Set Theory:

$\mathcal{P}(X)=\{E: E\subset X\}$, lim sup $E_n=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}E_n=\{x: x\in E_n \text{ for infinitely many } n\}$ lim inf $E_n=\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}E_n=\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}E_n=\{x: x\in E_n \text{ for all but finitely many } n\}$

$E\setminus F=\{x: x\in E\wedge x\notin F\}$, $E\Delta F=(E\setminus F)\cup(F\setminus E)$. De Morgan's $\left(\bigcup_{\alpha\in A}E_\alpha\right)^c=\bigcap_{\alpha\in A}E_\alpha^c$, $\left(\bigcap_{\alpha\in A}E_\alpha\right)^c=\bigcup_{\alpha\in A}E_\alpha^c$. $X\times Y=\{(x,y): x\in X, y\in Y\}$, $xRy \subset XY$, $(x,y)\in R$. Equivalence relation (on X): (i) $xRx\ \forall x\in X$, $xRy\iff yRx$, $xRz \overset{\exists y}{\Longleftarrow} xRy\wedge yRz$. Equivalence Class (x): $\{y\in X: xRy\}$. X: $\bigsqcup$ equivalence Classes. Mapping $f: X\to Y$ s.t. $\forall x\in X\ \exists! y\in Y$ s.t. xRy i.e. $y=f(x)$. $D\subset X\wedge E\subset Y$, image (D): $f(D)=\{f(x): x\in D\}$, Image$^{-1}(E)$: $f^{-1}(E)=\{x: f(x)\in E\}$, f injective $\Rightarrow f(x_1)=f(x_2)$ when $x_1=x_2$, surjective $\Rightarrow f(X)=Y$, f bijective if $\exists f^{-1}: Y\to X$ s.t. $f^{-1}\circ f \wedge f\circ f^{-1}$ id. maps. Sequence $f:\mathbb{N}\to X$, if $g:\mathbb{N}\to\mathbb{N}$ s.t. $g(n)<g(m)$ ($\overset{n<m}{\Downarrow}$), $(f\circ g)$: subsequence of f. Let $\{X_\alpha\}_{\alpha\in A}$. Cartesian P.: $\prod_{\alpha\in A}X_\alpha=\{\text{maps}: f: A\to\bigcup_{\alpha\in A}X_\alpha \text{ s.t. } f(\alpha)\in X_\alpha\ \forall \alpha\in A\}$. If $X=\prod_{\alpha\in A}X_\alpha$, αth-projection: $\pi_\alpha \triangleq X\to X_\alpha$, $\pi_\alpha(f)=f(\alpha)$. $\therefore$ If $X_\alpha=Y\ \forall\alpha\in A$, $\prod_{\alpha\in A}X_\alpha=Y^A:\{\text{maps}: A\to Y\}$

Partial Ordering on $X\neq\emptyset$: R on X s.t. (i) $xRy\wedge yRz\Rightarrow xRz$, (ii) $xRy\wedge yRx\Rightarrow x=y$, $xRx\ \forall x$. R total (linear) if: $x,y\in X\Rightarrow xRy \vee yRx$. X partially ordered by $\leq$ a (minimal) maximal element $x\in X \Rightarrow y\in X$ s.t. $x\leq y$ ($x\geq y$) $\Rightarrow y=x$. If $E\subset X$ (lower) upper bound for E: $x\in X$ s.t. $y\leq x$ ($x\geq y$) $\forall y\in E$. If X linear by $\leq$ and $\forall E\neq\emptyset\subset X\ \exists!$ minimal element, X well ordered by $\leq$. Hausdorff Maximal P.: Every Partially Ordered X has a maximal linearly ordered subset. Zorn's Lemma: If X partially o. and every linearly ordered subset of X has an upper bound $\Rightarrow X$ has a maximal element. Axiom of Choice: If $\{X_\alpha\}_{\alpha\in A}$ $X_\alpha, A\neq\emptyset\Rightarrow\prod_{\alpha\in A}X_\alpha\neq\emptyset$.

Well-Ordering P.: Every $X\neq\emptyset$ can be well ordered. Cardinality: $X,Y\neq\emptyset$, $\mathrm{Card}(X)\underset{(\geq)}{\overset{(\leq)}{(=)}}\mathrm{Card}(Y)\Rightarrow\exists f: X\to Y$ (injective) (bijective) (surjective). Schröder Bernstein: If $\mathrm{Card}(X)\leq\mathrm{Card}(Y)$ $\wedge\ \mathrm{Card}(Y)\leq\mathrm{Card}(X)\Rightarrow\mathrm{Card}(X)=\mathrm{Card}(Y)$, Prop: $\forall X$, $\mathrm{Card}(X)<\mathrm{Card}(\mathcal{P}(X))$. X countable if $\mathrm{Card}(X)\leq\mathrm{Card}(\mathbb{N})$, Prop: X, Y countable $\Rightarrow X\times Y$ countable (ii) $A\wedge X_\alpha$ countable, $\forall\alpha\in A\Rightarrow\bigcup_{\alpha\in A}X_\alpha$ countable, Cardinality Continuum: $\mathrm{Card}(X)=c\iff\mathrm{Card}(X)=\mathrm{Card}(\mathbb{R})$, $\mathrm{Card}(\mathcal{P}(\mathbb{N}))=c$. Let X well ordered $A\neq\emptyset\subset X$, A has minimal element: maximal lower bound (infimum) $\triangleq\inf(A)$. If A bounded above, A has minimal upper bound (supremum): $\sup A$

Extended $\mathbb{R}$: $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$: $\forall A\subset\overline{\mathbb{R}}\ \exists\inf A, \sup A$. $A=\{a_1,\dots,a_n\}\Rightarrow\max(a_1,\dots,a_n)=\sup A$, $\min(a_1,\dots,a_n)=\inf A$

lim sup $\{x_n\}=\inf_{k\geq1}\left(\sup_{n\geq k}x_n\right)$, lim inf $\{x_n\}=\sup_{k\geq1}\left(\inf_{n\geq k}x_n\right)$, lim inf $x_n=\lim_{n\to\infty}\left(\inf_{k\geq n}x_k\right)$, lim sup $x_n=\lim_{n\to\infty}\left(\sup_{k\geq n}x_k\right)$. $\{x_n\}$ converges in $\mathbb{R}$ iff they equal. For $f:\mathbb{R}\to\overline{\mathbb{R}}$ $\limsup_{x\to a}f(x)=\inf_{\delta>0}\left(\sup_{0<|x-a|<\delta}f(x)\right)$. Let X arbitrary, $f: X\to[0,\infty]$, $\sum_{x\in X}f(x)=\sup\left\{\sum_{x\in F}f(x): F\subset X, F \text{ finite}\right\}$

$U\subset\mathbb{R}$ open $\Rightarrow\forall x\in U$, U includes an I centered at x. Prop: Every open set $\subset\mathbb{R}$ is a countable disjoint union of open intervals.

Metric Space: A metric on X is a function $\rho: X\times X\to[0,\infty)$ s.t. (i) $\rho(x,y)=0\iff x=y$ (ii) $\rho(x,y)=\rho(y,x)\ \forall x,y\in X$ (iii) $\rho(x,z)\leq\rho(x,y)+\rho(y,z)\ \forall x,y,z\in X$ (X,ρ) metric space. Let (X_1,ρ_1), (X_2,ρ_2) The product metric ρ on $X_1\times X_2$: $\rho((x_1,x_2),(y_1,y_2))=\max(\rho_1(x_1,y_1),\rho_2(x_2,y_2))$

Let (X,ρ), $x\in X$, $r>0$, the open ball $B(r,x)=\{y\in X, \rho(x,y)<r\}$. $E\subset X$ open if $\forall x\in E$, $\exists r>0$ s.t. $B(r,x)\subset E$, E closed if E^c open. $X,\emptyset$ open and closed. $\bigcup_{\alpha\in A}X_\alpha$ open if X_α open $\forall\alpha$, $\bigcap_{\alpha\in A}X_\alpha$ closed if X_α closed $\forall\alpha$, $\bigcap^n\{X_i\}$ open $\forall X_i$ open, $\bigcup^n\{X_i\}$ closed $\forall X_i$ closed.

E°: interior $\triangleq\{\bigcup U: U\subset E, U \text{ open}\}$, $\bar{E}$: closure $\triangleq\{\bigcap F: F\supset E, F \text{ closed}\}$. E dense in X if $\bar{E}=X$, nowhere dense if $(\bar{E})^\circ=\emptyset$.

X separable if $\exists E\subset X$ countably dense. Prop: (X,ρ), $E\subset X$, $x\in X$, TFAE: (a) $x\in\bar{E}$ (b) $B(x,r)\cap E\neq\emptyset\ \forall r>0$. (c) $\exists\{x_n\}\in E$ s.t. $x_n\to x$

Let (X_1,ρ_1), (X_1,ρ_2), $f: X_1\to X_2$ continuous at $x\in X$ if $\forall\varepsilon>0\ \exists\delta_x>0$ s.t. $\rho_2(f(y),f(x))<\varepsilon$ whenever $\rho_1(x,y)<\delta$. (i.e. $f^{-1}(B(\varepsilon,f(x)))\supset B(\delta_x,x)$)

f continuous if cont. $\forall x\in X_1$, uniformly continuous if $\delta\perp\!\!\!\perp x$. Prop: $f: X_1\to X_2$ continuous $\iff f^{-1}(U)$ open in X_1 $\forall$ open $U\subset X_2$. A sequence $\{x_n\}\in(X,\rho)$ is Cauchy if $\rho(x_n,x_m)\longrightarrow0$ $n,m\to\infty$ ($\forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $\forall n,m>N\Rightarrow\rho(x_n,x_m)<\varepsilon$). $E\subset X$ complete if $\forall$ Cauchy sequence $\in E$ converges and its limit $\in E$.

Prop: X complete $\wedge$ F(closed)$\subset X\Rightarrow F$ complete, F complete $\subset X$ (arbitrary) $\Rightarrow F$ closed. (X,ρ), $\rho(x,E)=\inf\{\rho(x,y): y\in E\}$, $\rho(E,F)=\inf\{\rho(x,y): x\in E, y\in F\}$. $\mathrm{Diam}(E)=\sup\{\rho(x,y): x,y\in E\}$, E bounded if $\mathrm{diam}(E)<\infty$. Cover: $E\subset X$, $\{V_\alpha\}_{\alpha\in A}$ s.t. $E\subset\bigcup_{\alpha\in A}V_\alpha$, $\{V_\alpha\}_{\alpha\in A}$ cover of E

E totally-bounded if $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $E\subset\bigcup^n B(\varepsilon,z_j)$ Theorem: Let $E\subset(X,\rho)$: TFAE (a) E complete $\wedge$ totally bounded (b) (Bolzano-Weierstrass): $\forall\{x_n\}\in E\ \exists x_{n_k}\longrightarrow x_0\in E$ (c) (Heine-Borel): If $\{V_\alpha\}_{\alpha\in A}$ cover of E by open sets $\therefore\exists$ finite $F\subset A$ s.t. $\{V_\alpha\}_{\alpha\in F}$ covers E, (a)$\wedge$(b)$\wedge$(c) $\Rightarrow E$ compact

Prop: Every closed $\wedge$ bounded $S\subset\mathbb{R}^n$ is compact. (X,ρ) Let $\{f_n\}$ sequence of functions $f_n: X\to Y$. $\{f_n\}\longrightarrow f: X\to Y$ pointwise if $\forall x\in X$ $\wedge\ \forall\varepsilon>0\ \exists N(x,\varepsilon)\in\mathbb{N}$ s.t. $\forall n\geq N$, $\rho(f_n(x),f(x))<\varepsilon$. $\{f_n\}\rightrightarrows f$ (uniformly) if $\forall\varepsilon>0$, $\exists N=N(\varepsilon)$ s.t. $\forall n\geq N\wedge\forall x\in X: \rho(f_n(x),f(x))<\varepsilon$.

Measures: σ-Algebras: Let $X\neq\emptyset$, an Algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X closed under finite unions and complements $\therefore$ If $E_1,\dots,E_n\in\mathcal{A}\Rightarrow\bigcup_1^n E_j\in\mathcal{A}$, $E\in\mathcal{A}\Rightarrow E^c\in\mathcal{A}$. σ-Algebra: Algebra closed under countable unions. Note $\bigcap_j E_j=\left(\bigcup_j E_j^c\right)^c\therefore$ σ-Algebras (Algebras) closed under countable (finite) $\bigcap$'s $\emptyset\in\mathcal{A}$, $X\in\mathcal{A}$: if $E\in\mathcal{A}$, $\emptyset=E\wedge E^c$, $X=E\cup E^c$. Note: $\mathcal{A}$ algebra is σ-Algebra provided it's closed under countable disjoint unions. Suppose $\{E_j\}_1^\infty\subset\mathcal{A}$. Set $F_k=E_k\setminus\left[\bigcup_1^{k-1}E_j\right]=E_k\cap\left[\bigcup_1^{k-1}E_j\right]^c\therefore F_k\in\mathcal{A}\wedge F_i\cap F_j=\emptyset\wedge\bigcup_1^\infty E_j=\bigcup_1^\infty F_k$. Ex: X, $\mathcal{P}(X)$, $\{\emptyset,X\}$, $\mathcal{A}=\{E\subset X: E \text{ countable} \vee E^c \text{ countable}\}$ σ-Algebras. If $\mathcal{E}$ any subset of $\mathcal{P}(X)$ $\exists!$ smallest σ-Algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, $\mathcal{M}(\mathcal{E})$: σ-Algebra gen. by $\mathcal{E}$. Lemma: $\mathcal{E}\subset\mathcal{M}(\mathcal{F})\Rightarrow\mathcal{M}(\mathcal{E})\subset\mathcal{M}(\mathcal{F})$. Borel σ-Algebra: ($\mathcal{B}_X$) σ-Algebra generated by family of open sets in X. Prop: $\mathcal{B}_{\mathbb{R}}$ generated by: (a) $\mathcal{E}_1=\{(a,b): a<b\}$, $\mathcal{E}_2=\{[a,b]: a<b\}$, $\mathcal{E}_3=\{(a,b]: a<b\}$, $\mathcal{E}_4=[a,b)$, $\mathcal{E}=\{(a,\infty): a\in\mathbb{R}\}$ $(-\infty,a)$, $\mathcal{E}=\{[a,\infty): a\in\mathbb{R}\}$ $(-\infty,a]$

$(a,b]=\bigcap_1^\infty\left(a,b+\frac{1}{n}\right)$, $(a,b)=\bigcup_1^\infty\left[a+\frac{1}{n},b-\frac{1}{n}\right]\in\mathcal{M}(\mathcal{E}_2)$. Let $\{X_\alpha\}_{\alpha\in A}$, $X=\prod_{\alpha\in A}X_\alpha$, $\pi_\alpha: X\to X_\alpha$. If $\mathcal{M}_\alpha$ σ-Algebra on $X_\alpha\ \forall\alpha$, The product-σ-Algebra on X

Let $(X,\mathcal{M},\mu)$: $\mu(X)<\infty$: μ-finite; $X=\bigcup_1^\infty E_j$, $E_j\in\mathcal{M}$, $\mu(E_j)<\infty\ \forall j$: μ-σ-finite; $E\in\mathcal{M}$, $E=\bigcup_1^\infty E_j$, $\mu(E_j)<\infty$: E-σ-finite for μ; $\forall E\in\mathcal{M}$ s.t. $\mu(E)=\infty\ \exists F\subset E$, $0<\mu(F)<\infty$

μ: semifinite, F_σ sets: $\bigcup^n F_j$ (F_j closed), G_δ set: $\bigcap^n U_j$ (U_j open)

is the σ-Algebra gen. by $\{\pi_\alpha^{-1}(E_\alpha): E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$, denote by $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. Prop: A countable $\therefore \bigotimes_{\alpha\in A}\mathcal{M}_\alpha$: σ-Algebra gen. by $\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$

Prop: $X_1,\dots,X_n$ metric space, $X \triangleq \prod_1^n X_j \therefore \bigotimes_1^n B_{X_j} \subset B_X$. If X_j separable $\Rightarrow \bigotimes_1^n B_{X_j} = B_X \Rightarrow B_{\mathbb{R}^n} = \bigotimes^n B_{\mathbb{R}}$

Let $(X, \mathcal{M})$. A measure on $\mathcal{M}$: $\mu: \mathcal{M} \to [0,\infty]$ s.t. (i) $\mu(\emptyset)=0$ (ii) if $\{E_j\}^\infty$ disjoint sets in $\mathcal{M} \Rightarrow \mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$ (Countable-additivity) $\Rightarrow$ (iii) If $E_1,\dots,E_n$ disjoint in $\mathcal{M}$, $\mu(\bigcup^n E_j) = \sum^n \mu(E_j)$ (finite additivity). $\mathcal{M} \subset \mathcal{P}(X)$, $(X,\mathcal{M})$ measurable space, sets $\in \mathcal{M}$ measurable sets, $(X,\mathcal{M},\mu)$ measure space. Theorem: $(X,\mathcal{M},\mu)$ (a) (Monotonicity) if $E,F \in \mathcal{M}$, $E \subset F \therefore \mu(E) \le \mu(F)$ (b) (Subadditivity) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \le \sum_1^\infty \mu(E_j)$ (c) Continuity from below: If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \subset E_2 \subset \dots$: $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$ (Continuity from above) $\{E_j\}_1^\infty$, $E_1 \supset E_2 \supset \dots \wedge \mu(E_1) < \infty \Rightarrow \mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$. A set $E \subset \mathcal{M}$ s.t. $\mu(E)=0$ null-set. If $\mu(E)=0$, $F \subset E \Rightarrow \mu(F)=0$ $F \overset{?}{\in} \mathcal{M}$. A measure whose domain includes all subsets of null-sets is complete. Theorem: $(X,\mathcal{M},\mu)$. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N)=0\}$, $\bar{\mathcal{M}} = \{E \cup F,\ E \in \mathcal{M} \wedge F \subset N$ for some $N \in \mathcal{N}\} \therefore \bar{\mathcal{M}}$ σ-Algebra $\wedge\ \exists!$ extension $\bar{\mu}$ of μ to a complete measure on $\bar{\mathcal{M}}$.

Outer Measure: on $X \neq \emptyset$ is $\mu^*: \mathcal{P}(X) \to [0,\infty]$ s.t. (i) $\mu^*(\emptyset)=0$, (ii) $\mu^*(A) \le \mu^*(B)$ if $A \subset B$, (iii) $\mu^*(\bigcup_1^\infty A_j) \le \sum_1^\infty \mu^*(A_j)$. Prop: Let $\mathcal{E} \subset \mathcal{P}(X) \wedge \rho: \mathcal{E} \to [0,\infty]$ s.t. $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, $\rho(\emptyset)=0$. $\forall A \subset X$, define $\mu^*(A) = \inf\{\sum_1^\infty \mu(E_j): E_j \in \mathcal{E} \wedge A \subset \bigcup_1^\infty E_j\} \therefore \mu^*$: outer measure. If μ^* outer measure on X $A \subset X$ is μ^*-measurable if $\mu^*(E) = \mu^*(E\cap A) + \mu^*(E \cap A^c)$ $\forall E \subset X$ $\therefore A$ is μ^*-measurable $\Longleftrightarrow \underset{(\le)(\text{Trivial})}{\mu^*(E) \ge \mu^*(E\cap A) + \mu^*(E \cap A^c)}$ $\forall E \subset X$ s.t. $\mu^*(E) < \infty$

Carathéodory's Theorem: If μ^* outer measure on X, the collection $\mathcal{M}$ of μ^* measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure. Let $\mathcal{A} \subset \mathcal{P}(X)$ an Algebra, $\mu_0: \mathcal{A} \to [0,\infty]$ premeasure if $\mu_0(\emptyset)=0$, $\{A_j\}^\infty$ sequence of disjoint sets in $\mathcal{A}$ s.t. $\bigcup^\infty A_j \in \mathcal{A} \Rightarrow \mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$. μ_0 induces an outer measure on X : $\mu^*(E) = \inf\{\sum^\infty \mu_0(A_j): A_j \in \mathcal{A}, E \subset \bigcup^\infty A_j\}$

Prop: If μ_0 premeasure on $\mathcal{A}$, $\wedge\ \mu^*(E) \triangleq \{\sum^\infty \mu_0(A_j): A_j \in \mathcal{A}, E \subset \bigcup^\infty A_j\}$: (a) $\mu^*|\mathcal{A} = \mu_0$ (b) Every set $\subset \mathcal{A}$ μ^* measurable. Fact: $\forall \varepsilon > 0$ $\exists \{B_j\} \subset \mathcal{A}$ s.t. $E \subseteq \bigcup_{j=1}^\infty B_j$ and $\sum_{j=1}^\infty \mu_0(B_j) \le \mu^*(E) + \varepsilon$. Prop: Let $F: \mathbb{R} \to \mathbb{R}$ increasing and right continuous. If $(a_j, b_j]$ $j=1,\dots,n$, are disjoint h-intervals, let $\mu_0(\bigcup_1^n (a_j,b_j]) = \sum_1^n [F(b_j) - F(a_j)]$, let $\mu_0(\emptyset) = 0 \Rightarrow \mu^0$ premeasure on Algebra $\mathcal{A}$. Theorem: If $F: \mathbb{R} \to \mathbb{R}$ any right continuous function, $\exists!$ Borel Measure μ_F on $\mathbb{R}$ s.t. $\mu_F((a,b]) = F(b) - F(a)$ $\forall a,b$. If G another such function, $\mu_F = \mu_G \Longleftrightarrow F - G$ constant. Conversely, if μ a Borel measure on $\mathbb{R}$, finite on all bounded Borel sets and we define $F(x) = \begin{cases} \mu((0,x]), & x>0 \\ 0, & x=0 \\ -\mu((-x,0]), & x<0 \end{cases} \Rightarrow F$ increasing, right-continuous, and $\mu = \mu_F$

Let $\bar{\mu}_F$ completion of μ_F : Lebesgue-Stieltjes measure associated to F. Fix a complete Lebesgue-Stieltjes measure μ on $\mathbb{R}$ associated to F (increasing, r.c.) and $\mathcal{M}_\mu$ (domain of μ) $\therefore \forall E \in \mathcal{M}_\mu$, $\mu(E) = \inf\{\sum^\infty (F(b_j) - F(a_j)] : E \subset \bigcup_1^\infty (a_j,b_j]\} = \inf\{\sum^\infty \mu((a_j,b_j]): E \subset \bigcup^\infty (a_j,b_j]\}$. Lemma: $\forall E\, \mathcal{M}_\mu$ $\mu(E) = \inf\{\sum_1^\infty \mu((a_j,b_j)): E \subset \bigcup^\infty (a_j,b_j)\}$. Theorem: If $E \in \mathcal{M}_\mu \Rightarrow \mu(E) = \inf\{\mu(U): U \supset E, U \text{ open}\} = \sup\{\mu(K): K \subset E, K \text{ compact}\}$

Prop: If $E \subset \mathcal{M}_\mu\ \wedge\ \mu(E) < \infty \Longrightarrow \forall \varepsilon > 0\ \exists A$: finite union of open intervals s.t. $\mu(E \Delta A) < \varepsilon$.

Lebesgue Measure: Complete measure associated to the function $F(x) = x \Longrightarrow m((a,b]) = b - a$. Domain: $\mathcal{L}$. The restriction of m to $B_{\mathbb{R}}$: Lebesgue Measure. If $E \subset \mathbb{R}$, $s,r \in \mathbb{R}$: $E + s = \{x + s : x \in E\}$, $rE = \{xr : x \in E\}$. Theorem: If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ $\forall s,r \in \mathbb{R}$. Moreover $m(E+s) = m(E)$, $m(rE) = \|r\| m(E)$, $m(\{x_0\}) = 0$, $m(\mathbb{Q}) = 0$ (i.e.) Proof: Let $\{r_j\}_1^\infty$ enumeration of $\mathbb{Q}$ in $(0,1)$, given $\varepsilon > 0$ let I_j: interval centered at r_j and $|I_j| = \varepsilon 2^{-j}$ $\therefore U \triangleq (0,1) \cap \bigcup_1^\infty I_j$ open and dense in $[0,1]$ but $m(U) \le \sum_1^\infty \varepsilon 2^{-j} = \varepsilon$, its complement $K = [0,1] \setminus U$ closed and nowhere dense but $m(K) \ge 1 - \varepsilon$. Prop: Cantor Set: $\{x \in [0,1] : x = \sum^\infty a_j 3^{-j}, a_j \in \{0,2\}\}$

a) C compact, nowhere dense, totally disconnected, C has no isolated points, b) $\underset{(\text{length})}{m(C) = 0}$, c) $\text{Card}(C) = \mathfrak{c}$, $m(C) = 1 - \sum_0^\infty \frac{2^j}{3^{j+1}} = 0$

Given $E \subset \mathbb{R}$, $m(E)$: Lebesgue's Measure. (1) $\forall$ Interval, $m(I) = \ell(I) := b - a$, Monotone: If $A \subset B \subset \mathbb{R}$ $0 \le \mu(A) \le \mu(B) \le \infty$ Translational Invariant, $\mu(A + x_0) = \mu(A)$, Countably Additive: $A \cap B = \emptyset \subset \mathbb{R}$. If $\{A_i\}^\infty$ disjoint $\Longrightarrow \mu(\bigcup_1^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$

Lebesgue outer measure: $\mu^*(E) = \inf\{\sum_{k=1}^\infty \ell(I_k) : \{I_k\}$ open intervals s.t. $E \subset \bigcup_{k=1}^\infty I_k$. Theorem: E countable if $\mu^*(E) = 0$, extends length, monotone, translational invariant, countable subadditive : $\forall E_i \subset \mathbb{R}$, $\mu^*(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu^*(E_i)$. $E \subset \mathbb{R}$ Lebesgue Measurable if $\forall A \in \mathbb{R}$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. If E Lebesgue Measurable, then Lebesgue Measure of E is defined to be $\mu^*(E)$ written as $\mu(E)$. Theorem: Collection $\mathcal{M}$ of measurable sets : (i) $\emptyset$, $\mathbb{R}$ measurable, $\mu(\emptyset) = 0$, $\mu(\mathbb{R}) = \infty$, (ii) If E measurable so is E^c, (iii) If $\mu^*(E) = 0 \Rightarrow E$ measurable, (iv) $E_1 \wedge E_2$ measurable $\therefore E_1 \cup E_2$, $E_1 \cap E_2$ measurable, (v) E m. $\Rightarrow E + x_0$ m. f) $\forall I$ measurable and $\mu(I) = \mu^*(I) = \ell(I)$ (vi) If $\{E_i\}^n$ disjoint measurable $\Longrightarrow \forall A \subset \mathbb{R}$ $\mu^*(\bigcup_{i=1}^n A \cap E_i) = \mu^*(A \cap (\bigcup_{i=1}^n E_i)) = \sum_{i=1}^n \mu^*(A \cap E_i)$, If $A = \mathbb{R}$: $\mu(\bigcup^n E_i) = \sum^n \mu(E_i)$, (vii) If $\{E_i\}$ measurable sets $\therefore \bigcup_{i=1}^\infty E_i$, $\bigcap_{i=1}^\infty E_i$ m. (viii) $\{E_i\}^\infty$ disjoint m. $\therefore \mu(\bigcup^\infty E_i) = \sum^\infty \mu(E_i)$

Measurable Functions: Any mapping $f: X \to Y$ induces $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$, $f^{-1}(E) \triangleq \{x \in X : f(x) \in E\}$, preserves $(\cap, \cup, {}^c)$ $\therefore$ If N

σ-Algebra on Y, $\{f^{-1}(E): E:\mathcal{N}\}$ σ-Algebra on X. If $(X,\mathcal{M})$, $(Y,\mathcal{N})$ measurable spaces, $f: X\to Y$ is called $(\mathcal{M},\mathcal{N})$-measurable if $f^{-1}(E)\in\mathcal{M}$ $\forall E\in\mathcal{N}$. **Prop:** If $\mathcal{N}$ generated by $\mathcal{E}$ $\Rightarrow$ $f: X\to Y$ $(\mathcal{M},\mathcal{N})$-measurable $\Longleftrightarrow f^{-1}(E)\in\mathcal{M}$ $\forall E\in\mathcal{E}$, **Corollary:** X, Y metric spaces, every continuous $f: X\to Y$ is $(\mathbb{B}_X,\mathbb{B}_Y)$-measurable. Let $(X,\mathcal{M})$, f ℝ ∨ ℂ-valued measurable if $(\mathcal{M},\mathbb{B}_{\mathbb{R}})$ or $(\mathcal{M},\mathbb{B}_{\mathbb{C}})$ measurable. $f:\mathbb{R}\to\mathbb{C}$ Lebesgue (Borel) measurable if $(\mathcal{L},\mathbb{B}_{\mathbb{C}})$ (($\mathbb{B}_{\mathbb{R}},\mathbb{B}_{\mathbb{C}}$))-measurable. **Corollary:** A function is $\mathcal{M}$-measurable $\Longleftrightarrow Re(f)$, $Im(f)$ are $\mathcal{M}$-measurable.

$(X,\mathcal{M},\mu)$, $f: X\to\mathbb{R}$, f-measurable $\Leftrightarrow \forall\alpha\in\mathbb{R}$, $\underbrace{\{x\in X, f(x)>\alpha\}}_{}$ (pre-image) $\in\mathcal{M}$ $\to \forall\alpha$, $f\geq\alpha$, $f<\alpha$, $\alpha<f<\beta$ measurable. **Prop:** f, g measur. $\therefore f+c$, cf, $f+g$, $f\cdot g$ measur. **Prop:** If $\{f_j\}$ sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X,\mathcal{M})$ $\Rightarrow g_1(x)=\sup_j f_j(x)$, $g_2(x)=\inf_j f_j(x)$, $g_3(x)=\limsup_{j\to\infty} f_j(x)$, $g_4(x)=\liminf_{j\to\infty} f_j(x)$ measurable. **Prop:** If $(X,\mathcal{M})$ measurable space, $f: X\to\mathbb{R}$: TFAE (a) f $\mathcal{M}$-measurable (b) $f^{-1}((a,\infty))\in\mathcal{M}$ $\forall a\in\mathbb{R}$, (b) $f^{-1}([a,\infty))\in\mathcal{M}$ $\forall a\in\mathbb{R}$, (c) $f^{-1}((-\infty,a))\in\mathcal{M}$ $\forall a\in\mathbb{R}$, (e) $f^{-1}((-\infty,a]))\in\mathcal{M}$ $\forall a\in\mathbb{R}$. **Corollary:** $f, g: X\to\overline{\mathbb{R}}$ measurable $\Longrightarrow$ $\max\{f,g\}$, $\min\{f,g\}$ measurable. If $f: X\to\overline{\mathbb{R}}$ positive-part: $f^+(x)=\max(f(x),0)$, negative-part: $f^-(x)=\max\{-f(x),0\}$ $\therefore$ $f=f^+-f^-$

Suppose $(X,\mathcal{M})$. $E\subset X$, Characteristic-function χ_E: $\mathbb{1}_E$, $\chi_E\triangleq\begin{cases}1, & x\in E\\ 0, & x\notin E\end{cases}$. χ_E measurable $\Longleftrightarrow E\in\mathcal{M}$. A simple-function on X, ϕ: $\phi=z_1\chi_{E_1}+\dots+z_n\chi_{E_n}$, $z_i\neq z_j$, $\forall E_i\cap E_j=\phi$ $i\neq j$, $E_j=f^{-1}(\{z_j\})$, range$(f)=\{z_1,\dots,z_n\}$. **Theorem:** $(X,\mathcal{M})$ measurable space

(a) If $X\to[0,\infty]$ measurable, $\exists$ sequence $\{\phi_n\}$ of simple functions s.t. $0\leq\phi_1\leq\phi_2\leq\dots\leq f$, $\phi_n\longrightarrow f$ pointwise, and $\phi_n\rightrightarrows f$ uniformly on any set on which f is bounded. (b) $\dots$ $0\leq|\phi_1|\leq|\phi_2|\leq\dots\leq|f|$, $\dots$. **Construction of ϕ_n:** $n=0,1,2,\dots$, $0<k\leq 2^{2n}-1$, $E_n^k\triangleq f^{-1}((k2^{-n},(k+1)2^{-n}])=\{x: k2^{-n}<f(x)\leq(k+1)2^{-n}$, $0\leq k\leq 2^{2n}-1\}$, $F_n\triangleq f^{-1}((2^n,\infty])=\{x: f(x)\geq 2\}$, $\phi_n\triangleq\sum_{k=0}^{2^{2n}-1}k2^{-n}\chi_{E_n^k}+2^n\chi_{F_n}$.

Integration of Nonnegative F: Fix $(X,\mathcal{M},\mu)$, $\mathcal{L}^+=\{$space of all measurable functions from $X\to[0,\infty]\}$. If ϕ: simple function $\in\mathcal{L}^+$ s.t. $\phi=\sum_1^n a_j\chi_{E_j}$ $\therefore$ integral of ϕ with respect to μ: $\int\phi\,d\mu=\sum_1^n a_j\mu(E_j)$. If $A\in\mathcal{M}$, $\phi\chi_A$ simple, $\phi\chi_A=\sum a_j\chi_{A\cap E_j}$, $\int_A\phi\,d\mu=\int\phi\chi_A\,d\mu$. **Prop:** Let ϕ,ψ simple-functions $\in\mathcal{L}^+$ (a) If $c\geq 0$, $\int c\phi=c\int\phi$, (b) $\int(\phi+\psi)=\int\phi+\int\psi$ (c) $\phi\leq\psi\Rightarrow\int\phi\leq\int\psi$. (d) $A\longmapsto\int_A d\mu$ a measure on $\mathcal{M}$

Fact: $\inf_n\{f_n\}=-\sup_n\{-f_n\}$, $\sup_n\{f_n\}=-\inf_n\{-f_n\}$, $\limsup_n\{f_n\}=-\limsup_n\{-f_n\}$, $\liminf_n\{f_n\}=-\limsup_n\{f_n\}$. **Approximation of measurable sets:** (1) $\forall E$, $\forall\varepsilon>0$, $\exists$ Open set $\mathcal{O}$ s.t. $E\subseteq\mathcal{O}$, $\mu(\mathcal{O})\geq\mu^*(E)\geq\mu(\mathcal{O})-\varepsilon$ (2) $\forall E\subset\mathcal{M}$, $\forall\varepsilon>0$ $\exists$ closed $K\subset E$, $\mu(K)\leq\mu(E)\leq\mu^*(K)+\varepsilon$

Monotone Subsequence Theorem: Let $\{S_n\}\subset\mathbb{R}$. S_m a peak if $S_m>S_n$ $\forall n>m$. Let $\mathbb{P}=\{S_m \mid S_m:$ a peak$\}$. If Card$(\mathbb{P})=\infty$. Construct: Let m_1 s.t. S_{m_1} a peak, $m_2>m_1$ s.t. S_{m_2} a peak $\dots\Longrightarrow m_1<m_2<\dots<m_k<\dots$ and $S_{m_1}>S_{m_2}>\dots$ $\therefore\{S_{m_k}\}$ monotone (1) P finite: $P=\{S_{m_1},S_{m_2},\dots,S_{m_r}\}$ Let $n_1>\max\{m_1,\dots,m_2\}$ $\therefore S_{n_1}\neg$peak and $S_k\neg$peak $\forall k>n_1$ $\therefore\exists n_2>n_1$ s.t. $S_{n_1}\leq S_{n_2}$ $\therefore\exists n_3>n_2$ s.t. $S_{n_2}\leq S_{n_3}$, $\dots$, $\Rightarrow S_{n_1}\leq S_{n_2}\leq\dots$ $\{S_{n_k}\}$ monotone-inc

A point $x\in\mathbb{R}$ accumulation point of $S\subset\mathbb{R}$ if $\forall U$ neighborhood of x, $S\cap U$ is infinite. Isolated point if $x\in S$, $x\neg$ accumulated point

Prop: If f measurable, $f=g$ μ-a.e. $\Rightarrow g$ measurable, if $\{f_n\}$ measurable $\forall n\in\mathbb{N}$, $f_n\to f$ μ-a.e. $\Rightarrow f$ measurable $\Longleftrightarrow$ μ-complete

Prop: $(X,\mathcal{M},\mu)$ let $(X,\overline{\mathcal{M}},\overline{\mu})$ be its completion. If $\overline{\mathcal{M}}$-measurable on X, $\exists$ $\mathcal{M}$-measurable g s.t. $f=g$ $\overline{\mu}$-a.e.

Integral: $\forall f\in\mathcal{L}^+$, $\int f\,d\mu=\sup\{\int\phi\,d\mu: 0\leq\phi\leq f$, ϕ simple$\}$ $\therefore f\leq g\Rightarrow\int f\leq\int g$. **Monotone-Convergence Theorem:** If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ s.t. $f_j\leq f_{j+1}$ $\forall j$, and $f=\lim_{n\to\infty}f_n(=\sup_n f_n)\Longrightarrow\int f=\lim_{n\to\infty}\int f_n$. **Theorem:** If $\{f_n\}\subset\mathcal{L}^+$, $f=\sum_n f_n\Rightarrow\int f=\sum_n\int f_n$. **Prop:** $f\in\mathcal{L}^+$, $\int f=0\Longleftrightarrow f=0$ a.e. Corollary: If $\{f_n\}\subset\mathcal{L}^+$, $f\in\mathcal{L}^+$, $f_n(x)\nearrow f(x)$ for a.e. $x\Longrightarrow\int f=\lim\int f_n$. **Fatou's Lemma:** If $\{f_n\}\subset\mathcal{L}^+$, $\int(\liminf f_n)\leq\liminf\int f_n$, **Reverse Fatou's:** $\{f_n\}$ bounded below, $\lim_{n\to\infty}\sup\int_X f_n\,d\mu\leq\int_X f\,d\mu$. **Corollary:** $\{f_n\}\subset\mathcal{L}^+$, $f\in\mathcal{L}^+$, $f_n\to f$ a.e. then $\int f\leq\liminf\int f_n$. **Prop:** $f\in\mathcal{L}^+$, $\int f<\infty$, then $\{x: f(x)=\infty\}$ null-set, $\{x: f(x)>0\}$ σ-finite. **Integration of Complex Functions:** $(X,\mathcal{M},\mu)$, $f=f^+-f^-$ $\therefore\int f\triangleq\int f^+-\int f^-$ if $\int f^+,\int f^-<\infty$ f integrable, $|f|=f^++f^-\Rightarrow f$ integrable iff $\int|f|<\infty$. **Prop:** The set of real-valued functions on X is a $\mathbb{R}$-vector space, and the integral is a linear functional of it. If $E\in\mathcal{M}$, f integrable on E if $\int_E|f|<\infty$. Since $|f|\leq|Re f|+|Im f|\leq 2|f|$, f integrable iff $Re(f)$, $Im(f)$ integrable and we define $\int f=\int Re f+i\int Im f$. $\mathcal{L}^1(X,\mu)$:= Space of Complex Valued integrable functions (Vector Space). **Prop:** $f\in\mathcal{L}^1\Rightarrow|\int f|\leq\int|f|$, **Prop:** (a) If $f\in\mathcal{L}^1$, $\{x: f(x)\neq 0\}$ σ-finite (b) If $f,g\in\mathcal{L}^1$ then $\int_E f=\int_E g$ $\forall E\in\mathcal{M}\Leftrightarrow\int|f-g|=0\Leftrightarrow f=g$ a.e. Redefine $\mathcal{L}^1(\mu)$: Set of equivalence classes of a.e.-defined integrable functions on X, $f\sim g$ iff $f=g$ a.e. If $\overline{\mu}$: μ-completion $\exists$ one-to-one correspondence $\mathcal{L}^1(\overline{\mu})\to\mathcal{L}^1(\mu)$, $\mathcal{L}^1$ a metric space with $\rho(f,g)=\int|f-g|$ $\therefore$ **Convergence in $\mathcal{L}^1$:** $f_n\longrightarrow f$ iff $\int|f_n-f|\longrightarrow 0$. **Dominated-Convergence Theorem:** $\{f_n\}\subset\mathcal{L}^1$ s.t. (a) $f_n\to f$ a.e. (b) $\exists$ nonnegative $g\in\mathcal{L}^1$ s.t. $|f_n|\leq g$ a.e. $\forall n\Longrightarrow f\in\mathcal{L}^1$, $\int f=\lim_{n\to\infty}\int f_n$

Theorem: Suppose $\{f_j\}\subset\mathcal{L}^1$ s.t. $\sum_1^\infty\int|f_j|<\infty$ $\therefore\sum_1^\infty f_j\longrightarrow$ a.e. to a function in $\mathcal{L}^1$, and $\int\sum_1^\infty f_j=\sum_1^\infty\int f_j$ **Theorem:** If $f\in\mathcal{L}^1(\mu)$, $\epsilon>0$, $\exists\phi=\sum a_j\chi_{E_j}$ s.t. $\int|f-\phi|\,d\mu<\epsilon$ (Integrable Simple Functions are dense in $\mathcal{L}^1$ in the $\mathcal{L}^1$ metric). If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, E_j can be finite unions of open intervals and $\exists$ continuous g that vanishes outside a bounded interval s.t. $\int |f-g|\, d\mu < \epsilon$ **Theorem**: Suppose $f: X \times [a,b] \longrightarrow \mathbb{C}$ and $f(\cdot, t): X \longrightarrow \mathbb{C}$ integrable $\forall t \in [a,b]$. Let $F(t) = \int_X f(x,t)\, d\mu(x)$ (a) Suppose $\exists g \in \mathcal{L}^1(\mu)$ s.t. $|f(x,t)| \leq g(x)\ \forall x, t$. If $\lim_{t \to t_0} f(x,t) = f(x,t_0)\ \forall x$, then $\lim_{t\to t_0} F(t) = F(t_0)$; in particular, if $f(x,\cdot)$ continuous $\forall x$ $\Rightarrow$ F continuous (b) Suppose $\frac{\partial f}{\partial t}$ exists and $\exists g \in \mathcal{L}^1(\mu)$ s.t. $\left|\left(\frac{\partial f}{\partial t}\right)(x,t)\right| \leq g(x)$ $\forall x,t$. Then F is differentiable and $F'(x) = \int \left(\frac{\partial x}{\partial t}\right)(x,t)\, d\mu(x)$; **Lebesgue Integral**: μ - Lebesgue, $X = \mathbb{R}$. **Riemann Integral**: Let $[a,b]$ compact

A partition of $[a,b]$: $P = \{t_j\}_0^n$ s.t. $a = t_0 < t_1 < \ldots < t_n = b$. Let f be an arbitrary real-valued function on $[a,b]$, Define $\forall P$: $S_P f \triangleq \sum_1^n M_j (t_j - t_{j-1})$ $s_P f \triangleq \sum_1^n m_i (t_j - t_{j-1})$, M_j : sup f, m_j : inf f on $[t_{j-1}, t_j]$ $\therefore$ $\overline{I}_a^b(f) = \inf_P S_P f$, $\underline{I}_a^b(f) = \sup_P s_P f$. If $\overline{I}_a^b(f) = \underline{I}_a^b(f) = \int_a^b f(x)dx$ (Riemann Integral), f: Riemann Integrable

Theorem: Let f bounded-real-valued on $[a,b]$ (a) If f Riemann Integrable $\Rightarrow f$ Lebesgue Integrable, and $\int_a^b f(x)dx = \int_{[a,b]} f\, dm$ (b) f is Riemann Integrable $\Leftrightarrow \{x \in [a,b] : f \text{ discontinuous at } x\}$ has Lebesgue-measure zero. **Gamma-Function**: Γ. $z \in \mathbb{C}$, $Re(z) > 0$, $f_z : (0,\infty) \to \mathbb{C}$ by $f_z(t) = t^{z-1} e^{-t}$, $t^{z-1} = \exp[(z-1)\log(t)]$, $|t^{z-1}| = t^{Re(z-1)}$, $|f_z(t)| \leq t^{Re\, z-1}$ $t \leq 1$, $|f_z(t)| \leq C_z e^{-t/2}$, $\int_0^1 t^a dt < \infty$ $a > -1$, $\int_1^\infty e^{-t/2} dt < \infty$ $\therefore f_z \in \mathcal{L}^1((0,\infty))$ for $Re(z) > 0$

$\therefore$ $\Gamma(z) \triangleq \int_0^\infty t^{z-1} e^{-t} dt\ (Re(z) > 0)$ ($\mathbb{R}$ with m), $\int_\epsilon^N t^z e^{-t} dt = -t^z e^{-t}\Big|_\epsilon^N + z\int_\epsilon^N t^{z-1} e^{-t} dt$ $\underset{N \to \infty}{\epsilon \to 0}$ $\Gamma(z+1) = z\Gamma(z)$, for $-1 < Re\, z \leq 0$ $\Gamma \triangleq \frac{\Gamma(z+1)}{z}$, $\Gamma(1) = 1 \Rightarrow \Gamma(n+1) = n!$ for $Re(z) > -n$

Modes of Convergence: (i) $f_n = n^{-1}\chi_{(0,n)}$ (ii) $f_n = \chi_{(n,n+1)}$ (iii) $f_n = n\chi_{[0,1/n]}$ (iv) $f_n = \chi_{[j/2^k, (j+1)/2^k)}$ $n = 2^k + j$, $0 \leq j < 2^k$ (i-iii) $f_n \to 0$ uniformly, pointwise, a.e. $f_n \not\to 0 \in \mathcal{L}^1$ $(\int |f_n| = \int f_n = 1\ \forall n)$, (iv) $f_n \to 0$ in $\mathcal{L}^1$ since $\int |f_n| = 2^{-k}$ but $f_n(x) \not\to \forall x \in [0,1]$. $\{f_n\}$ C.V. functions on (X, M, μ) is Cauchy in Measure if $\forall \epsilon > 0$ $\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \xrightarrow[m,n \to \infty]{} 0$ $\{f_n\} \xrightarrow{m} f$ (in measure) if $\forall \epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \xrightarrow[n\to\infty]{} 0$ **Prop**: $f_n \to f$ in $L_1 \Rightarrow f_n \xrightarrow{m} f$

Theorem: Suppose $\{f_n\}$ Cauchy in Measure $\therefore \exists$ measurable f s.t. $f_n \xrightarrow{m} f$ and $\exists \{f_{n_j}\} \subset \{f_n\}$ s.t. $f_{n_j} \to f$ a.e.. Moreover, if also $f_n \xrightarrow{m} g \Rightarrow g = f$ a.e.

Corollary: If $f_n \to f$ in L_1, $\exists \{f_{n_j}\} \subset \{f_n\}$ s.t. $f_{n_j} \to f$ a.e. **Ergoff's Theorem**: Suppose $\mu(X) < \infty$, $\{f_n\}$ C.V. functions on X s.t. $f_n \to f$ a.e. Then $\forall \varepsilon > 0\ \exists E \subset X$ s.t. $\mu(E) < \varepsilon$ and $f_n \rightrightarrows f$ uniformly on E^c (almost uniform convergence $\Longrightarrow$ a.e. convergence, convergence in measure)

Lusin's Theorem: If $f : [a,b] \longrightarrow \mathbb{C}$ Lebesgue measurable and $\epsilon > 0$, $\exists$ compact set $E \subset [a,b]$ s.t. $\mu(E^c) < \epsilon$ and $f|_E$ continuous.

Product Measure: $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$. A measurable rectangle: $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$. Clearly $(A\times B) \cap (E \times F) = (A \cap E) \times (B \cap F)$, $(A \times B)^c = (A\times B)^c = (X \times B^c) \cup (A^c \times B)$, $\mathcal{A} = \{$collection of finite disjoint union of rectangles$\}$ an algebra, the σ-Algebra gen by $(\mathcal{A}) = \mathcal{M} \otimes \mathcal{N}$. Suppose $A \times B = \bigsqcup_1^n (A_j \times B_j)$ $\therefore$ for $x \in X, y \in Y$, $\chi_A(x)\chi_B(y) = \chi_{A\times B}(x,y) = \sum \chi_{A_j \times B_j}(x,y) = \sum \chi_{A_j}(x)\chi_{B_j}(y)$, $\mu(A)\chi_{B_j}(y) = \int \chi_A(x)\chi_B(y)\, d\mu(x) = \sum \int \chi_{A_j}(x)\chi_{B_j}(y)\, d\mu(x) = \sum \mu(A_j)\chi_{B_j}(y)$. Similarly, $\mu(A)\nu(B) = \sum_1 \mu(A_j)\nu(E_j)$ $\therefore$ If $\mathcal{A} \ni E = \bigsqcup_1^n (A_j \times B_j)$, $\pi(E) = \sum_1^n \mu(A_j)\nu(B_j)$ π: premeasure on $\mathcal{A}$ yields product measure: $\mu \times \nu$. Suppose we have $(X_j, \mathcal{M}_j, \mu_j)_1^n$, Rectangle: $A_1 \times \ldots \times A_n$, $A_j \in \mathcal{M}_j$ $\therefore$ product measure $\mu_1 \times \ldots \times \mu_n(A_1 \times \ldots \times A_n) = \prod_1^n \mu_j(A_j)$. $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$. If $E \subset X \times Y$, for $x \in X$, $y \in Y$ x-section: E_x, y-section E^y, $E_x = \{y \in Y : (x,y) \in E\}$, $E^y = \{x \in X : (x,y) \in E\}$, if f a function on $X \times Y$ $f_x(X) = f^y(y) = f(x,y)$ **Prop**: $E \in \mathcal{M} \times \mathcal{N} \Rightarrow E_x \in \mathcal{N}$, $E_y \in \mathcal{M}$ $\forall x, y$. (b) if f is $\mathcal{M} \otimes \mathcal{N}$-measurable, f_x $\mathcal{N}$-measurable, f^y $\mathcal{M}$-measurable $\forall x \forall y$

Monotone Class: on X, subset $\mathcal{C} \subset \mathcal{P}(X)$ s.t. if $E_j \in \mathcal{C}$ and $E_1 \subset \ldots \subset E_j \therefore \bigcup E_j \in \mathcal{C}$, if $E_1 \supset \ldots \supset E_n \therefore \bigcap E_j \in \mathcal{C}$. Every σ-Algebra is a monotone class $\therefore \forall \mathcal{E} \subset \mathcal{P}(x)\ \exists!$ smallest $\mathcal{C} \supset \mathcal{E}$: Monotone Class gen$(\mathcal{E})$. **Monotone Class Lemma**: If $\mathcal{A}$ an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-Algebra $\mathcal{M}$ gen. by $\mathcal{A}$. **Theorem**: Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \mu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$ are measurable on X and Y. and $\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$. **Fubini-Tonelli Theorem**: Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ σ-finite measure space

(a) **Tonelli**: If $f \in \mathcal{L}^+(X \times Y)$, then the functions $g(x) = \int f_x\, d\nu \in \mathcal{L}^+(X)$, $h(y) = \int f^y d\mu \in \mathcal{L}^+(Y)$ and $\int f\, d(\mu \times \nu) = \int \left[\int f(x,y)\, d\nu(y)\right] d\mu(x) = \int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y)$ (b) **Fubini's**: If $f \in \mathcal{L}^1(\mu \times \nu)$, then $f_x \in \mathcal{L}^1(\nu)$ for a.e. $x \in X$, $f^y \in \mathcal{L}^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x\, d\nu \in \mathcal{L}^1(\mu)$, $h(x) = \int f^y d\mu \in \mathcal{L}^1(\nu)$. **Fubini-Tonelli (Complete Measure)**: Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{M}, \nu)$ complete-σ-finite, Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in \mathcal{L}^1(\lambda)$, then f_x $\mathcal{N}$-measurable for a.e. x f^y $\mathcal{M}$-measurable for a.e. y and if (b): f_x, f^y are also integrable for a.e. x, y. Moreover, $x \mapsto \int f_x\, d\nu$, $y \mapsto \int f^y d\mu$ measurable, and in case (b) also integrable, and $\int f\, d\lambda = \iint f(x,y)\, d\mu(x)\, d\nu(y) = \iint f(x,y)\, d\nu(y)\, d\mu(x)$. **The n-Dimensional Lebesgue Integral**:

Lebesgue Measure m^n on $\mathbb{R}^n$: completion of $m \times \ldots \times m$ on $\mathcal{B}_{\mathbb{R}} \otimes \ldots \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n} \Longleftrightarrow$ completion of $m \times \ldots \times m$ on $\mathcal{L} \otimes \ldots \otimes \mathcal{L}$, $E = \prod_1^n E_j$ is a rectangle $\subset \mathbb{R}^n$, $E_j \subset \mathbb{R}$ side **Theorem**: $E \in \mathcal{L}^n$ (a) $m(E) = \inf\{m(U) : U \supset E, U \text{ open}\} = \sup\{m(K) : K \subset E, K \text{ compact}\}$ (b) $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1 an F_σ set, A_2 a G_δ set, $m(N_1) = m(N_2) = 0$ (c) If $m(E) < \infty$, $\forall \varepsilon > 0\ \exists \{R_j\}_1^N$ disjoint rectangles whose sides are intervals s.t. $m(E \Delta \bigcup_1^N R_j) < \epsilon$. By definition of product measure if $E \in \mathcal{L}^n$, $\epsilon > 0$ $\exists \{T_j\}$ rectangles s.t. $E \subset \bigcup_1^\infty T_j$ and $\sum_1^\infty m(T_j) \leq m(E) + \varepsilon$. **Theorem**: $f \in \mathcal{L}^1(m)$, $\epsilon > 0 \therefore \exists$ simple function $\phi = \sum_1^N a_j \chi_{R_j}$, each R_j = product of intervals s.t. $\int |f - \phi| < \epsilon$

and $\exists$ continuous g that vanishes outside a bounded set s.t. $\int|f-g|<\varepsilon$. **Theorem:** Lebesgue measure is Translation Invariant (invariant): For $a\in\mathbb{R}^n$ define $\tau_a:\mathbb{R}^n\to\mathbb{R}^n$, $\tau_a(x)=x+a$. (a) If $E\in\mathcal{L}^n$, then $\tau_a(E)\in\mathcal{L}^n$ and $m(\tau_a(E))=m(E)$ (b) $f:\mathbb{R}^n\to\mathbb{C}$ is Lebesgue measurable, then so is $f\circ\tau_a$. Moreover if $f\geq 0$ or $f\in\mathcal{L}^1(m)$, then $\int(f\circ\tau_a)\,dm=\int f\,dm$. A cube $\subset\mathbb{R}^n$ is a rectangle with all equal sides. For $k\in\mathbb{Z}$, $\mathcal{Q}_k=\{\text{cubes with side } 2^{-k}\}$ i.e. $\prod_1^n[a_j,b_j]\in\mathcal{Q}_k$ iff $2^k a_j \wedge 2^k b_j$ are integers and $b_j-a_j=2^{-k}\ \forall j$. If $E\subset\mathbb{R}^n$, $\underline{A}(E,k)=\bigcup\{Q\in\mathcal{Q}_k : Q\subset E\}$, $\bar{A}(E,k)=\bigcup\{Q\in\mathcal{Q}_k : Q\cap E\neq\emptyset\}$, $\underline{\kappa}(E)=\lim_{k\to\infty}m(\underline{A}(E,k))\nearrow$ (inner Content) $\bar{\kappa}(E)=\lim_{k\to\infty}m(\bar{A}(E,k))\searrow$ (outer content). If $\bar{\kappa}=\underline{\kappa}$ $\therefore$ $\kappa(E)=\bar{\kappa}=\underline{\kappa}$ Jordan Content. Let $\underline{A}(E)=\bigcup_1^\infty\underline{A}(E,k)$, and $\bar{A}(E)=\bigcap_1^\infty\bar{A}(E,k)$ $\therefore$ $\underline{A}(E)\subset E\subset\bar{A}(E)$, $\underline{A}$, $\bar{A}$ Borel sets, $\underline{\kappa}(E)=m(\underline{A}(E))$, $\bar{\kappa}(E)=m(\bar{A}(E))$ $\therefore$ Jordan Content of E exists iff $m(\bar{A}(E)\setminus\underline{A}(E))=0 \Rightarrow E$ Lebesgue Measurable and $m(E)=\kappa(E)$. **Lemma:** $U\subset\mathbb{R}^n$ open, then $U=\underline{A}(U)$. Moreover, U is a countable union of cubes with disjoint interiors. **Lemma:** $U\subset\mathbb{R}^n$ open, $F\subset\mathbb{R}^n$ compact $\therefore$ $m(U)=$ inner content, $m(F)=$ outer content. **Linear Transformations:** $T:\mathbb{R}^n\to\mathbb{R}^n$, $(T_{ij})=(e_i\cdot Te_j)$, $\{e_j\}$ standard-basis of $\mathbb{R}^n$, $\det(T\circ S)=\det(T)\det(S)$ $\therefore$ $GL(n,\mathbb{R})$ general-linear group: group of invertible transformations of $\mathbb{R}^n$, $\forall T\in GL(n,\mathbb{R})$, $T=$ finitely many transformations $T_1(x_1,\dots,x_j,\dots,x_n)=(x_1,\dots,cx_j,\dots,x_n)$, $T_2(x_1,\dots,x_j,\dots,x_n)=(x_1,\dots,x_j+cx_k,\dots,x_n)$, $T_3=(x_1,\dots,x_j,\dots,x_k,\dots,x_n)=(x_1,\dots,x_k,\dots,x_j,\dots,x_n)$. **Theorem:** Suppose $T\in GL(n,\mathbb{R})$, (a) if f Lebesgue Measurable on $\mathbb{R}^n$, so is $f\circ T$. If $f\geq 0$ or $f\in\mathcal{L}^1(m)\Rightarrow\int f(x)\,dx=|\det T|\int f\circ T(x)\,dx$ (b) If $E\in\mathcal{L}^n$, then $T(E)\in\mathcal{L}^n$ and $m(T(E))=|\det T|\,m(E)$. **Corollary:** Lebesgue measure is invariant under rotations. Let $G=(g_1,\dots,g_n)$ $G:\Omega\overset{(\text{open})}{\subset}\mathbb{R}^n\to\mathbb{R}^n$, g_j of class C^1 $\therefore$ D_xG defined by $\left(\left(\frac{\partial g_i}{\partial x_j}\right)(x)\right)$, G: C^1 = diffeomorphism if G injective and D_xG invertible $\forall x\in\Omega$. **Theorem:** Suppose $\overset{\text{open}}{\Omega}\subset\mathbb{R}^n$, $G:\Omega\to\mathbb{R}^n$ a C^1-diffeomorphism (a) If f Lebesgue-measurable on $G(\Omega)$, then $f\circ G$ Lebesgue-measurable on Ω. If $f\geq 0$ or $f\in\mathcal{L}^1(G(\Omega),m)$ then $\int_{G(\Omega)}f(x)\,dx=\int_\Omega f\circ G(x)\,|\det D_xG|\,dx$ (b) If $E\subset\Omega$, $E\in\mathcal{L}^n$, then $G(E)\in\mathcal{L}^n$ and $m(G(E))=\int_E|\det D_xG|\,dx$. **Polar Coordinates:** Unit Sphere: $\{x\in\mathbb{R}^n : |x|=1\}:=S^{n-1}$. If $x\in\mathbb{R}^n\setminus\{0\}$ the polar coordinates of x are $r=|x|\in[0,\infty)$, $x'=\frac{x}{|x|}\in S^{n-1}$, $\Phi:\mathbb{R}^n\setminus\{0\}\to(0,\infty)\times S^{n-1}$, $\Phi(x)=(r,x')$ $\Phi^{-1}(r,x')=rx'$ (continuous bijection). denote $m_*(E)=m(\Phi^{-1}(E))$ (Borel measure on $(0,\infty)\times S^{n-1}$; define measure $\rho=\rho_n$ on $(0,\infty)$, $\rho(E)=\int_E r^{n-1}\,dr$ **Theorem:** $\exists!$ Borel Measure $\sigma=\sigma_{n-1}$ on S^{n-1} s.t. $m_*=\rho\times\sigma$. If f is Borel-Measurable on $\mathbb{R}^n$, $f\geq 0$ or $f\in\mathcal{L}^1(m)$ then $\int_{\mathbb{R}^n}f(x)\,dx=\int_0^\infty\int_{S^{n-1}}f(rx')r^{n-1}\,d\sigma(x')$ **Corollary:** f measurable on $\mathbb{R}^n$, nonnegative or integrable s.t. $f(x)=g(|x|)$ for some g on $(0,\infty)$ then $\int f(x)\,dx=\sigma(S^{n-1})\int_0^\infty g(r)r^{n-1}\,dr$. **Corollary:** Let $c, C>0$, $B=\{x\in\mathbb{R}^n : |x|<c\}$. Suppose f measurable on $\mathbb{R}^n$ $\therefore$ a) If $|f(x)|\leq C|x|^{-a}$ on B for some $a<n$, then $f\in\mathcal{L}^1(B)$. However if $|f(x)|\geq C|x|^{-n}$ on $B\Rightarrow f\notin\mathcal{L}^1(B)$ (b) If $|f(x)|\leq C|x|^{-a}$ on B^c for some $a>n$, then $f\in\mathcal{L}^1(B^c)$. However, if $|f(x)|\geq C|x|^{-n}$ on $B^c\Rightarrow f\notin\mathcal{L}^1(B^c)$. **Prop:** $a>0$, $\int_{\mathbb{R}^n}\exp(-a|x|^2)\,dx=\left(\frac{\pi}{a}\right)^{n/2}$, $\sigma(S^{n-1})=\frac{2\pi^{n/2}}{\Gamma(n/2)}$, If $B^n=\{x\in\mathbb{R}^n : |x|=1\}\Rightarrow m(B^n)=\frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)}$, $\Gamma(n+\frac{1}{2})=(n-\frac{1}{2})(n-\frac{3}{2})\dots(\frac{1}{2})\Gamma(\frac{1}{2})$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$. **Signed Measures:** Let $(X,\mathcal{M})$. A signed measure on $(X,\mathcal{M})$ is a function $\nu:\mathcal{M}\to[-\infty,\infty]$ s.t. $\nu(\emptyset)=0$, ν assumes at most one of $\{\pm\infty\}$, If $\{E_j\}\subset\mathcal{M}$ disjoint sets, $\nu(\bigcup_1^\infty E_j)=\sum_1^\infty\nu(E_j)$, if $\nu(\bigcup_1^\infty E_j)<\infty$, $\sum_1^\infty|\nu(E_j)|<\infty$ **Prop:** ν signed measure on $(X,\mathcal{M})$. If $\{E_j\}\subset\mathcal{M}$ increasing $\Rightarrow\nu(\bigcup_1^\infty E_j)=\lim_{j\to\infty}\nu(E_j)$. If $\{E_j\}\subset\mathcal{M}$ decreasing $\Rightarrow\nu(\bigcap_1^\infty E_j)=\lim_{j\to\infty}\nu(E_j)$ $E\in\mathcal{M}$ positive for ν if $\nu(F)\geq 0\ \forall F\subset E$, negative if $\nu(F)\leq 0$, null if $\nu(F)=0$). **Lemma:** Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive. **Hahn Decomposition Theorem:** If ν signed on $(X,\mathcal{M})$ $\exists P$(positive), $\exists N$(negative) for ν s.t. $P\cup N=X$, $P\cap N=\emptyset$, If (P',N') another pair $\Rightarrow P\Delta P'=N\Delta N'$ null for ν. The decomposition $X=P\cup N$: Hahn decomposition for ν. Two signed measures μ,ν mutually singular if $\exists E,F\in\mathcal{M}$ s.t. $E\cap F=\emptyset$, $E\cup F=X$, E null for μ, F null for ν: $\mu\perp\nu$ **Jordan Decomposition Theorem:** ν signed-measure, $\exists!$ positive measures ν^+,ν^- s.t $\nu=\nu^+-\nu^-$ and $\nu^+\perp\nu^-$, $\nu=\nu^+-\nu^-$ Jordan, ν^+,ν^- (positive, negative) variations, total variation $|\nu|=\nu^++\nu^-$, $E\in\mathcal{M}$ ν-null iff $|\nu|(E)=0$, $\nu\perp\mu\Leftrightarrow|\nu|\perp\mu\Leftrightarrow\nu^+\perp\mu$ and $\nu^-\perp\mu$. ν is of the form $\nu(E)=\int_E f\,d\mu$ where $\mu=|\nu|$ and $f=\chi_P-\chi_N$, $X=P\cup N$, $\mathcal{L}^1(\nu)=\mathcal{L}^1(\nu^+)\cap\mathcal{L}^1(\nu^-)$ and $\int f\,d\nu=\int f\,d\nu^+-\int f\,d\nu^-$. **Lebesgue-Radon-Nikodym Theorem:** Suppose ν signed-measure, μ positive-measure on $(X,\mathcal{M})$ ν is absolutely continuous w.r.t. μ $(\nu\ll\mu)$ if $\nu(E)=0\ \forall E\in\mathcal{M}$ s.t. $\mu(E)=0$, $\nu\ll\mu$ iff $|\nu|\ll\mu$ iff $\nu^+,\nu^-\ll\mu$ **Theorem:** Let ν signed-measure, μ positive-measure on $(X,\mathcal{M})$ then $\nu\ll\mu\Leftrightarrow\forall\varepsilon>0\ \exists\delta>0$ s.t. $|\nu(E)|<\varepsilon$ whenever $\mu(E)<\delta$. **Corollary:** If $f\in\mathcal{L}^1(\mu)$, $\forall\varepsilon>0\ \exists\delta>0$ s.t. $|\int_E f\,d\mu|<\varepsilon$ whenever $\mu(E)<\delta$, $\nu(E)=\int_E f\,d\mu\ll\mu$ $\therefore$ we express this relationship as $d\nu=f\,d\mu$. **Lemma:** Suppose ν,μ finite-measures on $(X,\mathcal{M})$. Either $\nu\perp\mu$, or $\exists\varepsilon>0$ and $E\in\mathcal{M}$ s.t. $\mu(E)>0$ and $\nu>\varepsilon\mu$ on E (i.e. E positive for $\nu-\varepsilon\mu$). **Lebesgue-Radykon-Nikodym:** Let ν σ-finite-signed-measure and

1 If μ measure on $\mathcal{M}$ and $f: X \longrightarrow [-\infty, \infty]$ measurable s.t. at least one of $\int f^+ d\mu$, $\int f^- d\mu$ is finite, f: extended μ-integrable, (signed) $\nu(E) := \int_E f \, d\mu$

μ σ-finite-positive measure on $(X, \mathcal{M})$. $\exists!$ σ-finite-signed measures λ, ρ on $(X, \mathcal{M})$ s.t. $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$. Moreover, $\exists$ extended-μ-integrable function $f: X \longrightarrow \mathbb{R}$ s.t. $d\rho = f \, d\mu$, and any two such functions are equal a.e. The decomposition $\nu = \lambda + \rho$ where $\lambda \perp \mu$, $\rho \ll \mu$: Lebesgue Decomposition of ν w.r.t. μ. $\Longrightarrow$ Radon-Nikodym: In the case where $\nu \ll \mu$, $d\nu = f \, d\mu$ for some f, and f is called Radon-Nikodym derivative of ν w.r.t. μ, denoted by $f = \frac{d\nu}{d\mu}$ $\therefore$ $d\nu = \frac{d\nu}{d\mu} d\mu$. Prop: Suppose ν σ-finite-signed measure, μ, λ σ-finite measures on $(X, \mathcal{M})$ s.t. $\nu \ll \mu$, $\mu \ll \lambda$ (a) If $g \in \mathcal{L}^1(\nu)$ then $g\left(\frac{d\nu}{d\mu}\right) \in \mathcal{L}^1(\mu)$ and $\int g \, d\nu = \int g \frac{d\nu}{d\mu} d\mu$ (b) We have $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ-a.e. Corollary: If $\mu \ll \lambda$, $\lambda \ll \mu$, $\Longrightarrow \left(\frac{d\lambda}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) = 1$ a.e. w.r.t. (λ or μ). Prop: If $\mu_1, \dots, \mu_n$ measures on $(X, \mathcal{M})$, $\exists$ measure μ s.t. $\mu_j \ll \mu$ $\forall j$, namely, $\mu = \sum_1^n \mu_j$. Complex Measure: A complex measure on $(X, \mathcal{M})$, $\nu: \mathcal{M} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$, if $\{E_j\} \subset \mathcal{M}$ disjoint $\therefore \nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$. $\nu = \nu_r + i\nu_i$ where ν_r, ν_i signed measures that don't assume $\pm\infty$, $\mathcal{L}^1(\nu) = \mathcal{L}^1(\nu_r) \cap \mathcal{L}^1(\nu_i)$ for $f \in \mathcal{L}^1(\nu)$, $\int f \, d\nu = \int f \, d\nu_r + i \int f \, d\nu_i$. Differentiation on Euclidean Space: $(X, \mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, $\mu = m$. ν complex measure w.r.t. $\mu = m$. Let $B(r,x)$ open ball $\subset \mathbb{R}^n$ $\therefore$ consider $F(x) = \lim_{r \to 0} \frac{\nu(B(r,x))}{m(B(r,x))}$ If $\nu \ll m$ s.t. $d\nu = f \, dm$ then $\nu(B(r,x))/m(B(r,x))$: average-value of f on $B(r,x)$. Lemma: $\mathcal{C}$ a collection of open balls in $\mathbb{R}^n$, let $U = \bigcup_{B \in \mathcal{C}} B$. If $c < m(U)$, $\exists$ disjoint $B_1, \dots, B_k \in \mathcal{C}$ s.t. $\sum_1^k m(B_j) > 3^{-n} c$. A measurable function $f: \mathbb{R}^n \to \mathbb{C}$ is called locally integrable (w.r.t. Lebesgue measure) if $\int_K |f(x)| \, dx < \infty$ $\forall$ bounded measurable set $K \subset \mathbb{R}^n$. $\mathcal{L}^1_{loc}$: Space of locally integrable functions, if $f \in \mathcal{L}^1_{loc}$, $x \in \mathbb{R}^n$, $r > 0$, the average value of f on $B(r,x)$: $A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) \, dy$. Lemma: If $f \in \mathcal{L}^1_{loc}$, $A_r f(x)$ is jointly continuous in r and x ($r > 0$, $x \in \mathbb{R}^n$). If $f \in \mathcal{L}^1_{loc}$, Hardy-Littlewood Maximal Function: $Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \, dy$. The Maximal Theorem: $\exists$ constant $C > 0$ s.t. $\forall f \in \mathcal{L}^1$, $\forall \alpha > 0$, $m(\{x : Hf(x) > \alpha\}) \le \frac{C}{\alpha} \int |f(x)| \, dx$ $\limsup_{r \to R} \phi(r) = \lim_{\epsilon \to 0} \sup_{0 < |r - R| < \epsilon} \phi(r) = \inf_{\epsilon > 0} \sup_{0 < |r-R| < \epsilon} \phi(r)$, $\lim_{r \to R} \phi(r) = c \iff \limsup_{r \to R} |\phi(r) - c| = 0$. Theorem: If $f \in \mathcal{L}^1_{loc}$ then $\lim$ $\lim_{r \to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$. Lebesgue set of $f \in \mathcal{L}^1_{loc}$: $L_f = \left\{ x : \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| \, dy = 0 \right\}$. Theorem: $f \in \mathcal{L}^1_{loc}$ $\Longrightarrow m((L_f)^c) = 0$. A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to shrink nicely to $x \in \mathbb{R}^n$ if $E_r \subset B(r,x)$ $\forall r$. and $\exists \alpha > 0$, $\alpha \perp\!\!\!\perp r$ s.t. $m(E_r) > \alpha m(B(r,x))$. Lebesgue Differentiation theorem: $f \in \mathcal{L}^1_{loc}$. $\forall$ almost every $x \in L_f$ we have $\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| \, dy = 0$ and $\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, dy = f(x)$ $\forall$ family $\{E_r\}_{r>0}$ that shrinks nicely to x. A Borel measure ν on $\mathbb{R}^n$ is called regular if $\nu(K) < \infty$ $\forall$ compact K, $\nu(E) = \inf\{\nu(U) : U \text{ open}, E \subset U\}$ $\forall E \in \mathbb{R}^n$. A signed or complex Borel measure ν regular if $|\nu|$ is regular. e.g. $f \in \mathcal{L}^+(\mathbb{R}^n)$, measure $f \, dm$ regular $\iff f \in \mathcal{L}^1_{loc}$. $\nu(E) = \inf\{\nu(U) : U \text{ open}$ $E \subset U\}$ $\forall E \in \mathbb{R}^n$: Suppose E bounded Borel set. Given $\delta > 0$ $\exists$ bounded open $U \supset E$ s.t. $m(U) < m(E) + \delta$ and $m(U \setminus E) < \delta$ $\therefore$ given $\varepsilon > 0$ $\exists$ open $U \supset E$ s.t. $\int_{U \setminus E} f \, dm < \varepsilon$ $\therefore$ $\int_U f \, dm < \int_E f \, dm + \varepsilon$. If E unbounded. Let $E = \bigcup_1^\infty E_j$ where E_j bounded and find $U_j \supset E_j$ s.t. $\int_{U_j \setminus E_j} f \, dm < \varepsilon 2^{-j}$. Theorem: Let ν be a regular signed-complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f \, dm$ be its Lebesgue-Radon-Nikodym representation. Then, for m-almost every $x \in \mathbb{R}^n$, $\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ $\forall$ family $\{E_r\}_{r>0}$ that shrinks nicely to x. (a) Any nonempty $E \subset \mathbb{R}$ with an upper bound has a least-upper bound $\iff$ $\forall \{a_j\}_{n \in \mathbb{N}} \subset \mathbb{R}$ monotone and bounded converges $\iff$ $\forall \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$ bounded $\exists$ convergent $\{a_{j_k}\} \subset \{a_j\}$ $\iff$ $F \subset \mathbb{R}$ closed (continuous) and bounded, $\forall$ open cover $\mathcal{F}$, $\exists$ finite subcover $\iff$ Any Cauchy Sequence in $\mathbb{R}$ converges. Theorem: $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n \longrightarrow f$ $\therefore$ f continuous. Theorem: If f continuous over $F \subset \mathbb{R}$ closed-bounded $\therefore$ f attains its maximum and minimum. Prop: A countable set $\therefore$ the set of all finite sequences from A, $X \equiv \{(a_1 a_2 a_3 \dots a_n) : a_i \in A, \text{ fixed } n \ge 1\}$ is uncountable. Theorem. Let $p > 1 \in \mathbb{N}$, $x \in (0,1)$ $\exists$ $\{a_n\} \subset \mathbb{Z}$ with $0 \le a_n < p$ s.t. $x = \sum_{n=1}^\infty \frac{a_n}{p^n}$. Theorem: The set of infinite sequences from $\{0, 1\}$: $E\{(a_1 a_2 a_3 \dots) : a_i \in \{0, 1\}\}$ not countable. Theorem: X: set of all continuous f over $[0,1]$. Define $\rho_1(f,g) \equiv: \int_0^1 |f(x) - g(x)| \, dx$ (i) (X, ρ_1) metric space, (X, ρ_1) not complete. Theorem: $E \subset (X, \rho)$ metric space: TFAE (i) E complete and totally bounded (ii) $\forall \{a_n\} \subset E$ $\exists \{a_{n_k}\} \longrightarrow x \in E$. (iii) If $\{V_\alpha\}_{\alpha \in A}$ open covering of E, $\exists$ finite $\mathcal{F} \subset A$ s.t. $\{V_\alpha\}_{\alpha \in \mathcal{F}}$ cover E. Theorem: An Algebra $\mathcal{A}$ is σ $\iff$ $\{E_j\}_1^\infty \subset \mathcal{A}$ and

$E_1 \subset E_2 \subset E_3 \dots$ then $\bigcup_1^\infty E_j \in \mathcal{A}$. Theorem: $\mathcal{B}_{\mathbb{R}}$ generated by $\{(a,b] : a<b\}$ or $\{[a,b) : a<b\}$, $\{[a,\infty) : a\in\mathbb{R}\}$ or $\{(-\infty,a)\}$ Prop: $(X,\mathcal{M},\mu)$, $E,F\in\mathcal{M}$, then $\mu(E)+\mu(F)=\mu(E\cup F)+\mu(E\cap F)$ Theorem: $(X,\mathcal{M},\mu)$, $\{E_j\}_1^\infty \subset \mathcal{M} \Rightarrow$ $\mu(\liminf E_j) \le \liminf \mu(E_j)$, if $\mu(\bigcup_1^\infty E_j)<\infty \Rightarrow \mu(\limsup E_j) \ge \limsup \mu(E_j)$ Prop: Let μ^* an outer measure Let $\{E_k\}_1^\infty$ s.t. $\sum_1^\infty \mu^*(E_k)<\infty$, then $\mu^*(\limsup E_k)=0$. Prop: μ Lebesgue measure in $\mathbb{R}$, $\mu(E)=0 \Rightarrow \mu(\{x^2 : x\in E\})=0$ Theorem: (Construction of Lebesgue m) : n-dimensional open intervals $I=\{x : a_j < x_k < b_j, j=1,\dots,n\}$ with volume $\rho(I)=\prod_1^n (b_j-a_j)$. By constructing an outer measure μ^* and Caratheodory Procedure: $\mu^*(I)=\rho(I)$, I is measurable μ^* is the same if we choose closed cubes with length $<$ fixed $\varepsilon>0$ Theorem: (X,ρ) metric space, Define $\mathcal{H}_\alpha^\varepsilon(A) = \inf_{\substack{A\subset \bigcup_1^\infty A_k \\ \operatorname{diam} A_k \le \varepsilon}} \sum_k \{\operatorname{diam} A_k\}^\alpha$ an outer measure, $\operatorname{diam} A = \sup_{x,y\in A} \rho(x,y)$. Moreover $\mu^*(A)=\lim_{\varepsilon\to 0} \mathcal{H}_\alpha^\varepsilon$ is an outer measure. (Hausdorff Measure)

Theorem: (Lebesgue-Stieltjes) Let f monotone increasing function ($f(x)\le f(y)\ \forall x\le y$). Define $\rho((a,b])=f(b)-f(a)$. Define $\mu^*(A)$: $\mu^*(A)=\inf\{\sum \rho((a,b]) : A\subset \bigcup_1^\infty (a_k,b_k]\}$ is an outer measure and the μ from Caratheodory construction is called Lebesgue-Stieltjes Measure. Prop: The characteristic function $\chi_E = \mathbb{1}_E$ measurable $\Leftrightarrow$ E measurable. Theorem: $\{f_n\}$ measurable functions on X then $\{x : \lim f_n(x)\}$ is a measurable set. Theorem: Let $E\subset\mathbb{R}$ Lebesgue Measurable, $\mu(E)>0 \therefore \forall \alpha<1\ \exists$ open interval I_α s.t. $\mu(E\cap I_\alpha)>\alpha\mu(I_\alpha)$. Theorem: $f\ge 0$ measurable function on $(X,\mathcal{A},\mu)$, f integrable $\Leftrightarrow \sum_{n=-\infty}^{+\infty} 2^n \mu(\{x : f(x)>2^n\})<+\infty$ Prop: Define $f\in\mathcal{L}^p(X,\mathcal{A},\mu)$ if $\int |f|^p d\mu<+\infty$, $1\le p\le\infty$. If $\mu(X)<\infty \Rightarrow f\in\mathcal{L}^q(X,\mathcal{A},\mu)\ \forall 1\le q\le p$. Theorem: $\mu(X)<+\infty$ $f_n\to f$ a.e. $\therefore$ Assume $\forall\varepsilon>0\ \exists\delta>0$ s.t. $\sup_n \sup_{E:\mu(E)<\delta} \int_E |f_n|\, d\mu<\varepsilon \Rightarrow \int |f_n-f|\, d\mu \xrightarrow{n\to\infty} 0$. Theorem: Assume $f_n\to f$ a.e. then if $\lim_{n\to\infty}\int|f_n|\to\int|f|\,d\mu \Rightarrow \lim_{n\to\infty}\int|f_n-f|\,d\mu\to 0$. Theorem: Let $f\ge 0$ measurable and define its distribution function for $\lambda\ge 0$ $d_f(\lambda)=\mu(x : |f(x)|>\lambda) \therefore \int|f|\,d\mu=\int_0^\infty d_f(\lambda)\,d\lambda$. Theorem: $f_n \xrightarrow{m} f \therefore \liminf_n \int|f_n|\,d\mu \ge \int|f|\,d\mu$, moreover, if $|f_n|\le g\in\mathcal{L}^1 \therefore f_n\to f$ in $\mathcal{L}^1$ Theorem: Let $f_n\ge 0\ \forall n$, $\lim_{n\to\infty} f_n\to f$ a.e., $\lim_n \int_X f_n=\int_X f \therefore \forall E\subset X$, $\lim_{n\to\infty}\int_E f_n=\int_E f$. (Riemann-Lebesgue) Theorem: $f\in\mathcal{L}^1(\mathbb{R}^d) \Rightarrow \lim_{n\to\infty}\int f(x)\cos(nx)\,dx=0$ Theorem: $f\neq 0\in\mathcal{L}^1(\mathbb{R}^d)$, $\exists R_f>0$ s.t. $\sup_{r>0} \frac{1}{m(B(r,x))}\int_{B(r,x)}|f(y)|\,dy \ge C|x|^{-d}$ for $|x|>R_f$. Theorem: $f_n\xrightarrow{m} f$ $g_n\xrightarrow{m} g \Rightarrow (f_n+g_n)\xrightarrow{m}(f+g)$, $f_n g_n\xrightarrow{m} fg$ if $\mu(X)<\infty$. Theorem: $f\in\mathcal{L}^1(\mathbb{R}^d)$, Lebesgue Integrable $\forall\varepsilon>0\ \exists\delta>0$ s.t. if $|h|<\delta$ $\Rightarrow \int_{\mathbb{R}}|f(x+h)-f(x)|\,dx<\varepsilon$. Theorem: $f\in\mathcal{L}^1(\mathbb{R})$ Lebesgue Integrable $\therefore F(t):=\int_{-\infty}^x f(t)\,dt$ continuous. Theorem: ν signed measure on $(X,\mathcal{M})$, $E\in\mathcal{M} \therefore \nu^+(E)=\sup\{\nu(F) : F\in\mathcal{M}, F\subset E\}$, $\nu^-(E)=-\inf\{\nu(F) : F\in\mathcal{M}, F\subset E\}$, $|\nu|(E)=\sup\{\sum_1^\infty |\nu(E_j)| : n\in\mathbb{N}, \bigsqcup_1^n E_j=E\}$. Theorem: Assume μ positive measure, $f_n\to f$ in $\mathcal{L}^1(\mu)$. $\Rightarrow$ $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $|\int_E f_n\,d\mu|<\varepsilon\ \forall n$ and any $\mu(E)<\delta$.