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Fundamental Nonconvexities in Arrovian Markets and a Coasian Solution to the Problem of Externalities*

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D. Starrett [*J. Econ. Theory* **4** (1972), 180–199] argues that the presence of externalities implies fundamental nonconvexities which cause Arrow markets to fail. While this is true, we argue this failure is due to the structure of the Arrovian markets that Starrett uses, and not to the presence of externalities as such. We provide an extension of a general equilibrium public goods model in which property rights are explicitly treated. Nonconvexities are not fundamental in this framework. We define a notion of Coasian equilibrium for this economy, and show first and second welfare theorems. In this context, the first welfare theorem is a type of Coase theorem. *Journal of Economic Literature* Classification Numbers: D62, H41.

1. INTRODUCTION

One of the basic programs in economics is to show that markets can decentralize essentially any Pareto efficient allocation given the right set of endowments. As is well known, the presence of externalities causes this type of market decentralization to fail in ordinary competitive economies. In his influential paper, Arrow [2] suggested that since this failure was due to missing externality markets, the solution was to extend the commodity space in a way that would permit such markets to exist. In particular, he proposed that each agent's observations of every other agent's consumption and production choices be treated as "artificial" commodities and included as arguments in utility and production. For example, if a particular firm's production also generates smoke, observations of the firm's

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production activity would enter netatively in the preferences or production functions of the remaining agents. Extending the market to include these artificial commodities makes it possible to treat economies with externalities as a special case of the standard Arrow–Debreu–McKenzie general equilibrium model.

Since second welfare theorems depend on separation arguments, it is critical that the feasible set be convex, and that the efficient allocations lie on its boundary. Starrett [23] argued it is far from trivial to assume that these conditions will be satisfied in an economy with externalities.[1] He provided an example which seems to show that points in the interior of the production set can be Pareto optimal in nonpathological situations. Of even greater concern is his claim that the presence of externalities necessarily implies the existence of a "fundamental nonconvexity" in the underlying Arrow markets. If this is true then there is little hope of solving the problem of externalities with the traditional Pigovian tax approach or indeed any approach based on Arrovian prices.

While agreeing with much of what Starrett says, we disagree with his dire conclusion. Starrett shows by example that interior points in the original, non-Arrovian, production set may be optimal. However, the real issue is whether they are in the interior of the production set in the extended Arrow commodity space. This is the space where equilibria must be supported, and Starrett's example does not address the question of failure in these markets. When we look specifically at the Arrow production sets, it turns out that efficient allocations will never be in the interior. Very simply, this is because Arrow's "observations" are jointly produced commodities. The joint production implies that the Arrow production set is a manifold of lower dimension than the Arrow commodity space and thus can have no interior in this space.

Starrett's concern about nonconvexity is much more troubling. He makes his case through the classic example of a laundry and a steel mill by considering the market for the Arrow commodity "observations of steel production by the laundry." This is meant to be a proxy for the external damage caused by the smoke the steel mill emits. Starrett points out that at any positive price the laundry will find it optimal to close down and sell an infinite number of rights to pollute, while the steel mill will demand only a finite number of rights. On the other hand, at a price of zero the laundry will supply no rights, while the steel mill's demand will be positive. Thus, no price equates supply and demand, and the market necessarily fails.

Starrett's conclusion that Arrow markets fail because of this fundamental nonconvexity is important, but has led to a great deal of confusion in the literature. It is not immediately clear whether the presence of externalities

[1] See also the exposition in Heller and Starrett [13] and discussion by Ledyard [19].

will always lead to nonconvexities, or if these nonconvexities should be attributed to an aspect of Starrett's model not essentially tied to the economies of externalities. Our position is the latter. We argue that it is the unboundedness of the endowments of property rights in the Arrow externality markets rather than the presence of externalities as such which drives the market failure pointed out by Starrett. While this unboundedness does seem to be fundamental to the Arrovian approach to externalities, it is not fundamental, and is in fact quite unnatural, in real economies with externalities.

The main contribution of this paper is to provide a new solution in the spirit of Coase [6]. The problem with Arrow markets is that there is no natural way to introduce the notion of a bounded set of property rights endowments. What is required is a model in which there are a certain number of rights to emit smoke,[2] and a market in which these rights are efficiently allocated between public uses (abatement of smoke which takes the form of publicly held unused rights) and production uses (in which rights are used by firms to allow the generation of smoke). We show that an extension of the Samuelson [21] public goods model with the addition of property rights provides just such a market. Given that property rights are bounded, there is no longer any fundamental nonconvexity. Moreover, both the first and second welfare theorems hold in our model. The first welfare theorem can be viewed as a Coase theorem. Once property rights are established, the equilibrium is efficient in the absence of transactions costs. The second welfare theorem addresses Starrett's concern that markets may not be able to decentralize all the efficient allocations.

The existing literature on externalities falls into two broad categories. The first is market oriented. Two recent contributions are Laffont [18] and DeSerpa [10]. In Chapter 1 of his book, Laffont discusses a model which suggests the one presented in this paper. He argues that taking externality markets directly into account may lead to Pareto efficient outcomes. He does not seem to believe, however, that these markets are any more immune from Starrett's fundamental nonconvexities than Arrow markets.[3] DeSerpa, on the other hand, specifies a simple model in which land plays an essential role. He explicitly rejects Coasian externality markets and focuses on various liability rules and systems of Pigovian taxes on existing

[2] This bound can take the form of an *ad hoc* policy choice or some natural limit imposed by the physical world (the volume of the earth's atmosphere, for example).

[3] There are several important differences between Laffont's model and the one presented in this paper. In particular, Laffont does not make a distinction between directed and public externality rights, does not discuss markets which make Lindahlian allocations of externality rights, and studies only the case in which the pollutee is endowed with all the property rights. Also see Laffont [17] for a more traditional theoretical treatment of externalities and further discussion of these issues.

output markets to achieve efficiency.[4] Other contributions in this vein include Baumol and Bradford [4], Otani and Sicilian [20], and Cooter [7].

Another approach to the Coase theorem uses game-theoretic models in the spirit of Shapley and Shubik [22]. This branch of the literature undoubtedly received encouragement from Starrett's claim that fundamental nonconvexities doomed market mechanisms to failure. Nevertheless, the results obtained with this approach are often negative. For example, Aivazian *et al.* [1] show that for games with externalities, Pareto inefficient allocations may be stable in the sense of Aumann–Mashler Bargaining sets. Bigelow also finds, in simple bimatrix games with side payments, that the absence of markets may cause inefficiency or nonexistence of Nash equilibrium. Other papers include Cooter and Marks [8], Harrison and McKee [12], and Hoffman *et al.* [14].

The plan of this paper is as follows. In Section 2 we recapitulate the Arrovian model, and show that all the efficient allocations will be on the boundary of the Arrovian production sets. In Section 3 we restate Starrett's example which shows Arrovian production sets must be nonconvex, and give a simple example of a non-Arrovian externality market which does not fail. In Section 4 we point out some other limitations of the Arrovian approach and give a general equilibrium model in which property rights and externality markets are modeled explicitly, as in Section 3's example. We show both the first and second welfare theorems. Concluding remarks, including a discussion of how the model may be extended, are contained in Section 5.

2. ARROVIAN COMMODITIES AND LOCAL PARETO SATIATION

We begin by restating the Arrow–Starrett model of an economy with externalities. Consider an economy with F firms and N private goods. The ordinary production sets of firms are subsets of $\mathfrak{R}^N$. Arrow's insight is to define a set of "artificial" commodities as follows. Suppose that firm j produces a level x_k of commodity k. We can think instead that the firm *jointly* produces observations of its production for all other firms. Formally, instead of producing x_k, the firm produces a set of F commodities denoted x_{ijk} where $x_{ijk} = x_k$ for all $i = 1, ..., F$. The Arrow commodity x_{ijk} is interpreted as firm i's observation of firm j's production of commodity k. Commodity x_{jjk} is interpreted as firm j's observation of its own production

[4] Like us, DeSerpa attributes the existence of Starrett's fundamental nonconvexities to his choice of stating the problem in the artificial Arrow commodity space.

of commodity k, which is what it sells to the consumer. The firms' production sets are now contained in $\Re^{FN}$. Thus, firm j is affected by its observations of each firm's production of each commodity, and inversely. If firm j generates smoke as a result of producing commodity k, for example, then all firms suffer production losses every time they observe the production of k by j. The entire Arrow commodity space is $\Re^{F^2N}$. It is easy to extend this model to include consumers and general consumption and production externalities by enlarging the commodity space to include every firm's and consumer's observations of every other firm's and consumer's production and consumption of every commodity. The appeal of this approach is that it shows that an economy with arbitrary production and consumption externalities is isomorphic to an ordinary economy with joint production. Thus, we can simply appeal to the existing literature for first and second welfare theorems, existence results, and so on.

One of Starrett's concerns is that Pareto efficient production plans may not lie on the boundary of the aggregate production set. To illustrate, consider an economy with two agents, Jones and Smith, and two goods, acres of lawn for Jones and acres of lawn for Smith. Denote these two goods X_J and X_S, respectively. Let the feasible allocations over these two goods be described by $X_J + X_S \leqslant 4$. Suppose each of these goods has an associated negative consumption externality: Smith envies Jones' lawn and Jones envies Smith's. This external effect might be so strong that even though two acres of lawn for each agent is a feasible allocation, they would both be better off if they had only one acre of lawn each.[5] This is despite the fact that for each agent, having a bigger lawn is always better for any fixed size of his neighbor's lawn. In Starrett's language, one acre of lawn for each is a point of Local Pareto Satiation (LPS). He concludes that an assumption that no points of LPS exist is needed in order to guarantee that all optimal plans are on the boundary of the feasible set and so can be supported by market prices.

We agree that points of LPS may exist, but argue that this does not imply that Arrow markets will fail. It is not really relevant that we cannot find a set of prices which support optimal production plans in the ordinary aggregate production set. Even if we could, they would not, in general, support efficient consumption by consumers. We know that the presence of externalities in fact *requires* that producer prices differ from consumer prices. This is the whole point of the Pigovian tax approach. What we really want are prices that support the efficient production plan in the extended Arrow commodity space. The real question is: will optimal production plans be on the boundary of the Arrow production sets even if there are points of LPS.

[5] If we want to maintain the production flavor of Starrett's paper, we can imagine that Smith and Jones are utility producers, and that we are trying to maximize the sum of utilities.

Returning to the example, if we define the extended Arrow commodity space in the usual way, two goods are added: Smith's observation of Jones' lawn and Jones' observation of Smith's lawn. Let the four commodities be denoted by X_{ij} where $i, j \in \{S, J\}$ and we interpret this as i's observation of j's lawn. The extended production set can be described by three expressions: $X_{JJ} + X_{SS} \leqslant 4$, $X_{SS} = X_{JS}$, and $X_{JJ} = X_{SJ}$. Here is the essential point: these equations describe a 2-dimensional manifold in a 4-dimensional space. In general, if there are N goods, F firms, and C consumers, the feasible set will always be an $(F+C)\,N$-dimensional manifold in an $(F+C)^2\,N$-dimensional Arrow commodity space. Since a submanifold can never have an interior in a larger dimensional space, it follows that no matter which production plan is optimal, it will necessarily be on the boundary of the extended production set. Then by the Hahn–Banach theorem, any optimal production plan can be supported by prices under the right convexity conditions.[6]

Let us briefly consider the interpretation of supporting prices in this economy when a point in the interior of the ordinary aggregate production set is optimal. For simplicity let α be the marginal external cost of observing a neighbor's lawn at an optimal (interior) production plan. We claim that $p_{ii} = \alpha$ and $p_{ij} = -\alpha$ for $i \neq j$ are supporting Arrow prices. To see this, note that in the context of a competitive Arrow market, these prices induce both agents to choose the optimal size of lawn by equating their marginal private benefits with their marginal private costs ($p_{ij} = -\alpha$). On the other hand, producing an acre lawn for agent j gives firms an additional revenue of α, but also increases costs by α since this is the price of a unit of the Arrovian input X_{ij} which must be used when producing a unit of commodity X_{jj}. Thus, marginal cost and revenue are equal at the optimal level, and net profit is zero. If instead we consider these prices as defining a Pigovian tax equilibrium, then producer price is zero, and a tax of α brings the consumer price of lawn up to the apropriate level. In both cases the profit from producing any amount of output is identically zero. Producing in the interior of the ordinary production set is a profit maximizing choice (as would be every other feasible plan). We conclude that local Pareto satiation does not affect the existence of equilibrium in Arrow markets.

3. FUNDAMENTAL NONCONVEXITIES

Starrett's claim that nonconvexities are always present when an economy is subject to externalities is of much greater concern. He gives the following example: suppose a laundry suffers a negative externality from smoke put

[6] Of course, these are prices for all the commodities, including the artificial ones.

into the air when steel is produced. The laundry's observation of steel production is then an input which affects laundry production negatively. If we graph the laundry's production *given a fixed quantity of all the other inputs* versus steel output, the frontier slopes downward. That is, more steel leads to less laundry, all else equal. But then, either the frontier must intersect the laundry axis and then continue along the axis, or become asymptotic to the axis as steel production increases. This is because the laundry can shut down, and so it is always possible to produce zero laundry with a given quantity of labor regardless of the level of steel production. In either case, the production set is nonconvex (see Fig. 1 in Starrett [23].)

This leads to a market failure. To see this, suppose the steel mill is required to buy the right to pollute from the laundry. If these rights are priced at any positive level, the laundry would choose to shut down and sell an infinite quantity of rights. This would certainly exceed the demand for rights. On the other hand, at a price of zero, the laundry would choose to supply no rights to the market, while demand would be positive.

We agree with Starrett's example; however, it does not seem to us that there is anything essential about externalities that drives this result. Suppose, for example, that one agent is endowed with an infinite quantity of any input, but that there are no externalities in the economy whatsoever. Suppose also that the first unit of the input has positive value to both the agent with the endowment, and at least one other agent. Then again at any positive price, supply is infinite, and at price zero, supply is zero and there is excess demand. Thus, there do not exist supporting prices. In other words, it is really the unboundedness of the endowment, and not the presence of externalities *per se*, which drives Starrett's market failure.

The question now becomes whether or not the presence of externalities implies a fundamental unboundedness of endowments of certain types of pollution rights which would in turn imply a nonconvexity. A little reflection shows that this is not the case. Suppose an agent owns a piece of land which is being polluted by toxic waste flowing from the factory next door. If we think about the associated externality market, it is clearly not plausible that the agent would be able to sell the right to dump an infinite quantity of toxic waste. The land has an inherent limit on the amount of waste that can be stored. It is important to emphasize that this limit is not imposed by any human agency. A stream can only hold so much effluent. An airport can generate noise pollution at most 24 hours in a given day. Even the atmosphere is finite. The most ambitious factory cannot pollute an infinite quantity of air, because only a finite amount exists to be polluted.

An example will help make this clear. Suppose that the pollution level is initially zero, and can rise to 100. We might think of this as the percentage of air that consists of pollutants. Suppose that we have two industries, laundry and steel with output levels denoted ℓ and s, respectively. Let the labor inputs for these two industries be denoted by x_ℓ, and x_s, and externality rights be denoted e_ℓ and e_s. We interpret the consumption of externality rights differently depending on the firm. When the laundry consumes rights, it is consuming clean air as an input. When the steel mill consumes rights, on the other hand, it is generating smoke.

Both goods are produced by linear technology:

$$\ell = x_\ell + e_\ell \qquad \text{and} \qquad s = x_s.$$

In addition, the production of steel generates an externality according to the following equation:

$$e_s = 0.5s.$$

Let the price of the input be normalized to one. Given the linear technology, the prices of laundry, steel, and externality rights in equilibrium must be:

$$p_\ell = 1, \qquad p_s = 1.5, \qquad p_e = 1.$$

Now, let us take initial endowments of property rights into account. Let the η_i denote the endowment of externality rights to firm i. Thus, $\eta_\ell \geqslant 0$, $\eta_s \geqslant 0$, and $\eta_\ell + \eta_s = 100$. This means that the steel mill can sell at most η_s rights to clean air if it goes out of business, and the laundry can sell at most $\eta_\ell = 100 - \eta_s$ rights to pollute. In a sense, η_s is the "benchmark" level of pollution, and rights must be traded to move away from this point. Take an arbitrary optimal program of production for these two firms: (ℓ, s). The steel mill must use up $x_s = s$ units of input and generate $e_s = 0.5s$ units of externality to produce s units of steel. The steel mill, therefore, must make a net purchase of $0.5s - \eta_s$ (which may be positive or negative) externality rights. This gives the steel mill a profit of:

$$\pi_s = 1.5s - x_s - (0.5s - \eta_s) = 1.5s - s - 0.5s + \eta_s = \eta_s.$$

Similarly, when the pollution level is $0.5s = 100 - e_\ell$, the laundry must use up $x_\ell = \ell - (100 - 0.5s)$ units of input, and make a net purchase of $100 - 0.5s - \eta_\ell$ externality rights. Thus, the laundry's profit is:

$$\pi_\ell = \ell - x_\ell - (100 - 0.5s - \eta_\ell) = \ell - (\ell - (100 - 0.5s)) - (100 - 0.5s - \eta_\ell) = \eta_\ell.$$

We discover that the profits of the two firms under these prices are positive and equal to exactly the value of the endowments regardless of the level of output.[7] It does not matter, of course, that the endowments are property rights, the same result would hold for CRS firms endowed with any type of input. Note in particular that the presence of externalities does not generate a market failure in this type of Coasian model. Going out of business and selling all of the property rights gives exactly as much profit as any other output choice.

4. A COASIAN SOLUTION TO THE PROBLEM OF EXTERNALITIES

The main conclusion to be drawn from the example above is that the presence of externalities does not necessarily imply market failure. This does not mean, however, that Starrett's arguments are in some way flawed. If we accept the Arrow model as the right abstraction of economies with externalities, then Starrett is correct that there is an unboundedness which must lead to a nonconvexity. This market failure could be resolved if we were somehow able to place a bound on the Arrow production sets. It is very hard to see, however, how such a bound could be introduced. It is not very appealing to claim that there is a limit on the capacity of firm i to observe the production by firm j of commodity k. In any event, it would be completely counter to the well-established tradition in general equilibrium theory to incorporate such bounds in the production sets themselves. We usually define production sets such that it is possible to contemplate arbitrarily high levels of input. The fact that there exists only a finite amount of labor (of possibilities to pollute in the world) does not enter into their definition. Such constraints are more appropriately treated through the endowments of agents. The Arrow model does not seem to have sufficient flexibility to allow for this. In addition, there are several other complexities in real economies with externalities which seem difficult or impossible to handle in Arrow's model. In particular:

1. There may not be a fixed relationship between production and effluent. Different ways of making paper, for example, may put more or less toxic waste into a river. It is not the observation that paper is produced,

[7] This point causes some confusion in Starrett's Proposition 4. There, he tries to show that markets always fail no matter where the benchmark level of externality is set when the technology is CRS. His proof is logically correct and depends critically on CRS firms making zero profits. The addition of benchmarks or property rights to the model, however, essentially causes a linear technology to become an affine one. The production set remains a convex cone, but the apex is no longer at zero. Changing the allocation of property rights changes the apex for each firm. Thus, Proposition 4 is correct as stated, but it is not clear the "constant returns to fixed bads" is an interesting case. In any event it does not imply that the market will fail when the firm operates under ordinary constant returns to scale and we add an externality rights market, as Starrett seems to imply.

but the fact of pollution that causes damage to other firms. If there is not a linear correlation between production and effluent, then the Arrovian model misses an important richness of economic possibilities. When firms take Arrovian observation prices as given, firms do not receive any benefit from reducing the damage that these observations do to other agents. Thus, Arrow prices give no incentive for firms to switch to an effluent reducing technology. This same argument holds for Pigovian tax systems or any other plan that works through prices of existing outputs. Solving this problem requires that the externalities be taken directly into account through markets.

2. The Arrovian model does not distinguish between directed and public externalities. Smelting steel produces slag and smoke. These are fundamentally different types of "bads." Slag is a directed externality. It only inflicts harm on the agent who has to live with it. For example, the disutility I get from observing the production of steel depends on whether the slag is dumped in my backyard or kept on the premises of the mill. On the other hand, the damage generated by smoke is experienced by all when steel is produced. This important structural difference is not captured by Arrow commodities.

3. As we mentioned above, total quantity of rights to pollute is bounded either by nature or statute. Not only should this bound be taken into account, but an explicit rights market must be established which efficiently allocates these rights between the two competing uses: pollution and pollution abatement (unused rights retained by the public). To do so, abatement should be treated as a form of public good. Note that this is a bit different from the effluent markets currently under discussion in policy circles. These markets are one-sided in the sense that they allocate a fixed quantity of rights over firms, but the public is not allowed to participate in order to gain additional abatement. This generally will lead to a second best outcome unless some other mechanism is employed to determine the first best level of pollution rights.

Our solution is to define an extension of the Lindahl equilibrium with the addition of a property rights market. We consider a model with I individual consumers and F firms. We use the convention $\mathscr{I} \equiv \{1, ..., I\}$ and $\mathscr{F} \equiv \{1, ..., F\}$. Subscripts are used to denote firms and consumers and superscripts to denote types of commodities. There are N^c *private consumer goods*, N^d *directed externalities*, N^g *public goods*, and N^r *public externality rights*, for a total of $N = N^c + N^d + N^g + N^r$ goods. Directed externalities are a type of private bad (garbage, for example). The only reason to separate them from other private goods is to highlight their production technology. We are most interested in the public externality rights market. These rights may be put to two uses. First, the public can buy rights and use them up in the form of a public good we can think of as abatement of externalities. All consumers experience the same level of this public good. Second, the individual firms can buy rights and use them up by generating public bads (smoke for example). The purpose of the externality market described below is to divide the total endowment of externality rights between these two competing uses.

A typical consumption bundle will be written $x_i = (x_i^c, x_i^d, x_i^g, x_i^r)$ where $x_i^c \in \Re^{N^c}$, $x_i^d \in \Re^{N^d}$, $x_i^g \in \Re^{N^g}$, and $x_i^r \in \Re^{N^r}$ denote bundles of private goods, private bads, public goods, and externality rights (abatement of public bads), respectively. Each agent $i \in \mathscr{I}$ is characterized by an endowment $\omega_i = (\omega_i^c, 0, 0, \omega_i^r)$ and a preference relation $\succcurlyeq_i$ over a consumption set $X_i \subset \Re^N_+$. Consumers are endowed with private goods and public externality rights. The aggregate endowment is $\omega = \sum_i \omega_i$.

We make the following assumptions on preferences for all $i \in \mathscr{I}$.

(A1) $\succcurlyeq_i$ is complete and transitive.

(A2) $\succcurlyeq_i$ is continuous (the upper and lower contour sets are closed relative to X_i).

(A3) If $x_i \succcurlyeq_i \tilde{x}_i$, then for all $\lambda \in [0, 1]$, $\lambda x_i + (1-\lambda)\,\tilde{x}_i \succcurlyeq_i \tilde{x}_i$ (weak convexity).

(A4) For all $x_i \in X_i$, and for all $\varepsilon \geqslant 0$ there exists $\tilde{x}_i \in X_i$ such that $\|x_i - \tilde{x}_i\| \leqslant \varepsilon$ and $\tilde{x}_i \succ_i x_i$ (local nonsatiation).[8]

We represent each firm $f \in \mathscr{F}$ by a production set $Y_f \subset \Re^N$. A typical production plan will be written $y_f = (y_f^c, y_f^d, y_f^g, y_f^r)$ where these represent net output vectors of private and public commodities. Note in particular that firms which generate externalities must consume externality rights, and so y_f^r will be negative in these cases. Firms also have endowments of property rights denoted $\eta_f^r \in \Re^{N^r}_+$. Let $\eta_f = (0, 0, 0, \eta_f^r)$ and $\eta = \sum_f \eta_f$. Endowing firms with property rights is not necessary for our model to work, but we include it in order to show that efficiency obtains in equilibrium regardless of how property rights are initially distributed. Note that firms' production sets do not include the outputs of public goods or bads of the remaining firms. The externalities we model in this paper are all generated by firms and experienced by consumers. This is done in the interest of simplifying notation. There should be no technical difficulty in extending the model. We assume for all $f \in \mathscr{F}$:

[8] Assuming local nonsatiation is very different from assuming that no points of local Pareto satiation exist, and is not particularly stronger in this context than in a private goods economy.

(B1) Y_f is a nonempty, closed set.

(B2) For all y_f, $\tilde{y}_f \in Y_f$ and all $\lambda \in [0, 1]$, $\lambda y_f + (1-\lambda)\, \tilde{y}_f \in Y_f$ (weak convexity).

Formally, we have added public and private bads to a standard Samuelson [21] model of public and private goods. There is no fundamental reason that public bads should generate a nonconvexity, as the example in the previous section demonstrates. The nonconvexity Starrett discusses comes from the unboundedness of endowments in the rights market. In our model, the rights market is bounded by construction. On the other hand, the nonfundamental nonconvexities discussed in Baumol [3] may still occur. We should always bear in mind that convexity is a much stronger assumption when externalities are present.

We define the *global production set* in the usual way:

$$Y \equiv \left\{ y \in \Re^N \;\middle|\; y = \sum_f y_f \text{ and } \forall f \in \mathscr{F},\ y_f \in Y_f \right\}.$$

We make the additional assumption:

(B3) Y is closed.

Notice that Y inherits convexity from the individual Y_f sets.

An allocation is a list $a = (x_1, ..., x_I, y_1, ..., y_F)$. The set of feasible allocations A consists of all allocations a such that:

1. For all $i \in \mathscr{I}$, $x_i \in X_i$.
2. For all $f \in \mathscr{F}$, $y_f \in Y_f$.
3. $\sum_i x_i^c = \sum_i \omega_i^c + \sum_f y_f^c$.
4. $\sum_i x_i^d = \sum_f y_f^d$.
5. For all $i \in \mathscr{I}$, $x_i^g = \sum_f y_f^g$.
6. For all $i \in \mathscr{I}$, $x_i^r = \sum_f \eta_f^r + \sum_i \omega_i^r + \sum_f y_f^r$.

Conditions 1 and 2 require that the allocation be feasible for each consumer and producer. Condition 3 requires that the net production of private goods equals the consumption. Condition 4 requires that the production and consumption of directed externalities be equal. In this simple model, consumers must accept all the garbage that firms generate. We discuss how this might be generalized in the conclusion. Condition 5 requires that each consumer consumes the total amount of public goods produced by firms. Finally, condition 6 requires that the total endowment of property rights is divided between externality and abatement uses, and that each consumer experiences the total level of externality generated by all the firms.

The set of *Pareto efficient allocations* is defined as

$$PE \equiv \{a \in A \mid \text{there is no } \hat{a} \in A \text{ with } \hat{x}_i \succcurlyeq_i x_i \text{ for all } i \in \mathscr{I} \text{ and } \hat{x}_j \succ_j x_j \text{ for some } j\}.$$

The price space is

$$\Pi \equiv \{\mathbf{p} \equiv (p^c, p^d, p^g, p^r) \in \Re^{N^c + N^d + IN^g + IN^r} \mid \mathbf{p} \neq 0\}.$$

We denote elements of $\Re^{N^c + N^d + IN^g + IN^r}$ by boldface. Notice that the private commodities have one price which is common across agents, while there are personalized prices for public commodities. In addition, we do not assume that prices are positive, but we exclude the zero vector from the price space. Given a price vector $\mathbf{p} \in \Pi$, denote the personalized prices faced by individual i by $p_i \equiv (p^c, p^d, p^g_i, p^r_i) \in \Re^N$, and the prices faced by the firms as $p \equiv (p^c, p^d, \sum_i p^g_i, \sum_i p^r_i)$. Note that all firms face the same prices, and that for both kinds of public commodities, the firms' prices are just the sums of the consumers' personalized prices. The value of a net output y_f is py_f. It follows that profits of firm f are $\pi_f(y_f, p) = p(y_f + \eta_f)$. We will denote a vector of profits for all firms as $\pi \equiv (\pi_1, ..., \pi_F)$.

Let Δ^{I-1} denote the $I-1$-dimensional simplex:

$$\Delta^{I-1} \equiv \left\{\theta \in \Re^I \,\middle|\, \sum_i \theta_i = 1, \text{ and } \theta_i \geqslant 0 \ \forall i \in \mathscr{I}\right\}.$$

We denote a profit share system for a private ownership economy by $\theta = (\theta_1, ..., \theta_f) \in \Delta^{I-1} \times \cdots \times \Delta^{I-1} \equiv \Theta$ where θ_{if} is interpreted as consumer i's share of the profits of firm f.

The budget set of agent i depends on the endowment of goods and firm shares, profits, and prices. Omitting the arguments in the profit function, this is given by:

$$B_i(\omega_i, \theta_i, \pi, \mathbf{p}) \equiv \left\{x_i \in X_i \,\middle|\, p_i x_i \leqslant p\omega_i + \sum_f \theta_{if} \pi_f\right\}.$$

A feasible allocation and price vector $(a, \mathbf{p}) \in A \times \Pi$ is said to be a *Coasian equilibrium relative to endowments* ω *and* η, *and profit shares* $\theta \in \Theta$ if and only if:

a. For all $i \in \mathscr{I}$, $x_i \in B_i(\omega_i, \theta_i, \pi, \mathbf{p})$ and $x_i \succcurlyeq_i \hat{x}_i$ for every $\hat{x}_i \in B_i(\omega_i, \theta_i, \pi, \mathbf{p})$ where $\pi = \pi(a, \mathbf{p})$.

b. $py_f \geqslant p\hat{y}_f$ for all $f \in \mathscr{F}$ and all $\hat{y}_f \in Y_f$.

c. For all $f \in \mathscr{F}$, $\pi_f = p(y_f + \eta_f)$.

We are now ready to state the first welfare theorem.

THEOREM 1. *If* $(a, \mathbf{p})$ *is a Coasian equilibrium, then* $a \in PE$.

Proof. Suppose not. Then there exists a feasible allocation $\hat{a} \in A$ with $\hat{x}_i \succcurlyeq_i x_i$ for all $i \in \mathscr{I}$ and $\hat{x}_j \succ_j x_j$ for some $j \in \mathscr{I}$.

First observe that if $\hat{x}_i \succcurlyeq_i x_i$ then $p_i \hat{x}_i \geqslant p_i x_i$. Suppose instead that

$$p_i \hat{x}_i < p_i x_i.$$

Then by local nonsatiation, for all $\varepsilon > 0$ there exists $\bar{x}_i \in X_i$ such that $\|\bar{x}_i - \hat{x}_i\| \leqslant \varepsilon$ and $\bar{x}_i \succ_i \hat{x}_i$. But then, for small enough ε, $p_i \bar{x}_i < p_i x_i$, contradicting part (a) of the definition of Coasian equilibrium. Part (a) also implies for agent j that $p_j \hat{x}_j > p_j x_j$.

We now sum over agents, to find

$$\sum_i p_i \hat{x}_i > \sum_i p_i x_i.$$

Because $x_i^g = x_1^g$ and $x_i^r = x_1^r$ for all i, it holds that

$$\sum_i p_i x_i = p\left(\sum_i x_i^c, \sum_i x_i^d, x_i^g, x_i^r\right).$$

Substituting the feasibility conditions yields

$$\sum_i p_i x_i = p\left(\omega + \eta + \sum_f y_f\right).$$

Similar arguments apply to $\hat{x}_i$, yielding

$$p \sum_f \hat{y}_f > p \sum_f y_f.$$

But this violates profit maximization, which is impossible as we started from an equilibrium. Thus no Pareto improvements are possible. ∎

Theorem 1 is essentially a Coase Theorem for convex economies. Given the structure of this general equilibrium economy, transactions costs are zero. Thus, we can read the first welfare theorem as saying "when transactions costs are zero, any endowment of property rights (including private goods endowments and profit shares) leads to Pareto efficient outcomes through market exchanges." It should be noted that the assumption of price taking behavior in our model is quite strong, and that our equilibrium notion is subject to exactly the same limitations in this regard as the Lindahl equilibrium.

If we drop convexity, a form of the Coase Theorem very different from Theorem 1 is still true. Coase did not necessarily require that property rights be traded on markets through linear price systems (as we describe above). Trade could also take place through bundled offers of the form: a payment of x dollars in exchange for a total of y rights to pollute. This point is made in an editorial addendum in the *Journal of Economic Theory* [16], for example. But as Cooter [7] points out, the Coase Theorem is almost a tautology at this level. If transactions costs are zero and there are gains from trade, then of course the trades will take place. Since Pareto optimality means that there are no further gains from trade, we get the Coase Theorem.

In a more recent paper, Hurwicz [15] has nicely summed up the source of this confusion. Hurwicz points out that to say the equilibrium is efficient regardless of allocation of property rights is vacuous until we agree on an equilibrium concept. Depending on this institutional choice, it is possible to construct Coase theorems which are false (for example in the game-theoretic approaches where the core or Nash equilibrium is used) or internally inconsistent (as Starrett points out is the case with Arrow market equilibrium). One way to view the contribution of the current paper is as providing a definition of a sensible equilibrium concept for which a Coase Theorem may be proven.

We give a second welfare theorem next. The proof follows the basic strategy of Foley [11]. It would not be difficult to generalize this theorem to include Pareto optimal allocations on the boundary along the lines of Conley and Diamantaras [9].

THEOREM 2. *Suppose that* $a \in PE$, *and for all* $i \in \mathcal{I}$, $x_i \in \text{interior}(X_i)$. *Then there exists a set of endowments* $(\hat{\omega}, \hat{\eta})$ *such that* $\sum_i \hat{\omega}_i + \sum_f \hat{\eta}_f = \sum_i \omega_i + \sum_f \eta_f$, *a set of profit shares*, $\hat{\theta} \in \Theta$, *and prices*, $\mathbf{p} \in \Pi$, *such that* $(a, \mathbf{p})$ *is a Coasian equilibrium relative to* $\hat{\omega}$, $\hat{\eta}$, *and* $\hat{\theta}$.

Proof. Following Foley, we define an artificial production set in which public commodities and externalities are treated as strictly jointly produced private commodities:[9]

$$AP \equiv \{\mathbf{z} \in \Re^{N^c + N^d + IN^g + IN^r} \mid z_i^g = z^g, \text{ and } z_i^r = z^r \text{ for all } i \in \mathcal{I},$$
$$\text{and } z \equiv (z^c, z^d, z^g, z^r) \text{ obeys } z - (\omega + \eta) \in Y\}.$$

[9] This artificial production set is defined such that x_i^r is interpreted as externality rights left unused by the firms, and so $x_i^r \geqslant 0$.

Notice AP is closed, convex, and nonempty because Y possesses these properties. Next we define the socially preferred set for an allocation a:

$$SP(a) \equiv \left\{ \mathbf{z} \in \mathfrak{R}^{N^c + N^d + IN^g + IN^r} \;\middle|\; \text{there are } \tilde{x}_i \in X_i \text{ with} \right.$$

$$z^c = \sum_i \tilde{x}_i^c,\ z^d = \sum_i \tilde{x}_i^d,\ z_i^g = \tilde{x}_i^g \text{ and } z_i^r = \tilde{x}_i^r$$

$$\left. \text{obeying } \tilde{x}_i \succcurlyeq_i x_i \text{ for each } i, \text{ with } \tilde{x}_j \succ_i x_j \text{ for some } j \right\}.$$

The socially preferred set inherits convexity, and by continuity and local nonsatiation has a nonempty interior.

Since $a \in PE$ by assumption, $SP(a) \cap AP = \varnothing$. Then by the Minkowski Separation Theorem,[10] there exists a price vector $\mathbf{p} \neq 0$ with $|\mathbf{p}| < \infty$, and a scalar r, such that:

(i) For all $\mathbf{z} \in \text{closure}(SP(a))$, $\mathbf{p} \cdot \mathbf{z} \geqslant r$.

(ii) For all $\mathbf{z} \in AP$, $\mathbf{p} \cdot \mathbf{z} \leqslant r$.

Let $\mathbf{z} = (\sum_i x_i^c, \sum_i x_i^d, x_1^g, ..., x_I^g, x_1^r, ..., x_I^r)$. By continuity of preferences, $\mathbf{z} \in \text{closure}(SP(a))$. By hypothesis, $\mathbf{z} \in AP$. It follows from (i) and (ii) that $\mathbf{p} \cdot \mathbf{z} = \sum_i p_i x_i = r$.

a. Now we define the endowments and profit shares needed to support the allocation a. For all $i \in \mathscr{I}$ and $f \in \mathscr{F}$, let

$$\hat{\omega}_i \equiv \frac{p_i x_i}{\sum_j p_j x_j} \left(\sum_j \omega_j + \sum_f \eta_f \right) = \frac{p_i x_i}{\sum_j p_j x_j} (\omega + \eta).$$

Also, set

$$\hat{\theta}_{if} \equiv \frac{p_i x_i}{\sum_j p_j x_j} \qquad \text{and} \qquad \hat{\eta}_f \equiv 0.$$

By construction, $\sum_i \theta_{if} = 1$ for all $f \in \mathscr{F}$. Also, $\sum_i \hat{\omega}_i = \sum_j \omega_j + \sum_f \eta_f$. We now show for all $i \in \mathscr{I}$, $x_i \in B_i(\hat{\omega}_i, \hat{\theta}_i, py_1, ..., py_f, \mathbf{p})$.[11] At these prices and endowments, agent i has income

[10] See, for example, Takayama [24, p. 44].

[11] Since the firms have zero endowments py_f is the profit of firm f under prices P at allocation a.

$$p\hat{\omega}_i+\sum_f \hat{\theta}_{if}\, py_f=\frac{p_i x_i}{\sum_j p_j x_j}\left[p^c\omega^c+\left(\sum_j p_j^r\right)(\omega^r+\eta^r)+\sum_f py_f\right]$$
$$=\frac{p_i x_i}{\sum_j p_j x_j}\left[p^c\sum_j x_j^c+p^d\sum_j x_j^d+\left(\sum_j p_j^g\right)x_i^g+\left(\sum_j p_j^r\right)x_i^r\right]$$
$$=p_i x_i,$$

where we have substituted from the feasibility conditions:

$$\sum_f py_f=p^c\left(\sum_j (x_j^c-\omega_j^c)\right)+p^d\left(\sum_j x_j^d\right)+\left(\sum_j p_j^g\right)x_i^g$$
$$+\left(\sum_j p_j^r\right)(\omega_i^r-\omega^r-\eta^r).$$

Thus, x_i is in the budget set of agent i.

b. Next we show that for all $i\in\mathscr{I}$, $\hat{x}_i\in B_i(\hat{\omega}_i, \hat{\theta}_i, py_1, ..., py_f, \mathbf{p})$ implies $x_i \succsim_i \hat{x}_i$. Suppose instead that for some $j\in\mathscr{I}$ there exists $\hat{x}_j\in B_j(\hat{\omega}_i, \hat{\theta}_i, py_1, ..., py_f, \mathbf{p})$, such that $\hat{x}_j\succ_j x_j$. By assumption $x_j\in \text{interior}(X_j)$ and so by convexity of preferences, somewhere on the line joining x_j and $\hat{x}_j$ there is a consumption bundle in the interior of the consumption set which is also strictly preferred to x_j and is in the budget set. Then by continuity of preferences and the fact that $p\neq 0$, there exists $\bar{x}_j\in X_j$ such that $\bar{x}_j\succ_j x_j$ and $p_j\bar{x}_j<p_j x_j$. Hence,

$$\left(\sum_{i\neq j} x_i^c+\bar{x}_j^c,\ \sum_{i\neq j} x_i^d+\bar{x}_j^d,\ x_1^g, ..., \bar{x}_j^g, ..., x_I^g, x_1^r, ..., \bar{x}_j^r, ..., x_I^r\right)\in SP(a)$$

and

$$\sum_{i\neq j} p_i(x_i^c, x_i^d, x_i^g, x_i^r)+p_j(\bar{x}_j^c, \bar{x}_j^d, \bar{x}_j^g, \bar{x}_j^r)<\sum_i p_i x_i=r$$

contradicting (i) above. We conclude that part (a) of the definition of Coasian equilibrium is satisfied at these endowments and prices.

c. We now need only show that all firms maximize profit at y_f given prices $\mathbf{p}$. Let $\hat{y}_l\in Y_l$. Define $\hat{z}^c=\omega^c+\sum_{f\neq l} y^c+\hat{y}_l^c$, and $\hat{z}^d=\sum_{f\neq l} y^d+\hat{y}_l^d$, and for all $i\in\mathscr{I}$, $\hat{z}_i^g=\sum_{f\neq l} y_f^g+\hat{y}_l^g$ and $\hat{z}_i^r=\omega^r+\eta^r+\sum_{f\neq l} y_f^r+\hat{y}_l^r$. Then $\hat{z}\in AP$, so $\mathbf{p}\cdot\hat{z}\leqslant r$. Expanding $p\hat{z}$, we find:

$$p\hat{z}=p\hat{y}_l+\sum_{f\neq l} p\hat{y}_f+p(\omega+\eta)\leqslant r.$$

But

$$r = \sum_f p\hat{y}_f + p(\omega + \eta)$$

so $p\hat{y}_l \leqslant py_l$. Thus profits are maximized at y_f for each $f \in \mathcal{F}$. We conclude that part (b) of the definition of Coasian equilibrium is satisfied at these endowments and prices, and the theorem is true. ▮

The second welfare theorem above says that every Pareto efficient allocation not on the boundary of the agent's consumption sets can be achieved as an equilibrium. This directly puts to rest Starrett's concern that some efficient allocations will not be supportable by prices in economies with externalities.

5. CONCLUSIONS

We have made several simplifications in the interest of clarity. The most important is that only consumers, and not firms, are affected by public externalities. Generalizing this would require that the production sets of firms be expanded to include the abatement level, and that a set of individualized Lindahl prices for these public goods be defined for firms (as they already are for consumers). A notational distinction would have to be made between externality rights which are privately consumed (and fully paid for) by firms, and those which are publicly owned and used for abatement (for which the firms contribute a share of the cost proportional to their marginal benefit). Ordinary public goods could also be made into inputs for firms' production possibilities in the same way. Another possible extension would be to allow individuals to consume externality rights for private purposes (leaf burning, for example). The treatment of directed externalities could be made more elaborate. For example, we could endow agents with dumping rights which we might think of as land capacity. These endowments would place an upper bound on consumption of directed externalities. We might include specialized firms with dumping endowments who use inputs like transportation and labor to place directed externalities out of sight.[12] None of these extensions present any theoretic difficulty as far as we can see, but do require greater complexity in the notation of the model.

In conclusion, we believe that Starrett's major contribution was to point out the weakness of the Arrovian approach to understanding economies with externalities. Not only are nonconvexities fundamental in Arrow

[12] The model already allows for incinerator companies which convert private inputs and garbage into public externalities (smoke). Private dumping companies of the form described above are not included, however.

markets, but the very generality of the Arrow model misses many important institutional structures. We provide a general equilibrium model and a definition of Coasian equilibrium which address these problems. The main advantages of our approach are that nonconvexities are not fundamental, directed and public externalities are distinguished, firms have an incentive to use the most efficient technology available to reduce negative externalities, physical or statutory limits on externalities can be accounted for, and the market efficiently divides the total property rights endowment between abatement and externality generation uses.

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