Probability Axioms: (i) $\mathbb{P}(\Omega)=1 \wedge \mathbb{P}(\emptyset)=0$ (ii) $0 \leq \mathbb{P}(A) \leq 1 \ \forall A \subset \Omega$ (iii) $\forall \{A_n\}_1^\infty, A_i \cap A_j = \emptyset \ \forall i \neq j \Longrightarrow \mathbb{P}(\bigcup_{n=1}^{n} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

Lemma: (i) $\mathbb{P}(A)+\mathbb{P}(A^c)=1$; (ii) $\mathbb{P}(A_1 \cup \ldots \cup A_n) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i A_j) + \sum_{i<j<k} \mathbb{P}(A_i A_j A_k) + \ldots + (-1)^{n+1} \mathbb{P}(A_1 \cdots A_n)$ (iii) $\forall \{A_1 \subseteq A_2 \subseteq \ldots\}$ $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \uparrow \mathbb{P}(A_n)$ (iv) $\forall \{A_1 \supseteq A_2 \supseteq \ldots\}$ $\mathbb{P}(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \downarrow \mathbb{P}(A_n)$. **Independence:** $A_1, \ldots, A_n$ ind. if $\mathbb{P}(A_{k_1} A_{k_2} \cdots A_{k_m}) = \prod_i^m \mathbb{P}(A_{k_i})$

Conditional Probability: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \Longrightarrow \mathbb{P}(\bigsqcup_1^\infty A_n | B) = \sum_1^\infty \mathbb{P}(A_n|B)$, $A \perp B \Longleftrightarrow \mathbb{P}(A|B)=\mathbb{P}(A)$ **Theorem:** Suppose $\{B_n\}$ (partition) i.e. $\bigsqcup_n B_n = \Omega \therefore \forall A \subset \Omega$: (i) **Law of Total Probability:** $\mathbb{P}(A) = \sum \mathbb{P}(A|B_n)\mathbb{P}(B_n)$, **Bayes:** $\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_n \mathbb{P}(A|B_n)\mathbb{P}(B_n)}$

Random variable $X: \Omega \to \mathbb{R}$ $F(x) \triangleq \mathbb{P}(X \leq x)$. X **discrete** if $X \in \{x_1, x_2, \ldots\}$ $p(x_i) \triangleq \mathbb{P}(X=x_i)$ (**pmf**), $\mathbb{E}[X] = \sum_i x_i p(x_i)$, $Var[X] = \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ $\forall$ function h $\mathbb{E}[h(X)] = \sum_i h(x_i) p(x_i)$, $Var[aX+b] = a^2 Var[X]$.

Bernoulli R.V.: $\begin{cases} \mathbb{P}(X=1)=p \\ \mathbb{P}(X=0)=1-p \end{cases}$, $\mathbb{E}[X]=p$, $Var[X]=p(1-p)$. **Binomial R.V.:** $X \in \{0,1,\ldots,n\}$ $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $\mathbb{E}[X]=np$, $Var[X]=np(1-p)$; **Geometric R.V.:** $\mathbb{P}(X=k) = p(1-p)^{k-1}$, $\mathbb{E}(X) = \frac{1}{p}$, $Var[X] = \frac{1-p}{p^2}$ **Poisson R.V.** $X \in \{0,1,2,\ldots\}$ $\mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $\mathbb{E}[X]=\lambda$, $Var[X]=\lambda$. Indicator R.V. $\mathbb{1}_A(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases}$ (Bernoulli) $\mathbb{P}(\mathbb{1}_A=1)=\mathbb{P}(A)$, $\mathbb{P}(\mathbb{1}_A=0)=\mathbb{P}(A^c)$

Continuous R.V. X continuous if $\exists f(x) \geq 0$ (density) s.t. $\mathbb{P}(X \in B) \triangleq \int_B f(x)dx \ \forall B \subset \mathbb{R} \therefore \int_{\mathbb{R}} f(x)dx = 1$ s.t. $f(x) = F'(x)$, $F(x) = \int_{-\infty}^{x} f(t)dt$, $\mathbb{E}[X] \triangleq \int_{\mathbb{R}} x f(x) dx$, $\forall$ function h $\mathbb{E}[h(x)] = \int_{\mathbb{R}} h(x) f(x) dx$. **Uniform R.V.** $X \sim Uniform(a,b)$: $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0, & \ldots \end{cases}$, $\mathbb{E}[X] = \frac{b+a}{2}$, $Var[X] = \frac{(b-a)^2}{12}$. **Exponential R.V.** $f(x) = \begin{cases} \lambda e^{-\lambda}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ $\mathbb{E}[X] = 1/\lambda$, $Var[X] = 1/\lambda^2$ (memoryless) $\mathbb{P}(X \geq t+s | X \geq s) = \mathbb{P}(X \geq t)$. **Normal R.V.:** $X \sim N(\mu, \sigma^2)$ $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $\mathbb{E}[X]=\mu$, $Var[X]=\sigma^2$, $Z \triangleq \frac{X-\mu}{\sigma} \sim N(0,1)$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$; $X \in \{0,1,2,\ldots\}$: $\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$.

Random Vector: Collection of random variables on Ω (i) **Discrete:** **joint pmf** $p(x_i, y_i) \triangleq \mathbb{P}(X=x_i, Y=y_i)$ **marginal pmf:** $p_X(x_i) = \sum_j p(x_i, y_j)$, $p_Y(y_j) = \sum_i p(x_i, y_j)$. $\forall h: \mathbb{R}^2 \to \mathbb{R}$ $\mathbb{E}[h(X,Y)] \triangleq \sum_{i,j} h(x_i, y_j) p(x_i, y_j)$ (ii) **Continuous:** X and Y are said to be **jointly continuous** if $\exists$ nonnegative joint density function, say $f(x,y)$ s.t. $\mathbb{P}(X \in A, Y \in B) = \iint_{A \times B} f(x,y) dxdy$ $\forall A, B \subset \mathbb{R}$. The densities for X, Y equal the **marginal density functions** $f_X(x) = \int_{\mathbb{R}} f(x,y) dy$, $f_Y(y) = \int_{\mathbb{R}} f(x,y) dx$, $\forall h: \mathbb{R}^2 \to \mathbb{R}$: $\mathbb{E}[h(X,Y)] \triangleq \iint_{\mathbb{R}^2} h(x,y) f(x,y) dxdy$ **Theorem:** $\forall X, Y$ $\mathbb{E}[ax+by] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.

Covariance: $Cov(X,Y) \triangleq \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \therefore Cov(X,X) = Var(X)$, $Cov(X,Y) = Cov(Y,X)$, $Cov(a,X)=0 \ \forall a \in \mathbb{R}$, **Bilinearity** (i) $Cov(aX+bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$ (ii) $Cov(X, aY+bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$

Lemma: $\forall$ random variables $X_1, \ldots, X_n$: $Var(\sum^n X_i) = \sum_{i=1}^{n} Var X_i + \sum_{i \neq j} Cov(X_i, X_j) = \sum_{i=1}^{n} Var X_i + 2\sum_{i<j} Cov(X_i, X_j)$. **Independence:** A collection of random variables $X_1, X_2, \cdots, X_n$ are independent if $\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \cdots \mathbb{P}(X_n \in A_n)$ $\forall A_i \subseteq \mathbb{R}$. **Lemma:** Assume $X \perp Y$. Then (i) $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ $Cov(X,Y)=0$, $Var(X+Y) = Var(X) + Var(Y)$. Theorem: X, Y r.v. with densities $f(x), g(y)$ $\therefore$ the density of $X+Y$: $\mathbb{P}(X+Y \leq t) = \int_{-\infty}^{t} f_{X+Y}(s) ds$ where $f_{X+Y}(s) = \int_{-\infty}^{\infty} f(s-y) g(y) dy$ (**convolution**).

Conditional Distribution: $\mathbb{P}(X \leq x | Y=y) = \frac{\mathbb{P}(X \leq x, Y=y)}{\mathbb{P}(Y=y)}$ **Theorem:** X, Y continuous r.v. joint density $f(x,y)$ The conditional density of X given $Y=y$ is $f(x,y) = \frac{f(x,y)}{f_Y(y)} \therefore \forall A \subset \mathbb{R}$ $\mathbb{P}(X \in A | Y=y) = \int_A f(x|y) dx$. **Lemma:** (**LTP**) $\forall$ r.v. X, Y, $A \subseteq \mathbb{R}$ (i) Y **discrete**: $\mathbb{P}(X \in A) = \sum_j \mathbb{P}(X \in A | Y=y_j) \mathbb{P}(Y=y_j)$ (ii) Y **continuous**: $\mathbb{P}(X \in A) = \int_{\mathbb{R}} \mathbb{P}(X \in A | Y=y) f_Y(y) dy$

Conditional Expectation: X, Y r.v. $\Longrightarrow \mathbb{E}[X|Y]$ r.v. **Theorem:** $\forall X, Y$ r.v. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$

Moment Generating Function:

Random Walk: $\{Y_1, Y_2, \dots\}$ iid on $(\Omega, \mathbb{P})$ s.t. $\mathbb{P}(Y_i=1)=p$, $\mathbb{P}(Y_i=-1)=1-p$ $\therefore \{S_n : n \geq 0 \mid S_n \triangleq x + \sum_{i=1}^{n} Y_i,\ S_0 \triangleq x \in \mathbb{Z}\}$. $\therefore S=\{S_n\}$ simple random walk. Gambler's Ruin: $T \triangleq \inf\{n \geq 0 : S_n = 0\}$, $W \triangleq \inf\{n \geq 0 : S_n = b\}$. Ruin Probability: $\forall 0 \leq x \leq b$ $h(x) \triangleq \mathbb{P}^x(T<W)$, $g(x) \triangleq \mathbb{P}^x(W<T)$

First Step Analysis: $g(x) = \mathbb{P}^x(W<T \mid Y_1=1)\mathbb{P}^x(Y_1=1) + \mathbb{P}(W<T \mid Y_1=-1)\mathbb{P}^x(Y_1=-1) = g(x+1)\cdot p + g(x-1)(1-p) \Longrightarrow$ Characteristic Sol: $s = ps^2+q$ roots $\begin{cases} S_1 = 1 \\ S_2 = q/p \end{cases}$, (i) $p \neq q$: $g(x) = A(s_1)^x + B(s_2)^x = A + B(q/p)^x$, $\begin{cases} g(0)=0 \\ g(1)=1 \end{cases} \Longrightarrow g(x) = \frac{1-(q/p)^x}{1-(q/p)^b}$ (ii) $p=q$: $g(x) = AS_1^x + BxS_1^x = A+Bx$, $g(x)=\frac{x}{b}$

(i) $p \neq q$: $h(x) = \frac{(q/p)^x - (q/p)^b}{1-(q/b)^b}$ (ii) $p=q$: $h(x) = \frac{b-x}{b}$ $\therefore g(x)+h(x)=1$. Duration: $D = \min(T,W) < +\infty$ $d(x) \triangleq \mathbb{E}^x[D]$, $d(x) = p\,d(x+1) + q\,d(x-1)+1$

(i) $p \neq q$: $d(x) = \frac{x}{q-p} - \frac{b}{q-p} \cdot \frac{1-(q/p)^x}{1-(q/b)^b}$ (ii) $p=q$: $d(x) = x(b-x)$. Recurrence & Transience: $(S_0 = x \geq 0)$ $\therefore \mathbb{P}^x[T<\infty] = \lim_{b \uparrow \infty} \mathbb{P}^x[T<W] = \lim_{b \uparrow \infty} h(x) = \begin{cases} 1 & p \leq 1/2 \\ \left(\frac{q}{p}\right)^x & p > 1/2 \end{cases}$

$R \triangleq \inf\{n \geq 1 : S_n = 0\}$, $\mathbb{P}^0[R<\infty] = 1-|p-q|$. Def: A simple random walk recurrent if returns to S_0 with $P=1$, Otherwise Transient.

Wald's Equation: $\mathbb{E}^x[S_D] = \begin{cases} x & p=q \\ b\,g(x) & p \neq q \end{cases}$. Def: (Stopping Time) $\{X_1, X_2, \dots\}$ r.v.. A r.v. $N \in \{0,1,2,\dots\} \cup \{\infty\}$ is a stopping time If the event $\{N=n\}$ is determined by $X_1, X_2, \dots, X_n$ $\therefore \mathbb{1}_{\{N=n\}} = F_n(X_1, \dots, X_n)$ for some F_n Theorem: $\{X_1, X_2, \dots\}$ iid r.v. $\mathbb{E}[X_i] < +\infty$, N stopping time $\mathbb{E}[N] < \infty \Longrightarrow \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[X]\mathbb{E}[N]$. Combinatorial Problems: (Simple random walks with $S_0 = 0$) Lemma: $\forall n \geq 1$, $S_n \in \{n-2k : k = 0,1,\dots,n\}$ and $\mathbb{P}(S_n = n-2k) = \binom{n}{k} p^{n-k} q^k$

Markov Chains: Def: A matrix $\mathcal{P} = [P_{ij}]_{i,j \in \mathcal{S}}$ is a stochastic Matrix on state space $\mathcal{S}$ if $P_{ij} \geq 0$, $\sum_{j \in \mathcal{S}} P_{ij} = 1$ $\forall i \in \mathcal{S}$. Def: A stochastic process $X = \{X_0, X_1, \dots\}$ is a Markov Chain with transition probability matrix $\mathcal{P}$ if $\mathbb{P}(X_{n+1}=j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i) = P_{ij}$ $\forall n$ $\forall i_0, \dots, i_{n-1}, i \in \mathcal{S} \Longrightarrow \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_0 = i_0)\mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdots \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, \cdots, X_1 = i_1, X_0 = 0) = \mathbb{P}(X_0 = i_0)\mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdots \mathbb{P}(X_n = i_n \mid X_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$ Lemma: $(X, \mathcal{P} = [P_{ij}])$-MC $\therefore \forall n \geq 1, m \geq 0$: $\mathbb{P}(X_{m+n} = j_n, \dots, X_{m+1} = j_1 \mid X_m = i, X_{m-1} = i_{m-1}, \cdots, X_0 = i_0) = P_{ij_1} P_{j_1 j_2} \cdots P_{j_{n-1} j_n}$

Def: A state $i \in \mathcal{S}$ absorbing if $P_{ii} = 1$. Theorem: $\forall n \geq 1, m \geq 0$: $\mathbb{P}(X_{m+n} = j \mid X_m = i, X_{m-1} = i_{m-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{m+n} = j \mid X_m = i) = [\mathcal{P}^n]_{ij}$ $P_{ij}^{(n)} \triangleq [\mathcal{P}^n]_{ij}$: Chapman-Kolgomorov Equation: $P_{ij}^{(n+m)} = \sum_{k \in \mathcal{S}} P_{ik}^{(n)} P_{kj}^{(m)}$. Lemma: $(\mathcal{S}, \mathcal{P})$-MC and initial distribution $\Lambda = [\Lambda_i]_{i \in \mathcal{S}}$ s.t. $\mathbb{P}(X_0 = i) = \Lambda_i$ $\therefore \forall n \geq 0, i \in \mathcal{S}$ $\mathbb{P}(X_n = i) = [\Lambda \cdot \mathcal{P}^n]_i$. General Hitting Times: $(\mathcal{S}, \mathcal{P})$-MC, $A \subset \mathcal{S}$ $\therefore T^A \triangleq \inf\{n \geq 0 : X_n \in A\}$, $\inf\{\emptyset\} \triangleq \infty$, $h^A(x) \triangleq \mathbb{P}^x(T^A < \infty)$, $d^A(x) \triangleq \mathbb{E}^x[T^A]$

Prop: The vector $\{h^A(x) : x \in \mathcal{S}\}$ is the minimal nonnegative solution of the system of linear equations: $h(x) = \sum_{y \in \mathcal{S}} P_{xy} h(y)$, $x \notin A$ and $h(x) = 1$ $\forall x \in A$. Similarly, $\{d^A(x) : x \in \mathcal{S}\}$ is the minimal nonnegative solution to the system $d(x) = 1 + \sum_{y \in \mathcal{S}} P_{xy} d(y)$, $x \notin A$, $d(x) = 0$ $\forall x \in A$. Strong Markov Property: $\{X\}$ Markov Chain and any fixed n $\therefore$ given $X_n = i$, $X' = \{X_n, X_{n+1}, \dots\}$ is again Markov C. with $\mathcal{P}$ Theorem: Suppose $X = \{X_n\}$, $\mathcal{P}$, T: Arbitrary stopping Time. Conditioned on $\{T<\infty, X_T = i\}$, The process $\{X_T, X_{T+1}, X_{T+2}, \dots\}$-MC with $\mathcal{P} \perp \{X_0, \dots, X_T\}$ Classification of States: Irreducible MC. $X = \{X_n\}$. A state (j) is accesible from state (i): $i \longrightarrow j$ if $\mathbb{P}^i(X_n = j$, for some $n \geq 0) = \mathbb{P}\left(\bigcup_{n=0}^{\infty} \{X_n = j\} \mid X_0 = i\right) > 0$; i, j communicate: $i \longleftrightarrow j \equiv i \underset{\longleftarrow}{\overset{\longrightarrow}{\wedge}} j$. If all states communicate X irreducible

Lemma: For $i \neq j \in \mathcal{S}$, TFAE: (1) $i \longrightarrow j$ (2) $P_{ij}^{(n)} > 0$ for some $n \geq 0$ (3) $P_{ii_1} P_{i_1 i_2} \cdots P_{i_{n-1} j} > 0$ for some n and $i_1, \dots, i_{n-1} \in \mathcal{S}$. Lemma: Communication relation is an equivalence relation i.e. (1) (Reflexivity) $i \longleftrightarrow i$ (2) (Symmetry) $i \longleftrightarrow j \Longrightarrow j \longleftrightarrow i$ (3) (Transitivity) $i \longleftrightarrow j$, $j \longleftrightarrow k \Longrightarrow i \longleftrightarrow k$. $\therefore X$ irreducible has 1 comm. class

Periodicity: Period of state i: $d(i) \triangleq \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\}$. (i) aperiodic if $d(i) = 1$ Prop: Periodicity is a class property i.e. if $i \longleftrightarrow j \Longrightarrow d(i) = d(j)$ Rmk: An irreducible MC aperiodic if some (whence all states) have period 1. Lemma: Suppose state i has period $d(i)$ $\therefore \exists N$ s.t. $\forall n \geq N$ $P_{ii}^{(nd(i))} > 0$ Recurrence $\wedge$ Transience: $X = \{X_n\}$, $\mathcal{P}$: $\forall i \in \mathcal{S}$ define $f_i \triangleq \mathbb{P}^i(R_i < \infty) = \mathbb{P}(R_i < \infty \mid X_0 = i)$, $R_i \triangleq \inf\{n \geq 1 : X_n = i\}$, $f_i = 1$ (i is recurrent), $f_i < 1$ (i is transient) Theorem: A state i is recurrent $\Longleftrightarrow \sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$ (i) transient $\Longleftrightarrow \sum_{n=0}^{\infty} P_{ii}^{(n)} < \infty$. Moreover recurrence and transcience are class properties. Lemma: Every irreducible MC s.t. $|\mathcal{S}| < \infty$ is recurrent. Lemma: Suppose (i) recurrent $\wedge$ $i \longrightarrow j \Longrightarrow i \longleftrightarrow j \wedge \mathbb{P}^i(T_j < \infty) = 1$, $T_j \triangleq \inf\{n \geq 0 : X_n = j\}$. Stirling Formula: $n! = \sqrt{2\pi n} \cdot n^n e^{-n+\varepsilon_n}$, $\frac{1}{12n+1} < \varepsilon_n < \frac{1}{12n}$. For n large $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$. Lemma: A Random Walk on $\mathbb{Z}$ recurrent $\Longleftrightarrow p = q$. Stationary Distribution: $R_i \triangleq \inf\{n \geq 1 : X_n = i\}$ A recurrent state i is positive recurrent if $\mathbb{E}^i[R_i] < \infty$, null recurrent otherwise. Def: A stationary distribution of a MC with $\mathcal{P}$ is a (row) vector $\pi = \{\pi_i\}_{i \in \mathcal{S}}$ s.t. $\pi_i \geq 0$, $\sum_{i \in \mathcal{S}} \pi_i = 1$, $\pi \mathcal{P} = \pi$ Theorem: $X = \{X_n\}$-MC, $\mathcal{P}$, TFAE: (1) All states are positive recurrent (2) Some state is positive recurrent (3) $\exists$ stationary distribution π, i.e. system of equations $\pi_i \geq 0$, $\sum_{i \in \mathcal{S}} \pi_i = 1$, $\pi \mathcal{P} = \pi$ admits a solution. Furthermore, If X positive recurrent then $\exists! \pi$ s.t. $\pi_i = \frac{1}{\mathbb{E}^i[R_i]}$ Corollary: Positive recurrence and null recurrence are class properties Theorem: $X = \{X_n\}$ irreducible MC, $\mathcal{P}$ and $|\mathcal{S}| = \infty$ $\therefore$ TFAE: (1) MC transient (2) (For some) $\forall$ state, say i^*, the system of equations $\sum_{j \in \mathcal{S}} P_{ij} x_j = x_i$ $\forall i \neq i^*$ admits a bounded non-constant solution Convergence of Markov Chains: Convergence for positive r. MC: Theorem: $X = \{X_n\}$ irreducible-aperiodic-positive recurrent with $(\mathcal{P}, \Lambda, \pi) \Longrightarrow \lim_n \mathbb{P}(X_n = j) = \pi_j$ $\forall j$, In particular, $\lim_n P_{ij}^{(n)} = \pi_j$ $\forall i, j$, Lemma: $\{X_n\}$, $\{Y_n\}$ be two independent irreducible-aperiodic-recurrent MC $\therefore Z_n \triangleq \{X_n, Y_n\}$ irreducible aperiodic MC. If X, Y positive recurrent so is Z. Theorem: $X = \{X_n\}$ irreducible-aperiodic positive recurrent MC. with $\mathcal{S}, \pi$. $\therefore \forall$ function $h : \mathcal{S} \longrightarrow \mathbb{R}$ $\lim_n \frac{1}{n} \sum_{k=1}^{n} h(X_k) = \sum_{i \in \mathcal{S}} h(i)\pi_i$, Rmk: If $h(x) \triangleq \mathbb{1}_{\{x=i\}}$ $\therefore \lim_n \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{X_k = i\}} = \pi_i$ $\therefore$ In the long run π_i: proportion of time $|_{(i)}$. Convergence for null recurrent and transient chains: Theorem: $X = \{X_n\}$ irreducible-aperiodic either transient or null recurrent. Then $\lim_n \mathbb{P}^i[X_n = j] = \lim_n P_{ij}^{(n)} = 0$ $\forall i, j$. Corollary: An irreducible MC, $|\mathcal{S}| < \infty \Longrightarrow$ MC positive recurrent Lemma: (Fatou's Lemma): Suppose $a_i^{(n)} \geq 0$ $\forall i, \forall n$ Then $\liminf_n \sum_{i=1}^{\infty} a_i^{(n)} \geq \sum_{i=1}^{\infty} \liminf_n a_i^{(n)}$

Martingales: A stochastic Process $X=\{X_n\}$ Martingale w.r.t. $Z=\{Z_n\}$ if (i) (Adaptedness) $\forall n$, X_n is a function of $\{Z_0,\dots,Z_n\}$, (ii) $\mathbb{E}[X_{n+1}|Z_n,\dots,Z_0]=X_n$, $\mathcal{F}_n \triangleq \sigma\{Z_0,\dots,Z_n\}$. **Prop:** $\forall$ random v. X, $\forall$ random vectors Y,Z, $\mathbb{E}[\mathbb{E}[X|Y,Z]|Z]=\mathbb{E}[X|Z]$
Lemma: Let $X=\{X_n\}$ Martingale w.r.t. $Z=\{Z_n\}$ $\therefore \forall n, m\geq 0$ $\mathbb{E}[X_{n+m}|Z_n,\dots,Z_0]=\mathbb{E}[X_{n+m}|X_n,\dots,X_0]=X_n$, and in particular $\mathbb{E}[X_n]=\mathbb{E}[X_0]$ $\forall n\in\mathbb{N}$. **Wald's Martingale:** $\{Z_n\}\overset{iid}{\sim}$ with ϕ: MGF $\therefore \forall\theta$, $M_n \triangleq \frac{1}{\phi^n(\theta)}e^{\theta S_n}$; $S_n\triangleq\sum_1^n Z_i$ Martingale w.r.t. Z
Doob's Martingale: $Z=\{Z_n\}$, X any r.v. $\therefore Y_n \triangleq \mathbb{E}[X|Z_n,\cdots,Z_0]=\mathbb{E}[X|\mathcal{F}_n]$. $Y=\{Y_n\}$ Martingale w.r.t. Z. **Ex:** Let $Z=\{Z_n\}$ Markov chain, $P=[P_{ij}]$, S $\therefore h: S\to\mathbb{R}$ harmonic if $\forall i\in S$ $h(i)=\sum_{j\in S}P_{ij}h(j)$. If h harmonic $\Longrightarrow$ $X_n\triangleq h(Z_n)$ martingale w.r.t. Z. **Ex:** Likelihood Ratio $Z=\{Z_n\}\overset{iid}{\sim}$ with common density $f(z)$, let $h(z)$ any $\mathbb{P}$ density $\therefore X_n \triangleq \frac{\prod_0^n h(Z_i)}{\prod_0^n f(Z_i)} \Rightarrow \{X_n\}$ martingale w.r.t. $\{Z_n\}$. **Optional Sampling Thm:** $X=\{X_n\}$ marting. w.r.t. $Z=\{Z_n\}$. A r.v. $\sigma:\Omega\to\{0,1,\dots\}\cup\{\infty\}$ stopping time if $\forall n\geq 0$ the event $\{\sigma=n\}$ is determined by $\{Z_0,\dots,Z_n\}$ i.e. $\mathbb{1}_{\{\sigma=n\}}=F_n(Z_0,\dots,Z_n)$ for some F_n
Ex: $S=\{S_n\}$ symmetric simple random walk $|_0 \Longrightarrow S$ martingale. Let $\sigma\triangleq\inf_n\{n\geq 0: S_n=1\}$ Since $\mathbb{P}(\sigma<\infty)=1 \Longrightarrow \mathbb{E}[S_\sigma]=1\neq\mathbb{E}[S_0]=0$. Let $\sigma\wedge\tau\triangleq\min\{\sigma,\tau\}$
O.S. Theorem: Suppose $X=\{X_n\}$ martingale w.r.t. $Z=\{Z_n\}$, σ an arbitrary stopping time. Then the stopped process $Y=\{Y_n\}$, $Y_n\triangleq X_{n\wedge\sigma}$ martingale w.r.t. $Z=\{Z_n\}$
In particular, $\mathbb{E}[X_{n\wedge\sigma}]=\mathbb{E}[X_0]$ $\forall n\geq 0$. **Corollary:** $\mathbb{E}[X_\sigma]=\mathbb{E}[X_0] \Longleftarrow$ (i) $\exists T$ s.t. $\mathbb{P}(\sigma\leq T)=1$; or (ii) $\mathbb{P}(\sigma<\infty)=1$ and $\exists K$ s.t. $|X_{n\wedge\sigma}|\leq K$ $\forall n\in\mathbb{N}$

Supermartingales & submartingales: A stochastic process $X=\{X_n\}$ supermartingale w.r.t. $Z=\{Z_n\}$ (i) Adaptedness: $\forall n$ X_n a function of $\{Z_0,\dots,Z_n\}$ (ii) $\mathbb{E}[X_{n+1}|Z_n,\cdots,Z_0]\leq X_n$, Submartingale if: $\mathbb{E}[X_{n+1}|Z_n,\cdots,Z_0]\geq X_n$. **Limit Theorems:** **Fatou's Lemma:** $\forall$ nonnegative random variables $\{X_n\}$ $\mathbb{E}[\liminf_n X_n]\leq\liminf_n \mathbb{E}[X_n]$ **Monotone Convergence Theorem:** Suppose $\{0\leq X_1\leq X_2\leq\dots\}$ increasing nonnegative r.v. $\Longrightarrow \mathbb{E}[\lim_n X_n]=\lim_n\mathbb{E}[X_n]$
Dominated Convergence Theorem: Suppose $\{X_n\}$ r.v. s.t. $\lim_n X_n \longrightarrow X$. If $\exists Y$ r.v. with $\mathbb{E}[Y]<\infty$ s.t. $\forall n$ $|X_n|\leq Y$ $\therefore \lim_n\mathbb{E}[X_n]=\mathbb{E}[\lim_n X_n]=\mathbb{E}[X]$

Poisson Processes: Continuous time stochastic processes defined on $[0,\infty)$ with $S=\{0,1,2,\dots\}$: Counting process whose state $|_{T=t}$ represents # "events" occured up to time $T=t \Longrightarrow$ Poisson Process non-decreasing $\forall$ sample path. **Def:** A counting process $N=\{N_t: t\geq 0\}$ Poisson w/ rate $\lambda>0$ (i) $N_0\equiv 0$ (ii) N has independent increments: $\forall t_0<t_1<\cdots<t_n$, $\{N_{t_1}-N_{t_0}, N_{t_2}-N_{t_1},\cdots,N_{t_n}-N_{t_{n-1}}\}$ are independent (iii) $\forall s\geq 0, t>0$ $(N_{t+s}-N_s)\sim$ Poisson(λt): $\mathbb{P}(N_{t+s}-N_t=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$ $\forall n=0,1,\dots$ **Rmk:** A stochastic process $\{X_t: t\geq 0\}$ has stationary increments if $\forall s,t,h\geq 0$: $X_{t+h}-X_t$, $X_{s+h}-X_s$ have the same distribution: SRW, Poisson P. have stationary i. **Interarrival & Arrival Times:** Let S_n be the arrival time of the n-th event $(S_0\equiv 0)$. Interarrival time $T_n\triangleq S_n-S_{n-1}$: time between $(n-1)$-th event and (n)th. Clearly: $S_n=\sum_{i=1}^n T_i$ (i) $\{N_t\geq n\}=\{S_n\leq t\}$ (ii) $\{N_t=n\}=\{S_n\leq t\leq S_{n+1}\}$ (iii) $N_t=\max\{n\geq 0: S_n\leq t\}$ (iv) $N_t=\sum_{n=1}^\infty \mathbb{1}_{\{S_n\leq t\}}$
N_t: # "events" up to time t, S_n: time of the n-th event. **Prop:** For a Poisson P. w/ rate λ, the interarrival times $\{T_n: n\geq 1\}\overset{iid}{\sim} exp(\lambda): (\lambda e^{-\lambda})$
Corollary: Suppose $\{T_n\}\overset{iid}{\sim} exp(\lambda)$, the counting process $N_t\triangleq\max\{n\geq 0: S_n\leq t\}$, $S_n\triangleq\sum_{i=1}^n T_i$ is Poisson Process with rate λ
Corollary: Suppose $\{X_1,\dots,X_n\}\overset{iid}{\sim} exp(\lambda)$ $\therefore$ the density of $X_1+X_2+\dots+X_n$, denoted by $h_n(x)=\begin{cases}\frac{\lambda}{(n-1)!}e^{-\lambda x}(\lambda x)^{n-1}, & x\geq 0\\ 0, & x<0\end{cases}$, $Z\sim exp(\lambda)$, $W\sim exp(\mu)$ $\mathbb{P}(Z<W)=\frac{\lambda}{\lambda+\mu}$ **Conditional Distributions:** **Lemma:** Suppose $N=\{N_t\}$ Poisson Process with rate $\lambda \Longrightarrow \forall 0<s<t$, $n\geq 1$ the conditional distribution of N_s given $N_t=n$ is Binomial $Bin(n; s/t)$ i.e. $\mathbb{P}(N_s=k|N_t=n)=\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$ $\forall k=0,\dots,n$. **Def:** Suppose $\{X_1,\dots,X_n\}\overset{iid}{\sim}$. Rearrange in increasing order, denote $\{X_{(1)},\cdots,X_{(n)}\}$:= order statistics. If $\{X_1,\dots,X_n\}\overset{iid}{\sim}$ Uniform$([0,\theta])$ the joint density for $\{X_{(1)},\cdots,X_{(n)}\}$ is given by $f(y_1,\dots,y_n)=\frac{n!}{\theta^n}$ $0\leq y_1<\cdots<y_n\leq\theta$; **Lemma:** $N=\{N_t\}$ P.P. with rate λ. Fix $n\geq 1$, $t>0$. Given that $N_t=n$, the (n) arrival times $\{S_1,\dots,S_n\}$ have the same distribution as the order statistics of n independent random variables $\sim$ Uniform$([0,t])$.

Brownian Motion: Continuous time continuous state process. A stochastic process $W=\{W_t: t\geq 0\}$ Standard Brownian motion $|_x$ if (i) $\forall$ sample path of process W is continuous (ii) $W_0=x$ (iii) Independent increments: $\forall$ sequence $0=t_0<t_1<\cdots<t_n$, $W_{t_1}-W_{t_0}, W_{t_2}-W_{t_1},\cdots,W_{t_n}-W_{t_{n-1}}$ independent r.v. $\Longrightarrow$ Brownian motion is a Markov Process i.e. $\mathbb{P}(W_{t+s}\in A|W_u, 0\leq u\leq t)=\mathbb{P}(W_{t+s}\in A|W_t)$ $\forall s,t\geq 0$, $A\subseteq\mathbb{R}$. Moreover Brownian Motion is a martingale since $\mathbb{E}[W_{t+s}|W_u, 0\leq u\leq t)=W_t$ $\forall s,t\geq 0$. (i) (LLN) $\mathbb{P}(\lim_{t\to\infty}\frac{W_t}{t}=0)=1$ (ii) (Scaling) $\forall c>0$, the process $B=\{B_t\}$ where $B_t=\frac{1}{\sqrt{c}}W_{ct}$ is a Standard Brownian $|_{x=0}$. (iii) (Time Inversion) The Process $B=\{B_t\}$ Standard Brownian $|_{x=0}$ where $B_t\triangleq\begin{cases}tW_{1/t}, & t>0\\ 0, & t=0\end{cases}$
(Law of Iterated Logarithm): With $\mathbb{P}=1$: $\limsup_{t\to 0}\frac{W_t}{\sqrt{2t\log(1/t)}}=1$, $\limsup_{t\to\infty}\frac{W_t}{\sqrt{2t\log(t)}}=1$, $\liminf_{t\to 0}\frac{W_t}{\sqrt{2t\log(1/t)}}=-1$, $\liminf_{t\to}\frac{W_t}{\sqrt{2t\log(t)}}=-1$,
with $\mathbb{P}=1$: Brownian Motion sample path (a) not monotone $\forall$ time interval (b) Nowhere differentiable. (Strong Markov Property): $\forall$ stopping time τ, $B=\{B_t\}$ with $B_t\triangleq W_{t+\tau}-W_\tau$ Standard Brownian $|_0$. Moreover, $B\perp\{W_s: 0\leq s\leq\tau\}$ **Hitting times and running maxima:** Let $W=\{W_t\}$ standard Brownian $|_0$. its running maximum $M=\{M_t\}$, $M_t\triangleq\max\{W_s: 0\leq s\leq t\}$, for a fixed $b\in\mathbb{R}$ we denote hitting time to level b by T_b i.e. $T_b\triangleq\inf\{t\geq 0: W_t=b\}$. **Lemma:** $\forall b\in\mathbb{R}$, the density of T_b is given by $f(t)=\frac{|b|}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}$
Corollary: $\forall$ fixed t, the distribution of M_t is given by $\mathbb{P}(M_t\geq b)=\mathbb{P}(T_b\leq t)=2\Phi(-\frac{b}{\sqrt{t}})$ $\forall b\geq 0$ **Lemma:** The joint distribution of Brownian motion W

$\forall b \geq 0, a \leq b$

and its running maximum M is given by $\mathbb{P}(W_t \leq a, M_t \geq b) = \Phi\left(\frac{a-2b}{\sqrt{t}}\right)$ Martingales associated with Brownian Motion: Lemma: $W = \{W_t\}$ standard Brownian $|_x$, arbitrary $x \in \mathbb{R} \implies$ (i) $W = \{W_t\}$ martingale, (ii) The process $\{W_t^2 - t\}$ martingale w.r.t. $W = \{W_t\}$ (iii) $\forall$ fixed $\theta \in \mathbb{R}$ the process $X = \{X_t\}$, $X_t \triangleq \exp\{\theta W_t - \frac{1}{2}\theta^2 t\}$ (exponential) martingale w.r.t. $\{W_t\}$