MATHEMATICS

Fourier Analysis and Laplace's Equation

What is the importance of Fourier Analysis when deriving and analyzing a complete solution space of Laplace's Equation in two dimensions with boundary conditions.

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Contents

1 Introduction

This extended essay will analyze the solution space of Laplace´s equation and will do it using linear algebra, calculus and real analysis. In particular, the solution to Laplace´s equation in two dimensions will be studied and derived.

The solutions to Laplace's equation, are harmonic functions, extensively used in physics that model the behavior of electrical, gravitational and magnetic potentials and hydrodynamics. Harmonic series are a concept used in different disciplines, such as music, physics and acoustics. These series help model periodic signals in form of sinusoidal waves. In mathematics, a so-called harmonic function is a real multivariable differentiable function, this means functions of the form $u : \mathbb{R}^n \to \mathbb{R}$ that satisfy Laplace´s equation in a domain $O \subseteq \mathbb{R}^n$,

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} = 0 \tag{1}$$

also known as,

$$\nabla^2 u = 0$$

where ∇^2 is the Laplace differential operator, going by the name of the Laplacian. At every point of the domain O the Laplacian of the function is equal to zero.

On the other hand, Fourier Analysis expands functions on closed bounded intervals in terms of series of harmonic functions, such as the cosine and sine functions. Harmonic analysis deals with functions or signals and the superposition of waves, where these periodic functions can be expressed as the infinite series of sines and cosines.

2 Linear Algebra

Linear algebra is the area of mathematics that allows us to derive the properties of linear spaces and linear transformations. Notwithstanding, we will us this properties to create a thorough understanding of the complex mathematics underlying Fourier Analysis which will be introduced in one of the upcoming sections.

2.1 Vector Spaces

Vectors are defined as the elements in a vector space V on a field F on which the addition and the scalar multiplication are defined for every element x, y, and z. The elements of a field are called scalars and they take place on any field such as $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$.

For V to be a vector space, the elements $\vec{x}, \vec{y}, \vec{z} \in V$ and the scalars $\alpha \in F$ must be closed under addition and scalar multiplication. In other words, for any given vectors $\vec{x}$, $\vec{y}$,$\vec{z}$ and any scalars α and β, the following axioms hold: [1]

1. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
2. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
3. There is a unique zero element in V such that $\vec{x} + 0 = \vec{y} + 0 = \vec{z} + 0 = 0$
4. For every $\vec{x}$ there is a vector $\vec{y}$ such that $\vec{x} + \vec{y} = 0$
5. For each $\vec{x}$ in V $1\vec{x} = \vec{x}$, where 1 is the unit in F
6. $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x})$
7. $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$
8. $(\alpha + \beta)x = \alpha\vec{x} + \beta\vec{y}$.

[1] Friedberg, S., Insel, A., & Spence, L. (1979). Linear algebra (pp. 1-60, 231-250, 321-330, 379-400). Englewood Cliffs, N.J.: Prentice-Hall.

For example lets prove that the solution space of an ordinary differential homogeneous equation is a vector space, and thus that it holds the 8 axioms.

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$$

Let the zeros of this differential equation be $A = \{\varphi : \varphi \text{ is a solution }\}$ with $\varphi_1, \varphi_2 \in A$ where A is a vector space. Hence A must be closed under addition and the scalar multiplication. This is:

$$\frac{d}{dx}(\varphi_1 + \varphi_2) = (\frac{d^2\varphi_1}{dx^2} - 7\frac{d\varphi_1}{dx} + 2\varphi_1) + (\frac{d^2\varphi_2}{dx^2} - 7\frac{d\varphi_2}{dx} + 2\varphi_2) = 0.$$

Similarly, the remaining axioms can be proved and we have that

$\varphi_1 + \varphi_2 \in A$

$\alpha\varphi_1 \in A$

$\beta\varphi_2 \in A$

$0 \in A$.

We infer that if the differential equation is linear and homogeneous, then its solution space is vectorial.

2.2 Linear dependence and Independence

In order to generate V it is optimal to find a finite, or in other cases infinite, set $S :=\{v_1, v_2, v_3 ... v_n\} \subset V$ with the smallest number of vectors that generate V. The linear combination of vectors in S it is said to be the linear span of V. S is considered to be linearly independent if,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_n v_n = 0$$

implies,

$$\alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_n = 0$$

Henceforth, a set of vectors is said to be linearly dependent if there exist at least one scalar different than zero such that the linear combination is equal to 0.

2.3 Bases and Dimension

Linearly independent set of vectors are the building blocks of vector spaces. A basis β is a subset of vectors in V such that its vector´s linear combinations generate V, that is, if each vector $v \in V$ can be expressed as a unique linear combination of vectors in $\beta = \{u_1, u_2, u_3, ..., u_n\}$. This is,

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + ... + \alpha_n u_n.$$

Moreover, there is a number $n \in \mathbb{N} \cup \{\infty\}$ such that a subset of n linearly independent vectors form a basis. In this manner, a vector space is either finite or infinite dimensional depending on the number of vectors contained on the basis of V, and this number is denoted as $dim(V)$, since every basis has the same number of vectors.

Zorn´s Lemma, equivalent in the Zermelo–Fraenkel set theory to the Axiom of Choice, proves the existence of a basis, it states: every partially ordered set in which every chain has an upper bound, has at least one maximal element[2]. For every vector space V there exists a maximal set β of independent vectors, where $\beta \subseteq V$ is a basis and the cardinality of β is the dimension of V.

If $|\beta|$ is not finite, $dim(V) = \infty$.

If $|\beta|$ is a finite positive integer, $dim(V) = n$.

[2] Gunderson, D., & Rosen, K. (2014). Handbook of Mathematical Induction (pp. 62-67). Boca Raton: Chapman and Hall/CRC.

3 Real Analysis I

Previously we proved that the solution space of an ODE is a vector space. Henceforth, this section will introduce the renowned principle of superposition for differential equations. Consequently we will derive different possible solutions to linear ODE's while simultaneously proving the linear independence between its solution functions.

An ordinary differential equation (ODE) of order n is an equation of the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0$$

where y is a function of x and $y^{(n)}$ is the n-th derivative of y. The order of a differential equation is given by the highest derivative.

A first order linear ODE of the first order has the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are real functions. If $q(x) = 0$ the ODE is said to be homogeneous and if $q(x) \neq 0$ is said to be non-homogeneous. Thus,a linear ODE of order n has the form

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{x-1}} + p_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + ... + p_1(x)\frac{dy}{dx} + p_0(x)y = q(x).$$

Now we will consider the functions $p_k(x)$, $k = 0, 1, 2..., n$, as constants, and let $n = 2$. Without loss of generality we assume $p_2(x) = 1$. Consider the equation:

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0 \qquad \alpha, \beta \in \mathbb{R}$$

In order to solve it, we propose $y = e^{kx}$ as a solution. Substituting into the equation, we have

$$e^{kx}(k^2 + \alpha k + \beta) = 0.$$

Since the exponential function e^{kx} is never zero, we conclude that

$$k^2 + \alpha k + \beta = 0.$$

This polynomial equation is called the characteristic equation. When solving for k we face three different cases.

Case 1): If $\alpha^2 - 4\beta > 0$

$$k_1 = -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 4\beta}}{2}$$

$$k_2 = -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2 - 4\beta}}{2}$$

Since $y_1(x) := e^{k_1 x}$ and $y_2(x) := e^{k_2 x}$ are solutions of the equation, the principle of superposition of linear homogeneous ODE´s states that any linear combination of such solutions is again a solution to the differential equation. Thus, we see that the solution set to this differential equation constitutes a vector space. Henceforth, we take a general solution of the form

$$y(x) = Ay_1(x) + By_2(x), \qquad A, B \in \mathbb{R}.$$

That means that if V is the solution space, it is spanned by the functions $y_1(x) = e^{k_1 x}, y_2(x) = e^{k_2 x}$. We denote this as

$$V = \langle e^{k_1 x}, e^{k_2 x} \rangle = \{Ae^{kx} + Be^{kx} : A, B \in \mathbb{R}\}.$$

Both functions $e^{k_1 x}, e^{k_2 x}$ are linearly independent and they form a basis of V, and $dim(V) = 2$.

Case 2): If $\alpha^2 - 4\beta = 0$, then

$$k_1 = k_2 = -\frac{\alpha}{2},$$

and we have that $y_1 = y_2 = e^{k_1 x} = e^{k_2 x}$. In order to obtain two linearly independent solutions we take

$$y_1(x) := e^{kx} \text{ and } y_2(x) := xe^{kx}.$$

The linear independence between y_1 and y_2 is proved with Wronskian determinant which states that y_1 and y_2 are independent functions if

$$W(y_1, y_2) := \begin{vmatrix} y_1 & y_2 \\ & \\ y_1' & y_2' \end{vmatrix} = y_1(x)y_2'(x) - y_1(x)'y_2(x) \neq 0.^3$$

[3]Kiseliov, A., Kiseljak, M., & Makarenko, G. (1973). Problemas de ecuaciones diferenciales ordinarias (3rd ed., pp. 41-54, 97-120). Moscow: MIR.

In our case, we see that

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) = e^{2kx} \neq 0$$

then,

$$V = \langle e^{kx}, xe^{kx} \rangle$$

Case 3): if $\alpha^2 - 4\beta < 0$

We assume that k is a complex number of the form $a \pm bi$. From the theory of algebraic equations we know that the solutions are complex conjugates, this is

$$k_1 = a + ib \text{ and } k_2 = a - ib.$$

Then our solutions are

$$y_1 = e^{(a+ib)x} = e^{ax}e^{ibx} \text{ and } y_2 = e^{(a-ib)x} = e^{ax}e^{-ibx}.$$

From Euler´s identity

$$e^{iz} = \cos z + i \sin z \tag{2}$$

we have that

$$y_1(x) = e^{ax}\cos bx + ie^{ax}\sin bx$$

similarly,

$$y_2(x) = e^{ax}\cos bx - ie^{ax}\sin bx.$$

The linear combination of both solutions can be expressed as follows:

$$Ay_1 + By_2 = (A+B)e^{ax}\cos bx + (iA - iB)e^{ax}\sin bx,$$

and since $A+B$ and $i(A-B)$ are constants, we can take $V = \langle e^{ax}\sin bx, e^{ax}\cos bx\rangle$.

Later on when solving Laplace's equation we will encounter this particular differential equation of the form

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0, \qquad y(z_0) = 0, \tag{3}$$

where z_0 is a constant point inside the domain of the solution and λ is a constant. Solving this particular equation as we did before, we have a solution:

$$y(x) = Ae^{\lambda x} + Be^{-\lambda x}. \tag{4}$$

(It should be noted that this solution will be employed in the last section of this paper). When applying the condition $y(z_0) = 0$ we obtain

$$y(z_0) = Ae^{\lambda z} + Be^{-\lambda z_0} = 0$$

$$A = -Be^{-2\lambda z_0}$$

We now substitute A into the original solution and factor out B, obtaining

$$y(x) = B(-e^{-2\lambda z_0}e^{\lambda x} + e^{-\lambda x})$$

Since $\langle v \rangle = \langle Bv \rangle$ for any constant B, we can conveniently take $B = \frac{e^{\lambda z_0}}{2}$ and obtain:

$$\begin{aligned} y(x) &= \frac{-e^{-\lambda z_0} e^{\lambda x} + e^{-\lambda x} e^{\lambda z_0}}{2} \\ y(x) &= \frac{-e^{-\lambda(z_0 - x)} + e^{\lambda(z_0 - x)}}{2} \end{aligned} \tag{5}$$

obtaining the simplified solution:

$$y(x) = \sinh \lambda(z_0 - x) \tag{6}$$

3.1 Eigenvalues and Eigenfunctions

Consider a function

$$T : \mathbb{R}^n \to \mathbb{R}^n.$$

We say that T is a linear operator if for every $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, T satisfies

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

The operator T has a matrix representation associated with a certain basis of $\mathbb{R}^n$ denoted as $[T]$. Let T then be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an eigenvector of T if there exists a scalar λ such that

$$T(v) = \lambda v.$$ [4]

In this case, λ is called an eigenvalue, and V is an eigenvector associated with λ. For example, let V be the set of functions of the form $f : [0, 2\pi] \to \mathbb{R}$ that

[4] Friedberg, S., Insel, A., & Spence, L. (1979). Linear algebra (pp. 1-60, 231-250, 321-330, 379-400). Englewood Cliffs, N.J.: Prentice-Hall.

possess derivatives of all orders, usually denoted as $C^\infty([0,2\pi],\mathbb{R})$. Consider the operator $D^2 : V \to V$ given by

$$D^2(f) = \frac{d^2 f}{dx^2}.$$

We can form differential equations with this operator. Suppose we have the next differential equation

$$\frac{d^2 f}{dx^2} + w^2 f = 0.$$

We can represent this differential equation with the operator D^2 as

$$D^2(f) = (-w^2)f.$$

We observe that this turns the equation into an eigenvalue problem. Solving the ODE by the classical method stated previously, we obtain a general solution

$$y(x) = A\cos wx + B\sin wx, \qquad A, B \in \mathbb{R}. \tag{7}$$

The boundary conditions of an ODE are additional constraints that the solutions to the equation must fulfill. For example the boundary values that we assign to the solution of this equation could be

$$y(0) = 0$$

$$y(2\pi) = 0.$$

If $y(0) = 0$ we have that

$$y(0) = A\cos 0 + B\sin 0 = 0,$$

therefore, $A = 0$. Then $y(x) = B\sin wx$. Applying the second condition $y(2\pi) = B\sin 2\pi w = 0$. Since we seek for non-zero solution we ask that $\sin 2\pi w = 0$. Solving for w we observe that $2\pi w = n\pi$ where n is a positive integer. From this, we obtain the following eigenvalues:

$$w_n = \frac{n}{2}, \qquad n \in \mathbb{N}.$$

Thus, the eigenfunctions have the form:

$$y_n(x) = \sin\frac{nx}{2}, \qquad n \in \mathbb{N}.$$

The principle of superposition tells us that any solution is a linear combination of the eigenfunctions $y_n(x)$, thus we can express it as

$$y(x) = \sum_{n=1}^{\infty} b_n \sin\frac{nx}{2}. \tag{8}$$

4 Real Analysis II

So far we have observed an evident relation between the eigenvalues and differential equations. Now, we will deal with sequences of functions and some convergence criteria, which is a crucial mathematical concept when it comes to understanding the Fourier Series and the modeling of continuous, differentiable-integrable functions.

4.1 Convergence

A sequence $X = (x_n)$ converges to $Z \in \mathbb{R}$ if for every error $\mathcal{E} > 0$ there is an index $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$\mid x_n - Z \mid < \mathcal{E}.$$ [5]

If a sequence has a limit, the sequence is said to be convergent. If a sequence is not convergent, it is said to be divergent.

4.2 Sequences of functions

Let $A \subset \mathbb{R}$. Let (f_n) be a sequence of functions in A, that means, for every $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$. This sequence of functions generates a sequence of real numbers for every x in A of the form $(f_n(x))$ obtained by evaluating every function at the same x.

For instance, let $t_n(x) := x^n, x \in [-1, 1]$ and $n \in \mathbb{N}$. If $x = 1$, the sequence $t_n(1) = 1$ converges to 1. On the other hand, the limit of (x^n) is 0 for

[5] Friedberg, S., Insel, A., & Spence, L. (1979). Linear algebra (pp. 1-60, 231-250, 321-330, 379-400). Englewood Cliffs, N.J.: Prentice-Hall.

$-1 < x < 1$. Thus, $t_n(x) \to g(x)$, where

$$g(x) := \begin{cases} 0, & x \in (-1,1) \\ \\ 1, & x = 1 \end{cases}$$

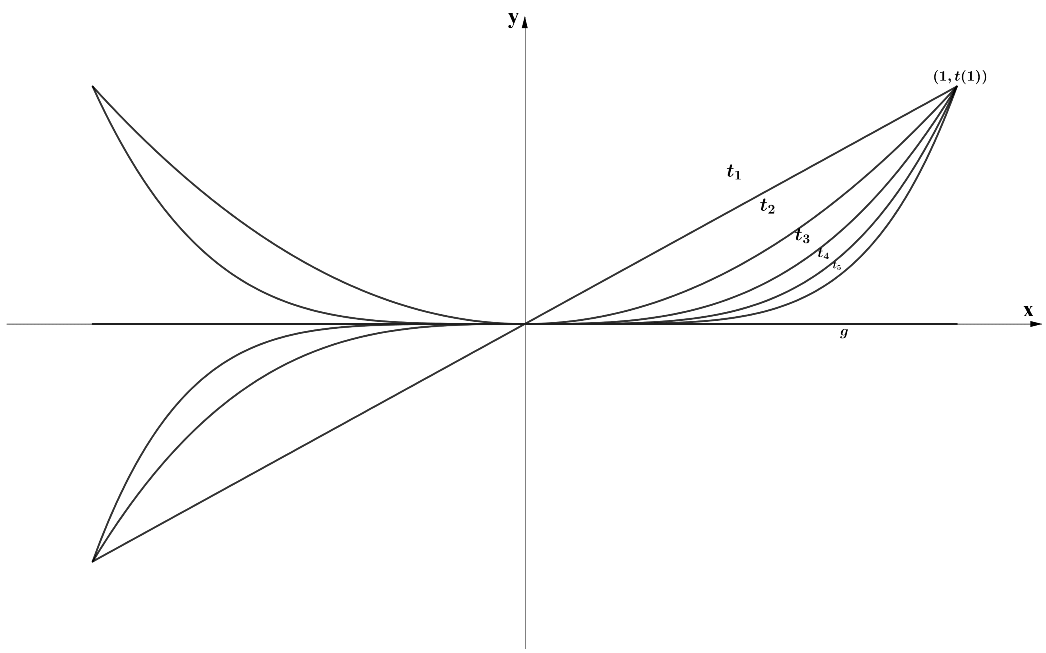

Figure 1: Some functions $t_n := x^n$

It can be seen in Figure 1 how the sequence t_n converges to g in the interval (-1,1].

4.3 Pointwise and Uniform Convergence

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of functions, $t_n : [a,b] \to \mathbb{R}$ for every $n \in \mathbb{N}$ and let $g : [a,b] \to \mathbb{R}$ be another function. It is said that t_n converges pointwise on the interval $[a,b]$ to g if for every $x \in [a,b]$, we have that $t_n(x) \to g(x)$. In this manner, g is called the limit of the sequence t_n. Equivalently, t_n

is said to converge pointwise on $g : [a, b] \to \mathbb{R}$, if for every $\mathcal{E} > 0$ and for every $x \in [a, b]$ there is an index $N \in \mathbb{N}$ such that for $n \geq N$ we have that $\mid t_n(x) - g(x) \mid < \mathcal{E}$.[6] We denote this as

$$t_n \to g.$$

Likewise, we say that $t_n : [a, b] \to \mathbb{R}$ converges uniformly on $g : [a, b] \to \mathbb{R}$ if for every $\mathcal{E} > 0$ there is a $n \in \mathbb{N}$ such that for $N \geq n$, we have

$$\mid t_n(x) - g(x) \mid < \mathcal{E}$$

for each $x \in [a, b]$ and we denote this as $t_n \rightrightarrows g$. If the sequence t_n of continuous functions converges uniformly to the function $g(x)$, then $g(x)$ is necessarily continuous.

Equivalently, t_n converges uniformly to g $t_n \rightrightarrows g$ if and only if

$$\lim_{n\to\infty} \|f_n - g\|_A = 0$$

with $A \subset \mathbb{R}$ where the latter means

$$\lim_{n\to\infty} \sup\{\mid f_n(x) - g(x) \mid : x \in A\} = 0$$ [7]

[6] Friedberg, S., Insel, A., & Spence, L. (1979). Linear algebra (pp. 1-60, 231-250, 321-330, 379-400). Englewood Cliffs, N.J.: Prentice-Hall.

[7] Friedberg, S., Insel, A., & Spence, L. (1979). Linear algebra (pp. 1-60, 231-250, 321-330, 379-400). Englewood Cliffs, N.J.: Prentice-Halle.

Let use the previous example with $t_n := x^n$ and a limit function

$$g(x) := \begin{cases} 0, & x \in (-1,1) \\ \\ 1, & x = 1 \end{cases}$$

In order to prove if t_n converges uniformly to g on the arbitrary interval $(-1,1]$ the supreme of the algebraic subtraction of the supreme limit when n tends to infinity of t_n and $g(x)$ must be equal to zero. We have then that,

$$\mid f_n(x) - g(x) \mid = \mid x^n - g(x) \mid = \begin{cases} \mid x^n - 0 \mid, x \in (-1,1) \\ \mid 1 - 1 \mid, x = 1 \end{cases}$$

$$\mid f_n(x) - g(x) \mid = \begin{cases} x^n, & x \in (-1,1) \\ \\ 0, & x = 1 \end{cases}$$

$$\lim_{n\to\infty} \sup\{f_n(x) - g(x) \mid : x \in (-1,1]\} = 1$$

Hence, t_n does not converge uniformly to g in $(-1,1]$.

5 Fourier Analysis

During his early years, Jean-Baptiste Fourier conducted endless contributions to mathematics, namely: new methods to solve the equation for heat conduction. Nevertheless his works where firmly criticized by Pierre-Simon Laplace who invalidated the renowned Fourier Series as it lacked "thoroughness". It was inconceivable that an infinite series of $\cos(x)$ could be replaced by a series of $\sin(x)$. Other famous mathematicians like Lagrange refuted the validity of the trigonometric series proposed by Fourier[8]. Nevertheless, the objections where withdrawn, and Fourier contributions where applied to many different physics problems.

We will begin to introduce the theory underlying the trigonometric Fourier series. Let V be a vector space where $\vec{x} = (a, b, c)$ and $\vec{y} = (c, d, e)$. The inner product of $\vec{x}$ and $\vec{y}$ is denoted as $\langle \vec{x} \mid \vec{y} \rangle$ and it is a function of the form $\langle \vec{x} \mid \vec{y} \rangle \to \mathbb{F}$ defined as

$$\sum_{i=1}^{n} x_i y_i. \tag{9}$$

An inner product holds that $\langle \alpha\vec{x} + \beta\vec{y} \mid \vec{z} \rangle = \alpha\langle \vec{x} \mid \vec{z} \rangle + \beta\langle \vec{y} \mid \vec{z} \rangle$ and if $\langle \vec{x} \mid \vec{y} \rangle = 0$, then $\vec{x}$ and $\vec{y}$ are orthogonal[9]. The magnitude of a vector $\vec{x}$ is the square root of the inner product between two vectors and it is expressed as follows,

$$|| \vec{x} ||= \sqrt{\langle \vec{x} \mid \vec{x} \rangle}.$$

[8] Korner, T. (2002). Fourier analysis (pp. 470-483). Cambridge [u.a].: Cambridge Univ. Press.

[9] Bartle, R., & Sherbert, D. (2010). Introduccion al analisis matematico de una variable (pp. 85-160, 309-365, 343-361). Mexico: Limusa Wiley.

If $|| \vec{x} ||= 1$, $\vec{x}$ is a normal (unitary) vector. If $V = \{v_1, v_2, ..v_n, ...\}$ and it holds that $\langle v_i \mid v_j \rangle = \delta_{ij}$, it is called Kronecker Delta, defined as:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ \\ 0, & i \neq j \end{cases}$$

Hence, we say that V is an orthonormal set of vectors.

Npw if V is a space of functions, we define the inner product of two functions to be:

$$\langle \vec{f} \mid \vec{g} \rangle := \int_a^b f(x)\overline{g(x)} \tag{10}$$

where $\overline{g(x)}$ is the conjugate complex of $g(x)$. Moreover, $\vec{f}, \vec{g} \in V$ are orthogonal functions if

$$\int_a^b f(x)\overline{g(x)}dx = 0.$$

In Fourier Analysis, we define the function space as the continuous functions that go from 0 to 2π in the real field. $V = \mathbb{C}^0([0, 2\pi], \mathbb{R})$.

A function $f : [a, b] \to \mathbb{R}$ is continuous in $c \in [a, b]$ if for every $\mathcal{E} > 0$ there is a $\delta > 0$ such that if $| x - c |< \delta$, then $| f(x) - f(c) |< \mathcal{E}$.[10]

In the vector space V, let $(f_n)n \in N$ be a sequence of functions in V. We

[10] Bartle, R., & Sherbert, D. (2010). Introduccion al analisis matematico de una variable (pp. 85-160, 309-365, 343-361). Mexico: Limusa Wiley.

say that the sequence f_n converges to f in the mean $f_n \to f$, if

$$\| f_n - f \| \to 0,$$

where

$$\| f_n - f \| = \sqrt{\int_a^b | f_n(x) - f(x) |^2 \, dx} \quad ^{11}$$

Henceforth, $f_n \to f$ in mean, if

$$\lim_{n\to\infty} \left(\int_a^b | f_n(x) - f(x) |^2 \right)^{\frac{1}{2}} = 0.$$

A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if for every $\mathcal{E} > 0$ there is a $N \in \mathbb{N}$ such that for every $n, m \geq N$, it is held

$$\| f_n - f_m \| < \mathcal{E}$$

equivalent to

$$\sqrt{\int_a^b | f_n(x) - f_m(x) |^2 \, dx} < \mathcal{E}.$$

A space V is a complete space if every Cauchy sequence in V converges. This space is called a Hilbert Space[12]. In the space V a set $S = \{\phi_1, \phi_2, ..., \phi_n, ...\}$ is considered to be complete if every function $f \in V$ we have that

$$f = \sum_{n=1}^{\infty} C_n \phi_n, \qquad C_n \in \mathbb{R}. \tag{11}$$

[11] Bartle, R., & Sherbert, D. (2010). Introduccion al analisis matematico de una variable (pp. 85-160, 309-365, 343-361). Mexico: Limusa Wiley.

[12] Bartle, R., & Sherbert, D. (2010). Introduccion al analisis matematico de una variable (pp. 85-160, 309-365, 343-361). Mexico: Limusa Wiley.

If $S = \{\phi_1, \phi_2, ..., \phi_n, ...\}$ is a complete set for every $f \in V$ we have that f is a linear combination of the functions ϕ_n that converge in the mean and $C_n = \langle f \mid \phi_n \rangle$ are the Fourier Coefficients[13] which will be later introduced.

If $f(x)$ is a continuous function for $-\pi \leq x \leq \pi, \quad f(-\pi) = f(\pi)$, and

$$\int_{\pi}^{\pi} f'^2(x)dx \tag{12}$$

is finite, then the Fourier Series of $f(x)$ will converge to $f(x)$ uniformly—concept to be shortly introduced. Then, if the series converges uniformly, the Fourier Series converges to $f(x)$ pointwise and in the mean, we know for example, that the set $\{1, \cos x, \sin x, \cos 2x...\}$ is a complete set on the interval $-\pi \leq x \leq \pi$. [14]

Let $S = \{\phi_1, \phi_2, ..., \phi_n, ...\}$. If S is an orthonormal set of vectors then the vectors are linearly independent. This is,

$$\alpha_1\phi_1 + ... + \alpha_n\phi_n = 0.$$

Let ϕ_k be any vector in S and α_k any scalar in a field F. The inner product

[13] Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods (pp. 29-36, 48-54, 63-126, 141-145, 160-169). New York: Dover Publications.

[14] Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods. New York: Dover Publications.

of any vector and zero is always zero, thus we have that

$$\begin{aligned} 0 &= \langle 0 \mid \phi_k \rangle \\ &= \langle \alpha_1\phi_1 + ... + \alpha_n\phi_n \mid \phi_k \rangle \\ &= \alpha_1\langle \phi_1 \mid \phi_k \rangle + ... + \alpha_k\langle \phi_k \mid \phi_k \rangle + ... + \alpha_n\langle \phi_n \mid \phi_k \rangle \end{aligned} \tag{13}$$

With the previously introduced Kronecker´s Delta property of orthonormal vectors, we simplify the inner products and thus, it shows that the values vary discretely with 0 or 1.

$$\begin{aligned} &= \alpha_1(0) + \alpha_k(1) + \alpha_n(0) \\ &= \alpha_k = 0 \end{aligned} \tag{14}$$

Thus, the vectors in S are linearly independent and form a basis of the space f.

In $V = \mathbb{C}^0([0, 2\pi], \mathbb{R})$ the functions $\{1, \cos nx, \cos mx : n, m \in \mathbb{N}\}$ and $\{\sin nx, \sin mx : n, m \in \mathbb{N}\}$ form orthogonal systems in the interval $[0, 2\pi]$.

$$\begin{aligned} \langle 1 \mid \cos(nx) \rangle &= \int_0^{2\pi} \cos(nx)(\overline{1})dx \\ &= \frac{\sin nx}{n}\Big|_0^{2\pi} \\ &= 0 \quad , \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{15}$$

Equivalently,

$$\begin{aligned}\langle \sin(nx) \mid \sin(mx)\rangle &= \int_0^{2\pi} \sin(nx)\sin(mx)dx \\ &= \frac{\sin(n-m)x}{2n-2m} - \frac{\sin(n+m)x}{2n+2m}\Big|_0^{2\pi} \\ &= 0 \quad , \qquad \forall n \neq m, m \in \mathbb{N}.\end{aligned} \tag{16}$$

We find the norm of the orthogonal functions, to obtain the magnitude of each function. This is:

$$||\, 1 \,||, \quad ||\, \sin nx \,|| \;\text{ and }\; ||\, \cos nx \,||\ .$$

$$||\, 1 \,||= \sqrt{\langle 1 \mid 1\rangle} = \sqrt{\int_0^{2\pi} dx} = \sqrt{2\pi}$$

$$||\, \sin nx \,||= \sqrt{\langle \sin nx \mid \sin nx\rangle} = \sqrt{\int_0^{2\pi} \sin(nx)^2 dx} = \sqrt{\frac{2nx - \sin 2nx}{4n}\Big|_0^{2\pi}} = \sqrt{\pi}$$

$$||\, \cos nx \,||= \sqrt{\langle \cos nx \mid \cos nx\rangle} = \sqrt{\int_0^{2\pi} \cos(nx)^2 dx} = \sqrt{\frac{2nx + \sin 2nx}{4n}\Big|_0^{2\pi}} = \sqrt{\pi}.$$

From this, we have that the system

$$\{1, \cos nx, \sin mx : n, m \in \mathbb{N}\}$$

is a set of orthogonal functions. Furthermore, with the normalization process of vectors, that states that a non-zero vector is normalized by multiplying it by the reciprocal of its norm, we obtain that

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\sin mx : n, m \in \mathbb{N}\}$$

is then an orthonormal set of vectors. The orthonormal system is called the trigonometric system and it is complete in the space $\mathcal{L}^2([0, 2\pi] \to \mathbb{R})$ where the functions in this space hold

$$f : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} \mid f \mid^2 < +\infty.$$

Then, given $f \in \mathcal{L}^2$, there exist coefficients a_0, a_n and b_n that correspond to every specific function $f(x)$ such that

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{17}$$

and it is called the classic Fourier Series[15] , in some cases also expressed as

$$f = \sum_{n=-\infty}^{\infty} C_n e^{inx}.$$

In order to obtain an expression for the unknown coefficient a_0, a_n, and, b_n, we will derive three distinct inner products between orthonormal vectors and the classic Fourier Series, these being:

$$\langle f \mid \frac{1}{\sqrt{2\pi}} \rangle, \langle f \mid \frac{1}{\sqrt{\pi}} \cos mx \rangle, \langle f \mid \frac{1}{\sqrt{\pi}} \sin mx \rangle$$

Now, we begin to derive a_0 by simplifying the first inner product and by

[15] Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods (pp. 29-36, 48-54, 63-126, 141-145, 160-169). New York: Dover Publications.

employing the orthogonality properties of inner products:

$$
\begin{aligned}
\langle f \mid \frac{1}{\sqrt{2\pi}} \rangle &= \langle \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_m \cos nx + b_m \sin nx) \mid \frac{1}{\sqrt{2\pi}} \rangle \\
&= \langle \frac{a_0}{2}(1) \mid \frac{1}{\sqrt{2\pi}}(1) \rangle + \sum_{n=1}^{\infty} \langle a_m \cos nx \mid \frac{1}{\sqrt{2\pi}} \rangle + \sum_{n=1}^{\infty} \langle b_m \sin nx \mid \frac{1}{\sqrt{2\pi}} \rangle \\
&= (\frac{a_0}{2})(\frac{1}{\sqrt{2\pi}}) \langle 1 \mid 1 \rangle + \sum_{n=1}^{\infty} a_m \langle \cos nx \mid \frac{1}{\sqrt{2\pi}} \rangle + \sum_{n=1}^{\infty} b_n \langle \sin nx \mid \frac{1}{\sqrt{2\pi}} \rangle \\
&= \frac{a_0}{2} \sqrt{2\pi}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
a_0 &= \frac{2}{\sqrt{2\pi}} \langle f \mid \frac{1}{\sqrt{2\pi}} \rangle \\
a_0 &= \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \frac{1}{\sqrt{2\pi}} dx \\
a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx
\end{aligned}
\tag{19}
$$

Similarly, we follow the same procedure a_n with the second inner product:

$$\begin{aligned}
\langle f \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle &= \langle \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle \\
&= \langle \frac{a_0}{2} \mid \frac{1}{\sqrt{\pi}}(\cos mx)\rangle + \sum_{n=1}^{\infty}\langle a_n \cos nx \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle + \sum_{n=1}^{\infty}\langle b_n \sin nx \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle \\
&= (\frac{a_0}{2})(\frac{1}{\sqrt{\pi}})\langle 1 \mid \cos mx\rangle + \sum_{n=1}^{\infty}(\frac{a_n}{\sqrt{\pi}})\langle \cos nx \mid \cos mx\rangle + \sum_{n=1}^{\infty} b_n\langle \sin nx \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle \\
&= \sum_{n=1}^{\infty}(\frac{a_n}{\sqrt{\pi}})\langle \cos nx \mid \cos mx\rangle \\
&= \frac{a_1}{\sqrt{\pi}}\langle \cos x \mid \cos mx\rangle + \frac{a_2}{\sqrt{\pi}}\langle \cos 2x \mid \cos mx\rangle + ... + \frac{a_n}{\sqrt{\pi}}\langle \cos mx \mid \cos mx\rangle + ... \\
&= \frac{a_n}{\sqrt{\pi}}\langle \cos mx \mid \cos mx\rangle \\
&= \frac{a_n}{\sqrt{\pi}} \mid\mid \cos mx \mid\mid^2 \\
&= \frac{a_n}{\sqrt{\pi}}(\sqrt{\pi})^2 \\
&= a_n\sqrt{\pi}
\end{aligned} \tag{20}$$

$$\begin{aligned}
a_n\sqrt{\pi} &= \langle f \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle \\
a_n &= \frac{1}{\sqrt{\pi}}\langle f \mid \frac{1}{\sqrt{\pi}}\cos mx\rangle \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx \ dx
\end{aligned} \tag{21}$$

Following the same procedure with $\langle f \mid \frac{1}{\sqrt{\pi}}\sin mx\rangle$ we obtain that

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx \ dx. \tag{22}$$

Therefore we have that the classic Fourier Series are equivalent to

$$f = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx + \sum_{m=1}^{\infty}\Big[\Big(\frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\ dx\Big)\cos mx + \Big(\frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\ dx\Big)\sin mx\Big]. \tag{23}$$

For example we can obtain the Fourier series of the function $f(x) = x$ by integrating the Fourier coefficients a_0, a_n, b_n. Thereafter we obtain:

$$x = \pi + \sum_{n=1}^{\infty} -\frac{2}{n}\sin nx.$$

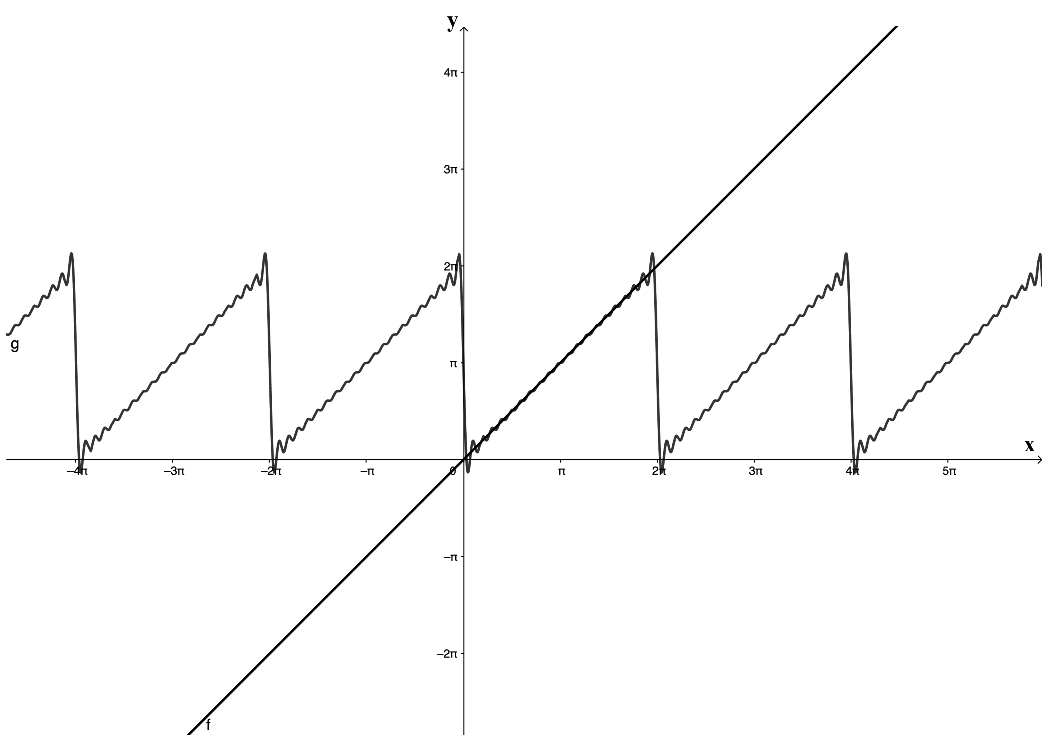

Figure 2: $f(x) = x$ and $g(x) = \pi + \sum_{n=1}^{20} -\frac{2}{n}\sin nx$

It can be observed in Figure 2 how the Fourier Series converges in the mean to the function $f(x) = x$ from 0 to 2π as n tends to infinity.

5.1 Sine and Cosine Series

In this section we will prove how the trigonometric series $\cos x$ can be expressed as a series of $\sin(x)$ and vice versa.

A function $f(x)$ is called to be periodic if there is a constant $C > 0$ such that $f(x + C) = f(x)$. If C is equal to the period of the function we can agree that $f(x) = f(x+C) = f(x+2C) = f(x+nC) = ..., n \in \mathbb{N}$ [16]. In some cases we want to periodically extend functions to intervals of different length, and as a result of this we obtain the Fourier Series of sines and cosines.

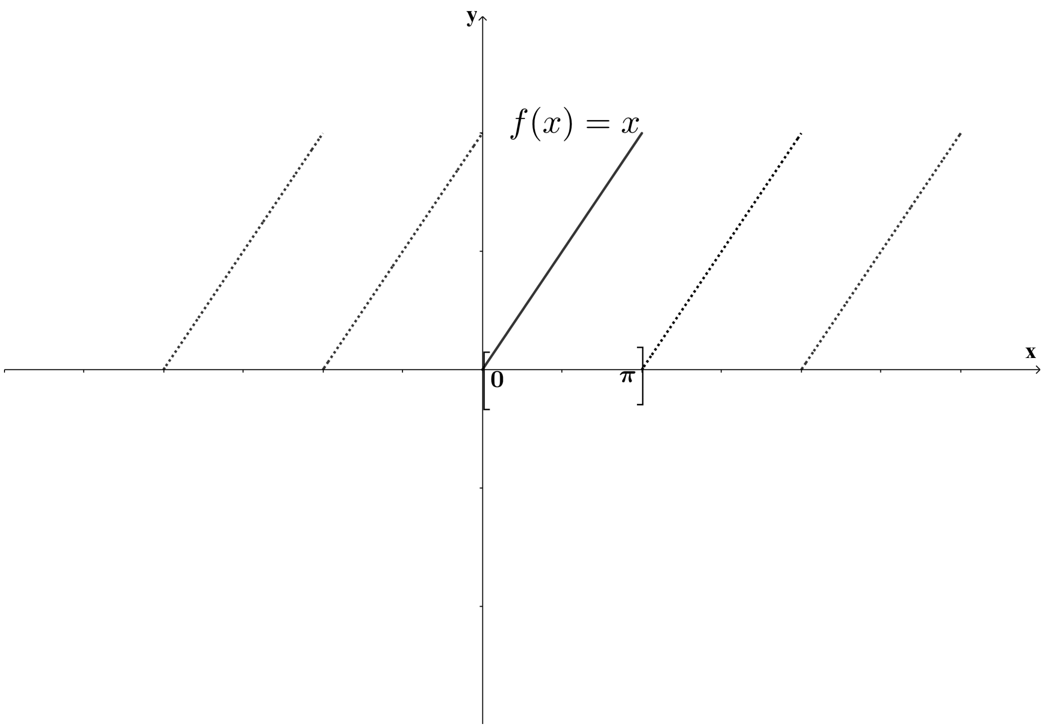

Figure 3: Periodic extension of $f(x) = x, \qquad x \in [0, \pi]$

In Fourier series, even and odd functions satisfy symmetry relations and allow to create a new type of series. A function is said to be even if $f(x) =$

[16]Tolstov, G. (1976). Fourier series (pp. 1-40). Estados Unidos: Dover Publications.

$f(-x)$, and the function is said to be odd if $f(-x) = -f(x)$. The periodic extension of a function consists on repeating periodically a segment of the function in a restricted interval until infinity. For example, in Figure 3 the line segment of $f(x) = x$ is repeated periodically until infinity.

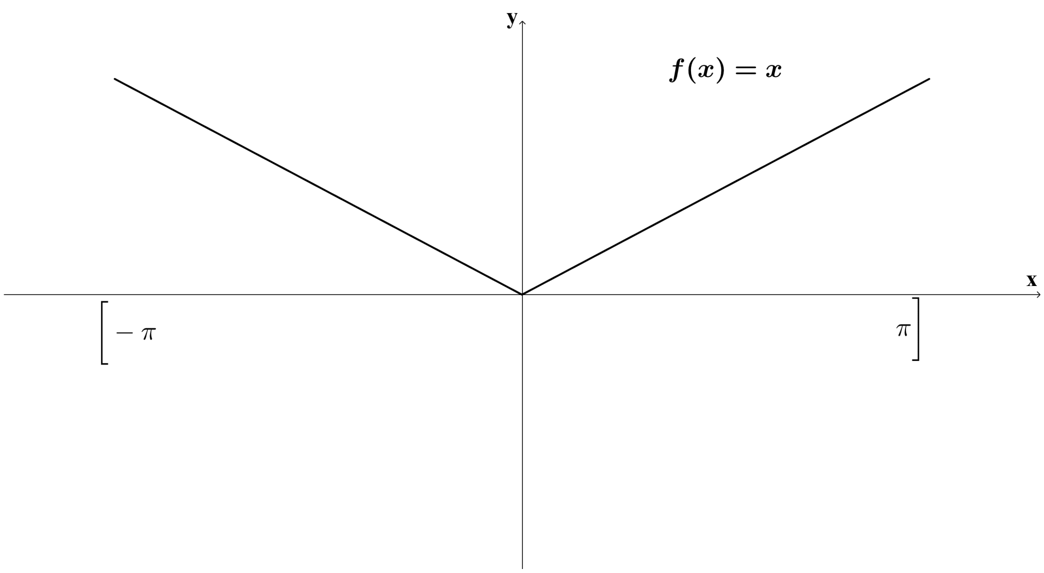

Figure 4: Even extension of $f(x) = x, \qquad x \in [-\pi, \pi]$

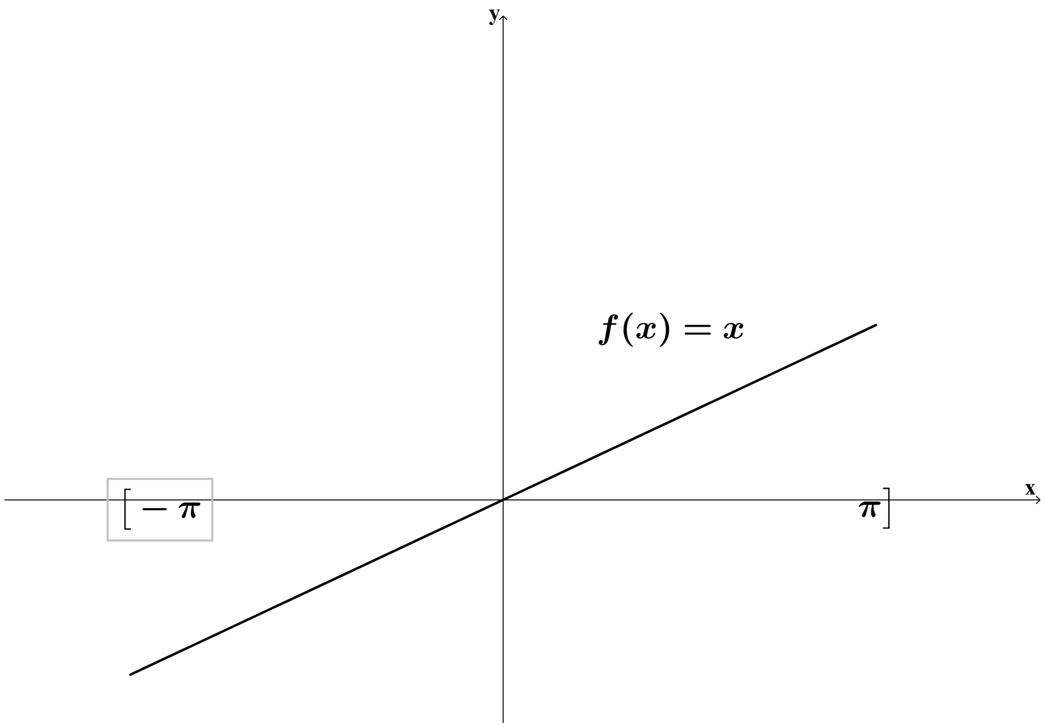

Figure 5: Odd extension of $f(x) = x, \qquad x \in [-\pi, \pi]$

Figure 4 and Figure 5 show two different ways to periodically extend $f(x) = x$. The even extension of $f(x) = x$ is precisely the function $f(x) = \mid x \mid$ restricted to $[-\pi, \pi]$.

Let f and g be two different functions. If f is even and g is even, fg is even; if f is odd and g is even, fg is odd; if f is odd and g is odd, then fg is even. Consequently, let $f : [-a, a] \to \mathbb{R}$ an integrable function. We can simplify the integrals of the coefficients depending on even and odd properties of integration. If f is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

and if f is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

For example, let $f : [0, \pi] \to \mathbb{R}$ be a well-behaved function. Lets suppose we evenly expand $f(x)$ to $f : [-\pi, \pi] \to \mathbb{R}$. If

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

we have that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx.$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0.$$

When the function f is evenly expanded we obtain a Fourier cosine series.

$$f = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_n \cos nx.$$

The cosine series of $f(x) = x$ equals

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos{(2n-1)x}$$

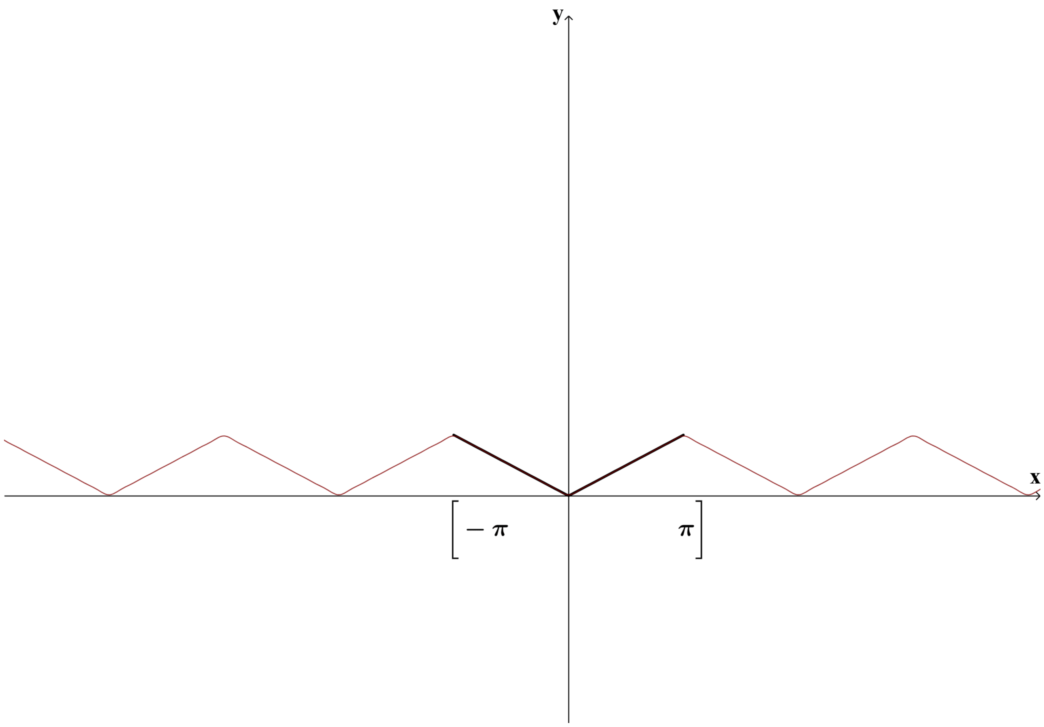

Figure 6: Cosine series $f(x) = x$, $\qquad g(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{5} \frac{1}{(2n-1)^2} \cos{(2n-1)x}$

Analogously, when oddly expanding $f(x)$ to $f : [-\pi, \pi] \to \mathbb{R}$ we obtain the coefficients.

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_{0\pi}^{\pi} f(x) \sin nx dx$$

. We then obtain the Fourier sine series of a function of the form.

$$f = \sum_{m=1}^{\infty} b_n \sin nx.$$

The sine series of $f(x) = x$ equals

$$f(x) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

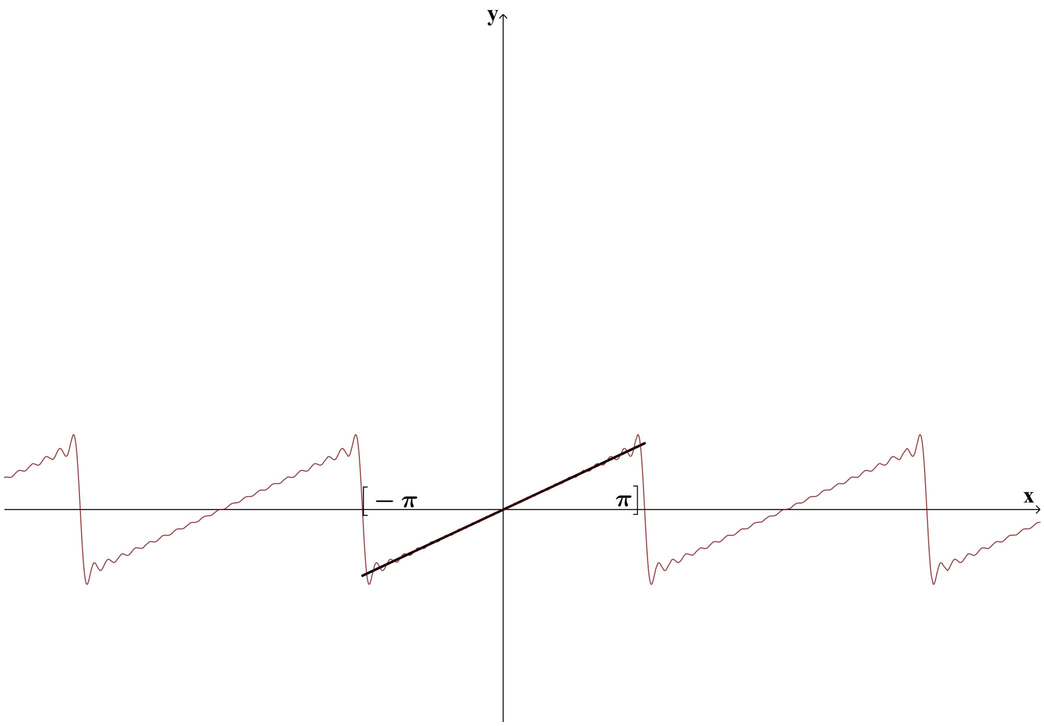

Figure 7: Sine series of $f(x) = x$, $\quad g(x) = -2\sum_{n=1}^{20} \frac{(-1)^n}{n} \sin nx$

6 Real Analysis III

In this last section, partial differential equations and the method of separation of variables are introduced. Analogously, Laplace´s equation in a 2-Dimension Rectangle and constrained by 4 boundary conditions will be thoroughly solved with the help of some interesting trigonometric identities, and by using the mathematical concepts and tools that have been previously analysed and derived along the paper.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The i-th partial derivative is defined as

$$\frac{\partial f}{\partial x_i}(x_1, x_2, ..., x_n) = \lim_{h\to 0} \frac{f(x_1, x_2, ..., x_i + h, ..., x_n) - f(x_1, x_2, ..., x_n)}{h} \tag{24}$$

A second order partial differential equation (PDE) with variables x, y, z is an equation

$$f\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial x \partial z}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}\right) = 0. \tag{25}$$

6.1 Separation of Variables, Laplace in a 2-Dimension Rectangle

One of the most famous second order partial differential equation is Laplace's equation, which is a prototype for linear-elliptic partial differential equations and has been thoroughly studied during the last fifty years[17].

Laplace´s equation in two dimensions has the following form.

[17] Lindqvist, P. (2006). Notes on the p-Laplace equation (pp. 1-18). Jyvaskyla: University of Jyvaskyla, Dept. of Mathematics and Statistics.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ [18]

When we attempt to solve this equation, we propose a solution of the form $u(x,y) = \varphi(x)\psi(y)$. We have that

$$\frac{\partial^2 u}{\partial x^2} = \psi(y)\frac{d^2\varphi}{dx^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \varphi(x)\frac{d^2\psi}{dy^2},$$

we substitute the partial derivatives into the equation and obtain

$$\psi(y)\frac{d^2\varphi}{dx^2} + \varphi(x)\frac{d^2\psi}{dy^2} = 0$$

$$\psi(y)\frac{d^2\varphi}{dx^2} = -\varphi(x)\frac{d^2\psi}{dy^2}$$

$$\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2} = -\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}$$

If noted, the left hand of the equation depends solely on x. If we then differentiate both sides we observe the following

$$\frac{d}{dx}\Big(\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2}\Big) = \frac{d}{dx}\Big(-\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}\Big)$$

$$\frac{d}{dx}\Big(\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2}\Big) = 0$$

$$\frac{d}{dx}\Big(-\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}\Big) = 0$$

Similarly,

$$\frac{d}{dy}\Big(-\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}\Big) = \frac{d}{dy}\Big(\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2}\Big)$$

[18] Arfken, G., & Weber, H. (1998). Mathematical methods for physicists (4th ed., pp. 320-48, 535-554, 635-649, 880-903). San Diego, Calif.: Acad. Press.

$$\frac{d}{dy}\Big(-\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}\Big) = 0$$

$$\frac{d}{dy}\Big(\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2}\Big) = 0$$

This behavior tells us that both

$$\frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2} \text{ and } \frac{1}{\psi(y)}\frac{d^2\psi}{dy^2}$$

are a constant that we will denote as λ. Hence, we can rearrange the equations as follows:

$$\begin{aligned} \frac{1}{\varphi(x)}\frac{d^2\varphi}{dx^2} &= \lambda \\ -\frac{1}{\psi(y)}\frac{d^2\psi}{dy^2} &= \lambda \\ \frac{d^2\varphi}{dx^2} - \lambda\varphi(x) &= 0 \\ \frac{d^2\psi}{dy^2} + \lambda\psi(y) &= 0 \end{aligned} \tag{26}$$

Now we consider the different cases that we previously solved, so that we can solve for $\varphi(x)$ and $\psi(y)$. If $\lambda > 0$ we have that

$$\varphi(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}, \qquad \psi(y) = C\sin\sqrt{\lambda}y + D\cos\sqrt{\lambda}y$$

Therefore, the solution to the equation is

$$u(x,y) = \varphi(x)\psi(y) = (Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x})(C\sin\sqrt{\lambda}y + D\cos\sqrt{\lambda}y)$$

In this case, the solution space is then the span of the four independent functions

$$e^{\pm\sqrt{\lambda}x}\sin\sqrt{\lambda}y, \quad e^{\pm\sqrt{\lambda}x}\cos\sqrt{\lambda}y.$$

Similarly, if $\lambda < 0$,

$$u(x,y) = (A\sin\sqrt{-\lambda}x + B\cos\sqrt{-\lambda}x)(Ce^{\sqrt{-\lambda}y} + De^{-\sqrt{-\lambda}y}).$$

Thus, the solution space is the span of the four independent functions

$$e^{\pm\sqrt{-\lambda}x}\sin\sqrt{-\lambda}y, \quad e^{\pm\sqrt{\lambda}x}\cos\sqrt{\lambda}y.$$

Finally, if $\lambda = 0$,

$$\frac{d^2\varphi}{dx^2} = 0, \qquad \frac{d^2\psi}{dy^2} = 0$$

This means that $\varphi(x)$ and $\psi(x)$ can only be linear functions and we express the solution space as

$$u(x,y) = (Ax+B)(Cy+D). \tag{27}$$

Henceforth,this solution space is the span of the four independent functions $1, x, y, xy$.

Lets consider Laplace equation in two dimensions and restrict it with four boundary conditions[19] .

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in [0,\pi], \quad y \in [0, z_0],$$

[19]Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods (pp. 29-36, 48-54, 63-126, 141-145, 160-169). New York: Dover Publications.

$$u(0,y) = u(\pi,y) = u(x,z_0) = 0,$$

$$u(x,0) = f(x) \quad ^{20}$$

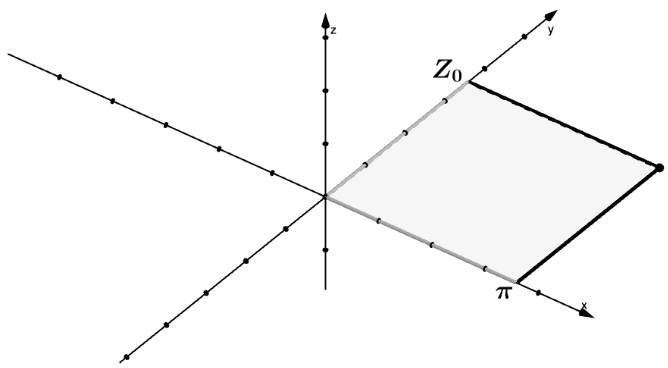

Figure 8: Boundary conditions restrict the domain to a plane

We evaluate the first three boundary conditions, $u(0,y) = \varphi(0)\psi(y) = 0$ since y is a variable we conclude that $\varphi(0) = 0$; $u(\pi,y) = \varphi(\pi)\psi(y) = 0$ then $\varphi(\pi) = 0$; $u(x,z_0) = \varphi(x)\psi(z_0)$ x is a variable, thus $\psi(z_0) = 0$. Separating the variables we obtain

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi(x) = 0$$

$$\frac{d^2\psi}{dy^2} - \lambda\psi(y) = 0.$$

The solutions of these two ODE´s were previously obtained. Now we begin to substitute the boundary conditions and we have

$$\varphi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x = 0$$

[20]Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods (pp. 29-36, 48-54, 63-126, 141-145, 160-169). New York: Dover Publications.

We evaluate the solution of $\varphi(x)$ so that we and obtain a value for the constant A. $\varphi(0) = A = 0$, from $A = 0$ we have that $\varphi(x) = B \sin \sqrt{\lambda} x$ and

$$\varphi(\pi) = B \sin \sqrt{\lambda} \pi = 0.$$

For our solution space we seek for $B \neq 0$ in order to have a solution other than the constant 0, and hence

$$\sin \sqrt{\lambda} \pi = 0$$

$$\sqrt{\lambda} \pi = n\pi, \qquad n \in \mathbb{N}$$

Hence, the eigenvalues of the function are

$$\lambda_n = n^2$$

and the eigenfunctions are calculated to be

$$\varphi_n(x) = \sin nx.$$

We follow the same procedure to obtain the eigenalues and eigenfunctions of the function $\psi(x)$. Since the solution to $\psi(x)$ was also previously shown we have

$$\psi(y) = \sinh \sqrt{\lambda}(z_0 - y).$$

This is the moment where we introduce the eigenvalues of the solution space,

$$\psi_n(y) = \sinh n(z_0 - y), \qquad n \in \mathbb{N}.$$

Consequently the solution to Laplace equation with the first three boundary conditions is

$$u_n(x,y) = \varphi_n(x)\psi_n(y). \tag{28}$$

The principle of superposition of a differential equation allows us to represent $u(x.y)$ as the next infinite series

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} b_n \varphi_n(x)\psi_n(y) \\ &= \sum_{n=1}^{\infty} b_n \sin nx \sinh n(z_0 - y) \end{aligned} \tag{29}$$

Now we will assess the fourth boundary condition that implies $u(x,0) = f(x)$ and obtain the following:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx \sinh{(nz_0)} = f(x).$$

Taking $C_n := b_n \sinh{(nz_0)}$ we obtain,

$$f(x) = \sum_{n=1}^{\infty} C_n \sin nx,$$

From the Fourier sine series we have that

$$C_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

and thus b_n

$$b_n = \frac{2}{\pi \sinh{(nz_0)}} \int_0^{\pi} f(x) \sin nx \, dx.$$

From this, we have that the solution to Laplace´s equation with these four boundary conditions is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \left(\frac{\sin(nx) \sinh n(z_0 - y)}{\sinh(nz_0)} \right).$$

We observe that

$$\frac{\sinh n(z_0 - y)}{\sinh(nz_0)} = \frac{e^{-ny}}{1 - e^{2z_0}}.$$

Henceforth, the series of $u(x,y)$ is dominated by a constant times the series $\sum_{n=1}^{\infty} e^{-ny}$.

The series $u(x,y)$ converges for $y > 0$, and uniformly in x and y for $y \geq y_0$ with any positive y_0. If the series of $u(x,y)$ converge uniformly to $f(x)$, $u(x,y)$ is a continuous function in x and y for $y > 0$. Now, we assume that

$$\int_0^{\pi} f'^2 dx$$

is finite for this sine series and $u(x,y)$ converges uniformly to $f(x)$. Since each of the terms in the series is continuous, the limit function $u(x,y)$ is continuous [21].

For example, lets propose the next boundary conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in [0,\pi], \quad y \in [0,\pi],$$

$$u(0,y) = u(\pi,y) = u(x,\pi) = 0,$$

$$u(x,0) = \sin x^2$$

[21] Weinberger, H. (1995). A first course in partial differential equations with complex variables and transform methods (pp. 29-36, 48-54, 63-126, 141-145, 160-169). New York: Dover Publications.

We have that

$$C_n = \frac{2}{\pi}\int_0^{\pi} \sin{(x)}^2 \sin nx \, dx,$$

using the trigonometric identity $\sin{(x^2)} = \frac{1-\cos 2x}{2}$ we have

$$C_n = \frac{1}{\pi}\int_0^{\pi} (1-\cos 2x)\sin nx \, dx$$

$$C_n = -\frac{2\sin x^2 \cos nx(n^2-4) - \cos{(nx-2x)}(n+2) + \cos{(nx+2x)}(n-2)}{\pi n^3 - 4\pi n}\Bigg|_0^{\pi}$$

substituting the limits of integration we obtain

$$C_n = -\frac{-\cos \pi n(n+2) + \cos \pi n(n-2) + 4}{\pi n^3 - 4\pi n}$$

we simplify using the identity $\cos nx = (-1)^n$

$$C_n := \begin{cases} \frac{2(2(-1)^n-2)}{\pi n(n^2-4)}, & n \neq 2 \\ \\ 0, & n = 2 \end{cases}$$

Therefore, when n is an even number the Fourier coefficient equals zero, $C_n = 0$. Thus, for convenience we take n to be the odd integers of the form $n = 2k-1$. From this we have the solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{C_n \sin nx \sinh n(\pi - y)}{\sinh n\pi}$$

Taking C_n not equal to zero,

$$C_n = \frac{-8}{\pi(2k-1)((2k-1)^2-4)}$$

we obtain

$$u(x,y) = \frac{-8}{\pi} \sum_{k=1}^{\infty} \frac{\sinh{(2k-1)(\pi - y)} \sin x(2k-1)}{\sinh{(\pi(2k-1))}(4k^2-1)(2k-3)}. \tag{30}$$

We finally observe that this series uniformly converges in $x \in [0,\pi], \quad y \in [0,\pi]$.

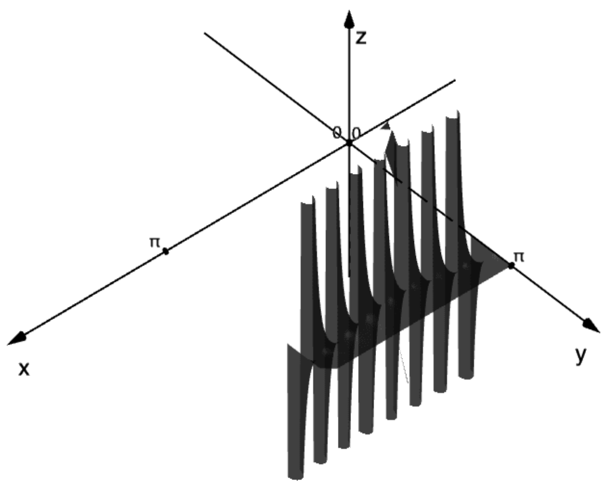

Figure 9: Solution of $u(x,y)$, with $k=8$

It is observed that the number of maxima equals to $k-1$ and the number of minima equals k with $k \in \mathbb{N}$. A sinusoidal behavior is observed at $y=0$ where $u(x,0) = \sin{(x^2)}$. This solution space $u(x,y)$ is a harmonic function whose Laplacian at any point in the domain of $u(x,y)$ is equal to zero.

7 Conclusion

To conduct this extended essay, many different mathematical resources were used, especially mathematical analysis. The relationship between Fourier series and Laplace´s equation was derived from scratch with the objective to obtain a completely defined solution to Laplace´s equation and to observe it in an analytic approach.

Laplace´s equation is satisfied by harmonic functions. In this case the method started off with partial differential equations that when are simplified through the method of separation of variables we obtain simpler differential equations of single variables: ordinary homogeneous differential equations. After the separation of variables in Laplace´s equation, the solution to both ODE's can be obtained thanks to the principle of superposition of ODE´s that involves the concepts of linear Independence and linear combination. From the boundary conditions of the ODE's the eigenvalues were obtained and thus its corresponding eigenfunctions that form a basis to the solution vector space of the differential equations in a specific dimension.The principle of superposition of ODE's affords us the possibility to express the solution space as a series of eigenfunctions that are an extension of the Fourier series. These Fourier series are a sequences of functions that converge always in the mean to limit functions. The Kronecker Delta and the inner product properties of vectors and functions allow us prove a set of vectors to be orthonormal, allowing the construction of the Fourier series. On the other hand, a Hilbert space V is constructed with Cauchy sequences that always converge in the mean and so does the Fourier Series. In the Hilbert space a set of vectors is said to be complete if every function can be expressed as the linear combination of the vectors in V and a Fourier coefficient. Uniform convergence involves convergence in the mean and point wise and thus, some Fourier Series can converge uniformly. The functions $1, \sin nx$, and $\cos nx$ are orthogonal functions in the interval $[0, 2\pi]$. The functions are then normalized to obtain a unit magnitudes and to obtain an orthonormal set of functions. The Fourier coefficients are derived with the aforementioned process. With the eigenvalues and eigenfunctions of the now separated ODE´s from the partial differential equations $\varphi(x)$ and $\psi(y)$ are obtained. The solution function of the Laplace equation $u(x, y)$ is an infinite series spanned by the linear combination of $\varphi(x)$ and $\psi(y)$ times a constant b_n that can only be obtained through calculus and the Fourier coefficients from the Fourier sine or cosine series. Henceforth the solution space of Laplace´s equation is only complete and with a defined value for b_n that later on tells us the convergence and behavior of the solution space to Laplace's equation. Every different boundary condition defines distinct behaviors that can be observed graphically or mathematically.

We now conclude and affirm that there is an crucial relationship between Fourier Analysis and Laplace´s equation that allows us to define a complete solution space without unknown coefficients and further assumptions. Moreover this Fourier analysis can be analogously employed to solve for a complete solution space of a plethora of different partial differential equations by following the same derivation method, mathematical tools and concepts conducted in this extended essay.

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