$\mathcal{L}^+ = \{f$ measurable , $(X, \mathcal{M}, \mu)$, $f: X \to [0,\infty]\}$, $\phi \in \mathcal{L}^+$ s.t. $\phi = \sum_1^n a_n \chi_{E_j}$ $\therefore \int \phi \, d\mu \triangleq \sum_1^n a_j \mu(E_j)$; $\int f \, d\mu \triangleq \sup\{\int \phi \, d\mu : 0 \le \phi \le f\}$

Monotone Convergence Theorem: $\{f_n\} \subset \mathcal{L}^+$, $f_j \le f_{j+1} \ \forall j$, $f = \lim_{n\to\infty} f_n (= \sup f_n) \Longrightarrow \int f = \lim_{n\to\infty} \int f_n$, **Theorem:** $\{f_n\} \subset \mathcal{L}^+$, $f = \sum_n f_n \Longrightarrow \int f = \sum_n \int f_n$. **Fatou's Lemma:** $\forall \{f_n\} \subset \mathcal{L}^+ \Longrightarrow \int (\liminf f_n) \le \liminf \int f_n$, **Def:** $f: X \to \mathbb{C}$, $f = f^+ - f^-$ $\therefore \int f = \int f^+ - \int f^-$, f integrable if $\int f^+ \wedge \int f^- < \infty \Longrightarrow \int |f| < \infty$. **Def:** $\mathcal{L}^1(\mu) \equiv \mathcal{L}^1(X,\mu) \equiv \mathcal{L}^1(X)$, $\mathcal{L}^1(X)$: {Complex valued integrable functions}, vector space , $\int \cdot$ linear functional. **Dominated Convergence Theorem:** $\{f_n\} \subset \mathcal{L}^1$ s.t. (a) $f_n \xrightarrow{a.e.} f$ (b) $\exists g^+ \in \mathcal{L}^1$, $|f_n| \le g$ a.e. $\forall n \Longrightarrow f \in \mathcal{L}^1 \wedge \int f = \lim_{n\to\infty} \int f_n$

Signed Measures: $(X, \mathcal{M})$ measurable space, signed measure on $(X,\mathcal{M})$: $\nu: \mathcal{M} \to [-\infty, \infty]$ s.t. (i) $\nu(\emptyset) = 0$ (ii) ν assumes $\begin{pmatrix} +\infty \\ \vee \\ -\infty \end{pmatrix}$ (one finite), $\{E_j\} \subset \mathcal{M}$, $E_i \cap E_j = \emptyset \ \forall i,j \Longrightarrow \nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ and $\sum_1^\infty \nu(E_j) \longrightarrow$ absolutely if $\nu(\bigcup_1^\infty E_j) < \pm\infty$

Ex: If $\mathcal{M}$–μ_1, μ_2 measure $\Longrightarrow \nu = \mu_1 - \mu_2$ (signed), $\mathcal{M}$–μ, $f: X \to [-\infty,\infty]$ measurable s.t. $\int f^+ d\mu \vee \int f^- < \infty$ $\therefore \nu(E) \triangleq \int_E f \, d\mu$ (signed)

Hahn Decomposition: ν (signed) on $(X,\mathcal{M})$ $\therefore \exists$ positive P, negative N s.t. $P \cup N = X$, $P \cap N = \emptyset$, if (P', N') another pair $\Longrightarrow P' \Delta P (= N \Delta N')$ null for ν

Def: Two signed measures μ, ν on $(X,\mathcal{M})$ are mutually singular if $\exists E, F \in \mathcal{M}$ s.t. $E \cap F = \emptyset$, $E \cup F = X$, E: μ-null, F: ν-null : $\mu \perp \nu$.

Jordan Decomposition: ν signed, $\exists! \nu^+, \nu^-$ s.t. $\nu = \nu^+ - \nu^- \wedge \nu^+ \perp \nu^-$ (i.e.) $\nu^+(E) = \nu(E \cap P)$, $\nu^-(E) = -\nu(E \cap N)$. $|\nu| = \nu^+ + \nu^-$, ν is of the form $\nu(E) = \int_E f \, d\mu$, $\mu = |\nu|$, $f = \chi_P - \chi_N$, $X = P \cup N$, $\mathcal{L}^1(\nu) = \mathcal{L}^1(\nu^+) \cap \mathcal{L}^1(\nu^-)$, $\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$, $f \in \mathcal{L}^1(\nu)$

Lebesgue-Radon-Nikodym: **Def:** ν (signed), μ (positive) on $(X,\mathcal{M})$, ν is absolutely continuous w.r.t. μ ($\nu \ll \mu$) if $\nu(E) = 0 \ \forall E \in \mathcal{M}$ s.t. $\mu(E) = 0$ **Theorem:** ν (finite-signed), μ (positive) on $(X,\mathcal{M})$. $\nu \ll \mu \Longleftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$. **Corollary:** If $f \in \mathcal{L}^1(\mu)$, $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\left|\int_E f \, d\mu\right| < \varepsilon$ whenever $\mu(E) < \delta$. **Def:** We express the relationship $\nu(E) = \int_E f \, d\mu$: $d\nu = f \, d\mu$

LRD Theorem: Let ν-signed-σ-finite, μ-positive-σ-finite on $(X,\mathcal{M})$. $\exists!$ signed-σ-finite λ, ρ on $(X,\mathcal{M})$ s.t. $\lambda \perp \mu$, $\rho \ll \mu$ $\wedge$ $\nu = \lambda + \rho$, $\exists$ extended μ-integrable $f: X \to \mathbb{R}$ s.t. $d\rho = f \, d\mu$ and any two such functions are equal μ-a.e. **Def:** The decomposition $\nu = \lambda + \rho$, $\lambda \perp \rho$, $\rho \ll \mu$ is called Lebesgue decomposition of ν w.r.t. μ. In the case $\nu \ll \mu \Longrightarrow d\nu = f \, d\mu$ for some f. f is called Radon-Nikodym derivative of ν w.r.t. μ. $f := \frac{d\nu}{d\mu}$ $\therefore d\nu = \frac{d\nu}{d\mu} d\mu$, $\frac{d\nu}{d\mu}$: (Class of functions equal to f μ-a.e.)

Proposition: ν-signed-σ-finite , μ, λ-σ-finite on $(X,\mathcal{M})$ s.t. $\nu \ll \mu$, $\mu \ll \lambda$: a) If $g \in \mathcal{L}^1(\nu) \Longrightarrow g\left(\frac{d\nu}{d\mu}\right) \in \mathcal{L}^1(\mu) \wedge \int g \, d\nu = \int g \frac{d\nu}{d\mu} d\mu$ b) $\nu \ll \lambda$, and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$ λ-a.e.

Complex Measures: Complex Measure on $(X,\mathcal{M})$: map $\nu: \mathcal{M} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ (ii) If $\{E_j\} \subset \mathcal{M}$ $E_i \cap E_j = \emptyset \ \forall i,j \Longrightarrow \nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ $\wedge$ converges absolutely, if μ (positive measure), $f \in \mathcal{L}^1(\mu) \Longrightarrow f \, d\mu$ complex measure. **L.R.N. Theorem:** ν–complex, μ-σ-finite-positive on $(X,\mathcal{M})$ $\therefore \exists$ complex λ $\wedge$ $\exists f \in \mathcal{L}^1(\mu)$ s.t. $\lambda \perp \mu$ and $d\nu = d\lambda + f \, d\mu$, If also $(\lambda' \perp \mu)$ and $d\nu = d\lambda' + f' d\mu \Longrightarrow \lambda = \lambda'$, $f = f'$ μ-a.e.

Differentiation on Euclidean Space: $(X,\mathcal{M}) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, $\mu = m$. **Def:** Pointwise derivative of ν w.r.t. m: Let $B(r,x) \subset \mathbb{R}^n$, $F(x) = \lim_{r\to 0} \frac{\nu(B(r,x))}{m(B(r,x))}$, if $\nu \ll m$ s.t. $d\nu = f \, dm$ then $\nu(B(r,x))/m(B(r,x))$: Average value of f on $B(r,x)$

Lemma: Let $\mathcal{C}$ a collection of open balls in $\mathbb{R}^n$, $U = \bigcup_{B \in \mathcal{C}} B$. If $c < m(U)$, $\exists$ disjoint $B_1, \dots, B_K \in \mathcal{C}$ s.t. $\sum^K m(B_j) > \frac{c}{3^n}$

Def: A measurable function $f: \mathbb{R}^n \to \mathbb{C}$ is locally integrable if $\int_K |f(x)| \, dx < \infty$ $\forall$ bounded measurable set $K \subset \mathbb{R}^n$ $\therefore \mathcal{L}^1_{loc} = \{$Locally integrable functions$\}$. **Def:** $f \in \mathcal{L}^1_{loc}$, $x \in \mathbb{R}^n$, $r > 0$, $A_r f(x)$: Average value of f on $B(r,x)$, $A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) \, dy$

Lemma: If $f \in \mathcal{L}^1_{loc}$, $A_r f(x)$ is jointly continuous in r and x ($r > 0$, $x \in \mathbb{R}^n$). **Def:** $f \in \mathcal{L}^1_{loc}$, Hardy-LittleWood Maximal function Hf: $Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \, dy$. **Maximal Theorem:** $\exists C > 0$ s.t. $\forall f \in \mathcal{L}^1 \ \forall \alpha > 0$, $m(\{x : Hf(x) > \alpha\}) \le \frac{C}{\alpha} \int |f(x)| \, dx$

$\limsup_{r \to R} \phi(r) = \lim_{\varepsilon \to 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r)$, $\lim_{r \to R} \phi(r) = c \Longleftrightarrow \limsup_{r \to R} |\phi(r) - c| = 0$ **Theorem:** If $f \in \mathcal{L}^1_{loc} \Longrightarrow \lim_{r\to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$. **Def:** Lebesgue Set (L_f) of f : $L_f = \{x : \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| \, dy = 0\}$ **Theorem:** $f \in \mathcal{L}^1_{loc} \Longrightarrow m((L_f)^c) = 0$ **Def:** A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to shrink nicely to $x \in \mathbb{R}^n$ if $E_r \subset B(r,x) \ \forall r$, $\exists \alpha > r$ s.t. $m(E_r) > \alpha m(B(r,x))$. **Lebesgue Differentiation Theorem:** Suppose $f \in \mathcal{L}^1_{loc}$, $\forall x \in L_f$ (a.e. x) : $\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| \, dy = 0$ and $\lim_{r\to 0} \frac{1}{m(E_r)} \int f(y) \, dy = f(x)$ $\forall$ family $\{E_r\}_{r>0}$ that shrinks nicely to x. **Def:** A Borel measure ν on $\mathbb{R}^n$ is regular if $\nu(K) < \infty$ $\forall$ compact K, $\nu(E) = \inf\{\nu(U) : U$ open, $E \subset U\}$ $\forall E \in \mathcal{B}_{\mathbb{R}^n}$ **Theorem:** Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, let $d\nu = d\lambda + f \, dm$ be its LRN representation. Then for m-a.e. $x \in \mathbb{R}^n$, $\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ $\forall$ family $\{E_r\}_{r>0}$ that shrinks nicely to x.

Functions of Bounded Variation: F increasing, right continuous on $\mathbb{R}$, $\mu_F((a,b]) = F(b) - F(a)$

Theorem: $F: \mathbb{R} \to \mathbb{R}$ increasing , $G(x) = F(x+)$: (a) $\{x : F$ discontinuous$|_x\}$ countable b) F, G differentiable a.e. $\wedge$ $F' = G'$ a.e. **Def:** $F: \mathbb{R} \to \mathbb{C}$

$x \in \mathbb{R}$, define $T_F(x) \triangleq \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \dots < x_n = x \right\}$ (Total Variation of F) $\Rightarrow T_F(b) - T_F(a) = \sup_{n \in \mathbb{N}} \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| \,\middle|\, a = x_0 < \dots < x_n = b \right\}$
If $T_F(\infty) = \lim_{x \to \infty} T_F(x) < \infty$: F of Bounded Variation. ∴ $BV = \{F : \text{Bounded Variation on } \mathbb{R}\}$. Lemma: $F : \mathbb{R} \to \mathbb{R}$ bounded and increasing $\Rightarrow F \in BV$, $T_F(x) = F(x) - F(-\infty)$
b) If $F, G \in BV$, $a, b \in \mathbb{C} \Rightarrow aF + bG \in BV$ (c) F differentiable on $\mathbb{R}$, F' bounded $\Rightarrow F \in BV([a,b])$ for $-\infty < a < b < \infty$ (by mean value theorem)
d) $F(x) = \sin(x)$ ∴ $F \in BV([a,b])$, $F \notin BV$. Lemma: If $F \in BV$ real-valued, then $T_F + F$ and $T_F - F$ are increasing. Theorem: a) $F \in BV$ iff $\operatorname{Re} F \wedge \operatorname{Im} F \in BV$ (b) $F : \mathbb{R} \to \mathbb{R}$, $F \in BV \Leftrightarrow F$ is the difference between two bounded increasing functions such as $\frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ (c) If $F \in BV$ then $F(x^+) = \lim_{y \searrow x} F(y)$ and $F(x^-) = \lim_{y \nearrow x} F(y)$ exist $\forall x \in \mathbb{R}$, as do $F(\pm\infty) = \lim_{y \to \pm\infty} F(y)$ (d) $F \in BV$, $G(x) = F(x^+) \Rightarrow F', G'$ exist and equal a.e.
Def: Normalized BV $NBV = \{F \in BV : F \text{ right continuous} \wedge F(-\infty) = 0\}$. ∴ $G(x) \triangleq F(x+) - F(-\infty) \in NBV$ and $G' = F'$ a.e. Lemma: If $F \in BV$ then $T_F(-\infty) = 0$. If F also right continuous so is T_F. Theorem: μ complex-Borel measure on $\mathbb{R}$, $F(x) = \mu((-\infty, x]) \Rightarrow F \in NBV$. Conversely, if $F \in NBV$ $\exists!$ complex Borel μ_F s.t. $F(x) = \mu_F((-\infty, x])$; moreover $|\mu_F| = \mu_{T_F}$. Prop: $F \in NBV$, then $F \in L^1(m)$. Moreover: $\mu_F \perp m \Leftrightarrow F' = 0$, $\mu_F \ll m \Leftrightarrow F(x) = \int_{-\infty}^x F'(t)\,dt$
$F'(x) = \lim_{r \to 0} \frac{\mu_F(E_r)}{m(E_r)}$, $E_r = (x, x+r]$ Def: The condition $\mu_F \ll m$ can be expressed in terms of F. A function $F : \mathbb{R} \to \mathbb{C}$ absolutely continuous if $\forall \varepsilon > 0\ \exists \delta$ s.t. $\forall$ set of disjoint $\{(a_1, b_1), \dots, (a_N, b_N)\}$: $\sum_1^N (b_j - a_j) < \delta \Rightarrow \sum_1^N |F(b_j) - F(a_j)| < \varepsilon$, Clearly, F absolutely continuous $\Rightarrow F$ uniformly continuous
If F everywhere differentiable and F' bounded $\Rightarrow F$ absolutely continuous, for $|F(b_j) - F(a_j)| \leq \max |F'| (b_j - a_j)$ by mean value theorem
Prop: If $F \in NBV$, then F absolutely continuous $\Leftrightarrow \mu_F \ll m$. Corollary: If $f \in L^1(m)$, then the function $F(x) = \int_{-\infty}^x f(t)\,dt$ is in NBV and absolutely continuous, and $f = F'$ a.e.. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)\,dt$. Lemma: If F absolutely continuous on $[a,b]$, then $F \in BV[a,b]$. Fundamental Theorem of Calculus for Lebesgue Integrals: If $-\infty < a < b < \infty$, $F : [a,b] \to \mathbb{C}$ TFAE (a) F absolutely continuous on $[a,b]$ (b) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a,b], m)$ (c) F differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], m)$ and $F(x) - F(a) = \int_a^x F'(t)\,dt$.
Def: $F \in NBV$, the integral of g w.r.t. μ_F: $\int g\,dF$ (Lebesgue-Stieltjes Integrals) Theorem: If $F, G \in NBV$ and at least one is continuous then for $-\infty < a < b < \infty$ $\int_{(a,b]} F\,dG + \int_{(a,b]} G\,dF = F(b)G(b) - F(a)G(a)$. Def: ν complex, the characterization $|\nu|(E) = \sup \left\{ \sum_1^n |\nu(E)| : n \in \mathbb{N}, E_1, \dots, E_n \text{ disjoint}, E = \bigcup_1^n E_j \right\}$ total variation. Vitali-Covering Lemma: Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed bounded intervals that cover E ∴ $\forall \varepsilon > 0$, $\exists$ finite disjoint subcollection $\{I_k\}_{k=1}^n \subset \mathcal{F}$ for which $m\left(E \setminus \bigcup_{k=1}^n I_k\right) < \varepsilon$

Functional Analysis: Normed Vector Spaces: $K = (\mathbb{R} \vee \mathbb{C})$, $\mathcal{X}$: K-vector space. If $x \in \mathcal{X}$, Kx: one-dimensional subspace spanned by x. If $\mathcal{M}, \mathcal{N} \subset \mathcal{X}$ (subspaces) $\mathcal{M} + \mathcal{N} = \{x + y : x \in \mathcal{M}, y \in \mathcal{N}\}$. Def: A seminorm on $\mathcal{X}$ is a function $x \longmapsto \|x\|$ from $\mathcal{X} \to [0, \infty)$ (i) s.t. $\|x + y\| \leq \|x\| + \|y\|\ \forall x, y \in \mathcal{X}$ (triangle Ineq.) (ii) $\|\lambda x\| = |\lambda| \|x\|\ \forall x \in \mathcal{X}, \lambda \in K$. A seminorm s.t. $\|x\| = 0$ only when $x = 0$ is a norm. If $\mathcal{X}$ is a normed vector space, $\rho(x, y) = \|x - y\|$ is a metric on $\mathcal{X}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathcal{X}$ are equivalent norms if $\exists C_1, C_2 > 0$ s.t. $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ $(x \in \mathcal{X})$, Equivalent norms induce equivalent metrics and same Cauchy Sequences. Def: A normed vector space that is complete w.r.t. norm metric is a Banach Space. Def: $\{x_n\}$ sequence in $\mathcal{X}$, $\sum_1^\infty x_n$ converges to x if $\sum_1^N x_n \to x$ as $N \to \infty$ and is absolutely convergent if $\sum_1^\infty \|x_n\| < \infty$ Theorem: $(\mathcal{X}, \|\cdot\|)$ complete $\Leftrightarrow$ every absolutely convergent series in $\mathcal{X}$ converges
Ex: if $(X, \mathcal{M}, \mu)$ is a measure space, $(L_1(\mu), \|f\|_1 = \int |f|\,d\mu)$ is Banach. Def: $(\mathcal{X}, \|\cdot\|_x), (\mathcal{Y}, \|\cdot\|_y)$, $\mathcal{X} \times \mathcal{Y}$-normed with $\|(x,y)\| = \max(\|x\|, \|y\|)$
Def: If $\mathcal{M}$ (vector subspace) $\subset \mathcal{X}$, it defines an equivalence relation on $\mathcal{X}$: $x \sim y \Leftrightarrow x - y \in \mathcal{M}$. The equivalence class of $x \in \mathcal{X}$ is denoted by $x + \mathcal{M}$, the set of equivalence classes (quotient space) is denoted by $\mathcal{X}/\mathcal{M}$. $(\mathcal{X}/\mathcal{M})$ is a vector space with vector operations $(x + \mathcal{M}) + (y + \mathcal{M}) = (x + y) + \mathcal{M}$ and $\lambda(x + \mathcal{M}) = (\lambda x) + \mathcal{M}$. If $\mathcal{X}$ normed, $\mathcal{M}$ (closed), $\mathcal{X}/\mathcal{M}$ inherits the quotient norm $\|x + \mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x + y\|$ Def: A linear map $T : \mathcal{X} \to \mathcal{Y}$ ($\mathcal{X}, \mathcal{Y}$ normed) is bounded if $\exists C \geq 0$ s.t. $\|Tx\|_y \leq C \|x\|_x\ \forall x \in \mathcal{X}$ (T bounded on bounded sets of $\mathcal{X}$) Prop: $\mathcal{X}, \mathcal{Y}$ normed, $T : \mathcal{X} \to \mathcal{Y}$ linear: TFAE (a) T continuous (b) T continuous$|_0$ (c) T bounded Def: $\mathcal{X}, \mathcal{Y}$ normed, $\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \{$bounded linear maps $T : \mathcal{X} \to \mathcal{Y}\}$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ vector space and $T \longrightarrow \|T\| \triangleq \sup\{\|Tx\| : \|x\| = 1\} = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} = \inf\{C : \|Tx\| \leq C\|x\|\}$ a norm on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\|T\|$: operator norm. Prop: If $\mathcal{Y}$ complete (Banach) so is $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. Lemma: $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$, $\|Tx\| \leq C\|x\|\ \forall x \in \mathcal{X}$ i.e. $\|Tx\| \leq \|T\| \|x\|$; $\|STx\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|$ ∴ $ST \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $\|ST\| \leq \|S\| \|T\|$. Def: $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, T invertible (or isomorphism) if T bijective and T^{-1} bounded (i.e. $\|Tx\| \geq C'\|x\|$ for some $C' > 0$). T isometry if $\|Tx\| = \|x\|\ \forall x \in \mathcal{X}$. An isometry is injective but not necessarily surjective but is an isomorphism onto its range. Linear Functionals: $\mathcal{X}$ K-vector space, $K = \mathbb{R} \vee \mathbb{C}$, $T : \mathcal{X} \to K$ linear functional, If $\mathcal{X}$ normed, $\mathcal{L}(\mathcal{X}, K)$: dual space $(\mathcal{X}^*)$ ∴ $\mathcal{X}^*$ Banach with operator norm Prop: Let $\mathcal{X}$ vector space over $\mathbb{C}$. If

$\operatorname{sgn}(z) = \begin{cases} \frac{\bar{z}}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ (complex), $\operatorname{sgn}(z) \in T = \{z \in \mathbb{C} : |z| = 1\}$, $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ | Normed v.s. (Complete) if $\forall \{f_n\} \subset X$, $(\|f_n - f_m\| \xrightarrow{n,m \to \infty} 0) \Longrightarrow \exists f \in X$ s.t. $\|f_n - f\| \to 0$

f linear functional on X, $u = Re(f)$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ $\forall x \in X$. Conversely, if u real functional on X and $f : X \to \mathbb{C}$ defined by $f(x) = u(x) - iu(ix)$, then f complex linear. If X normed: $\|u\| = \|f\|$. Def: X real vector space, a sublinear functional on X is a map $p : X \to \mathbb{R}$ s.t. $p(x+y) \leq p(x) + p(y)$, $p(\lambda x) = \lambda p(x)$ $\forall x, y \in X, \lambda \geq 0$. Hahn-Banach Theorem: X real vector s. p sublinear f. on X, $M \subset X$, f linear functional on M s.t. $f(x) \leq p(x)$ $\forall x \in M$ $\therefore \exists$ linear functional F on X s.t. $F(x) \leq p(x)$ $\forall x \in X$ and $F|_M = f$. Complex Hahn-Banach Theorem: X complex v.s., p seminorm, $M \subset X$, f complex linear f. on M s.t. $|f(x)| \leq p(x)$ $\forall x \in M$ $\therefore \exists$ complex linear f. F on X s.t. $|F(x)| \leq p(x)$ $\forall x \in X$, $F|_M = f$. Theorem: X normed v.s. (a) If M (closed) $\subset X$, $x \in X \setminus M$, $\exists f \in X^*$ s.t. $f(x) \neq 0$ and $f|_M = 0$. In fact, if $\delta = \inf_{y \in M} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $\|f\| = 1$ (b) if $x \neq 0 \in X$ $\exists f \in X^*$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$ (c) The bounded linear functionals on X separate points (d) If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**} (dual of X^*) Def: $\hat{X} \triangleq \{\hat{x} : x \in X\}$. Since X^{**} always complete, the closure $\overline{\hat{X}}$ of $\hat{X}$ in X^{**} is Banach and $x \mapsto \hat{x}$ embeds X into $\overline{\hat{X}}$ as a dense subspace. Baire Category Theorem: Let X be a complete metric space (a) If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X (b) X is not a countable union of nowhere dense sets. Remark: Baire theorem is topological $\therefore$ If $X \cong$ Complete metric space, the theorem applies. Def: $E \subset X$ "of the first Category" (meager) if $E = \bigcup_1^\infty$(nowhere dense sets), otherwise E "of the second category". $\therefore$ $\forall$ complete metric space is second category. If X, Y topological spaces, a map $f : X \to Y$ is open if $f(U)$ open in Y whenever U open in X $\Longleftrightarrow$ $B_x \subset X \Rightarrow f(B) \supset B_{f(x)}$. If X, Y normed, f linear, f commutes with dilations and translations $\therefore$ f open $\Longleftrightarrow f(B)$ contains a ball centered$|_0$ in Y when $B = B(1,0) \subset X$. Open Mapping Theorem: X, Y Banach. If $T \in \mathcal{L}(X,Y)$ surjective $\Longrightarrow T$ is open. Corollary: If X, Y Banach, $T \in \mathcal{L}(X,Y)$ bijective, then T is an isomorphism i.e. $T^{-1} \in \mathcal{L}(Y, X)$. Def: X, Y normed, T linear, $T : X \to Y$, graph of T: $\Gamma(T) = \{(x,y) \in X \times Y : y = Tx\} \subset X \times Y$. We say T is closed if $\Gamma(T)$ is a closed subspace of $X \times Y$. If T continuous $\Longrightarrow T$ closed; If X, Y complete $\wedge$ T closed $\Longrightarrow T$ continuous. Closed Graph theorem: If X, Y Banach, $T : X \to Y$ (closed linear) $\Longrightarrow T$ bounded. Continuity of a linear map $T : X \to Y$ means if $x_n \to x$ then $Tx_n \to Tx$, whereas closedness means that if $x_n \to x$ and $Tx_n = y \Longrightarrow y = Tx$. Uniform Boundedness Theorem: Suppose X, Y normed v.s. $A \subset \mathcal{L}(X, Y)$ a) If $\sup_{T \in A} \|Tx\| < \infty$ $\forall x$ in some nonmeager subset of $X \Longrightarrow \sup_{T \in A} \|T\| < \infty$ b) If X Banach and $\sup_{T \in A} \|Tx\| < \infty$ $\forall x \in X \Longrightarrow \sup_{T \in A} \|T\| < \infty$

Lemma: If X, Y normed v.s., $\{T_n\} \subset \mathcal{L}(X, Y)$, $T_n \to T$. If $x_n \to x$ in $X \Longrightarrow T_n x_n \to Tx$. Lemma: A linear function $f \in X^* \Longleftrightarrow f^{-1}(\{0\})$ closed. Vitali's Covering: $A \subset \mathbb{R}^n$. A collection $\mathcal{F}$ of balls is Vitali Cover of A if $\forall x \in A, \delta > 0$ $\exists B(a,r) \in \mathcal{F}$ s.t. $r < \delta$, $x \in B$. Covering Theorem: $A \subseteq \mathbb{R}^n$, $\mathcal{F}$ vitali of A $\therefore \exists \mathcal{G} \subseteq \mathcal{F}$ of disjoint balls s.t. $m(A \setminus \bigcup \mathcal{G}) = 0$. Point Set Topology: A topology on $X \neq \emptyset$ is a family $\{\mathcal{T}\}$ of subsets of X s.t. $\emptyset, X \in \{\mathcal{T}\}$ if $\{U_\alpha\}_{\alpha \in A} \subset \{\mathcal{T}\} \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ and if $U_1, \ldots, U_n \in \mathcal{T} \Rightarrow \bigcap_1^n U_j \in \mathcal{T}$ $\therefore (X, \mathcal{T})$ topological space. Def: $(X, \mathcal{T})$, $Y \subset X$ $\therefore \mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y (relative topology induced by $\mathcal{T}$). Def: If $A \subset X$, the union of all open sets contained in A are the interior of A (A°), the $\bigcap$ of all closed sets containing A: closure of A ($\bar{A}$), $(A^\circ)^c = \overline{A^c}$, $(\bar{A})^c = (A^c)^\circ$, $\bar{A} \setminus A^\circ = \bar{A} \cap \overline{A^c}$ boundary of A (∂A). If $\bar{A} = X$ dense, if $(\bar{A})^\circ = \emptyset$ nowhere dense. If $x \in X$, $A \subset X$ a neighborhood if $x \in A^\circ$. $x \in X$ accumulation point of A if $A \cap (U \setminus \{0\}) \neq \emptyset$ $\forall$ neighborhood U of x. Prop: If $A \subset X$, $\bar{A} = A \cup acc(A)$; A closed $\Longleftrightarrow acc(A) \subset A$. Def: $(X, \mathcal{T})$. a neighborhood base for $\mathcal{T}|_{x \in X}$ is a family $\mathcal{N} \subset \mathcal{T}$ s.t. $x \in V$ $\forall V \in \mathcal{N}$; If $U \in \mathcal{T} \wedge x \in U$, $\exists V \in \mathcal{N}$ s.t. $x \in V \wedge V \subset U$. A base for $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ that contains a neighborhood base for $\mathcal{T}$ $\forall x \in X$. Prop: $(X, \mathcal{T})$ $\mathcal{E} \subset \mathcal{T}$, then $\mathcal{E}$ a base for $\mathcal{T} \Longleftrightarrow \forall \emptyset \neq U \in \mathcal{T}$ is a $\bigcup$ of members of $\mathcal{E}$. Continuous Maps: X, Y topological spaces, $f : X \to Y$ $\therefore f$ continuous if $f^{-1}(V)$ open in X $\forall$ open $V \subset Y$ f continuous$|_x$ if $\forall V \ni f(x)$ neighborhood $\exists U \ni x$ s.t. $f(U) \subset V$. Def: If $f : X \to Y$ bijective and $f \wedge f^{-1}$ continuous, f homeomorphism $X \cong Y$ (homeomorphic). If $f : X \to Y$ injective $\wedge \neg$ surjective, and $f : X \to f(X) \subset Y$ homeomorphism, f embedding. Def: X any set, $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ (Y_α topological spaces), $\exists!$ weakest topology $\mathcal{T}$ on X s.t. f_α continuous $\forall \alpha$. (weak topology gen. by $\{f_\alpha\}_{\alpha \in A}$ i.e. $\mathcal{T}$: topology gen. by sets of form $f_\alpha^{-1}(U_\alpha)$ where $\alpha \in A$, U_α open in Y_α Topological Vector Spaces: X normed vector space, The weak topology gen. by X^* is the weak topology on X. $\therefore$ If $x_n \xrightarrow{\text{weakly}} x \in X$ if $f(x_n) \to (x)$ $\forall f \in X^*$. The weak topology on X^* is the topology gen. by X^{**} (weak* topology) on X^* which is the topology of pointwise convergence: $f_\alpha \to f \Longleftrightarrow f_\alpha(x) \to f(x)$ $\forall x \in X$.

$ab \le \frac{1}{2}(a^2+b^2)\ \forall a,b \ge 0$ $T \in \mathcal{L}(X,Y)$ $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}$

The two coincide when $X = X^{**}$. $T_\alpha \to T$ strongly $\iff T_\alpha x \xrightarrow{s} Tx$ in norm topology of $Y\ \forall x \in X$, whereas $T_\alpha \xrightarrow{w} T$ weakly in the weak topology of $Y\ \forall x \in X$. Prop: Suppose $\{T_n\}_1^\infty \subset \mathcal{L}(X,Y)$, $\sup_n \|T_n\| < \infty$, $T \in \mathcal{L}(X,Y)$. If $\|T_n x - Tx\| \to 0$ $\forall x$ in a dense $D \subset X$, then $T_n \to T$ strongly. Alaoglu's Theorem: If X normed-vector-space, the closed unit ball $B^* = \{f \in X^* : \|f\| \le 1\}$ in X^* is compact in the weak* topology. Hilbert Spaces: Let $\mathcal{H}$ complex-vector-space. An inner product on $\mathcal{H}$ is a map $(x,y) \to \langle x,y\rangle$, from $\mathcal{H}\times\mathcal{H} \to \mathbb{C}$ s.t. (i) $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle\ \forall x,y,z \in \mathcal{H}, a,b \in \mathbb{C}$ (ii) $\langle y,x\rangle = \overline{\langle x,y\rangle}\ x,y \in \mathcal{H}$ (iii) $\langle x,x\rangle \in (0,\infty)\ \forall x \ne 0 \in \mathcal{H}$. (i)$\wedge$(ii) $\Rightarrow \langle x, ay+bz\rangle = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z\rangle$ Def: A c.v.s. with inner product, is called pre-Hilbert. If $\mathcal{H}$ is pre-Hilbert, for $x \in \mathcal{H}$, $\|x\| \triangleq \sqrt{\langle x,x\rangle}$. Schwartz Inequality: $|\langle x,y\rangle| \le \|x\|\|y\|$ $\forall x,y \in \mathcal{H}$ and $|\langle x,y\rangle| = \|x\|\|y\| \iff x,y$ linearly dependent. Prop: $x \mapsto \|x\| = \sqrt{\langle x,x\rangle}$ $\mathcal{H}$ norm. $\langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) + \|y\|^2$

Def: If $\mathcal{H}$ complete w.r.t. $\|x\| = \sqrt{\langle x,x\rangle}$: $\mathcal{H}$ Hilbert Space. Prop: $x_n \to x, y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x,y\rangle$. Parallelogram Law: $\forall x,y \in \mathcal{H}$: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. Def: $x,y \in \mathcal{H}$, x orthogonal to y $(x \perp y)$ if $\langle x,y\rangle = 0$. If $E \subset \mathcal{H}$, define $E^\perp = \{x \in \mathcal{H} : \langle x,y\rangle = 0\ \forall y \in E\}$

Pythagorean Theorem: $x_1,\dots,x_n \in \mathcal{H}$, $\overset{j \ne k}{x_j \perp x_k}$, $\|\sum_1^n x_j\|^2 = \sum_1^n \|x_j\|^2$. Theorem: M closed subspace of $\mathcal{H}$, then $\mathcal{H} = M \oplus M^\perp$: $\forall x \in \mathcal{H}\ \exists! y,z$, $y \in M, z \in M^\perp$, $\overset{(x-y)}{x = y+z}$ and $y = \inf_{m \in M}\|x-m\|$, $z = \inf_{m \in M^\perp}\|x-m\|$. Rmk: If $y \in \mathcal{H}$, $f_y(x) = \langle x,y\rangle$ bounded linear functional s.t. $\|f_y\| = \|y\|$ $\therefore$ $y \mapsto f_y$ an isometry of $\mathcal{H}$ into $\mathcal{H}^*$. Riesz-Fréchet Representation Thm: $f \in \mathcal{H}^*$, $\exists! y \in \mathcal{H}$ s.t. $f(x) = \langle x,y\rangle\ \forall x \in X$, $\|f\| = \|y\|$. $\therefore \mathcal{H} \cong \mathcal{H}^{**}$

If X Banach, $x \to \hat{x}$ not surjective i.e. $X \le X^{**}$. Def: $\{u_\alpha\}_{\alpha \in A} \subset \mathcal{H}$ orthonormal if $\|u_\alpha\| = 1$, $\overset{\alpha \ne \beta}{u_\alpha \perp u_\beta}$. Gram-Schmidt process: $\{x_n\}_1^\infty \subset \mathcal{H}$ lin. independent, Gram Schmidt converts $\{x_n\}$ into an orthonormal sequence s.t. $\mathrm{span}(\{x_n\}_1^N) = \mathrm{span}(\{u_n\}_1^N)\ \forall N$. (i) $u_1 := \frac{x_1}{\|x_1\|}$ (ii) Having defined $u_1,\dots,u_{N-1}$, set $v_N = x_N - \sum_1^{N-1}\langle x_N, u_n\rangle u_n$. $v_N \ne 0$ since $x_N \notin \mathrm{span}(x_1,\dots,x_{N-1})$, and $\langle v_N, u_m\rangle = \langle x_N, u_m\rangle - \langle x_N, u_m\rangle\ \forall m < N$ $\therefore$ take $u_N := \frac{v_N}{\|v_N\|}$. Bessel's Inequality: $\{u_\alpha\}_{\alpha \in A}$ orthonormal in $\mathcal{H}$, $\forall x \in \mathcal{H}$ $\sum_{\alpha \in A}|\langle x, u_\alpha\rangle|^2 \le \|x\|^2$. In particular $\{\alpha : \langle x, u_\alpha\rangle \ne 0\}$ is countable.

Theorem: If $\{u_\alpha\}_{\alpha \in A}$ orthonormal in $\mathcal{H}$: TFAE (a) (Completeness) If $\langle x, u_\alpha\rangle = 0\ \forall \alpha \Rightarrow x = 0$ (b) Parseval's Identity $\|x\|^2 = \sum_{\alpha \in A}|\langle x, u_\alpha\rangle|^2\ \forall x \in \mathcal{H}$ (c) $\forall x \in \mathcal{H}$, $x = \sum_{\alpha \in A}\langle x, u_\alpha\rangle u_\alpha$ and $\{u_\alpha\}_{\alpha \in A}$ orthonormal basis. Prop: Every Hilbert space has an orthonormal basis. Prop: $\mathcal{H}$ separable $\iff$ it has a countable orthonormal basis, in which case every o.n. basis for $\mathcal{H}$ is countable. Def: $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_1)$, $(\mathcal{H}_2, \langle\cdot,\cdot\rangle_2)$ a unitary map $U: \mathcal{H}_1 \to \mathcal{H}_2$ invertible linear map s.t. $\langle Ux, Uy\rangle_2 = \langle x,y\rangle_1\ \forall x,y \in \mathcal{H}_1$, $y = x \Rightarrow U$ an isometry $\|Ux\|_2 = \|x\|_1$. Prop: Let $\{e_\alpha\}_{\alpha \in A}$ orthonormal for $\mathcal{H}$. Then the correspondence $x \mapsto \hat{x}$: $\hat{x}(\alpha) = \langle x, u_\alpha\rangle$ unitary map from $\mathcal{H}$ to $\ell^2(A)$. Projection on Closed Convex: A set C convex if $\forall x,y \in C, \lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$. Thm: $\mathcal{H}$ pre-hilbert, $C \subset \mathcal{H}$ convex $\therefore \forall x \in \mathcal{H}\ \exists! y \in C$ (orthogonal projection) s.t. $\rho(x,A) = \inf_{z \in A}\|x-z\| = \|x-y\|$. $y := P_C(x)$ and $\forall z \in A$ $\mathrm{Re}(\langle x-y, z-y\rangle) \le 0$. Prop: $C \subset \mathcal{H}$, $\forall x_1, x_2 \in \mathcal{H}$ $\|P_C(x) - P_C(y)\| \le \|x-y\|$. Def: $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$. $x_n \xrightarrow{w} x$ (weakly) iff $\forall y \in \mathcal{H}$ $\lim_{n\to\infty}\langle y, x_n\rangle = \langle y,x\rangle$ $\|x\| \le \liminf \|x_n\|$. Lax-Milgram - Stampachia Theorems: Def: Let $a(\cdot,\cdot): \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ a bilinear form (i) Continuous if $\exists C > 0$ s.t. $|a(x,y)| \le C\|x\|\|y\|\ \forall x,y \in \mathcal{H}$ (ii) α-elliptic (coercive) if $\exists \alpha > 0$ s.t. $a(x,x) \ge \alpha\|x\|^2\ \forall x \in \mathcal{H}$. Thm (Lax-Milgram): $a(\cdot,\cdot): \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ continuous and coercive. $\forall \ell \in \mathcal{H}^*\ \exists! x \in \mathcal{H}$ s.t. $a(x,y) = \ell(y)\ \forall y \in \mathcal{H}$. Moreover if $a(\cdot,\cdot)$ symmetric, then $\frac{1}{2}a(x,x) - \ell(x) = \min_{y \in \mathcal{H}}\left(\frac{1}{2}a(y,y) - \ell(y)\right)$. Stampacchia: Let C closed convex. $\because \forall \ell \in \mathcal{H}^*\ \exists! x \in C$ s.t. $a(x, y-x) \ge \ell(y-x)\ \forall y \in C$, Moreover if $a(\cdot,\cdot)$ symmetric, $\frac{1}{2}a(x,x) - \ell(x) = \min_{y \in C}\left(\frac{1}{2}a(x,y) - \ell(y)\right)$ applies to

$$\begin{cases} -\Delta u(x) = -\sum_1^d \frac{\partial^2 u(x)}{\partial x_i^2} = f(x) \in L^2(\Omega), \forall x \in \Omega \subset \mathbb{R}^d \\ u(x) = 0\ \forall x \in \partial\Omega \end{cases}$$

equivalent to the variational formulation: $a(u,v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = \dots = \int_\Omega f(x)v(x)\,dx$ $\forall v \in E := H_0^1(\Omega)$ = closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ i.e. functions $u \in L^2(\Omega)$ with weak derivatives $\partial_{x_i}u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$. i.e. : $-\int \Delta u(x)v(x)\,dx = \int_\Omega f(x)v(x)\,dx$ $v \in H_0^1(\Omega)$ IBP: $\int_\Omega -\Delta u \cdot v\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}\frac{\partial u}{\partial n}v\,dS \Rightarrow \int_\Omega -\Delta u(x)v(x)\,dx = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx \Rightarrow \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx\ \forall v \in H_0^1(\Omega)$ $\therefore$ define $a(u,v) \triangleq \int_\Omega \nabla u\cdot\nabla v\,dx$ $\ell(v) \triangleq \int_\Omega fv\,dx$ $\therefore$ We want to find $u \in H_0^1(\Omega)$ s.t. $a(u,v) = \ell(v)\ \forall v \in H_0^1(\Omega)$. Sobolev Spaces: Def: $\Omega \subset \mathbb{R}^d$ Sobolev S. (1°) $H^1(\Omega) = \{v \in L^2(\Omega), \partial_{x_i}v \in L^2(\Omega), 1 \le i \le d\}$, $\langle u,v\rangle_{1,\Omega} = \int_\Omega (uv + \sum_1^d \partial_{x_i}u\,\partial_{x_i}v)\,dx$ $\therefore \|v\|_{1,\Omega} = \left(\int_\Omega |v|^2dx + \int|\nabla v|^2dx\right)^{1/2}$. Def: $m \in \mathbb{N}$, $v \in L^2(\Omega)$ in $H^m(\Omega)$ if all derivatives of v up to order $m \in L^2(\Omega)$, $H^0(\Omega) = L^2(\Omega)$. Thm: $H^m(\Omega)$, $m \ge 0$ endowed with the inner product $\langle u,v\rangle_{m,\Omega}$ is Hilbert: $\langle u,v\rangle_{m,\Omega} = \sum_{|\alpha|\le m}\int_\Omega \partial^\alpha u(x)\overline{\partial^\alpha v(x)}\,dx \hookrightarrow \|u\|_{m,\Omega} = \sqrt{\sum_{|\alpha|\le m}\|\partial^\alpha u\|^2_{L^2(\Omega)}}$. Def: $\overset{1 \le p \le \infty}{m \ge 1}$ $W^{m,p} \triangleq \{u \in L^p(\Omega), \partial^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^d, |\alpha| \le m\}$ with norm $\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|\le m}\int_\Omega |\partial^\alpha u|^p dx\right)^{1/p}$ Banach. L^p Spaces (Banach): $(X, \mathcal{M}, \mu)$, f measurable on X, $0 < p < \infty$, $\|f\|_p \triangleq \left[\int|f|^p d\mu\right]^{1/p}$ $\therefore L^p(X,\mathcal{M},\mu) = \{f: X \to \mathbb{C} : f \text{ measurable and } \|f\|_p < \infty\}$, $\ell^p(A) := L^p(\mu)$ (μ: counting measure). L^p vector space i.e. $f,g \in L^p$, $|f+g|^p \le [2\max(|f|,|g|)]^p \le 2^p(|f|^p + |g|^p)$ Lemma: $a \ge 0$, $b \ge 0$, $0 < \lambda < 1$ $\therefore a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b$. Hölder's Inequality: $1 < p < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ i.e. $q = \frac{p}{(p-1)}$. If f,g measurable on $X \Rightarrow \|fg\|_1 \le \|f\|_p\|g\|_q$. In particular if $f \in L^p$, $g \in L^q \Rightarrow fg \in L^1$ and $\|fg\|_1 = \|f\|_p\|g\|_q \iff \alpha|f|^p = \beta|g|^q$ a.e. for some α,β with $\alpha\beta \ne 0$ Minkowski's Inequality: If $1 \le p < \infty$, $f,g \in L^p$ then

$\|f+g\|_p \le \|f\|_p + \|g\|_p$ ∴ For $p\ge 1$ $\mathcal{L}^p$ normed v.s. Thm: $1\le p<\infty$, $\mathcal{L}^p$ Banach. Prop: ($1\le p<\infty$) The set of simple functions $\{f=\sum_1^n a_j\chi_{E_j}, \mu(E_j)<\infty\ \forall j\}$ dense in $\mathcal{L}^p$

Def: $\|f\|_\infty \triangleq \inf\{a\ge 0: \mu(\{x:|f(x)|>a\})=0\}$, $\inf\emptyset=\infty$ ∴ $\{x:|f(x)|>a\}=\bigcup_1^\infty\{x:|f(x)|>a+\frac{1}{n}\}$ ∴ $\mathcal{L}^\infty(X,\mathcal{M},\mu)=\{f:X\to\mathbb{C}, f \text{ measurable} \wedge \|f\|_\infty<\infty\}$

Thm: (a) f,g measurable on X, then $\|fg\|_1\le\|f\|_1\|g\|_\infty$. If $f\in\mathcal{L}^1, g\in\mathcal{L}^\infty$, $\|fg\|_1=\|f\|_1\|g\|_\infty \iff |g(x)|=\|g\|_\infty$ a.e. on $\{x: f(x)\ne 0\}$. (b) $\|\cdot\|_\infty$ a norm on $\mathcal{L}^\infty$
(c) $\|f_n-f\|_\infty\to 0 \iff \exists E\in\mathcal{M}$ s.t. $\mu(E^c)=0$ and $f_n \rightrightarrows f$ on E (d) $\mathcal{L}^\infty$ Banach (e) simple functions dense in $\mathcal{L}^\infty$.

Prop: $0<p<q<r\le\infty$ then $\mathcal{L}^q\subset\mathcal{L}^p+\mathcal{L}^r$ i.e. $f\in\mathcal{L}^q$, $f=f_p+f_r$, $f_p\in\mathcal{L}^p$, $f_r\in\mathcal{L}^r$. Prop: $0<p<q<r\le\infty$ then $\mathcal{L}^p\cap\mathcal{L}^r\subset\mathcal{L}^q$ and $\|f\|_q\le\|f\|_p^\lambda\|f\|_r^{1-\lambda}$, ($\lambda\in(0,1)$) s.t. $\frac{1}{q}=\frac{\lambda}{p}+\frac{(1-\lambda)}{r}$
$\lambda=\frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$ Prop: If A any set, $0<p<q\le\infty$ then $\ell^p(A)\subset\ell^q(A)$ and $\|f\|_q\le\|f\|_p$ Prop: $\mu(X)<\infty$, $0<p<q\le\infty$, then $\mathcal{L}^p(\mu)\supset\mathcal{L}^q(\mu)$ and $\|f\|_p\le\|f\|_q\,\mu(X)^{(\frac{1}{p})-(\frac{1}{q})}$

Dual of $\mathcal{L}^p$: $(\mathcal{L}^p)^*$: $\frac{1}{p}+\frac{1}{q}=1$. $g\in\mathcal{L}^q$ defines a bounded linear functional $\phi_g(f)=\int fg$ on $\mathcal{L}^p$ s.t. $\|\phi_g\|\le\|g\|_q$. The map $g\longmapsto\phi_g$ is almost always isometry from $\mathcal{L}^q$ into $(\mathcal{L}^p)^*$. Prop: $\frac{1}{p}+\frac{1}{q}=1$, $1\le q<\infty$. If $g\in\mathcal{L}^q \Rightarrow \|g\|_q=\|\phi_g\|=\sup\{|\int fg|:\|f\|_p=1\}$ Thm: $\frac{1}{p}+\frac{1}{q}=1$
g measurable on X s.t. $fg\in\mathcal{L}^1$ $\forall f\in\Sigma$ = {simple functions that vanish outside a set of finite measure}, $M_q(g)=\sup\{|\int fg|: f\in\Sigma \text{ and } \|f\|_p=1\}$
finite. Also, suppose $S_g=\{x:g(x)\ne 0\}$ σ-finite or μ semifinite $\Rightarrow g\in\mathcal{L}^q$ and $M_q(g)=\|g\|_q$ Thm: $\frac{1}{p}+\frac{1}{q}$. If $1<p<\infty$ $\forall\phi\in(\mathcal{L}^p)^*\ \exists g\in\mathcal{L}^q$ s.t. $\phi(f)=\int fg\ \forall g\in\mathcal{L}^p$ ∴ $\mathcal{L}^q$ isometrically-isomorphic to $(\mathcal{L}^p)^*$. Corollary: If $1<p<\infty$, $\mathcal{L}^p$ reflexive. Chebyshev's Inequality: $f\in\mathcal{L}^p$ $(0<p<\infty)$ ∴ $\forall\alpha>0$
$\mu\{x:|f(x)|>\alpha\}\le\left[\frac{\|f\|_p}{\alpha}\right]^p$. Minkowski's Inequality: $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$, f: $(\mathcal{M}\otimes\mathcal{N})$-measurable on $X\times Y$ (a) If $f\ge 0$, $1\le p<\infty$
$\Rightarrow \left[\int\left(\int f(x,y)\,d\nu(y)\right)^p d\mu(x)\right]^{1/p}\le\int\left[\int f(x,y)^p\,d\mu(x)\right]^{1/p}d\nu(y)$. Convolutions: f,g measurable on $\mathbb{R}^n$. Def: Convolution of f and g: $f*g(x)\triangleq\int f(x-y)g(y)\,dy$
Prop: a) $f*g=g*f$ (b) $(f*g)*h=f*(g*h)$ (c) For $z\in\mathbb{R}^n$, $\tau_z f(x)\triangleq f(x-z)$, $\tau_z(f*g)=(\tau_z f)*g=f*(\tau_z g)$ Young's Inequality: $f\in\mathcal{L}^1$, $g\in\mathcal{L}^p$ $(1\le p\le\infty)$
$f*g(x)$ exists a.e. x; $f*g\in\mathcal{L}^p$, and $\|f*g\|_p\le\|f\|_1\|g\|_p$. Elements of Fourier Analysis: U open in $\mathbb{R}^n$, $C^k(U)=\{f:$ partial derivatives of order $\le k$ exist and are continuous$\}$, $C^\infty(U)=\bigcap_1^\infty C^k(U)$, $\forall E\subset\mathbb{R}^n$, $C_c^\infty(E)=\{f\in C^\infty$ with support $K\subset E\}$. If $x,y\in\mathbb{R}^n$
$x\cdot y\triangleq\sum_1^n x_jy_j$, $|x|=\sqrt{x\cdot x}$. Thm: If ϕ measurable on $\mathbb{R}^n$ s.t. $\phi(x+y)=\phi(x)\phi(y)$, $|\phi|=1$ $\exists\xi\in\mathbb{R}^n$ s.t. $\phi(x)=e^{2\pi i\xi\cdot x}$. Def: Fourier Transform
of $f\in\mathcal{L}^1(\mathbb{R}^n)$ by $\mathcal{F}f(\xi)=\hat f(\xi)=\int_{\mathbb{R}^n}f(x)e^{-2\pi i\xi\cdot x}dx$, $\|\hat f\|_u\le\|f\|_1$, $\mathcal{F}:\mathcal{L}^1(\mathbb{R}^n)\to BC(\mathbb{R}^n)$ Thm: $f,g\in\mathcal{L}^1(\mathbb{R}^n)$, (a) $(\tau_y f)^\wedge(\xi)=e^{-2\pi i\xi\cdot y}\hat f(\xi)$ and $\tau_\eta(\hat f)=\hat h$ where $h(x)=e^{2\pi i\eta\cdot x}f(x)$ (b) $(f*g)^\wedge=\hat f\hat g$ (c) Rieman-Lebesgue Lemma: $\mathcal{F}(\mathcal{L}^1(\mathbb{R}^n))\subset C_0(\mathbb{R}^n)$. Lemma:
If $f,g\in\mathcal{L}^1$ then $\int\hat f g=\int f\hat g$.

Geometric Hahn-Banach: Def: Let X n.v.s. an affine hyperplane is a subset $H\subset X$, $H=\{x\in X; f(x)=\alpha\}$
f linear functional Def: $A,B\subset X$. $H=[f=\alpha]$ separates A,B if $f(x)\le\alpha\ \forall x\in A$, $f(x)\ge\alpha\ \forall x\in B$. strictly separates If $\exists\varepsilon>0$ s.t. $\underset{x\in A}{f(x)\le\alpha-\varepsilon}$, $\underset{x\in B}{f(x)\ge\alpha+\varepsilon}$
$C\subset X$ convex if $tx+(1-t)y\in C\ \forall x,y\in A, \forall t\in[0,1]$ Theorem: (Geometric Hahn-Banach): $C_1,C_2\subset X$ convex s.t. $C_1\cap C_2=\emptyset$. Assume C_1 or C_2 closed
∴ ∃ closed hyperplane separating C_1,C_2. Lemma: $C\subset X$ convex s.t. $0\in C$. $\forall x\in X$ set $p(x)=\inf\{\alpha>0:\frac{x}{\alpha}\in C\}$ (Minkowski functional on C)
p satisfies $p(\lambda x)=\lambda p(x)$, $p(x+y)\le p(x)+p(y)$, $\exists M$ s.t. $0\le p(x)\le M\|x\|\ \forall x\in X$, $C=\{x\in X; p(x)<1\}$. Lemma: Let $C\subset X$ open convex, $x_0\in X$ s.t.
$x_0\notin C$ ∴ $\exists f\in X^*$ s.t. $f(x)<f(x_0)\ \forall x\in C$. (i.e.) $[f=f(x_0)]$ separates $\{x_0\}$ and C. Corollary: Let $F\subset X$ linear subspace s.t. $\bar F\ne X$.
∴ $\exists f\in X^*$ s.t. $f\not\equiv 0$, s.t. $\langle f,x\rangle=0\ \forall x\in F$.

Unbounded L.O. & Adjoint: X, Y Banach. An unbounded linear operator
$A: D(A)\subset X\to Y$ defined on $D(A)\subset X$. A bounded (continuous) if $D(A)=Y$ and $\exists c\ge 0$ s.t. $\|Ax\|\le c\|x\|\ \forall x\in X$.
$\|A\|_{\mathcal{L}(X,Y)}\triangleq\sup_{u\ne 0}\frac{\|Au\|}{\|u\|}$. Graph: $G(A)=\{[u,Au]:u\in D(A)\subset X\times Y\}$ Range: $R(A)=\{Au: u\in D(A)\}\subset Y$, Kernel: $N(A)=\{u\in D(A): Au=0\}\subset E$, A closed if $G(A)$ closed
Def: Adjoint A^*: $A: D(A)\subset X\to Y$, s.t. $\overline{D(A)}=E$. $A^*: D(A^*)\subset Y^*\to X^*$, $D(A^*)=\{v\in Y^*; \exists c\ge 0$ s.t. $|\langle v,Au\rangle|\le c\|u\|\ \forall u\in D(A)\}$. $D(A^*)$ linear
subspace of Y^*. Given $v\in D(A^*)$, consider $g: D(A)\to\mathbb{R}$, $g(u)=\langle v,Au\rangle\ \forall u\in D(A)$, $|g(u)|\le c\|u\|$. By Hahn-Banach ∃ linear map $f: X\to\mathbb{R}$ s.t. $|f(x)|\le c\|x\|\ \forall x\in X$
∴ $f\in E^*$. Set $A^*v=f$. $\langle v,Au\rangle_{Y^*,Y}=\langle A^*v,u\rangle_{X^*,X}$, $|\langle A^*v,u\rangle|\le\|A\|\|u\|\|v\|\Longrightarrow\|A^*v\|\le\|A\|\|v\|$ ∴ $\|A^*\|\le\|A\|$, $|\langle v,Au\rangle|\le\|A^*\|\|u\|\|v\|\Longrightarrow$
$\|A^*\|_{\mathcal{L}(Y^*,X^*)}=\|A\|_{\mathcal{L}(X,Y)}$.

Weak Topology & Convex Sets: For convex sets "weakly closed = strongly closed Thm: C convex $\subset X$. C closed in weak topology $\sigma(X,X^*)\iff$ closed in strong topology. Prop: Let $\{x_n\}\subset X$ then (i) $x_n\xrightarrow{W}x$ in $\sigma(X,X^*)\iff\langle f,x_n\rangle\to\langle f,x\rangle\ \forall f\in X^*$ (ii) If $x_n\to x$ strongly $\Longrightarrow$
$x_n\xrightarrow{W}x$ in $\sigma(X,X^*)$ (iii) $x_n\xrightarrow{W}x$ in $\sigma(X,X^*)\Longrightarrow\|x_n\|$ bounded and $\|x\|\le\liminf\|x_n\|$ (iv) $x_n\xrightarrow{W}x$ in $\sigma(X,X^*)$ and $f_n\xrightarrow{S}f$ in X^* (i.e. $\|f_n-f\|_{X^*}\to 0$)
$\Longrightarrow\langle f_n,x_n\rangle\to\langle f,x\rangle$. Prop: If $\dim(X)<+\infty$, $\sigma(X,X^*)$ = Usual Topology. Corollary: (Mazur's) $x_n\xrightarrow{W}x$, $\forall\varepsilon>0\ \exists\, 0\le\lambda_n\le 1$
$\sum_j^n\lambda_j=1$ s.t. $\|\sum_j^n\lambda_jx_j-x\|\le\varepsilon$ i.e. $x\in\overline{\{\sum^n\lambda_jx_j,\sum\lambda_j=1\}}$. Thm: The closed unit ball $B_{X^*}=\{f\in X^*:\|f\|\le 1\}$ compact in the weak* topology of $\sigma(X,X^*)$.

Reflexive Spaces: Let X Banach, $J: X\to X^{**}$ be the cannonical injection. X reflexive if J surjective i.e. $X=X^{**}$. Finite dim n.v.s. are reflexive, Hilbert Spaces are reflexive. Theorem: (Kakutani) Let X Banach,
$\Longrightarrow X$ reflexive $\iff B_X=\{x\in X:\|x\|\le 1\}$ compact in the weak topology $\sigma(X,X^*)$. If $X=X^{**}$, bounded sequences $\{x_n\}$ are
weakly compact ∴ $\exists\{x_{n_k}\}$ s.t. $x_{n_k}\xrightarrow{W}x_0$ $\forall f\in X^*$ $\langle f,x_{n_k}\rangle\longrightarrow\langle f,x_0\rangle$ Goldstine Lemma: Let X Banach ∴ $J(B_X)$ dense
in $B_{X^{**}}$ w.r.t. $\sigma(X^{**},X^*)$ ∴ $J(X)$ dense in X^{**} in the topology $\sigma(X^{**},X^*)$ Thm: X reflexive-Banach, $\{x_n\}$ bounded ∴

2.6, 3.3, 3.4, 3.5, 3.6, 3.7, S.1, S.2

$\exists (x_{n_k}) \xrightarrow{W} x_0$ in $\sigma(X, X^*)$ Thm (Eberlein–Šmulian): Assume X Banach s.t. every bounded sequence in X admits a weakly convergent subsequence (in $\sigma(X^*, X)$) $\Longrightarrow X = X^{**}$ Prop: Assume X reflexive-Banach and $M \subset X$ closed linear subspace of $X \Longrightarrow M$ is reflexive. Corollary: Banach-X is reflexive $\Longleftrightarrow E^*$ reflexive Corollary: X reflexive-Banach. $K \subset X$ bounded, closed, convex $\therefore K$ compact in $\sigma(X, X^*)$. Separable Spaces: Def: Metric Space X separable if $\exists D \subset E$ countable and dense Prop: Let E separable metric space, let $F \subset E$ any subset $\Longrightarrow F$ separable Thm: X Banach s.t. X^* separable $\Longrightarrow$ X separable. Corollary: X Banach. Then $\{X$ reflexive $\wedge$ separable$\} \Longleftrightarrow \{X^*$ reflexive and separable$\}$. Def: A Banach space is uniformly convex if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\{x, y \in X,\ \|x\| \le 1,\ \|y\| \le 1$ and $\|x - y\| > \varepsilon\} \Longrightarrow \{\|\frac{x+y}{2}\| < 1 - \delta\}$. Milman–Pettis: Every uniformly convex Banach space is reflexive. Proposition: A hilbert Space $\mathcal{H}$ is uniformly convex $\Longrightarrow \mathcal{H} = \mathcal{H}^{**}$. Projection onto convex set: Let $C \subset \mathcal{H}$ closed convex $\therefore \forall f \in \mathcal{H}\ \exists!\, u \in C$ s.t. $|f - u| = \min_{v \in C} |f - v| = \operatorname{dist}(f, C)$, Moreover $u \in C$ and $\langle f - u, v - u \rangle \le 0\ \forall v \in K$

Hilbert Sums: Let $(E_n)_{n \ge 1}$ sequence of closed subspaces of $\mathcal{H}$. $\mathcal{H} = \bigoplus_n E_n$ if (a) $\langle u, v \rangle = 0\ \forall u \in E_n\ \forall v \in E_m,\ m \ne n$ (b) The linear space spanned by $\bigcup_{n=1}^{\infty} E_n$ is dense in $\mathcal{H}$. Theorem: Assume $\mathcal{H} = \bigoplus_n E_n$. Given $u \in \mathcal{H}$ set $u_n \triangleq P_{E_n} u$ (projection) and $S_n = \sum_{k=1}^{n} u_k \Longrightarrow \lim_{n \to \infty} S_n = u$ and $\sum_{k=1}^{\infty} |u_k|^2 = |u|^2$ (Bessel–Parseval's Identity). Def: A sequence $\{e_n\}_{n \ge 1} \subset \mathcal{H}$ has an orthonormal basis if (i) $|e_n| = 1\ \forall n$ and $\langle e_n, e_m \rangle = \delta_{ij}$ (ii) linear $\overline{\operatorname{span}(\{e_n\}_n)} = \mathcal{H}$. Corollary: Let (e_n) orthonormal basis $\therefore \forall u \in \mathcal{H}: u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k$ i.e. $u = \lim_{n \to +\infty} \sum_{k=1}^{n} \langle u, e_k \rangle e_k$, $|u|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$. Thm: Every Separable Hilbert Space has an orthonormal basis Theorem: Assume a norm $\|\cdot\|$ satisfies parallelogram Law $\therefore \|\cdot\|$ is a Hilbert norm. Ex: $L^2(0, \pi)$ has orthonormal basis $\{e_n(x)\} = \sqrt{\frac{2}{\pi}} \sin nx,\ n \ge 1$ or $e_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx)\ n \ge 0$. Given $u \in L^2(0, \pi)$, Fourier Series $S_n = \sum_{k=1}^{n} \langle u, e_k \rangle e_k$, $S_n \to u$, $S_{n_k} \to u$ a.e. on $(0, \pi)$. Lemma: If F increasing on $\mathbb{R}$, $F(b) - F(a) \ge \int_a^b F'(t)\,dt$, Operator Norm: $\|T\| = \sup\{\|Tx\| : \|x\| = 1\} = \sup\{\frac{\|Tx\|}{\|x\|} : x \ne 0\} = \inf\{C : \|Tx\| \le C\|x\|\ \forall x\}$. Lemma: X, Y n.v.s. $T \in \mathcal{L}(X, Y)$, s.t. $T_n \to T$. If $x_n \to x$ in $X \Longrightarrow T_n x_n \to Tx$. Lemma: A linear function $f \in X^* \Longleftrightarrow f^{-1}(\{0\})$ is closed. Lemma: If M closed subspace of c.n.v.s. X and $x \in X \setminus M \Longrightarrow M + \mathbb{C}x$ closed. Thm: X, Y Banach, $A \in \mathcal{L}(X, Y)$ surjective. If $y_n \to y_0$ in $Y \Longrightarrow \exists C > 0$ and $x_n \to x_0$ s.t. $Ax_n = y_n$ and $\|x_n\| \le C\|y_n\|$. Lemma: X, Y Banach. If $T : X \to Y$ linear s.t. $f \circ T \in X^*\ \forall f \in Y^* \Longrightarrow T$ bounded. Lemma: $M \subset \mathcal{H}$ hilbert $\{M^\perp\}^\perp = \overline{\operatorname{span} M}$ Lemma: $(\ell^p)^* = \ell^q$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$, $C = \{$All convergent sequences $[a_n]_1^\infty\}$ with $\|[a_n]_1^\infty\| \equiv \sup_n |a_n|$, $C^* = \ell^1$. Thm: X n.v.s. $T : X \to X^{**}$ canonical map $\langle T(x), f \rangle = f(x)\ \forall f \in X^*$, $R(T)$ closed $\Longleftrightarrow X$ Banach. Lemma: $f_n \in C[a,b]$, $f_n \xrightarrow{W} f \in C[a,b] \therefore f_n(x) \to f(x)\ \forall x \in [a,b]$. Lemma: $f_n \in BV[0,1]$, $f_n \xrightarrow{W} f \Longrightarrow f_n \to f$ in L^1. Theorem: $\mathcal{H}$ hilbert. $x_n \to x$ in $\mathcal{H} \Longleftrightarrow x_n \xrightarrow{W} x$ and $\|x_n\| \to \|x\|$ Theorem: $X^{**} = X$. Let C bounded-closed-convex. $\forall f \in X^*$, f attains its maximum $\wedge$ minimum on C. Lemma: $f \in L^1 \cap L^2(\mathbb{R}^n)$, $\tau_h f = f(x - h)$, $(\delta_\lambda f) = f\left(\frac{x}{\lambda}\right) \Longrightarrow \widehat{\tau_h f}(\xi) = e^{-2\pi i} \hat{f}(\xi)$ and $\widehat{\delta_h f}(\xi) = \lambda^n \hat{f}(\lambda \xi)$. Theorem: $f \in L^2([-\pi, \pi])$, fourier coefficient: $C_n(f)$ $C_n(f) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$, $S_m(f) = \sum_{n=-m}^{n=m} C_n e^{inx} \therefore \lim_{m \to \infty} S_m(f) = f$ in L^2