1}\hat{u})(x):=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-ikx}\hat{u}(k)dk$. IBP: $\int_{\mathbb{R}}uv'dx=-\int_{\mathbb{R}}u'vdx+u(\infty)v(\infty)-u(-\infty)v(-\infty)$. Heat eq. $u_t=Du_{xx}$, $u(x,0)=f(x)$

$\implies\hat{u}_t(k,t)=\hat{u}(k,0)e^{-Dk^2t}=\hat{f}(k)e^{-Dk^2t}\implies u(x,t)=\mathcal{F}^{-1}(\hat{f}(k)e^{-Dk^2t})=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ikx}\hat{f}(k)e^{-Dk^2t}dk$. Convolution: $u(x),v(x)\in\mathcal{S}$, $(u*v)(x):=\int_{\mathbb{R}}u(x-y)v(y)dy$: $\mathcal{F}^{-1}(\hat{u}(k)\hat{v}(k))(x)=(u*v)(x)$ ⟺

$[\mathcal{F}(u*v)(x)](k)=\hat{u}(k)\hat{v}(k)$ ∴ $u(x,t)=\mathcal{F}^{-1}(e^{-Dk^2t}\hat{f}(k))=(G*f)(x,t)$, $G(x,t)=\mathcal{F}^{-1}(e^{-Dk^2t})$. Recall: $\mathcal{F}(e^{-ax^2})=\sqrt{\frac{\pi}{a}}e^{-k^2/4a}\implies\mathcal{F}^{-1}(e^{-k^2/4a})=\sqrt{\frac{a}{\pi}}e^{-ax^2}$, need: $\frac{1}{4a}=Dt\implies a=\frac{1}{4Dt}$

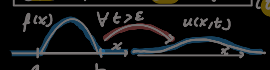
$\implies G(x,t)=\sqrt{\frac{a}{\pi}}e^{-a^2x}\Big|_{a=\frac{1}{4Dt}}=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}\implies$ Sol. Heat Eq $u(x,t)=(G*f)(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{4Dt}}f(y)dy$. Derivatives: (i) $(\mathcal{F}u_x)(k)=(-ik)\hat{u}(k)$ (ii) $(\mathcal{F}u_t)(k)=\frac{\partial}{\partial t}\hat{u}(k)$

$$\begin{array}{ccc} \text{PDE in }(x,t) & \xrightarrow{\mathcal{F}} & \text{ODE in }(k,t) \\ \downarrow\S & & \downarrow \\ \text{Sol } u(x,t) & \xleftarrow{\mathcal{F}^{-1}} & \text{sol. }\hat{u}(k,t) \end{array}$$

Sol. Heat Prop: $u(x,y)$, (i) Conserv. of mass: $\int_{\mathbb{R}}u(x,t)dx=\int_{\mathbb{R}}f(x)dx$, (ii) $\exists!\,u(x,t)$, (iii) Bounded. Let $\mathcal{M}=\max|f(x)|\implies|u(x,t)|\le\mathcal{M}$ $\forall x\in\mathbb{R},t>0$

$\forall t\ge0$ $\tau\to0$ in $x\in\mathbb{R}$ Infinite S. of Prop. $\forall x\in\mathbb{R},t>0$

(iv) Decay: $|u(x,t)|\le\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}|f(y)|dy$ ∴ $|u(x,t)|\to0$ uniformly; (v) If $\exists a,b\in\mathbb{R}$ s.t $f(x)>0$ in (a,b), $f(x)=0$ elsewhere $\implies u(x,t)>0$



Proofs: (i) $\int_{\mathbb{R}}u_t(x,t)dx=D\int_{\mathbb{R}}u_{xx}(x,t)dx\stackrel{IBP}{=}-D\int_{\mathbb{R}}\partial_x(1)u_x(x,t)dt=0$ ∴ $\frac{d}{dt}\int_{\mathbb{R}}u(x,t)dx=0\implies\int_{\mathbb{R}}u(x,t)dx=\int_{\mathbb{R}}u(x,0)dx=\int_{\mathbb{R}}f(x)dx$ (ii) Let

$u_1(x,t)$, $u_2(x,t)$ sol. $v(x,t):=u_1(x,t)-u_2(x,t)$ ∴ v sol. $\begin{cases}v_t=Dv_{xx}\\v(x,0)=0\end{cases}$ $(\cdot v)$: $\int_{\mathbb{R}}v_t(x,t)v(x,t)dx=D\int_{\mathbb{R}}v_{xx}(x,t)v(x,t)dx$, LHS $=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}v(x,t)^2dx$, RHS $\stackrel{IBP}{=}-D\int_{\mathbb{R}}(v_x(x,t))^2dx\le0$

∴ $\int_{\mathbb{R}}v(x,t)^2dx$ non-inc. $\implies0\le\int_{\mathbb{R}}v(x,t)^2dx\le\int_{\mathbb{R}}v(x,0)^2dx=0\implies v(x,t)=0$ $\forall x\in\mathbb{R}$. Energy Method: Show $E(t):=\int_{\mathbb{R}}v(x,t)^2dx$ is not decreasing in time. (iii) Given (ii)

$\exists!\,u(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{4Dt}}f(y)dy$ ∴ $|u(x,t)|\le\frac{\mathcal{M}}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{4Dt}}dy$, $z=y-x$ ∴ $|u(x,t)|\le\mathcal{M}\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}e^{-z^2/4Dt}dz=\mathcal{M}$. (iv) $e^{-\frac{(x-y)^2}{4Dt}}\le1$ ∴ $|u(x,t)|\le\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}|f(y)|dy$. (v) $u(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{4Dt}}f(y)dy$

$=\frac{1}{\sqrt{4\pi Dt}}\int_a^be^{-\frac{(x-y)^2}{4Dt}}f(y)dy>0$. Wave eq. $u_{tt}=c^2u_{xx}$, $u(x,t)$ displacement of a vib. at (x,t). Damped Wave Eq. $u_{tt}=c^2u_{xx}-ku_t$. $\frac{\partial^2}{\partial t^2}u-c^2\frac{\partial^2}{\partial x^2}u=(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x})(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x})u$, change var.

$\begin{cases}\xi=x+ct\\\eta=x-ct\end{cases}\implies\begin{cases}x=\frac{\xi+\eta}{2}\\t=\frac{\xi-\eta}{2c}\end{cases}$; $\frac{\partial}{\partial\xi}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{1}{2c}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial\eta}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{1}{2c}\frac{\partial}{\partial t}$. chain Rule: ξ → x, t; η → x, t. Let $v(\xi,\eta)=u(x,t)=u(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})$. C.R: $v_\xi(\xi,\eta)=u_x\frac{\partial x}{\partial\xi}+u_t\frac{\partial t}{\partial\xi}$

$=\frac{1}{2}u_x(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})+\frac{1}{2c}u_t(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})$ ∴ $v_{\xi\eta}(\xi,\eta)=\frac{1}{2}u_{xx}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})\frac{\partial x}{\partial\eta}+\frac{1}{2}u_{xt}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})\frac{\partial t}{\partial\eta}+\frac{1}{2c}u_{tx}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})\frac{\partial x}{\partial\eta}+\frac{1}{2c}u_{tt}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})\frac{\partial t}{\partial\eta}=0$ ⟹

$v_{\xi\eta}=\frac{1}{4}u_{xx}-\frac{1}{4c}u_{xt}+\frac{1}{4c}u_{tx}-\frac{1}{4c^2}u_{tt}=\frac{1}{4c^2}(c^2u_{xx}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c})-u_{tt}(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}))=0$ ∴ $v_{\xi\eta}=0\implies v_\xi$ ind. of $\eta\implies v_\xi=f(\xi)\implies v(\xi,\eta)=\int f(\xi)+G(\eta)=F(\xi)+G(\eta)$

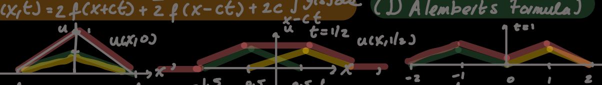
∴ $u(x,t)=v(\xi,\eta)=F(x+ct)+G(x-ct)$ $\forall F,G$. gen. sol. wave Eq. Wave-like Eq: $u_{tt}+Au_{xt}+Bu_{xx}=0$ If $\exists a,b\in\mathbb{R}$ s.t $a\neq b$ and $(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x})(\frac{\partial}{\partial t}-b\frac{\partial}{\partial x})u=0$

⟹ gen. sol. is: $F(x+at)+G(x+bt)$. $\begin{cases}u_{tt}-c^2u_{xx}=0\\u(x,0)=f(x) & \text{init. position}\\u_t(x,0)=g(x) & \text{init. velocity}\end{cases}$ ∴ $u(x,0)=F(x)+G(x)=f(x)$, $u_t(x,t)=cF'(x+ct)-cG'(x-ct)$ ⟹

$u_t(x,0)=cF'(x)-cG'(x)=g(x)$ ∴ $\int_0^xF'(s)-G'(s)ds=\frac{1}{c}\int_0^xg(s)ds$ | $F(x)-G(x)-\underbrace{(F(0)-G(0))}_{A}=\frac{1}{c}\int_0^xg(s)ds$ ∴ $\begin{cases}F(x)+G(x)=f(x)\\F(x)-G(x)=A+\frac{1}{c}\int_0^xg(s)ds\end{cases}$ ⟹

$F(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_0^xg(s)ds+\frac{A}{2}$; $G(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_0^xg(s)ds-\frac{A}{2}$ ⟹ $u(x,t)=F(x+ct)+G(x-ct)$. Sol. $u(x,t)=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds$ (D'Alembert's Formula)

If init veloc. $g(x)=0$, $f(x)$ splitted evenly into 2: ←, →, $u(x,t)=\frac{1}{2}f(x+ct)-\frac{1}{2}f(x-ct)$

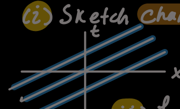


Conservation of Energy: Define $E(t)=\frac{1}{2}\int_{\mathbb{R}}(u_t(x,t))^2dx+\frac{c^2}{2}\int_{\mathbb{R}}(u_x(x,t))^2dx$ (K.E.) (P.E.). If $u(x,t)\in\mathcal{S}\implies E(t)=E(0)$ $\forall t\in\mathbb{R}$. Proof: $(\cdot,u_t)$ ∴ $\int_{\mathbb{R}}u_{tt}u_tdx-c^2\int_{\mathbb{R}}u_{xx}u_tdx=0$

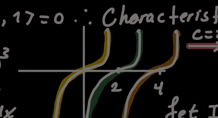
LHS: $\int_{\mathbb{R}}\frac{\partial}{\partial t}(\frac{1}{2}(u_t)^2)dx+c^2\int_{\mathbb{R}}u_xu_{xt}dx$ (IBP) $=\frac{d}{dt}(\frac{1}{2}\int_{\mathbb{R}}(u_t)^2dx+\frac{c^2}{2}\int_{\mathbb{R}}(u_x)^2dx)=\frac{d}{dt}(E(t))=0\implies E(t)=E(0)$. Remark: For energy method for (i) Heat Eq. $(\cdot,u)$

(ii) Wave Eq $(\cdot,u_t)$. Theorem: Let $a,b\in\mathbb{R}$. If $g(x)=f(x-a)\implies\hat{g}(k)=e^{ika}\hat{f}(k)$

(i) Sketch characteristic lines, solve Transport Eq. $\begin{cases}u_t+2u_x=0\\u(x,0)=x^3\end{cases}$: $u_t+2u_x=0\iff(u_x,u_t)\cdot\langle2,1\rangle=0$ ∴ Characteristic lines ∥ $\langle2,1\rangle$



Sol: $u(x,t)=u(x-2t,0)=(x-2t)^3$. ∴ $u(x,0)=x^3$, $u(x,1)=(x-2)^3$, $u(x,2)=(x-4)^3$



(ii) Let $u(x,t):=\frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt}$. Verify $u(x,t)$ sol. to Heat Eq: $u_t=Du_{xx}$. (iii) $a>0$, $\int_{\mathbb{R}}e^{-ax^2}dx$. Let $I=\int_{\mathbb{R}}e^{-ax^2}dx$

∴ $I^2=\int_{\mathbb{R}}e^{-ax^2}dx\int_{\mathbb{R}}e^{-ay^2}dy=\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-a(x^2+y^2)}dxdy=\int_{\mathbb{R}^+}\int_0^{2\pi}e^{-ar^2}rd\theta dr=2\pi\int_{\mathbb{R}^+}e^{-ar^2}rdr=\pi\int_{\mathbb{R}^+}e^{-ar^2}dr^2=\frac{\pi}{-a}e^{-ar^2}\Big|_{r^2=0}^{\infty}=0-\frac{\pi}{-a}=\frac{\pi}{a}$ ∴ $I=\sqrt{\frac{\pi}{a}}$

(iv) Derive transp. Eq. Assume $u(x,t)$: $\#p\,|(x,t)$. Let $0<h,\tau<<1$. $\mathbb{R}/=\dots,x-2h,x-h,x,x+h,x+2h,\dots$ Assume $\forall p\in[t,t+\tau]$

$p\to$ nearest interval to the right. (a) Balance Eq.

$\underset{1°}{hu(x,t+\tau)}=\underset{2°}{hu(x,t)}+\underset{3°}{hu(x-h,t)}-\underset{4°}{hu(x,t)}$

$1°: \#p \in [x-\frac{h}{2}, x+\frac{h}{2}]\Big|_{t+\tau}$, $2°: \#p \in [x-\frac{h}{2}, x+\frac{h}{2}]\Big|_{t}$, $3°: \#p \in [x-\frac{3h}{2}, x-\frac{h}{2}]$, $4°$: $\#p$ going out. (b) Rearrange terms and use Taylor Expansions:

§ $\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{h}{\tau}\left(\frac{u(x-h,t)-u(x,t)}{h}\right)$. Taylor E.: $u(x,t+\tau) = u(x,t) + u_t(x,t)\tau + O(\tau^2)$; $u(x-h,t) = u(x,t) - u_x(x,t)h + O(h^2)$ $\Longrightarrow$ plug into §

§ $u_t(x,t) + O(\tau) = \frac{h}{\tau}(-u_x(x,t) + O(h))$ (c) Choose h, τ s.t $h, \tau \to 0$ and § $\to$ Transp. Eq: $\tau = \frac{h}{c}$, $h \to 0$ $\therefore$ $u_t(x,t) = -c\,u_x(x,t) \Longleftrightarrow u_t + cu_x = 0$

(v) If $g(x) = f(x-a) \Longrightarrow \hat{g}(k) = e^{ika}\hat{f}(k)$. Use $\mathcal{F}$ and theorem to solve Transp. $\begin{cases} u_t + cu_x = 0 \\ u(x,0) = f(x) \end{cases}$: $\hat{u}_t - ikc\hat{u} = 0$, $\hat{u}(k,0) = \hat{f}(k)$

$\therefore$ $\hat{u}(k,t) = \hat{u}(k,0)e^{ikct} = \hat{f}(k)e^{ikct} \Longrightarrow u(x,t) = \mathcal{F}^{-1}(\hat{f}e^{ikct}) = f(x-ct)$. (vi) Consider $u_t = u_{xxxx}$ for $x \in \mathbb{R}$ with $u(x,0) = f(x) \in \mathcal{S}$.

Derive an expression for F.T. $\hat{u}(k,t)$ of the solution $u(x,t)$ in terms of k, t and $\hat{f}(k)$: $\hat{u}_t = \widehat{u_t} = \widehat{u_{xxxx}} = (-ik)\widehat{u_{xxx}} = (-ik)^2\widehat{u_{xx}} = (-ik)^3\widehat{u_x} = (-ik)^4\hat{u} = k^4\hat{u}$

$\therefore$ $\hat{u}(k,t) = \hat{u}(k,0)e^{k^4 t} = \hat{f}(k)e^{k^4 t}$ (vii) For $D>0$, $a, c \in \mathbb{R}$, consider $u_t = Du_{xx} + cu_x - au$ for $x \in \mathbb{R}$, $t>0$. Assume $u(x,t)$ sol. Verify

$v(y,t) := u(y-ct,t)e^{at}$ sol. to $v_t = Dv_{yy}$. : $v_t = \frac{\partial}{\partial t}(u(y-ct,t)e^{at}) = -c\,u_x(y-ct,t)e^{at} + u_t(y-ct,t)e^{at} + a\,u(y-ct,t)e^{at}$

$= -cu_x e^{at} + (Du_{xx} + cu_x e^{at} - au)e^{at} + aue^{at} = Du_{xx}(y-ct,t)e^{at}$. $Dv_{yy} = D\frac{\partial^2}{\partial y^2}(u(y-ct,t)e^{at}) = D\frac{\partial}{\partial y}(u_x(y-ct,t)e^{at}) = D\,u_{xx}(y-ct,t)e^{at}$

(viii) $v(y,t) := \frac{1}{\sqrt{4\pi t}}e^{-y^2/4t}$ sol. $v_t = v_{yy}$. Use (vii) to find $u(x,t)$ sol. to $u_t = u_{xx} + 2u_x - 0.5u$. $a = 1/2$, $c = 2$ $\therefore$ $v(y,t) = u(y-2,t)e^{1/2\,t}$

let $y = x+2t$ $\therefore$ $v(x+2t,t) = u(x,t)e^{1/2\,t}$ $\therefore$ $u(x,t) = v(x+2,t)e^{-1/2\,t} = \frac{1}{\sqrt{4\pi t}}e^{(x+2t)^2/4t} \cdot e^{-1/2\,t}$. (ix) Solve $\begin{cases} u_t - Du_{xx} = 0 \\ u(x,0) = e^x \end{cases}$

By sol. Formula $u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}}e^{y}\,dy$. Let $z = y-x$ $\therefore$ $= \frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}} e^{-\frac{z^2}{4Dt}}e^{z+x}\,dz = e^x\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}} e^{-\frac{z^2-4Dtz}{4Dt}}\,dz = e^x\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}} e^{-\frac{(z-2\sqrt{Dt})^2}{4Dt}}e^{Dt}\,dz$

Let $s = z - 2\sqrt{Dt} \Longrightarrow = e^{x+Dt}\frac{1}{\sqrt{4\pi Dt}}\int_{\mathbb{R}} e^{-s^2/4Dt}\,ds = e^{x+Dt}$ (x) Use $\mathcal{F}$ to find gen. sol to Wave Eq $u_{tt} = c^2u_{xx}$: $\hat{u}_{tt} = c^2(ik)^2\hat{u}(x,t)$

$\therefore$ $\hat{u}_{tt}(k,t) = -c^2k^2\hat{u}(k,t)$ $\therefore$ $\hat{u}(k,t) = A(k)e^{ickt} + B(k)e^{-ickt}$. Let $F(x) = \mathcal{F}^{-1}(A(k))$, $G(x) = \mathcal{F}^{-1}(B(k)) \Longrightarrow u(x,t) = \mathcal{F}^{-1}(\hat{u}(k,t))$

$= \mathcal{F}^{-1}(A(k)e^{ickt}) + \mathcal{F}^{-1}(B(k)e^{-ickt}) = F(x-ct) + G(x+ct)$. (xi) Solve $\begin{cases} u_{tt} - 3u_{xt} - 4u_{xx} = 0 \\ u(x,0) = 2x^2 \\ u_t(x,0) = e^{4x} \end{cases} \Longrightarrow (\frac{\partial}{\partial t} - 4\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u = 0$ $\therefore$ gen. sol: $u(x,t) = F(x+4t) + G(x-t)$

$u_t(x,t) = 4F'(x+4t) - G'(x-t)$ $\therefore$ $\begin{cases} F(x) + G(x) = 2x^2 \\ 4F'(x) - G'(x) = e^{4x} \end{cases} \Longrightarrow 4F(x) - G(x) = \frac{1}{4}e^{4x} + C$ $\therefore$ $\begin{cases} F(x) + G(x) = 2x^2 \\ 4F(x) - G(x) = \frac{1}{4}e^{4x} + C \end{cases}$ $\therefore$ $F(x) = \frac{2}{5}x^2 + \frac{1}{20}e^{4x} + \frac{C}{5}$

$G(x) = \frac{8}{5}x^2 - \frac{1}{20}e^{4x} - \frac{C}{5}$ $\therefore$ $u(x,t) = \frac{2}{5}(x+4t)^2 + \frac{1}{20}e^{4(x+4t)} + \frac{8}{5}(x-t)^2 - \frac{1}{20}e^{4(x-t)}$ (xii) Choose $\alpha > 0$. Let $u(x,t)$ be a sol. $\in \mathcal{S}$ to damped

Wave Eq. $u_{tt} + \alpha u_t = u_{xx}$. Show that energy $E(t) = \frac{1}{2}\int_{\mathbb{R}}[(u_t(x,t))^2 + (u_x(x,t))^2]\,dx$ is non-increasing ($\frac{d}{dt}E(t) \le 0$)

1) $(\cdot, u_t) \Longrightarrow \underbrace{\int_{\mathbb{R}} u_{tt}u_t\,dx}_{= \frac{d}{dt}(\frac{1}{2}\int_{\mathbb{R}}(u_t)^2dx)} + \alpha\int_{\mathbb{R}}(u_t)^2dx = \underbrace{\int_{\mathbb{R}} u_{xx}u_t\,dx}_{\overset{IBP}{=} -\int_{\mathbb{R}} u_x u_{xt}\,dx = -\frac{d}{dt}(\frac{1}{2}\int_{\mathbb{R}}(u_x)^2dx)} \Longrightarrow \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}(u_t)^2 + (u_x)^2dx\right) = -\alpha\int_{\mathbb{R}}(u_t)^2dx \le 0 \Longrightarrow \frac{d}{dt}E(t) \le 0$

Heat Eq: Bounded Intervals : $u(x,t): u_t = Du_{xx}$, $a<x<b$ $\Rightarrow$ (i) $u(a,t)=T_1$, $u(b,t)=T_2$ $\forall t>0$ (Dirichlet Conditions), T_1, T_2 constants.

(ii) $u_x(a,t)=u_x(b,t)=0$ $\forall t>0$ (Neumann Conditions) $\overset{a-h/2 \quad a \quad a+h/2}{\longleftrightarrow}$ $\forall t$, $\#p\in[a-h,a]=\#p\in[a,a+h]$. Balance Eq. of $x=a$:

$hu(a-\frac{h}{2},t)=hu(a+\frac{h}{2},t)$ $\therefore \lim_{h\to 0}\frac{hu(a-\frac{h}{2},t)-hu(a+\frac{h}{2},t)}{h^2}=0 \Rightarrow u_x(a,t)=0$. Homogeneous Dirichlet Conditions: $[a,b]=[0,\pi]$, $D=1$

(1) $\begin{cases} u_t=u_{xx}, & 0<x<\pi, t>0 \\ u(0,t)=u(\pi,t)=0, & t>0 \\ u(x,0)=f(x), & 0<x<\pi \end{cases}$. Recall $V'=AV \in \mathbb{R}^2$. (i) Sol: $v(t)=v_0 e^{\lambda t}$ (ii) $(A-\lambda)V_0=0$ (iii) $\exists \lambda_1,\lambda_2$ sol. $\det(A-\lambda I)=0$ (iv) eigenvectors:

$\vec{v}_1,\vec{v}_2$ sol. $(A-\lambda_i)\vec{v}_i=0$, $\lambda_i \rightsquigarrow \vec{v}_i$ (v) $v(t)=A_1e^{\lambda_1 t}\vec{v}_1+A_2e^{\lambda_2 t}\vec{v}_2$ (vi) A_1,A_2, from $V(0)$. (1): Look solutions of form $u(x,t)=e^{\lambda t}\sin(nx)$

$LHS=\frac{\partial}{\partial t}(e^{\lambda t}\sin(nx))=\lambda e^{\lambda t}\sin(nx)$, $RHS=\frac{\partial^2}{\partial x^2}(e^{\lambda t}\sin(nx))=-n^2e^{\lambda t}\sin(nx)$ $\therefore \lambda=-n^2 \Rightarrow LHS=RHS$, $u(0,t)=e^{-n^2t}\sin(0)=0$ $\forall t$, $u(\pi,t)=e^{-n^2t}\sin(n\pi)=0$ $(n\in\mathbb{N})$

$\Rightarrow e^{-n^2t}\sin(n\pi)$ sol. $\forall n\in\mathbb{N}$. Superposition: If u_1,u_2 sol $\Rightarrow (a_1u_1+a_2u_2)$ sol. $\forall a_1,a_2$ $\therefore u(x,t)=\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)$. Separation of Variables:

$u(x,t)=X(x)T(t)$. (1) $\Rightarrow X(x)T'(t)=X''(x)T(x)$, $\frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}=\lambda\in\mathbb{R}$, $u(0,t)=X(0)T(t)=0 \Rightarrow X(0)=0$, $u(\pi,t)=0 \Rightarrow X(\pi)=0$ $\therefore \begin{cases} X''-\lambda X=0 \\ X(0)=X(\pi)=0 \end{cases}$ set $V(x)=\begin{pmatrix} x'(x) \\ x(x) \end{pmatrix}$

$\therefore V'(x)=\begin{pmatrix} x''(x) \\ x'(x) \end{pmatrix}=\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x'(x) \\ x(x) \end{pmatrix}$ $\therefore V'=AV$, $A=\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$ $\therefore$ eigenvalues: $\lambda_1=\sqrt{\lambda}$, $\lambda_2=-\sqrt{\lambda}$. Case $\lambda>0$: $\exists \omega\in\mathbb{R}^+$ s.t $\lambda=\omega^2$ $\therefore \lambda_1=\omega$, $\lambda_2=-\omega$. $A-\lambda_1 I=\begin{pmatrix} -\omega & \omega^2 \\ 1 & -\omega \end{pmatrix}$

$(A-\lambda_i I)\vec{v}_i=\vec{0} \Rightarrow \lambda_i \rightsquigarrow V_i$ $\therefore \begin{pmatrix} -\omega & \omega^2 \\ 1 & -\omega \end{pmatrix}\begin{pmatrix} \omega \\ 1 \end{pmatrix}=\vec{0}$ $\therefore V_1=\begin{pmatrix} \omega \\ 1 \end{pmatrix}$; $(A-\lambda_2 I)=\begin{pmatrix} \omega & \omega^2 \\ 1 & \omega \end{pmatrix} \rightsquigarrow \vec{V}_2=\begin{pmatrix} -\omega \\ 1 \end{pmatrix}$ $\therefore V(x)=\begin{pmatrix} x'(x) \\ x(x) \end{pmatrix}=Ae^{\omega x}\begin{pmatrix} \omega \\ 1 \end{pmatrix}+Be^{-\omega x}\begin{pmatrix} -\omega \\ 1 \end{pmatrix}$ $\therefore X(x)=Ae^{\omega x}+Be^{-\omega x}$ $\forall A,B$.

$X(0)=a_1+a_2=0 \Rightarrow a_1=-a_2$, $X(\pi)=a_1(e^{\omega\pi}-e^{-\omega\pi})=0 \Rightarrow a_1=0$ $\therefore X(x)=0$ Only solution. Case $\lambda=0$: $\lambda_1=\lambda_2=0$, $A-\lambda_1 I=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\therefore \lambda_1 \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

generalized eigenvector $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\vec{V}_{gen}=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow V_{gen}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\therefore V(x)=\begin{pmatrix} x'(x) \\ x(x) \end{pmatrix}=a_1e^{0x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+a_2\left(xe^{0x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+e^{0x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \Rightarrow X(x)=a_1+a_2x$ $\forall a_1,a_2$. $X(0)=X(\pi)=0$

$\Rightarrow X(x)=0$. Case 3: $\lambda<0$: $\lambda=-\omega^2 \Rightarrow \lambda_1=i\omega$, $\lambda_2=-i\omega \Longrightarrow X(x)=a_1\cos(\omega x)+a_2\sin(\omega x)$ $\forall a_1,a_2$, $X(0)=a_1=0$, $X(\pi)=a_2\sin(\omega\pi)=0 \Rightarrow \omega=n\in\mathbb{N}$. $X(x)=a_2\sin(nx)$

$\lambda=-n^2 \Rightarrow \frac{T'(t)}{T(t)}=-n^2$ $\therefore T'(t)=-n^2T(t) \Rightarrow T(t)=ce^{-n^2t}$ $\therefore X(x)T(t)=Ce^{-n^2t}\sin(nx), n\in\mathbb{N}$. By Superposition: $u(x,t)=\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)$. a_n determined from

$u(x,0)=f(x) \Longrightarrow \sum_{n=1}^{\infty}a_n\sin(nx)=f(x)$. Ex: $u(x,0)=3\sin(x)+7\sin(3x) \Rightarrow a_1=3$, $a_3=7$, $a_n=0$ for $n\neq 1,3$. $\therefore u(x,t)=3e^{-t}\sin x+7e^{-9t}\sin(3x)$. Solution is unique.

Assume u_1,u_2 sol. to (1). Let $V=u_1-u_2 \Rightarrow \begin{cases} V_t=V_{xx} \\ V(0,t)=V(\pi,t)=0 \\ V(x,0)=0 \end{cases}$ $\therefore \int_0^{\pi}V_tV\,dx=\int_0^{\pi}V_{xx}V\,dx$, $\frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^{\pi}V^2dx\right)=V_xV\Big|_{x=0}^{\pi}-\int_0^{\pi}V_x^2dx \Rightarrow 0\le E(t)\le E(0)=0$, $E(t)=\frac{1}{2}\int_0^{\pi}V^2(x,t)dx$

$\Rightarrow V(x,t)=0$ $\forall t \Rightarrow u_1(x,t)=u_2(x,t)$ $\therefore \exists!\, u(x,t)$. Inhomogeneous B.C. $\begin{cases} u_t=u_{xx} \\ u(0,t)=T_1 \\ u(\pi,t)=T_2 \end{cases}$ $U_s:=ax+b$ sol. $\therefore U_s(0)=T_1$, $U_s(\pi)=T_2 \Rightarrow b=T_1$, $a=\frac{T_2-T_1}{\pi}$ $\therefore V(x,t)=u(x,t)-U_s(x)$

$\therefore u(x,t)=U_s(x)+V(x,t)=\frac{T_2-T_1}{\pi}x+T_1+\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)$. Wave Eq. $\begin{cases} u_{tt}=u_{xx} \\ u_x(0,t)=0 \\ u_x(\pi,t)=0 \end{cases} \Rightarrow \frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=\lambda$. $X''(x)=\lambda X(x)$, $X'(0)=X'(\pi)=0$. Case $\lambda>0$: $X(x)=a_1e^{\omega x}+a_2e^{-\omega x}$

$X(x)=0$

$X'(x)=a_1\omega e^{\omega x}-a_2\omega e^{-\omega x}$, $X'(0) \Rightarrow a_1=a_2$, $X'(\pi)=a_1(\omega e^{\omega\pi}-\omega e^{-\omega\pi})=0 \Rightarrow a_1=a_2=0$. Case $\lambda=0$: $X(x)=a_1+a_2x$, $X'(0) \Rightarrow a_2=0$ $\therefore X(x)=a_1$; $T''=0 \Rightarrow T=a_0+b_0t$

Case 3: $\lambda=-\omega^2$: $X(x)=a_1\cos(\omega x)+a_2\sin(\omega x)$, $X'(x)=-a_1\omega\sin(\omega x)+a_2\omega\cos(\omega x)$, $X'(0)=a_2\omega=0 \Rightarrow a_2=0$, $X'(\pi)=-a_1\omega\sin(\omega x)=0 \Rightarrow \omega=n\in\mathbb{N}$

$\therefore X(x)=a_1\cos(nx)$, $T''(t)=-n^2T(t) \Rightarrow T(t)=c_1\cos(nt)+c_2\sin(nt)$ $\therefore X(x)T(x)=c_1\cos(nt)\cos(nx)+c_2\sin(nt)\cos(nx)$

$\therefore u(x,t)=a_0+b_0t+\sum_{n=1}^{\infty}a_n\cos(nt)\cos(nx)+b_n\sin(nt)\cos(nx)$. $\therefore$ Dirichlet B.C. $(X''=\lambda X)$ $(X(0)=X(\pi)=0) \xRightarrow{\lambda=-n^2} X(x)=C\sin(nx)$ $\forall c$. $n\geq 1\in\mathbb{N}$, $k\geq 1\in\mathbb{N}$

Neumann B.C. $(X'(0)=X'(\pi)=0)$: $\xrightarrow{\lambda=-n^2,\ n\geq 0\in\mathbb{Z}}$ $X(x)=C\cos(nx)$ $\forall c$. From (1) $u(x,t)=\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)$, a_n s.t $f(x)=\sum_{n=1}^{\infty}a_n\sin(nx)$ $\therefore \int_0^{\pi}f(x)\sin(kx)dx=\int_0^{\pi}\sum_{n=1}^{\infty}a_n\sin(nx)\sin(kx)dx$

$=\sum_{n=1}^{\infty}a_n\int_0^{\pi}\sin(nx)\sin(kx)dx$. $\cos(a+b)=\cos a\cos b-\sin a\sin b$, $\cos(a-b)=\cos a\cos b+\sin a\sin b \Rightarrow \sin a\sin b=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$ $\therefore \sum_{n=1}^{\infty}a_n\int_0^{\pi}\sin(nx)\sin(kx)dx=$

$\sum\frac{a_n}{2}\int_0^{\pi}\cos((n-k)x)-\cos((n+k)x)dx=\frac{a_k}{2}\int_0^{\pi}(1-\cos(2kx))dx+\sum_{n\neq k}\frac{a_n}{2}\int_0^{\pi}\cos((n+k)x)-\cos((n+k)x)dx=\frac{\pi}{2}a_k-\frac{a_k}{2}\frac{1}{2k}\sin(2kx)\Big|_{x=0}^{\pi}+\sum_{n\neq k}\frac{a_n}{2}\left[\frac{\sin(n-k)x}{n-k}-\frac{\sin(n+k)x}{n+k}\right]\Big|_0^{\pi}=\frac{\pi}{2}a_k$

$\Rightarrow a_k=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(kx)dx$. Fourier Series: $\mathcal{L}^2(a,b):=\left\{f:[a,b]\to\mathbb{R}=\int_a^b|f(x)|^2dx<\infty\right\}$, $\|f\|_2:=\sqrt{\int_a^b f(x)^2dx}$ (norm of $f\in\mathcal{L}^2(a,b)$, $\langle f,g\rangle$ Scalar Product

$\langle f,g\rangle:=\int_a^b f(x)g(x)dx$, $f,g\in\mathcal{L}^2(a,b) \Rightarrow f,g$ orthogonal if $\langle f,g\rangle=0$. Ex: $V_n(x)=\sin(nx)\in\mathcal{L}^2$, $\|V_n\|_2=\sqrt{\int_0^{\pi}\sin(nx)^2dx}=\sqrt{\int_0^{\pi}\frac{1}{2}\cos(0)-\frac{1}{2}\cos(2nx)dx}=\sqrt{\frac{\pi}{2}}$

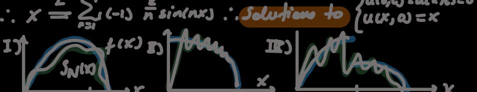
$\langle V_n,V_m\rangle=\int_0^{\pi}\sin(mx)\sin(nx)=0$ for $n\neq m \Rightarrow V_n\perp V_m$ $n\neq m\in\mathbb{Z}$; Let $W_n(x)=\cos(nx)\in\mathcal{L}^2$ $\therefore \|W_n\|_2=\sqrt{\frac{\pi}{2}}$ $n\neq 0$, $\|W_0\|_2=\sqrt{\pi}$, $W_n\perp W_m$ $n\neq m$

$V_n\perp W_m$, $n,m\in\mathbb{Z}$. Definition: Let $f,f_n\in\mathcal{L}^2(a,b)$, $n=1,2,\ldots$ we say $f\overset{L^2}{=}\sum f_n$ if $\|f-\sum_{n=1}^{N}\|_2\to 0$ as $N\to\infty$. Theorem: $\forall f\in\mathcal{L}^2(0,\pi)$, we have

(i) $f\overset{L^2}{=}\sum_{n=1}^{\infty}a_n\sin(nx)$, $a_n=\frac{\langle f,\sin(nx)\rangle}{\|\sin(nx)\|_2^2}$ (ii) $f\overset{L^2}{=}\sum_{n=0}^{\infty}a_n\cos(nx)$, $a_n=\frac{\langle f,\cos(nx)\rangle}{\|\cos(nx)\|_2^2}$ Remark: for $n\neq 0$: $\|\sin(nx)\|_2^2=\|\cos(nx)\|_2^2=\frac{\pi}{2} \Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)dx$ or

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos(nx)dx$. For $n=0$: $\|\cos(nx)\|_2^2=\int_0^{\pi}dx=\pi \Rightarrow a_0=\frac{1}{\pi}\int_0^{\pi}f(x)dx$. Ex: $f(x)=x$, $a_n=\frac{2}{\pi}\int_0^{\pi}x\sin(nx)dx=(-1)^{n+1}\frac{2}{n}$ $\therefore x\overset{L^2}{=}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)$ $\therefore$ Solution to $\begin{cases} u_t=u_{xx} \\ u(0,t)=u(\pi,t)=0 \\ u(x,0)=x \end{cases}$

$u(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}e^{-n^2t}\sin(nx)$. $f(x)\overset{L^2}{=}\sum a_n\sin(nx) \not\Rightarrow f(x)=\sum_{n=1}^{\infty}a_n\sin(nx)$ $\forall 0\le x\le\pi$. $x=\pi\neq\sum(-1)^{n+1}\frac{2}{n}\sin(n\pi)$



I): $\max_{0\le x\le\pi}|S_N(x)-f(x)|\to 0$, $N\to\infty$: Uniform Convergence to $f(x)$ on $[0,\pi]$, II): Fix $0<x\le\pi$, then $|S_N(x)-f(x)|\to 0$, $N\to\infty$: Pointwise convergence

to $f(x)$, III) For $0\le x\le\pi$, we have $S_N(x)\to g(x):=\begin{cases} f(x), & x\neq d \\ \frac{1}{2}f(d^-)+\frac{1}{2}f(d^+), & x=d \end{cases}$. Def: $f(x)$ on $[a,b]$ called piecewise continuous, if has at most finitely

many points of discontinuity, and $f(x)$ has left and right limits at each point. Theorem: $f(x)$ piecewise cont. on $[0,\pi]$. fix $0<x<\pi$

$\lim_{N\to\infty}S_N(x)\longrightarrow\frac{f(x^-)+f(x^+)}{2}$. Application: $x=\frac{\pi}{2}=\frac{2}{1}\sin(\frac{\pi}{2})-\frac{2}{2}\sin(\frac{2\pi}{2})+\frac{2}{3}\sin(\frac{3\pi}{2})-\ldots \Rightarrow \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ Parseval's Id: Assume $f(x)\overset{L^2}{=}\sum_{n=1}^{\infty}a_n\sin(nx)$

$\|f(x)\|_2^2=\langle f(x),f(x)\rangle=\langle\sum_{n=1}^{\infty}a_n\sin(nx),\sum_{m=1}^{\infty}a_m\sin(mx)\rangle=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle a_n\sin(nx),a_m\sin(mx)\rangle \overset{\text{By orthogon.}}{=\!=\!=} \sum_{n=1}^{\infty}\langle a_n\sin(nx),a_n\sin(nx)\rangle=\sum_{n=1}^{\infty}a_n^2\|\sin(nx)\|^2 \Longrightarrow \|f(x)\|_2^2=\frac{\pi}{2}\sum_{n=1}^{\infty}a_n^2$

If $f(x)=\sum_{n=0}^{\infty}a_n\cos(nx)$, $\|f\|_2^2=\pi a_0^2+\frac{\pi}{2}\sum_{n=1}^{\infty}a_n^2$. Heat Eq. $\begin{cases} u_t=u_{xx} \\ u(0,t)=u(\pi,t) \\ u(x,0)=f(x) \end{cases}$, $u(x,t)=\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)$, $a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)$, $f(x)$ bounded p.c.; wave Eq: $\begin{cases} u_{tt}=u_{xx} \\ u(0,t)=u(\pi,t)=0 \\ u(x,0)=f(x) \\ u_t(x,0)=g(x) \end{cases}$, f,g bounded p.c.

$u(x,t)=\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt))\sin(nx)$, $u_t(x,t)=\sum_{n=1}^{\infty}(-a_n\sin(nt)+nb_n\cos(nt))\sin(nx) \Rightarrow u(x,0)=\sum a_n\sin(nx)=f(x) \Rightarrow a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)dx$, $u_t(x,0)=\sum nb_n\sin(nx)=g(x)$

$\Rightarrow b_n=\frac{2}{n\pi}\int_0^{\pi}g(x)\sin(nx)dx$. Q: $\sum_{n=1}^{\infty}a_ne^{-n^2t}\sin(nx)<\infty$. $|a_n|=\frac{2}{\pi}\left|\int_0^{\pi}f(x)\sin(nx)dx\right|\le\frac{2}{\pi}\int_0^{\pi}|f(x)||\sin(nx)|dx\le 2\max_{0<x<\pi}|f(x)|:=M$. Ratio Test: $\exists\{b_n\}\geq 0$, $\exists 0<\delta<1$ s.t $\frac{b_{n+1}}{b_n}\le\delta$

pointwise c. $\forall x: \lim_{n\to\infty}f_n(x)=f(x)$. Uniform C. $\forall\varepsilon>0$ $\exists N: |f_n(x)-f(x)|<\varepsilon$ $\forall n\geq N$, $\forall x$.

$n \geq n_0 \Rightarrow \sum_{}^{\infty} b_n$ converges, and $\sum_{N}^{\infty} b_n \to 0,\ N \to \infty$ $\therefore |a_n e^{-n^2 t}\sin(nx)| \leq M e^{-n^2 t} := b_n$, $\frac{b_{n+1}}{b_n} = e^{-(2n+1)t} \longrightarrow 0,\ n \to \infty$ $\therefore \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin(nx)$. $\forall m \in \mathbb{N}\ \sum_{n=1}^{\infty} \frac{\partial^{(m)}}{\partial x^{m}}\left(a_n e^{-n^2 t}\sin(nx)\right) < \infty\ \forall t > 0$

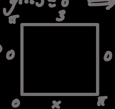
Heat Eq. (Insights): (i) Differentiability: Solutions can be differentiated arbitrarily many times in x or t $\forall t > 0$, even though $f(x)$ is p.c. (ii) Time Decay:
$|u(x,t)| \leq \sum_{n=1}^{\infty} |a_n| e^{-n^2 t} |\sin(nx)| \leq M e^{-t} \sum^{\infty} e^{-(n^2-1)t} \longrightarrow 0,\ t \to \infty$ (Solutions to H.E. with homogeneous Dirichlet B.C. will vanish. Wave Eq (Insights) $\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

$u(x,t) = \sum^{\infty} (a_n\cos(nt) + b_n\sin(nt))\sin(nx)$. (i) Differentiability: Assume f, g p.c. $\Rightarrow u(x,t)$ may not be differentiable. $u(x,t) = \sum_{1} (a_n\cos(nt) + b_n\sin(nt))\sin(nx)$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$, $b_n = \frac{2}{n\pi}\int_0^{\pi} g(x)\sin(nx)dx$ $\therefore \frac{\partial^2}{\partial x^2} u(x,t) = \sum^{\infty} (-n^2 a_n\cos(nt) - n^2 b_n\sin(nt))\sin(nx) = \frac{\partial^2}{\partial t^2} u(x,t)$ Converges if $\sum_{n=1}^{\infty} n^2(|a_n| + |b_n|)$ converges

$\therefore |a_n| + |b_n| \leq \frac{C}{n^4}$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi n^4}\int_0^{\pi} f'''(x)\sin(nx)$ Assuming $f(0) = f(\pi) = f''(0) = f''(\pi) = 0$ $\therefore |a_n| \leq \frac{2}{n^4}\max_{0<x<\pi} |f'''(x)| \leq \frac{M}{n^4}$. Similarly, assuming

$g(0) = g(\pi) = g''(0) = g(\pi)'' = 0 \Rightarrow |b_n| \leq \frac{2}{n^4}\max_{0<x<\pi} |g'''(x)| \leq \frac{\tilde{M}}{n^4}$. Laplace's Equation: $u(x,y)$. $\nabla u := u_{xx} + u_{yy}$, if $\nabla u = 0 \longrightarrow u(x,y)$ harmonic

$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\pi,y) = 0 \\ u(x,0) = x \\ u(x,\pi) = 3 \end{cases}$  Set $u(x,y) = X(x)Y(y) \Rightarrow X''(x)Y(y) + XY''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \lambda$, From B.C: $X(0) = X(\pi) = 0$ $\therefore$ start with $X(x)$ $\therefore \begin{cases} X''(x) = \lambda(x) \\ X(0) = X(\pi) = 0 \end{cases}$

$\Rightarrow X(x) = \sin(nx)$, $\lambda = -n^2$, $n = 1, 2, \ldots$; Y: $Y''(y) = -\lambda Y(y) = n^2 Y(y) \Rightarrow Y(y) = Ae^{ny} + Be^{-ny}$, $\forall A, B$; $\cosh(x) = \frac{e^x + e^{-x}}{2}$; $\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\therefore e^x = \cosh(x) + \sinh(x)$, $e^{-x} = \cosh(x) - \sinh(x)$ $\therefore Y(y) = A(\cosh(ny) + \sinh(ny)) + B(\cosh(ny) - \sinh(ny)) = (A+B)\cosh(ny) + (A-B)\sinh(ny) = \tilde{A}\cosh(ny) + \tilde{B}\sinh(ny)$

$\therefore$ (given $u(0,y) = u(\pi,y) = 0$) $u(x,y) = \sum^{\infty} (a_n\cosh(ny) + b_n\sinh(ny))\sin(nx)$. $\rightarrow u(x,0) = x = \sum_{n=1}^{\infty} a_n\sin(nx)$ $\therefore a_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx = (-1)^{n+1}\frac{2}{n}$, when $y = \pi$, $u(x,\pi) =$

$\sum_{n=1}^{\infty} (a_n\cosh(n\pi) + b_n\sinh(n\pi))\sin(nx)$. $\therefore \tilde{b}_n := a_n\cosh(n\pi) + b_n\sinh(n\pi) \Rightarrow \tilde{b}_n = \frac{2}{\pi}\int_0^{\pi} 3\sin(nx)dx = \frac{-6}{n\pi}[(-1)^n - 1] \Rightarrow b_n = \frac{\tilde{b}_n - a_n\cosh(n\pi)}{\sinh(n\pi)}$ $\begin{cases} u_{1xx} + u_{1yy} = 0 \\ u_1(0,y) = u_1(\pi,y) = 0 \\ u_1(x,0) = x \\ u_1(x,\pi) = 3x^2 \end{cases}$

$b_n = \frac{\frac{6}{n\pi}[(-1)^{n+1} + 1] - \frac{2}{n}(-1)^{n+1}\cosh(n\pi)}{\sinh(n\pi)}$. Inhomogeneous B.C. $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = \sin(y) \\ u(\pi,y) = y^2 \\ u(x,0) = x \\ u(x,\pi) = 3x^2 \end{cases}$. By Superposition: $u(x,y) = u_1(x,y) + u_2(x,y)$ s.t.

$\begin{cases} u_{2xx} + u_{2yy} = 0 \\ u_2(0,y) = \sin(y) \\ u_2(\pi,y) = y^2 \\ u_2(x,0) = u_2(x,\pi) = 0 \end{cases}$ Solve u_1, u_2 by separation of variables. $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} \tilde{x} = x\cos\theta - y\sin\theta \\ \tilde{y} = x\sin\theta + y\cos\theta \end{cases}$ $\therefore u_x = \left(\frac{\partial u}{\partial \tilde{x}}\right)\left(\frac{\partial \tilde{x}}{\partial x}\right) + \left(\frac{\partial u}{\partial \tilde{y}}\right)\left(\frac{\partial \tilde{y}}{\partial x}\right)$

$\therefore u_x = u_{\tilde{x}}\cos\theta + u_{\tilde{y}}\sin\theta$, $u_{xx} = \left(\frac{\partial u_{\tilde{x}}}{\partial \tilde{x}}\right)\left(\frac{\partial \tilde{x}}{\partial x}\right)\cos\theta + \left(\frac{\partial u_{\tilde{x}}}{\partial \tilde{y}}\right)\left(\frac{\partial \tilde{y}}{\partial x}\right)\cos\theta + \left(\frac{\partial u_{\tilde{y}}}{\partial \tilde{x}}\right)\left(\frac{\partial \tilde{x}}{\partial x}\right)\sin\theta + \left(\frac{\partial u_{\tilde{y}}}{\partial \tilde{y}}\right)\left(\frac{\partial \tilde{y}}{\partial x}\right)\sin\theta$

$\therefore u_{xx} = u_{\tilde{x}\tilde{x}}(\cos\theta)^2 + 2u_{\tilde{x}\tilde{y}}\sin\theta\cos\theta + u_{\tilde{y}\tilde{y}}(\sin\theta)^2$; $u_y = -u_{\tilde{x}}\sin\theta + u_{\tilde{y}}\cos\theta \Rightarrow u_{yy} = u_{\tilde{x}\tilde{x}}(\sin\theta)^2 - 2u_{\tilde{x}\tilde{y}}\sin\theta\cos\theta + u_{\tilde{y}\tilde{y}}(\cos\theta)^2$ $\therefore u_{xx} + u_{yy} = u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} \Rightarrow$

Ex 1: $\begin{cases} u_t - u_{xx} = 0 \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = 1 + 2\cos(3x) \end{cases} \Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda$ $\therefore X(x) = a_1\cos(wx) + a_2\sin(wx)$, $X'(x) = -a_1 w\sin(wx) + a_2 w\cos(wx)$, $X'(0) = a_2 w = 0$ $\therefore X'(x) = a_1 w\sin(wx)$

$X'(\pi) = a_1 w\sin(w\pi) = 0 \Rightarrow w = n \in \mathbb{Z}^+$ $\therefore X(x) = a_1\cos(nx)$, $\lambda = -n^2$ $\therefore T'(t) = -n^2 T(t) \Rightarrow T(t) = Ce^{-n^2 t}$ $\therefore u(x,y) = \sum_{n=0} a_n\cos(nx)e^{-n^2 t}$, $u(x,0) \Rightarrow \sum_{n=0}^{\infty} a_n\cos(nx) = f(x)$

$\sum_{n=0}^{\infty} a_n\cos(nx) = 1 + 2\cos(3x) \Rightarrow a_0 = 1,\ a_3 = 2,\ a_n = 0,\ n \neq 1,2$ $\therefore u(x,y) = 1 + 2\cos(3x)e^{-9t}$. Ex (2): $\begin{cases} tu_t - u_{xx} = 0 \\ u(0,t) = u(\pi,t) = 0 \end{cases} \Rightarrow tX(x)T'(t) = X''(x)T(t)$ $\therefore \frac{tT'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$

$\Rightarrow X(x) = C\sin(nx),\ n \in \mathbb{N}$, $\frac{T'(t)}{T(t)} = \frac{-n^2}{t}$ $\therefore \ln|T(t)| = -n^2\ln|t| + c = \ln(t^{-n^2}) + c$ $\therefore T(t) = Ct^{-n^2}$ $\therefore X(x)T(t) = Ct^{-n^2}\sin(nx) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n t^{-n^2}\sin(nx)$. Ex (3):

$\begin{cases} u_t - u_{xx} = 0 \\ u_x(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \Rightarrow X(x) = a_1\cos(wx) + a_2\sin(wx)$, $X'(x) = -a_1 w\sin(wx) + a_2 w\cos(wx)$, $X'(0) \Rightarrow a_2 = 0$ $\therefore X(\pi) = a_1\cos(w\pi) = 0 \Rightarrow w = \left(\frac{2k+1}{2}\right) k \in \mathbb{Z}^+$, $\lambda = -\left(\frac{2k+1}{2}\right)^2$ $\therefore T'(t) = -w^2 T(t)$

$\therefore T(t) = Ce^{-\left(\frac{2k+1}{2}\right)^2 t}$ $\therefore u(x,t) = \sum_{n=0}^{\infty} a_n\cos\left(\frac{2n+1}{2}\right)x)e^{-\left(\frac{2n+1}{2}\right)^2 t}$. Ex (4): Verify $\cos(nx) \perp \cos(mx)$, $x \in [0,\pi]$, $n \neq m \in \mathbb{N}$ $\therefore \int_0^{\pi}\cos(nx)\cos(mx)dx$, $\cos(nx)\cos(mx) =$

$\frac{1}{2}(\cos(n+m) + \cos(n-m))$ $\therefore \frac{1}{2}\int_0^{\pi}\cos((n+m)x) + \cos((n-m)x) = \frac{1}{2}\left(\frac{-\sin((n+m)x)}{(n+m)} - \frac{\sin((n-m)x)}{n-m}\right)\Big|_0^{\pi} = 0$; $\sin(nx) \perp \cos(mx)$, $x \in [-\pi,\pi]$ $n, m \in \mathbb{N}$ $\therefore \int_{-\pi}^{\pi}\sin(nx)\cos(mx)dx$

let $f(x) = \sin(nx)\cos(mx) \Rightarrow f(-x) = \sin(-nx)\cos(-mx) = -\sin(nx)\cdot\cos(mx) = -f(x) \Rightarrow \int_{-a}^{a} f(x) = 0$. Ex (5): Fourier Sine Series of $f(x) = \begin{cases} -1 & 0 < x < \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(x)dx = \frac{2}{\pi}\int_0^{\pi/2} -\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\sin(nx)dx = \frac{2}{\pi n}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \frac{2}{\pi n}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) = \frac{4}{\pi n}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{\pi n}(1 + \cos(n\pi))$, for $n = 4k \Rightarrow a_{4k} = 0$, $n = 4k+1 \Rightarrow a_{4k+1} = 0$

$\ldots$, $n = 4k+3 \Rightarrow a_{4k+3} = 0$ $\therefore a_n\sin\left(\frac{n\pi}{2}\right) = 0\ \forall n \in \mathbb{N}$. Ex (6): $\begin{cases} u_t - u_{xx} = 0 \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases} \Rightarrow X(x) = C\cos(nx)$, $T(t) = C'e^{-n^2 t}$, $n \in \mathbb{Z}$, $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2 t}\cos(nx)$, $u(x,0) \Rightarrow$

$\sum_{n=0}^{\infty} a_n\cos(nx) = x^2$ $\therefore a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$, $a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx = \frac{4}{n^2}\cos(n\pi) = \frac{4}{n^2}(-1)^n$, $n \geq 1$ $\therefore u(x,t) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}e^{-n^2 t}\cos(nx)$ Ex (7): $\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\pi) = u(\pi,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = \sin(2x) \end{cases}$

$X(x) = C\sin(nx),\ n \in \mathbb{N}$, $\therefore T''(t) = -n^2 T(t) \Rightarrow T(t) = A\cos(nt) + B\sin(nt)$ $\therefore \sum_{n=1}^{\infty} a_n\cos(nt)\sin(nx) + b_n\sin(nt)\sin(nx)$, $u(x,0) \Rightarrow$

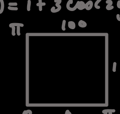
$\sum_{n=1}^{\infty} a_n\sin(nx) = x^2$ $\therefore a_n = \frac{2}{\pi}\int_0^{\pi} x^2\sin(x) = (-1)^n\left(\frac{4}{\pi n^3} - \frac{2\pi}{n}\right) - \frac{4}{\pi n^3}$, $u_t(x,t) = \sum_{n=1}^{\infty} -na_n\sin(nt)\sin(nx) + nb_n\cos(nt)\sin(nx)$ $\therefore u_t(x,0) = \sum_{n=1}^{\infty} nb_n\sin(nx)$

$\therefore \sum_{n=1}^{\infty} nb_n\sin(nx) = \sin(2x) \Rightarrow nb_n = 1$ for $n = 2$ $\therefore b_n = \frac{1}{2}$, $b_n = 0$ $n \neq 2$ $\therefore u(x,t) = \frac{1}{2}\sin(2t)\sin(2x) + \sum_{n=1}^{\infty}\left[(-1)^n\left(\frac{4}{\pi n^3} - \frac{2\pi}{n}\right) - \frac{4}{\pi n^3}\right]\cos(nt)\sin(nx)$

Ex (8): $\begin{cases} u_{xx} + u_{yy} = 0 \\ u_y(x,0) = 0 \\ u_y(x,\pi) = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 1 + 3\cos(2y) \end{cases} \Rightarrow \frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = \lambda$ $\therefore Y(y) = C\cos(nx)$ $n \in \mathbb{Z}^+$, $\therefore X''(x) = n^2 X(x) \Rightarrow$ for $n = 0 \Rightarrow X(x) = a_0 + b_0 x$
for $n = 1, 2, 3, \ldots$, $X(x) = a_n e^{nx} + b_n e^{-nx}$

$X(x) = A(\cosh(nx) + \sinh(nx)) + B(\cosh(nx) - \sinh(nx)) = A\cosh(nx) + B\sinh(nx)$ $\therefore u(x,y) = a_0 + b_0 x + \sum_{n=}^{\infty}((a_n\cosh(nx) + b_n\sinh(nx))\cos(ny)$

$u(0,y) = a_0 + \sum_{n=1}^{\infty} a_n\cos(ny) = 0$ $\therefore a_n = 0\ \forall n$; $u(\pi,x) = b_0\pi + \sum_{n=1}^{\infty} b_n\sinh(n\pi)\cos(ny) = 1 + 3\cos(2y)$ $\therefore b_0 = \frac{1}{\pi}$, $b_2\sinh(2\pi) = 3$ $\therefore b_2 = \frac{3}{\sinh(2\pi)}$, $b_n = 0\ \forall n \neq 2$

$u(x,y) = \frac{x}{\pi} + \frac{3}{\sinh(2\pi)}\sinh(2x)\cos(2y)$. Ex (9): $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = 0 \\ u(x,\pi) = 100 \\ u(0,y) = 0 \\ u(\pi,y) = 100 \end{cases}$  Set $u(x,y) = u_1(x,y) + u_2(x,y)$ s.t $\begin{cases} u_{1xx} + u_{2yy} = 0 \\ u_1(x,0) = 0 \\ u_1(x,\pi) = 100 \\ u_1(0,y) = u_1(\pi,y) = 0 \end{cases}$

$\begin{cases} u_{2xx} + u_{yy} = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 100 \\ u(x,0) = u(x,\pi) = 0 \end{cases}$, u_1: $X(x) = C\sin(nx)$, $n \in \mathbb{N} \Rightarrow$ $Y(y) = A\cosh(ny) + B\sinh(ny)$, $n \in \mathbb{N}$

$\therefore u(x,y) = \sum_{n=1}^{\infty}(a_n\cosh(ny) + b_n\sinh(ny))\sin(nx)$, $u(x,0) = \sum_{n=1}^{\infty} a_n\sin(nx) = 0 \Rightarrow a_n = 0$ $\therefore u(x,\pi) = \sum_{n=1}^{\infty} b_n\sinh(n\pi)\sin(nx) = 100$

$\therefore b_n = \frac{200}{\pi\sinh(n\pi)}\int_0^{\pi}\sin(nx)dx = \frac{-200}{n\pi\sinh(n\pi)}((-1)^n - 1)$; u_2: $Y(y) = C\sin(ny)$, $n \in \mathbb{N}$, $X(x) = A\cosh(nx) + B\sinh(nx)$, $u_2(x,y) = \sum_{n=1}^{\infty}(a_n\cosh(nx) + b_n\sinh(nx))\sin(ny)$

$u_2(0,y) = \sum_{n=1}^{\infty} a_n\sin(ny) = 0$ $\therefore a_n = 0$, $u_2(\pi,y) = \sum_{n=1}^{\infty} b_n\sinh(n\pi)\sin(ny) = 100 \Rightarrow b_n = \frac{-200\cdot((-1)^n - 1)}{n\pi\sinh(n\pi)}$ $\therefore u(x,y) = u_1(x,y) + u_2(x,y)$

$u(x,y) = \sum b_n\sinh(nx)\sin(ny) + b_n\sinh(ny)\sin(nx) = \sum_{n=1}^{\infty}\left(\frac{-200((-1)^n - 1)}{n\pi\sinh(n\pi)}\right)[\sinh(ny)\sin(nx) + \sinh(nx)\sin(ny)]$ (10) $\begin{cases} u_t = 4u_{xx} - 3u_x - 5UV^2 \\ V_t = -3UV^2 \end{cases}$

$\begin{cases} u(x,t) = U(x-ct) = U(z) \\ v(x,t) = V(x-ct) = V(z) \end{cases} \Rightarrow u_t = \frac{dU}{dz}\cdot\frac{dz}{dt} = -cU'(z)$, $v_t = \frac{dV}{dz}\cdot\frac{dz}{dt} = -cV'(z)$, $u_x = \frac{dU}{dz}\cdot\frac{dz}{dx} = U'(z)$, $u_{xx} = \frac{dU'}{dz}\frac{dz}{dx} = U''(x)$ $\therefore \begin{cases} -cU'(z) = 4U''(z) - 3U'(z) - 5UV^2 \\ -cV'(z) = -3UV^2 \end{cases}$