Field ($\mathbb{F}$): set containing elements 0 and 1, together with operations + and $\cdot$ on $\mathbb{F}$, satisfying:

(1 & 2) Commutativity on + and $\cdot$

$\forall x, y \in \mathbb{F}$, $x+y=y+x$, $xy=yx$

(3 & 4) Associativity on + and $\cdot$.

$\forall x,y,z \in \mathbb{F}$, $x+(y+z)=(x+y)+z$, $x\cdot(y\cdot z)=(x\cdot y)\cdot z$

(5) Distributive property

$\forall x,y,z \in \mathbb{F}$ $\quad x\cdot(y+z)=xy+xz$

(6) Additive identity:

$\forall x \in \mathbb{F}$, $x+0=x$

(7) Multiplicative identity:

$\forall x \in \mathbb{F}$, $x \cdot 1 = x$

(8) Additive Inverse

$\forall x \in \mathbb{F}$, $\exists!\, y \in \mathbb{F}$, $x+y=0$ $(-x)$

(9) Multiplicative Inverse:

$\forall x \in \mathbb{F}, x \neq 0$, $\exists!\, y \in \mathbb{F}$, $xy=1$. $1/(x)$

$U_1+\dots+U_n := \{\vec{v} \in V : \vec{v}=\vec{v_1}+\dots+\vec{v_n}$, where $\vec{v_i} \in U_i\ \forall i\}$

Direct Sum: Let $U_1,\dots,U_n$ be subspaces of V. Then $U_1+\dots+U_n$ is a direct sum if $\forall \vec{y} \in U_1+\dots+U_n$, $\exists!\, y_1 \in U_1,\dots,y_n \in U_n$, $\vec{y}=\vec{y_1}+\dots+\vec{y_n}$. Then $U_1 \oplus U_2 \oplus \dots \oplus U_n$.

Prop: $U_1 \oplus U_2 \oplus \dots \oplus U_n$ iff $\vec{0}=\vec{0}+\dots+\vec{0}$ is the only way to represent $\vec{0} \in V$ as a sum $\vec{y_1}+\dots+\vec{y_n}$, $\vec{y_i} \in U_i$

Linear combination of a list of vectors $\vec{v_1},\dots,\vec{v_n} \in V$ is $a_1\vec{v_1}+\dots+a_n\vec{v_n}$, $a_i \in \mathbb{F}\ \forall i$: $\operatorname{span}(\vec{v_1},\dots,\vec{v_n})$

PROP: $\operatorname{span}(v_1,\dots,v_n)$: smallest subspace containing $\vec{v_1},\dots,\vec{v_n}$

$\vec{v_1},\dots,\vec{v_n}$ is linearly independent if $a_1\vec{v_1}+\dots+a_n\vec{v_n}=\vec{0}$ implies $a_i=0$. is linearly dependent if $\exists\, a_i \neq 0$ s.t. $a_1\vec{v_1}+\dots+\dots=\vec{0}$.

PROP: $\operatorname{span}(\vec{v_1},\dots,\vec{v_n})=\operatorname{span}(\vec{v_1},\dots,\vec{v_n},\vec{w})$ iff $\vec{w} \in \operatorname{span}(\vec{v_1},\dots,\vec{v_n})$

PROP: every vector in $\operatorname{span}(\vec{v_1},\dots,\vec{v_n})$ has a unique repres. iff $\vec{v_1},\dots,\vec{v_n}$ is linearly independent

PROP: if $\vec{v_1},\dots,\vec{v_n}$ are lin. dependent then $\exists j \in \{1,\dots,n\}$ $v_j \in \operatorname{span}(\vec{v_1},\dots,\vec{v_{j-1}})$

$V=U \oplus W$ show $V=U+W$, $U \cap W=\{0\}$

theorem: Let V be a finite dim'l V. If $\vec{v_1},\dots,\vec{v_n}$ are l. i. in V and $\operatorname{span}(\vec{w_1},\dots,\vec{w_m}) = V$, then $n \le m$

TIP:

$f(x)=-f(-x)$ $\qquad (f+g)(x) = -(f+g)(-x)$

$(af)(x) = -(af)(-x)$

A vector space V over a field $\mathbb{F}$ is a set V together with

vector addition: $\forall \vec{v},\vec{w} \in V$, $\vec{v}+\vec{w} \in V$

scalar multiplication: $\forall \vec{v} \in V, \lambda \in \mathbb{F}$, $\lambda\vec{v} \in V$

$\vec{0}\,!$, and satisfy comm, assoc, add. inv, distr.

Prop: $0 \cdot \vec{v} = \vec{0}$

$0\cdot\vec{v}=(0+0)\vec{v}$ by add. id on $\mathbb{F}$

$= 0\vec{v}+0\vec{v}$ by distr.

$\vec{0}=0\cdot\vec{v}+(-0\cdot\vec{v})$ by add. inv on V

$=(0\cdot\vec{v}+0\cdot\vec{v})+(-0\cdot\vec{v})$

$=(0\cdot\vec{v})+(0\cdot\vec{v}-0\cdot\vec{v})$ By assoc. on V

$= 0\vec{v}+\vec{0}$ by add inv on V

$0\vec{v}$ by add id on V

Proof: $a\cdot\vec{0}=\vec{0}$

$a\cdot\vec{0}=a(\vec{0}+\vec{0})$ By add. id on V

$= a\vec{0}+a\vec{0}$ By distr.

$\vec{0} = a\vec{0}+(-a\vec{0})$ By add inv on V

$=(a\vec{0}+a\vec{0})+(-a\vec{0})$

$= a\vec{0}+(a\vec{0}+(-a\vec{0}))$ By assoc on V

$= a\vec{0}+\vec{0}$ By add id on V

$\vec{0}=a\vec{0}$

Proof. $(-1)\vec{v}=-\vec{v}$

$\vec{0}=0\cdot\vec{v}=(1+(-1))\vec{v}$ by add inv on $\mathbb{F}$

$=1v+(-1)\vec{v}$ by distr

$\vec{0}=\vec{v}+(-1)\vec{v}$ by scalar mult. id

$\therefore (-1)\vec{v}$ is $-\vec{v}$

Ex: Continuous $f: \mathbb{R} \to \mathbb{R}$ is a V

$\vec{0} := \vec{0}(\mathbb{R})=\{0\}$

V. add $:= (f+g)(x) := f(x)+g(y)$

s. mult $:= (\lambda f)(x) := \lambda(f(x))$

Polynomials in x over $\mathbb{R}$: $\mathbb{R}[x]$

A subset of $W \subseteq V$ is a subspace of V if

(1) $\vec{0} \in W$, (2) $\vec{u},\vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$ (closed under a.)

(3) $a \in \mathbb{F}$, $\vec{u} \in W$, $a\vec{u} \in W$ (closed under s. m.)

Ex: $\{f \in \mathbb{R}[x] : f(3)=0\}$ is a subspace of $\mathbb{R}[x]$:

(1) $0(3)=0$, so $\vec{0} \in W$, (2) $f,g \in W$, $(f+g)(3)=f(3)+g(3)=\vec{0}+\vec{0}=\vec{0}$

(3) $f \in W, \lambda \in \mathbb{F}$, $(\lambda f)(3)=\lambda(f(3))=\lambda(0)=\vec{0}$.

$\dim(U+V)=\dim(U)+\dim(V)-\dim(U \cap V)$

$\mathbb{F}^{m,n}$: set of all $m\times n$ matrices with entries in $\mathbb{F}$. $\dim(\mathbb{F}^{m,n}) = mn$

$\lambda \in \mathbb{F}$, $T \in \mathcal{L}(V,W)$. Then $M(\lambda T) = \lambda M(T)$

$S, T \in \mathcal{L}(V,W)$. Then $M(S+T) = M(S) + M(T)$

$M(ST) = M(S)M(T)$. $A \in \mathbb{F}^{m,n}$, $C \in \mathbb{F}^{n,p}$: $AC \in \mathbb{F}^{m,p}$. $(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$ row $\cdot$ column

if $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$, then $M(ST) = M(S)M(T)$

$V = U + W$ if $v = u + w$, $u \in U, w \in W$

$\text{span}(v_1,\dots,v_n) = \text{span}(v_1,\dots,v_n, w)$ iff $w \in \text{span}(v_1,\dots,v_n)$

$\forall v \in \text{span}(v_1,\dots,v_n)$ has a unique representation iff $v_1,\dots,v_n$ is l. i.

If $v_1,\dots,v_n$ l. d., $\exists\, j \in \{1,\dots,n\}$ st $v_j \in \text{span}(v_1,\dots,v_{j-1})$

V: f.d.v.s. If $v_1,\dots,v_n$ l.i in V and $\text{span}(w_1,\dots,w_m) = V$, $n \le m$

$v_1,\dots,v_n$ is a basis iff $\forall v \in V$, $\exists!\, a_1,\dots,a_n \in \mathbb{F}$ st $v = a_1v_1 + \dots + a_nv_n$

V: f.d.v.s., $U \subseteq V$. Then $\exists\, W \subseteq V$ s.t. $V = U \oplus W$

Show $V = U \oplus W$: $V = U + W$, $\vec{0} = \vec{v} + \vec{w} \Longrightarrow \vec{u} = \vec{w} = \vec{0}$.

$f: S \to T$ if $\forall s \in S\ \exists!\, t \in T$ s.t $f(s) = t$

S: domain, $f(S) := \{t \in T : \exists s \in S, f(s) = t\}$: range

T: codomain

INJECTIVE if $f(s_1) = f(s_2) \Longrightarrow s_1 = s_2$

surjective if $f(S) = T$ (codomain = range)

$f: \mathbb{R}^2 \to \mathbb{R}^2$: $f(x_1, x_2) = f(x_1, x_1 + x_2)$

INJECTIVE? $f(x_1, x_2) = f(y_1, y_2)\dots$

Surjective? Let $(a,b) \in \mathbb{R}^2\dots$

$T(\vec{0}) = \vec{0}$.

$f(v_1 + v_2) = f(v_1) + f(v_2)$, $f(\lambda v) = \lambda f(v)$

$\mathcal{L}(V,W)$: set of all linear maps $V \to W$

STRUCTURE THEOREM: $B_V = v_1,\dots,v_n$, V, $w_1,\dots,w_n$ be any list $\in W$. $\therefore \exists!\, T \in \mathcal{L}(V,W)$ s.t. $T(v_i) = w_i$

$\forall T \in \mathcal{L}(V,W)$ $\dim(V) = n$, $\dim(W) = m$

$$M(T, B_V, B_W) := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$
k-th column: coeff. rep. of $T(v_k)$ in B_W

$T \in \mathcal{L}(V,W)$. T injective iff $T(\vec{v}) = \vec{0} \Rightarrow \vec{v} = \vec{0}$.

T surjective $\Longleftrightarrow$ range$(T) = W$

$\text{null}(T) := \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$. $\text{null}(T) = \{\vec{0}\} \Longleftrightarrow T$ is injective

$B_V = v_1,\dots,v_n$, V, $T(\vec{v}_i) = w_i \in W$. T injective iff $\vec{w}_1,\dots,\vec{w}_n$ l.i.

surjective iff $\text{span}(\vec{w}_1,\dots,\vec{w}_n) = W$. bijective $B_W = \{w_1,\dots,w_n\}$

Rank-Nullity Theorem: $T \in \mathcal{L}(V,W)$

$\dim(V) = \text{rank}(T) + \text{nullity}(T)$

if $U \subseteq V$ B_U can be extended to B_V.

$T \in \mathcal{L}(V,W)$

$\dim(V) > \dim(W)$ T can be surjective but no injective

$\dim(V) = \text{nullity}(T) + \text{rank}(T) \le \text{nullity}(T) + \dim(W)$

$\dim(V) < \dim(W)$ T can be injective but...not S.

$\dim(V) = \dim(V)$ $\therefore$ T injective $\Longleftrightarrow$ T surjective.

$T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$ $ST \in \mathcal{L}(U,W)$ $U \xrightarrow{T} V \xrightarrow{S} W$

$ST(\vec{u}) = S(T(\vec{u}))$. $\text{null}(T) \subseteq \text{null}(ST)$

$\text{range}(ST) \subseteq \text{range}(S)$

The k-th column of $M(T)$ consists of the scalars needed to write $T(v_k)$ as a linear combination of $(w_1, w_2, \dots, w_m)$: $T \in \mathcal{L}(V,W)$

Given FDVS U, V over $\mathbb{F}$, B_U, B_V then $\forall T \in \mathcal{L}(U,V)$, $\exists!\, M(T, B_U, B_V) \in \mathbb{F}^{m,n}$ describing the transformation

Given T, $\vec{u} \in U$ how to use $M(T, B_U, B_V)$ to find $T(\vec{u})$

Given V, $B_V = v_1,\dots,v_n$. the matrix of $\vec{v}$ in B_V is $M(\vec{v}, B_V)$

$M(\vec{v}, B_V) := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ if $\vec{v} = c_1\vec{v}_1 + \dots + c_n v_n$

if $V = \mathbb{R}^2$, $B_V = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{8}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \therefore M(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_V) = \begin{bmatrix} 3/8 \\ -1/8 \end{bmatrix}$

$B_V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \therefore M(\begin{bmatrix} a \\ b \end{bmatrix}, B_V) = \begin{bmatrix} a \\ b \end{bmatrix}$

The k-th column of $M(T, B_U, B_V)$ is $M(T(\vec{u}_k), B_V)$

$T \in \mathcal{L}(U,V)$, B_U, B_V. $\forall \vec{u} \in U$

$M(T(\vec{u}), B_V) = M(T, B_U, B_V)\, M(\vec{u}, B_U)$

$T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$

$M(ST, B_U, B_W) = M(S, B_V, B_W)\, M(T, B_U, B_V)$

$D: P_m(\mathbb{R}) \longrightarrow P_m(\mathbb{R})$, $D(f) = f' + f$

$M(D, B_{P_3(\mathbb{R})}, B_{P_3(\mathbb{R})})$ in standard basis. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$M(D\dots)M(D\dots)$

$D(x^3) = x^3 + 3x^2$, $D(x^2) = x^2 + 2x$, $D(x) = x + 1$, $D(1) = 1$, $M(D^2, \dots) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

$I_V \in \mathcal{L}(V,V)$, $M(I_V, B, B) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & 1 \end{pmatrix}$

$T \in \mathcal{L}(V,W)$ invertible if $\exists\, S \in \mathcal{L}(W,V)$ s.t. $ST = I_V \in \mathcal{L}(V,V)$, $TS = I_W \in \mathcal{L}(W,W)$

T invertible $\Longleftrightarrow$ T bijective.

$T: P_4(\mathbb{R}) \longrightarrow \mathbb{R}^5$ invertible equal dimension

$T: \mathcal{L}(V^n, W^m) \longrightarrow \mathbb{R}^{mn}$. $S \in \mathcal{L}(V,W)$ bijective $\Longleftrightarrow$ $\dim(V) = \text{nullity}(S) + \text{rank}(S)$
$= 0 + \dim(W)$
(inj.) (sur.)

$V \cong W$ if $\exists$ invertible $T \in \mathcal{L}(V,W)$. T: isomorphism.

$\forall V$, s.t. $\dim(V) = n$, V over $\mathbb{F}$, $V \cong \mathbb{F}^n$. [$U \cong V$ iff $\dim(U) = \dim(V)$]

Let $T \in \mathcal{L}(V)$, $\dim(V) \ne \infty$. T is inj. surj. inv. [$\mathcal{L}(V,W) \cong \mathbb{F}^{m,n}$ $\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W)$]

$n\times n$ M invertible if $\exists$ $n\times n$ M', $MM' = M'M = I_{n\times n}$

$V = \mathbb{R}^2$. $M_1(I_V, (\vec{e}_1, \vec{e}_2), (\vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2))$: $\Longrightarrow M(\dots) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$I_V(\vec{e}_1) = 1/2(\vec{e}_1 + \vec{e}_2) + 1/2(\vec{e}_1 - \vec{e}_2)$, $I_V(\vec{e}_2) = 1/2(\vec{e}_1 + \vec{e}_2) - 1/2(\vec{e}_1 - \vec{e}_2)$

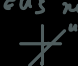
$M_2(I_V, (\vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2), (\vec{e}_1, \vec{e}_2)) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \therefore \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_V$

$V_1,\dots,V_n$ vector spaces over $\mathbb{F}$. $V_1 \times V_2 \dots \times V_n := \{(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) : \vec{v}_i \in V_i\}$

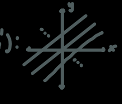
$\dim(V_1 \times V_2 \times \dots \times V_n) = \dim(V_1) + \dim(V_2) + \dots + \dim(V_n)$

$U_1,\dots,U_n \subseteq V$. $U_1 \oplus \dots \oplus U_n \Longleftrightarrow \dim(U_1 + \dots + U_n) = \dim(U_1) + \dots + \dim(U_n)$

V, $\mathbb{F}$, $U \subseteq V$, $\vec{v} \in V$, $\vec{v} + U := \{\vec{v} + \vec{u} : \vec{u} \in U\}$ not a subspace, affine subset.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + U = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + U = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + U = \{(x,y) : y = x - 1\}$ 

$U \subseteq V$, $\vec{v}, \vec{w} \in V$. (1) $\vec{v} + U = \vec{w} + U$. (2) $\vec{v} - \vec{w} \in U$. (3) $(\vec{v} + U) \cap (\vec{w} + U) \ne \emptyset$

$V/U := \{\vec{v} + U : \vec{v} \in V\}$ ($\forall$ parallel translates of U): $\mathbb{R}^2/\text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: 

$\dim(V/U) = \dim(V) - \dim(U)$

Eigenvector of $T \in \mathcal{L}(V)$: $\vec{v} \ne 0 \in V \to T(\vec{v}) = \lambda\vec{v}$, $\lambda \in \mathbb{F}$. λ: eigenvalue

$T \in \mathcal{L}(V,W)$. Define $\tilde{T}: V/\text{null}(T) \to W$ by $\tilde{T}(v + \text{null}\,T) = Tv$

$T \in \mathcal{L}(V)$. $U \subseteq V$ invariant under T if $u \in U \Rightarrow Tu \in U$

affine subset $A = x + U$. $x \in V$, $U \subseteq V$.

Eigenvectors

$\mathbb{R}$ spectral Theorem:

$T \in \mathcal{L}(V)_{\mathbb{R}}$. TFAE: T s.a., $\exists B_{\perp V} = \{\text{eigenvectors } T\}$

$T \in \mathcal{L}(V)$. $U \subseteq V$ invariant under T if $u \in U \Rightarrow Tu \in U$ | for $v \in V$, norm: $\|v\| = \sqrt{\langle v, v\rangle}$, $\|v\| = 0$ iff $v = 0$, $\|\lambda v\| = |\lambda|\|v\|$ | T has diagonal $M(T)$ with respect to $\exists B_{\perp V}$

$\forall$ fixed $u \in V$, the function $v \to \langle v, u\rangle$: lin. map from V to $\mathbb{F}$ | $E(\lambda$ $\{v : Tv = \lambda v\}$, $null(T - \lambda I) = E(\lambda, T)$, λ e.v. of T iff $T - \lambda I$ not inv

$T \in \mathcal{L}(V)$. $\lambda \in \mathbb{F}$ eigenvalue of T if $\exists v \in V$ s.t. $v \neq 0$ and $Tv = \lambda v$ v: eigenvector | $\langle 0, u\rangle = 0$, $\langle u, 0\rangle = 0$, $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$, $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$ | $\lambda_1, \dots, \lambda_m$ ($\lambda_i \neq \lambda_j$ if $i \neq j$). $v_1, \dots, v_m$ s.t. $Tv_i = \lambda_i v_i$. $v_1, \dots, v_m$ is lin. ind.

$u, v \in V$ orthogonal if $\langle u, v\rangle = 0$, | $E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \dots \oplus E(\lambda_m, T) \subseteq V$ direct sum of invariant subspaces

TFAE: λ e.v. of T, $T - \lambda I$ not injective, $T - \lambda I$ not surjective, $T - \lambda I$ not invertible | $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ | λ e.v for $T \in \mathcal{L}(V) \Leftrightarrow \exists \vec{v} \neq 0$ s.t $T\vec{v} = \lambda\vec{v} \Leftrightarrow null(T - \lambda I) \neq \{\vec{0}\}$, $T - I\lambda$ not inv., $E(\lambda, T) = \{\vec{v} : T\vec{v} = \lambda\vec{v}\} \neq \{\vec{0}\}$

$u, v \in V$, $|\langle u, v\rangle| \leq \|u\|\|v\|$ | Euclidean Norm $\vec{v} \in \mathbb{R}^n$: $\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. multilinear $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $(\vec{v_1}, \vec{v_2}) \mapsto v_1 \cdot v_2$ $(a\vec{v_1}) \cdot \vec{v_2} = a(\vec{v_1} \cdot \vec{v_2})$, $(v_1 + v_1') \cdot v_2 = v_1 \cdot v_2 + v_1' \cdot v_2$

$1 \Rightarrow 2$: $Tv = \lambda v \equiv (T - \lambda I)v = 0$ | $v_1, \dots, v_n$ orthogonal if $\|v_i\| = 1$ and $v_i \perp$ to all vectors in the list. | $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \geq 0$. $v \cdot v = 0$ iff $v = \vec{0}$. For $z = x + iy \in \mathbb{C}$, $\bar{z} = x - iy$. $|z|^2 = z \cdot \bar{z}$

$Tv = \lambda v$ iff $(T - \lambda I)v = 0$. $v \in V$, $v \neq 0$ | $e_1, \dots, e_m$ orthogonal if $\langle e_j, e_k\rangle \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ | I.P. V over $\mathbb{F}$. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$: $\langle v, v\rangle \in \mathbb{R}$, $\langle v, v\rangle \geq 0$ $\forall v \in V$. $\langle a_1 v_1 + a_2 v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$ $F = \mathbb{C}$

Orthonormal list

eigenvector of T corresponding to λ iff $v \in null(T - \lambda I)$ | $\|a_1 e_1, \dots, a_m e_m\|^2 = |a_1|^2 + \dots + |a_m|^2$ | $\langle v, \alpha w\rangle = \overline{\langle \alpha w, v\rangle} = \bar{\alpha}\overline{\langle w, v\rangle} = \bar{\alpha}\langle v, w\rangle$.

$e_1, \dots, e_n$ orthonormal basis in V, let $v \in V$

$v = \langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n$, $\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_n\rangle|^2$ | For I.P.S. $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v}\rangle}$; $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$. $V \perp W$ if $\langle V, W\rangle = 0$

$T \in \mathcal{L}(V)$. $\lambda_1, \dots, \lambda_m$ distinct eigenvalues of T $v_1, \dots, v_m$ corresponding eigv. $\therefore v_1, \dots, v_m$ lin. ind.

V has at most $\dim(V)$ distinct eigenvalues

Gram Schmidth:

$v_1, \dots, v_m$ lin. ind. in V. $e_1 = \frac{v_1}{\|v_1\|}$. e_j, $j = 2, \dots, m$

$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \dots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}$

if $\vec{u} \perp \vec{v} \Rightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|u+v\|^2$

$\langle w, v\rangle = 0$ $\vec{u} = c\vec{v} + \vec{w}$ $\vec{u} = \frac{\langle u, v\rangle}{\langle v, v\rangle}\vec{v} + \left(\vec{u} - \frac{\langle u, v\rangle}{\langle v, v\rangle}\vec{v}\right)$.

Cauchy-Schwarz

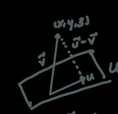
$T \in \mathcal{L}(V)$, $T^m := T^m = T \cdot T \dots T$ m times | $e_1, \dots, e_m$ is an orthonormal list in V, $span(v_1, \dots, v_j) = span(e_1, \dots, e_j)$, $j = 1, \dots, m$ | $|\langle u, v\rangle| \leq \|u\|\|v\|$ (equality iff $\vec{u}, \vec{v}$ lin. dep.

Triangle

$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$. $p(T) = a_0 I + a_1 T + \dots + a_m T^m$ | $\forall$ orthonormal list can be extended to orthonormal Basis | $\|u+v\| \leq \|u\| + \|v\|$ (eq. iff $\vec{v} = c\vec{u}$, $c > 0$)

Orth. Basis (ind. list)

$D \in \mathcal{L}(P(\mathbb{R}))$, $Dq = q'$. $p(x) = 7 - 3x + 5x^2$. $p(D) = 7I - 3D + 5D^2$; | $T \in \mathcal{L}(V)$. if $\exists M(T, B_V)$ uΔ $\Rightarrow \exists M(T, B_{orthonormal})$ uΔ | $v_1, \dots, v_n$ ON if $\|v_i\| = 1$, $\langle v_i, v_j\rangle = 0$ ($i \neq j$)



(x, y, z) $\vec{u} - \vec{v}$ $\vec{v}$ u U

$\therefore (p(D))q = 7q - 3q' + 5q''$ $\forall q \in P(\mathbb{R})$ | Linear Functional: element of $\mathcal{L}(V, \mathbb{F})$ Ex: $\varphi : P_2(\mathbb{R}) \to \mathbb{R}$, $\varphi(p) = \int p(t)\cos(\pi t)\, dt$ | distance (x, y, z) to $U \in V$ $\forall \vec{v} \in U$. Find $\|\vec{u} - \vec{v}\|$. $\min\|\vec{u} - \vec{v}\| = dist(\vec{v}, u)$

Ex: $\varphi \in \mathcal{L}(P_2(\mathbb{R}), \mathbb{R})$, $\varphi(p) = p'(2)$. Ex: $p(\vec{v}) = \langle v, \vec{v}\rangle$ is a linear F. | $v_1, \dots, v_n$ & $w_1, \dots, w_n$ O.N.B. of V, W

$*$ $\forall T \in \mathcal{L}(\mathbb{C}^n)$ has an eigenvalue $\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ | φ lin. functional of V. $\exists! u \in V$ s.t. $\varphi(v) = \langle v, u\rangle$ $\forall v \in V$. $u = \overline{\varphi(e_1)}e_1 + \dots + \overline{\varphi(e_n)}e_n$ $e_1, \dots, e_n$ B.o.N. | $M(T^*) = \overline{M(T)}^T$ (take $\forall A_{jj}$ conjugate, then transpose.

$M(T)$ upperΔ if $\forall A_{jk}$ below diagonal $= 0$. | $U \subseteq V$. $U^{\perp} = \{v \in V : \langle v, u\rangle = 0\ \forall u \in U\}$. $U^{\perp} \subseteq V$. $\{0\}^{\perp} = V$, $V^{\perp} = \{0\}$

$T \in \mathcal{L}(V)$, $B_V = v_1, \dots, v_n$. TFAE: $M(T, B_V)$ uΔ, V over $\mathbb{C}$ $Tv_j \in span(v_1, \dots, v_j)$ $\forall j$, $span(v_1, \dots, v_j)$ invariant under T $\forall j$ | $U \cap U^{\perp} \subset \{0\}$. $U, W \subset V$, $U \subset W$, $W^{\perp} \subset U^{\perp}$ | "Self-adjoint" "equivalent to normal and eigenval. $\in \mathbb{R}$"

$V = U \oplus U^{\perp}$. $\dim U^{\perp} = \dim V - \dim U$. $U = (U^{\perp})^{\perp}$

$*$ V over $\mathbb{C}$, $T \in \mathcal{L}(V)$. $\exists B_V$ s.t. $M(T)$ is uΔ. | Orth. Projection $U \subseteq V$. O.P. of V onto U: $P_U \in \mathcal{L}(V)$. $v \in V$. write $\underbrace{v = u + w}_{V = U \oplus U^{\perp}}$. $u \in U$. $w \in U^{\perp}$. $\therefore P_U v = u$

$T \in \mathcal{L}(V)$. $M(T)$ uΔ. T invertible iff $\forall A_{jj} \in$ diagonal $\neq 0$. | $U \subseteq V$. $v \in V$. $P_U \in \mathcal{L}(V)$; $P_U u = u$ $\forall u \in U$; $P_U w = 0$ $\forall w \in U^{\perp}$; range $P_U = U$; null $P_U = U^{\perp}$

$T \in \mathcal{L}(V)$. $M(T, B_V)$ uΔ. λ_i are precisely $A_{jj} \in$ diagonal | $v - P_U v \in U^{\perp}$; $P_U^2 = P_U$; $\|P_U v\| \leq \|v\|$; $P_U v = \langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m$ $e_1, \dots, e_m$ $B_{orth}U$. $\vec{v} = P_U(\vec{v}) + \vec{w}$

$M(T) = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda \in \mathbb{F}$: $M(T - \lambda I) = \begin{bmatrix} \lambda_1 - \lambda & & * \\ & \ddots & \\ 0 & & \lambda_n - \lambda \end{bmatrix}$ $\therefore T - \lambda I$ not invertible iff $\lambda = \lambda_i$. $\therefore \lambda$ e.v. iff λ equals an λ_i. | $U \subseteq V$. $v \in V$, $u \in U$. $\therefore \|v - P_U v\| \leq \|v - u\|$ | "Equal dimension means existence of isomorphism."

adjoint, T^*: $T \in \mathcal{L}(V, W)$. adjoint T: $T^* : W \to V$. s.t. $\langle Tv, w\rangle_W = \langle v, T^* w\rangle_V$, $v \in V$, $w \in W$

Eigenspace of T corresponding to λ: $E(\lambda, T) = null(T - \lambda I)$ | If $T \in \mathcal{L}(V, W)$, $T^* \in \mathcal{L}(W, V)$. $(ST)^* = T^* S^*$

$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. $E(8, T) = span(v_1)$, $E(5, T) = span(v_2, v_3)$ | $T \in \mathcal{L}(V, W)$. null $T^* = (\text{range } T)^{\perp}$; range $T^* = (\text{null } T)^{\perp}$; null $T = (\text{range } T^*)^{\perp}$; range $T = (\text{null } T^*)^{\perp}$

$\lambda_1, \dots, \lambda_m$ eigenvalues of $T \in \mathcal{L}(V)$. $E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$ $\therefore$ $\dim(E(\lambda_1, T)) + \dots + \dim(E(\lambda_m, T)) = \dim[E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)] \leq \dim V$ | $T \in \mathcal{L}(V, W)$, $B_{\perp V} = \{e_1, \dots, e_n\}$, $B_{\perp W} = \{f_1, \dots, f_m\}$. $M(T^*, B_{\perp W}, B_{\perp V})$: conjugate transpose of $M(T, B_{\perp V}, B_{\perp W})$

$T \in \mathcal{L}(V)$ self-adjoint if $T = T^*$ iff $\langle Tv, w\rangle = \langle v, Tw\rangle$ $\forall v, w \in V$

$T \in \mathcal{L}(V)$ diagonizable if $\exists$ diagonal $M(T, B_V)$ | Every eigenvalue of self-adjoint operator $\in \mathbb{R}$

$T \in \mathcal{L}(V)$, $\lambda_1, \dots, \lambda_m$ e.v.'s $\in T$. TFAE: T diagonizable, $\exists B_V = \{\text{eigenvectors of } T\}$, $\exists$ 1-d'l $U_1, \dots, U_m$ U_i: invariant under T s.t. $V = U_1 \oplus \dots \oplus U_n$, $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$, $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$ | $\forall$ $\mathbb{C}$, I.P.S., $T \in \mathcal{L}(V)$. if $\langle Tv, v\rangle = 0$ $\forall v \in V$. $\therefore T = 0$ (0 operator)

$\forall$ $\mathbb{C}$, I.P.S., $T \in \mathcal{L}(V)$. T self adjoint iff $\langle Tv, v\rangle \in \mathbb{R}$ $\forall v \in V$.

$T \in \mathcal{L}(V)$ normal if $TT^* = T^*T$. or iff $\|Tv\| = \|T^*v\|$ $\forall v \in V$

$T \in \mathcal{L}(V)$ normal, $v \in V$ eigenv. of T with λ. $\therefore v$ eigenvector of T^* with $\bar{\lambda}$

If $T \in \mathcal{L}(V)$ has $\dim(V)$ eigenvalues. T is diagonizable | $T \in \mathcal{L}(V)$ normal. $\therefore$ orthogonal eigenvectors for λ_i's

$\mathbb{C}$ spectral theorem

$T \in \mathcal{L}(\mathbb{F}^3)$, $T(x, y, z) = (2x + y, 5y + 3z, 8z)$ $M(T) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ | $T \in \mathcal{L}(V)$, $\mathbb{C}$. TFAE: T is normal; $\exists B_{\perp V}\{\text{eigenvectors of } T\}$ T has diagonal $M(T)$ with respect to $\exists B_{\perp V}$

$\lambda_1 = 2$, $\lambda_2 = 5$, $\lambda_3 = 8$ λ_1: $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 5-2 & 3 \\ 0 & 0 & 8-2 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

λ_2: $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2-5 & 1 & 0 \\ 0 & 5-5 & 3 \\ 0 & 0 & 8-5 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $(1/3y, y, 0)$ ... λ_3: | $T \in \mathcal{L}(V)$ self-adjoint. $b, c \in \mathbb{R}$ s.t. $b^2 < 4c$, then $T^2 + bT + cI$ invertible

EXAMPLE:

$\langle Tv, w\rangle_W = \langle v, T^* w\rangle_V$. $T : \mathbb{C}^3 \to \mathbb{C}^2$ $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \to \begin{bmatrix} z_1 + 2z_3 \\ 3iz_2 \end{bmatrix}$ $M(T) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3i & 0 \end{bmatrix}$

$\left\langle \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, T^*\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle_{\mathbb{C}^3} = \left\langle \begin{bmatrix} z_1 + 2z_3 \\ 3iz_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle_{\mathbb{C}^2}$

$(z_1 + 2z_3) \cdot \overline{w_1} + (3iz_2)\overline{w_2} = z_1\overline{w_1} + z_2\overline{(-3iw_2)} + z_3\overline{w_1} = \left\langle \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ -3iw_2 \\ 2w \end{bmatrix} \right\rangle_{\mathbb{C}^3}$ $\therefore T^*\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ 3iw_2 \\ 2w_1 \end{bmatrix}$, $M(T) = \begin{bmatrix} 1 & 0 \\ 0 & -3i \\ 2 & 0 \end{bmatrix}$

inner product: $(u, v) \to \langle u, v\rangle \in \mathbb{F}$. $\langle v, v\rangle \geq 0$ | $V \neq \{0\}$, $T \in \mathcal{L}(V)$ self-adjoint. T has an eigenvalue

I.P.S := V over $\mathbb{F}$ equipped with I.P. on V $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$. I.P. space a vector space V along with an IP on V | $T \in \mathcal{L}(V)$ s.a. $U \subseteq V$ invariant under T. $\therefore U^{\perp}$ invariant u. T; $T|_U \in \mathcal{L}(U)$ s.a.; $T|_{U^{\perp}} \in \mathcal{L}(U^{\perp})$ s.a.

The k-th column of $M(T)$ consists of the scalars needed to write $T(v_k)$ as a linear combination of $(w_1, w_2, \dots, w_m)$: $T \in \mathcal{L}(V, W)$