(n) — Order of ODE C.C. ODE Linear ODE Homogeneous $\begin{cases} y' = f(y,t) \\ y(0)=y_0 \end{cases}$ if f, $\frac{\partial f}{\partial y}$ continuous $\Longrightarrow$ ODE $\exists!$ solution $y=g(t)$ for $t \in (-\delta + y_0, y_0+\delta)$, $\delta > 0$

$y' = ky \Rightarrow y = Ce^{kt}$; $ay^{(n)} + \dots + cy = f(t)$; $ay'' + by' + cy = f(t)$; $a(t)y'' + b(t)y' + c(t)y = f(t)$; if $f(t) = 0$; STABLE BI-ST. UNST.

Separation of V. Hidden Solutions I.F

$y' = f(y,t) \therefore \int_0^t y'(s)\,ds = \int_0^t f(y,s)\,ds \therefore y(t) = y_0 + \int_0^t f(y,s)\,ds$; $dy/dt = f(t)/g(y) \therefore g(y)dy = f(t)dt \Longrightarrow G(t) = F(t) + C$. when dividing by y; For $y' + ay = f(t)$, $I(t) = e^{at}$ [make sure y' coeff $= 1$.]

GRADIENT

For $y' + P(t)y = f(t)$, $I(t) = e^{\int P(t)dt}$; $\int \tan(t)\,dt = \ln|\sec(t)|$; If $f = f(x,y) \Rightarrow \nabla f = (f_x, f_y)$; $F = \langle P, Q\rangle$ conservative $\Longleftrightarrow P_y = Q_x \therefore \exists f$ s.t $F = \nabla f$; $F(x,y) = \langle 2xy + y^2, x^2 + 2xy\rangle$ $P_y = Q_x = 2x + 2y$ EXACT EQ.

$\therefore F = \nabla f \therefore \langle P, Q\rangle = \langle f_x, f_y\rangle \therefore f_x = 2xy + y^2 \Rightarrow f(x,y) = \int 2xy + y^2\,dx = x^2y + y^2x + g(y)$, $f_y = x^2 + 2xy \therefore f(x,y) = \int x^2 + 2xy\,dy = x^2y + y^2x + h(x) \therefore f(x,y) = x^2y + y^2x$; $Pdx + Qdy = 0$, Let $F = \nabla f$ Solutions

CER EXACT ODE Lin. Approx.

$f(x,y) = c$ If $P_y = Q_y \Rightarrow Pdx + Qdy = 0$. If $P_y \neq Q_x$ then $g = I(x,y) \therefore g(Pdx + Qdy) = (Pg)dx + (Qg)dy = 0 \therefore (Pg)_y = (Qg)_x$; $y'(t) = \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \therefore y(t+h) \approx y(t) + hy'(t)$

Euler M. $h = \frac{b-a}{N}$, $t_n = a + nh$ Error: $\lim_{h\to 0} E(h) = 0$ $\begin{cases} y'' = f(y,y',t) \\ y(0) = y_0 \\ y'(0) = v_0 \end{cases}$ if f and p. derivatives continuous

$[a,b]$, N steps: ; $y_{n+1} = y_n + hf(y_n, t_n)$, $f(y,t) = y'$; $E(h) = \max\{|y_1 - y(t_1)|, \dots, |y_n - y(t_n)|\}$; $\exists C$ s.t $\forall h\ |E(h)| \leq Ch$ Theorem: $\Longrightarrow \exists!$ solution $y = g(t)$ for t near 0

Diff. Operators s.t Let $z = (D-d)y$ Aux. Eq:

$Dy = y'$, $D^2y = y''$: $y'' \pm ay' + by = 0 \therefore (D-c)(D-d)y = 0 \therefore (D-c)z = 0$: $z' - cz = 0 \therefore z = Ae^{ct} \therefore (D-d)y = Ae^{ct}$: $y' - dy = Ae^{ct} \therefore y = Ae^{ct} + Be^{dt}$; $A(r) = (r-c)(r-d) \Rightarrow y = Ae^{ct} + Be^{dt}$

$\therefore r = a$ $A(r) = r^2 + 1$ $\therefore y = A\cos(t) + B\sin(t)$

$ay^{(n)} + \dots + by \therefore A(r) = ar^n + \dots + b$ if $A(r) = (r-a)\dots(r-b)$, $y = Ae^{at} + \dots + Be^{bt}$; if $A(r) = (r-a)^2 = 0$ $y = Ae^{at} + Bte^{at}$; $e^{it} = \cos(t) + i\sin(t)$; $y'' + y = 0 \therefore y = Ae^{it} + Be^{-it} = (A+B)\cos(t) + i(A-B)\sin(t)$

$y = 0$ $y = 0$

$A(r) = 0$ s.t. $r = a + bi \therefore y = Ae^{at}\cos(bt) + Be^{at}\sin(bt)$; $\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$ $\therefore r^2 = \lambda$: $\lambda > 0$: $\exists \omega > 0$ s.t $\lambda = \omega^2 \therefore r = \pm\omega$, $y = Ae^{\omega t} + Be^{-\omega t}$; $\lambda = 0$: $r^2 = 0$: $y = A + Bt$, $\lambda < 0$: $\lambda = -\omega^2 \therefore r = \pm\omega i$: $y = A\cos(\omega t) + B\sin(\omega t)$

$B \neq 0$ $k \in \mathbb{Z}$ $\therefore$ Inhomog. eq:

$y(0) = A = 0 \therefore y(t) = B\sin(\omega t)$; $y(\pi) = \sin(\pi\omega) = 0 \therefore \omega\pi = \pi k$. $\omega = k$, $\lambda = -k^2$, $y = B\sin(kt)$: Eigenvalue: $\lambda = -k^2$, Eigenfunctions: $y = \sin(kt)$; $y'' + ay' + by = f(t)$; $y'' + ay' + by = 0$

inh.

$\therefore$ Let $y_0(t)$ solution, Find 1 sol. to eq. $y_p(t)$ $\therefore$ general solution: $y = y_0 + y_p$. y_p: $y'' + ay' + by = Be^{rt}$, $y_p = Ae^{rt}$; if $f(t) = at^n + \dots + b$, $y_0 = At^n + \dots + B$ (plug and solve coeffic.)

Resonance:

if $f(t) = \cos(at) \,\|\, \sin(at)$, $y_p = A\cos(at) + B\sin(at)$, if $f(t) =$ sum of terms, guess y_p separately. If guessing $y_p = Ae^{rt}$, Ae^{rt} is sol. to $f(t) = 0$ then guess $y_p = Ate^{rt}$

$y'' + 3y' - 4y = t^3e^{-4t}$; $e^{-4t} \rightsquigarrow r = -4$ (resonance) $\therefore y_p = (t)(At^3 + Bt^2 + Ct + D)e^{-4t}$, $y'' + 3y' - 4y = e^{-4t}\cos(t)$ $(r = -4 \pm i)$ (no resonance); $y'' - 4y' + 4y = e^{2t}$ $e^{2t} \rightsquigarrow r = 2 \therefore y_p = At^2e^{2t}$

Mass spring equation: $my'' + \gamma y' + ky = F(t)$; m = mass; γ = damping c.; k = spring c.; $F(t)$ = external force. $\int \sec(t) = \ln|\sec t + \tan t|$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. $\begin{bmatrix} \alpha & \beta \\ \sigma & \theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$: $x = \begin{bmatrix} \lambda_1 & \beta \\ \lambda_2 & \theta \end{bmatrix} \Big/ \begin{bmatrix} \alpha & \beta \\ \sigma & \theta \end{bmatrix} = c_1$ $y = \begin{bmatrix} \alpha & \lambda_1 \\ \sigma & \lambda_2 \end{bmatrix} \Big/ \begin{bmatrix} \alpha & \beta \\ \sigma & \theta \end{bmatrix} = c_2 \therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$; Wronskian M of $f(t), g(t)$ $= \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix}$ Ex: $\begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix}$

W.M.

$M(t)\begin{bmatrix} \mu'(t) \\ \beta'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \text{Inhomo} \end{bmatrix}$. $ay'' + by' + c = f(t)$, $ay'' + by' + c = 0 \Longrightarrow y_0 = g(t)$, $y_p = \mu(t)g(t)$, $y(t) = y_0(t) + y_p(t) \therefore y'' + y = \sec(t)$; h: $y'' + y = 0 \Rightarrow y_0 = A\cos(t) + B\sin(t)$

guess $y_p \approx y_0$

$y_p = u(t)\cos(t) + v(t)\sin(t) \Rightarrow \begin{cases} \cos(t)u'(t) + \sin(t)v'(t) = 0 \\ (-\sin(t))u'(t) + \cos(t)v'(t) = \sec(t) \end{cases} \therefore \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(t) \end{bmatrix} \Longrightarrow$ $u'(t) = -\tan(x)$ $u(t) = -\ln(\sec t)$, $v'(t) = 1 \Longrightarrow t \therefore y_p = -\ln|\sec t|\cos t + t\sin t$.

$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$; $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \Longrightarrow \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$; Step functions: $u_c(t) = \begin{cases} 0, & \text{if } t < c \\ 1, & \text{if } t \geq c \end{cases}$; $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$

$\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs}\mathcal{L}\{f(t)\}$, $e^{-cs}\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t-c)u_c(t)\}$; Shifting: $\mathcal{L}\{f(t)\} = F(s)$, $\exists F \therefore \mathcal{L}\{e^{ct}f(t)\} = F(s-c)$

$f(t) = \begin{cases} a(t), & 0 \leq t < \lambda_1 \\ b(t), & \lambda_1 \leq t < \lambda_2 \\ c(t) & t \geq \lambda_2 \end{cases} \therefore f(t) = a(t) + (b(t) - a(t))u_{\lambda_1}(t) + (c(t) - b(t))u_{\lambda_2}(t)$. Dirac Delta: $\exists\,\delta(t)$ s.t. (i) $\delta(t) = 0$, $\forall t \neq 0$. s.t. $\delta(0) = \infty$ (ii) $\int_{-\infty}^\infty \delta(t)\,dt = 1$

Consider $\{f_n(t)\}$, $f_1(t) > 0$, $-1 \leq t \leq 1$, $f(t) = 0$ otherwise s.t. $\int_{-\infty}^\infty f(t) = 1$, $f_2(t) > 0$, $-1/2 \leq t \leq 1/2$ s.t $\int_{-\infty}^\infty f_2(t)\,dt = 1$, …, $f_n(t) > 0$, $-1/n \leq t \leq 1/n$ s.t $\int_{-\infty}^\infty f_n(t)\,dt = 1$

$f(t)$, $2f(2t)$, $3f(3t)$, …, $nf(nt)$ $\therefore \delta(t) = \lim_{n\to\infty} nf(nt)$. $\forall f(t)$, $\int_{-\infty}^\infty \delta(t)f(t)\,dt = f(0) \therefore \mathcal{L}\{\delta(t)\} = 1$; $\int_{\mathbb{R}} \delta(t-c)f(t)\,dt = f(c)$. $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

In general: $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$. Def: (Convolution): $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$. $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$

$\phi(t) + \int_0^t (t-\tau)\phi(\tau)\,d\tau = f(t) \Longrightarrow \mathcal{L}\{\phi(t)\} + \mathcal{L}\{t * \phi(t)\} = \mathcal{L}\{\phi(t)\} + \mathcal{L}\{t\}\mathcal{L}\{\phi(t)\} = \mathcal{L}\{f(t)\}$; $\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$

If $Av = \lambda v$ then: λ an eigenvalue of A, v an eigenvector of A corresponding to λ. $\therefore \vec{v}$ and $A\vec{v}$ lie on the same line. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A\vec{v} = \lambda\vec{v} \therefore A\vec{v} - \lambda\vec{v} = 0 \therefore A\vec{v} - \lambda I\vec{v} = 0 \therefore (A - \lambda I)\vec{v} = 0 \therefore \det(A - \lambda I) = 0$. $\forall$ eigenvalue λ, solve $(A - \lambda I)\vec{v} = 0$, $Null(A - \lambda I)$. After row-reduction: $\left[\begin{array}{c|c} ax + by & 0 \\ 0 \quad 0 & 0 \end{array}\right] \therefore$ $ax + by = 0$, $x = -b$, $y = a \therefore \begin{bmatrix} -b \\ a \end{bmatrix}$

Diagonalization: Find D diagonal and P s.t $A = PDP^{-1}$ where $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\lambda = 7$, $\lambda = -4 \therefore D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \lambda = 7$, $\begin{bmatrix} -6 \\ 5 \end{bmatrix} \rightsquigarrow \lambda = -4 \therefore P = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$, $AP = PD \Longrightarrow APP^{-1} = PDP^{-1} = A$

ODE Sys. Phase P.

In general $AB \neq BA$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $x' = Ax$ $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ s.t $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \rightsquigarrow \lambda_1$, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \rightsquigarrow \lambda_2 \therefore x(t) = c_1e^{\lambda_1 t}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2e^{\lambda_2 t}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Draw axes with eigenvectors if $\lambda_i < 0$ then

$\lambda > 0$ Complex. e. $x' = Ax$

arrows towards origin; Outwards. Solutions by following arrows. $A = \begin{bmatrix} -2 & 4 \\ -2 & 2 \end{bmatrix}$ $\lambda = \pm 2i$ $Null(A - (2i)I) = \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$ (Keep one row) $\therefore Null(A - 2iI) = \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \therefore (1+i)x - 2y = 0$

$\vec{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix} \rightsquigarrow \lambda = 2i \therefore e^{2it}\begin{bmatrix} 2 \\ 1+i \end{bmatrix}$ sol. $\therefore (\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left(\cos(2t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + i\left(\cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(2t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \therefore x(t) = c_1(\dots) + c_2(\dots)$ (one complex eigenvalue suffices)

Direction: direction Repeated eig.

$x(t) = c_1\begin{bmatrix} 2\cos(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$ $\forall \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $x'(t) = \begin{bmatrix} -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \therefore$ sol. through $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix}$; $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, $\lambda = 2$; $Null(A - 2I) \longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \therefore \lambda = 2 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, Now solve:

M. Expon.

$(A - 2I)w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \therefore \left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \therefore$ $-x + y = 1$, $y = 1 + x$ $\therefore w = \begin{bmatrix} x \\ 1+x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \therefore c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$; Find A^n where $A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$, $A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$

$A = PDP^{-1}$, $P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow D^n = \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} \therefore A^n = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix}\frac{1}{2\cdot 2 - 1\cdot 3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ (-3)2^n & 2^{n+1} \end{bmatrix} = \begin{bmatrix} 4 - 3(2^n) & -2 + 2^{n+1} \\ 6 - 6(2^n) & -3 + 2^{n+2} \end{bmatrix} \therefore A^{1/2} = \dots$ Find e^A, $A = \begin{bmatrix} -2 & 2 \\ 6 & 5 \end{bmatrix}$, $e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \therefore e^A = \sum \frac{A^n}{n!}$

$\therefore$ if $A = PDP^{-1} \Rightarrow e^A = Pe^DP^{-1}$. $e^A = \sum_{n=0}^\infty \frac{A^n}{n!} = \sum_{n=0}^\infty \frac{PD^nP^{-1}}{n!} = P\left(\sum_0^\infty \frac{D^n}{n!}\right)P^{-1}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \therefore e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix} \therefore e^D = \sum_0^\infty \frac{D^n}{n!} = \sum \frac{1}{n!}\begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} = \sum \begin{bmatrix} \frac{1^n}{n!} & 0 \\ 0 & \frac{2^n}{n!} \end{bmatrix} = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix} \therefore e^A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$

$e^A = \begin{bmatrix} 4e - 3e^2 & -2e + 2e^2 \\ 6e - 6e^2 & -3e + 4e^2 \end{bmatrix}$, find e^{At} where $A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$; $e^{At} = \sum \frac{(At)^n}{n!}$, $e^{At} = \begin{bmatrix} 4e^t - 3e^{2t} & -2e^t + 2e^{2t} \\ 6e^t - 6e^{2t} & -3e^t + 4e^{2t} \end{bmatrix}$; $x' = Ax \Rightarrow x = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \therefore x(t) = \begin{bmatrix} 4e^t - 3e^{2t} & -2e^t + 2e^{2t} \\ 6e^t - 6e^{2t} & -3e^t + 4e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1e^t\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

M. exp. rep. eigen. $\lambda = 3$

given $x(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $x(t) = e^{At}x(0) = \begin{bmatrix} 4e^t - 3e^{2t} & -2e^t + 2e^{2t} \\ 6e^t - 6e^{2t} & -3e^t + 4e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^t\begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $x' = Ax$, $A = \begin{bmatrix} -1 & 4 \\ -4 & 7 \end{bmatrix}$ $\lambda_{1,2} = 3$; $(A - 3I)^2 = 0$. $e^{At} = e^{(3t)I + (A - 3I)t} = e^{3t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1-3 & 4 \\ -4 & 7-3 \end{bmatrix}t\right) = e^{3t}\begin{bmatrix} 1 - 4t & 4t \\ -4t & 1 + 4t \end{bmatrix}$

Und. coeff. $\lambda = 1$ find x_p.

$x(t) = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1e^{3t}\begin{bmatrix} 1 - 4t \\ -4t \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 4t \\ 1 + 4t \end{bmatrix}$; $x' = Ax + f$. $A = \begin{bmatrix} 7 & -3 \\ 8 & -3 \end{bmatrix}$, $f = \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix}$ $\lambda_2 = 3$, guess $x_p = e^{2t}\begin{bmatrix} A \\ B \end{bmatrix} \therefore \begin{bmatrix} Ae^{2t} \\ Be^{2t} \end{bmatrix}' = \begin{bmatrix} 7 & -3 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} Ae^{2t} \\ Be^{2t} \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix} \therefore \begin{bmatrix} 2Ae^{2t} \\ 2Be^{2t} \end{bmatrix} = \begin{bmatrix} (7A - 3B + 1)e^{2t} \\ (8A - 3B + 4)e^{2t} \end{bmatrix} \therefore x_p = e^{2t}\begin{bmatrix} 7 \\ 12 \end{bmatrix}$

Part. Sol. $\lambda = -1, 3$

$x' = Ax + f$, $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$, $f = \begin{bmatrix} e^{2t} \\ e^{2t} + 4e^{3t} \end{bmatrix}$ $\lambda = \pm 1$; $x_0 = c_1e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $x_p' = Ax_p + f \therefore \begin{bmatrix} Ae^{2t} + Ce^{3t} \\ Be^{2t} + De^{3t} \end{bmatrix}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} Ae^{2t} + Ce^{3t} \\ Be^{2t} + De^{3t} \end{bmatrix} + \begin{bmatrix} e^{2t} \\ e^{2t} + 4e^{3t} \end{bmatrix} \therefore x_p = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \therefore x = x_0 + x_p$. guess x_p, $x' = Ax + f$, $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$\lambda = \pm 3i$

$f = \begin{bmatrix} 2e^{2t} - 3e^{4t} \\ e^{2t} + 4e^{4t} \end{bmatrix} = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{4t}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\lambda = 2$, $\lambda = 4$ $\therefore x_p = e^{2t}\begin{bmatrix} A \\ B \end{bmatrix} + e^{4t}\begin{bmatrix} C \\ D \end{bmatrix}$, $f = \begin{bmatrix} \cos 3t \\ 2\cos 3t \end{bmatrix}$, $x_p = \cos(3t)\begin{bmatrix} A \\ B \end{bmatrix} + \sin(3t)\begin{bmatrix} C \\ D \end{bmatrix}$; $f = \begin{bmatrix} 2t + 3 \\ 5t^2 + 4t \end{bmatrix}$, $x_p = \begin{bmatrix} A \\ B \end{bmatrix}t^2 + \begin{bmatrix} C \\ D \end{bmatrix}t + \begin{bmatrix} E \\ F \end{bmatrix}$; $f = \begin{bmatrix} (2t + 3)e^{2t}\cos(3t) \\ 4te^{2t}\sin(3t) \end{bmatrix}$ $\lambda = 2 \pm 3i$; $x_p = \left(\begin{bmatrix} A \\ B \end{bmatrix}t + \begin{bmatrix} C \\ D \end{bmatrix}\right)e^{2t}\cos(3t) + \left(\begin{bmatrix} A \\ B \end{bmatrix}t + \begin{bmatrix} C \\ D \end{bmatrix}\right)e^{2t}\sin(3t)$

Resonance: $\lambda = 3$ REP $\lambda = 2, 2$ $\lambda = 2 \pm 3i$

$f = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$; $x_p = \left(\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} C \\ D \end{bmatrix}t\right)e^{3t}$; let A: $f = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} \therefore x_p = \left(\begin{bmatrix} A \\ B \end{bmatrix}t^2 + \begin{bmatrix} C \\ D \end{bmatrix}t + \begin{bmatrix} E \\ F \end{bmatrix}\right)e^{2t}$, A: $f = \begin{bmatrix} t^3e^{2t}\cos(3t) \\ e^t \end{bmatrix} \therefore \left(\begin{bmatrix} A \\ B \end{bmatrix}t^5 + \dots + \begin{bmatrix} K \\ L \end{bmatrix}\right)e^{2t}\cos(3t) + \left(\begin{bmatrix} M \\ N \end{bmatrix}t^5 + \dots + \begin{bmatrix} W \\ X \end{bmatrix}\right)e^{2t}\sin(3t) + \begin{bmatrix} Y \\ Z \end{bmatrix}e^t$

Var. Param.

$x' = Ax + f$, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $f = \begin{bmatrix} 2e^{t} \\ 4t \end{bmatrix}$, $\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = 3 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $x_0(\tau) = c_1 e^{\tau}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3\tau}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, guess $x_p = u(t)\begin{bmatrix} e^{\tau} \\ e^{\tau} \end{bmatrix} + v(\tau)\begin{bmatrix} e^{3\tau} \\ -e^{3\tau} \end{bmatrix}$ $\therefore \begin{bmatrix} e^{\tau} & e^{3\tau} \\ e^{\tau} & -e^{3\tau} \end{bmatrix}\begin{bmatrix} u'(\tau) \\ v'(\tau) \end{bmatrix} = \begin{bmatrix} 2e^{\tau} \\ 4\tau \end{bmatrix}$ Den $= e^{\tau}(-e^{3\tau}) - e^{\tau}e^{3\tau} = -2e^{4\tau}$

$u'(\tau) = \begin{bmatrix} 2e^{\tau} & e^{3\tau} \\ 4\tau & -e^{3\tau} \end{bmatrix} / -2e^{4\tau} = 1 + 2\tau e^{-\tau}$, $v'(\tau) = \begin{bmatrix} e^{\tau} & 2e^{\tau} \\ e^{\tau} & 4e^{\tau} \end{bmatrix} / -2e^{4\tau} = -2\tau e^{-3\tau} + e^{-2\tau}$ $\therefore$ $u(\tau) = \tau - 2\tau e^{-\tau} - 2e^{-\tau}$, $v(\tau) = \frac{2}{3}\tau e^{-3\tau} + \frac{2}{9}e^{-3\tau} - \frac{1}{2}e^{-2\tau}$ $x_p = u(\tau)\begin{bmatrix} e^{\tau} \\ e^{\tau} \end{bmatrix} + v(\tau)\begin{bmatrix} e^{3\tau} \\ -e^{3\tau} \end{bmatrix}$, $x(\tau) = x_0(\tau) + x_p(\tau)$

Non Lin. syst. $\begin{cases} x'(\tau) = x^2 + xy + e^{y} \\ y'(\tau) = \cos(x) + 2y \end{cases}$, $x'(\tau) = F(x(\tau))$, $x(\tau) = \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}$, $F(x,y) = \begin{bmatrix} x^2 + xy + e^{y} \\ \cos(x) + 2y \end{bmatrix}$, $\nabla F = \begin{bmatrix} \partial/\partial x\,(x^2+xy+e^{y}) & \partial/\partial y\,(x^2+xy+e^{y}) \\ \partial/\partial x\,(\cos(x)+2y) & \partial/\partial y\,(\cos(x)+2y) \end{bmatrix} = \begin{bmatrix} 2x+y & x+e^{y} \\ -\sin(x) & 2 \end{bmatrix}$.

theorem If F, ∇F cont. then $\begin{cases} x' = F(x) \\ x(0) = x_0 \end{cases}$ Has a unique sol. $x(\tau)$ for τ close enough to 0.

Equilibrium sol. **Equil. Points**

$\begin{cases} x' = -x + xy \\ y' = -8y + 4xy \end{cases}$ set $x' = y' = 0 \Longrightarrow (0,0), (2,1)$, $\nabla F(x,y) = \begin{bmatrix} -1+y & x \\ 4y & -8+4x \end{bmatrix}$ $\therefore \nabla F(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} = A$, eigenvalues of A $\lambda = -1 < 0$, $\lambda = -8 < 0$ $\therefore (0,0)$ STABLE

CLASSIFICATION:

Let (a,b) eq. sol. and $A = \nabla F(a,b)$, if eigenvalues of A: All negative (or $R(z)$ neg) $\Longrightarrow (a,b)$ (STABLE); All positive (or $R(z)$ pos.) $\Longrightarrow (a,b)$ UNSTABLE; Positive and negative $\Longrightarrow (a,b)$ SADDLE