$(\mathbb{C},+,\cdot)$: Field, C.D.: Complex Differentiable, n'd': neighborhood, CRE: Cauchy–Riemann Equations

$|z|=\sqrt{x^2+y^2}$, Triangle Ineq. ($z,w\in\mathbb{C}$): $|z+w|\le|z|+|w|$, $|z-w|\ge|z|-|w|$, $\frac{1}{z}=\frac{x-iy}{x^2+y^2}=\frac{\bar z}{|z|^2}$. Complex Polynomial $z\in\mathbb{C}$: $p(z)=a_nz^n+a_{n-1}z^{n-1}+\dots+a_0$. Fundamental Theorem of Algebra: $\forall p(z)$, $n\ge1$, $\exists!$ $p(z)=c(z-z_1)^{m_1}\cdots(z-z_k)^{m_k}$, $m_j\ge1$, $z_j\ne z_k$. Polar Form: $z\ne0$, $z=x+iy=r(\cos\theta+i\sin\theta)$, $r>0$, $\theta\in\mathbb{R}$, $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$, $e^{i\theta}=\cos\theta+i\sin\theta$ (De Moivre), $r=|z|$, $\theta=\arg(z)\triangleq\{\phi: z=|z|e^{i\phi}\}=\{\mathrm{Arg}(z)+2\pi k, k=0,\pm1,\dots\}$, $-\pi<\mathrm{Arg}(z)\le\pi$, $z=re^{i\theta}$, $e^{i(\theta+2\pi)}=e^{i\theta}$, $e^{i(\theta+\varphi)}=e^{i\theta}e^{i\varphi}\Rightarrow\begin{cases}\cos(\theta+\varphi)=\cos\theta\cos\varphi-\sin\theta\sin\varphi\\ \sin(\theta+\varphi)=\cos\varphi\sin\varphi+\sin\theta\cos\varphi\end{cases}$

$\therefore (e^{i\theta})^n=(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$. Open ε-disc ($z_0\in\mathbb{C}$, $\varepsilon>0$): $D_\varepsilon(z_0)\triangleq\{z\in\mathbb{C}\,|\,|z-z_0|<\varepsilon\}$. $U\subseteq\mathbb{C}$ open if $\forall z_0\in U\ \exists\varepsilon>0$ s.t. $D_\varepsilon(z_0)\subseteq U$. $F\subset\mathbb{C}$ closed if $\mathbb{C}\setminus F$ open. A point z_0: limit point of $S\subseteq\mathbb{C}$ if $\forall\varepsilon>0$, $D_\varepsilon(z_0)\cap S\ne\emptyset$. Closure of S: $\bar S:=\{z_0\in S\,|\,z_0\in S\vee z_0\ \text{l.p.}\}$. S closed $\Leftrightarrow S=\bar S$. S bounded: $\exists M>0$ s.t. $\forall z\in S$, $|z|\le M$. $\{z_n\}_{n\in\mathbb{N}}\subset\mathbb{C}$, $z_n\xrightarrow{n\to\infty}z$ if $\forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $\forall n>N$, $|z_n-z|<\varepsilon$. $z_n\to z\Leftrightarrow \mathrm{Re}(z_n)\to\mathrm{Re}(z)$, $\mathrm{Im}(z_n)\to\mathrm{Im}(z)$. Lemma: $z\in\bar S\Leftrightarrow\exists\{z_n\}\subseteq S$ s.t. $z_n\xrightarrow{n\to\infty}z$. Cauchy Sequence: $\{z_n\}_{n\in\mathbb{N}}$ Cauchy if $\forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $\forall n,m>N$ $|z_n-z_m|<\varepsilon$. Lemma: $\{z_n\}$ has a limit $\Leftrightarrow$ Cauchy (Bolzano-W) Let $\{z_n\}$ bounded $\therefore\exists\{z_{n_k}\}\xrightarrow{k\to\infty}z$. Complex Valued F.: $f:\mathbb{C}\to\mathbb{C}$, $f(z)=f(x+iy)=\mathrm{Re}(f(x+iy))+\mathrm{Im}(f(x+iy))=u(x,y)+iv(x,y)$, $u,v:\mathbb{R}^2\to\mathbb{R}$. Limit: $S\subset\mathbb{C}$, $f:S\to\mathbb{C}$, $z_0\in\bar S$, $a_0\in\mathbb{C}=\lim_{z\to z_0}f(z)$ if $\forall\varepsilon>0\ \exists\delta>0$ s.t. $|f(z)-a_0|<\varepsilon$ whenever $|z-z_0|<\delta$. Continuity: $S\subseteq\mathbb{C}$, $f:S\to\mathbb{C}$ $\therefore f$ continuous at $z_0\in S$ if $\forall\varepsilon>0\ \exists\delta>0$ s.t. $|f(z_0)-f(z)|<\varepsilon$ whenever $|z_0-z|<\delta$. Lemma: $\lim_{z\to z_0}f(z)=a_0\Leftrightarrow \mathrm{Re}(a_0)=\lim_{(x,y)\to(x_0,y_0)}u(x,y)$, $\mathrm{Im}(a_0)=\lim_{(x,y)\to(x_0,y_0)}v(x,y)$. Lemma: $f:\mathbb{C}\to\mathbb{C}$ then f continuous $\Leftrightarrow f^{-1}(U)=\{z\in\mathbb{C}\,|\,f(z)\in U\}$ open $\forall$ open $U\subset\mathbb{C}$. Def: $z_0\in\mathbb{C}$, U neighborhood of z_0, $f:U\to\mathbb{C}$ $\therefore f$ complex-different$|_{z_0}$ if $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f'(z_0)$ exists. Def: f holomorphic$|_{z_0}$ if $\exists$ n'd U of z_0 on which f defined and C.D. If $f:U\to\mathbb{C}$ holomorphic$|_{z_0}$ $\forall z_0\in U$, f holomorphic on U. Lemma: $z_0\in U$ (n'd), $g:U\to\mathbb{C}$ s.t. $g(z_0)\in g(U)$ and g C.D.$|_{z_0}$, f C.D.$|_{g(z_0)}\Rightarrow f\circ g:U\to\mathbb{C}$ C.D.$|_{z_0}\wedge(f\circ g)'(z_0)=f'(g(z_0))g'(z_0)$. Cauchy–Riemann Eq: $z_0=x_0+iy_0\in\mathbb{C}$, $U\ni z_0$ (n'd), $f:U\to\mathbb{C}$ C.D.$|_{z_0}$, $f=u+iv$. $\therefore \frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)\wedge\frac{\partial v}{\partial x}(x_0,y_0)=\frac{-\partial u}{\partial y}(x_0,y_0)$, Given f C.D. $\lim_{n\to\infty}\frac{f(z_n)-f(z_0)}{z_n-z_0}=f'(z_0)$ $\forall\{z_n\}\to z_0$ (e.g.) $z_n=z_0+\frac1n$, $\tilde z_n=z_0+i\frac1n$. Def: $U\subseteq\mathbb{C}$ open, $\tilde U:=\{\binom{x}{y}\in\mathbb{R}^2, z=x+iy\in U\}$, $f:U\to\mathbb{C}$, $f=u+iv$. Let $\tilde f:\tilde U\to\mathbb{R}^2$, $\tilde f(x,y)=\binom{u(x,y)}{v(x,y)}$ $\therefore f$ real-differentiable (at (x_0,y_0)) If $\tilde f$ total-differentiable: $\exists$ real 2×2 matrix T and continuous $\varphi:\tilde U\to\mathbb{R}^2$ with $\varphi(x_0,y_0)=0$ s.t. $\forall\binom{x}{y}\in\tilde U$, $\tilde f(x,y)=\tilde f(x_0,y_0)+T\cdot\binom{x-x_0}{y-y_0}+\left\|\binom{x-x_0}{y-y_0}\right\|\cdot\varphi(x,y)$, $T=\begin{pmatrix}\frac{\partial u}{\partial x}(x_0,y_0)&\frac{\partial u}{\partial y}(x_0,y_0)\\ \frac{\partial v}{\partial x}(x_0,y_0)&\frac{\partial v}{\partial y}(x_0,y_0)\end{pmatrix}=D\tilde f(x_0,y_0)$, $\tilde f$ has continuous partial derivatives in $(x_0,y_0)\Longrightarrow\tilde f$ total diff.$|_{(x_0,y_0)}\Longrightarrow\tilde f$ has partial derivatives$|_{(x_0,y_0)}$. Theorem: $f:U\to\mathbb{C}$, $f=u+iv$, $z_0=x+iy\in U$. TFAE: (i) f C.D.$|_{z_0}$ (ii) f real diff.$|_{z_0}$ and CRE hold $\Rightarrow$ If $\tilde f$ has continuous partial derivatives$|_{z_0}$ (i.e. $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ continuous$|_{z_0}$) and CRE hold $\Rightarrow f$ C.D.$|_{z_0}$. Remark: f C.D.$|_{z_0}\Rightarrow f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)=i\left(\frac{\partial u}{\partial y}(x_0,y_0)+i\frac{\partial v}{\partial y}\right)$. Def: $h:\mathbb{R}^2\to\mathbb{R}^2$, h harmonic if $\forall(x,y)\in\mathbb{R}^2$, $\frac{\partial^2h}{\partial x^2}(x,y)+\frac{\partial^2h}{\partial y^2}(x,y)=\Delta h=0$ (Laplacian). Lemma: Let $u,v:\mathbb{R}^2\to\mathbb{R}$ twice continuously differentiable, and $f(x+iy)=u(x,y)+iv(x,y)$ holomorphic on $\mathbb{C}\Longrightarrow u,v$ harmonic. Def: $U\subseteq\mathbb{R}^2$ open, $u:U\to\mathbb{R}$ harmonic, $v:U\to\mathbb{R}$ harmonic conjugate of u if $f=u+iv$ holomorphic on U. Lemma: $P(z)=\sum_{n=0}^{N}a_nz^n$ ($a_n\in\mathbb{C}$) holomorphic on $\mathbb{C}$. Let $P:\mathbb{C}\to\mathbb{C}$, $P(z)=\sum^{N}r_nz^n$, $r_n\in\mathbb{R}$ and let z_0 s.t. $P(z)=0\Rightarrow P(\bar z_0)=0$. Def: $P,Q:\mathbb{C}\to\mathbb{C}$, $\therefore R:\mathbb{C}\setminus\{z\in\mathbb{C}, Q(z)=0\}\to\mathbb{C}$ $\therefore R(z)\triangleq\frac{P(z)}{Q(z)}$ rational function $\Longrightarrow R(z)$ holomorphic on $\{z\in\mathbb{C}\,|\,Q(z)=0\}$

Def: Exp: $\mathbb{C}\to\mathbb{C}$, $\exp(z)\triangleq\exp(x+iy):=e^x(\cos(x)+i\sin(y))$. Prop: Exp holomorphic on $\mathbb{C}$, $\forall z\in\mathbb{C}$ $\exp'(z)=\exp(z)$, $\forall z,w\in\mathbb{C}$ $\exp(z+w)=\exp(z)\cdot\exp(w)$, $\exp(z+2\pi i)=\exp(z)$. $\mathrm{Exp}(x+iy)=u(x,y)+iv(x,y)=e^x\cos(y)+ie^x\sin(y)$. Def: Complex Sine $\wedge$ Cosine $\cos(z)=\frac{\exp(iz)+\exp(-iz)}{2}$, $\sin(z)=\frac{\exp(iz)-\exp(-iz)}{2i}$. Prop: (i) $\cos(z),\sin(z)$ holomorphic at z with $\cos'(z)=-\sin(z)$ and $\sin'(z)=\cos(z)$, (ii) $\cos^2(z)+\sin^2(z)=1$ (iii) $\cos(z+2\pi)=\cos(z)$ (sin), (iv) $\sin(z+\frac{\pi}{2})=\cos(z)$. Def: $z\in\mathbb{C}$. Multivalued function $\log(z):=\{w\in\mathbb{C}\,|\,\exp(w)=z\}$. Lemma: $z,w\in\mathbb{C}$ ($\ne0$), $z=re^{i\theta}\Rightarrow$ (i) $\log(z)=\log(re^{i\theta})=\ln|z|+i\arg(z)=\{\ln|z|+i\varphi:\varphi\in\arg(z)\}=\{\ln|z|+i\,\mathrm{Arg}(z)+2\pi ik\}=\{\ln r+i\theta+i2\pi ik\}$ (ii) $\log(zw)=\log(z)+\log(w)$ (iii) $\log(\frac1z)=-\log(z)$. Def: Principal-Branch: $\mathrm{Log}:\mathbb{C}\setminus\{0\}\to\mathbb{C}$, $\mathrm{Log}(z):=\ln|z|+i\,\mathrm{Arg}(z)$ $\therefore\mathrm{Log}(z)$ discontinuous on $-x$-axis $\therefore\mathrm{Log}(\cdot)$ continuous on $\mathbb{C}_-:=\mathbb{C}\setminus\{-x:x\ge0\}$. Def: Branch-Cut: $z_0\in\mathbb{C}$, $\phi\in\mathbb{R}$. $L_{z_0,\phi}:=\{z\in\mathbb{C}:z=z_0+re^{i\phi}, r\ge0\}$, Cut-Plane: $D_{z_0,\phi}\triangleq\mathbb{C}\setminus L_{z_0,\phi}$. Def: $\phi\in\mathbb{R}$ define Branch $\mathrm{Arg}_\phi(z)$ of $\arg(\cdot)$: $\phi<\mathrm{Arg}_\phi(z)\le\phi+2\pi\Rightarrow$ Branch $\mathrm{Log}_\phi(z)=\ln|z|+i\,\mathrm{Arg}_\phi(z)$. Lemma: $\phi\in\mathbb{R}$ $\therefore\mathrm{Log}_\phi(z)$ holomorphic on $D_{0,\phi}$ and its derivative is $\mathrm{Log}_\phi'(z)=\frac1z$ $\forall z\in D_{0,\phi}$. Def: Let $\alpha,z\in\mathbb{C}$, $z\ne0$, α-th power of z: $z^\alpha=\{\exp(\alpha w):w\in\log(z)\}$, z^α a set $z^\alpha=\{\exp(\alpha\ln|z|+i\alpha\,\mathrm{Arg}(z)+i\alpha2\pi k:k\in\mathbb{Z}\}=\{\exp(\alpha\,\mathrm{Log}(z))\exp(i\alpha2\pi k):k\in\mathbb{Z}\}$ (e.g.) $i^i=\{\exp(i\,\mathrm{Log}(i))\exp(i2\pi k):k\in\mathbb{Z}\}=\{\exp(i(\frac{i\pi}{2}))\exp(-2\pi k)\}=\{e^{-\frac{\pi}{2}}e^{-2\pi k}:k\in\mathbb{Z}\}$. Theorem: $\alpha,z\in\mathbb{C}$ ($z\ne0$) $\therefore$ (i) If $\alpha\in\mathbb{Z}$ $\exists$ one value of z^α, (ii) If $\alpha=\frac{p}{q}$, p,q coprime ($q\ne0$), $\exists\,q$ values of z^α. i.e. $z^\alpha=\{\exp(\frac{p}{q}\mathrm{Log}(z))\exp(i2\pi k\frac{p}{q}):k=0,1,\dots,q-1\}$ (iii) $\alpha\notin\mathbb{Q}\Rightarrow z^\alpha$ infinite value. Roots of Unity Let $q\in\mathbb{N}$ $\therefore 1^{1/q}=1,w,w^2,\dots,w^{q-1}$, $w=\exp(i\frac{2\pi}{q})$. Def: For $z\in\mathbb{C}\setminus\{0\}$, principle-branch of $z^\alpha:=\exp(\alpha\,\mathrm{Log}(z))$. Lemma: Choose branch $\phi\in\mathbb{R}$, $z^\alpha=\exp(\alpha\,\mathrm{Log}_\phi(z))$ holomorphic on $D_{0,\phi}$, where $\frac{d}{dz}z^\alpha=\alpha z^{\alpha-1}$ $\forall z\in D_{0,\phi}$

Complex Integration: Def: Let $[a,b]\subseteq\mathbb{R}$, $f:[a,b]\to\mathbb{C}$, $f=u+iv$. f integrable if $u,v:[a,b]\to\mathbb{C}$ integrable in the real sense and $\int_a^bf(t)dt=\int_a^bu(t)dt+i\int_a^bv(t)dt$. f continuous, $f=\frac{dF}{dt}$ for differentiable $F:[a,b]\to\mathbb{C}$ $\therefore\int_a^bf(t)dt=F(b)-F(a)$, $\left|\int_a^bf(t)dt\right|\le\int_a^b|f(t)|dt$

Def: A parametrized curve connecting z_0-z_1 is a continuous function $\gamma:[t_0,t_1]$ s.t. $\gamma(t_0)=z_0$, $\gamma(t_1)=z_1$ $\begin{cases}z_0=x_0+iy_0\\ z_1=x_1+iy_1\end{cases}\Rightarrow\gamma(t)=x(t)+iy(t)$, $x,y:[t_0,t_1]\to\mathbb{R}$ $\begin{cases}x(t_0)=x_0\\ x(t_1)=x_1\end{cases}\begin{cases}y(t_0)=y_0\\ y(t_1)=y_1\end{cases}$. A curve γ regular if γ continuously differentiable and $\gamma'(t)\ne0$ $\forall t\in[t_0,t_1]$. Let $\gamma[t_0,t_1]:=\Gamma$. Def: $z_0,z_1\in\mathbb{C}$, Γ regular and $f:\Gamma\to\mathbb{C}$ continuous $\therefore\int_\Gamma f(z)dz=\int_{t_0}^{t_1}f(\gamma(t))\gamma'(t)dt$. Def: Γ regular, arc length $\ell(\Gamma):=\int_{t_0}^{t_1}|\gamma'(t)|dt=\int_{t_0}^{t_1}\sqrt{x'(t)^2+y'(t)^2}\,dt$

Lemma: $\left|\int_\Gamma f(z)dz\right|\le\max_{z\in\Gamma}|f(z)|\cdot\ell(\Gamma)$. Lemma: (i) $\gamma:[t_0,t_1]\to\mathbb{C}$, $\tilde\gamma:[\tilde t_0,\tilde t_1]\to\mathbb{C}$ s.t. $\Gamma=\gamma([t_0,t_1])=\tilde\gamma([\tilde t_0,\tilde t_1])$, $\exists$ injective differentiable $\lambda:[\tilde t_0,\tilde t_1]\to[t_0,t_1]$ with $\lambda'(t)>0$ $\forall t\in[\tilde t_0,\tilde t_1]$ s.t. $\tilde\gamma(t)=\gamma(\lambda(t))$ $\forall t\in[\tilde t_0,\tilde t_1]\Longrightarrow\int_{\tilde t_0}^{\tilde t_1}f(\tilde\gamma(t))\tilde\gamma'(t)dt=\int_{t_0}^{t_1}f(\gamma(t))\gamma'(t)dt$ (ii) If $\gamma:[0,1]\to\mathbb{C}$ and $\gamma([0,1])=\Gamma$, then $-\Gamma=\bar\gamma([0,1])$, $\bar\gamma:=\gamma(1-t)$ and $\int_{-\Gamma}f(z)dz=-\int_\Gamma f(z)dz$. Def: Γ from z_0 to z_1 is a contour if it is a

$i = 1, \dots, n-1$

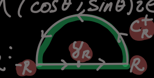
finite union of regular curves. i.e. $\exists \gamma_i : [t_0^i, t_1^i] \to \mathbb{C}$ s.t. $\forall \gamma_i([t_0^i, t_1^i]) = \Gamma_i$ regular and $\bigcup_i \Gamma_i = \Gamma$
and $\gamma_i(t_1^i) = \gamma_{i+1}(t_0^{i+1})$, $\gamma_1(t_0^1) = z_0$ and $\gamma_n(t_1^n) = z_1$. Γ_i: regular component of Γ. For a
continuous $f : \Gamma \to \mathbb{C}$, the contour integral $\int_\Gamma f(z)\,dz = \sum_{i=1}^{N} \int_{\Gamma_i} f(z)\,dz$. Def: $D \subseteq \mathbb{C}$. D a domain if D open
and $\forall$ two points can be connected by a contour $\Gamma \subset D$. Let $f : D \to \mathbb{C}$ continuous, f has an antiderivative F
if $\exists F : D \to \mathbb{C}$ s.t. $F' = f$. Theorem: (Fundamental T. of Calculus) $D \in \mathbb{C}$ domain, Γ contour $\subset D$ joining z_0, z_1, $f : D \to \mathbb{C}$ have
an antiderivative F on D $\therefore \int_\Gamma f(z)\,dz = F(z_1) - F(z_0)$. Def: Γ closed if $\gamma(t_0) = \gamma(t_1)$ $\forall \gamma([t_0, t_1]) = \Gamma$ $\therefore$ If $f : D \to \mathbb{C}$
continuous with F in D $\therefore \int_\Gamma f(z)\,dz = 0$. Lemma: (Path-Indepence) $D \subseteq \mathbb{C}$ domain, $f : D \to \mathbb{C}$ continuous: TFAE: (i) $\exists$ antiderivative F on D
(ii) $\int_\Gamma f(z)\,dz = 0$ $\forall$ closed $\Gamma \subset D$. (iii) $\int_\Gamma f(z)\,dz$ independent of Γ and depend only on endpoints.
Cauchy's Integral Theorem: Def: Γ contour, Γ simple if $\gamma(t) \neq \gamma(s)$ for $(s,t) \in [t_0, t_1]^2 \setminus \{(t_0,t_1), (t_1,t_0)\}$. If Γ closed $\wedge$ simple $\Longrightarrow$ loop
Theorem: (Jordan-Curve) Γ-loop. Then Γ defines one unbounded component, $\underset{\text{Exterior}}{\mathrm{Ext}}(\Gamma)$, and one bounded Component $\underset{\text{Interior}}{\mathrm{Int}}(\Gamma)$ of $\mathbb{C} \setminus \Gamma$
Def: Γ-loop, Γ positively-oriented if as we move along the curve in the direction of parametrization, $\mathrm{int}(\Gamma)$ on left-hand. Def: $D \subseteq \mathbb{C}$
D simply-connected if $\mathrm{Int}(\Gamma) \subset D$ $\forall \Gamma$-loops $\subseteq D$. Theorem: Γ-Loop, f holomorphic inside and on Γ $\therefore \int_\Gamma f(z)\,dz = 0$. Corollary: $D \subseteq \mathbb{C}$
simply-connected, f holomorphic on $D \Longrightarrow f$ has antiderivative on D. Theorem: $z_0 \in \mathbb{C}$, Γ loop in $\mathbb{C}$ which does not pass through z_0
then $\int_\Gamma \frac{1}{z - z_0} = \begin{cases} 2\pi i, & \text{if } z_0 \in \mathrm{Int}(\Gamma) \\ 0, & \text{otherwise} \end{cases}$. Theorem: (Deformation) Γ_1, Γ_2 - Loops, f holomorphic on Γ_1, Γ_2 and $\mathrm{Int}(\Gamma_1) \setminus \mathrm{Int}(\Gamma_2) \cup$
$\mathrm{Int}(\Gamma_2) \setminus \mathrm{Int}(\Gamma_1) \Longrightarrow \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$. Cauchy's Integral Formula: Theorem: Γ loop, $z_0 \in \mathrm{Int}(\Gamma)$, f holomorphic on Γ
and $\mathrm{Int}(\Gamma)$, $f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\,dz$. Theorem: Let $D \subseteq \mathbb{C}$ domain, Γ contour in D, suppose $g : D \to \mathbb{C}$
continuous on Γ. Then the function $G : D \setminus \Gamma \longrightarrow \mathbb{C}$, $G(z) = \int_\Gamma \frac{g(w)}{(w - z)^n}\,dw$ holomorphic, and $G'(z) = n \int_\Gamma \frac{g(w)}{(w - z)^{n+1}}\,dw$
Corollary: $D \subseteq \mathbb{C}$ domain, f holomorphic on D $\therefore f$ infinitely differentiable on D, and all derivatives holomorphic
on D. Theorem: (Generalized Cauchy I.F.) Γ-loop, f holomorphic inside and on Γ, $z \in \mathrm{Int}(\Gamma)$ $\therefore$
f infinitely differentiable at z and $\forall n \in \mathbb{N}$ $f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)^{n+1}}\,dw$. Ex: $\int_{C_1(0)} \frac{\exp(5z)}{z^3}\,dz = \int_{C_1(0)} \frac{\exp(5z)}{(z-0)^3}\,dz =$
$2\pi i \cdot \left(\frac{1}{2!} \frac{d^2}{dz^2} (\exp(5z))\right)\Big|_{z=0} = 2\pi i \cdot \frac{1}{2!} \cdot 25 = 25\pi i$. Liouville's Theorem: Let $D \subseteq \mathbb{C}$ domain, $z_0 \in D$, $R > 0$ s.t. $D_R(z_0) \subseteq D$, f holomorphic
on D, $M \geq 0$ s.t. $|f(z)| \leq M$, $\forall z \in D$ $\therefore \forall n \in \mathbb{N}$, $|f^{(n)}(z_0)| \leq \frac{n!\,M}{R^n}$, Theorem: (Liouville's) f holomorphic on $\mathbb{C}$, bounded i.e.
$\exists M > 0$ s.t. $|f(z)| \leq M$ $\forall z \in \mathbb{C}$ $\therefore$ f is constant. Theorem: (Fundamental Theorem of Algebra) Let $P : \mathbb{C} \to \mathbb{C}$ be a
polynomial. If $P \neq$ constant, P has at least one root $\therefore \exists z \in \mathbb{C}$ s.t. $P(z) = 0$. Def: Let $\{z_n\}_{n \in \mathbb{N}} \in \mathbb{C}$ sequence
$\sum_{n=1}^{\infty} z_n$ infinite series, converges $\Longleftrightarrow$ $S_k := \sum_1^k z_n$ convergent. Lemma: Comparison Test, Let $\{z_n\} \subset \mathbb{C}$
s.t. $|z_n| \leq M_n$, $\{M_n\} \subset \mathbb{R}$ and $\sum_{j=0}^{\infty} M_j < \infty \Longrightarrow \sum_{j=0}^{\infty} z_j$ converges. Lemma: Ratio Test
Let $\{z_n\}_{n \in \mathbb{N}}$ s.t. $\lim_{n \to \infty} \left|\frac{z_{n+1}}{z_n}\right| = L \Longrightarrow$ (i) $L < 1$: $\sum_0^\infty z_j < \infty$ (ii) $L > 1$: $\sum_0^\infty z_j = \infty$, $L = 1$: Inconclusive
Def: Let $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$, $\{f_n\}$ sequence of functions $\therefore f_n \longrightarrow f$ pointwise if $\forall z \in S\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}$:
$|f_n(z) - f(z)| < \varepsilon$. Def: $S \subseteq \mathbb{C}$, $\{f_n\} : S \to \mathbb{C}$ $\therefore f_n \rightrightarrows f$ uniformly if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall z \in S : |f_n(z) - f(z)| < \varepsilon$. Def:
Lemma: (Weierstraß M-Test) Let $S \subset \mathbb{C}$, $\{f_n\} : S \to \mathbb{C}$, $\{M_n\} \geq 0 \subset \mathbb{R}^+$ s.t. $|f_n(z)| \leq M_n$ $\forall n \in \mathbb{N}\ \forall z \in S$
and $\sum_0^\infty M_j < \infty \Longrightarrow \sum_{j=0}^{\infty} f_j(z) \rightrightarrows$ uniformly on S. Lemma: $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$ s.t. $f_n \rightrightarrows f : S \to \mathbb{C}$
$\therefore$ (i) If $\forall n \in \mathbb{N}$ f_n continuous $\Longrightarrow f$ continuous (ii) f_n continuous and Γ a contour inside of $S \Longrightarrow$
$\int_\Gamma f_n(z)\,dz \longrightarrow \int_\Gamma f(z)\,dz$ for $n \longrightarrow \infty$ (iii) If f_n holomorphic, S simply-connected $\Longrightarrow f$ holomorphic on S
Lemma: Let $S \subset \mathbb{C}$, $\{f_n\} : S \to \mathbb{C}$ continuous s.t. $\sum_0^\infty f_j(z) \rightrightarrows$ uniformly on S, Γ contour $\subset S \Longrightarrow$
$\int_\Gamma \sum_0^\infty f_j(z)\,dz = \sum_0^\infty \int_\Gamma f_j(z)\,dz$. Def: Let $z_0 \in \mathbb{C}$, $\{a_n\} \in \mathbb{C}$. A power series: $\sum_0^\infty a_j (z - z_0)^j$.
Theorem: Let $X = \sum_0^\infty a_j (z - z_0)^j \Longrightarrow \exists R \in [0, \infty) \cup \{\infty\}$ s.t. (i) X converges on $D_R(z_0)$ (ii) $X \rightrightarrows$ uniformly
on $D_R(z_0)$ $\forall r \in [0, R)$ (iii) $X \not\longrightarrow$ (diverges) on $\mathbb{C} \setminus \overline{D_R(z_0)}$ (R: radius of convergence) Theorem:
Let $\sum_0^\infty a_j (z - z_0)^j$ s.t. $\lim_{j \to \infty} \left|\frac{a_{j+1}}{a_j}\right| = L$ $\therefore R = 1/L$. Theorem: Let $\sum a_j (z - z_0)^j$ power series
with $R \Longrightarrow f$ holomorphic on $D_R(z_0)$ Def: (Taylor Series) Let $z_0 \in \mathbb{C}$, f holomorphic $|_{z_0}$ The
Taylor Series of $f |_{z_0}$: $\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j$ Theorem: $z_0 \in \mathbb{C}$, $R > 0$, f holomorphic on $D_R(z_0) \Longrightarrow$

$\sum_{0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j \xrightarrow{\text{pointwise}} f(z)$ on $D_R(z_0)$ and $\xRightarrow{\text{uniform}}$ on $\overline{D_r(z_0)}$ $\forall 0 \le r < R$. **Def:** $U \subseteq \mathbb{C}$ open, $f: U \to \mathbb{C}$, f (analytic) if $\forall z \in U$ f can be expressed as a convergent power series $|_z$ valid on $D_R(z)$, $R>0$. **Theorem:** $U \subseteq \mathbb{C}$ open, $f: U \to \mathbb{C}$ holomorphic $\Longrightarrow f$ analytic. **Prop:** $z_0 \in \mathbb{C}$, $R>0$, f holomor. on $D_R(z_0)$ $\therefore$ $\forall z \in D_R(z_0)$: $f'(z) = \sum_{0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z-z_0)^j$. **Lemma:** $z_0 \in \mathbb{C}$, $R>0$, $\alpha, \beta \in \mathbb{C}$ f, g, holomorphic on $D_R(z_0) \Longrightarrow$ (i) Taylor Series of $\alpha f + \beta g$ $|_{z_0}$, valid on $D_R(z_0)$: $\sum_{0}^{\infty} \frac{\alpha f^{(j)}(z_0) + \beta g^{(j)}(z_0)}{j!}(z-z_0)^j$ (ii) $f \cdot g$: $\sum_{j=0}^{\infty} \frac{1}{j!}\left(\sum_{k=0}^{j}\binom{j}{k} f^{(k)}(z_0)\, g^{(j-k)}(z_0)\right)(z-z_0)^j$. **Theorem:** (Uniqueness) Let $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j$ convergent power series with $R>0$ $\therefore$ Taylor Series of f $|_{z_0}$: $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ valid on $D_R(z_0)$. **Def:** $z_0 \in \mathbb{C}$, $\{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$ i.e. $\{a_n\} = \ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ A Laurent Series $|_{z_0}$: $\sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$, converges $\Longleftrightarrow \sum_{j=0}^{\infty} a_j (z-z_0)^j < \infty \wedge \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} < \infty$ Let R, S be the convergent radii i.e. $|z-z_0| < R$, $|(z-z_0)^{-1}| < S \Longrightarrow$ Laurent Series converges on $z \in \mathbb{C}$ s.t. $S^{-1} < |z-z_0| < R$. **Def:** (Annulus) $z_0 \in \mathbb{C}$, $r, R \in [0,\infty) \cup \{\infty\}$ $\therefore$ $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z-z_0| < R\}$ (open) and $\overline{A_{r,R}}(z_0) = \{z \in \mathbb{C} : r \le |z-z_0| \le R\}$ (closed). **Theorem:** $z_0 \in \mathbb{C}$, $0 \le r < R < \infty$, f holomorphic on $A_{r,R}(z_0)$ $\therefore$ f can be expressed as a Laurent Series $|_{z_0}$ which converges on $A_{r,R}(z_0)$, uniformly on $A_{r_1,R_1}(z_0)$ where $r < r_1 \le R_1 < R$. Moreover, $a_j := \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z-z_0)^{j+1}}\,dz$ s.t. $\forall \Gamma \subset A_{r,R}(z_0)$, $z_0 \in \mathrm{Int}(\Gamma)$. **Theorem:** Uniqueness: Let $z_0 \in \mathbb{C}$, $0 \le r < R < \infty$, and suppose $\sum_{j=-\infty}^{\infty} c_j (z-z_0)^j < \infty$ on $A_{r,R}(z_0) \Longrightarrow f(z) := \sum_{-\infty}^{\infty} c_j (z-z_0)^j$ holo. on $A_{r,R}(z_0)$ and it's a unique Laurent Series. **Def:** Let $D \subseteq \mathbb{C}$ a domain, $z_0 \in \mathbb{C}$, $f: D \to \mathbb{C}$. z_0 a singularity of f if f not holo. $|_{z_0}$. A singularity of f isolated if $\exists R>0$ s.t. f holo. on $D_R^{\circ}(z_0) = D_R(z_0) \setminus \{z_0\}$. **Def:** Let $z_0 \in \mathbb{C}$, $U \subset \mathbb{C}$ a neighborhood, f holo. on U $\therefore$ z_0 a zero of f if $f(z_0) = 0$. z_0 m°-zero if $f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$, $f^{(m)}(z_0) \neq 0$. A zero z_0 isolated if $\exists R>0$ s.t. $f(z) \neq 0$ $\forall z \in D_R(z_0)$ **Prop:** Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ n'h'd', f holo on U with a zero of finite order $|_{z_0} \Longrightarrow z_0$ isolated. **Corollay:** (Identity Theorem): Let $z_0 \in U$ $U \subseteq \mathbb{C}$ n'h'd', f holo. on U s.t. $f(\{z_n\}) = 0$ $\{z_n\} \subset U$ s.t. $z_n \to z_0 \Longrightarrow f$ identically 0 in some $D_R(z_0)$. **Corollay:** Let $z_0 \in \mathbb{C}$ a singularity of a rational function $f = P/Q \Longrightarrow z_0$ isolated. **Def:** $z_0 \in \mathbb{C}$ isolated singularity of f holo. on $D_R^{\circ}(z_0)$, $R>0$ $\therefore$ f has Laurent Expansion $|_{z_0}$ valid on $A_{0,R}(z_0)$. Suppose $f(z) = \sum_{-\infty}^{\infty} a_j (z-z_0)^j$ $\Longrightarrow$ (i) z_0 a removable-singularity if $a_j = 0$ $\forall j < 0$ i.e. $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j$ (ii) z_0 pole of order m if $\exists m \in \mathbb{N}$ s.t. $a_j = 0$ $\forall j < -m$, $a_{-m} \neq 0$, i.e. $f(z) = \sum_{j=-m}^{\infty} a_j (z-z_0)^j$ (iii) z_0 essential-singularity if $a_j \neq 0$ for infinitely many $-j$. **Theorem:** $z_0 \in \mathbb{C}$ be a removable singularity of f holo. on $D_R^{\circ}(z_0)$, $R>0 \Longrightarrow f$ can be (re)defined in z_0 s.t. f holo. on $D_R(z_0)$. **Lemma:** f, g holo. $|_{z_0}$, where z_0 m°-zero $\Longrightarrow$ (i) If z_0 not a zero of f, then f/g has a pole of order m $|_{z_0}$ (ii) If z_0 a k°-zero of $f \Longrightarrow f/g$ has a $(m-k)^{\circ}$-pole $|_{z_0}$ if $m > k$. If $m < k$: removable singularity. **Theorem:** (Riemann) Let $z_0 \in \mathbb{C}$ an isolated singularity of f, f holo. on a n'h'd of z_0 If $\exists r>0, M>0$ s.t. $|f(z)| \le M$ $\forall z \in D_r(z_0) \Longrightarrow z_0$ a removable singularity. **Def:** Let $D \subseteq \mathbb{C}$ domain, f holo on $D \setminus \{z_n\}$ where $\{z_n\}$ are isolated singularities. If f has poles of finite order $\forall z_k \Longrightarrow f$ is a meromorphic function. **Theorem:** (Casorati-Weierstraß) $z_0 \in \mathbb{C}$ an essential singularity of f (holo. on a n'h'd' of z_0) $\Longrightarrow \forall w_0 \in \mathbb{C}$, $\exists \{z_n\} \longrightarrow z_0$ s.t. $f(z_n) \longrightarrow w_0$. **Def:** (Analytic Continuation) Let $D \subseteq \tilde{D} \subseteq \mathbb{C}$ domains $f: D \longrightarrow \mathbb{C}$ holo. We say a holomorphic $F: \tilde{D} \longrightarrow \mathbb{C}$ is an analytic continuation of f if $F(z) = f(z)$ $\forall z \in D$. **Theorem:** (Identity) $D \subseteq \mathbb{C}$ domain, $z_0 \in D$, f holo. on D s.t. $f(z) = 0$ $\forall z \in D_R(z_0)$ for some $R>0$, then $f(z) = 0$ $\forall z \in D$. **Corollay:** $D \subseteq \mathbb{C}$, $\{z_n\} \subset D$ s.t. $z_n \to z_0 \in D$. If a holo. $f: D \longrightarrow \mathbb{C}$ vanishes $\forall z_n \Longrightarrow f = 0$ on D. **Corollary:** (Identity) $D \subseteq \mathbb{C}$, $z_0 \in D$, $f, g: D \to \mathbb{C}$ holo. s.t. $f(z) = g(z)$ $\forall z \in D_R(z_0)$ for some $R>0 \Longrightarrow f(z) = g(z)$. Alternatively if $f(z_n) = g(z_n)$ $\forall n$, $\{z_n\} \subseteq D$, $z_n \to z_0 \in D$ $\Longrightarrow f(z) = g(z)$. **Theorem:** $z_0 \in \mathbb{C}$, f holo. on $D_R^{\circ}(z_0)$ for some $R>0$, with isolated singularity $|_{z_0}$ Let $z_0 \in \Gamma$-loop $\subset D_R^{\circ}(z_0) \Longrightarrow \int_{\Gamma} f(z)\,dz = 2\pi i a_{-1}$; $a_{-1}(z-z_0)^{-1}$ in Laurent E, $\mathrm{Res}(f, z_0) \triangleq a_{-1}$ **Lemma:** $z_0 \in \mathbb{C}$, f holo. on $D_R^{\circ}(z_0)$, m°-pole $|_{z_0} \Longrightarrow \mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}}\left((z-z_0)^m f(z)\right)$. **Lemma:** $z_0 \in \mathbb{C}$ g, h holo. on $D_R(z_0)$ s.t. h has simple-zero $|_{z_0}$ but $g(z_0) \neq 0$. For $f = g/h$: $\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$. **Theorem:** Cauchy

Residue: Γ-loop, f holo. on $Int(\Gamma)\cup\Gamma$ except for $\{z_1,\dots,z_k\}\subset Int(\Gamma)$, z_j isolated-singularity $\Longrightarrow \int_\Gamma f(z)\,dz = 2\pi i\sum_{j=1}^{k} Res(f,z_j)$. **Def:** Γ-loop, f meromorphic on $int(\Gamma)$ $\therefore$ $N_0(f)$: # zeros with multiplicity, $N_\infty(f)$: # poles with multiplicity, $N_0(f)\triangleq\sum^{\ell} Order(w_j)$, $N_\infty(f)\triangleq\sum^{k} Order(z_j)$. **Theorem:** (Argument Principle) Γ-loop, f meromorphic on $Int(\Gamma)$, holomorphic $\wedge$ $f\neq 0$ on Γ $\therefore \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = N_0(f)-N_\infty(f)$. **Corollary:** Γ-loop, f holomorphic on $\overline{Int(\Gamma)}$, $f(z)\neq 0$ $\forall z\in\Gamma$ $\therefore \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = N_0(f)$. **Rouché's Theorem:** Γ-loop, f,g holomorphic on $\overline{Int(\Gamma)}$ s.t. $\forall z\in\Gamma: |f(z)-g(z)|<|f(z)| \Longrightarrow N_0(f)=N_0(g)$. Fundamental Theorem of Algebra: $P(z)=a_nz^n+\dots+a_0$, $a_{n-1},\dots,a_0\in\mathbb{C}$ $\therefore N_0(P)=n$. **Open Mapping Theorem:** $D\subseteq\mathbb{C}$, f non-constant holomorphic $\therefore f(D)=\{f(z):z\in D\}$ open $\subset\mathbb{C}$. **Corollary:** (Maximum Modulus Principle): $D\subseteq\mathbb{C}$, f holomorphic and non-constant on D $\Longrightarrow |f(z)|$ does not attain a maximum on D. If $|f(z)|$ has a maximum, occurs on ∂D

Trigonometric Integral: R: rational function, $\int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta$, for $z=e^{i\theta}\in C_1(0)$, $\cos(\theta)=Re(z)=\frac{z+\bar z}{2}=\frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin(\theta)=Im(z)=\frac{z-\bar z}{2i}=\frac{1}{2i}\left(z-\frac{1}{z}\right)$. Set $f(z)=\frac{1}{iz}R\left(\frac{z+1/z}{2},\frac{z-1/z}{2i}\right)\Rightarrow f(e^{i\theta})=\frac{1}{ie^{\theta}}R(\cos(\theta),\sin(\theta))$. Consider $C_1(0)$ parametrized by $\gamma:[0,2\pi]\to\mathbb{C}$, $\gamma(\theta)=e^{i\theta}$ $\therefore \oint_{C_1(0)} f(z)\,dz=\int_0^{2\pi} f(\gamma(\theta))\gamma'(\theta)\,d\theta=\int_0^{2\pi}\frac{1}{ie^{\theta}}R(\cos\theta,\sin\theta)ie^{\theta}=\int_0^{2\pi}R(\cos\theta,\sin\theta)\,d\theta$

Improper Integrals: $p.v.\int_{\mathbb{R}} f(x)\,dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx$. Let $T_R=C_R^+\oplus\gamma_R$:



$p.v.\int_{-\infty}^{\infty}f(x)\,dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx=\lim_{R\to\infty}\int_{\gamma_R}f(z)\,dz+\lim_{R\to\infty}\int_{C_R^+}f(z)\,dz$ $\left|\int_{C_R^+}f(z)\,dz\right|\overset{M\text{-}L}{\le}\ell(C_R^+)\cdot\max_{z\in T}f(z)$ If $\longrightarrow 0$ for $R\to\infty$ $\Longrightarrow p.v.\int_{\mathbb{R}}f(x)\,dx=\lim_{R\to\infty}\int_{T_R}f(z)\,dz$

Def: $U\subseteq\mathbb{C}$ open, $f:U\to\mathbb{C}$. f conformal if f preserves angles. **Theorem:** $U\subseteq\mathbb{C}$ open, $f:U\to\mathbb{C}$ holomorphic $\therefore f$ preserves angles $\forall z_0\in U$ s.t. $f'(z_0)\neq 0$. **Def:** A Möbius Transformation: $f(z)=\frac{az+b}{cz+d}$, $a,b,c,d\in\mathbb{C}$, $ad\neq bc$. **Lemma:** To a complex matrix $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $det(M)=1$ we associate $f_M(z)=\frac{az+b}{cz+d}\Longrightarrow f_{M_1M_2}=f_{M_1}\circ f_{M_2}$ and $f_{M^{-1}}=(f_M)^{-1}$. **Def:** We define $SL(2,\mathbb{C}):=\left\{M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathbb{C}^{2\times 2}; ad-bc=1\right\}$ (Special Linear Group). **Def:** The extended complex plane is $\tilde{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, $\infty\notin\mathbb{C}$ s.t. $\forall a\in\mathbb{C}$, $\forall b\neq 0\in\mathbb{C}$ $a+\infty:=\infty$, $b\cdot\infty:=\infty$, $\frac{b}{0}:=\infty$, $\frac{b}{\infty}:=0$ $\therefore f\left(\frac{-d}{c}\right)=\infty$, $f(\infty)=\frac{a}{c}$. **Def:** Consider the Riemann Sphere $S^2:=\{(x,y,z)\in\mathbb{R}^3, x^2+y^2+z^2=1\}$, $N=(0,0,1)$ North-Pole. Identify $z=x+iy\in\mathbb{C}$ with $(x,y,0)\in\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$ $\Phi:\mathbb{C}\to S^2$ s.t. $z=x+iy\in\mathbb{C}\mapsto(x,y,0)$ colinear with N and $\Phi(z)$. $\lim_{|z|\to\infty}\Phi(z)=\begin{pmatrix}0\\0\\1\end{pmatrix}=N\Longrightarrow\Phi(\infty)=N$ $\therefore\Phi:\tilde{\mathbb{C}}\to S^2$ $\Psi=\Phi^{-1}: S^2\to\mathbb{C}^2$ (stereographic Projection) bijective and $\Phi(z)=\Phi(x+iy)=\left(\frac{2x}{|z|^2+1},\frac{2y}{|z|^2+1},\frac{|z^2|-1}{|z^2|+1}\right)$, $\Psi(X,Y,Z)=\begin{cases}\frac{x+iy}{1-z}, & (x,y,z)\neq N\\ \infty, & N\end{cases}$

Def: A circline is a line $\vee$ circle. **Lemma:** Stereographic Projection maps a circle to circline. **Def:** Translation: Möbius T. $f_M(z)=z+b$, $M=\begin{pmatrix}1&b\\0&1\end{pmatrix}$ (ii) Rotation: $f_M(z)=az$, $|a|=1$, $M=\begin{pmatrix}e^{i\theta/2}&0\\0&e^{-i\theta/2}\end{pmatrix}$ $a=e^{i\theta}$ (iii) Dilation: $f_M(z)=rz$, $r>0$ $M=\begin{pmatrix}\sqrt r&0\\0&1/\sqrt r\end{pmatrix}$ (iv) Inversion: $f_M=\frac{1}{z}$, $M=\begin{pmatrix}0&i\\i&0\end{pmatrix}$ **Theorem:** f -Möbius-T. If $f(\infty)=\infty\Longrightarrow f$ is a composition of a finite number of translations, rotations and dilations, If $f(\infty)\neq 0$ ... and one inversion. **Corollary:** Möbius T. map circlines to circlines. **Corollary:** Möbius Transformation are conformal. **Lemma:** Suppose f a Möbius T. $z_2,z_3,z_4\in\tilde{\mathbb{C}}$ with $f(z_2)=z_2\neq f(z_3)=z_3\neq f(z_4)=z_4\Longrightarrow f=$ identity. **Theorem:** Let $z_2,z_3,z_4\in\tilde{\mathbb{C}}$ distinct $\therefore\exists! f$ -Möbius s.t. $f(z_2)=1$, $f(z_3)=0$ $f(z_4)=\infty$. Set $f(z)=\lambda\frac{z-z_3}{z-z_4}$ and $\lambda=\frac{z_2-z_4}{z_2-z_3}$. **Corollary:** Let $(z_2,z_3,z_4),(w_2,w_3,w_4)\in\tilde{\mathbb{C}}$ be two triplets of distinct points $\therefore\exists! f$ - Möbius s.t. $f(z_2)=w_2$, $f(z_3)=w_3$, $f(z_4)=w_4$. Let $z_1,z_2,z_3,z_4\in\tilde{\mathbb{C}}$ distinct. The Cross-Ratio: $[z_1,z_2,z_3,z_4]=f(z_1)$ where $f:(z_2,z_3,z_4)\longmapsto(1,0,\infty)$ $\therefore\forall z_1,z_2,z_3,z_4\in\tilde{\mathbb{C}}$, $[z_1,z_2,z_3,z_4]=\frac{z_2-z_4}{z_2-z_3}\cdot\frac{z_1-z_3}{z_1-z_4}$, $[\infty,z_2,z_3,z_4]=\frac{z_2-z_4}{z_2-z_3}$, $[z_1,\infty,z_3,z_4]=\frac{z_1-z_3}{z_1-z_4}$; $[z_1,z_2,\infty,z_4]=\frac{z_2-z_4}{z_1-z_4}$, $[z_1,z_2,z_3,\infty]=\frac{z_1-z_3}{z_2-z_3}$. **Theorem:** Let $z_1,z_2,z_3,z_4\in\tilde{\mathbb{C}}$, f - Möbius $\Rightarrow[f(z_1),f(z_2),f(z_3),f(z_4)]=[z_1,z_2,z_3,z_4]$. **Lemma:** (Complex Bolzano-Weierstraß): Let $\{z_n\}\subset\mathbb{C}$ bounded $\therefore\{z_k\}$ has a convergent subsequence. **Casorati-Weierstraß:** $U\subset\mathbb{C}$ open, $z_0\in U$, f holo on $U\setminus\{z_0\}$ z_0 essential singularity $|_{z_0}$ $\therefore\forall V$ neighborhood of $z_0\in V\subset U\Longrightarrow f(V\setminus\{z_0\})$ dense in $\mathbb{C}$