Q2 Given an Adjacency matrix A
and a diagonal degree matrix D

$$A \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} \qquad D \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_m \end{pmatrix}$$

every component is connected with at least one edge hence graph Laplacian $L = D - A$

$$L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_m \end{pmatrix}$$ now solve $v^T L_i v$

$$v^T L_i v = v^T (D_i - A_i) v = 0 \quad = v^T D_i v - v^T A_i v$$
$$= \sum_{i=1}^{n} d_i v_i^2 - \sum_{ij} a_{ij} v_i v_j \quad = \frac{1}{2}\left[\sum_{i=1}^{n} d_i v_i^2 - 2 \sum_{ij} a_{ij} v_i v_j + \sum_{j=1}^{n} d_j v_j^2 \right]$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (v_i - v_j)^2 = 0$$ $\Rightarrow$ This means that $v_i = v_j \; \forall_{i \text{ and } j}$

~~General~~ In general that means, thatt this concept applies to all components:

$$v^T L v = \begin{pmatrix} v_1 \mathbb{1}_{A_1} & & \\ & v_2 \mathbb{1}_{A_2} & \\ & & v_M \mathbb{1}_{A_m} \end{pmatrix}$$

what shows, that the eigenvectors to eigenvalue zero are linear combis of $\mathbb{1}_A \ldots \mathbb{1}_{A_m}$