(1.1) **Euler's Theorem:** Let P be a polyhedron s.t. (i) any two vertices of P can be connected by a chain of edge, (ii) Any loop in P made of straight line segments separates P in 2 pieces $\therefore$ $v - e + f = 2$

(1.2) Topologically equivalent polyhedra have the same euler number.

Def. Let f be a function between two euclidean spaces. $f: \mathbb{E}^m \longrightarrow \mathbb{E}^n$. f continuous at $x \in \mathbb{E}^m$ if given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\|f(y) - f(x)\| < \varepsilon$, whenever $\|y - x\| < \delta$. f continuous if this holds $\forall x \in \mathbb{E}^m$.

Call a subset of N of $\mathbb{E}^m$ a neighborhood of the point $p \in \mathbb{E}^m$ if for some $r \in \mathbb{R}$, $r > 0$ the closed disc centre p radius r lies entirely inside N. $\therefore$ f continuous if given any $x \in \mathbb{E}^m$ and any neighborhood N of $f(x)$ in $\mathbb{E}^n$, then $f^{-1}(N)$ is a neighborhood of x in $\mathbb{E}^m$. If A open, $A = \mathring{A}$. $x \in \partial A$ if $\forall$ open N of x, $N \cap A$ and $N \cap (X - A) \neq \emptyset$

(1.3) X set. $\forall x$ of X a nonempty collection of subsets of X, called neighborhoods of x. These satisfy: (i) $x \in$ each of its n'h's (ii) Intersection of 2 n'h's of x is a n'h' of x. (iii) if N n'h' of x, if $U \subset X$ which contains N, then U n'h' of x. (iv) if N n'h' of x, if $\mathring{N}$ denotes the set $\{z \in N \mid N \text{ n'h' of } z\}$, then $\mathring{N}$ n'h' of x. ($\mathring{N}$:= interior of N). (i—iv) := topological space, n'h's of x, $\forall x \in X$:= topology on set X.

Def: Let X, Y t.s. $f: X \to Y$ continuous if $\forall x \in X$ and $\forall$ n'h' N of $f(x)$ in Y, the set $f^{-1}(N)$ n'h' of x in X.

Def: $h: X \to Y$ a homeomorphism if one-to-one, onto, continuous, and has continuous inverse. X, Y are homeomorphic (Topologically equivalent)

Def: A surface is a topological space in which each point has a neighborhood homeomorphic to the plane and for which any two points have disjoint n'h's.

(1.5) Classification Th'm: Any closed surface is homeomorphic either to the sphere, or to the sphere with n handles added, or sphere with n discs removed and replaced by Möbius Strips. No two of these $\cong$. In discrete τ every subset open

Def! Let X topological space, call $O \subseteq X$ open if its n'h' $\forall x \in O$. (ii) $\bigcup$ any collection of open sets is open. $\bigcap$ same but has to be finite X itself, $\emptyset$ open. Given N n'h of x, $\mathring{N}$ open and $x \in \mathring{N}$. In $\mathbb{E}^3$ a set is open if $\forall x$, $\exists B(x,r)$ s.t. $B(x,r) \in$ the set.

Def! Given a point $x \in X$, we call a subset N of x a n'h' of x if $\exists$ open set O s.t. $x \in O \subseteq N$. $A \subseteq Y \subseteq X$. A (closed) open in Y. $\therefore A = Y \cap U$, U (closed) open in X

2.1) A topology on set X := nonempty collection of open subsets of X s.t, any union of open sets is open, any finite intersection of open sets is open, $X, \emptyset$ open. A set together with a topology on it is called topological space.

Def: U open if given $x \in U$ we can always find $\varepsilon > 0$ s.t $B(x, \varepsilon)$ lies entirely in U.

Def: X topological s., $Y \subseteq X$. The open sets of the subspace or induced topology on Y are obtained by intersecting all open sets of X with Y. $\therefore$ A subset U of Y open in the subspace t. if $\exists$ open set O of X s.t. $U = O \cap Y$. Subspace $\tau := \{Y \cap U \mid U \in X\}$

Def: A subset of a topological space is closed if its compliment is open. A closed if $X \setminus A$ open.

Def: $\bigcap$ of any family of closed sets is closed (ii) $\bigcup$ of any finite family of closed sets is closed

Def: Let $A \subseteq X$, call point p of X limit point of A if $\forall$ n'h' of p contains at least 1 point of $A - \{p\}$

(2.2) A set is closed iff it contains all its limit points

(2.3) Union of A and all its limit points is called closure of $A = \bar{A}$. $\bar{A}$ is the smallest set containing $A \equiv \bigcap$ all closed sets containing A.

(2.4) Set is closed iff it is equal to its closure $\therefore A = \bar{A}$. A set whose closure is the whole space is said to be "dense" in the space

(2.5) Let β be a nonempty collection of subsets of a set X. If the intersection of any finite number of members of β is always in β and if $\bigcup \beta = X$, then β is a base for a topology on X.

(2.6) A function from $X \longrightarrow Y$ continuous iff the inverse image of each open set of Y is open in X. Composition of two maps is a map

(2.8) Suppose $f: X \to Y$ continuous, let $A \subseteq X$ have the subspace topology. Then $f|A: A \longrightarrow Y$ is continuous

(2.9) TFAE: (i) $f: X \to Y$ a map; (ii) if β base for the topology of Y, the inverse image of $\forall$ member of β is open in X; (iii) $f(\bar{A}) \subseteq \overline{f(A)}$ $\forall A$, $A \subseteq X$ (iv) $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ $\forall B$, $B \subseteq Y$. (v) The inverse image of each closed set in Y is closed in X.

(2.12) Metric on a set X is a real-valued function d on $X \times X$ s.t. $\forall x, y, z \in X$: (i) $d(x,y) \geq 0$; (ii) $d(x,y) = d(y,x)$; (iii) $d(x,y) + d(y,z) \geq d(x,z)$. Set + metric: metric space

Def: Let X topolog. space. Let $\mathcal{F}$ be a family of open subsets of X whose $\bigcup$ is all of X. $\mathcal{F}$ is an open cover of X. If $\mathcal{F}' \subset \mathcal{F}$ and $\bigcup \mathcal{F}' = X$. $\mathcal{F}'$:= subcover of X.

(3.1) A subset X of $\mathbb{E}^n$ closed and bounded iff $\forall$ open cover of X (with the induced top.) has a finite subcover $\equiv$ subset of $\mathbb{E}^n$ compact iff closed and bounded

(3.2) A Topological space X is compact if every open cover of X has a finite subcover

Def: A continuous real-valued f. defined on a compact space is bounded and attains its bounds. An infinite set of points in a compact s. has a limit point

Def: Compactness is a topological property of a space $\therefore$ If X compact, $X \simeq Y \implies Y$ is compact. If E closed-bounded, and $E \subset \mathbb{R}^k \implies E$ compact.

(3.3) Heine-Borel Theorem: A closed interval of the real line is compact. $[0,1]^n \subseteq \mathbb{R}^n$ is compact

def: A sequence of points is Cauchy iff $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ $\|x_n - x_m\| < \varepsilon$. Cauchy seq. converge $\therefore \exists x \in \mathbb{R}^n$ s.t. $\{x_n\} \longrightarrow x$.

(3.4) The continuous image of a compact space is compact. **(3.5)** Let $A \subseteq X$, X compact, $\implies A$ compact

def: A Topological s. with property that two diff. points can always be surrounded by disjoint open sets is called Hausdorff space

(3.6) If A compact subset of a Haussdorf s. X, and if $x \in X - A$, $\exists$ disjoint n'h's of x and A. $\therefore$ a compact subset of a Housdorff s. is closed.

(3.7) A one-one, onto, continuous function from a compact s. X to Hausdorff s. Y is a homeomorphism

(3.8) Bolzano-Weierstrass property: An infinite subset of a compact space must have a limit point. $\therefore$ to ∞ in some direction in Euclidean s.

(3.9) A compact subset of a euclidean space is closed and bounded. Due (3.8) A subspace of Euclidean s. cannot be stretched to ∞ in 1 direction.

(3.10) A continuous real-valued function defined on a compact space is bounded and attains its bounds.

(3.11) Lebesgue's Lemma: X compact metric space. $\mathcal{F}$ open cover of X. $\exists \delta \in \mathbb{R} > 0$ s.t. $\forall C \subseteq X$ with diameter $< \delta$, $C \subseteq$ some member of $\mathcal{F}$.

Def: X, Y top. s., let $\mathcal{B}$ denote the family $\forall$ subset of $X \times Y$ of the form $U \times V$, where U is open in X, V open in Y. $\therefore \bigcup \mathcal{B} = X \times Y$ and the intersection of any two members of $\mathcal{B}$ lie in $\mathcal{B}$. $\therefore \mathcal{B} :=$ base t. on $X \times Y$ (product t.). Set $X \times Y$ equipped with $\mathcal{B}$ is called a product space.

Def: Functions $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ s.t. $p_1(x,y) = x$, $p_2(x,y) = y$ are called projections.

(3.12) if $X \times Y$ has product t. then p_1, p_2 continuous and take open sets to open sets. $\mathcal{B}$: smallest topology on $X \times Y$ s.t both p_1, p_2 continuous

(3.13) $f : Z \to X \times Y$ is continuous iff $p_1 f : Z \to X$, $p_2 f : Z \to Y$ are both continuous.

(3.14) $X \times Y$ is Hausdorff iff X, Y Hausdorff. **(3.15)** $X \times Y$ compact iff X, Y compact

(3.16) X top. s., $\mathcal{B}$ a base t. for X. Then X compact iff $\forall$ open cover of X by members of $\mathcal{B}$ has a finite subcover.

(3.17) A space X is connected if whenever it is decomposed as the union $A \cup B$, $A, B \neq \emptyset$ then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. **(3.18)** $\mathbb{R}$ is a connected space

(3.19) A nonempty subset of the real line is connected iff it's an interval. **(3.21)** Continuous image of connected space is connected.

(3.20) TFAE: (i) X connected (ii) Only subsets of X both open and closed are $X, \emptyset$. (iii) $X \neq \bigcup 2$ nonempty disjoint open sets (iv) $\nexists f$ onto, continuous from X to a discrete space which contains more than one point. **(3.22)** if $h : X \to Y$ homeomorphism then X connected iff Y connected.

(3.23) Let X top. space, let $Z \subseteq X$. If Z connected, and Z dense in X. Then X connected.

(3.24) If Z connected, $Z \subseteq X$, and if $Z \subseteq Y \subseteq \bar{Z}$, then Y connected. In particular, $\bar{Z}$ is connected.

(3.25) Let $\mathcal{F}$ family of subsets of X whose union $= X$. If each member of $\mathcal{F}$ is connected, and if not two members of $\mathcal{F}$ are separated from another then X connected. If $A, B \subseteq X$, and if $\bar{A} \cap \bar{B} = \emptyset$ then A, B separated in X. **(3.26)** if X, Y connected $\implies X \times Y$ is connected.

(3.27) If X not connected, it breaks up as a union of connected pieces any two of which separated from each other. Components: A component of X is a maximal connected subset of X. Each component is closed and distinct ones are separated.

(3.28) A path in t.s. $X :=$ continuous $\gamma : [0,1] \longrightarrow X$. $\gamma(0), \gamma(1)$ beginning and endpoint joined by γ. If $\gamma^{-1} := \gamma^{-1}(t) = \gamma(1-t)$, $0 \leq t \leq 1$ then γ^{-1} is a path in X joining $\gamma(1)$ to $\gamma(0)$. A space path-connected if any two points can be joined by a path. γ path in X, if $f : X \to Y$ continuous $\implies [0,1] \xrightarrow{\gamma} X \xrightarrow{f} Y$ path in Y

(3.28.1) Then if $h : X \to Y$ a homeomorphism, X path connected, then Y path connected. path-connectedness: t. property of a space.

(3.29) A path connected space is connected. **(3.30)** A connected open subset of euclidean space is path-connected. P. component := maximal p. connected subset

Def: X t.s. let $\mathcal{P}$ family of disjoint nonempty subsets of X s.t. $\bigcup \mathcal{P} = X$. Form new space Y (identification space) := points of Y are the members of $\mathcal{P}$ and, if $\pi : X \to Y$ sends each point of X to the subset of $\mathcal{P}$ containing it, the topology of Y is the largest for which π is continuous

$X \xrightarrow{\pi} X/\sim := \{[x]_\sim \mid x \in X\}$

$\therefore$ A subset O of Y is open iff $\pi^{-1}(O)$ is open in X. This top. called identification topology on Y. $X \xrightarrow{\pi} [x]$ (π surjective)

Think of Y as the space obtained from X by identifying each subset of $\mathcal{P}$ to a single point. $x \sim x'$ iff x and $x' \in \mathcal{P}_i$ (same partition)

(4.1) Let Y identif. space, Z arbitrary top. s. $f : Y \to Z$ continuous iff composition $f\pi : X \longrightarrow Z$ continuous. $[x] = \{z \mid z \sim x\}$ equiv. class

Def: Let $f : X \to Y$ onto map, and Y largest for which f continuous. f: id. map. $\forall f : X \to Y$ gives rise to a partition of X whose members are subsets $\{f^{-1}(y)\}$, $y \in Y$. Let Y_* denote id. map with this partition, and $\pi : X \longrightarrow Y_*$ the usual map.

Cone on X: $X\times I/\sim\; = CX = X\times I/X\times\{0\}$, $\sim := ((x,0)\sim(y,0)\ \forall x,y\in X)$ $\therefore CX = X\times I/((x,0)\sim(y,0))$

Prop: Let $q: X\to Y$ open quotient map, then Y Hausdorff $\iff R=\{(x_1,x_2)\mid q(x_1)=q(x_2)\}$ closed in $X\times X$.

Def: Let $q: X\to Y$ be any map. A subset of the form $q^{-1}(y)$ for $y\in Y$ is called a fiber of q. A subset $U\subseteq X$ is called saturated w.r.t. q if $U=q^{-1}(V)$ for some $V\subseteq Y$. **Prop:** A continuous surjective map $q: X\to Y$ quotient map iff takes saturated (closed) open subsets to (closed) open subsets.

(i) Any composition of quotient maps is a quotient map; (iii) Injective quotient map is a homeomorphism. $K\subseteq Y$ closed $\iff q^{-1}(K)$ closed; (iv) If $U\subseteq X$ saturated (closed) (open) then $q|_U: U\to q(U)$ quotient map. (iv) If $\{q_\alpha: X_\alpha\to Y_\alpha\}$ family of q. m., then $\tilde{q}: \coprod_\alpha X_\alpha\to\coprod_\alpha Y_\alpha$ a quotient map.

Let $f: X\to Y$ be continuous map open or closed: (a) If f injective topological embedding; b) f surjective, quotient map; c) f bijective, homeomorphism.

Lemma: Let $q: X\to Y$ quotient map. $\forall$ space Z, a map $f: Y\to Z$ continuous iff $f\circ q$ continuous

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f\circ q"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Theorem: Let $q: X\to Y$ quotient map, $f: X\to Y$ continuous s.t. $q(x)=q(x')\Rightarrow f(x)=f(x')$ $\therefore \exists!$ continuous map $\tilde{f}: Y\to Z$ satisfying $f=\tilde{f}\circ q$

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f"] & \\ Y \arrow[r, dotted, "\tilde{f}"'] & Z \end{tikzcd}$$

Suppose $q_1: X\to Y_1$, $q_2: X\to Y_2$ quotient maps s.t. $q_1(x)=q_1(x')\iff q_2(x)=q_2(x')$ $\therefore Y_1\cong Y_2$

$$\begin{tikzcd} & X \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ Y_1 \arrow[rr, dotted, bend left, "\tilde{q}_1"] & & Y_2 \arrow[ll, dotted, bend left, "\tilde{q}_2"] \end{tikzcd}$$

$\tilde{q}_1\circ q_2 = q_1$, $q_2=\tilde{q}_2\circ q_1$, $\tilde{q}_2\circ\tilde{q}_1 = id_{Y_2}$

(4.2) If f id. map then: (i) Y and Y_f are homeomorphic; (ii) a func. $g: Y\to Z$ continuous iff $g\circ f: X\to Z$ continuous

(4.3) Let $f: X\to Y$ onto map. If maps open sets in X to o.s. in Y or closed sets in X to c.s. in Y, then f: identification map

(4.4) Let $f: X\to Y$ onto map. If X compact and Y Hausdorff, then f: identification map.

Def: X, Y subsets of t.s. give X, Y, $X\cup Y$ induced t. If $f: X\to Z$, $g: Y\to Z$ agree on $X\cap Y$ then we define $f\cup g: X\cup Y\to Z$ by: $f\cup g(x)=f(x)$, $f\cup g(y)=f(y)$ ($x\in X$, $y\in Y$)

(4.6) Glueing Lemma: If X, Y closed in $X\cup Y$, and if both f, g continuous then $f\cup g$ continuous

Def: Introduce Disjoint union $X+Y$ of X, Y, and function $j: X+Y\to X\cup Y$ which when restricted to X or Y is just the inclusion in $X\cup Y$

(i) This function is continuous (ii) Composition $(f\cup g)j: X+Y\to Z$ continuous iff f, g continuous

(4.7) If j is id. map, and both $f: X\to Z$, $g: Y\to Z$ continuous, then $f\cup g: X\cup Y\to Z$ continuous

Def: Define $F := \bigcup X_\alpha\to Z$ (X_α, $\alpha\in A$ family of subsets) by glueing together the f_α, i.e., $F(x)=f_\alpha(x)$ if $x\in X_\alpha$. Let $\bigoplus X_\alpha$ denote the disjoint union of the spaces X_α, let $j: \bigoplus X_\alpha\to\bigcup X_\alpha$ s.t. when restricted to X_α is the inclusion $\bigcup X_\alpha$.

(4.8) If j id. map, and if each f_α continuous then F continuous.

(4.9) A Top. Group G is both a Hausdorff t.s. and a group. $m: G\times G\to G$, $i: G\to G$ continuous (i: sends group element to its inverse)

Def: Group of invertible $n\times n$ M with $\mathbb{R}$ entries. Group structure: M multiplication. For the topology identify each $n\times n$ matrix $A=(a_{ij})$ with the point $(a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, a_{31}, \dots, a_{nn})$ of E^{n^2} and take subspace topology. This group: General linear group: $GL(n)$ is a topological group

Def: Orthogonal group $O(n)$ consisting of $n\times n$ orthogonal matrices with $\mathbb{R}$ entries. $O(n)$ has topology and group structure induced from $GL(n)$. The subgroup of $O(n)$ consisting of M with $\det M=1$ is called the special orthogonal group: $SO(n)$

(4.10) Let G t.g. Let K: connected component of G which contains the identity element. $\therefore K$ is a closed normal subgroup of G. $G=O(n)\to K=SO(n)$

(4.11) In a connected topological group $\forall$ N'd of the identity element is a set of generators for the whole group. **(4.13)** $O(n)$ and $SO(n)$ compact)

Def: If $h\in G$ then $R_h: G\to G$ is defined by $R_h(x)=xh$ $\forall x\in G$. $L_h: G\to G := L_h(x)=hx$ $\forall x\in G$. L_h and R_h are homeomorphisms

Lemma: $\forall$ pair $g, h\in G$ t.g. $\exists$ homeomorphism $\varphi: G\to G$ mapping g to h: $\varphi(g)=h$. $L_a L_b = L_{ab}$, $R_a R_b = R_{ba}$ $\forall a, b\in G$.

Lemma: Let G t.g. with identity e. Then G is Hausdorff iff $\{e\}$ is closed. Group G: $\cdot: G\times G\to G$ (Binary op. closed), (i) $\exists$ id $e\in G$, (ii) $\forall a\in G$, $\exists a^{-1}\in G$ s.t. $a\cdot a^{-1}=e$, $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

(Def) A t.g G is said to act as a group of homeomorphisms on a space X if each group element induces a homeomorphism of the space s.t. i) $hg(x)=h(g(x))$ $\forall g, h\in G$, $\forall x\in X$; (ii) $e(x)=x$, (e id. element $\in G$); (iii) function $G\times X\to X :=$ by $(g,x)\mapsto g(x)$ is continuous.

Def: If x point $\in X$, then $\forall g\in G$, the corresponding homeom. either fixes x or maps x to $g(x)$. The subset of X consisting of all images $g(x)$ as g varies through G, is called orbit of x: $O(x)$. If two orbits intersect: the relation defined by $x\sim y$ iff $x=g(y)$ $\exists g\in G$ is an equivalence relation on X whose equivalence classes are precisely the orbits of the given action. So the orbits define a partition of X. The corresponding id. space is called the orbit space: X/G. In constructing X/G we 'divide' by G s.t. we identify two points of X iff they differ by one of the homeomorphisms $x\mapsto g(x)$.

(4.15) Let G act on X and suppose that both G and X/G are connected, then X is connected. Def: $H\subseteq G$.g. if $(H, \cdot_G)$ satisfies group axioms, H subg.

(Def) By a loop in a space X, map $\alpha: I\to X$ s.t. $\alpha(0)=\alpha(1)$, based at point $\alpha(0)$. If α, β loops based at same point of X, define $\alpha\cdot\beta :=$ $\alpha\cdot\beta(s) = \begin{cases}\alpha(2s), & 0\le s\le 1/2\\ \beta(2s-1) & 1/2\le s\le 1.\end{cases}$ **(Def)** If $f, g: X\to Y$, to deform f continuously into g is called a homotopy

(Def) $f, g: X\to Y$ maps. f homotopic to g if $\exists$ a map $F: X\times I\to Y$ s.t $F(x,0)=f(x)$ and $F(x,1)=g(x)$ $\forall$ points $x\in X$. F: homotopy from f to g: $f\underset{F}{\simeq} g$

Homomorphism: map $f: G \to H$ (between groups) that respects binary operations: $f(a)\cdot f(b) = f(a\cdot b)$, $a,b \in G$. If bijective, f: isomorphism.

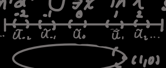
Def: G isomorphic to H iff $\exists$ homom. $\phi: G \to H$ and $\chi: H \to G$ s.t. $\phi\chi = 1_H$, $\chi\phi = 1_G$ $\equiv$ $\exists$ bijective homomorphism $\phi: G \to H$.

Th) let $f:(X,x_0) \to (Y,y_0)$, $y_0 = f(x_0)$ be continuous, $\therefore$ we get induced homomorphism $f_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$. f sends loops to loops

Lemma) Let C be a convex subset of $\mathbb{R}^n$, then $\pi_1(X,C) \simeq 1$ (trivial group). $H(x,t) = (1-t)\gamma(x) + tc$ $\therefore$ $[\gamma] = [C]$

Def: let $p: C \to X$ s.t. $\forall x \in X$, $\exists$ a (open) n'd' $U \ni x$ in X so that $p^{-1}(U) = \coprod_{\alpha \in \beta} \tilde{U}_\alpha$ where $p|_{\tilde{U}}: \tilde{U}_\alpha \to U$ is a homeomorphism. p is Covering Spaces.

$\mathbb{R} \xrightarrow{p} \mathbb{S}^1$: $p(x) = e^{2\pi i x}$: $(\cos(2\pi x), \sin(2\pi x))$



$p|_{\tilde{U}_2}$ h'm'm $\hookrightarrow$ $]^{(1,0)}$. **Corollary:** Coverings are quotients., Coverings are local h'm'm's.

Homotopy lifting Lemma: let $F: Y \times I \to X$ be a homotopy of maps. If $\exists$ a lift of F_0 to a cover $C \xrightarrow{p} X$, $\exists$ a map $\tilde{F}: Y \times 0 \to C$ satisfying

$$\begin{tikzcd} & & C \arrow[d, "p"] \\ Y\times 0 \arrow[r, hook] \arrow[urr, dotted, "\tilde{F}_0"] & Y\times I \arrow[ur, dotted, "\tilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

$\Rightarrow \exists!\ \tilde{F}: Y \times I \to C$ that lifts F and satisfies $\tilde{F}|_{Y\times 0} = \tilde{F}_0$,

$$\begin{tikzcd} & C \arrow[d] \\ Y\times I \arrow[ur, dotted, "\tilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

P.L.L: For each $\gamma: I \to X$ and initial lift of $\gamma(0)$ to $c \in C$. $\exists!$ path $\tilde{\gamma}: I \to C$ s.t. $p \circ \tilde{\gamma} = \gamma$ $\quad I \xrightarrow{\gamma} (X, \gamma(0))$

Lemma. The relation of 'homotopy' is an equivalence relation on the set of all maps from X to Y. $[f]$ 'homotopy class of f".

Lemma: The relation of 'homotopy relative to a subset A of X' is an equiv. relation on the set of all maps $X \to Y$ which agree with some map A

5.5) The set of homotopy classes of loops in X based at p forms a group under the mult. $\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha\cdot\beta\rangle$

Def) The group constructed in (5.5) is called fundamental group of X based at p: $\pi_1(X,p)$. Since any loop based at p must lie entirely inside the path component of X which contains p, we restrict to path-connected spaces: $\pi_1(X)$.

(5.6) If X is path-connected then $\pi_1(X,p)$, $\pi_1(X,q)$ are isomorphic $\forall p,q \in X$. $\exists\, \varphi: \pi_1(X,p) \to \pi_1(X,q)$ $\quad [f] \longmapsto [\rho\cdot f\cdot\bar{\rho}]$ (ρ path $p - q$)

(5.10) Path-Lifting Lemma: If σ is a path in S^1 which begins at point 1, $\exists!$ path $\tilde{\sigma}$ in $\mathbb{R}$ which begins at 0 and $\pi \circ \tilde{\sigma} = \sigma$

Th) Homotopy lifting Lemma: if $F: I\times I \to S^1$ map s.t. $F(0,t) = F(1,t) = 1$ for $0 \le t \le 1$, $\exists!$ map $\tilde{F}: I \times I \to \mathbb{R}$ s.t. $\pi \circ \tilde{F} = F$ and $\tilde{F}(0,t) = 0$, $0 \le t \le 1$

Def The fundamental group of a space X := its elements are loops in X starting and ending at a fixed basepoint $x_0 \in X$, but two such loops

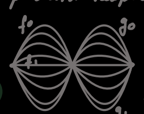
Def: regarded as determining the same element of the f. group if one loop can be continuously deformed to the other within X.

(ii) Path in X: $f: I \to X$, $I: [0,1]$. Homotopy of paths in X is a family $f_t: I \to X$, $0 \le t \le 1$, s.t. (i) endpoints $f_t(0) = x_0$, $f_t(1) = x_1$ are independent of t

Def: (ii) associated map $F: I \times I \to X := F(s,t) = f_t(s)$ continuous. When two paths f_0, f_1 connected likewise by homotopy f_t, they are said to be homotopic

(Prop) $\pi_1(X, x_0)$ $f: I \to X$ s.t. $f(0) = f(1)$ (loops). The set of all homotopy classes $[f]$ of loops $f: I \to X$ at basepoint x_0 is denoted $\pi_1(X,x_0)$ is a group with respect to the product $[f][g] = [f\cdot g]$: fundamental group. Given two paths $f,g: I \to X$ s.t. $f(1) = g(0)$

$f\cdot g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$



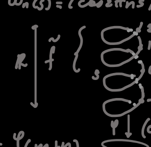
$f_0 \underset{f_t}{\simeq} f_1$, $g_0 \underset{g_t}{\simeq} g_1$, $f_0(1) = g_0(0)$ s.t. $f_0\cdot g_0$ is defined $\therefore$ $f_t \cdot g_t$ is defined and provide a homotopy $f_0\cdot g_0 \simeq f_1 \cdot g_1$

Def) linear homotopy $\quad f_t(s) = (1-t)f_0(s) + tf_1(s)$. **Prop:** Map $\beta_h: \pi_1(X,x_1) \to \pi_1(X,x_0)$ is an isomorphism

Def) A space is called simply-connected if (i) path-connected (ii) has trivial fundamental group.. **Th)** $\pi_1(\mathbb{S}^1)$ is an infinite cyclic group

$[f] = [g]$, $f \underset{h_t}{\simeq} g$, $[f'] = [g']$ the $\simeq$ of the loop $\omega(s) = (\cos(2\pi s), \sin(2\pi s))$ based at $(1,0)$. Note $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ $n \in \mathbb{Z}$

Th) $\pi_1(X,x_0)$ with $[f][f'] = [f\circ f']$ is' $\simeq$ $[f][f'] = [g][g']$, $[f\cdot f'] = [g\cdot g']$ $\quad f\cdot f' = h_0 \cdot h_0'$, $g \cdot g' = h_1 \cdot h_1'$ $\therefore$ $f\circ f' \simeq g \circ g'$



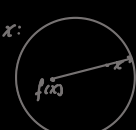
$\varphi(n) = [\ell(f_n)]$

Th) $\pi(\mathbb{S}^1) \approx \mathbb{Z}$ (isomorphic): $\omega_m \cdot \omega_n \simeq \omega$ group. id: $[C]$ where $C: [0,1] \to x_0$; inv: $[f]^{-1} = [\bar{f}]$ where $\bar{f}(t) = f(1-t)$.

Surjective: let $[f] \in \pi_1(\mathbb{S}^1)$. $\tilde{f} \simeq \tilde{\omega}_2$, $m+n \to [\omega_m][\omega_n] \to [\omega_{m+n}]$ $\therefore$ $\varphi: \mathbb{Z} \to \pi_1(\mathbb{S}^1)$ $\quad n \longmapsto [\omega_n]$ $\therefore$ $\varphi(m)\varphi(n) = [\omega_m][\omega_n] = [\omega_{m+n}] = \varphi(m+n)$ homomorphism.

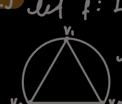
$h_t(s) = \{(1-t)\tilde{f}(s) + t\tilde{\omega}_2(s)$ $\quad\therefore f \simeq \omega_2$. Then in $\mathbb{R}$, $\tilde{f} \simeq \tilde{\omega}_n$, $f \simeq \omega_n$, thus $[f] = [\omega_n] = \varphi(n)$. –

Inj: Suppose $\varphi(m) = \varphi(n) \Rightarrow [\omega_m] = [\omega_n]$ $\therefore \exists h_t$ s.t. $\omega_m \underset{h_t}{\simeq} \omega_n$. Lift homotopy since $\tilde{h}_t$ preserve endpoints, $m = n$. $\quad \forall x \in D^2$, if $f(x) \ne x$:



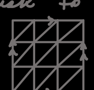
def: A space X simply connected iff its connected, and $\pi_1(X,p) = 1$, e. $\mathbb{R}^n$, convex subsets of $\mathbb{R}^n$ simply connected

Th) let $f: D^2 \to D^2$ be continuous map from disk to itself, then f has a fixed point (Brouwer's fixed p. theorem)



Triang. a circle.

min
Triang. $\mathbb{R}P^2$



Triang $T^2 \subseteq \mathbb{R}^2$;

Triang Annulus.

def) A collection S of simplices (in $\mathbb{R}^n$) is a simplicial complex iff (i) if $\forall \sigma \in S$ any (face $\in \sigma$) $\in S$; (ii) if $\sigma, \tau \in S$, and $\sigma \cap \tau \ne 0$ then $\sigma \cap \tau \in S$

def) let X t.s., $S \subseteq \mathbb{R}^n$ simplicial complex. We say a homeomorphism $h: |S| \to X$ is a triang. of X. def) Let K, L simplicial complexes we say K, L isomorphic iff $\exists$ bijection on vertex sets $\{v_1, v_2, \dots, v_k\}$ to $\{w_1, \dots, w_k\}$ s.t. $\{v_1, \dots, v_i\}$ simplex iff $\{\phi(v_1), \dots, \phi(v_i)\}$ is a simplex.
If so then $|K| \simeq |L|$.

def: Let $v_0, v_1, \dots, v_k$ points in $\mathbb{E}^n$. **Hyperplane** (of vertices) spanned by s. points consists of all linear combinations $\lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_k v_k$, $\lambda_i \in \mathbb{R}$, $\sum_{i=0}^{k} \lambda_i = 1$.
Points are in **general position** if any subset of them spans a **strictly smaller hyperplane**. Given $k+1$ points in general position, we call the smallest convex set containing them a **simplex of dimension k** **(k-simplex)**. $v_0, v_1, \dots, v_k$ vertices of the simplex. A point x lies in such convex set iff $x = \lambda_0 v_0 + \lambda_1 v_1 + \dots + \lambda_k v_k$. $\sum_{i=0}^{k} \lambda_i = 1$ 0-simplex $= \cdot v_0$; 1-simplex $= v_0 — v_1$; 2-simplex: v_0, v_1, v_2; 3-simplex $= v_0, v_1, v_2, v_3$; ... If two simplexes intersect they do so at common face

The set $\{v_0, v_1, \dots, v_k\}$ in **general position** iff $\{v_k - v_0, v_{k-1} - v_0, \dots, v_1 - v_0\}$ is **lin. ind. subset** of $\mathbb{R}^n$. **interior**$(\sigma) = \{ \sum_{i=0}^{k} \lambda_i v_i \mid \lambda_i > 0, \sum_{i=0}^{k} \lambda_i = 1 \}$

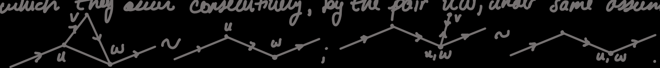
Def: A space X **triangulable** if homeomorphic to the union of fin. collection of simplexes which fit together nicely in some $\mathbb{E}^n$.

6.1) A fin. collection of simpl. in $\exists \mathbb{E}^n$ is called a **simplicial complex** if whenever a simplex lies in the collection so does each of its faces and whenever two simpl. in the collection intersect, they do so in a common face. A complex K when regarded as a t.S. called **polyhedron** and den. $|K|$.

6.2) A **triangulation** of a top. space X consists of a **simplicial complex** K and a **homeomorphism** $h: |K| \longrightarrow X$. **Triangulations** are not unique

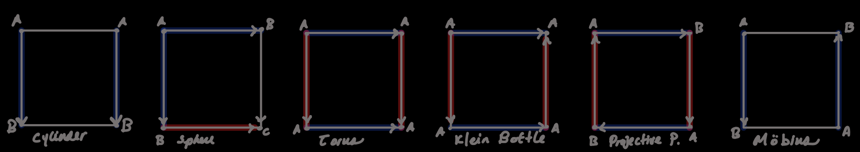
Def: A **face** of a **simplex** $\sigma = [v_0, v_1, \dots, v_k]$ is a simplex made of a subset of the vertices of σ

6.3) Lemma: Let K be a **simplicial complex** in $\mathbb{E}^n$. (i) $|K|$ is a **closed bounded** subset of $\mathbb{E}^n$ $\therefore$ $|K|$ **compact space** (ii) Each point of $|K|$ lies in the **interior** of exactly one simplex of K; (iii) If we take the **simplexes** of K separately and give their union the **identification topology** then we obtain $|K|$. (iv) if $|K|$ **connected, then path-connected**. **Def)** An **edge path** in a complex K is a seq. $v_0, v_1, \dots, v_k$ of vertices in which each consecutive pair $v_i v_{i+1}$ spans a simplex of K. If $v_0 = v_k = v_0$ we have an **edge loop based at v**. We consider two edge **paths equivalent** if we can obtain one from the other by a fin. # operations of the form: If three vertices uvw span a simplex of K, they may be replaced, if any edge path in which they occur consecutively, by the pair uw; under same assumption, uw may be replaced by uvw. Change repeated uu for u and vice versa.



. **Equivalence class** of edge path $v_0 v_1 \dots v_k := \{v_0 v_1 \dots v_k\}$. **Set of equivalence classes** of e.l. based at a vertex v forms **group** under mult. $\{v v_1 \dots v_{k-1} v\} \cdot \{v w_1 \dots w_{l-1} v\} = \{v v_1 \dots v_{k-1} v w_1 \dots w_{l-1} v\}$. id $e := \{v\}$; $\{v v_1 \dots v_{k-1} v\}^{-1} = \{v v_{k-1} \dots v_1 v\}$: **Edge Group** of K based at v: **$E(K,v)$**

6.10) $E(K,v)$ **isomorphic** to $\pi_1(|K|, v)$. **6.11)** **A maximal tree** contains all vertices of K



A **topological space** is a pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of **open subsets** of X so that:

1. $X \in \mathcal{O}, \varnothing \in \mathcal{O}$
2. For any collection $\{O_\alpha\}_{\alpha \in \Lambda}$ of open sets, $\bigcup_{\alpha \in \Lambda} O_\alpha$ is an open set.
3. For any finite collection $O_1, \dots, O_N$ of open sets, $\bigcap_{i=1}^{N} O_i$ is an open set.

A **base** (or **basis**) for the toplogical space $(X, \mathcal{O})$ is a collection $\mathcal{B} \subseteq \mathcal{O}$ such that any $O \in \mathcal{O}$ has $O = \bigcup_{\alpha \in \Lambda} B_\alpha$ for some $\{B_\alpha\}_{\alpha \in \Lambda} \subset \mathcal{B}$.

The following properties are finite-product-herediary, i.e. if the topological spaces $X_1, \dots, X_N$ have the property, then so does $X_1 \times \dots \times X_N$:

1. connectedness
2. path-connectedness
3. local path-connectedness
4. compactness
5. Hausdorffness
6. second-countability