# Vectors and Geometry of Space

**Distance between two points**

$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ $\qquad |k\vec{u}| = |k||\vec{u}|$

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A point $P(x,y,z)$ lies on the sphere $r=a$, centered at $C(x_0,y_0,z_0)$ if $|P_0P|=a$.

Sphere of radius $a$, center $P_0(x_0,y_0,z_0)$

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2$

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$P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ $\therefore$ $\vec{PQ} = \langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$

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unit vector $|\vec{u}|=1$, $i=\langle 1,0,0\rangle$, $j=\langle 0,1,0\rangle$, $k=\langle 0,0,1\rangle$

$\frac{\vec{v}}{|\vec{v}|}=1$ , $v=|v|\cdot\frac{v}{|v|}$ expresses $v$ as length $\times$ direction

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$\theta=\arccos\left(\frac{u\cdot v}{|u||v|}\right)$ $\quad u\cdot v = u_1v_1+u_2v_2+u_3v_3$

$\qquad = |u||v|\cos\theta$

$\theta<90 \Longrightarrow u\cdot v>0$ $\qquad \theta>90^\circ \Longrightarrow u\cdot v<0$

$u\perp v$ if $u\cdot v=0$ , $u\cdot u=|u|^2$

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$\text{proj}_v u$ "the projection of $u$ onto $v$"

$\text{proj}_v u = (|u|\cos\theta)\frac{v}{|v|} = \frac{u\cdot v}{|v|^2}\cdot v = \frac{u\cdot v}{v\cdot v}v$

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$u \parallel v \Longleftrightarrow u\times v=0$ , Area of parallelogram:

Triangle $\qquad |u\times v| = |u||v|\sin\theta$

$1/2|u||v|\sin\theta = 1/2|u\times v|$ , $u\times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} u_2v_3-u_3v_2 \\ -(u_1v_3-u_3v_1) \\ u_1v_2-u_2v_1 \end{pmatrix}$

Box Product $= |(u\times v)\cdot w|$

$\qquad = |u\times v||w|\cos\theta$ volume of parallelepiped

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$r(\lambda)=r_0+\lambda\vec{v}$, $\lambda\in\mathbb{R}$ , $r(\lambda)=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}+\lambda\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$

$r(\lambda)=\begin{cases} x=x_0+\lambda v_1 \\ y=y_0+\lambda v_2 \\ z=z_0+\lambda v_3 \end{cases}$ , $\lambda\in\mathbb{R}$

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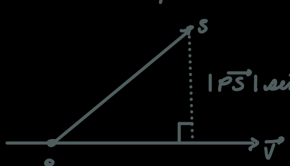
Distance from $S$ to line in space through $P$ $\parallel$ to $\vec{v}$




$|\vec{PS}|\sin\theta = d = \frac{|\vec{PS}||\vec{v}|\sin\theta}{|\vec{v}|} = \frac{|PS\times v|}{|v|}$

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Plane in Space through $P(x_0,y_0,z_0)$ $\qquad \pi_1 \parallel \pi_2$ iff

$n := Ai+Bj+Ck$ $\perp \pi$ $\quad \therefore n\cdot\vec{P_0P}=0$ $\qquad n_1 = kn_2$

$(Ai+Bj+Ck)\cdot(x-x_0, y-y_0, z-z_0)=0$

$\pi: \quad A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$\qquad Ax+By+Cz=\lambda.$

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Line of intersection between $\pi_1, \pi_2$ is perpendicular to both $n_1, n_2$.

Vector parallel to line of intersection is $\alpha(n_1\times n_2)$

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Distance from point $S$ to a plane at $P$. $\quad d=\left|\vec{PS}\cdot\frac{\vec{n}}{|\vec{n}|}\right|$

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Angle between planes: $\theta<90$ $n_1, n_2$

$\theta=\arccos\left(\frac{n_1\cdot n_2}{|n_1||n_2|}\right)$

# Vector-Valued Functions

$x = f(t),\ y = g(t),\ z = h(t) \quad t \in I$

$P(x, y, z) = (f(t), g(t), z(t))$

$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k$

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$\lim_{t \to t_0} r(t) = L \qquad D$: domain

if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\forall t \in D \quad |r(t) - L| < \varepsilon$ whenever $0 < |t - t_0| < \delta$.

$\lim_{t \to t_0} r(t) = \lim_{t \to t_0} f(t)i + \dots + \lim_{t \to t_0} h(t)k$

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$r(t)$ is continuous at $t = t_0$ if $\lim_{t \to t_0} r(t) = r(t_0)$. $r(t)$ is continuous if it is continuous $\forall$ in its domain

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$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$

$r'(t) = \frac{df}{dt} i + \frac{dg}{dt} j + \frac{dh}{dt} k$

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If $r$ is the position vector

$v(t) = \frac{dr}{dt}$, velocity vector tangent to the curve.

$|v(t)|$ speed

$a = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$

$\frac{v}{|v|}$ = direction of motion.

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$(\cdot)$ Rule:

$d/dt [u(t) \cdot v(t)] = u'(t) \cdot v(t) + v'(t) \cdot u(t)$

$(\times)$ Rule:

$d/dt [u(t) \times v(t)] = u'(t) \times v(t) + v'(t) \times u(t)$

chain Rule:

$d/dt [u(f(t))] = f'(t)\, u'(f(t))$

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$\int r(t)\, dt = R(t) + \vec{C}$

$\int_a^b r(t)\, dt = R(b) - R(a)$

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Projectile motion: $v_0 = |v_0| \cos\alpha\, i + |v_0| \sin\alpha\, j$

$\frac{d^2 r}{dt^2} = -g j \quad \therefore \quad \frac{dr}{dt} = -gt j + v_0$

$r = -\frac{1}{2} g t^2 j + v_0 t + r_0$

$r = (v_0 \cos\alpha) t i + \left((v_0 \sin\alpha) t - \frac{1}{2} g t^2\right) j$

$$\left.\begin{aligned} x &= v_0 \cos\alpha\, t \\ y &= v_0 \sin\alpha\, t - \tfrac{1}{2} g t^2 \end{aligned}\right\}$$

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Arc length in Space: $\qquad r'(t) = v$

$r(t) = x(t)i + y(t)j + z(t)k$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |v|\, dt$$

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$v = dr/dt, \qquad T = \frac{v}{|v|}$ unit tangent vector.

$dt/ds = 1 / ds/dt = 1/|v| \qquad \frac{dr}{ds} = T$

$\frac{dr}{ds} = dr/dt \cdot dt/ds = v \cdot 1/|v| = T$

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# Partial Derivatives:

Suppose $D$ is a set of $n$-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued function $f$ on $D$ is a rule that assigns a unique real number $w = f(x_1, x_2, \ldots, x_n)$ to each element in $D$ (domain). The set of $w$'s is the range of $f$.

A region is bounded if it lies inside a disk of finite radius. A region is open if it contains entirely of interior points. closed if it contains all its boundary points.

$f(x,y) = c$ is a level curve. surface $z = f(x,y)$

$f(x,y,z) = c$ is called a level surface

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall\, (x,y)$ in the domain of $f$,

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Path Test: If a function $f(x,y)$ has different limits along two different paths in the domain of $f$ as $(x,y)$ approaches $(x_0,y_0)$ then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

the slope of the curve $z = f(x, y_0)$ at $P(x_0, y_0, f(x_0,y_0))$ in the plane $y = y_0$ is the value of $\frac{\partial f}{\partial x}$ at $(x_0,y_0)$.

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$ , $f_{yx} = (f_y)_x$

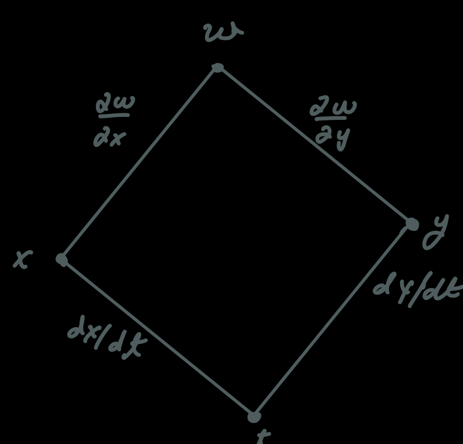
$f_{yx}(a,b) = f_{xy}(a,b)$

Functions of two variables: $w = f(x,y)$ , $x = x(t)$, $y = y(t)$

$w = f(x(t), y(t))$ is a differentiable function of $t$.

$$\frac{dw}{dt} = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

$$dw/dt = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$




$w = f(x,y,z)$ $x = x(t)$, $y = y(t)$, $z = z(t)$

$$dw/dt = \frac{\partial w}{\partial x}\, dx/dt + \frac{\partial w}{\partial y} \cdot dy/dt + \frac{\partial w}{\partial z}\, dz/dt$$

$w = f(x,y,z)$ , $x = g(r,s)$ , $y = h(r,s)$ , $z = k(r,s)$

$$\frac{dw}{dr} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r} \quad , \quad \frac{dw}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} + \frac{\partial z}{\partial s}$$

Implicit differentiation : $\frac{dy}{dx} = \frac{-F_x}{F_y}$ , $F(x,y) = 0$

$\boxed{\text{The gradient vector } \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j}$ its value at $P(x_0,y_0) = \nabla f(x_0,y_0)$

Directional Derivative is a dot product

$$\left(\frac{df}{ds}\right)_{u,P_0} = \nabla f|_{P_0} \cdot u = D_u f = \nabla f \cdot u$$

where $u = \frac{v}{|v|}$
$|u| = 1$

1) $D_u f$ increases fastest when $\cos\theta = 1$ ( $D_u f = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta$ , so $\theta = 1$
so $u$ is the direction of $\nabla f$ $\therefore u = \frac{\nabla f}{|\nabla f|} \longrightarrow D_u f = |\nabla f|$

2) Similarly, $f$ decreases most rapidly in the direction $-\nabla f$.
$u = -\frac{\nabla f}{|\nabla f|}$ so $D_u f = -|\nabla f|$

3) Any direction $u$ orthogonal to $\nabla f \neq 0$ is a direction of zero change in $f$ because $\theta = \pi/2$ , $D_u f = 0$.

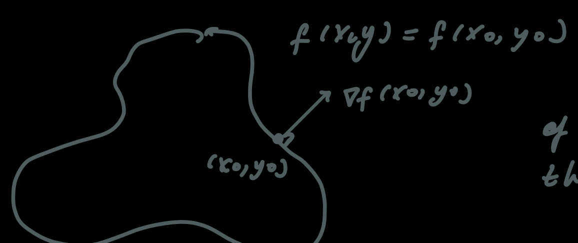
$f(x,y) = c$ along curve $r = g(t) i + h(t) j$ , then $f(g(t), h(t)) = c$

$$d/dt\ f(g(t), h(t)) = d/dt\,(c)$$

$$\frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dh}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right)\cdot\left(\frac{dg}{dt} i + \frac{dh}{dt} j\right) = 0$$

$\nabla f \cdot dr/dt = 0$ so $\nabla f$ is tangent to the normal vector $dr/dt$. so it is normal to the curve.




At every point $(x_0,y_0)$ in the domain of $f(x,y)$, $\nabla f$ is normal to the level curve through $(x_0,y_0)$

Tangent lines to level curves are normal to gradients, therefore:

$$N = Ai + Bj \quad , \quad N = \nabla f = f_x(x_0, y_0)i + f_x(x_0, y_0)j$$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Derivatives along a path

$$d/dr\, f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

Tangent plane of a level surface $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t \quad , \quad y = y_0 + f_y(P_0)t \quad , \quad z = z_0 + f_z(P_0)t$$

Tangent plane to surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Estimating change in a specific direction:

- For a single variable function: $df = f'(P_0)\, ds$

- For a function of two or more variables:

$$df = (\nabla f|_{P_0} \cdot u)\, ds$$ where $u$ is the direction of motion away from $P_0$.

Linearization:

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

## Extreme Values and Saddle Points:

$f(a,b)$ local maximum if $f(a,b) \geq f(x,y)$ $\forall$ $(x,y)$ in an open disk centered at $(a,b)$. Min if "$\leq$".

If $f(x,y)$ has a local maxima or min at an interior point $(a,b) \Rightarrow f_x(a,b)=0$ and $f_y(a,b)=0$

A saddle point is a critical point $(a,b)$ if $\exists$ points $f(x,y) \geq$ or $\leq$ $(a,b)$

$f$ has a local maximum at $(a,b)$ if $f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at $(a,b)$ $\quad \therefore \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ = Hessian

$f$ has a local minimum at $(a,b)$ if $f_{xx}>0$ and $f_{xx}f_{yy}\ f_{xy}>0$ at $(a,b)$

$f$ has a saddle point at $(a,b)$ if $f_{xx}f_{yy}-f_{xy}^2<0$ at $(a,b)$. Inconclusive test if $f_{xx}f_{yy}-f_{xy}^2=0$

## Lagrange Multipliers:

Local extreme values of $f(x,y,z)$ whose variables are subject to a constraint $g(x,y,z)=0$ are to be found on the surface $g=0$ among the points where $\nabla f = \lambda \nabla g$.

Suppose $f$ has interior curve $C: r(t) = x(t)i + y(t)j + z(t)k$. If $p_0$ is a point on $C$ where $f$ has a local max or min. $\nabla f$ is orthogonal to $C$ at $p_0$.

Suppose $f(x,y,z)$, $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$. To find the local maximum and minimum values of $f$ subject to a constraint $g(x,y,z)=0$, find values of $x,y,z$ and $(\lambda)$ that simultaneously satisfy the equations $\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$.

If two or more constraints: $\nabla f = \lambda \nabla g_1 + \alpha \nabla g_2$, $g_1(x,y,z)=0$, $g_2(x,y,z)=0$

## Multiple Integrals:

$f(x,y)$ defined on rectangular region $R$, $R: a \leq x \leq b$, $c \leq y \leq d$, $\Delta A = \Delta x \Delta y$

$$S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k, \quad \iint_R f(x,y)\,dA \Rightarrow \iint f(x,y)\,dx\,dy$$

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA. \text{ if, } R: a \leq x \leq b,\ c \leq y \leq d, \quad \int_a^b\!\!\int_c^d f(x,y)\,dx\,dy = \int_c^d\!\!\int_a^b f(x,y)\,dy\,dx$$

$$\text{If } R := a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x) \quad \iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\text{If } R := c \leq y \leq d,\ h_1(y) \leq x \leq h_2(x) \quad \iint_R f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

**Area by double integration:** if $f(x,y)=1$. $S_n = \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k \quad \therefore A = \iint_R dA$

**Average Value of $f$ over $R$** $= \dfrac{1}{A_R} \iint_R f\,dA$

**Integral in Polar Coordinates:** $f(r,\theta)$, $S_n = \sum_{k=1}^{n} f(r_k,\theta_k)\,\Delta A_k$, $\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\,dA$

$$S_n = \sum_{k=1}^{n} f(r_k,\theta_k)\,r_k\,\Delta r\,\Delta\theta \quad \therefore \lim_{n\to\infty} S_n = \iint_R f(r,\theta)\,r\,dr\,d\theta = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta \qquad A = \iint_R r\,dr\,d\theta$$

**Cartesian integrals into Polar integrals:**

$$x = r\cos\theta,\ y = r\sin\theta \quad \iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \quad r = \sqrt{x^2+y^2}$$

**Triple Integrals in cartesian coordinates:** $S_n = \sum_{k=1}^{n} F(x_k,y_k,z_k)\,\Delta V_k \quad \therefore \lim_{n\to\infty} S_n = \iiint_D F(x,y,z)\,dV$

if $F(x,y,z)=1$, $S_n = \sum_{k=1}^{n} \Delta V_k \quad \therefore \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV$. Volume $= \iiint_D dV$

$$\iiint_D F(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} F(x,y,z)\,dz\,dy\,dx. \text{ Average Value of } F \text{ over } D = \frac{1}{V_D} \iiint_D F\,dV$$

**Applications:** Masses and First movements. If $\delta(x,y,z)$ is a density function of an object a region $D$ in space. The integral of $\delta$ over $D$ gives the mass of the object.

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\Delta V_k = \iiint_D \delta(x,y,z)\,dV$$

The first movement of a solid region $D$ is defined as the triple integral over $D$ of the distance from a point $(x,y,z)$. For instance, the first movement about the $yz$-plane is:

$M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$. The $x$-coordinate of the center of mass is $\bar{x} = M_{yz}/M$

3D solid:

Mass: $M = \iiint_D \delta\,dV$

First movements: $M_{yz} = \iiint_D x\,\delta\,dV$, $M_{xz} = \iiint_D y\,\delta\,dV$ $M_{xy} = \iiint_D z\,\delta\,dV$

Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}(M_{yz}, M_{xz}, M_{xy})$

2D plate:

Mass: $M = \iint_D \delta\,dA$

First movements: $M_y = \iint_R x\,\delta\,dA$, $M_x = \iint_R y\,\delta\,dA$

center of mass: $(\bar{x}, \bar{y}) = \frac{1}{M}(M_y, M_x)$

**Triple integrals in cylindrical and spherical coordinates:** **Cylindrical coordinates** represent a point $P$ in space by ordered triples $(r, \theta, z)$ $r \geq 0$. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $r^2 = x^2 + y^2$, $\tan\theta = y/x$

$\theta = \theta_0$ describes the plane that contains the $z$-axis and makes an angle $\theta_0$ with the positive $x$-axis. ($r, z$ vary)

$r = a$ describes a cylinder of radius $a$ ($\theta, z$ vary)

$z = z_0$ describes a plane perpendicular to $z$-axis at $z_0$. ($r, \theta$ vary)

$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\,\Delta z_k\, r_k\,\Delta r_k\,\Delta\theta_k$. $\Delta V = \Delta z\, r\,\Delta r\,\Delta\theta$ $\therefore$ $dV = dz\,r\,dr\,d\theta$

$\lim_{n\to\infty} S_n = \iiint_D f\,dV = \iiint_D f\,dz\,r\,dr\,d\theta$

$$\iiint_D f(r,\theta,z)\,dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

**Spherical Coordinates:** Spherical coordinates represent a point $P$ in space by ordered triples $(\rho, \phi, \theta)$. 1) $\rho$ is the distance from $P$ to the origin.

2) $\phi$: angle that $\overrightarrow{OP}$ makes with the positive $z$-axis. $0 \leq \phi \leq \pi$

3) $\theta$: angle from cylindrical coordinates

$\rho = a$: describes sphere of radius $a$, centered at origin ($\phi, \theta$ vary)

$\phi = \phi_0$: cone with vertex in origin and axis = $z$-axis ($\rho, \theta$ vary)

$\theta = \theta_0$: half plane that contains $z$-axis and makes $\theta_0$ angle with $x$-positive-axis ($\rho, \phi$ vary)

$$\left.\begin{array}{ll} r = \rho\sin\phi, & x = r\cos\theta = \rho\sin\phi\cos\theta \\ z = \rho\cos\phi, & y = r\sin\theta = \rho\sin\phi\sin\theta \\ \rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} & \end{array}\right\} \text{Equations relating Spherical c. to cartesian and cylindrical coordinates}$$

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\,\rho_k^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$$

$$\lim_{n\to\infty} S_n = \iiint_D f(\rho,\phi,\theta)\,dV = \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\alpha}^{\beta} \int_{\phi_{min}}^{\phi_{max}} \int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

**Substitution in Double Integrals:** $x = g(u,v)$ , $y = h(u,v)$

The jacobian of the coordinate transformation $x = g(u,v)$, $y = h(u,v)$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$ also denoted: $\frac{\partial(x,y)}{\partial(u,v)}$

$$dx\,dy \Longrightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\iint_R f(x,y)\, dx\, dy = \iint_G f(g(u,v), h(u,v)) \left|\frac{\partial(x,v)}{\partial(u,v)}\right| du\, dv$$

$$J(r,\theta) = r \;\therefore\; \iint_R f(x,y)\, dx\, dy = \iint f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

# Integrals and Vector Fields.

## Line integrals of scalar functions:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k.$$

$f :=$ curve $C := r(t) = \langle g(t), h(t), k(t)\rangle \quad a \le t \le b$

line integral of $f$ over $C$ is

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t))\,|v(t)|\,dt$$

$$\int_C f\,ds = \int_{C_1} f\,ds + \dots + \int_{C_n} f\,ds$$

$$M = \int_a^b \delta(x(t), y(t), z(t))\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\,dt$$

## Vector Fields:

$$F(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

Gradient fields: $\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$ : gives the direction of greatest increase of $f$.

Let $F$ be a vector field along $C := r(t)$, $a \le t \le b$. the line integral of $F$ along $C$:

$$\int_C F\cdot T\,ds = \int_C F\cdot\frac{dr}{ds}\,ds = \int_C F\cdot dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$$

$$\int_C M(x,y,z)\,dx = \int_a^b M(g(t), h(t), k(t))\,g'(t)\,dt$$

$$\int_C N(x,y,z)\,dy = \int_a^b N(g(t), h(t), k(t))\,h'(t)\,dt$$

$$\int_C P(x,y,z)\,dz = \int_a^b P(g(t), h(t), k(t))\,k'(t)\,dt$$

$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} F(x_k, y_k, z_k)\cdot T(x_k, y_k, z_k)\,\Delta s_k$$

$$W = \int_C F\cdot T\,ds = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$$

$$\text{Flow} = \int_C F\cdot T\,ds$$

If $C$ is a simple closed curve in the domain of a continuous vector field $F = M(x,y)i + N(x,y)j$ and $n$ is the outward pointing unit vector.

Flux of $F$ across $C = \int_C F\cdot n\,ds$. $n = T\times k = dy/ds\,i - dx/ds\,j$

$$\int F\cdot n\,ds = \int\left(M\,dy/ds - N\,dx/ds\right)ds = \oint_C M\,dy - N\,dx$$

## Path ind., Conservative Fields, Potential Functions:

Suppose that $\forall\, a,b \in R$, $\int_C F\cdot dr$ along $C$ from $a$ to $b$ in an open region $D$ is the same over all paths. Then $\int_C F\cdot dr$ is path ind. in $D$ and $F$ is conservative on $D$

given $F$ defined on $D$ and $F = \nabla f$ for some scalar function $f$ on $D$, then $f$ is called a potential function for $F$.

## Fundamental theorem of Line Integrals

Let $C$ be a curve joining $A$ and $B$ parametrized by $r(t)$. Let $f$ be a diff. function with $F = \nabla f$ on a domain containing $C$. Then

$$\int_C F\cdot dr = f(B) - f(A)$$

**Theorem 2:** Let $F = Mi + Nj + Pk$. $F$ conservative iff $F$ is a gradient field $\exists$ diff. $f$.

TFAE: $\oint_C F\cdot dr = 0$ ($\forall$ loop (closed c.) in $D$)

$F$ is conservative on $D$

## Green's Theorem in the plane:

Circulation density of $F = Mi + Nj$ at $(x,y)$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (\operatorname{curl} F)\cdot k$$

if $(\operatorname{curl} F)\cdot k > 0$ counterclockwise rotation.

divergence (flux density): $F = Mi + Nj$ : $\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

$$\underbrace{\oint_C F\cdot T\,ds}_{\text{Circulation}} = \oint M\,dx + N\,dy = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{Curl}}dx\,dy$$

$$\underbrace{\oint_C F\cdot n\,ds}_{\text{outward Flux}} = \oint M\,dy - N\,dx = \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\text{Divergence}}dx\,dy$$

## Parametrization of Surface:

$$r(u,v) = f(u,v)i + g(u,v)j + h(u,v)k$$

$a \le u \le b$, $c \le v \le d$.

## Area of smooth surface:

$$r(u,v) = f(u,v)i + g(u,v)j + h(u,v)k$$

$a \le u \le b$, $c \le v \le d$.

$$A = \iint_R |r_u \times r_v|\,dA = \int_c^d\int_a^b |r_u \times r_v|\,du\,dv$$

$$d\sigma = |r_u \times r_v|\,du\,dv \quad \therefore \iint_S d\sigma$$

## Surface area of graph $z = f(x,y)$

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

## Surface Integrals:

$$\iint_S G(x,y,z)\,d\sigma =$$

$$\iint_R G(f(u,v), g(u,v), h(u,v))\,|r_u \times r_v|\,du\,dv$$

$$\iint_R G(x,y,z)\frac{|\nabla F|}{|\nabla F\cdot p|}\,dA$$

$$\iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

## Stokes theorem:

$$\operatorname{Curl} F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$\operatorname{Curl} F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k$$

$$\oint F\cdot dr = \iint_S (\nabla\times F)\cdot n\,d\sigma$$

## Divergence Theorem:

$$\operatorname{div} F = \nabla\cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$\underbrace{\iint_S F\cdot n\,d\sigma}_{\text{outward Flux}} = \iiint_D \nabla\cdot F\,dV$$

$n =$ outer unit normal to $S$. $f(x,y,z) = x^2 + y^2 + z^2 - a^2$

$$n = \frac{2xi + 2yj + 2zk}{\sqrt{4(x^2+y^2+z^2)}} = \frac{xi + yj + zk}{a}$$

$$\operatorname{div}(\operatorname{curl} F) = \nabla\cdot(\nabla\times F) = 0$$