The Optimal Depletion of Exhaustible Resources

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The Optimal Depletion of Exhaustible Resources [1, 2]

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0. INTRODUCTION

0.1. In any given economy there must be a number of commodities that enter into production and which are used up, and whose available stock cannot be increased. Examples like fossil fuels come readily to mind. It would seem reasonable to argue that in the long run the limited availability of these commodities, together with their technological importance, would begin to act as a constraint on the economy's growth potential. In fact several recent studies have laid great emphasis on this possibility.[3] Although the point is an obvious one, most economic studies of the properties of long-run plans neglect it.[4] In this paper, therefore, we explore in a rather preliminary way the problems that appear to arise naturally when the existence of exhaustible resources is incorporated into the study of intertemporal plans.

It appears that questions in this area are hard to answer. For even in the simplest of environments where it is supposed that there is perfect foresight, one is concerned not merely with the optimal depletion of exhaustible resources but with the optimal rate of investment as well. The two must plainly be interrelated. The latter problem on its own is hard, and the combined problem is very complex. In fact it will appear in the course of the arguments that follow that intuition is not a very good guide for this joint problem. In any case, the assumption of perfect foresight is particularly dubious here since it would appear almost immediate that an investigation of intertemporal plans in the presence of exhaustible resources readily invites consideration of the possibilities of large-scale alterations in technology at dates in the future that are inherently uncertain. Moreover, it is clear that such extensive technological changes would not be achievable costlessly. In this paper, therefore, we attempt to demonstrate how one might, in a relatively simple manner, bring such considerations as these to bear on a set of questions that have generated a considerable amount of interest in recent years.

0.2. It is plain of course that the mere *existence* of a resource that is exhaustible is not a

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2 Among the many to whom acknowledgement is due for their comments we would like to mention Christopher Bliss, Steve Glaister, Terence Gorman, Frank Hahn, Tjalling Koopmans, Bill Nordhaus, Robert Solow, Joseph Stiglitz, Niel Vousden and, in particular, Harl Ryder.

3 We have in mind the works of Forrester [8] and Meadows *et al.* [20]. But see also Cole *et al.* [4] for a penetrating commentary on the Forrester analysis; as well as the lively comments of Beckerman [3].

4 Thus the extensive literature on optimal planning is clearly concerned with horizons long enough for such constraints to become effective, and yet it rarely mentions them. Exceptions are Anderson [1], Ingham and Simmonds [15], Vousden [27], and the seminal and rather neglected work of Hotelling [14]. Professors Koopmans [18], Solow [25], and Stiglitz [26] have recently, and independently of us, explored different aspects of the question under review here. Their findings are complementary to ours and, in what follows, we shall have occasion to refer to the tasks that they have undertaken in their contributions. For a detailed survey of recent modelling experiences in the area of exhaustible resources, see Dasgupta [7].

sufficient basis for apocalyptic visions. Even if we assumed away the possibility of technological change, exhaustible resources would pose a " problem " only if they are, in some sense, essential in production. Intuition suggests that one regards a resource as being *essential* if output of final consumption goods is nil in the absence of the resource. Otherwise, call the resource inessential.[1] It is then natural to ask whether feasible output must eventually decline to zero in an economy that possesses an essential exhaustible resource. Rather surprisingly, perhaps, the answer turns out to be " no ", and this even if there is no technological progress.[2] The point, of course, is that the possibilities open to such an economy depend crucially on the ease with which reproducible inputs can be substituted for the exhaustible resources. It follows, then, that the effect that the existence of an exhaustible resource will have on the characteristics of an *optimal* plan will depend on the extent of such substitution possibilities. In particular we demonstrate below that the *elasticity of substitution* between reproducible inputs and exhaustible resources plays an important and a rather direct role in the properties of an optimal plan. A related question that arises is whether one should, as intuition suggests, deplete an *essential* exhaustible resource slowly over the planning horizon, or whether one should exhaust it in finite time, closing down and living off the existing capital stock. In the same way one wants to know whether it is optimal to deplete an *inessential* exhaustible resource in finite time and, subsequently, rely solely on reproducible capital to continue production. In Section 1 we investigate this range of issues under the assumption that future technology is known with certainty to be the same as that available at present. As one would expect, the analysis will give an indication of the structure of relative prices that would sustain an optimal policy.

While intuition suggests that the elasticity of substitution will be an important parameter, it also suggests that it would be inappropriate to regard the technology of the economy as unchanging over time. Even if under present technological knowledge an exhaustible resource is judged essential it would be unwise to rule out the possibility of substitutes being discovered in the future. In one sense this is at the core of the controversy between Forrester and his critics. In their critiques of the results published by Forrester [8], both Maddox [19] and Nordhaus [21] argue that as an essential exhaustible resource is depleted its market price will rise, forcing entrepreneurs to search for cheaper substitutes. This argument clearly has force, though in the absence of a well-articulated intertemporal plan or a satisfactory set of forward markets it is not at all plain that market prices will be providing the correct signals.[3] Moreover, there are some additional problems that arise in this context, for many of the difficulties that are involved in making policy recommendations about the rate of depletion of exhaustible resources stem from the fact that crucial aspects of this problem are inherently uncertain, and it is not clear that an adequate class of contingent markets exists. There is, of course, some uncertainty about the amounts of resources actually available at any given date, but one would be inclined to feel that this is not the most interesting source of uncertainty. Rather, it would seem plausible that the really important source of uncertainty is connected with future technology. There is, for example, a chance that the discovery of substitutes will render previously essential resources inessential. This has happened a number of times in the past, and one would expect that the best policy towards resource depletion would depend on the probability of such an occurrence. In Section 2 we attempt to analyse this aspect of the problem. We assume in particular that technical change is costlessly attained and that we know in advance the form of the technical change (i.e. the nature of future technology and resource availability) but that the date of its appearance is random.

[1] Later we shall note that this method of partitioning resources that are exhaustible is not as helpful as might appear at first sight.

[2] See Solow [25] for a demonstration of this.

[3] Stiglitz [26] and Dasgupta [7] have investigated the properties of a competitive system where futures markets do not exist and where expectations are not necessarily fulfilled. As would be suspected, it is very easy in such an economy to have a systematic over or under utilization of the resource: for an analysis of the manner in which they have actually allocated resources over time see Heal [10, 12].

In point of fact, of course, substitutes are not obtained at zero cost. One would like (finally) to analyse a problem where resources are deployed in the search for substitutes and where, at the same time, there is no guarantee that the search will be successful. In such a situation the problem is not merely one of obtaining the optimal depletion rate for the exhaustible resources; it also involves finding the correct allocation of the resources used between the production of goods and expenditure on research. In a subsequent paper we shall analyse this aspect of the problem.

The problems that we are considering in this paper arise from the implications that a finite earth has for intertemporal planning. In this context the treatment of population poses considerable difficulties. It would clearly be unreasonable to consider an exogenously given rate of population increase, because the very factors that we are attempting to allow for must eventually force a voluntary or involuntary reduction in rates of population growth on society. There is thus, in principle, a clear need for an optimal population policy, together with an optimum consumption and depletion policy. Professor Koopmans [18] has recently investigated a special form of this problem which consists in assuming a *constant* and *given* population size, with the interesting twist that the planning horizon is regarded as choice variable. The Koopmans problem in effect consists of locating an optimal depletion policy and the optimal survival period for an economy that contains a given and fixed population. It looks as though the population problem in its general setting is particularly complex and we have, so far, been able to obtain some preliminary results only.[1] In this paper, therefore, we assume a constant level of population through time. We are, then, analysing a world in which the objective of zero population growth has been acccepted and achieved. Such a device poses its own interpretive problems, since one has to assume away the existence of a level of consumption below which life cannot be sustained. We can, of course, by suitable assumptions, avoid *zero* consumption along the optimum programme. But in effect we shall be supposing that the subsistence level of consumption is nil.

0.3. In order to clarify ideas it will prove convenient to review briefly the simplest of the models that attempt to capture the presence of exhaustible resources. What we review here is a slight generalization of the well-known " cake-eating " problem analysed first by Hotelling [14] and later by Gale [9]. It would seem plausible that the " cake-eating " problem would be a basic building block of any production model that is to catch the issues that we are concerned with in this paper. In subsequent sections of this paper we shall establish the precise sense in which this claim is true.

But for the moment assume that there is no production. The economy possesses, to begin with, a finite stock (inventory), S_0, of a homogeneous consumption good. By an absence of production in this economy we mean that the social rate of return to investment is zero. Our simple generalization consists of the supposition that a substitute for the resource is already available, in the sense that there is a steady flow of the consumption good that is fed into the economy at the rate M. Denote by C_t the rate of consumption at time t, and by $U(C_t)$ the instantaneous utility of consuming C_t. We assume, as is usual, that $U(\cdot)$ is monotonically increasing, strictly concave, and twice differentiable everywhere. We also suppose, as seems natural, that

$$\operatorname*{lt}_{C\to 0} U'(C) = +\infty.^2 \qquad \text{...(0.1)}$$

Write $\eta(C) = -CU''(C)/U'(C)$, for the *elasticity of marginal utility*. We take it that

$$+\infty > \operatorname*{lt}_{C\to 0} \eta(C) = \eta > 0. \qquad \text{...(0.2)}$$

[1] For an analysis of optimum population policies in the presence of a fixed factor (land), and external diseconomies through large population sizes, see Dasgupta [5].

[2] We follow the by now standard notation: $\dot{x}_t = dx_t/dt$ and $f'(x) = df/dx$. In what follows we shall often drop the time subscript. This should not cause any confusion.

Social welfare over the interval [0, T] will be taken to be of a utilitarian form, yielding an amount

$$\int_0^T e^{-\delta t} U(C_t)dt, \qquad \text{...(0.3)}$$

where δ is strictly positive.

One is therefore concerned with obtaining that consumption profile, C_t, which will maximize

$$\left.\begin{array}{l} \displaystyle\int_0^\infty e^{-\delta t} U(C_t)dt \text{ subject to} \\ \dot{S}_t = M - C_t, \\ \quad \text{where } C_t,\ S_t \geqq 0,\ M \geqq 0, \text{ and where } S_0 \text{ is given} \end{array}\right\}. \qquad \text{...(0.4)}$$

In exploring the problem as presented in (0.4) the natural thing to do is to introduce multipliers for the various constraints and write down the Hamiltonian of the system as

$$\mu = e^{-\delta t} U(C_t) + e^{-\delta t} p_t (M - C_t) + e^{-\delta t} q_t S_t \qquad \text{...(0.5)}$$

where

$$q_t \geqq 0 \text{ and } q_t S_t = 0. \qquad \text{...(0.6)}$$

We have not explicitly introduced the multiplier associated with the non-negativity of C_t simply because we know in advance that given (0·1) the multiplier will throughout be zero. It is then immediate that for a programme to be optimal it is necessary that

$$p_t = U'(C_t) \qquad \text{...(0.7)}$$

and also that p_t, the spot price of consumption, should satisfy the differential equation

$$\dot{p}_t = -q_t + \delta p_t. \qquad \text{...(0.8)}$$

Using (0.7) in (0.8) one obtains

$$\frac{\dot{C}}{C} = -\frac{\delta}{\eta(C)} + \frac{q_t}{\eta(C)U'(C)}. \qquad \text{...(0.9)}$$

From (0.6) and (0.9) it is clear that there are two (possibly repeated) phases, namely: *Phase A*. During which $S_t > 0$ and hence

$$\frac{\dot{C}}{C} = -\frac{\delta}{\eta(C)} < 0 \qquad \text{...(0.10)}$$

and *Phase B*. During which equation (0.9) holds with $q_t > 0$, so that $S_t = 0$.

The solution to problem (0.4) for the case $M = 0$ is well known (see e.g. Heal [11]). Consequently we sketch the argument for the case $M > 0$. One begins by noting from equation (0.9) that phase A cannot continue for all $t \geqq 0$ along an optimal policy. For then by (0.2) and (0.10), given any $\varepsilon > 0$ and C_0 there will exist a T such that $C_t \leqq \varepsilon$ for $t \geqq T$, and this would imply that $S_t \to \infty$, which is plainly inefficient. From (0.4) it is clearly feasible to have $C_t \geqq M$ for $t \geqq 0$. This suggests a policy of following phase A for an initial period [0, T] followed by phase B for $t \geqq T$. During phase A equation (0.10) holds. Both C_0 and T are determined by the requirements

$$\int_0^T C_t dt = S_0 + MT \text{ and } \operatorname*{lt}_{t \to T^-} C_t = C_T = M.$$

For $t \geqq T$ (i.e. during phase B) we have $C_t = M$. Thus one sets $q_t = 0$ for $0 \leqq t < T$ and

$$q_t = \frac{\delta}{U'(M)} \text{ for } t \geqq T.$$

The proposal satisfies the necessary conditions for an optimal policy. The Hamiltonian (0.5) is concave. It is also the case that along the proposed policy

$$\operatorname*{lt}_{t\to\infty} e^{-\delta t}p_tS_t = 0.$$

It follows that we have

Proposition 1. *An optimal policy for the problem* (0.4) *exists and it is uniquely given.*

Proposition 2. *If $M>0$ the optimal policy consists of precisely two phases. It consists initially of phase A until a time T at which the stock, S_t, is nil. From T onwards the policy consists of phase B during which $C_t = M$.* (Figure 1 illustrates the policy described in Proposition 2.)

The following proposition is well known.

Proposition 3. *If $M = 0$, the optimal policy consists of only one phase, namely phase A. The initial consumption rate, C_0, is so chosen that*

$$\operatorname*{lt}_{t\to\infty} S_t = 0. \quad \text{(See Figure 1.)}$$

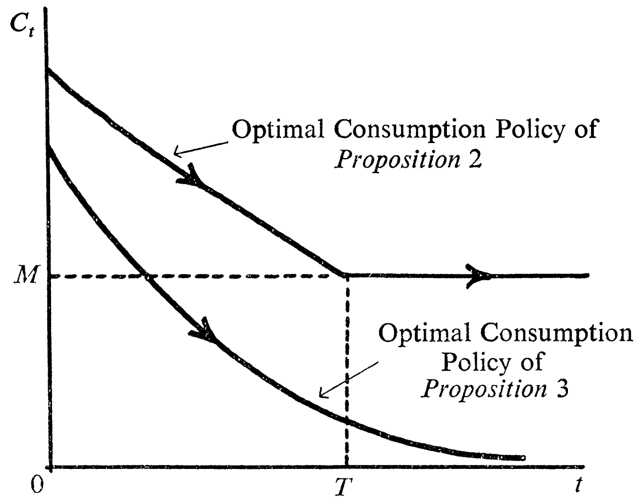

FIGURE 1

It is an implication of both Propositions 2 and 3 that the consumption rate is a non-increasing function of time. Furthermore, Proposition 2 might suggest that if a resource is inessential the economy should not carry a positive inventory of it indefinitely. It turns out that neither of these implications necessarily carries over into a model with production.

1. PRODUCTION

1.1. The economy introduced in Section 0.3 was instrumental merely in fixing ideas and was in itself of limited interest, as there was no production. Consequently we introduce production now. But before introducing exhaustible resources into a production model in a fully-fledged way we present, in this sub-section, a quick review of a production model that incorporates one aspect of the economy discussed in the previous section—namely that a steady stream of a resource that enters in production is fed into the economy from outside. By this device we are, of course, assuming away the essence of the problem.

There are, nevertheless, two reasons why we shall wish to review such a model here. First, it will allow one to see the sharp differences in the properties of an optimal path in an economy that has no exhaustible resources, and one in which such a resource constraint bites in the long run. Second, we shall use the model presented in this sub-section in our discussion of the specific issues that arise when one considers technological change.

We suppose now that there is a single non-deteriorating consumption good which, in conjunction with the service provided by a perfectly durable commodity (e.g. an energy source), can reproduce itself. The quantity of the durable commodity providing service is given and cannot be augmented. It is assumed to provide service at the constant rate M. We can, if we like, assume that the flow of service from the durable good can be stored costlessly, but we shall wish to suppose here that at the start of the planning period there is no inventory of the service.

Denote by K_t the stock of the reproducible composite commodity and by Z_t the rate of utilization of the service at time t. Efficient output possibilities are represented by the production function $G(K, Z)$. It is supposed that G is increasing, twice differentiable, strictly concave and that

$$G(0, Z) \geqq 0 \qquad \text{...(1.1}a\text{)}$$

$$\operatorname*{lt}_{K\to\infty} \frac{\partial G}{\partial K}(K, M) < \delta \qquad \text{...(1.1}b\text{)}$$

$$\operatorname*{lt}_{K\to 0} \frac{\partial G}{\partial K}(K, M) > \delta. \qquad \text{...(1.1}c\text{)}$$

It follows that consumption possibilities can be predicted by the equation

$$\dot{K} = G(K, Z) - C. \qquad \text{...(1.2)}$$

Denote by V_t the inventory of the service of the durable commodity at t. It follows that the rate, Z_t, at which one utilizes the service can be predicted by the equation

$$\dot{V}_t = M - Z_t, \text{ where } V_t \geqq 0 \text{ all } t \geqq 0. \qquad \text{...(1.3)}$$

The problem we are interested in is one of obtaining time profiles of $\tilde{C}_t$ and $\tilde{Z}_t$ which will maximize

$$\left.\begin{array}{l} \displaystyle\int_0^\infty e^{-\delta t} U(C_t) dt \text{ subject to equations (1.2), (1.3) and the constraints} \\ C_t, Z_t, V_t, K_y \geqq 0, \text{ and where } K_0 (>0) \text{ is given and } V_0 = 0. \end{array}\right\}. \qquad \text{...(1.4)}$$

The problem, though cast here in a rather unusual form, is a familiar one. Consequently we merely state the results and furnish no proofs. Write $g(K) = G(K, M)$ and define $\bar{K}$ as the solution of $g'(K) = \delta$. One has then

Proposition 4. *Problem* (1.4) *possesses a unique solution. Along the optimal policy* $Z_t = M$ *for all* $t \geqq 0$ (*i.e.* $V_t = 0$ *for all* $t \geqq 0$), *and the economy tends in the long run to the stationary-state consumption rate* $\bar{C} = g(\bar{K})$, *and therefore the capital stock level* $\bar{K}$. *If* $K_0 < \bar{K}$, *then along the optimal path* $\dot{C} > 0$ *and* $\dot{K} > 0$. *If* $K_0 > \bar{K}$ *then along the optimal path* $\dot{C} < 0$ *and* $\dot{K} < 0$.

(Figure 2 presents a typical consumption profile of Proposition 4 for the more plausible case of $K_0 < \bar{K}$.)

Consider now a special case of the model just discussed. Assume that the production function G satisfies (1.1b)-(1.1c) but now strengthen (1.1a) to the form

$$G(0, Z) > 0 \text{ for } Z > 0. \qquad \ldots(1.1a')$$

Consider problem (1.4) for this economy, but with the difference that now $K_0 = 0$ as well. That is, the economy begins with no capital stock and no inventory of the service. For completeness we express this special case of Proposition 4 as

Proposition 4′. *If G satisfies* (1.1a') *in addition to its other properties and if $K_0 = V_0 = 0$, then Problem* (1.4) *possesses a unique solution. Along the optimal path $Z_t = M$ for all $t \geqq 0$,* (*i.e. $V_t = 0$ for all $t \geqq 0$*), *and the economy tends, in the long run, to the stationary state consumption rate $\bar{C} = g(\bar{K})$, and therefore, to the capital stock level $\bar{K}$. Along the optimal path $\dot{C} > 0$ and $\dot{K} > 0$.*

(Figure 2 presents a typical consumption profile of Proposition 4′.)

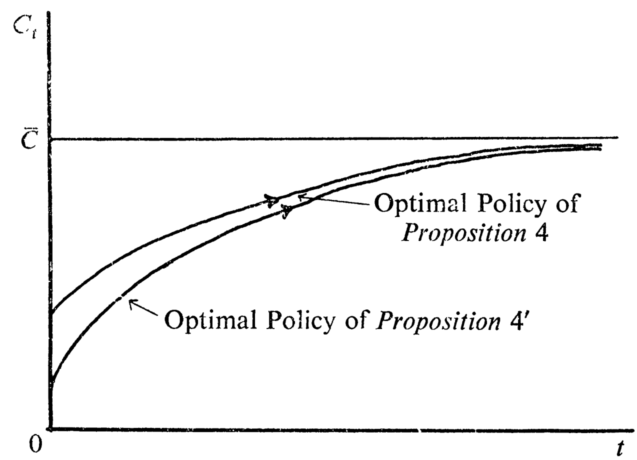

FIGURE 2

1.2. The economy discussed in the previous sub-section possessed no resource that was exhaustible. But it was suggestive, in that the technical possibilities in it were, albeit in a pristine form, the kind that one might be inclined to contemplate when reflecting on the circumstance that would prevail if, to take an example, some totally different source of energy (e.g. the sun) is harnessed. But presumably these are not the conditions that prevail as yet. Consequently we introduce exhaustible resources explicitly into production.

We suppose now that there is a non-deteriorating composite consumption good which, in conjunction with an exhaustible resource in production, can reproduce itself. We continue to denote by K_t the stock of the reproducible composite commodity at time t, but by R_t the flow of the exhaustible resource into production at t. Efficient output possibilities are, therefore, represented by a production function $F(K, R)$. It will be supposed that F is increasing, *strictly* concave, twice differentiable, and homogeneous of degree unity. Consumption possibilities are thus predicted by the condition

$$\dot{K}_t = F(K_t, R_t) - C_t. \qquad \ldots(1.5)$$

One assumes that the economy begins with a stock, S_0, of the exhaustible resource, and a stock, K_0, of the composite commodity. The planning problem then can be represented as being one of

$$\left.\begin{array}{l} \text{maximizing } \displaystyle\int_0^\infty e^{-\delta t}U(C_t)dt \text{ subject to equation (1.5) and the constraint}^1 \\ \displaystyle\int_0^\infty R_t dt \leqq S_0, \text{ where} \\ C_t,\ K_t,\ R_t \geqq 0 \text{ and } K_0(>0) \text{ is given} \end{array}\right\}. \qquad \ldots(1.6)$$

As before, it would be pointless to introduce the multipliers associated with the non-negativity of C and K, since we have supposed (0.1). Consequently we express the Hamiltonian associated with problem (1.6) as

$$H = e^{-\delta t}U(C_t) + e^{-\delta t}p_t(F(K_t, R_t) - C_t) - \lambda R_t + e^{-\delta t}\mu_t R_t, \qquad \ldots(1.7)$$

where

$$\mu_t \geqq 0 \text{ and } \mu_t R_t = 0 \qquad \ldots(1.8)$$

and

$$\lambda \geqq 0 \text{ and } \lambda\left(S_0 - \int_0^\infty R_t dt\right) = 0. \qquad \ldots(1.9)$$

One expects (and this will be confirmed) that $\lambda > 0$. It should be noted that λ is independent of time. Maximizing (1.7) with respect to C yields the condition

$$p_t = U'(C_t). \qquad \ldots(1.10)$$

Likewise, maximizing (1.7) with respect to R_t yields the condition

$$\lambda = e^{-\delta t}\left(\mu_t + p_t\frac{\partial F}{\partial R_t}\right). \qquad \ldots(1.11)$$

Write $F_R = \partial F/\partial R$, $F_K = \partial F/\partial K$, etc. Equations (1.8) and (1.11) imply that when $R_t > 0$ along an optimal policy

$$\lambda = e^{-\delta t}p_t F_{R_t}. \qquad \ldots(1.12)$$

But when $R_t > 0$, λ denotes the present value shadow price of the exhaustible resource in utility numeraire. It follows that during an interval along an optimal path when $R_t > 0$ this present value shadow price remains constant. However, during this interval the price of the resource relative to the composite commodity numeraire is not, of course, constant, but equal to the marginal product of the resource. A question of importance is the behaviour of this relative price along an optimal programme.[2]

1 We are thus supposing that extraction of the exhaustible resource is costless. Extraction costs do not appear to introduce any great problem, provided that we assume away non-convexities. One might for example, wish to assume that to extract the resource at a rate R when the remaining stock is of size S requires $E(R, S)$ units of the composite good. One might suppose that E is a convex function and that, in particular, $E(0, S) = 0$, $\partial E/\partial R > 0$ and $\partial E/\partial S \leqq 0$, the last inequality indicating that one is, as it were, digging deeper when the stock has shrunk. To have an interesting problem one would, in addition, need to assume that $\underset{R\to 0}{\text{lt}}\ \partial F/\partial R > \underset{R\to 0}{\text{lt}}\ \partial E/\partial R$. The planning problem (1.6) would in this event be modified only to the extent that equation (1.5) would be replaced by the equation,

$$\dot{K} = F(K, R) - C - E(R, S).$$

2 If extraction costs were introduced (see previous footnote) equation (1.12) would be replaced by the condition

$$\lambda = e^{-\delta t}p_t[F_R - E_R].$$

Thus, in fact, during an interval when $R_t > 0$ the shadow price of the unextracted exhaustible resource relative to the composite commodity is the *difference* between the marginal product of the resource and its marginal extraction cost. But the price of the extracted resource relative to the composite commodity is of course its marginal product. The extraction cost could be regarded as a pure transport cost. We have found that even amongst professional economists there is often the belief that the price of an exhaustible resource should be its marginal extraction cost. It is, of course, rather obvious why this belief is false.

A related question is whether or not the exhaustible resource ought to be depleted in *finite* time. Towards this consider an economy for which

$$\operatorname*{lt}_{R\to 0} F_R = \infty. \qquad \text{...(1.13)}$$

It is an immediate implication of equation (1.11) that we shall have

Proposition 5. *If* $F(K, R)$ *satisfies* (1.13), *then along an optimal policy, if one exists,* $R_t > 0$ *for all* $t \geqq 0$.

The sufficient condition (1.13) in Proposition 5 is, of course, unduly restrictive. One can relatively easily, as we shall presently, weaken it. But it is suggestive in that a condition that prohibits the depletion of the stock of the exhaustible resource in finite time rests not so much on whether the resource is essential (in the sense of output being nil in its absence), but rather on the behaviour of the marginal product of the resource for low rates of usage of the resource. To an economist this is very intuitive.

Proposition 5 is analogous to Proposition 3 of the " cake-eating " problem. It is tempting to construct an analogue of Proposition 2. Towards this we complete the set of conditions necessary for optimality by noting that one must have as well that

$$\frac{d}{dt}(e^{-\delta t}p) = -e^{-\delta t}pF_K, \qquad \text{...(1.14)}$$

or

$$-\dot{p}/p = F_K - \delta. \qquad \text{...(1.15)}$$

Now, equation (1.15), which is the familiar Ramsey condition can, in turn, be re-expressed on using (1.10) as

$$\frac{\dot{C}}{C} = (F_K - \delta)/\eta(C). \qquad \text{...(1.16)}$$

Consider now a time interval of positive measure, if one exists, during which it is optimal to set $R_t > 0$. During this interval equation (1.12) is operative. It follows that on differentiating (1.12) with respect to time one obtains the condition

$$\frac{\partial(F_R)}{\partial t}\frac{1}{F_R} = \frac{-\dot{p}}{p} + \delta,$$

which, on using equation (1.15) reduces to the form

$$\frac{\partial(F_R)}{\partial t}\frac{1}{F_R} = F_K. \qquad \text{...(1.17)}$$

Condition (1.17) is really rather obvious, for it is a statement concerning the equality of the rates of return on the two assets (the exhaustible resource and reproducible capital).

Write $x \equiv K/R$ and $f(x) \equiv F(K/R, 1)$. It follows from our assumptions on F that $F_K = f'(x) > 0$, $F_R = f(x) - xf'(x) < 0$, and $f''(x) < 0$. Thus one has as well that $d(F_R)/dx > 0$, for $x > 0$. Write

$$\sigma = \frac{-f'(x)(f(x) - xf'(x))}{xf(x)f''(x)}, \quad (\infty > \sigma > 0)$$

for the elasticity of substitution between K and R. It is then immediate that equation (1.17) reduces to the form

$$\frac{\dot{x}}{x} = \sigma\frac{f(x)}{x}. \qquad \text{...(1.18)}$$

Equations (1.16) and (1.18) govern an optimal programme, if one exists, during an interval when $R_t > 0$. Moreover, we have already established (Proposition 5) that if F satisfies (1.13) then equations (1.16) and (1.18) govern an optimal programme for all $t \geqq 0$.

It is remarkable that the conditions governing an optimal programme of aggregate consumption and resource depletion should be so simple in form. In particular, equation (1.18) makes the role of the elasticity of substitution very clear. The capital-resource ratio changes at a percentage rate equal to the product of the elasticity of substitution and the average product per unit of fixed capital. The former gives an indication of the ease with which substitution can be carried out, and the latter can be regarded as an index of the importance of fixed capital in production. Thus, the easier it is to substitute, and the more important is the reproducible input, the more one wants to substitute the reproducible resource for the exhaustible one.

We have already established that if F satisfies (1.13) then along an optimal policy, if one exists, $R_t > 0$ for all $t \geqq 0$. The following Lemma yields considerable information regarding the broad characteristics of an optimal depletion policy.

Lemma. *Along an optimal programme, if one exists, an interval during which $R = 0$ cannot be followed by an interval during which $R > 0$.*

Proof. Recall that $d(F_R)/dx > 0$. Assume that sup $(f(x) - xf'(x))$ is finite (for if not, the Lemma is trivially true in view of Proposition 5). Suppose then the contrary and let there be a change in phase at T. Assume first that the interval during which $R = 0$ is non-degenerate and that $R_T > 0$. Then since K_t is continuous one has

$$\operatorname*{lt}_{t \to T^-} (f(x_t) - x_t f'(x_t)) = \sup (f(x) - xf'(x)) > (f(x_T) - x_T f'(x_T)).$$

But $\operatorname*{lt}_{t \to T^-} e^{-\delta t} p_t = e^{-\delta T} p_T$. Such a phase change, therefore, is impossible since, given equations (1.11) and (1.12), one must have

$$\operatorname*{lt}_{t \to T^-} e^{-\delta t}(\mu_t + p_t(f(x_t) - x_t f'(x_t))) = e^{-\delta T} p_T(f(x_T) - x_T f'(x_T)),$$

with $\mu_t \geqq 0$.

The other possibility is that $R_T = 0$ but that during an interval (T, T_1) one sets $R_t > 0$. But if this is so then during (T, T_1) equation (1.18) holds, and in order for (1.18) to hold it must be the case that $\operatorname*{lt}_{t \to T^+} R_t > 0$. In other words, there must be a discontinuity in R at T. Thus $f(x_T) - x_T f'(x_T) > \operatorname*{lt}_{t \to T^+} (f(x_t) - x_t f'(x_t))$. One argues now, as earlier, that such a phase change is impossible since equations (1.11) and (1.12) dictate that

$$e^{-\delta T}(\mu_T + p_T(f(x_T) - x_T f'(x_T))) = \operatorname*{lt}_{t \to T^+} e^{-\delta t} p_t(f(x_t) - x_t f'(x_t))$$

and $\mu_T \geqq 0$. ‖

The Lemma is powerful in its implications, for we can now assert

Proposition 6. *Along an optimal policy, if one exists, R_t is a continuous function of time. Furthermore, either $R_t > 0$ for all $t \geqq 0$ or there exists a finite $T(> 0)$ such that $R_t > 0$ for $0 \leqq t < T$ and $R_t = 0$ for $t \geqq T$.*

Proof. The second part (with the exception of the demonstration that $T > 0$) of the proposition is a direct implication of the Lemma, and a simple argument (as in the proof of the Lemma) showing that we cannot have $R_t = 0$ for $0 \leqq t \leqq T$ and $R_t > 0$ for $t > T$. Therefore we begin by proving that R_t must be continuous if $T > 0$. Thus if $R_t > 0$ for all $t \geqq 0$ then the continuity of R_t is trivially true since x_t must satisfy (1.18) for all $t \geqq 0$ and since K_t must be a continuous function. Suppose then that there exists a finite $T(> 0)$ at which there is a phase change from $R > 0$ to $R = 0$. During the interval $[0, T)$ equation (1.18) holds. Since $\dot{x} > 0$ during $[0, T)$, $\operatorname*{lt}_{t \to T^-} x_t$ exists. But it cannot be finite. For suppose it is. By hypothesis $x_T = \infty$. This implies that we would violate the condition necessary at the switch point, namely that

$$\operatorname*{lt}_{t \to T^-} e^{-\delta t} p_t(f(x_t) - x_t f'(x_t)) = e^{-\delta T}\{\mu_T + p_T[f(x_T) - x_T f'(x_T)]\},$$

where $\mu_T \geqq 0$.

It follows that during $[0, T)$, x_t increases continuously to ∞. That is $\operatorname*{lt}_{t \to T^-} x_t = x_T = \infty$. Since $K_t > 0$ for all $t \geqq 0$ this implies that $\operatorname*{lt}_{t \to T^-} R_t = R_T = 0$.

What remains to be proved is that $T > 0$. So suppose on the contrary that the phase change occurs at $t = 0$. That is, assume $T = 0$ (i.e. that R_t is a δ-function). Once again we appeal to the assumption that $\frac{d}{dx}(f(x) - xf'(x)) > 0$ and note that if there is a singularity at $t = 0$ one would violate a condition necessary at this switch point, which is

$$p_0\{f(x_0) - x_0 f'(x_0)\} = \operatorname*{lt}_{t \to 0^+} e^{-\delta t} p_t\{f(x_t) - x_t f'(x_t)\} + \mu_t,$$

where $\mu_t \geqq 0$, $x_0 = 0$ and $\operatorname*{lt}_{t \to 0^+} x_t = \infty$.

We are now in a position to construct a proposition analogous to Proposition 2 of the previous section. Towards this consider an economy for which

$$\operatorname*{lt}_{x \to \infty} (f(x) - xf'(x)) = \gamma < \infty$$

and ...(1.19)

$$F(K, 0) > 0 \text{ for } K > 0.$$

It is then simple to confirm that it cannot be optimal to maintain $R_t > 0$ for all $t \geqq 0$ in an economy that satisfies (1.19). For suppose the contrary. From (1.18) one has $x_t \to \infty$. Furthermore, from the second part of (1.19) it follows that

$$\operatorname*{lt}_{x \to \infty} f(x)/x = \operatorname*{lt}_{x \to \infty} f'(x) = \rho > 0.$$

Therefore, from (1.14) one has that the present value price of consumption ($e^{-\delta t} p_t$) goes to zero, and from the first part of (1.19) the fact that the marginal product of the exhaustible resource is bounded above by γ. But these would contradict the requirement that the economy satisfies (1.12). Thus one has

Proposition 7. *If $F(K, R)$ satisfies* (1.19) *then along an optimal policy, if one exists, $R_t > 0$ for $0 \leqq t < T$ where T is finite and $R_t = 0$ for $t \geqq T$. Moreover R_t is continuous for all $t \geqq 0$.*[1]

1.3. The arguments of the previous sub-section were fairly general, and were concerned with glancing at the conditions that an optimal programme must necessarily satisfy. In particular, we noted the rather direct way in which the elasticity of substitution enters into the characterization of an optimal programme during a phase when $R_t > 0$. In this section we parametrize the economy somewhat further to obtain some definite answers to questions that one would like to ask. One would, for example, like to know under what circumstances an optimal programme exists, and whether the rate of consumption along an optimal policy necessarily falls if the resource is essential and indeed, whether the rate of consumption necessarily rises if the resource is inessential. Furthermore, one

[1] An example of an economy, satisfying (1.19) is one where $f(x) = \rho(1 + x + \phi(x))$ with $\rho > 0$, $\phi(0) = 0$, $\phi'(x) > 0$, $\phi''(x) < 0$ and $\operatorname*{lt}_{x \to \infty} \phi(x) < \infty$. One should note that we have avoided the case $f(x) = \rho(1 + x)$ (i.e. $F(K, R) = (K + R)\ \rho$, $\sigma = \infty$) by our assumption of strict concavity of $F(K, R)$. We shall have occasion to comment on this case in Section 1.4.

would like to know if the optimal utilization rate of the resource is non-increasing over time, as well as conditions under which the capital stock ought to be built up over time.

Towards this we turn to the simplest laboratory that might provide answers, namely the class of production functions for which the elasticity of substitution is constant. It might seem plausible that the class of CES production functions captures the variety of the issues under review. In fact, the CES production functions (with the exception of those for which $\sigma = 1$ and ∞) have rather unusual properties at the " corners ". Consequently some of the characteristics of an optimal policy in a CES world are, at first blush, counter-intuitive. It will be plain that one can establish rather easily conditions under which an optimal programme exists for a much larger environment than the CES world provides. Since the arguments turn out to be rather similar to the ones that we shall provide for the CES case, we do not elaborate on them here.

Recall that the CES production functions is of the form

$$F(K, R) = [\beta K^{(\sigma-1)/\sigma} + (1-\beta)R^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)},$$

where $0 < \beta < 1$ and $\infty \geqq \sigma \geqq 0$.

The analysis of the previous sub-section implied that one is particularly interested in the properties of the production function for $R = 0$. It is convenient, then, to catalogue the following properties:

If $\sigma = 1$ (i.e. the Cobb-Douglas form)

$$F(K, 0) = 0 \qquad \text{...(1.20}a\text{)}$$

$$\operatorname*{lt}_{x\to\infty} f'(x) = \operatorname*{lt}_{x\to\infty} \frac{f(x)}{x} = 0 \qquad \text{...(1.20}b\text{)}$$

$$\operatorname*{lt}_{x\to\infty} (f(x) - xf'(x)) = \operatorname*{lt}_{x\to\infty} f(x) = \infty. \qquad \text{...(1.20}c\text{)}$$

If $0 \leqq \sigma < 1$ then

$$F(K, 0) = 0 \qquad \text{...(1.2}a\text{)}$$

$$\operatorname*{lt}_{x\to\infty} f'(x) = \operatorname*{lt}_{x\to\infty} \frac{f(x)}{x} = 0 \qquad \text{...(1.21}b\text{)}$$

$$\operatorname*{lt}_{x\to\infty} (f(x) - xf'(x)) = \operatorname*{lt}_{x\to\infty} f(x) = (1-\beta)^{\sigma/(\sigma-1)}. \qquad \text{...(1.21}c\text{)}$$

If $\infty > \sigma > 1$ then

$$F(K, 0) = \beta^{\sigma/(\sigma-1)} K \qquad \text{...(1.22}a\text{)}$$

$$\operatorname*{lt}_{x\to\infty} f'(x) = \operatorname*{lt}_{x\to\infty} \frac{f(x)}{x} = \beta^{\sigma/(\sigma-1)} > 0 \qquad \text{...(1.22}b\text{)}$$

$$\operatorname*{lt}_{x\to\infty} (f(x) - xf'(x)) = \operatorname*{lt}_{x\to\infty} f(x) = \infty. \qquad \text{...(1.22}c\text{)}$$

An exhaustible resource is essential to production only when $\sigma \leqq 1$. Otherwise it is inessential. The pair (1.22c) and (1.21c) are curious: an inessential resource is infinitely valuable and an essential resource is finitely valuable at the margin when the rate of utilization of the resource is zero. Only the Cobb-Douglas form may be said to have properties that are reasonable at the corner.

From (1.20c) and (1.22c) and Proposition 5 it is immediate that when $\infty > \sigma \geqq 1$, along an optimal policy, if one exists, $R_t > 0$ for all $t \geqq 0$. We demonstrate now that this must be true for the case $0 < \sigma < 1$ as well. Recall Proposition 6. Assume that there exists a finite $T(>0)$ such that $R_t = 0$ for $t \geqq T$. During the interval $[0, T)$ equation (1.18) must be satisfied. Moreover R_t must be continuous at T. But this is impossible, for the constancy of σ and condition (1.21c) imply that x cannot increase to infinity via equation (1.18) in finite time.[1]

We have yet to demonstrate that an optimal programme exists. To keep the *presentation* simple we restrict ourselves to the case of utility functions that are iso-elastic. That is, assume $\eta(c) = \eta > 0$. Write

$$\rho = \lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{f(x)}{x} \text{ and } S_t = S_0 - \int_0^t R_t d\tau.$$

We can then establish

Proposition 8. *If* $\sigma(\infty > \sigma > 0)$ *is constant,* $\eta(>0)$ *is constant, and* $\delta > \rho(1-\eta)$, *a unique optimal policy exists and it satisfies equations* (1.5), (1.16) *and* (1.18)*for all* $t \geqq 0$. *Moreover, the optimal programme* $(\tilde{C}_t, \tilde{R}_t, \tilde{K}_t, \tilde{x}_t, \tilde{S}_t)$ *satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} \tilde{x} = \infty \qquad \text{...(1.23)}$$

$$\lim_{t\to\infty} \tilde{C} = \lim_{t\to\infty} \tilde{K} = \infty \text{ if } \rho > \delta \qquad \text{...(1.24)}$$

$$\lim_{t\to\infty} \tilde{C} = \lim_{\to\infty} \tilde{K} = 0 \text{ if } \rho < \delta \qquad \text{...(1.25)}$$

$$\lim_{t\to\infty} \frac{\dot{\tilde{C}}}{\tilde{C}} = \lim_{t\to\infty} \frac{\dot{\tilde{K}}}{\tilde{K}} = \frac{\rho-\delta}{\eta} \qquad \text{...(1.26)}$$

$$\lim_{t\to\infty} \frac{\tilde{C}}{\tilde{K}} = \rho + \frac{\delta-\rho}{\eta} \qquad \text{...(1.27)}$$

$$\lim_{t\to\infty} \frac{\dot{\tilde{R}}}{\tilde{R}} = \lim_{t\to\infty} \frac{\dot{\tilde{S}}}{\tilde{S}} = \frac{\rho-\delta}{\eta} - \sigma\rho < 0 \qquad \text{...(1.28)}$$

$$\lim_{\to\infty} \frac{\tilde{R}}{\tilde{S}} = \sigma\rho + \frac{\delta-\rho}{\eta} > 0. \qquad \text{...(1.29)}$$

Proof. It has already been established that the conditions necessary for optimality are that equations (1.16) and (1.18) must hold for all $t \geqq 0$. Condition (1.23) is then immediate from (1.18). From equation (1.18) one recognizes as well that for any pair of paths $x_t^{(1)}$ and $x_t^{(2)}$ one has $x_t^{(1)} > x_t^{(2)}$ for all $t > 0$ if $x_0^{(1)} > x_0^{(2)}$. From (1.16) it follows that $C_0^{(1)} > C_0^{(2)}$ if, and only if, $C_t^{(1)}(C_0^{(1)}, x_0) > C_t^{(2)}(C_0^{(2)}, x_0)$. Now write equation (1.5) as

$$\frac{\dot{K}}{K} = \frac{f(x)}{x} - \frac{C}{K}. \qquad \text{...(1.30)}$$

It follows from (1.30) that $K_t^{(1)}(C_0^{(1)}, x_0) < K_t^{(2)}(C_0^{(2)}, x_0)$ if, and only if, $C_0^{(1)} > C_0^{(2)}$.

Write $y = C/K$, and use equations (1.16) and (1.30) to obtain

$$\frac{\dot{y}}{y} = y - \frac{\delta}{\eta} + \frac{f'(x)}{\eta} - \frac{f(x)}{x}. \qquad \text{...(1.31)}$$

Recall that K_0 and S_0 are given and that we are to choose optimally C_0 (i.e. y_0) and R_0(i.e. x_0). The point is to observe that these choices can, in principle, be separated out

[1] It is now immediate that Proposition 5 can be generalized to

Proposition 5′. *Along an optimal policy, if one exists,* $R_t > 0$ *for all* $t \geqq 0$ *if either* $F(K, R)$ *satisfies* (1.13) *or* $F(K, 0) = 0$.

Since the sufficiency conditions in Propositions 5′ and 7 are not merely mutually exclusive but exhaustive as well, it follows that these two Propositions cover all the cases.

conveniently. We have, of course, assumed $\sigma>0$. Then, using (1.23) it follows from (1.31) that for large t

$$\frac{\dot{y}}{y} \simeq y - \frac{(\delta-\rho)}{\eta} - \rho. \qquad \ldots(1.32)$$

It follows from (1.31) and (1.32) that *given* x_0, if C_0 (i.e. y_0) is chosen large enough, then $y_t(x_0) \to \infty$ as $t \to \infty$. But this entails from (1.30) that the feasibility requirement, $K_t \geqq 0$, is violated in finite time. It follows as well from (1.31) and (1.32) that *given* x_0, if C_0 (i.e. y_0) is chosen positive, but low enough, then $y_t(x_0) \to 0$ as $t \to \infty$.

Thus let $\tilde{C}_0(x_0) = \sup [C_0 \mid y_t(C_0, x_0) \geqq 0 \text{ for } t \geqq 0]$. From (1.32) it is immediate that

$$\operatorname*{lt}_{t\to\infty} y_t(\tilde{C}_0(x_0), x_0) = \rho + (\delta-\rho)/\eta > 0,$$

confirming (1.27). From (1.16) and (1.30) condition (1.26) follows directly.

One can now verify that

$$\operatorname*{lt}_{t\to\infty} \frac{d(e^{-\delta t} p_t \tilde{K}_t)/dt}{e^{-\delta t} p_t \tilde{K}_t} = \frac{\rho-\delta}{\eta} - \rho < 0, \text{ and thus that } \operatorname*{lt}_{t\to\infty} e^{-\delta t} p_t K_t = 0. \qquad \ldots(1.33)$$

But $\dot{R}/R = \dot{K}/K - \dot{x}/x$. On using (1.26) and (1.18) it is immediate that

$$\operatorname*{lt}_{t\to\infty} \frac{\dot{R}}{R} = \frac{\rho-\delta}{\eta} - \sigma\rho < 0. \qquad \ldots(1.34)$$

It follows from (1.34) that $S(x_0) \equiv \int_0^\infty R_t(\tilde{C}_0(x_0), x_0)dt$ is well defined. It follows as well that we can choose x_0 large enough (i.e. R_0 small enough) so that $S(x_0) < S_0$, and x_0 small enough (i.e. R_0 large enough), so that $S(x_0) > S_0$. The former would be inefficient and the latter choice unfeasible. Thus write

$$\tilde{x}_0 \equiv \sup \{x_0 \mid S(x_0) \leqq S_0\}.$$

Then

$$S_t = \int_t^\infty \tilde{R}_\tau(\tilde{C}_0(\tilde{x}_0), \tilde{x}_0)d\tau \text{ and } \operatorname*{lt}_{t\to\infty} \tilde{S}_t = 0.$$

But $\dot{S}/S = -R/S$. Hence

$$\operatorname*{lt}_{t\to\infty} \dot{\tilde{S}}/\tilde{S} = \operatorname*{lt}_{t\to\infty} -\tilde{R}/\tilde{S} = \operatorname*{lt}_{t\to\infty} \dot{\tilde{R}}/\tilde{R} = (\rho-\delta)\eta - \sigma\rho < 0,$$

and conditions (1.28) and (1.29) are confirmed. Plainly

$$\operatorname*{lt}_{t\to\infty} \lambda \tilde{S}_t = 0. \qquad \ldots(1.35)$$

Conditions (1.24) and (1.25) are implied immediately by (1.26).

Finally, the Hamiltonian (1.7) is strictly concave. The two transversality conditions (1.33) and (1.35) have been verified. The proposition is thus established. ||.

Having established the main proposition of this section we present some specific commentaries in the following sub-section.

1.4. The case $0<\sigma<1$ is at once the simplest and the most pessimistic of all.[1] As total output is bounded if $\sigma<1$, it is plain that *feasibility* dictates that $C_t \to 0$ as $t \to \infty$. Since $\rho = 0$ it is clear that an optimum exists if $\delta>0$. This is so, even if $U(\cdot)$ is unbounded above (i.e. $0<\eta<1$). Given (1.23), the shadow price of the exhaustible resource relative to the composite is monotonically increasing. Given (1.21*c*) in addition, it is clear that

[1] We have, of course, assumed away the case $\sigma = 0$ by our assumptions on F. It is not a case that presents any special problems of analysis but as its analysis calls for some extra notation, we do not discuss it here.

this shadow price tends in the long run to the value $(1-\rho)^{\sigma/(\sigma-1)}$ (see equation (1.12)). This is mildly surprising, given that the resource is essential.

The Cobb-Douglas case is particularly interesting since the analysis can relatively easily be taken further. To begin with, since $\rho = 0$, an optimum exists if $\delta > 0$. Furthermore, given (1.20*c*) and (1.23), the price of the exhaustible resource relative to fixed capital is increasing and tends to infinity with time. Let $f(x) = x^{\alpha}$ with $0<\alpha<1$. Professor Solow [25] has demonstrated that consumption can be unbounded if $\alpha > 1/2$, but that feasible consumption must tend to zero if $\alpha < 1/2$. But from (1.25) we have that for all α, $0<\alpha<1$, $\tilde{C}_t \to 0$.

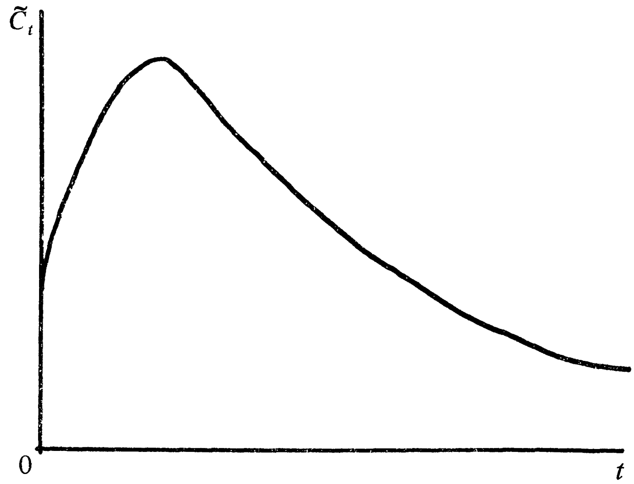

FIGURE 3

Now integrate (1.18) to obtain

$$\tilde{x}_t = [(1-\alpha)t + x_0^{(1-\alpha)}]^{1/(1-\alpha)}, \qquad \ldots(1.36)$$

and thus that in the long run the price of the exhaustible resource relative to the composite commodity should be approximately

$$(1-\alpha)^{1/(1-\alpha)} t^{\alpha/(1-\alpha)}.$$

It is also the case that in the long run the percentage rate of change in the rate of depletion of the exhaustible is approximately equal to $-\delta/\eta$ (see (1.28)) and, in particular, the rate of depletion, as a fraction of the then existing stock of the resource, is δ/η (see (1.29)).

Using (1.36) in equation (1.16) and integrating, the optimal consumption profile is

$$\tilde{C}_t = \tilde{C}_0 \tilde{x}_0^{-\alpha/\eta}[(1-\alpha)t + \tilde{x}_0^{(1-\alpha)}]^{\alpha/\eta(1-\alpha)} e^{(-\delta/\eta)t}. \qquad (\ldots 1.37)$$

From (1.37) it would appear that if the initial stocks K_0 and S_0 are " large " the optimal rate of consumption will, during an initial period, be rising, and will in fact be single peaked, with a maximum at T where [1]

$$T = \frac{\alpha}{\delta(1-\alpha)} - \frac{\tilde{x}_0^{\,1-\alpha}}{(1-\alpha)}.$$

A typical consumption profile is presented in Figure 3. It is routine to show that

$$\dot{\tilde{R}}_t = \tilde{C}_0 \tilde{x}_0^{-\alpha/\eta}[(1-\alpha)t + \tilde{x}_0^{1-\alpha}]^{\alpha/\eta} e^{(-\delta/\eta)t},$$

[1] It should, of course, be evident that T is not independent of η, since $\tilde{X}_0$ is dependent on η. Professor Solow ([25], appendix) considers the case where $\alpha > 1/2$ and $\delta = 0$. This last ensures that our existence theorem is not applicable to his case. He shows, in effect, that if η is large enough, then $T = \infty$. For this limiting case of $\delta = 0$ he assumes, therefore, that $\alpha > 1/2$ to ensure that consumption can be unbounded and large enough η to ensure the existence of an optimal policy.

from which

$$\text{sign}\,(\ddot{\tilde{R}}_t) = \text{sign}\left[\delta - \frac{\alpha(1-\alpha)}{(1-\alpha)t+\tilde{x}_0^{1-\alpha}}\right].$$

It follows that while $\dot{\tilde{R}}<0$ for all $t \geqq 0$ and $\ddot{\tilde{R}}_t>0$ for large enough t, one would have $\ddot{\tilde{R}}_t<0$ for an initial period for large values of K_0 and S_0, yielding a time profile of resource utilization as in Figure 4.

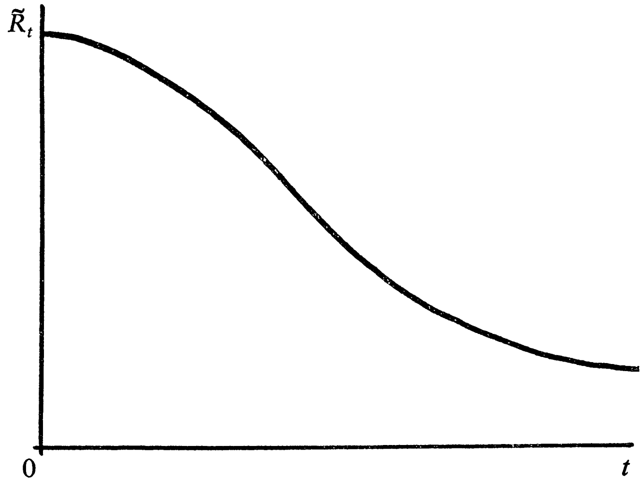

FIGURE 4

The case $\sigma>1$ is one in which exhaustible resources are not a fundamental problem, since they are inessential in production. Given (1.23) the shadow price of the exhaustible resource relative to fixed capital is monotonically increasing, but given (1.22*c*) one does have the surprising result that this price tends to infinity with time. Since $\rho>0$ it is plain that $\delta>0$ is not sufficient for the existence of an optimum policy. But if $U(\cdot)$ is bounded above (i.e. $\eta>1$) an optimum exists for $\delta \geqq 0$. As conditions (1.24) and (1.25) make clear, whether or not it is optimal to have a growing economy in the long run depends on whether ρ is greater than, or less than δ, a condition that is intuitively plain.

The case, $\sigma = \infty$, though silly, merits a brief glance, though we have so far assumed it away. Assume then that $f(x) = \rho(1+x)$, where $\rho>0$. It satisfies (1.19). But since f is linear, Proposition 7 is not strictly operative. Nevertheless, it is an easy matter to confirm that if $\delta>(1-\eta)\rho$ an optimal programme exists and that the optimal depletion rate, $\tilde{R}_t$, is a Dirac-δ-function with the singularity at $t = 0$. The point is that since the marginal product of the resource is constant, it does not pay to leave the resource lying idle. As we have imposed no costs of extraction one can, in principle, use up the entire initial stock, S_0, instantaneously to build up the capital stock, which is, in fact, the optimal strategy.[1]

2. TECHNICAL CHANGE AND UNCERTAINTY

2.1. Technological progress is, as usual, hard to model. We are concerned now with those situations where technical change causes a resource that was previously essential to become inessential. In practice this is most likely to be due to the discovery of a synthetic substitute (for example, being able to harness solar energy or nuclear fusion), and it is this type of alteration in the technological possibilities that provides the motivation

[1] The reader can easily check that if $\delta>(1-\eta)\rho$ then a unique optimum exists if the production function $f(\cdot)$ is of the form given in the footnote to Proposition 7. Given Proposition 7, the optimal depletion function, $\tilde{R}_t$, is continuous for this case, but the entire stock is exhausted in *finite* time.

for the discussion that follows. In such situations, although one may not know in advance exactly what the substitute will be like, there will probably be general information about its likely characteristics. Industrial research directed towards innovation is, after all, not carried out aimlessly. A more important source of uncertainty seems to be the exact *timing* of the availability of the substitute. It would seem reasonable to argue that we are much more uncertain about when, if ever, such an event will occur, than we are about the characteristics it will have if it does occur, and we shall focus attention on this aspect of the problem. Consequently we suppose that we know exactly the nature of the technical change that will occur, but we treat the date at which the event occurs as a random variable. In this paper, however, we suppose that the distribution of the random variable is exogenously given.

The basic framework that we shall present appears to us to be fairly robust in that it can accommodate quite diverse characterizations of the technological breakthrough. Furthermore, it would seem that one could, in principle, extend the analysis quite directly to allow for the possibility of a *finite* sequence of such events in the future. But it seems most interesting to conceive of the event as being the discovery of a substitute for an exhaustible resource which releases the economy from the resource constraints. Moreover, for the sake of simplicity we take it that only one such event is to be considered. Towards this we amalgamate the production model considered in Sections 1.1 and 1.2 in a sequential manner.

Assume that the economy is endowed initially with a stock, K_0, of the composite consumption good, and a stock, S_0, of an exhaustible resource. Production possibilities until the date of the technological breakthrough are represented, as in Section 1.2, by the production function $F(K_t, R_t)$, where, as before, R_t is the flow of resources into production at t. Suppose the technological breakthrough occurs at date $T>0$. Then the constraints that limit the choice of consumption till T can be denoted by the conditions

$$\dot{K}_t = F(K_t, R_t) - C_t,\ 0 \leqq t \leqq T; \int_0^T R_t dt \leqq S_0 \text{ and } K_0 \text{ and } S_0 \text{ are given.} \qquad \ldots(2.1)$$

We characterize the technological innovation as being the discovery of a perfectly durable commodity which provides a flow of service at the constant rate M. But it would seem hard to think that the production possibilities after the event would remain the same as those before. It is here that we revert to the economy discussed in Section 1.1, and suppose that the production function after the innovation can be regarded as being $G(K_t, Z_t)$, where Z_t is the rate of utilization of the service at date t. Now, there is of course nothing to suggest that the service provided by the durable commodity is a perfect substitute for the exhaustible resource in the production function, G. In fact there is every reason to suppose that it is not. Nevertheless, for the moment, to keep the presentation simple we suppose that they are perfect substitutes. It follows that the constraints that limit the choice of a consumption profile for the period *beyond* T are represented by the conditions

$$\left.\begin{array}{l} \dot{K} = G(K, Z) - C \\ \dot{V} = M - Z \\ \text{where } Z_t, K_t, C_t, V_t \geqq 0 \text{ and } V_T = S_0 - \displaystyle\int_0^T R_t dt \\ \text{and } K_T \text{ given} \end{array}\right\}. \qquad \ldots(2.2)$$

The date T, is assumed unknown. In fact suppose that T is random number with a probability density function ω_t. Thus we suppose that

$$\omega_t > 0 \text{ and } \int_0^\infty \omega_t dt = 1. \qquad \ldots(2.3)$$

The second condition in (2.3) supposes that the substitute source will certainly be discovered *sometime* in the future. One might wish to allow for the possibility of it *never* being discovered. Nothing substantial seems to emerge by supposing so. Therefore we suppose (2.3).[1]

We take it that the valuation function for planning is $E\int_0^\infty e^{-\delta t}U(C)dt$, where E is the expectation operator. An important question then arises as to what penalty one would wish to attach to depleting completely the exhaustible resource *before* the substitute is discovered. It would seem most interesting to formulate the problem in such a way as to make the penalty infinite. Thus we impose the restriction that along a feasible strategy the probability of being caught short with none of the exhaustible resource before the substitute is discovered, is nil.

Define $W(K_T, V_T)$ = maximum value of $\int_T^\infty e^{-\delta(t-T)}U(C_t)dt$ subject to (2.2). It follows that

$$E\int_0^\infty e^{-\delta t}U(C)dt = \int_0^\infty \omega_T\left\{\int_0^T e^{-\delta t}U(C)dt + W(K_T,\, V_T)e^{-\delta t}\right\}dT. \quad \ldots(2.4)$$

Write $\Omega_t = \int_t^\infty \omega_\tau d\tau$, and integrate the RHS of (2.4) by parts to obtain

$$E\int_0^\infty e^{-\delta t}U(C)dt = \int_0^\infty e^{-\delta t}\{U(C)\Omega_t + \omega_t W(K_t,\, V_t)\}dt. \quad \ldots(2.5)$$

The problem then reduces to one of

$$\left.\begin{array}{l} \text{maximizing expression (2.5) subject to the constraints} \\ \dot{K} = F(K,\, R) - C \\ \dot{S} = -R \\ \text{where } V_t = S_t \text{ and } K_t,\, C_t,\, R_t,\, S_t \geqq 0 \end{array}\right\}. \quad \ldots(2.6)$$

It is as well to point out that the optimal policy is to pursue the solution path of problem (2.6) only until the substitute is discovered. At the date the substitute is discovered there is a switch in regime and the optimal policy from then on is to follow the policy which yields $W(K_T,\, V_T)$.[2]

In order to analyse (2.6) it is convenient to express the Hamiltonian of the system as

$$H = e^{-\delta t}\{U(C)\Omega_t + \omega_t W(K_t,\, V_t)\} + e^{-\delta t}p_t\{F(K,\, R) - C\} - e^{-\delta t}q_t R_t + e^{-\delta t}\pi_t R_t + e^{-\delta t}\gamma_t S_t, \quad \ldots(2.7)$$

where
$$\left.\begin{array}{ll} & \pi_t \geqq 0 \text{ and } \pi_t R_t = 0 \\ \text{and} & \gamma_t \geqq 0 \text{ and } \gamma_t S_t = 0 \end{array}\right\}, \quad \ldots(2.8)$$

We shall assume $W(K_t, V_t)$ to be bounded and differentiable. The former can be justified if we assume that $U(\cdot)$ is bounded above, and we shall assume that this is so. The concavity of W follows from assumptions that we have already made. From (2.1) it is plain that $e^{-\delta t}p_t$ is the present value price of the composite consumption good and that $e^{-\delta t}q_t$ is the present value price of the exhaustible resource. One notes as well from (2.8) that $\gamma_t > 0$ implies that $\pi_t > 0$.

[1] For simplicity of exposition we shall take it, in fact, that $\omega_t > 0$ for $t \geqq 0$, though this is not at all essential.

[2] The solution to (2.6) is thus a conditional strategy, conditional upon the substitute not having been discovered.

For a programme to be optimal it is then necessary that it satisfies the following conditions

$$p = \Omega_t U'(C) \qquad \text{...(2.9)}$$

$$pF_R = q - \pi_t \qquad \text{...(2.10)}$$

$$\dot{p} = \delta p - \omega_t W_K - pF_K \qquad \text{...(2.11)}$$

and

$$\dot{q} = \delta q - \omega_t W_V - \gamma_t, \qquad \text{...(2.12)}$$

where

$$W_K = \frac{\partial W}{\partial K}, \text{ etc.}$$

Now $W_V \geqq 0$ and $W_K \geqq 0$. It follows from (2.10) to (2.12) that if F satisfies (1.13) then we shall have $R_t > 0$ for all $t \geqq 0$ along an optimal policy. We suppose, therefore, that this is so. Consequently $\gamma_t = \pi_t = 0$ for all $t \geqq 0$.

Write $\Psi_t = \omega_t/\Omega_t$, for the conditional probability of the substitute being discovered at t given that it has not been discovered earlier. From equations (2.9) and (2.11) one has [1]

$$\frac{\dot{C}}{C} = \frac{F_K - \delta + \Psi_t\{(W_K - U'(C))/U'(C)\}}{\eta(C)}. \qquad \text{...(2.13)}$$

Differentiate (2.10) with respect to time and use equations (2.12), (2.13) and the production constraint in (2.6) to obtain

$$\dot{x} = \sigma f(x)\left\{1 + \frac{W_K \Psi_t}{U'(C)f'(x)}\right\} + \frac{W_V \Psi_t}{xf''(x)U'(C)}, \qquad \text{...(2.14)}$$

where x, σ and $f(\cdot)$ are as in Section 1.2.[2] Equations (2.13) and (2.14) govern the solution to the problem expressed in (2.6). The nature of the path they define is far from obvious except in some special cases. We turn to these. Now the technology G subsequent to the technological breakthrough is likely to be considerably different from that prior to T. The simplification that we introduce lies in the supposition that at T (the date of the discovery), the then existing stocks of capital and the exhaustible resource come to have no economic value. This is, of course, a terribly strong assumption, but not totally unreasonable. It seems, for example, very likely that if fusion reactors ever become a commercial proposition, turning water into an abundantly available substitute for fossil fuel, then power stations generating electricity from fossil fuels will be rapidly phased out. Both the capital equipment that they comprise and the remaining fossil fuel stocks will have little economic value as sources of energy, although of course fuels would have value as sources of organic chemicals. This assumption implies

$$W_K = W_V = 0. \qquad \text{...(2.15)}$$

Given (2.15) we shall certainly have to suppose that $G(0, Z) > 0$ for $Z > 0$ (see condition (1.1a')). Using (2.15) we note that equations (2.13) and (2.14). reduce to the forms

$$\frac{\dot{C}}{C} = \frac{f'(x) - (\delta + \Psi)}{\eta(C)} \qquad \text{...(2.16)}$$

and

$$\dot{x} = \sigma f(x). \qquad \text{...(2.17)}$$

Assume that the utility function is iso-elastic. Having already assumed that $U(\cdot)$ is bounded above, this implies that $\eta > 1$. But the conditions (2.16) and (2.17) which an

1 As one would expect, equation (2.13) reduces to equation (1.16) when $\Psi_t = 0$.
2 As would be expected, (2.14) reduces to (1.18) if $\Psi_t = 0$.

optimal policy must satisfy differ from (1.16) and (1.24) only by the addition of an amount Ψ_t to the discount rate. Thus the optimal policy is to suppose that the substitute will *never* be discovered and to pursue the solution described in Section 1.4 with the single proviso that it be assumed that the utility rate of discount is not merely δ but $\delta+\Psi_t$. Since $\Psi_t>0$, one is in effect discounting at a higher rate in this uncertain problem than in the problem of Section 1.3. This is intuitively very reasonable. If, at T, the substitute appears, the economy switches regime and the optimal policy is to pursue the programme

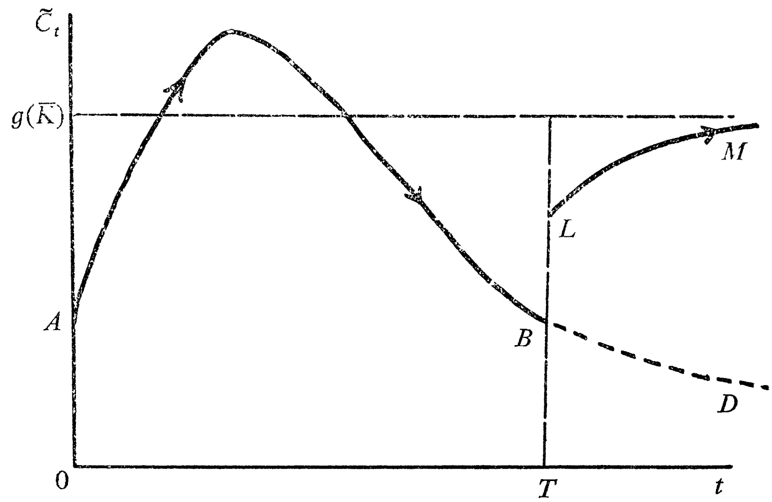

FIGURE 5

described in Proposition 4′. In fact Proposition 4′ is precisely what is required since assumption (2.15) entails a total break with the past at T. At T the economy is totally new, with no existing capital stock and no inventory of the resource. The break with the past is complete.[1] In Figure 5 we present the optimal consumption profile for this economy. *ABD* describes in broad terms the consumption profile of Figure 3. Throughout, however, the rate of discount is higher than δ by an amount Ψ_t. *LM* is the optimal consumption policy described in Proposition 4′.[2] We have then

Proposition 9. *Consider the two following problems:*

$$\left.\begin{array}{ll} \textit{maximize} & \displaystyle\int_0^\infty [U(C)\Omega_t+\omega_t W(K_t, S_t)]e^{-\delta t}dt \\ \textit{subject to} & \dot{K}=F(K, R)-C \\ & \dot{S}=-R \\ & C_t, K_t, R_t, S_t \geqq 0 \textit{ and } K_0, S_0 \textit{ given} \end{array}\right\} \quad \ldots(2.18)$$

and

$$\left.\begin{array}{ll} \textit{maximize} & \displaystyle\int_0^\infty e^{-\gamma_t t}U(C)dt \\ \textit{subject to the same constraints as in (2.18),} & \end{array}\right\} \quad \ldots(2.19)$$

where γ is a discount rate which is independent of the path followed by the economy.

[1] In effect, then, condition (2.15) reduces the model to one where the economy faces an uncertain terminal date, a case that has been explored in another context by Yaari [28].

[2] As would be expected, there is generally a discontinuity in consumption at T. By assumption (2.15) the height TL is independent of the value taken by T.

Then, necessary and sufficient conditions for Problems (2.18). *and* (2.19) *to have identical solutions are*

(i) $W_K = W_S = 0$ *for all* K, S.

(ii) $\gamma_t = \delta t + \int_0^t \Psi_\tau d\tau$.

Proof. Sufficiency is obvious. To establish necessity note that the conditions that a solution of problem (2.19) must satisfy are

$$\frac{\dot{C}}{C} = \frac{(F_K - \gamma)}{\eta(C)} \qquad \text{...(2.20)}$$

and that

$$\dot{x} = \sigma f(x). \qquad \text{...(2.21)}$$

It follows that if equation (2.13) is identical to equation (2.20) we must have $W_K = 0$. But if (2.14) is identical to (2.21) we must have

$$\frac{\sigma f(x) W_K \Psi}{U'(C) f'(x)} + \frac{W_V \Psi}{x f''(x) U'(C)} = 0.$$

But since $W_K = 0$ it follows that $W_V = 0$. ||

What is interesting about Proposition 9 is that it gives a precise statement of a set of conditions under which we can allow for uncertainty merely by solving a certain problem with a suitably chosen discount factor depending only on the nature of the uncertainty (a kind of certainty equivalent result), *and* where the equivalent certain problem is independent of the expected supply of the substitute. The amount Ψ_t that has to be added to the utility discount rate, δ, is of course generally time variant and equals the conditional probability of the substitute arriving at t, given no previous occurrence. We need hardly point out that non-constancy of Ψ_t, does *not* imply that the optimal policy is intertemporally inconsistent. In general one suspects that the conditional probability distribution is complex, but possibly single peaked. For example, it has been suggested that the conditional probability of harnessing nuclear fusion will rise to a peak at about the end of this century but will then fall, the argument being that expectations, though rising over the near future, will diminish if the discovery is not made by the turn of the century. A conditional distribution having such attributes would result if ω_t were log normally distributed. For the simple case where $\omega_t = \pi e^{-\pi t}$ (i.e. ω_t, is a Poisson distribution) Ψ_t is constant and equal to π and the planning problem becomes still easier.

Proposition 9 yielded a sufficient *and* necessary set of conditions under which the optimal depletion problem could be solved by pretending that the substitute will never appear and solving for the optimal depletion problem by a simple increase in the utility discount rate. Raising the utility discount rate to handle uncertainty has much intuitive appeal. Proposition 9 implies that an error will be committed in doing this if the conditions do not hold. It is then, naturally interesting to ask whether we have any reasons for believing that the error is as likely to go one way as the other. In fact, it is possible to establish the following proposition about the nature of the errors introduced if the " certainty equivalent " problem is used as a guide to decision-making when it should not be.

Proposition 10. *If we suppose* $W_K = W_V = 0$ *and solve the certainty equivalent problem, when in fact* $W_K = 0$ *but* $W_V > 0$, *then the values of* R_0 *and* C_0 *thus chosen will be greater than or equal to their correct values, and at least one will be strictly greater.*

2.3. The results established in the previous sub-section may seem to be dependent upon some of the unpalatable simplifying assumptions used there—a one-sector economy, a homogeneous capital stock, etc. The point of this section is to observe that in fact similar results hold in a far more general situation. In order to see this point, it is helpful to consider an economy whose evolution is characterized by three functions of time, $\underline{C}$, $\underline{K}$ and $\underline{S}$, which are respectively mappings to R^1, R^m and R^n: $\underline{C}_t$ is the consumption vector at t; $\underline{K}_t$ the vector of capital stocks, and $\underline{S}_t$ the vector of remaining resource stocks. Over any finite period $[0, T]$, and given the initial values $\underline{K}_0$ and $\underline{S}_0$, there exists a set $\mathscr{E}_T$ of triples $(\underline{C}_t, \underline{K}_t, \underline{S}_t)$ such that a triple is in $\mathscr{E}_T$ if and only if it describes an evolution of the economy from 0 to T which is feasible given the technology and initial conditions. Associated with each such path over $[0, T]$ is another triple $(\underline{C}_T, \underline{K}_T, \underline{S}_T)$ giving the terminal values of consumption, stocks of capital and resources. For such an economy, we consider the following finite-horizon problem:

$$\text{maximize} \qquad \int_0^T U(\underline{C})e^{-\delta t} + e^{-\delta T}W(\underline{K}_T, \underline{S}_T)$$

$$\text{subject to} \qquad (\underline{C}_t, \underline{K}_t, \underline{S}_t) \in \mathscr{E}_T.$$

This involves maximizing a utility integral and a terminal stock valuation function, subject to the usual technological constraints. It is assumed that a solution to this problem exists for each T.

Now suppose that the horizon T is a random variable with density function ω_T, and that planners seek a path of the economy which will maximize the expected value of the utility integral plus stock valuation function over this random horizon—i.e. they seek to

$$\text{maximize} \qquad \int_0^\infty \omega_T \left\{ \int_0^T U(\underline{C})e^{-\delta t}dt + e^{-\delta T}W(\underline{K}_T, \underline{S}_T) \right\} dT.$$

To make this a well-defined maximization problem, it is necessary to specify the operative constraints. A reasonable formulation would be to require the programme adopted to be feasible whatever the horizon should turn out to be: if $\omega_T > 0$ for all t, this gives as a constraint

$$(\underline{C}_T, \underline{K}_T, \underline{S}_T) \in \mathscr{E}_\infty$$

where $\mathscr{E}_\infty$ is the set of programmes feasible *ad infinitum*, given the technology and initial conditions. We can now prove the following simple but useful result:

Proposition 11. *The solution to the problem.*

$$\textit{Maximize} \qquad \int_0^\infty \omega_T \left\{ \int_0^T U(C)e^{-\delta t}dt + e^{-\delta T}W(K_T, S_T) \right\} dT$$

$$\textit{subject to} \qquad (C_t, K_t, S_t) \in \mathscr{E}_\infty$$

is identical to the solution to

$$\textit{maximize} \qquad \int_0^\infty U(C)e^{-\delta t - \int_0^t \Psi_\tau d\tau} dt + \int_0^\infty \omega_t e^{-\delta t}W(K_t, S_t)dt$$

subject to the same constraint, where

$$\Psi_\tau = \omega_\tau \Big/ \int_\tau^\infty \omega_t dt.$$

Proof. Consider the term

$$\int_0^\infty \omega_T \left(\int_0^T U(C)e^{-\delta t}dt \right) dT.$$

Defining $\Omega_t = \int_t^\infty \omega_\tau d\tau$ and integrating by parts, this equals

$$\int_0^\infty U(C)\Omega_t e^{-\delta t} dt.$$

Now $\dot{\Omega}_t = -\omega_t$ so that $\dot{\Omega}_t/\Omega_t = -\omega_t/\Omega_t = -\Psi_t$. Hence on integration

$$\Omega_t = e^{-\int_0^t \Psi_\tau d\tau}. \|$$

An immediate corollary of this is the following

Proposition 12. *Consider the problems*

maximize $\int_0^\infty \omega_T \left\{ \int_0^T U(C)e^{-\delta t}dt + e^{-\delta T} W(K_T, S_T) \right\} dT$

subject to $(C_t, K_t, S_t) \in \mathscr{E}_\infty$ *and*

maximize $\int_0^\infty U(C)e^{-\alpha_t}dt$ *subject to the same constraint.*

Then a sufficient condition for these two problems to have the same solution is that

(i) *$W(K_T, S_T)$ should be independent of K_T and S_T for all values of these variables, and*

(ii) $\alpha_t = \delta t + \int_0^t \Psi_\tau d\tau$.

Proposition 12 is clearly a generalisation of Proposition 9, in that it states a set of conditions under which it is possible to replace an uncertain problem (which is clearly a generalisation of the problem of the previous sub-section), by a certain problem derived from it by a modification of the discount rate that makes use solely of the probability function.[1]

3. CONCLUDING REMARKS

In this paper we have attempted to answer some questions that appear to arise rather naturally when one thinks of intertemporal planning in the presence of exhaustible resources. Since many of these questions are, in fact, independent of one's exact notion of intergenerational equity we have posed them within the context of the, by now standard, utilitarian framework.[2] In Section 1.2 we presented a simple production model with no uncertainty. While it is plain that the characteristics of an optimal *depletion policy* depend crucially on whether or not the resource is essential to the production of final goods, it is not *a priori* plain as to what constitutes essentiality. The analysis led rather naturally to a simple articulation of this notion, namely the sufficiency conditions in Proposition 5 (see footnote 1, p. 15): that is, that *either* the marginal product of the resource is unbounded *or* that ouput of final goods is nil in the absence of the resource. It is intuitively clear that the elasticity of substitution between reproducible capital and exhaustible resources is an important determinant of the characteristics of an optimal

[1] The " certainty equivalent " problem that we have discussed here should not be confused with the problem of determining the optimal depletion rate where it is supposed that the substitute *will* become available at its *expected date* of arrival. That the latter problem may, in some case, yield a bias towards excessive depletion rates is intuitively clear, and it is this issue that is discussed by Henry [13]. In spite of its shortcomings the certainty equivalent approach (of either variety) is adopted explicitly or implicitly in much of the more empirically oriented literature in this type of problem. On this see Posner [24].

[2] Dasgupta [6] and Solow [25] have explored some of these questions within the context of alternative notions of intergenerational equity.

policy. It turned out that the parameter enters in a natural and in fact strikingly simple way in the intertemporal plan (viz. equation (1.18)). One would judge that for the near future at least, one will have to resort to simple production models (such as, say, the class of CES functions) to obtain numerical results. It is partly for this reason (and partly also for pedagogic ones) that we have presented our existence theorem (Proposition 8) for a world that has a production function that is of the CES form, and a utility function that is iso-elastic. It is in fact relatively simple to extend Proposition 8 to cover the case of production functions that are not of the CES variety as well as utility functions that are not iso-elastic. We need hardly point out that one would need to make appropriate assumptions (such as (0.2)) about the behaviour of the elasticity of marginal utility for low and high rates of consumption and that it is these asymptotic values of the elasticity of marginal utility that would enter into the condition that guarantees the existence of an optimal policy, namely that $\delta > \rho\ (1-\eta)$. If, in particular $\delta > \rho$, then we would be interested in the value of the elasticity of marginal utility as consumption tends to zero. Likewise, if $\rho > \delta$ then we would be interested in the value of the elasticity of marginal utility as consumption tends to infinity.

The main novelty that exhaustible resources introduce in a planning exercise is that one has to be particularly conscious about the properties of production functions at the " corners ". The banality of this observation is matched only by the problems this poses in obtaining empirical estimates. Certainly it is possible that we live in a world where for " moderate " values of the capital-resource ratio, the elasticity of substitution between capital and the resource exceeds one.[1] The point of concern, of course, is its behaviour for large values of the capital-resource ratio, given that large values cannot be avoided in the long run.[2] For the purposes of planning one is particularly interested in the conditions under which it is optimal to spread the exhaustible resource thinly over the distant future and thereby never to exhaust it completely. In a *very* general sense the analysis of Section 1.2 has been re-assuring to intutition in that the conditions under which it is optimal to exhaust the resource in finite time are really rather stringent (see Proposition 7). Certainly, if we wish to rely on the working hypothesis that the elasticity of substitution is constant then irrespective of its value (except for the case where it is infinite, which is, of course, just silly) we should not exhaust the resource in finite time. But the analysis does suggest that if $\sigma \geqq 1$ but σ is not unduly large, the price of the exhaustible resource relative to output ought to be rising rather rapidly. Experience does not suggest that this has been the case.[3]

In Section 2 we presented a model incorporating technical progress under uncertainty. It seems entirely reasonable to suppose that the important uncertainty concerns the date of arrival of new knowledge. Furthermore, it seemed to us that for the kind of problem that we have in mind here, it would not be very appropriate to make use of the law of large numbers (as say, would be implicitly the case for models such as those of Arrow [2] and Kaldor and Mirrlees [16]) to generate technical progress in a continuous fashion. Rather, we are here trying to envisage it as coming in a discrete manner. The event (or perhaps we should say " Event "!) can, of course, be quite anything. It can, for instance, be the discovery of a new stock of the same resource (e.g. new beds of oil or natural gas). But in this event exhaustible resources would remain a fundamental problem if they were initially so. Certainly the discussions that are currently conducted amongst energy experts suggest that what they have in mind is the possibility of discovering a source that will provide, for all practical purposes, an unlimited flow of energy. It is for such reasons that we have explored the problems that arise if technological change consists in the economy being provided by a steady stream of an alternative resource which can replace

1 The important work by Nordhaus and Tobin [22] suggests that this may be true.
2 Either this or one runs down capital sufficiently rapidly as well, in which event consumption trivially goes to zero in the absence of technical progress.
3 For an analysis of the behaviour of oil prices in particular, see Heal [12].

a resource that was previously essential. The date of the discovery we have supposed uncertain, but in this paper we have supposed that the random variable is uninfluenced by policy; that is, that the acquisition of knowledge is costless.

While it appears to be customary in practice to cope with uncertainty by raising the rate of discount by an amount that merely reflects the uncertainty, it is generally recognized that it leads to errors. In Section 2.2 we explored the conditions under which no such error will in fact be committed. It emerged that this is so only when both the existing stock of capital and the stock of the exhaustible resource is judged totally devoid of value at the date of the technical change (Proposition 9). It appeared, not surprisingly perhaps, that in this instance the model bears a strong resemblance to one where the economy faces an uncertain terminal date; a case that has been explored at some length by Yaari [28]. That one will, in general, commit an error by this simple method of coping with uncertainty is not surprising. But it is not plain that there will be a bias. In fact, one can say something specific about such biases, and we have presented the result in the form of Proposition 10.

Finally, and at a more primitive level, while one can take the view (reflected in the model economy of Section 2) that even though exhaustible resources may be essential currently they will not remain so over the indefinite future, it is not plain that it is a particularly comforting view. The model of uncertainty that we have presented here certainly articulates the view that the probability of the new source being discovered *sometime* over the indefinite future is precisely one. But it may well be a long while coming. If in fact major discoveries occur only at a great distance in the future, the intervening generations will naturally be that much worse off. In particular, one can argue that one's conception of intergenerational justice ought to be influenced by the likelihood of the arrival of technological change. This may well be an appropriate view. In this paper we have not attempted to face this set of difficult questions. Our purpose has been to explore some of the immediate implications of the existence of exhaustible resources.

REFERENCES

[1] Anderson, K. P. " Optimal Growth when the Stock of Resources is Finite and Depletable ", *Journal of Economic Theory* (1972).

[2] Arrow, K. J. " The Economic Implications of Learning by Doing ", *Review of Economic Studies* (1962).

[3] Beckerman, W. " Economists, Scientists and Environmental Catastrophe ", *Oxford Economic Papers* (1972).

[4] Cole, H. S. D. *et al. Thinking about the Future: A Critique of the Limits to Growth* (Chatto and Windus, for Sussex University Press, 1973).

[5] Dasgupta, P. " On the Concept of Optimum Population ", *Review of Economic Studies* (1969).

[6] Dasgupta, P. " Some Alternative Criteria for Justice Between Generations ", Paper presented at the ISPE Conference on the Demand for Public Goods, Siena (1973).

[7] Dasgupta, P. " Some Theoretical Explorations in the Economics of Exhaustible Resources ", Paper presented at the Bayerisch Academy Symposium on Systems Analysis, Reisenburg (1973).

[8] Forrester, J. W. *World Dynamics* (Wright-Allen Press Inc., Cambridge, Mass., 1971).

[9] Gale, D. " Optimal Development in a Multi-Sector Economy ", *Review of Economic Studies* (1967).

[10] Heal, G. M. " The Depletion of Exhaustible Resources: Some Theoretical Issues ", in M. Parkin and R. Nobay (eds.) *Issues in Contemporary Economics*, (Manchester University Press, 1974).

[11] Heal, G. M. *The Theory of Planning* (North Holland, 1973).

[12] Heal, G. M. " Economic Aspects of Natural Resource Depletion ", in D. Pearce (ed.) *Economics of Natural Resource Depletion*, (Macmillan, London, 1975).

[13] Henry, C. " Irreversible Investment Decisions Under Uncertainty ", *American Economic Review*, forthcoming.

[14] Hotelling, H. " The Economics of Exhaustible Resources ", *Journal of Political Economy* (1931).

[15] Ingham, A. and Simmonds, P. " The Optimal Use of Natural Resources ", *Review of Economic Studies*, forthcoming.

[16] Kaldor, N. and Mirrlees, J. A. " A New Model of Economic Growth ", *Review of Economic Studies* (1962).

[17] Koopmans, T. C. " A Proof for the Case where Discounting Advances the Doomsday". *Review of Economic Studies* (this issue).

[18] Koopmans, T. C. " Some Observations on ' Optimal ' Economic Growth and Exhaustible Resources ", Cowles Foundation Discussion Paper, No. 356 (1973).

[19] Maddox, J. *The Doomsday Syndrome* (Macmillan, London, 1971).

[20] Meadows, D. H. *et al.* *The Limits to Growth* (Universe Books, New York, 1972).

[21] Nordhaus, W. D. " World Dynamics: Measurement without Data ", Cowles Foundation Discussion Paper (1972).

[22] Nordhaus, W. D. and Tobin, J. " Is Economic Growth Obsolete ", in *Economic Growth* (Fiftieth Anniversary Colloquium, V), National Bureau of Economic Research (1972).

[23] Pontryagin, L. S. *et al.* *The Mathematical Theory of Optimal Processes* (Interscience, New York, 1972).

[24] Posner, M. V. *Fuel Policy: A Study in Applied Economics* (MacMillan, London, 1972).

[25] Solow, R. M. " Intergenerational Equity and Exhaustible Resources ", *Review of Economic Studies* (this issue).

[26] Stiglitz, J. E. " Growth with Exhaustible Natural Resources ", *Review of Economic Studies* (this issue).

[27] Vousden, N, " Basic Theoretical Issues in Resource Depletion ", *Journal of Economic Theory* (1973).

[28] Yaari, M. E. " Uncertain Lifetime, Life Insurance, and the Theory of the Consumer ", *Review of Economic Studies* (1965).