Risky (Natural) Assets: Stochasticity and the Value of Natural Capital

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Abstract

There is renewed interest in valuing the natural assets that sustain human welfare and economic prosperity. However, there is little guidance in the literature, despite great interest in practice, concerning how to incorporate risk within capital dynamics into the valuation of nonmarket assets under real-world (often suboptimal) management. We extend and demonstrate the theory of natural capital valuation for smooth stochasticity (diffusions) with convex and nonconvex drift terms and for "stochastic nonconvexities," whereby a stock is subject to an endogenous risk of discrete collapse. We highlight pathways for risk to influence capital valuations and demonstrate common conditions where diffusions, added to otherwise convex capital dynamics, have negligible effects on shadow prices. However, we find that risk has large, complex, and stock-dependent effects in the context of nonconvexities. Overall, our findings suggest that efforts to incorporate risk into natural capital valuation should prioritize settings where nonconvex capital dynamics are likely.

Keywords: Natural capital, Stochasticity, Risk, Sustainability, Wealth Accounting, Green Accounting

1. Introduction

Non-declining wealth (Arrow et al., 2004; Hamilton and Clemens, 1999) or non-negative income (Weitzman, 1976; Sefton and Weale, 2006) are important indicators for assessing whether development is sustainable when the underlying accounts for wealth and income are sufficiently inclusive of the relevant assets and service flows. National accounts can, in theory,

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provide the underlying data for such assessments. Yet, many natural assets and their associated services are excluded or mis-attributed in practice—even if included conceptually—on national balance sheets and in income accounts (World Bank, 2024). This situation leads to a strong potential for national accounting to provide the wrong signals of sustainable economic development and growth (Dasgupta, 2021; Nordhaus and Tobin, 1972).

A common reason for exclusion, even for natural assets that have been in the scope of national accounts since 1993 (e.g., managed fish stocks, managed forests, managed water resources, minerals, land etc.), is the difficulty of measuring and pricing these assets. New technologies are making physical measurement feasible, and pricing theory has also evolved substantially (Fenichel, Abbott, and Yun, 2018). However, as World Bank (2024) emphasizes, a key remaining barrier is how pricing reflects risk and tipping points (nonconvexity).

There is new urgency to amend gaps in national accounts with reliable measurement and valuation for changes in natural capital, inclusive of risk and nonconvexity, as many countries implement standardized natural capital accounts.[1] While many countries are making progress with physical accounts, monetary accounts are often lagging.[2] Aside from official national accounting, the physical assessment and economic valuation of natural capital is increasingly important for international and national sustainability assessments (UNU-IHDP and UNEP, 2014; Dasgupta, 2021; World Bank, 2024); sustainability assessment for bounded systems such as cities (Dovern, Quaas, and Rickels, 2014), hydrological catchments (Pearson et al., 2013), and ecosystems (Yun et al., 2017); and the prospective valuation of conservation or remediation policies, including those guided by the recent Kunming-Montreal Global Biodiversity Framework (Fenichel, Dean, and Schmitz, 2024).

Theory and international standards are clear. Natural capital should be priced like other invested capital—as the change in the expected net present value to society associated with a change in the asset (Jorgenson, 1963; EC, IMF, OECD, UN, and World Bank, 2009), conditioning on expected capital dynamics under real-world management, optimal or otherwise (Fenichel and Abbott, 2014; Muller, Mendelsohn, and Nordhaus, 2011). Market asset prices

[1] The United States published a national strategy in 2023 (Office of Science and Technology Policy, Office of Management and Budget, and Department of Commerce, 2023). Canada contracted for a Comprehensive Wealth report in 2018 (https://www.iisd.org/library/comprehensive-wealth-canada-2018-measuring-what-matters-long-term), and the U.K. has developed a 25 year plan focused on natural capital (https://www.gov.uk/government/groups/natural-capital-committee), with a roadmap released in 2022 (https://www.gov.uk/government/statistics/uk-natural-capital-roadmap-2022). Australia has put out a national strategy and action plan for natural capital accounts (https://eea.environment.gov.au/about/national-strategy-and-action-plan). The EU has regulations to produce environmental-economic statistics (https://www.consilium.europa.eu/en/press/press-releases/2024/11/05/eu-adopts-rules-to-better-measure-the-environment-s-contribution-to-the-economy).

[2] See UN Statistics Division Report on Implementation of the System of Environmental Economic Accounts, https://seea.un.org/news/findings-2023-global-assessment-environmental-economic-accounting.

are frequently assumed to satisfy these conditions, although there are cases (e.g., owner occupied housing, quality adjustments for computers) where imputation methods are employed by national statisticians.[3] Prices for natural assets often must be imputed due to the lack of asset markets or significant market failures where asset prices do exist. Fenichel, Abbott, and Yun (2018) review methods for imputing appropriate accounting prices for natural assets. However, these methods mostly treat resource dynamics as deterministic and convex. This is a challenge, in practice, for natural capital where stochasticity and nonconvexity often are important elements of natural capital dynamics. Scientists' best models of ecological dynamics are imperfect, and natural stocks are often subject to unpredictable environmental perturbations (Hilborn and Mangel, 1997). Furthermore, aside from fossil fuel reserves, mineral resources, and some groundwater aquifers, many natural capital stocks are *renewable* in nature, and may exhibit non-linear accumulation or depletion dynamics capable of generating nonconvex production processes, leading to thresholds in system behavior and potentially multiple equilibria (Dasgupta and Maler, 2003).

Measurement practices for other national accounts offer little guidance for natural asset prices because, when asset prices are directly observed or imputed based on market data on rental flows, the assumptions underpinning the accounts imply that these prices accurately reflect uninsured risks, or there are other priced goods or services that manage risks. The overwhelming need to impute natural capital accounting prices elevates the importance of explicitly accounting for risk and nonconvexity (tipping points) in the imputation process. Nonconvexity is rarely a consideration for imputations outside of natural capital as nonrenewable capital stocks are generally assumed to be subject to simple depreciation dynamics.

Stochasticity and nonconvexity are addressed in the natural resource economics literature, but usually with the goal of characterizing optimal decision making rather than examining the effects of these features on valuation. Applications of stochastic optimal control theory that solve for *optimal* management of resources subject to probabilistic shocks (i.e., risk) are common (e.g., Pindyck, 1984; Reed, 1979; Sethi et al., 2005; Singh, Weninger, and Doyle, 2006).[4] In these studies, prices, represented by the adjoint variable in optimal control or the derivative of the value function with respect to the state variable in dynamic programming, are seldom the object of interest. A notable exception is the rapidly growing literature uti-

[3] Importantly, values in national accounts must incorporate risk into a *single* measure—as opposed to providing standard errors or other measures of uncertainty—analogous to how asset markets capitalize risk into a single price (Landefeld, Seskin, and Fraumeni, 2008).

[4] At times, the assumption of optimization over a known distribution of uncertainty is extended to consider the potential for reducing risk through active or passive learning (LaRiviere et al., 2017), while tools from robust control theory are sometimes used to address decision making under Knightian model uncertainty (e.g., Rodriguez et al., 2011; Roseta-Palma and Xepapadeas, 2004).

lizing integrated assessment models (IAMs) to assess the social cost of carbon (SCC) under stochasticity and various non-optimal abatement scenarios, typically under an explicit assumption of risk aversion.[5] A consistent finding of these models is that risk aversion combined with uncertainty, particularly uncertainty in damages and the potential for environmental tipping points, lead to much greater social costs of carbon, relative to deterministic models.

As with stochasticity, most research on nonconvexity and multiple equilibria in resource and environmental economics (reviewed in Fenichel, Gopalakrishnan, and Bayasgalan, 2015) focuses on implications of nonconvexity for optimal management programs rather than the influence of nonconvexity on the appropriate marginal valuation (e.g., Maler, Xepapadeas, and De Zeeuw, 2003; Brock and Starrett, 2003; Horan et al., 2011). A critical feature of these analyses is that they focus on how long-run optimal behavior and the transition path to steady states depend on history. This focus on optimal management is shared by models considering a stochastic form of nonconvexity through the endogenous probability of regime shift to alternative system dynamics (e.g., Polasky, De Zeeuw, and Wagener, 2011; Reed, 1984).[6]

The literature is mute on an important question: *how does the introduction of risk in resource dynamics, or a change in its magnitude, whether through stochastic processes or nonconvexity, alter the value of changes in the stocks of natural capital for the purposes of national accounting, sustainability assessment, and social benefit-cost analysis?* In other words, what are the qualitative and quantitative implications of stochastic and nonconvex capital dynamics for the expected present value of a marginal increment in the abundance of a natural asset? This realized "shadow price" functions analogously to the role of a market price in national accounts by capitalizing risk.

We provide a non-marginal extension to the theory of pricing natural capital that addresses the joint influence of stochastic and potentially nonconvex resource dynamics on revealed shadow prices for wealth accounting. We develop intuition through canonical applications from the resource economics literature, including examples with nonconvex drift

[5]Various IAMs consider uncertainty or stochasticity in consumption and emissions paths, the warming response from GHG concentrations, or economic damages from warming (Lemoine, 2021; Van den Bremer and Van der Ploeg, 2021; Cai and Lontzek, 2019); Knightian uncertainty in the presence of model misspecification and parameter ambiguity (Barnett, Brock, and Hansen, 2020); and stochastic environmental tipping points (Cai et al., 2015; Lontzek et al., 2015). The National Academies assessment on the social cost of carbon highlights the importance of risk, while acknowledging that the social cost of carbon literature cannot assume optimal emissions (National Academies of Sciences, Engineering, and Medicine, 2017).

[6]Fenichel, Abbott, and Yun (2018) provide a preliminary treatment of the theory of shadow prices for deterministic, nonconvex systems. Kvamsdal et al. (2020) offer an analysis of "ocean wealth" in the Barents sea for a three-species ecosystem that includes stochasticity. However, the role of risk is ancillary to the multi-species focus of the analysis.

terms and stochastic nonconvexities (i.e. probabilistic catastrophic shocks). To contain our analysis, we make two bounding assumptions. First, we only consider cases where the "resource allocation mechanism" or "economic program" (Dasgupta, 2021) linking current stocks of capital to consumption and investment behavior is continuous. This covers a large class of natural assets but excludes cases where management is characterized by intermittent action, such as forests subject to clear cutting (see Hashida and Fenichel, 2022, for treatment of nonconvex economic programs without stochasticity). Second, we avoid imposing an *a priori* capitalization effect of risk by assuming risk neutral preferences over income flows.[7]

We show that stochasticity in capital dynamics alone, specifically diffusions added to a convex drift term, have little effect on the appropriate imputed prices for natural capital. However, when stochasticity interacts with nonconvexity—either in the case of diffusions added to a nonconvex drift term or when a nonconvexity arises from the stochasticity itself (e.g., an endogenous risk of a catastrophic shock)—then stochasticity can have substantial price effects. One implication is that it is important to distinguish between cases where terms like "risk", "uncertainty" or "stochasticity" are being used as a shorthand to refer to the potential for nonconvexity and tipping points versus cases where they are referring to the presence of localized unpredictability in an otherwise well-behaved, convex system.

The organization of the paper is as follows. The following section develops the theoretical foundation for the paper by deriving the shadow price formulas for a single capital stock under real world (e.g., potentially suboptimal) management, generalized (potentially convex or nonconvex) continuous time stock dynamics, and with stochastic shocks characterized by a diffusion process. The subsequent sections explore the effects of stochasticity and nonconvexity in sequentially richer settings. Section 3 demonstrates the valuation of risk for a canonical renewable resource problem with a non-linear, but convex, drift term augmented by geometric Brownian motion stochasticity. We then expand this model to consider the effects of a range of economic programs and an alternative, arguably more realistic, form of stochasticity. Section 4 extends a real-world example from the Gulf of Mexico (Fenichel and Abbott, 2014) to nonconvex resource dynamics with multiple stochastic steady states, using the case of a harvested resource with a minimum viable population as an example—a model with a similar mathematical structure to the well-known "shallow lakes" problem (Maler, Xepapadeas, and De Zeeuw, 2003). Section 5 considers a case of a "stochastic nonconvexity" in which there is an endogenous probability of a discrete shock to otherwise deterministic and convex resource dynamics (e.g., Reed, 1984; Reed and Heras, 1992). Section 6 concludes

[7]National accounting, relying as it often does upon market prices, does not explicitly account for risk preferences or institutional factors that may affect the capitalization of risk in prices (i.e. insurance markets).

the paper.

2. Theory

In this section, we derive the asset price equations for capital stocks with stochastic dynamics captured by diffusion processes paired to deterministic drift terms that can exhibit either convex or nonconvex dynamics. We focus on the case of continuous economic programs. In Section 5, we develop parallel theoretical results for the case of an endogenous Poisson shock process.

2.1. Derivation of shadow pricing formula

Let $s(t)$ represent the known stock of a scalar asset at time t.[8] Suppose the dynamics of s are represented by a diffusion or Ito process with stationary infinitesimal functions $\mu\left(s(t), x\left(s(t)\right)\right)$ and $\sigma(s(t))$. The diffusion process is written as

$$ds(t) = \mu\left(s(t), x\left(s(t)\right)\right) dt + \sigma(s(t)) dZ(t) \tag{1}$$

where $dZ(t)$ is an increment of a Wiener process (Stokey, 2009). The drift of the diffusion $\mu\left(s, x\left(s\right)\right)$ is specified as a function of the current capital stock and as a function of the feedback control rule, known as the economic program, $x(s)$ (Dasgupta, 2001). Throughout we assume that $x(s)$ is continuous and differentiable. This implies a large and common class of economic programs but excludes programs such as rotational forestry. Once the substitution for the economic program is made, the drift is an explicit function of only s, $\mu(s)$. The volatility of the diffusion, governed by $\sigma(s)$, is only a function of the current state, and is not directly influenced by the economic program.

Let $W(s(t), x(s(t))$ be a measure of instantaneous contribution to welfare at time t, or the "instantaneous welfare."[9] Importantly, the concavity properties of this function are quite general, allowing for the full range of static risk preferences in "consumption" (x) and the resource stock (s). Define the intertemporal welfare function (i.e. the value function), evaluated along the economic program and along the stochastic capital trajectory given by (1), as

$$V\left(s\left(t\right)\right) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\delta(\tau - t)} W\left(s\left(\tau\right), x\left(s\left(\tau\right)\right)\right) d\tau\right] \tag{2}$$

[8] t is suppressed when doing so does not cause confusion.

[9] We use the term "instantaneous" because our model is in continuous time, whereas one could refer to "single period welfare" in a discrete time context. While we ground our theory of capital valuation in welfare-theoretic terms, our exposition would still hold if $W(\cdot)$ was redefined as a scoring function of "what counts" within an arbitrary accounting boundary.

where $\mathbb{E}_t$ is the expectations operator. The marginal value of an investment in the capital stock in expectation is defined as $p(s) \equiv V_s$. To derive the properties of $p(s)$, start by differentiating (2) with respect to t.

$$\frac{dV}{dt} = \mathbb{E}_t \left[\delta \int_t^{\infty} e^{-\delta(\tau - t)} W(\cdot)\, d\tau - W(s(t), x(s(t))) \right] = \delta V - W(s(t), x(s(t))) \qquad (3)$$

The first equality in (3) assumes that the derivative can be carried through the expectation operator, which is ensured by the stationarity of the infinitesimal parameters of (1). The second equality holds because the state of the system is known at $\tau = t$.

By Ito's Lemma

$$dV = \left[\mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \right] dt + \sigma(s) V_s dZ$$

Taking the expected value, and employing the property that all stochastic integrals are identically zero (Stokey, 2009):

$$\mathbb{E}_t[dV] = \left[\mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \right] dt$$

so that

$$\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt} = \mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \qquad (4)$$

Setting (3) equal to (4) and substituting $p(s) \equiv V_S$, we obtain the stochastic Hamilton-Jacobi-Bellman (HJB) equation:

$$\delta V(s) = W(s(t), x(s(t))) + p(s)\mu(s) + \frac{1}{2}\sigma^2(s) p_s(s) \qquad (5)$$

The first two terms on the RHS are the traditional deterministic current-value Hamiltonian. The third term captures the impact of risk through the curvature of the intertemporal welfare function. If the shadow price function is downward sloping, then $p_s < 0$ so that risk has a negative effect on the intertemporal welfare function.

Importantly, the derivatives p and p_s in (5) are evaluated after substitution of a particular feedback control rule $x(s)$, which need not optimize the HJB. The implication is that the slope, curvature, and higher derivatives of the intertemporal welfare function reflect, in part, the properties of this economic program. The one exception to this case is when the economic program is dynamically optimal.[10]

[10] See Appendix A for the mathematical and economic justification of this assertion.

Suppressing functional dependency on s, and differentiating (5) with respect to s yields:

$$\delta p = W_s + \mu_s p + \mu p_s + \sigma\sigma_s p_s + \frac{1}{2}\sigma^2 p_{ss}$$

Isolating p on the left-hand side we obtain the asset pricing equation:

$$p(s) = \frac{W_s + [\mu(s) + \sigma(s)\sigma_s(s)]p_s + \frac{1}{2}\sigma^2(s)p_{ss}}{\delta - \mu_s(s)} \tag{6}$$

Recognizing from Ito's Lemma that $\frac{1}{dt}\mathbb{E}_t[dp(s)] = \mu(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}$, (6) can be rewritten as follows:

$$p(s) = \frac{W_s + \sigma(s)\sigma_s(s)p_s + \frac{1}{dt}\mathbb{E}_t[dp(s)]}{\delta - \mu_s(s)}$$

It is then straightforward to rearrange this expression in "rate of return" form (cf. Pindyck, 1984):

$$\frac{\frac{1}{dt}\mathbb{E}_t[dp(s)]}{p(s)} = \delta - \mu_s(s) - \frac{W_s}{p(s)} + \sigma(s)\sigma_s(s)A(s) \tag{7}$$

where $A(s) = \frac{-p_s}{p} = \frac{-V_{ss}}{V_s}$ is the coefficient of absolute risk aversion defined over the intertemporal welfare function.

2.2. Intuition

From the standpoint of national accounting, equation (6) provides a single, theoretically-sound imputed price for natural capital under stochastic capital dynamics. It captures the expected increment to the forward-looking welfare function (2) from a small investment in the stock, integrating over the risk in the future path of natural capital. This distillation of risk into a single value mimics the operation of an idealized asset market. By contrast, a Monte Carlo approach over deterministic imputations, as might seem easier, provides a distribution of marginal valuations along a multiplicity of *particular* capital accumulation paths—as if all uncertainty concerning a path is already revealed.

In the case where capital dynamics are deterministic (6) reduces to:

$$p(s) = \frac{W_s + \mu(s)p_s}{\delta - \mu_s(s)} \tag{8}$$

which is the same as in Fenichel and Abbott (2014), who show that this equation is equivalent to Jorgenson (1963). To make the linkage to the rate of return formulation in (7), rearrange

(8) as follows:

$$\frac{dp}{dt} = p_s\mu(s) = \delta - \mu_s(s) - \frac{W_s}{p(s)} \tag{9}$$

This expression reveals that we can write the rate of return on a unit of the capital stock (i.e. capital gains) as the depreciation or appreciation adjusted discount rate less any immediate dividends from stock investment, $\delta - \mu_s(s) - W_s/p(s)$, whereas (7) contains one additional term $\sigma(s)\sigma_s(s)A(s)$. This term relates to the effect of investment on risk itself as valued by the revealed risk aversion to stock fluctuations $A(s)$. In the case where risk increases in the stock, $\sigma_s > 0$, and if risk aversion holds, $A(s) > 0$, then this term is positive. Intuitively, investment in natural capital increases risk, with the result that the expected price appreciation increases under stochasticity to compensate.

Importantly, the rate of return in (9) is deterministic, whereas (7) defines the *expected* rate of appreciation. Recall that the expected increment in price over time in the stochastic case is $\frac{1}{dt}\mathbb{E}_t[dp(s)] = \mu(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}$. The first of the two right-hand-side terms is the price appreciation due to physical capital appreciation under the deterministic drift term, and is the only driver of price appreciation in the deterministic case. The expected price appreciation includes an additional term, $\frac{1}{2}\sigma^2(s)p_{ss}$, that reflects the role of Jensen's inequality and causes expected price growth to be non-zero even if the stock is at its stochastic equilibrium ($\mu(s) = 0$). In particular, if the price curve is convex ($p_{ss} = V_{sss} > 0$) then stochasticity in the stock itself increases the expected rate of price appreciation over time, where the magnitude of this premium increases in the scale of risk $\sigma^2(s)$. Intuitively, if stock investments lead to lower levels of revealed risk aversion, then this increases the expected rate of return. Finally, note that if the shadow price curve is linear in s (i.e. the intertemporal welfare function is quadratic), then Jensen's inequality becomes irrelevant and expected price appreciation is the same as in the deterministic case.

Turning to the asset pricing equation (6), there are two additional terms relative to the deterministic case (8). The first, $\sigma(s)\sigma_s(s)p_s$, relates directly to the risk-specific term in (7) and depends on the extent of revealed dynamic "risk aversion" in stock levels ($p_s = V_{ss}$) and the extent to which the standard deviation of the diffusion is elastic with respect to s. If increasing investment in s increases the size of shocks, and if the shadow price function is decreasing in the stock (analogous to risk aversion), then this combines to create a price discount.[11] Note that this term is non-zero only if the variance depends on the capital stock, as in the case of geometric Brownian motion. Therefore, we frequently refer to $\sigma(s)\sigma_s(s)p_s$

[11]Note that the effect of risk on the accounting price is the opposite of its effect on the rate of return (7).

as the "endogenous risk effect."

Curvature of the intertemporal welfare function, which is only defined over the domain of the capital stock, depends on more than the curvature of the "social utility" or instantaneous welfare contribution, $W(\cdot)$. Indeed, the risk preferences over flows embodied in $W(\cdot)$ need not have a one-to-one relation to the curvature of $V(s)$. Curvature of the intertemporal welfare function can be inherited from the underlying biophysical dynamics in (1) or from non-optimal economic programs $x(s)$, suggesting that the risk premia embodied in the numerator of (6) are endogenous to policy and therefore reflect actual existing levels of self-insurance and self-protection (Ehrlich and Becker, 1972).

The second additional term in (6) $\frac{1}{2}\sigma^2(s)p_{ss}$ coincides to the aforementioned effect of Jensen's inequality on the rate of expected capital gains and is present with stochastic dynamics so long as $p_{ss} \equiv V_{sss}$ is non-zero. This expression can be interpreted as the effect of a marginal increase in the capital stock on risk aversion, holding risk constant—an "endogenous risk aversion" effect. Equivalently it can be viewed through the lens of "prudence," which is associated with precautionary savings (Kimball, 1990). Under imprudence ($V_{sss} \equiv p_{ss} < 0$), risk aversion is increased by the investment, lowering expected capital gains and the shadow price, whereas prudence ($V_{sss} \equiv p_{ss} > 0$) has the opposite effect. In other words, the pricing of risk into the capital asset depends on how additional investment affects the sensitivity to risk, given the biophysical dynamics and economic program in place. This can be viewed as a "self-insurance effect" since changes in the curvature of the intertemporal welfare function reflect the normative consequences of stochastic events rather than their probability (Shogren and Crocker, 1999).

Following Eq (6), risk has no effect on the *marginal* valuation of natural capital when two conditions hold: 1) the volatility of risk is not a function of the capital stock (i.e. risk is "exogenous"); and 2) the shadow price function is linear ($p_{ss} = 0$), so that the intertemporal welfare function is quadratic. These conditions are satisfied in a canonical problem in the literature: that of an optimizing agent with quadratic utility in consumption, a linear capital accumulation equation, and a mean preserving spread in risk. In this case (6) collapses to the co-state equation for the deterministic case, so that optimal savings behavior is invariant to risk (e.g., Kimball, 1990).[12]

[12]The fact that the conditions for the behavioral irrelevance of risk for investment behavior are typically discussed in terms of the third derivatives of the instantaneous utility function (in consumption) rather than the intertemporal welfare function (in unit of savings or capital) is because the linear properties of the growth function of capital in the simple savings problem, combined with the selection of the *optimal* investment rule ensure that the shape properties of the utility function are mirrored in the intertemporal welfare (value) function as well. This mapping does not carry over to the case of non-optimized control rules with non-linear drift terms, as in the case for most natural capital.

In the subsequent examples, it is sometimes useful to decompose the gap between the shadow price conditional on stochastic (i.e. diffusion-based) capital dynamics (6) relative to the shadow price with deterministic capital dynamics (8), holding constant the natural capital drift term and the economic program. Under these *ceteris paribus* assumptions $W(s(t), x(s(t)))$ and $\mu(s(t))$ do not vary between the stochastic and deterministic cases. Therefore, the shadow price under stochasticity can be written as the shadow price under deterministic dynamics plus adjustments that account for future stochasticity:

$$p^R(s) = p^D(s) + \frac{\sigma(s)\sigma_s(s)p_s^R + \frac{1}{2}\sigma^2(s)p_{ss}^R + \mu(s)(p_s^R - p_s^D)}{\delta - \mu_s(s)} \tag{10}$$

where $p^R(s)$ and $p^D(s)$ indicate the shadow prices under stochasticity (6) and the deterministic case (8), respectively. As already noted, the shadow price expression under stochasticity contains two additional terms, the endogenous risk and prudence effects, which are captured by the second and third terms on the RHS of (10). However, there is another, more subtle, wedge between the deterministic and stochastic shadow prices that arises because the slope of the shadow price function (i.e. the curvature of the intertemporal welfare function) may be affected by the introduction of risk. As a result, the valuation of the deterministic portion of capital gains from the drift term varies across the stochastic and deterministic cases. This "drift valuation effect" is seen in the final RHS term in (10): $\frac{\mu(s)(p_s^R - p_s^D)}{\delta - \mu_s(s)}$. Assuming greater concavity of the value function in the risky case ($p_s^R < p_s^D$) and a positive adjusted discount rate, this effect tends to decrease (increase) the stochastic shadow price relative to the risk-free case when the drift term yields capital appreciation (depreciation).

3. Example 1: optimized and non-optimized renewable resource management under convexity with stochasticity

To provide a straightforward test of the effect of risk on the accounting price of natural capital, we begin with an example with a non-linear, yet convex, stock production function (i.e. having a single stochastic steady state) using an optimized economic program developed by Pindyck (1984). By starting with an optimized problem, we can numerically compare the effects of stochasticity itself on shadow prices relative to the effects of departures from the optimized control rule. This example also has the benefit of yielding closed-form solutions for the co-state, which allows us to ensure that a numerical approximation approach used elsewhere in the paper performs adequately.

In this seminal contribution, Pindyck extends the canonical infinite horizon, continuous-time renewable resource model to allow for a stochastically growing resource stock that follows a geometric Brownian motion (GBM) process with three distinct drift functions.

The focus of the modeling is on revealing how the 'golden rule' of resource management is augmented by a risk premium term. He then explores how the biological and economic parameterization interacts with increases in risk to influence the optimal extraction rate and the stochastic steady state distribution. Pindyck's objective is to maximize the expected net present value of the combined consumer and producer surplus from harvest q of the fish stock s over an infinite horizon, where the demand function is isoelastic, $q(p) = bp^{-\eta}$, and the marginal cost of harvest is $cs^{-\gamma}$. By maximizing the NPV of monetary surplus, the social planner is explicitly risk-neutral in their evaluation of flows of monetary income; nevertheless, these assumptions imply an instantaneous welfare function that is strictly concave in harvest, exhibiting aversion to any risk-induced fluctuations in harvest levels.

Pindyck shows that stock stochasticity has three competing effects on the optimized co-state so that optimal harvest may be more or less aggressive (less or more precautionary) compared to the deterministic case. The first, a variance effect, encourages the manager to hold a smaller stock due to the fact that the variance of the stock increases in the stock size; this increased variance in turn lowers the accounting price of the stock due to the concavity of the intertemporal welfare function. The variance reduction effect works through the second numerator term in (6). The second effect of risk, a cost reduction effect, encourages the manager to hold a smaller stock as variance increases due to the cost-increasing effects of stochastic fluctuations on expected harvest costs because of the convexity of the marginal harvest cost function – an implication of Jensen's inequality. Third, a growth rate effect encourages managers to hold *more* stock as variance rises because stochasticity reduces the expected growth rate of the stock, which follows from the concavity of the growth function. The cost reduction and growth rate effects operate through the third numerator term in (6) as a form of "endogenous risk aversion." Pindyck shows through a series of examples how the combination of the three effects under different model parameterizations can lead to more or less aggressive harvest under risk, relative to the deterministic case. Of particular importance are the elasticity of demand and the skewness of the drift term in the growth function.

In general, this model requires a numerical solution. However, Pindyck (1984) demonstrates that a closed form solution to the HJB equation exists when $\eta = 1/2$ and $\gamma = 2$ and when the stock evolves according to a logistic drift function with GBM stochasticity: $ds = [rs\,(1 - s/K) - q]\,dt + \sigma s\,dZ$. Specifically, the optimized co-state (or rent) is:

$$V_s = \phi/s^2$$

and the optimized economic program (feedback control rule) is:

$$x(s) = q^*(s) = \frac{b}{(\phi + c)^{1/2}} s \tag{11}$$

where

$$\phi = \frac{2b^2 + 2b[b^2 + c(r + \delta - \sigma^2)^2]^{1/2}}{(r + \delta - \sigma^2)^2}$$

The economic program in (11) increases linearly in the stock. Such rules imply a constant percentage rate of fishing mortality (i.e. "constant-F") and are common in natural resource management (Deroba and Bence, 2008). Importantly, $\partial\phi/\partial\sigma^2 > 0$, implying that $\partial V_s/\partial\sigma^2 > 0$ and $\partial q^*/\partial\sigma^2 < 0$. Increasing stochasticity in this model always increases the accounting price of the stock, thereby *decreasing* the optimal rate of harvest at every stock level. Dynamic optimization yields an economic program that reflects "uniform precaution" (Figure 1, black line) since increases in stochasticity simply reduce the slope of the linear economic program, yielding a lower constant harvest rate at *all* stock levels and therefore a larger steady state biomass.

In practice, real-world resource managers may face regulatory mandates, competing objectives, or information limitations that preclude the adoption of the optimal management solution. This implies that realized economic program or actual resource management policy may appear, to the analyst, more (or less) precautionary than the level implied by the optimal rule.We reflect these adjustments through two scalar shifts of the optimal economic program under stochasticity, where harvests are either systematically lower (blue line, half the optimal harvest at every point) or greater (orange line, 1.5 times the optimal harvest at every point) than the optimizing program (black line) (Figure 1). These shifts lead to economic programs that are non-optimal everywhere and result in different stochastic equilibria. These deviations from optimality are reflected in the intertemporal welfare functions and accounting price functions.

All models utilize parameter values of $\sigma = 0.1, \delta = 0.05$, $b = 1$, $r = 0.5$, and $K = 1$. The level of stochasticity implied by $\sigma = 0.1$ is substantial, being almost 50% larger than the estimated value for the US Gulf of Mexico reef fish in Fenichel and Abbott (2014). The intertemporal welfare functions and shadow price functions lack analytical expressions for the non-optimal economic programs. We therefore approximate the intertemporal welfare function using an extension of the basis function approximation approach used in Yun et al. (2017) and detailed in Appendix B.We test these approximations against the closed form solutions in the optimized case. Figure B.1 shows that we are able to reproduce the analytical intertemporal welfare function and shadow price curves to a high degree of accuracy, with

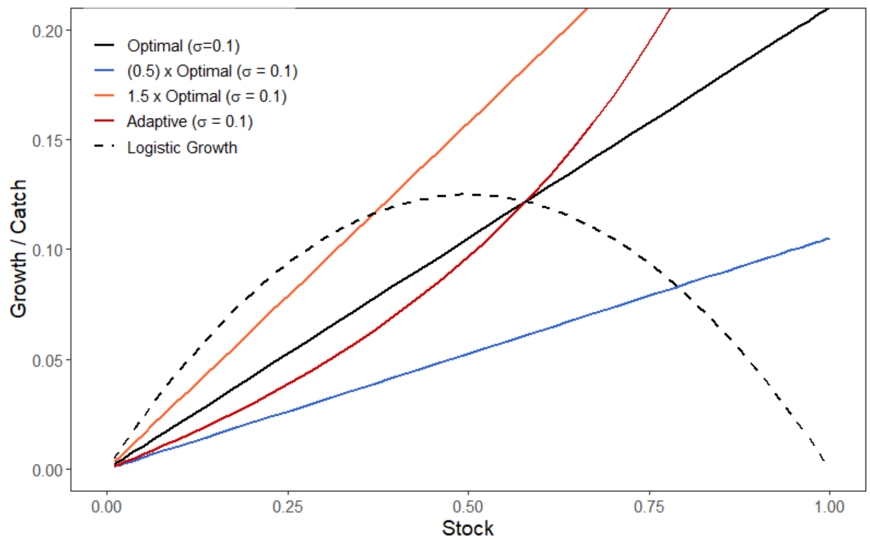


Figure 1: Stock-catch space showing the optimal economic program if $\sigma = 0.1$ and three alternative non-optimal economic programs

mean absolute percentage deviations of .0001% for V and .089% for p, respectively.

Figure 2 compares the intertemporal welfare and price functions with ($\sigma = 0.1$) and without ($\sigma = 0$) stochasticity for the optimal and suboptimal economic programs. Importantly, σ is allowed to vary between the deterministic and stochastic cases in the expression for the optimal economic program (as reflected by ϕ in (11)) to reflect the optimal response to the actual stochasticity level. By contrast the scalar transformations of the optimal economic program do not change across the deterministic vs. stochastic scenarios and are fixed at $\sigma = 0.1$. Panel A shows that risk strictly reduces intertemporal welfare and has a similar effect across all three economic programs. Stochasticity appears to translate the intertemporal welfare functions down in an approximately constant manner, although the shift is slightly smaller at low stock levels. This finding suggests that marginal or non-marginal *changes* in welfare for a change in the stock may be minimally affected by volatile stock dynamics.

The left-hand side of Table 1 considers the welfare change for a relatively large perturbation in stock from 0.35 to the optimal equilibrium stock when $\sigma = 0.1$ of 0.58. Under optimal management, which adjusts to changes in σ (i.e. is "σ varying"), the valuation of this investment is approximately 2.9% greater under stochasticity than in the deterministic case. Evaluating the same perturbation under the two scalar transformations of the optimal harvest rule results in valuations under stochasticity that are 2.3% and 4.5% greater than if we ignore risk, with the greatest deviation occurring for the least precautionary economic program (1.5 times optimal harvest).[13] Therefore, the degree of bias in valuation from ig-

[13]The optimal economic program (11) varies in σ such that both the economic program and the level

noring risk is affected, albeit to a mild extent in this case, by the economic program, and suboptimal economic programs may either amplify or diminish the effects of stochasticity on valuation compared to optimal management.

Given these findings for non-marginal changes in natural capital, it is no surprise that similar findings occur for the accounting prices applicable to marginal changes (Table 1). For the optimal program and its suboptimal scalar transformations, the shadow price always increases in stochasticity (Figure 2(b)). Nonetheless, stochasticity has only a second-order effect on marginal values.[14] This conclusion is robust across the full range of stock levels, with stochasticity premia of approximately 3% for the optimal program, 2.3% for the 0.5 optimal program, and 4.5% for the 1.5 optimal program, regardless of the stock level.

Importantly, the modest effects of stochasticity are not attributable to trivial magnitudes of the risk terms themselves (10). Table D.1 demonstrates across the spectrum of candidate economic programs that these terms can be large in relative terms, particularly at lower stock levels, yet the endogenous risk, prudence, and drift valuation effects approximately offset each other. For example, at $s = 0.2$ under the optimal program the endogenous risk effect is 52% of the deterministic shadow price but is offset by an even larger prudence effect in the opposite direction. Indeed, the prudence and endogenous risk effects consistently move in opposite directions in these cases, although the direction of influence for any given risk term in flips below a critical stock level as the adjusted discount rate $(\delta - \mu_s(s))$ becomes negative due to the high "natural interest rate" provided by natural capital at these stock levels.

Overall, this example demonstrates that, while ignoring stochasticity systematically undervalues changes in natural capital, this bias is relatively small across optimal and suboptimal economic programs. Indeed, when we compare the changes in welfare under stochasticity across different economic programs, we find that the welfare implications of the choice of economic program are greater than the effects of stochasticity itself for a given economic program. For example, the accounting price under stochasticity under the highly precautionary economic program (0.5 of optimal harvest) is 31% larger than under optimal management. An implication of this finding is that the appropriateness of assumptions about the economic program may have greater effects on the accuracy of natural capital valuation than the risk

of stochasticity change in the first row of Table 1. We therefore consider a better-controlled experiment where the economic program is frozen at the optimal feedback rule when $\sigma = 0.1$, even as the level of risk varies (second row of Table 1). We find that this change increases the difference between the stochastic and deterministic valuation by only 0.12%.

[14] Lest the reader suppose we are cherry-picking an extreme example to minimize the effect of stochasticity, Pindyck's example 2 replaces the symmetric logistic growth function with an asymmetric Gompertz growth curve, finding that risk has *no* effect on optimal management or shadow prices (Pindyck, 1984).

itself. For example, an assumption of optimal management when these assumptions fail to hold in the real world can swamp the effects of failing to account for stochasticity.

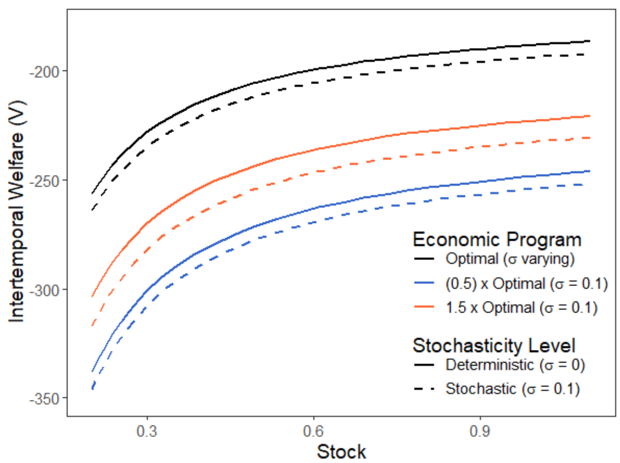
Table 1: Comparison of the change in intertemporal welfare and the accounting price for an investment in the stock.

Economic program	Change in intertemporal welfare (ΔV)			Accounting Price (p)		
	Deterministic ($\sigma = 0$)	Stochastic ($\sigma = 0.1$)	% difference	Deterministic ($\sigma = 0$)	Stochastic ($\sigma = 0.1$)	% difference
Optimal harvest (σ varying)	18.817	19.365	2.913%	50.911	52.460	3.044%
Optimal harvest ($\sigma = 0.1$)	18.795	19.365	3.030%	50.916	52.460	3.030%
0.5 Optimal harvest ($\sigma = 0.1$)	24.806	25.377	2.300%	67.201	68.747	2.300%
1.5 Optimal harvest ($\sigma = 0.1$)	22.255	23.246	4.453%	60.289	62.973	4.453%
Adaptive Rule	19.885	20.371	2.445%	51.353	52.607	2.441%

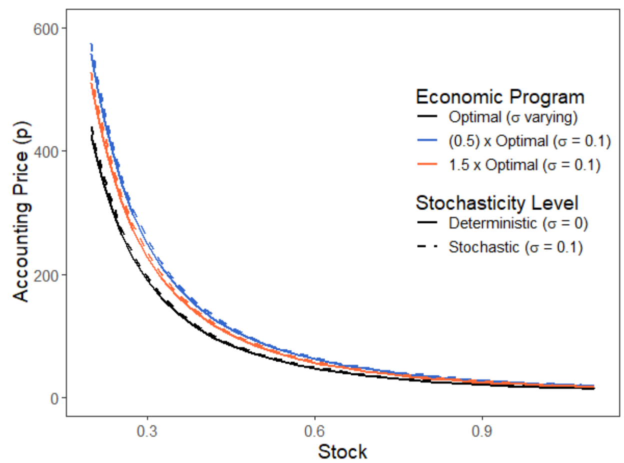
Note: The change in intertemporal welfare is calculated between $s_0 = 0.35$ and $s_1 = 0.58$ (the steady state for optimal management with $\sigma = 0.1$). Accounting prices are calculated at $s = 0.58$.

3.1. Increasing the realism of model assumptions

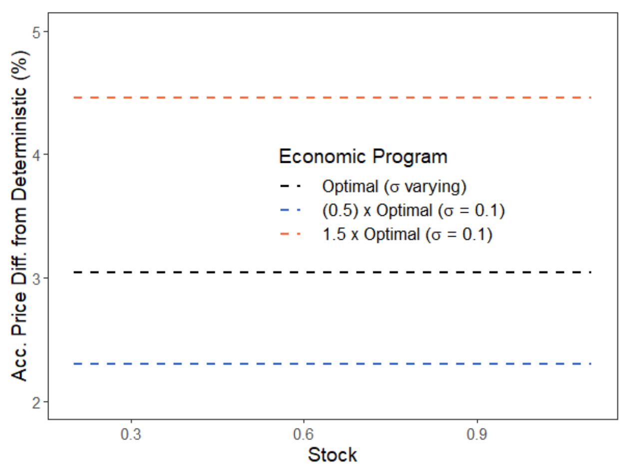
We now consider two extensions of the previous canonical example to more realistic settings. First, we expand our treatment of economic programs by considering an "adaptive" strictly convex program that is more conservative at low stocks while harvesting more liberally at large stocks. Second, Sims, Horan, and Meadows (2018) (SHM) argue that GBM—"logistic Brownian motion" in their parlance—is not the most reasonable stochastic process to use in ecological models, despite its widespread adoption. Instead, SHM propose a stochastic process where $\sigma^2(s)$ is non-monotonic in s.



(a) Intertemporal welfare function



(b) Accounting price



(c) Percentage difference of stochastic and deterministic accounting prices

Figure 2: The intertemporal welfare function (Equation 5) and acccounting price curves (Equation 6) for the optimal program and two scalar shifts of the optimal program. All welfare measures are evaluated under stochastic ($\sigma = 0.1$) and deterministic ($\sigma = 0$) dynamics.

3.1.1. Adaptive economic programs

We consider an economic program that deviates from the "constant-F" form by being a strictly convex function of the stock (Figure 1, red curve). This rule is "adaptive" in its degree of precaution by harvesting a lower fraction of the stock at low stocks and a higher fraction at high stocks.[15] For the sake of comparability, we calibrate this control rule to have the same stochastic equilibrium as the optimal program under stochasticity. Therefore, the adaptive program equals the optimal program only when the stock is at the stochastic equilibrium. The strong convexity of the adaptive program reflects an important feature of real natural resource management systems – a bias toward system stability. In this case, the steady state probability distribution will have a lower variance than the optimal program.[16]

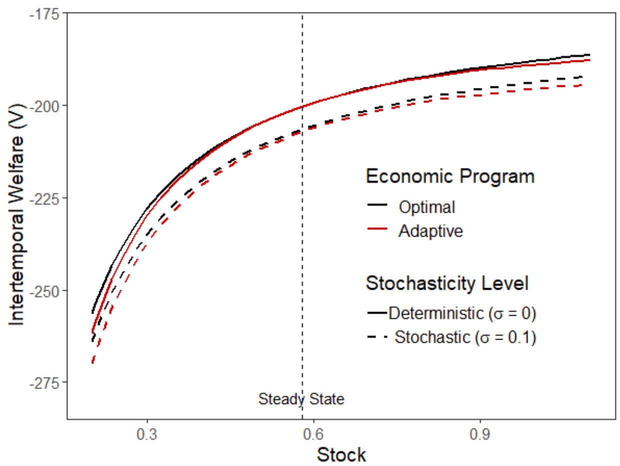
As noted for previous economic programs, the most apparent effect of introducing stochasticity to the adaptive economic program is a downward shift in the intertemporal welfare function (Figure 3(a)). However, there are also important differences. In the deterministic case (Figure 3, solid curves), the intertemporal welfare value in the region of the steady state is approximately the same under the adaptive and optimal programs and identical at the shared equilibrium stock level. Therefore, small changes in the stock in this region result in near-identical welfare changes under either program. It is only as the stock moves substantially from the equilibrium that there is a meaningful divergence between the intertemporal welfare functions in a deterministic system. By contrast, when the stock is stochastic (Figure 3, broken lines), the intertemporal welfare function under the adaptive program is always below that of the optimal program – even at the stochastic steady state biomass. This occurs because even at the stochastic equilibrium stock there is an expectation of a shock that will move the system to a region where the adaptive economic program is markedly suboptimal.

These features are reflected in the shadow price curves (Figure 3(b-c)). The shadow price curves for optimal and adaptive management cross at their respective steady state stocks in both the deterministic and stochastic cases. This must be the case for the adaptive program to be suboptimal everywhere except at the equilibrium. The implication, in the stochastic and deterministic case, is that the accounting price of natural capital under the adaptive control rule is distorted upward for stocks below the steady state level and distorted downward for stocks above the steady state compared to optimal management.

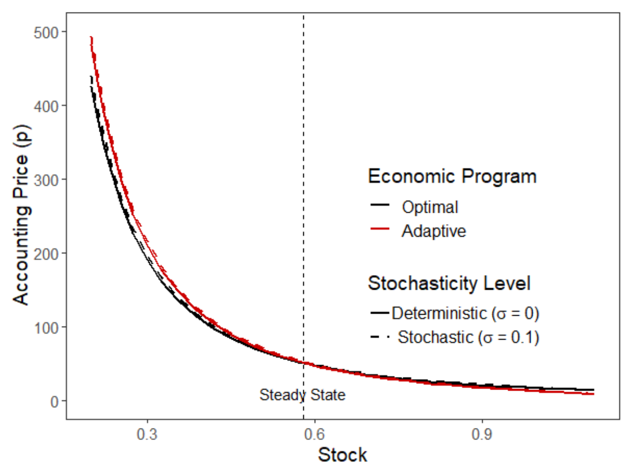
Notwithstanding these subtleties, the shadow price curves under deterministic versus

[15] In fisheries management, this adaptivity is often accomplished in practice by distinct linear harvest control rules that are each applicable within different ranges of the stock—in essence a linear spline function.

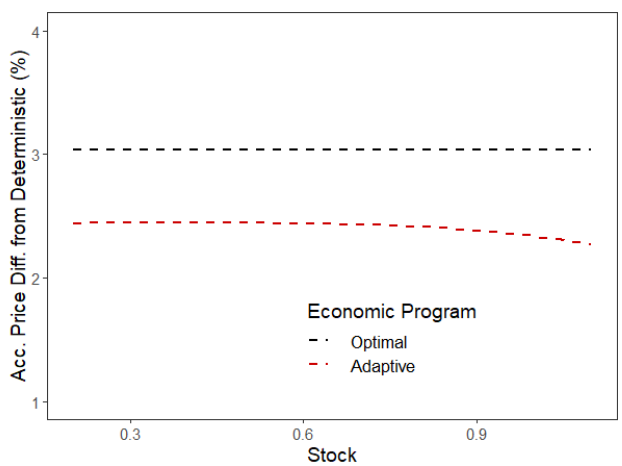
[16] The shape of this control rule, but not its anchoring on the optimal steady state, is similar to the feedback process that Zhang and Smith (2011) estimate and Fenichel and Abbott (2014) use in their application to the Gulf of Mexico reef fish fishery.



(a) Intertemporal welfare function

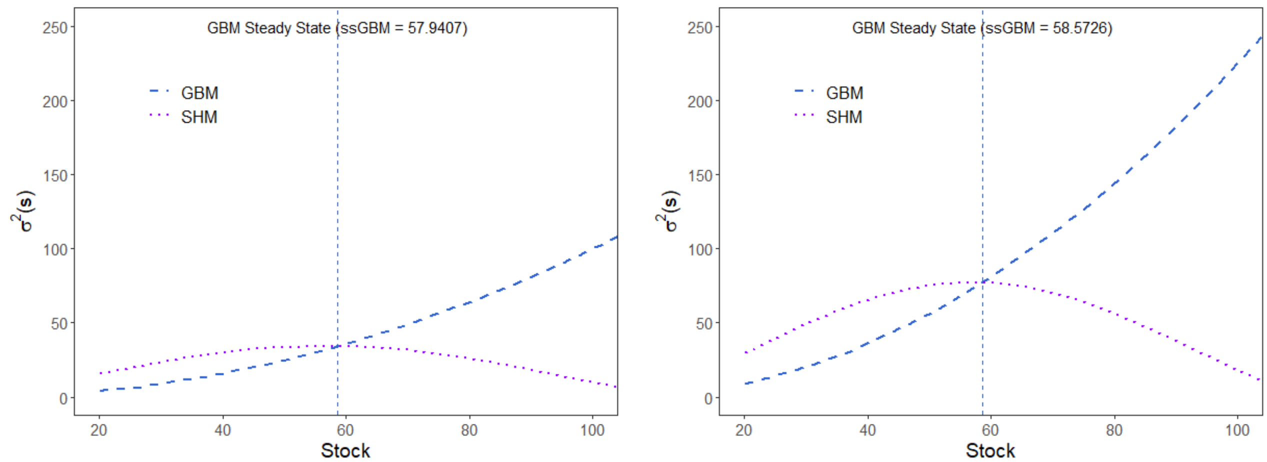


(b) Accounting price



(c) Percentage difference of stochastic and deterministic accounting prices

Figure 3: The intertemporal welfare function and accounting price curves for the optimal program and a non-optimal "adaptive" economic program with the same stochastic steady state under stochastic ($\sigma = 0.1$) and deterministic dynamics.



(a) $\sigma_0 = 0.1$ and $\sigma_M = 0.1482$

(b) $\sigma_0 = 0.15$ and $\sigma_M = 0.2747$

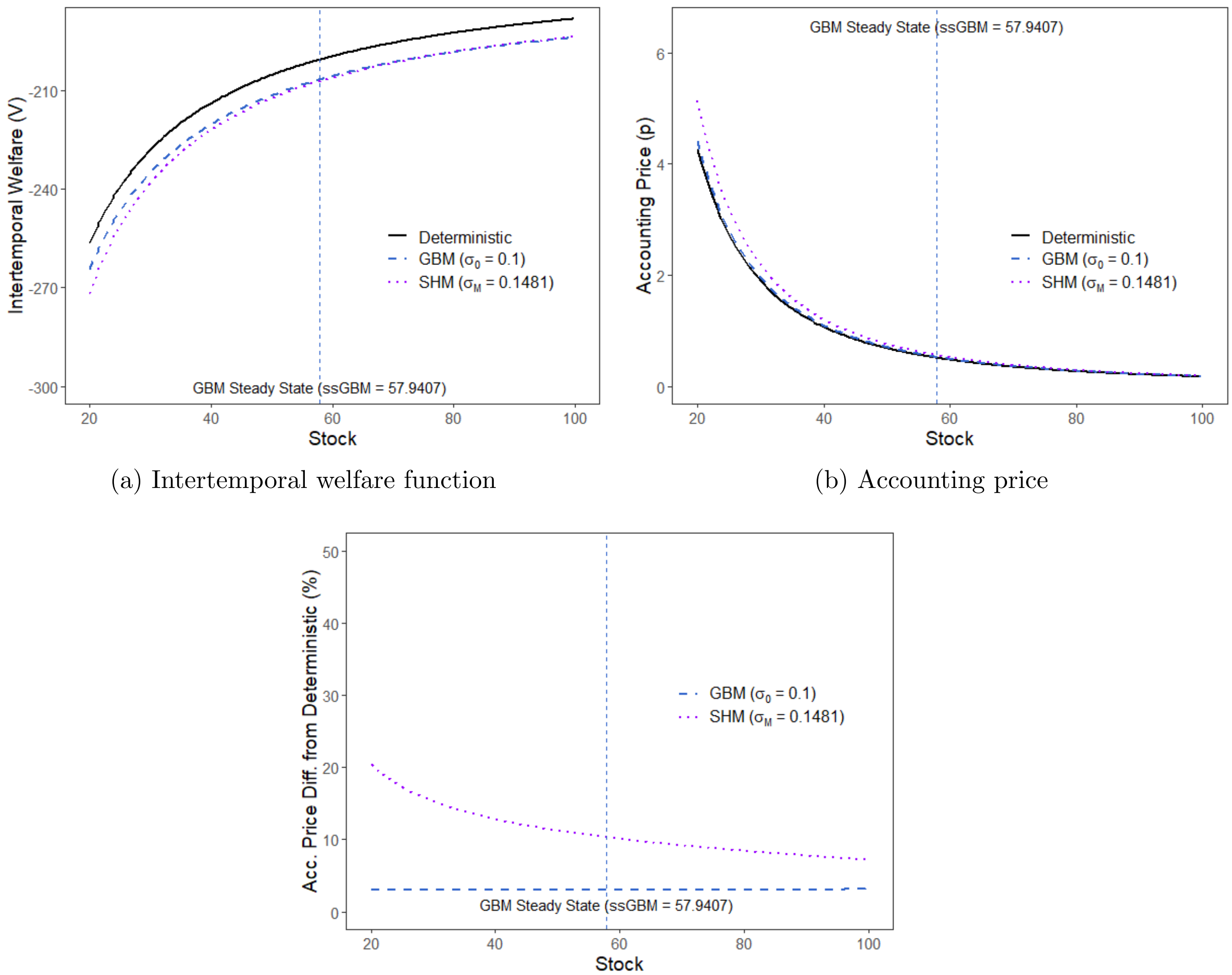
Figure 4: Volatility comparison of GBM and the Sims, Horan, and Meadows (2018) specification of demographic and environmental stochasticity (SHM)

stochastic stock dynamics are remarkably similar under the adaptive economic program. Table 1 shows that stochasticity imparts a 2.44% premium at the steady state. This premium is fairly consistent for smaller stock levels (Figure 3(c)), and actually declines further as stocks approach or exceed carrying capacity levels. Therefore, the adaptive economic program yields a risk premium for the shadow price that is lower than under optimal management, even though it shares the same steady state. The reduced effect of risk on valuation under the adaptive control rule arises due to the stability-enhancing nature of this control rule, which limits the effect of shocks on capital dynamics.

3.1.2. Non-GBM stochasticity

As a single-parameter specification of $\sigma^2(s)$, GBM requires that stochasticity increases in s at a quadratic rate. While GBM may approximate the true distribution of randomness at intermediate population levels, SHM argue that its properties may result in understating risk at low stock levels and overstating it at high abundances. Drawing upon the ecological literature, they derive a specification for $\sigma^2(s)$ that is composed of additive demographic (quadratic in s) and environmental (quartic in s) components, with the overall effect that stochasticity increases over low stocks before peaking at an intermediate level and then declining at sufficiently high stocks (see Appendix C).

Comparing these alternative volatility structures requires deciding how to calibrate them relative to each other—a decision that affects the quality of the approximation to the more complex SHM process by the one-parameter GBM process at different s values. Figure 4 illustrates one approach to this calibration for GBM specifications under two different levels



(a) Intertemporal welfare function

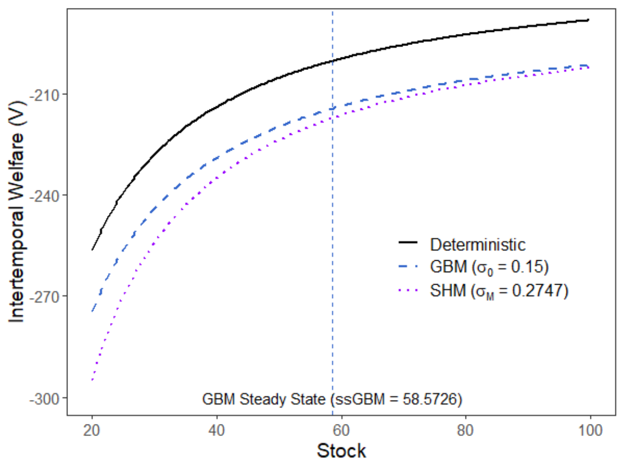
(b) Accounting price

(c) Percentage difference of stochastic and deterministic accounting prices

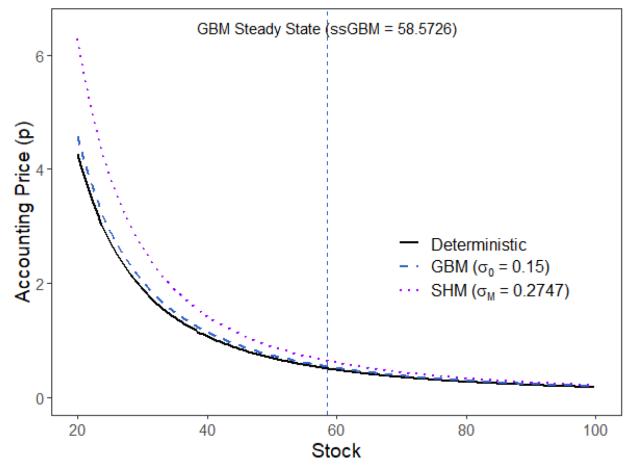
Figure 5: The intertemporal welfare and shadow price curves of the Pindyck example with moderate stochasticity ($\sigma_0 = 0.1$, $\sigma_M = 0.1482$). See Figure 4(a).

of σ_0. Here the environmental stochasticity parameter of the SHM model (σ_M) is set so that the peak of the volatility curve matches the volatility of GBM at the GBM stochastic steady state under the Pindyck model, with all other parameters of the SHM model being set at the values in Appendix C. From Figure 4 it seems possible that the SHM demographic and environmental volatility may be tolerably approximated by GBM to the left of the peak, but GBM will not adequately capture stochasticity in a system where carrying capacity exerts a strong stabilizing force.

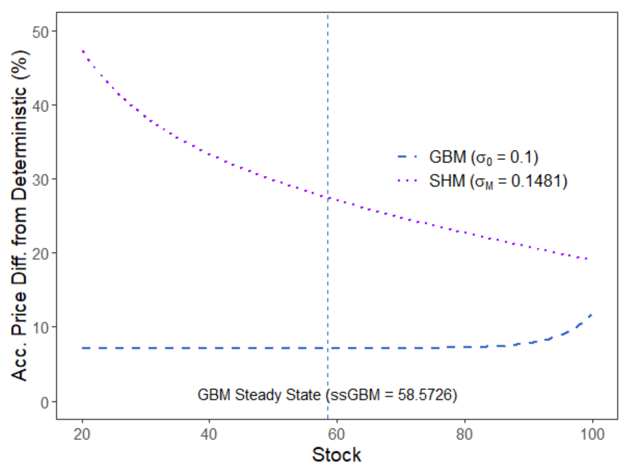
Figures 5 and 6 compare the intertemporal welfare and accounting price functions for the two GBM and SHM specifications of risk presented in Figure 4. Across the domain, we find that the risk premium to the accounting price is larger for the SHM specification than



(a) Intertemporal welfare function



(b) Accounting price



(c) Percentage difference of stochastic and deterministic accounting prices

Figure 6: The intertemporal welfare and shadow price curves of the Pindyck example with high stochasticity ($\sigma_0 = 0.15, \sigma_M = 0.2747$). See Figure 4(b).

in the GBM case, although this finding may be sensitive to the details of the calibration of the processes and shouldn't be considered a general finding. The most notable distinction is that the shadow prices rise more rapidly at small stocks when accounting for demographic and environmental stochasticity than for GBM. Some intuition for this finding can be found through examining the effects of risk in the numerator of Equation (6). Importantly, the endogenous risk effect is always negative when volatility is increasing in s (which is always true for GBM and for low stock levels for SHM volatility). This implies that the endogenous risk aversion effect, $\frac{1}{2}\sigma^2(s)p_{ss}$, which is positive given the convexity of the price curves, must overwhelm the endogenous risk effect in order for risk to create a price premium. The widening price premium for SHM volatility at low stocks arises primarily because $\sigma^2(s)$ is greater than under GBM for any value of $s(t)$, as well as due to the greater convexity of the price curve under SHM volatility at these stock levels.

The gap between the shadow prices under GBM and SHM volatility closes rapidly in absolute terms, and somewhat more slowly in percentage terms, as stock increases, even at high risk levels (Figures 5 and 6). This happens despite the widening gap in volatility between the two specifications at high stock levels and the fact that the decline in volatility at high stock levels for SHM (Figure 4) causes the endogenous risk effect to actually reinforce the risk premium rather than work against it. Overall, the GBM and SHM accounting prices show that the risk-focused terms in Equation (6) decline in empirical importance at stocks near or above the stochastic steady state. The hyperbolic approach of $p(s)$ to zero as stock increases implies that both p_s and p_{ss} approach zero as well. The result is that the dominant effect of risk at large stocks is to shift the intertemporal welfare function down in a nearly parallel fashion (e.g., Figs. 5(a) and 6(a)) while having small absolute effects on marginal valuation of capital investments. However, the *percentage* effects of risk on marginal valuation in the SHM case remain more than twice as high as in the GBM case (Figs. 5(c) and 6(c)).

Taken in its entirety, including the consideration of a variety of non-optimal economic programs as well as non-GBM risk, our analysis of Pindyck's model suggests that stochasticity may be a relatively minor concern for natural asset pricing. This finding is robust to significant deviations in the economic program from optimal management. Indeed, we find that common managerial responses to risk—either in the form of uniform precautionary control rules relative to the optimum or adaptive rules with a bias toward system stability—yield even smaller risk adjustments to accounting prices. Moving beyond GBM specifications of risk to explicitly account for the effect of demographic and environmental stochasticity may lead to larger effects of risk on valuation, although our results suggest ignoring stochasticity in valuation be most consequential for relatively low stocks and high levels of stochasticity.

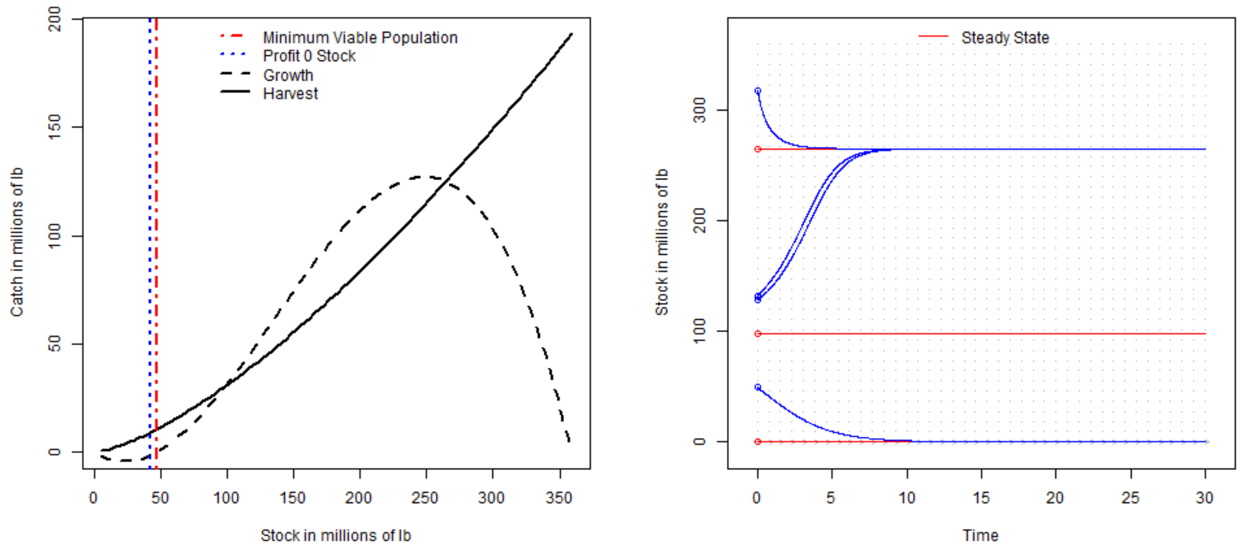
Our finding that risk has a small effect on valuation relies upon a consistent assumption that the stock follows a convex drift function yielding a single stochastic steady state—an assumption that we loosen in the following two sections.

4. Example 2: A stock with nonconvex drift and "smooth" stochasticity

The resource economics literature on nonconvex dynamical systems has primarily focused on their optimal management, especially the problem of selecting across multiple alternative stable states. Fenichel, Abbott, and Yun (2018) argue that the mathematical difficulties caused by multiple equilibria for valuation purposes—where the economic program is typically pre-determined and the relevant basin(s) of attraction are therefore known—are modest compared to optimally controlling a system in the presence of nonconvexities. However, introducing stochasticity to such a system brings with it the possibility of the system probabilistically entering or leaving one basin of attraction for another at a rate that is endogenous to the location in state space. The resulting qualitative and quantitative effects on the value of the capital stock are potentially complex and poorly understood.

As a simple illustration, we build upon the calibrated Gulf of Mexico example in Fenichel and Abbott (2014). This baseline Gulf of Mexico case, extended to introduce stochasticity, is detailed in Appendix E. The model is calibrated to reflect economic conditions in the fishery under the assumptions of logistic stock dynamics and an empirically estimated, non-optimized economic program that closely resembles the adaptive program of the previous section (Figure E.1). Importantly, income flows in the fishery are valued directly in terms of profits, so that instantaneous welfare $W(\cdot)$ is risk-neutral in monetary returns.

We alter the equation of motion of natural capital relative to Fenichel and Abbott (2014) in two ways: 1) adding geometric Brownian motion (GBM) to the deterministic drift, and 2) altering the drift function by replacing compensatory stock dynamics with dynamics reflecting critical depensation. Depensatory, as opposed to compensatory, growth occurs where the growth curve is convex in stock rather than concave at a low population level, implying that per-capita growth rates actually decline at low stocks rather than uniformly increasing (e.g., as in logistic growth). This may occur due to a variety of ecological mechanisms such as mate limitation or a breakdown of collective benefits in foraging or defense against predators (Gotelli et al., 2008). Critical depensation occurs when depensation is so strong that there is a minimum viable population; below this population the stock will not recover (in the absence of a positive stochastic shock to push the stock back over the threshold), even if



(a) Cubic growth and economic program

(b) Phase diagram of the drift term

Figure 7: The growth function (i.e. drift term) and economic program (a) and associated phase diagram (b) for the GOM reef fish model with critical depensation.

harvest were to cease. We adopt the following cubic growth function:[17]

$$ds = \left(rs(t) \left(\frac{s(t)}{K_1} - 1 \right) \left(1 - \frac{s(t)}{K_2} \right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right) \right) dt + \sigma s(t) dZ(t), \qquad (12)$$

where the values for parameters are given in Appendix E, with K_2 functioning as the same carrying capacity (3.59×10^8) as given in Equation (E.1). K_1 is the minimum viable population, which is set at 13% of K_2 for purely illustrative purposes. Figure 7 shows the growth function and economic program in panel (a) and the trajectories of stock, given different initial conditions, in panel (b) under deterministic conditions. Note that there are now three equlibria: one stable equilibrium at extinction, a stable steady state at about 265 million pounds, and an unstable steady state at about 98 million pounds. This unstable steady state—not the critical minimum viable population of K_1—marks the boundary between the basins of attraction for the two stable equilibria.

Prior to exploring the results for the depensatory case, it is useful to momentarily consider the effects of risk on natural capital valuation when stock dynamics are compensatory (logistic) but with all other dimensions of the parameterization unchanged (see Appendix E

[17]This example is mathematically similar to Maler, Xepapadeas, and De Zeeuw's (2003) shallow lakes problem, but in that problem the critical threshold occurs at high nutrient stocks rather than low stocks.

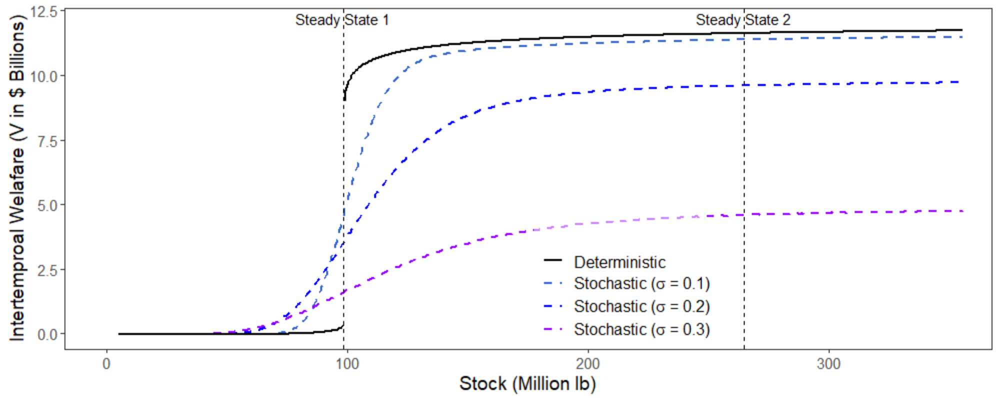
for details). Despite a tendency for risk to *lower* the shadow price, whereas in the Pindyck case risk increases it, the conclusion from the Pindyck model holds firm: *even very high levels of risk have only negligible effects on the shadow price of natural capital.*

Turning now to the decompensatory case, Figure 8(a) shows the intertemporal welfare function under varying degrees of risk.[18] To understand the effect of risk, it is useful to first understand the properties of the deterministic intertemporal welfare function. First, the intertemporal welfare function is indeterminate at the unstable equilibrium biomass, reflecting two potential trajectories in response to an infinitesimal shock. A "rightward" shock commits the system to harvests along the economic program as the stock builds to the upper stable equilibrium. The result is a concave and increasing intertemporal welfare function in this basin of attraction, which is reflected through a uniformly positive and downward sloping price curve. Conversely, a "leftward" shock commits the system to an inevitable drawdown of the stock to exhaustion.[19] Given greater harvests under the economic program at larger stocks, combined with decreasing marginal costs of harvest in stock levels, the intertemporal welfare function in this basin of attraction is convex and increasing in the stock. This is reflected through an increasing, albeit low, shadow price. Furthermore, the intertemporal welfare function experiences a dramatic decline to the left of the threshold biomass, reflecting a truly catastrophic devaluation. This devaluation is also experienced at the margin, with the marginal valuation of the stock as an investment plummeting once the system is committed to the "drawdown" equilibrium.

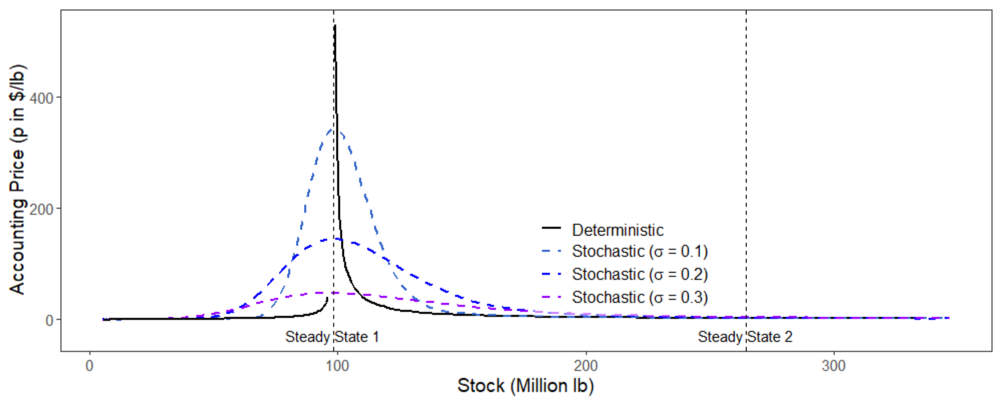
Now adding risk to the picture, it is apparent that the addition of stochasticity to this nonconvex system has qualitative and quantitative impacts on the valuation of natural capital—in total and at the margin. Focusing on the intertemporal welfare function, the former discontinuity at the unstable equilibrium biomass is replaced by functions that assume the shape of a logistic curve, maintaining the convexity properties of the deterministic intertemporal welfare function in each basin of attraction, with an inflection point located at the unstable equilibrium. The steepness of the central portion of the intertemporal welfare function declines as risk increases—acting to homogenize values on either side of the threshold relative to the deterministic case. Intuitively, this effect arises because in a small

[18] Approximation of intertemporal welfare functions and shadow prices using the same global approximation techniques employed in the previous examples is highly unstable for this highly nonconvex case. Therefore, we avoid these methods and instead utilize a robust forward-simulation approach (Appendix F).

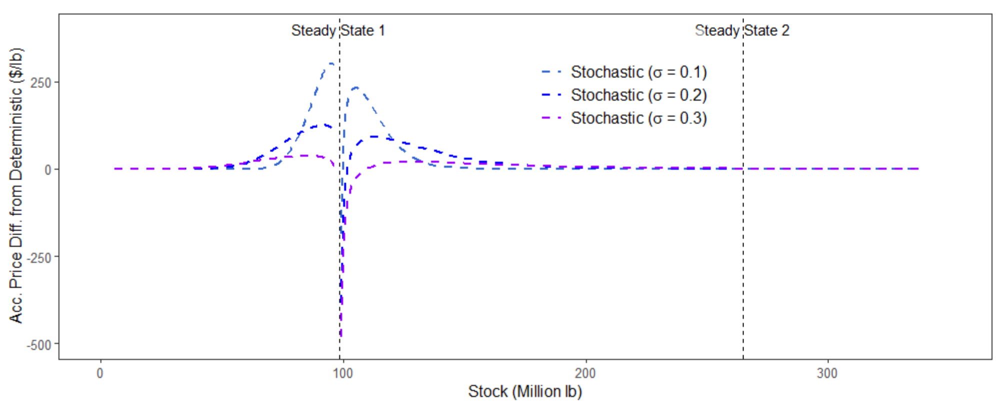
[19] Note that the economic program at very low stock levels is perhaps unrealistic, forcing fishing at low levels even when profits are negative (as in a heavily subsidized fleet), so that the intertemporal welfare function actually becomes negative, even though this is difficult to see from the figure. However, even if harvest is instead set to zero when profits are negative, the fact that the breakeven stock level occurs below K_1 (Figure 7a) still implies that extinction occurs in a deterministic system.



(a) Intertemporal welfare function



(b) Accounting price



(c) Difference of stochastic and deterministic accounting prices

Figure 8: Intertemporal welfare function and accounting price under critical depensation

neighborhood on either side of the threshold biomass, GBM stochasticity of the biomass acts roughly symmetrically, either to throw the system into the "drawdown" basin from the "sustainable fishing" basin or vice versa. The result is a smoothing of the intertemporal welfare function, where the extent of the smoothing increases in the level of stochasticity.

The qualitative effect of this risk-driven smoothing on the HJB equation is distinct depending upon the basin of attraction. For values of stock above the threshold value, risk interacts with the concavity of the intertemporal welfare function in this region to create a negative risk adjustment term in the HJB equation (5), therefore devaluing the fishery in present value terms. However, the opposite occurs below the threshold biomass, so that risk *increases* intertemporal welfare, given the convexity of the intertemporal welfare function. Intuitively, this "risk loving" valuation occurs because of the convexity of the original deterministic intertemporal welfare function in this region—reflecting that fluctuations in stock, even within the drawdown basin, generate increased expected profits relative to the deterministic case—but also because of the possibility of a "good" shock rescuing the fishery from the drawdown basin entirely. Importantly, the risk premia or discounts to the intertemporal welfare function, relative to the deterministic case, are at their largest in the immediate vicinity of the threshold and diminish as stocks move away from the threshold and the long-run value of the system is less influenced by the possibility of switching basins of attraction. This zone of influence grows in the magnitude of risk, and it persists over a wider range of stocks in the upper "sustainable fishing" basin due to the fact that the variance of GBM shocks increases in the stock size.

The effects of risk on shadow prices mirror those for the intertemporal welfare function (Figure 8(b)). Shadow prices under risk are now bell-shaped, albeit in a right-skewed manner, with risk serving to homogenize marginal valuations of investing in natural capital on either side of the threshold. Marginal valuations consistently peak at the threshold biomass itself and decline in either direction, with the rate of decline falling as stochasticity increases. Increasing levels of risk serve to homogenize marginal valuations over a wider range of stock values, as reflected in the expanding section of the intertemporal welfare function that is approximately linear in the region of the stock threshold (Figure 8(a)).

The smoothing of shadow prices on either side of the unstable equilibrium biomass can be intuited from the endogenous risk effect in equation (6). The sign of this term ($\sigma(s)\sigma_s(s)p_s = \sigma^2 s p_s$ in the GBM case) depends on the slope of the shadow price function $p_s \equiv V_{ss}$, which reverses on either side of the threshold. For stock levels in the "drawdown" basin, the endogenous risk effect increases shadow prices. Increasing natural capital increases the variance under GBM, which, given the "risk loving" convexity of the intertemporal welfare function in this region of state space, increases the marginal valuation of capital investments. The opposite effect holds for stock levels above the threshold level.

The qualitative effect of the second risk-related numerator term in the shadow price equation (6) varies across the domain of the stock. Each shadow price curve has a concave region, centered around the threshold biomass level, with two convex regions on either side. Within a zone of influence of the threshold $p_{ss} \equiv V_{sss} < 0$, so that risk (apart from the endogenous risk effect) tends to undermine the value of marginal investments in natural capital. In other words, the influence of the threshold implies a form of "imprudence" with respect to the forward-looking valuation of the stock. This apparent fatalism occurs because stochasticity, for values of stock sufficiently near the threshold, has a dominant effect on long-run stock dynamics, so that marginal investments in natural capital have relatively little effect relative to the lottery induced by stochasticity in the region of the threshold. Outside of the zone of influence of the unstable equilibrium biomass level, however, the sign of the prudence/self-insurance term in equation (6) reverses ($p_{ss} \equiv V_{sss} > 0$). Abstracting from the endogenous risk effect, this causes risk to increase the marginal valuation of capital, yielding higher marginal valuations at low and high stock levels, reflecting a degree of prudence—so that the qualitative effect of the endogenous risk aversion term now matches that seen in the convex Pindyck case.

The overall effect of stochasticity on the shadow price compared to the deterministic case is complex, depending strongly on the level of risk and the stock level. In a region just above the threshold biomass, adding risk, or increasing it from a lower level, unequivocally reduces the marginal valuation of capital. However, there is a biomass level at which risk has no effect on marginal valuation. For capital stocks above this point the shadow price under stochasticity always exceeds the deterministic shadow prices, and the gap between the stochastic and deterministic shadow price first expands and then contracts.

For stock levels below the threshold level, things are more straightforward. Stochasticity consistently increases the marginal value of the stock relative to the deterministic case, since holding more natural capital increases the chance of escaping the lower basin of attraction (i.e. the endogenous risk effect dominates). While the gap between the deterministic and stochastic shadow price may increase initially as stocks move further away from the unstable equilibrium, the general trend is for the effect of risk on marginal valuation to decrease as the stock falls and the potency of the endogenous risk effect decreases.

Increasing the level of risk does not have a monotonic effect on the marginal valuation of the stock. Figure 8 demonstrates that the stochastic shadow value curves cross in both basins of attraction. Near the threshold stock, the marginal valuation of natural capital falls as stochasticity increases, yet this effect is reversed at low and high stocks. It is possible to demonstrate that the marginal effect of risk on the shadow price is (Appendix G):

$$\frac{\partial p}{\partial \sigma} = \frac{\sigma s[2p_s + sp_{ss}]}{\delta - \mu_s(s)} \tag{13}$$

Given a positive adjusted discount rate in the denominator, the sign of (13) depends on the magnitude and sign of the impacts of increasing σ on the endogenous risk and endogenous risk aversion terms, as reflected in the two additive terms in the numerator. This tug of war can be collapsed into an inequality of a critical statistic—the coefficient of relative prudence (Kimball, 1990) of the intertemporal welfare function, $\frac{-sp_{ss}}{p_s} \equiv \frac{-sV_{sss}}{V_{ss}}$, which can be interpreted as the (negative) elasticity of the revealed (given the economic program) intertemporal risk aversion to a change in natural capital.

When stocks are above the critical threshold such that $p_s \equiv V_{ss} < 0$, the coefficient of relative prudence is positive (negative) in the case of prudence (imprudence). In this scenario $\frac{\partial p}{\partial \sigma} > 0$ only if the coefficient of relative prudence exceeds a value of 2 (Appendix G), showing that moderate prudence of the intertemporal welfare function is required for increases in risk to enhance shadow prices. Lower levels of relative prudence, or outright imprudence, result in risk lowering the marginal valuation of natural capital.[20]

When stocks are below the critical threshold such that $p_s \equiv V_{ss} > 0$, the coefficient of relative prudence reverses sign such that negative (positive) values are now consistent with prudence (imprudence). In this case $\frac{\partial p}{\partial \sigma} > 0$ as long as the coefficient of relative prudence is below 2 (Appendix G). This shows that prudence in the intertemporal welfare function, or even mild imprudence, will cause risk to inflate the marginal value of natural capital. On the other hand, a high degree of imprudence (relative prudence greater than 2) causes the shadow price to fall with increases in risk.

Finally, Figure 8(c) demonstrates that the magnitude of the errors in accounting prices from ignoring stochasticity can be extremely large, particularly in a region on either side of the threshold stock level. The direction and magnitude of the mismeasurement is also *highly* elastic to stock sizes as stocks approach the critical threshold from above, particularly for relatively low levels of stochasticity. Intriguingly, the absolute error induced by ignoring stochasticity is not monotonic in the level of volatility. Ignoring risk in valuation for relatively low levels of risk (e.g., $\sigma = 0.1$) implies a greater undervaluation of natural capital in a vicinity of stocks above the threshold biomass than for higher risk levels. However, as stocks approach closer to the threshold, the deterministic shadow price now *overvalues* stocks,

[20] While this result occurs in a non-optimized setting, it nevertheless corresponds to a finding of Mirman (1971) who showed in an optimal saving model under risk averse preferences with multiplicative shocks to wealth that precautionary saving occurs with an increase in risk if and only if relative prudence exceeds 2.

with higher σ levels leading to larger errors.[21] In general, the region affected by significant undervaluation increases in σ, although the maximum severity of undervaluation declines and the magnitude of undervaluation is less elastic to the location in state space as σ increases.

In summary, the presence of such large and complex valuation effects in such a simple model suggests that the interaction between stochasticity and nonconvex capital dynamics can make the inclusion of stochasticity in the valuation of natural capital important, with quantitative and qualitative implications that are not obvious in the absence of knowledge of the system dynamics, including the economic program, and the position in state space.

5. Example 3: A stock with convex drift and stochastic nonconvexity

The previous examples introduced stochasticity in the form of diffusions that are "smooth" in the sense that the stock does not jump discontinuously at an instant and where the distribution of stochasticity varies only through the effect of current stock on the volatility. In this context, the introduction of stochasticity to deterministic resource dynamics characterized by nonconvexity, as in Example 2, serves to "convexify" resource dynamics. Alternative stable states become stochastic steady states from whose basins of attraction it is possible to escape. The result for shadow prices is that increasing levels of stochasticity lead to smoothing of shadow prices across different stock levels, reducing the importance of the abundance of the resource as an influence on its economic scarcity.

While diffusions are an important form of risk, they are not exhaustive. An alternative notion of risk relates to the likelihood that there is an ongoing probability of an instantaneous, discontinuous shock to otherwise smooth, convex capital dynamics. In many environmental contexts these shocks are likely to have negative implications for society—crossing a climate tipping point, the eruption of a forest fire, or the (near) extinction of a beneficial species—and may result in a new regime of resource dynamics that is slow, if not impossible, to reverse. In this case, a nonconvexity in resource dynamics is induced by the stochastic process itself as opposed to the drift.[22] Importantly, the probability of "tipping" into the bad state of the world may be a function of the state of the resource; the accumulation of fuel in a forest may increase the probability of a catastrophic fire event, or the extent of overharvest of a resource may increase its probability of extinction. Such "stochastic nonconvexities" that introduce meaningful discontinuity, rather than forms of risk captured by diffusions, arguably lie at the heart of many concerns about risk in the environmental policy literature, including the classic

[21]For stocks in a region below the threshold biomass, the extent of undervaluation from using the deterministic accounting price is highest for lower σ values.

[22]Similarly, there may be a "true" deterministic threshold stock but with scientific uncertainty, characterized in terms of a probability distribution, about where this threshold is located.

precautionary principle (Gollier, Jullien, and Treich, 2000) and its intellectual descendants in the "safe operating space" literature (Steffen et al., 2015). Within resource economics, this concern has resulted in a substantial literature examining the optimal management of resources under endogenous catastrophic risk (e.g., Reed and Heras, 1992; Crocker and Shogren, 2002; Tsur and Zemel, 2021).

To focus our analysis, we consider the case of a single stock subject to risk of a catastrophic shock, where this shock can occur only once. In the absence of the shock, the stock grows deterministically according to $\frac{ds}{dt} = \dot{s} = \mu\left(s(t), x(s(t))\right)$. If the shock occurs, the infinite-horizon net benefit from that point forward is $Z(s(t))$, which allows the economic payoff to depend on the stock at the time of the arrival of the collapse.[23] Following Reed and Heras (1992), we assume that the arrival time of the shock follows a Poisson process, with an instantaneous hazard rate—the probability of a shock per increment of time, conditional on the shock not previously occurring—of $h(t)$. Importantly, while the hazard rate must be "memoryless" in the sense of not depending on the elapsed period of non-arrival or time paths of state variables, it *can* depend on current state variables, including the current resource stock, $h\left(s\left(t\right)\right)$.

Appendix H provides a derivation of the valuation of assets under these conditions. The HJB equation in this context, analogous to (5), is:

$$\delta V\left(s\left(t\right)\right) = W\left(s\left(t\right)\right) + V_s\dot{s} - h\left(s\left(t\right)\right)\left[V\left(s\left(t\right)\right) - Z\left(s\left(t\right)\right)\right] \tag{14}$$

This HJB equation is remarkably similar to that for a deterministic resource (e.g., Fenichel and Abbott, 2014). The current return on the value function is equal to the current value Hamiltonian minus one additional term, $h\left(s\left(t\right)\right)\left[V\left(s\left(t\right)\right) - Z\left(s\left(t\right)\right)\right]$. This adjustment reflects the gap in value between the continuation value in the absence of current catastrophe and the continuation value after the shock—the value at risk—multiplied by the hazard of collapse. The greater the economic cost of the shock or the greater its risk, the larger this adjustment to the current value Hamiltonian will be.

The accounting price of the resource is:

$$p(s) \equiv V_s = \frac{W_s + V_{ss}\dot{s} - h_s\left[V\left(s\right) - Z\left(s\right)\right] + h\left(s\right)Z_s}{\delta - \dot{s}_s + h\left(s\right)} \tag{15}$$

Consider the denominator of Eq (15). The first two terms are present in the deterministic

[23]This payoff function is sufficiently general to allow for an irreversible collapse (i.e. an "absorbing state") or for a shock followed by the discounted net benefits associated with the recovery trajectory and eventual return to "business as usual" management.

case (Fenichel and Abbott, 2014) and reflect the adjusted discount rate accounting for the marginal effect of investment on natural capital appreciation or depreciation. This discount rate is further adjusted by the addition of $h(s)$, reflecting the, potentially stock-dependent, risk of collapse. Eq (15) differs from Eq (6) in that the adjusted discount rate, the denominator, reflects the expected risk adjustment process, which does not happen with a diffusion. One effect of the potential for a discrete shock is to discount the benefits of stock investment more highly, lowering asset values, *ceteris paribus*.

Risk also enters into the last two numerator terms of (15). The first of these, $h_s\left[V\left(s\right)-Z\left(s\right)\right]$, reflects the marginal effect of an investment in the stock on the risk of collapse, h_s, multiplied by the value at risk, $\left[V\left(s\right)-Z\left(s\right)\right]$. This term bears some resemblance to the aforementioned "endogenous risk effect" for diffusions and is only present if the the hazard of collapse depends upon the stock, i.e. $h_s \neq 0$. In the event of a costly shock, so that $\left[V\left(s\right)-Z\left(s\right)\right] > 0$, and if an increase in the stock reduces the probability of collapse (consistent with an extinction through overharvest but not a forest fire), then this term will *increase* the shadow price of a resource, working contrary to the effect of the risk of collapse on the discount rate. The final numerator term $h\left(s\right)Z_s$ considers the possibility that the continuation value after collapse depends upon the stock level when the collapse occurs. If investments in natural capital are protective, reducing value at risk in the event of collapse, then the effect of this term will also be to increase the value of natural assets. In short, the qualitiative effect of the introduction of "stochastic nonconvexities" to an otherwise convex, deterministic setting on natural asset prices is unclear *a priori*.

We apply the same deterministic dynamics and economic specification as in our first example but now allow for the possibility of a stock collapse. The risk of collapse is decreasing in the stock and follows the rule, $h(s) = \frac{\alpha}{\alpha+s(t)}$.[24] The continuation value upon collapse $Z(s)$ is envisioned as resulting from an immediate collapse to a low stock level, followed by an infinite steady state harvest at this stock, $Z(s) = W(s = 10)/\delta$; therefore, the continuation value is invariant to the stock level upon collapse ($Z_s = 0$).

Figure 9 presents the intertemporal welfare function and accounting prices in the deterministic case and for three different scalings of endogenous risk of collapse, α. Specifically α is chosen where the probability of collapse is 0.5 at stocks of 0.5, 2, and 5, or, equivalently, where the instantaneous odds of collapse are four or ten times higher at any given stock level relative to the lowest risk case of $\alpha = 0.5$.[25] Figure 9(a) demonstrates that the risk of col-

[24]Collapse occurs with a hazard of one when s is zero and approaches zero as $s \to \infty$. α is the stock level at which the hazard of collapse is 1/2. The odds of collapse equal α/s and are, therefore, directly proportional to α.

[25]For comparison, the hazard of collapse at stocks of 20, 60 and 100 is 0.024, 0.008, and 0.005 for $\alpha = 0.5$;

lapse has a similar effect on the intertemporal welfare function as an increase in σ has in the simple diffusion case (e.g., Figure 2(a))—shrinking the expected value of the system given its management. However, the effect is not one of a pure downward shift of the intertemporal welfare function. Rather, for all three α values, the intertemporal welfare function varies much more substantially across stock levels, relative to the deterministic case. This sensitivity to stock is mirrored in the accounting prices (Figure 9(b)), which are all consistently higher than the deterministic prices across the domain of stock values. This shows that the endogenous risk effect in the numerator of (15) dominates the effect of collapse risk on the discount rate.

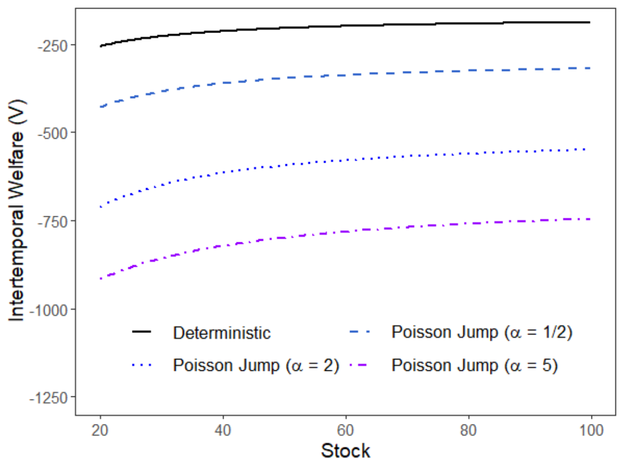
Comparing the accounting prices for $\alpha = 1/2$ and $\alpha = 2$, one might conclude that increasing the risk of collapse at any stock level will consistently increase the shadow price. However, comparing $\alpha = 2$ and $\alpha = 5$ shows this is not necessarily the case. Instead, increasing the risk of collapse can *raise* or *lower* the marginal value of investing in the stock, depending on the stock level and the size of risk increase. For stocks of roughly 28, the value-increasing endogenous risk effect in the numerator of (15), which is higher in the $\alpha = 5$ case than for $\alpha = 2$, is countervailed by the depreciating effect of the hazard of collapse in the denominator so that the marginal valuation of the stock is unchanged across these two α levels, despite the probability of collapse being more than twice as high when $\alpha = 5$. For larger stocks, the risk reducing effect of stock investments is sufficiently strong that shadow prices are higher for $\alpha = 5$ vs. $\alpha = 2$. For lower stock levels, the probability of imminent collapse at the higher α level is sufficiently extreme that this "fatalism effect" overwhelms the benefits of stock investment in reducing the hazard—a tendency that is strengthened by the fact that at low stock levels the value at risk, $V(s) - Z(s)$, also declines, reducing the value of risk-reducing investments in natural capital.

A key takeaway is that increasing the odds of collapse at any stock level may not yield consistent qualitative effects on accounting prices over the full range of stocks. Positive risk premia are likely, but risk discounts may occur at sufficiently low stocks.[26] Indeed, it can be shown, although we have not done so here for the sake of graphical clarity, that at very high levels of risk (e.g., $\alpha = 30$), the accounting price falls below the deterministic value over a range of stocks. These findings mirror those seen in Example 2 as stocks approach the critical threshold stock from above (Figure 8)—showing a parallel between the stochastic nonconvexity and nonconvex drift cases.

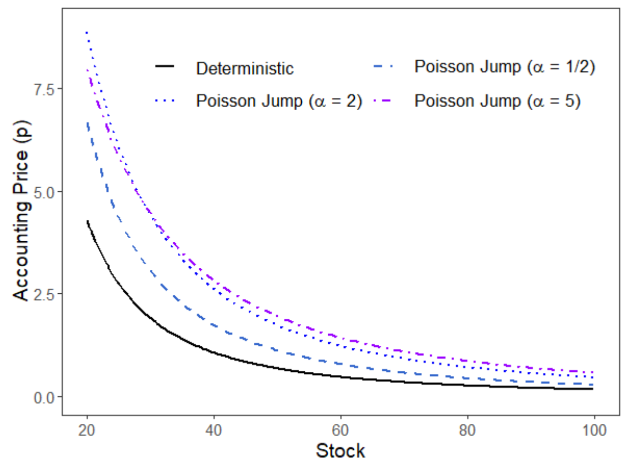
Finally, the presence of an endogenous risk of collapse can substantially affect accounting

0.091, 0.032, and 0.02 for $\alpha = 2$; and 0.2, 0.077, and 0.048 for $\alpha = 5$.

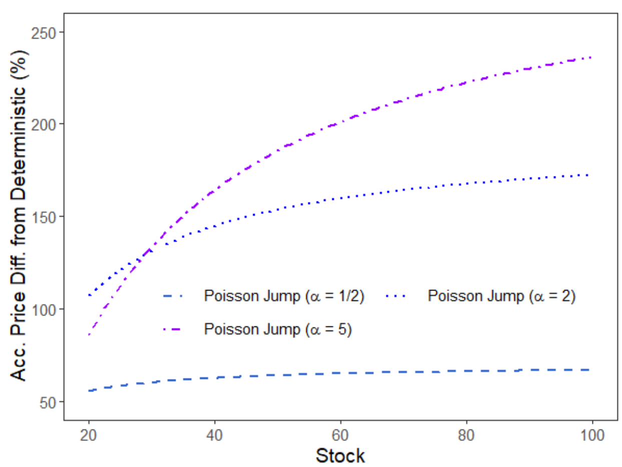
[26]This finding would reverse in the case of an endogenous risk that increases in the stock, yielding risk discounts at high stocks and premia at low stocks.



(a) Intertemporal welfare function



(b) Accounting Price



(c) Percentage difference of stochastic and deterministic accounting prices

Figure 9: The intertemporal welfare and shadow price curves under risk of a one-time Poisson shock.

prices. Consider the case of $\alpha = 1/2$. In this case, the deterministic shadow price undervalues the stock by almost 56% at $s = 20$, despite the relatively modest collapse risk. The undervaluation is driven by the effect of endogenous risk and can be substantial even when risk is relatively unresponsive to stock investments if value at risk is sufficiently high. The implication is that capital may be significantly undervalued, even at large, healthy stocks with minimal endogenous risk of collapse, if value at risk is high (Figure 9(b-c)). For example, in the case above $h_s = -4.95 \times 10^{-5}$ at $s = 100$, and yet ignoring stochasticity undervalues the accounting price by 67% given the large economic consequences of the shock. Indeed, the percentage error *increases* in stock for all α levels. This result differs from the continuous stochasticity case—even cases with nonconvex drift functions—where the valuation error from ignoring stochasticity dampens rapidly as stocks increase.

6. Conclusion

We have jointly considered two forms of "risk" that are seldom combined in the literature—stochasticity and nonconvexity in stock dynamics— to understand how they alter the realized shadow value of real assets, with a focus on natural capital. Stochasticity, as captured by diffusions with their fundamentally continuous and two-sided deviations of the capital dynamics from the deterministic drift term, are a staple of risk analysis. Indeed, the implications of diffusions for investment behavior have been extensively cataloged in the macroeconomic literature, albeit primarily in the special case of dynamically optimal investment behavior. The "dual" problem of understanding how risk influences the *valuation* of real assets subject to nonlinear growth, *conditional* on their potentially suboptimal management, is much less well understood.

We establish the theoretical channels for stochasticity to influence the value of natural capital investments: namely, 'endogenous risk' and 'endogenous risk aversion' (i.e. prudence) effects mediated through the second and third derivatives (in stock) of the intertemporal welfare function. In contrast with the optimized case with linear state equations that dominates the finance and macroeconomics literatures, the magnitude of these risk effects are not primitive features of underlying static risk preferences. Instead, they are a combined outcome of the instantaneous welfare function, the nonlinear dynamics of capital accumulation, and the feedback rule between capital stocks and consumption and investment behavior (i.e. the economic program). As a result, stochasticity can affect the shadow price of natural capital even when (as in all our examples) risk neutrality in income flows holds.

Despite the theoretical relevance of stochasticity for natural capital valuation, our investigation of a canonical resource model with a convex drift finds that this form of risk typically has only a minor influence on shadow prices. Investigating the components of the wedge between the deterministic and stochastic shadow price suggests that the risk-specific terms, while often empirically large on their own, frequently offset each other. The finding of a modest overall effect of risk on shadow prices is robust across a range of reasonable stochasticity levels and for both GBM and non-GBM stochasticity, although high levels of demographic stochasticity can have a sizable impact on shadow prices at sufficiently low stocks. Our findings hold for a range of plausible non-optimized feedback rules for the management of the resource.

By contrast, the introduction of nonconvexity *and* risk in capital accumulation—either in the form of a nonconvex drift term paired with stochastic shocks or through a probabilistic risk of "collapse" into an alternative basin of attraction—may have far more consequential impacts on the valuation of natural capital. When stochastic noise in stock growth is added to an otherwise deterministic nonlinear resource system with alternative stable states, we find strong and complex effects on the valuation of natural capital. These effects are especially large in the vicinity of the transition between basins of attraction, where the zone of influence increases with the magnitude of the stochasticity. The magnitude and direction of the bias from ignoring stochasticity depends on the location of the stock within state space.

Risk is especially important for valuing natural capital investments in the case of "stochastic nonconvexities", illustrated in our case by an endogenous (in stock) risk of collapse to an alternative, less valuable, state in an otherwise convex system. We find that the presence of this discontinuous and asymmetric risk dramatically affects the value of natural capital even when investments in the capital stock have a tiny influence on the odds of collapse. In short, while the literature on downside risk and insurance has emphasized that risk is comprised of the probability and the consequence (i.e., value at risk) of a negative shock (Ehrlich and Becker, 1972; Crocker and Shogren, 2002), the sheer scale of the negative consequences and long-lived nature of adverse shocks in natural resource systems can inflate the self-protective value of investments in natural capital. This mechanism is lacking in the case of pure stochasticity, where the combination of small and repeated two-sided risks, paired with convex, stabilizing drift dynamics, limits the possibility of truly catastrophic shocks—perhaps leading to a greater emphasis on insurance.

While we have significantly advanced the frontiers of understanding of how risk alters the valuation of natural capital, our analysis of this expansive domain of research is inherently incomplete. For example, there are multiple places where risk may be important to consider in valuation apart from the capital dynamics themselves. One may be concerned about "implementation error" in the economic program, so that responses to changes in the stock are understood only approximately. There may also be uncertainty about the static valuation of flows of benefits and costs stemming from natural capital (i.e. $W(\cdot)$ in our terminology), as in the climate change literature, where risk in the warming response and the economic damages from warming combine to make the "dividends" of human welfare uncertain (e.g., Van den Bremer and Van der Ploeg, 2021; Lemoine, 2021; Cai and Lontzek, 2019).

Second, the theory and practice of risk-adjusted pricing of natural capital should expand beyond the single stock paradigm to consider multiple capital stocks, linked through both their deterministic drift terms (e.g., through ecological interactions) and through correlated stochastic shocks. Doing so will enable decision makers to examine how linked capital stocks function as portfolios. Previous work has examined this question in the deterministic case for multiple ecologically-linked stocks of natural capital (Yun et al., 2017). It may also be important to consider the potential for uncertainty in the extent to which capital stocks are substitutable in either production or consumption (Gollier, 2019). An important special case of this may be interactions between the state of natural resources and the insurance market, e.g., how does the stock of forests and risk wildfire interact with the valuation of the insurance sector (Boomhower et al., 2024).

Looking beyond the domain of natural capital, our analytical approach may be useful for accounting for physical capital—in particular for valuing bespoke machinery or software that has thin or non-existent secondary markets. This approach provides an alternative to the perpetual inventory model for asset valuation, the go-to method in national accounts, and may be useful for assets that have not been traded in the market for a considerable period of time (e.g., some properties). The maintained assumption in national accounts is that there are insurance markets for risks to capital assets, yet in practice governments often act as insurers and distort insurance markets. Therefore, our insights may be relevant to a larger number of assets that are subject to uninsured risk—particularly those that exhibit nonlinear dynamics and nonconvexity.

Given these many opportunities for research and the growing demands for the valuation of natural capital for the purposes of sustainability assessment and program evaluation, we hope our analysis represents an early installment of much forthcoming research on the effects of risk on natural capital pricing. Altogether, our modeling highlights the importance of grounding natural capital valuation in the context of coupled natural-human systems models that clarify the nature of risk's influence on valuation and its interaction with features of the underlying system dynamics, including natural processes and the human investment and consumption decisions that affect them. Admittedly, such fully specified models may not be feasible or practical in many real-world cases. Nevertheless, we believe the insights derived from even stylized examples, such as those explored in this paper, can prove invaluable for researchers and national accountants as they seek to appropriately incorporate risk in the pricing of natural assets.

Acknowledgements

Joshua Abbott acknowledges funding from Lenfest Ocean Program Contract ID 00029728. Eli Fenichel acknowledges support from the Knobloch Family Foundation.

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Appendix A. Accounting prices under non-optimized economic programs

In this appendix we provide some insight into how non-optimal economic programs alter the marginal valuation of capital stocks, relative to the co-state evaluated along the optimal program. Let the non-optimal program be $x(s)$, while the optimized program is $x^*(s)$. The intertemporal welfare function along the non-optimal economic program is $V(s, x(s)) = V(s)$. The shadow price can therefore be written as:

$$V_s = \frac{dV}{ds} = \frac{\partial V}{\partial s} + \frac{\partial V}{\partial x}\frac{dx}{ds} \tag{A.1}$$

By the maximum principle and as a consequence of the envelope theorem $\frac{\partial V}{\partial x} = 0$ when $x(s) = x^*(s)\ \forall s$. In this case (A.1) collapses to $V_s = \frac{dV}{ds} = \frac{\partial V}{\partial s}$, such that the properties of the economic program play no role in the shape of the intertemporal welfare function when the economic program is chosen optimally. Equation A.1 suggests that the shadow price can be cleanly broken into two terms: 1) the partial derivative of the intertemporal welfare function w.r.t. s, ignoring any behavioral feedbacks on natural capital investment, which is present in both optimized and non-optimized cases; and 2) a component that is only present in non-optimized cases (i.e. when $\frac{\partial V}{\partial x} \neq 0$), therefore reflecting a form of distortion or bias from non-optimality. This interpretation is *almost* correct but fails to account for the fact that $\frac{\partial V}{\partial s}$ is evaluated at the non-optimized value of x for a given s, whereas it is evaluated at the optimal x in the optimized case.

To account for this, add zero to (A.1) in the form of $\frac{\partial V(s,x^*(s))}{\partial s} - \frac{\partial V(s,x^*(s))}{\partial s} = \frac{\partial V^*(s)}{\partial s} - \frac{\partial V^*(s)}{\partial s}$

$$V_s = \frac{\partial V^*(s)}{\partial s} + \left(\frac{\partial V}{\partial s} - \frac{\partial V^*(s)}{\partial s}\right) + \frac{\partial V}{\partial x}\frac{dx}{ds} \tag{A.2}$$

This equation tells us that we can write the shadow price as the optimal shadow price (which is, by definition, evaluated at the optimal value of the economic program) plus 1) a first distortion term that is the gap in the partial derivative of the intertemporal welfare function that arises solely through the gap between $x(s)$ and $x^*(s)$, and 2) a second distortion term that is a function of the sensitivity of the non-optimized intertemporal welfare function to the economic program $\frac{\partial V}{\partial x}$ and the slope of the non-optimized economic program.

The first distortion arises only because the partial derivative of V is evaluated at a level of x that does not maximize intertemporal welfare. If the optimal and non-optimal economic program coincide at a particular s then this term vanishes. For example, suppose that the optimized and non-optimized economic programs both achieve the same steady state, s^{SS}, so that $x(s^{SS}) = x^*(s^{SS})$, then this distortion would vanish at the steady state stock.

The second distortion arises because of the slope of the suboptimal economic program, rather than its level. Indeed, the more responsive the economic program is to the state, then the larger $\frac{dx}{ds}$ is in absolute terms and the greater this distortion will be. Moreover, the magnitude of distortion is magnified by the extent of the non-optimality of the economic program, so that as $\frac{\partial V}{\partial x}$ increases in absolute terms, the overall distortion increases. We may also be able to sign the distortion. For example, at the non-optimal level of $x(s)$, if it would improve intertemporal welfare to harvest less instantaneously ($\frac{\partial V}{\partial x} < 0$) and the economic program is upward sloping ($\frac{\partial x}{\partial s} > 0$), then we know that the distortion is negative, causing suboptimal management to lead to an undervaluation of the stock.

Appendix B. Numerical approximation of the HJB equation using Chebyshev polynomial basis functions

Fenichel, Abbott, and Yun (2018) and Yun et al. (2017) describe how the HJB equation can be combined with functional approximation approaches frequently used in numerical dynamic programming to approximate the entire shadow price *function* over a closed domain of capital stocks. In the deterministic setting, they advocate approximating $V(s(t))$ using the HJB equation, by replacing $V(s(t))$ on the LHS of the equation with a weighted sum of the Chebyshev basis functions in the stock vector $s(t)$ and replacing the partial derivatives of the value function on the RHS with the partial derivatives of this approximation. The coefficients that determine the weights on the basis functions can be solved analytically and are chosen (in a system with as many approximation points as coefficients) to make the LHS and RHS of the approximated HJB equation hold with equality.

This value (intertemporal welfare) function approximation technique can be adapted with relatively minor changes to the stochastic diffusion case. First, define the bounded approximation interval for the state variable. Then choose M evaluation points within this interval and calculate $W\left(s\left(t\right), x\left(s\left(t\right)\right)\right)$, $\mu\left(s(t), x\left(s(t)\right)\right)$, and $\sigma^2(s(t), x(s(t)))$ at each point s^m. We define $\Phi(s)$ as the $M \times (q+1)$ basis matrix of qth degree for the state variable. This is a matrix of $q+1$ basis functions— Chebyshev polynomials of ascending degree in our case—evaluated at the M evaluation points. We can now define our approximation to the intertemporal welfare function $V\left(s^m\right) \approx \Phi^m\left(s\right)\beta$ where $\Phi^m\left(s\right)$ is the mth row of $\Phi\left(s\right)$, and β is a $(q+1) \times 1$ vector of unknown approximation coefficients. Using the fact that $\frac{\partial V(s^m)}{\partial s} \approx \left(\frac{\partial \Phi^m(s)}{\partial s}\right)\beta$ we can replace the HJB equation in Equation (5) with the following approximation:

$$\delta\Phi^m\left(s\right)\beta = W\left(s^m\right) + \left[\mu\left(s^m\right)\left(\frac{\partial \Phi^m(s)}{\partial s}\right)\beta + \frac{1}{2}\sigma^2(s^m)\left(\frac{\partial^2 \Phi^m(s)}{\partial s^2}\right)\beta\right]$$

Collecting terms involving β yields:

$$\left[\delta\Phi^m\left(s\right) - \mu\left(s^m\right)\left(\frac{\partial \Phi^m(s)}{\partial s}\right) - \frac{1}{2}\sigma^2(s^m)\left(\frac{\partial^2 \Phi^m(s)}{\partial s^2}\right)\right]\beta$$

$$\equiv \Psi^m(s)\beta = W(s^m)$$

Stacking these M vector equations results in the equation $\Psi(s)\beta = W(s)$. If $M = (q+1)$ (i.e. the number of approximation points equals the number of unknown approximation coefficients) then the approximation coefficients can be calculated in a straightforward way through matrix inversion. Alternatively, if $M > (q+1)$ then the β can be found using least squares:

$$\beta = \left(\Psi(s)'\Psi(s)\right)^{-1} \Psi(s)'W(s)$$

After obtaining the approximation $\Phi(s)\beta$ it is straightforward to find the shadow values of any given capital stock by taking its partial derivative.

Figure B.1 demonstrates that this collocation approach approximates both the known value function and the shadow price functions to acceptable accuracy levels for the Pindyck model under optimal management.

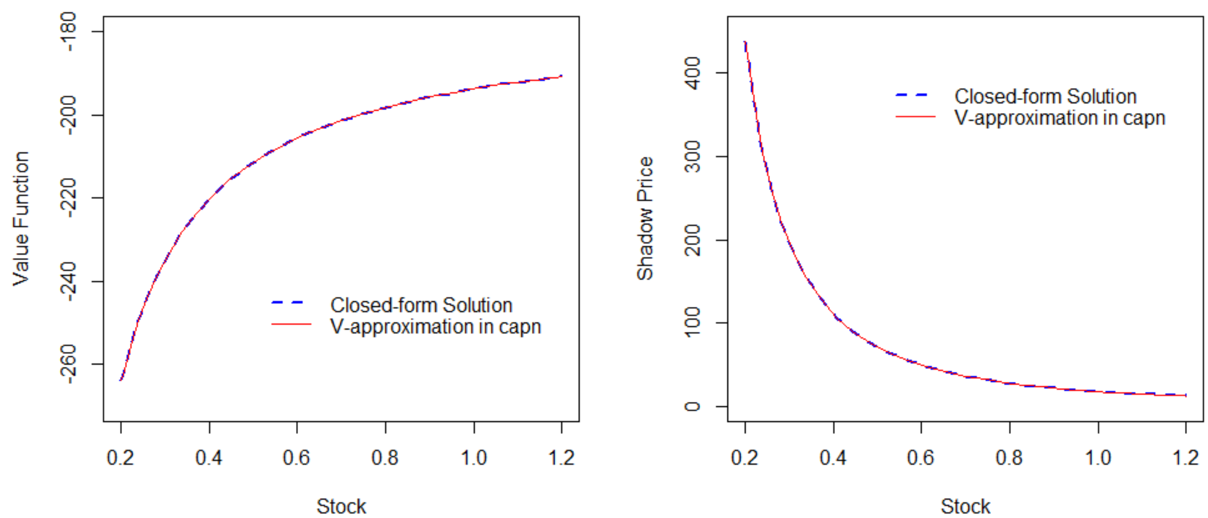


Figure B.1: Illustrating that the natural capital asset pricing approximation approach reproduces the known intertemporal welfare function and price curves for the Pindyck model with a known (optimal) economic program. The mean absolute percentage deviations are .0001% for V and .089% for p, respectively.

Appendix C. Extension of Example 1 with demographic and environmental stochasticity

We adopt the notation of Sims, Horan, and Meadows (2018). Specifically, $\sigma^2(s)$ in the case of GBM is

$$\sigma_0^2 s^2 \tag{C.1}$$

whereas $\sigma^2(s)$ with both demographic and environmental stochasticity is

$$\tilde{r}s\left(\frac{\tilde{M}+\tilde{C}}{\tilde{M}-\tilde{C}}+(1-2\mu)\frac{s}{\tilde{K}}\right)+\sigma_M^2\tilde{R}^2s^2\left[1-\frac{s}{K_1}\right]^2 \tag{C.2}$$

The first additive term reflects demographic stochasticity, while the second accounts for environmental stochasticity.

See Sims, Horan, and Meadows (2018) for a derivation of this expression and explanations of the parameters. We utilize the parameter values reported in Table C.1.

Table C.1: Parameter values

Economic parameters		Ecological parameters	
Parameter	Value	Paramter	Value
b	1	$\tilde{R}$	1
η	1/2	$\tilde{M}$	0.53
c	5	$\tilde{C}$	0.03
γ	2	$\tilde{r}$	0.5
δ	0.05	$\tilde{K}$	100
		K_1	114.3
		μ	1

Appendix D. Supplementary figures & tables

Table D.1: Accounting Price Decomposition

Case	Stock	Deterministic (a)	Endogenous Risk (b)	Prudence (c)	Drift Valuation (d)	All Risk Terms (e)=(b+c+d)	Total (a)+(e)
Optimal	0.2	427.334	221.792	-332.688	123.856	12.961	440.295
	0.4	106.833	-13.734	20.601	-3.627	3.240	110.074
	0.8	26.708	-0.9823	1.473	0.319	0.810	27.518
0.5 x Optimal	0.2	564.015	79.666	-119.498	52.803	12.970	576.985
	0.4	141.004	-52.312	78.468	-22.913	3.243	144.246
	0.8	35.251	-1.585	2.377	0.018	0.811	36.062
1.5 x Optimal	0.2	505.995	-161.517	242.268	-58.217	22.533	528.528
	0.4	126.499	-9.956	14.933	0.6556	6.633	132.132
	0.8	31.625	-0.992	1.489	0.912	1.408	33.033
Adaptive	0.2	483.256	128.608	-193.801	76.988	11.795	495.052
	0.4	114.877	-13.119	19.821	-3.894	2.809	117.686
	0.8	23.876	-0.660	0.952	0.284	0.576	24.452

Appendix E. Gulf of Mexico reef fish, with convex growth and non-optimal management

The Gulf of Mexico reef fish example presented in Fenichel and Abbott (2014) has many of the same properties as the Pindyck (1984) model. Zhang and Smith (2011) estimated a logistic growth equation for the stock, and Zhang (2011) estimated an empirically-grounded feedback rule with similar properties as the adaptive rule illustrated in Example 1—though Zhang's rule is based on real-world management rather than being designed to achieve the optimal equilibrium (Figure E.1).

We extend this deterministic model to the stochastic case while maintaining all other calibrated values. As in Pindyck, we augment the logistic stock dynamics with an additive geometric Brownian motion (GBM) noise term. Geometric Brownian motion is consistent with the assumptions of log-normal disturbances frequently used in biological population dynamic modeling and fisheries stock assessment. Utilizing the residuals from assessed biomass data relative to the deterministic model we estimate $\sigma = 0.067$; therefore the standard deviation from the deterministic drift given by the logistic growth equation with harvest is approximately 6.7 percent of the stock level. The stock dynamics are:

$$ds = \left(rs(t)\left(1 - \frac{s(t)}{K}\right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right)\right) dt + \sigma s(t) dZ(t), \qquad \text{(E.1)}$$

where the intrinsic growth rate $r = 0.3847$ and carrying capacity $K = 3.59 \times 10^8$. The economic program, the feedback relationship linking stock status (in pounds (lbs)) and effort (in crew-days) in the fishery, is provided by a power rule, $x(s) = ys^{\gamma}$, where $\gamma = 0.7882$ and $y = 0.157$. We assume that the valuation of income flows in the fishery is directly expressed in terms of monetary profits, with price-taking firms and costs that are linear in effort: $W = mh - cx$, with $m = \$2.70$/lb., $c = \$153$/crew-day. Instantaneous welfare is thus "risk-neutral" in monetary flows. The production function for harvests is of a generalized Schaefer form $h = qsx\left(s\right)^{\alpha}$, with $q = 3.17 \times 10^{-4}$ and $\alpha = 0.544$. $W(s)$ is a strictly *convex* function of the stock once the endogenous feedback from the stock level to harvest behavior $x(s)$ is incorporated, despite the linearity of harvests and costs for a fixed allocation of effort x. Abstracting from stochasticity, Figure E.1 shows the specification of the system is similar to the Pindyck model with the adaptive control rule.

Figure E.2 illustrates stochastic simulations of stock paths originating from the steady state biomass and harvest using four levels of stochasticity. The level of noise introduced by stochasticity in the base case (Fig. E.2a) is reminiscent of the noise seen in many ecological systems. Our numerical simulations nevertheless show that the number of paths that tend to a numerical zero level increase dramatically with increases in σ. Indeed, all paths reach

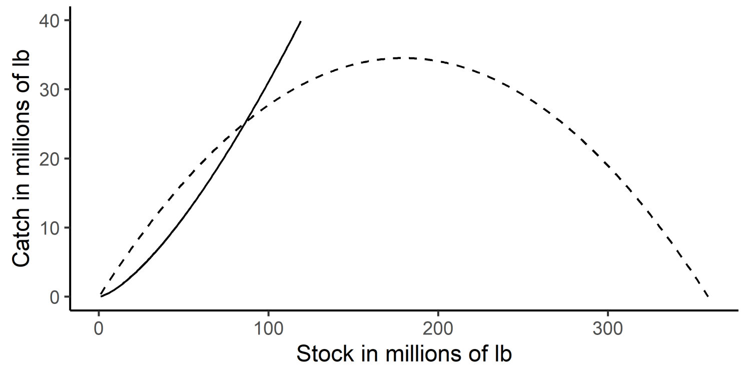
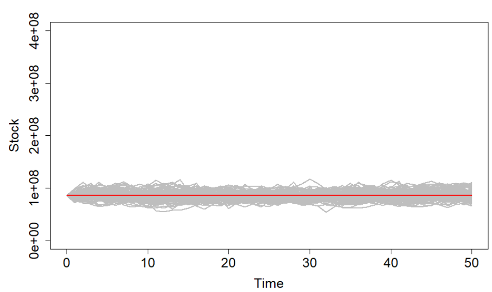
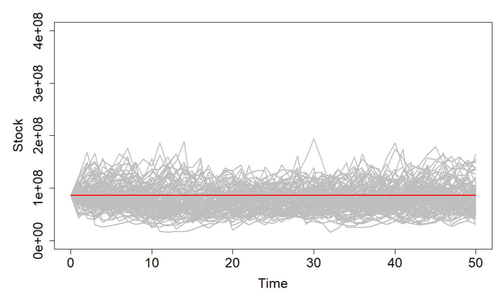


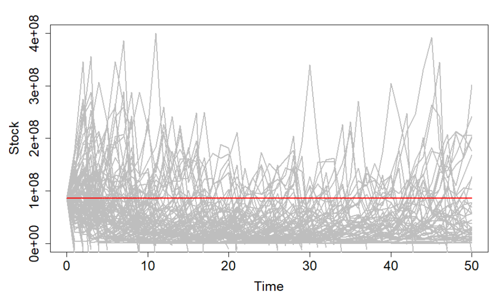
Figure E.1: The growth function and economic program for the Gulf of Mexico model with concave growth.



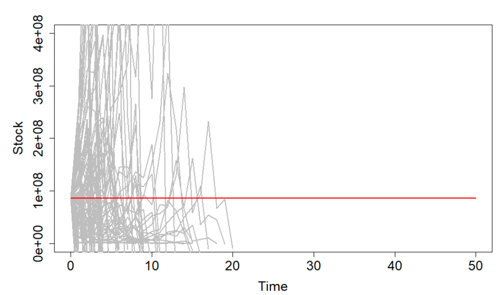
(a) $\sigma = .067$



(b) $\sigma = .2$



(c) $\sigma = .5$



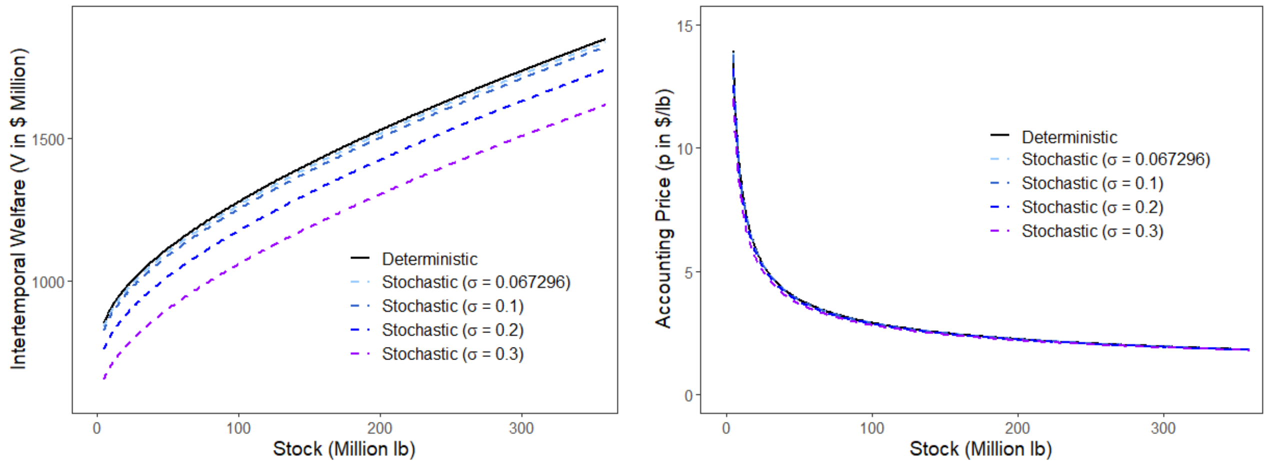
(d) $\sigma = 1$

Figure E.2: Stochastic simulations of stock dynamics for the Gulf of Mexico reef fish example with concave growth function over a range of values for σ. Note that the values for $\sigma = 1$ exceed the range of the graph on a number of runs.

numerical extinction within 20 periods when $\sigma = 1.0$. This suggests that levels of σ of 0.5 or 1.0 are likely inconsistent with the dynamics of most real-world species.[27]

Unsurprisingly, increasing levels of volatility reduce the intertemporal welfare function

[27] Dixit and Pindyck (1994) establish in an unharvested version of this model that the resource goes extinct with a probability of 1 if $\sigma^2 \geq 2r$. We thank Martin Quaas for bringing this to our attention.



(a) Intertemporal welfare function

(b) Accounting Price

Figure E.3: The intertemporal welfare function and shadow price function of the Gulf of Mexico reef fish example (concave growth function) with four different values of σ

(Figure E.3). Following Eq. (5) and the Pindyck example, the intertemporal welfare function in the stochastic case includes an additional risk term that serves, in part, to shift the intertemporal welfare function downward with increasing stochasticity. In the current case, the intertemporal welfare function is concave in s (i.e. the shadow price curve is downward-sloping) so that increasing σ reduces the expected net present value at any given stock level. This adjustment is relatively small – 0.9% for $\sigma = 0.067$ and 1.9%, 7.5%, and 16.2% for the successively higher levels of σ – suggesting that the economic program is fairly robust to the level of stochasticity in the system by maintaining stock levels in a relatively insensitive range of the profit function.

Higher-order effects on the shape of the intertemporal welfare function with increases in σ exist, but are small. Changes in the shadow prices (i.e. the derivative of the intertemporal welfare function) are hardly noticeable (Fig. E.3 (b)) and suggest the volatility is creating a nearly vertical shift in the intertemporal welfare function. Thus, while risk devalues the stock in total (albeit mildly), stochasticity has little appreciable effect on its *marginal* valuation. Lost in the small magnitude of these changes is a potentially interesting insight; whereas in the Pindyck example, increasing risk increases the marginal valuation of the stock, in this case risk *lowers* the shadow price.

Consider the value of a change from the observed equilibrium to the stock level supporting maximum sustained yield or half of carrying capacity. The change in the intertemporal welfare function for the deterministic case is $244 million, whereas in the stochastic case the value is $243 million.Therefore, using values from a deterministic model as a stand-in for the appropriate values under stochasticity appears to overvalue the change in welfare or wealth slightly. This differs from Pindyck's optimized example, where failure to consider risk leads to a slight *undervaluation* of the value of stock changes. Albeit, the undervaluation is slight, at only 0.4%.

Appendix F. Numerical approximation of the HJB equation using forward simulation

In principle, basis function approximation methods using collocation methods (i.e. where approximation nodes are equal to the number of basis coefficients to recover) exactly satisfy the HJB equation (5) at all approximation points and utilize the flexibility of the semi-parametric structure of the basis coefficients to interpolate between these points (Fenichel, Abbott, and Yun, 2018).[28] However, in practice we have found that basis function global approximation approaches of the HJB equation, at least those using Chebyshev polynomials, perform poorly at approximating the highly non-linear shape of the intertemporal welfare function for Example 2 while also providing approximations of its first and second derivatives. Approximations are particularly poor in the highly curved (and in the deterministic case discontinuous) region around the unstable equilibrium. Furthermore, attempts to improve the quality of approximation in this region through placement of nodes or the use of additional basis functions tends to degrade the quality of the approximation in the more distant portions of the approximation domain from the unstable equilibrium.

In Example 2 (to get the results of Figure 8), we employ a straightforward but computationally intensive forward simulation method over a dense grid of 600 evenly gridded starting stock values to estimate the intertemporal welfare function.[29] For each initial value of stock, we first generate 100,000 stochastic stock dynamics according to (12) using 800 time steps with $\Delta t = 0.01$ using the *Sim.DiffProc* package in R software (Guidoum and Boukhetala, 2020; R Core Team, 2021). We then calculate the numerical sum of the present value of profit for each trajectory, which is the Riemann sum to numerically approximate an integration with a midpoint rule. We then average over these 100,000 simulated value functions to obtain a Monte Carlo integration approximation to the intertemporal welfare function (2).

To approximate the shadow price functions, we adopt a two-step approach; first, we fit an analytically differentiable approximation of the value function to the simulated value function data, and, second, we calculate the first-order derivative of this fitted curve to obtain the accounting price function. In our experience, this approach is computationally more efficient and numerically more accurate than the direct calculation of derivatives of the value function by brute force. We produce the analytically differentiable approximation to the value function in Example 3 in two steps. First, we fit a smoothing spline to the 600 simulated

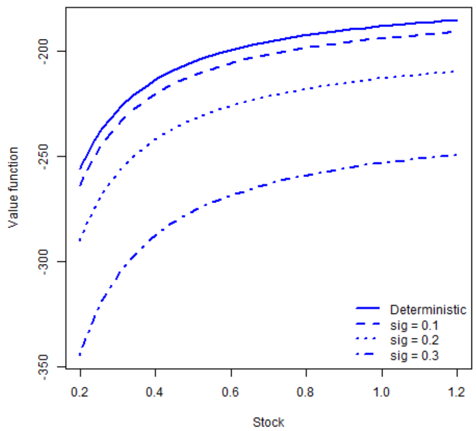
[28] Approximation using an overdetermined system (i.e. more nodes than coefficients) does not guarantee exact approximations at any point, but rather trades off approximation error at all points (Fenichel, Abbott, and Yun, 2018).

[29] A similar approach was suggested by Hamilton and Ruta (2009).

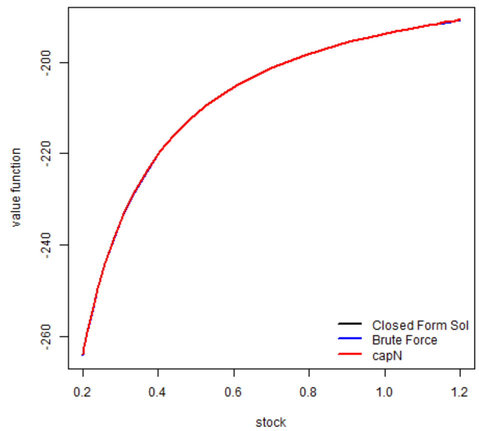
data points and generate predictions from this approximation at 300 Chebyshev polynomial nodes. We then utilize these nodes to fit a Chebyshev polynomial approximation. The Chebyshev polynomial approximation of the value function has the advantage of generating analytical derivatives and hence a direct approximation of $p(s)$. However, directly fitting a Chebyshev polynomial approximation to the brute force simulation data is not feasible for the reason that the number of polynomials required to produce a strong global approximation of the value function, in combination with the Monte Carlo simulation error, leads to significant instabilities in the implied first derivatives. Intuitively, the tendency of Chebyshev approximations to create errors that oscillate evenly across the approximation domain (Miranda and Fackler, 2004) leads to unstable derivative estimates as the approximation chases the simulation errors at particular data points. The smoothing spline provides an estimate of the value function from the brute force simulation data that does not suffer from this shortcoming; however, it has the flaw of not being analytically differentiable. Fitting the Chebyshev approximation to the data created by the smoothing spline provides a direct estimate of the accounting price function that doesn't rely upon secant or other numerical approximations.

The Pindyck example presented in Section 3 is useful for evaluating the properties of the brute force forward simulation since it provides a known closed-form solution for the intertemporal welfare function and shadow price function against which to compare our approximation. We use it to compare the accuracy of the numerical approximation of Chebyshev polynomial approximation and the "brute force" forward simulation approach. Table F.1 represents the error size in RMSE and running time with the different levels of stochasticity and approximation setup. The left panel of Table F.1 shows the Chebyshev polynomial approximation with 20, 35, and 45 Chebyshev polynomials in the approximation. As expected, more approximation nodes generally provide smaller RMSE with a slightly longer running time. In the case of $\sigma = 0.1$, the lowest RMSE is achieved with $n = 20$, with larger numbers of nodes being required as σ increases. With $\sigma = 0.3$, more than 41 approximation nodes face a singularity issue and the best fit we can achieve is with 41 nodes.

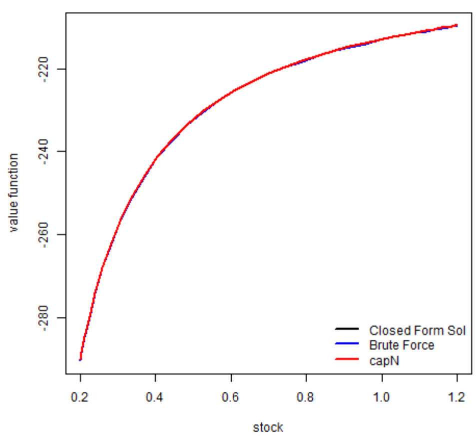
The right panel of Table F.1 demonstrates the numerical performance of brute force forward simulations. Results are produced on a Windows machine with Intel(R) Xeon(T) CPU E5-1603 (2.80 GHz) processor and 64GB RAM, with the simulations parallelized over 4 cores. In the brute force approximation, 0.01 time interval for the final time 200 are replicated for 1,000, 10,000, 50,000 and 100,000 on the 35 Chebyshev nodes used in the Chebyshev polynomial approximation. The graphical comparison across the closed form solution, Chebyshev polynomial approximations, and brute force forward simulation are shown in Figure (F.1). Panel (a) of Figure (F.1) is the closed form solution of the value function for the deterministic and stochastic models. The other panels in Figure (F.1) are a graphical comparison of the closed form solution (black), Chebyshev polynomial approximation (red), and brute force forward simulation (blue) across different volatility levels.



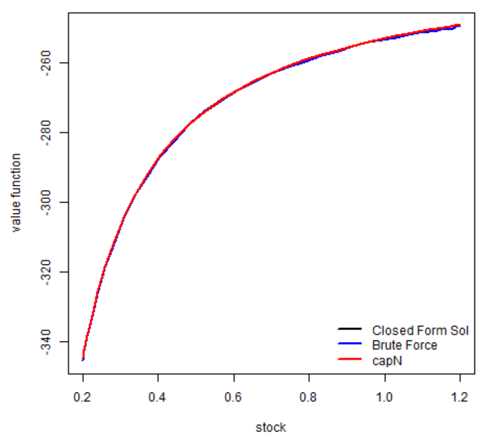
(a) Closed form solution



(b) $\sigma = 0.1$



(c) $\sigma = 0.2$



(d) $\sigma = 0.3$

Figure F.1: Comparisons of numerical approximations of the intertemporal welfare function for the Pindyck model: Closed form solution vs. Chebyshev polynomial approximation ($n = 35$ in the left panel of Table F.1) and brute force forward simulation with 100,000 replications (the right panel of Table F.1)

Table F.1: Comparison of numerical performances: Chebyshev polynomial and brute force approximation results are compared with the closed form solution.

	Chebyshev			Brute Force (n = 35)		
	no. nodes	RMSE	Running Time	no. rep.	RMSE	Running Time
$\sigma = 0.1$	n= 20	0.0005	0.0736 sec.	rep = 1,000	0.3171	4.9330 mins.
	n= 35	0.0006	0.1090 sec.	rep = 10,000	0.1170	48.7790 mins.
	n= 45	0.0063	0.1779 sec.	rep = 50,000	0.0913	4.0080 hrs.
				rep = 100,000	0.0705	8.0232 hrs.
$\sigma = 0.2$	n= 20	1.5941	0.0579 sec.	rep = 1,000	0.6600	4.9177 mins.
	n= 35	0.0044	0.1322 sec.	rep = 10,000	0.2673	48.6094 mins.
	n= 45	0.0002	0.1356 sec.	rep = 50,000	0.1557	3.9993 hrs.
				rep = 100,000	0.1131	7.9375 hrs.
$\sigma = 0.3$	n= 20	0.4894	0.0779 sec.	rep = 1,000	1.4715	4.8928 mins.
	n= 30	0.0639	0.1159 sec.	rep = 10,000	0.6785	48.6649 mins.
	n= 41*	0.0401	0.1243 sec.	rep = 50,000	0.3641	4.0589 hrs.
				rep = 100,000	0.3166	8.0619 hrs.

Note: 'no. nodes' is the number of Chebyshev approximation nodes and 'no. rep.' is the number of replication in each simulation.
* With $\sigma = 0.3$, $n > 41$ faces singularity issues and cannot be approximated.

Table F.1 presents the numerical performances of Chebyshev polynomial and brute force approximation. It is obvious that Chebyshev polynomial approximation dominates the error size and running time for all volatility cases. If Chebyshev polynomial approximation works, its approximation performance will dominate brute force approximation. However, Chebyshev polynomial approximation often fails to converge (e.g., Example 3) or produces overfitting issues (e.g., overly wavy value functions or shadow price functions). Despite its heavier computational burden, brute force approximation has the value of robustness relative to Chebyshev polynomial approximation and other global approximation approaches – particularly for highly non-linear intertemporal welfare function and shadow price curves as in Example 2.

Appendix G. The effect of risk on the accounting price (Example 2)

We start with the definition of the accounting price (6) and then substitute in for the fact that $\sigma(s) = \sigma s$ in the case of GBM.

$$p(s) = \frac{W_s + [\mu(s) + \sigma^2 s]p_s + \frac{1}{2}\sigma^2 s^2 p_{ss}}{\delta - \mu_s(s)} \tag{G.1}$$

Differentiating (G.1) with respect to the volatility parameter we obtain

$$\frac{\partial p}{\partial \sigma} = \frac{\sigma s[2p_s + sp_{ss}]}{\delta - \mu_s(s)} \tag{G.2}$$

Assuming that the modified discount rate, $\delta - \mu_s(s)$, is positive, the sign of (G.2) matches that of its numerator. The first numerator term reflects the effect on valuation through the "endogenous risk" effect – increasing σ raises the increment of risk from an investment in the stock by $2\sigma s$, which is valued through the curvature of the intertemporal welfare function ($V_{ss} \equiv p_s$). This term is positive (negative) if $p_s > 0$ (< 0). In other words, when $V(\cdot)$ is "risk-loving" (as below the unstable equilibrium in Example 3) then the endogenous risk effect causes increases in risk to raise the marginal valuation of the stock. This relationship reverses when $V(\cdot)$ is risk-averse.

The second numerator term reflects the effect on valuation through the "endogenous risk-aversion" or "self-insurance" effect. A marginal change in s increments risk aversion in the stock by $p_{ss} \equiv V_{sss}$ which then is used to evaluate the effect of an increase in the overall variance from an increase in σ of σs^2. If the intertemporal welfare function (IWF) reflects imprudence in the space of capital stocks ($p_{ss} < 0$), as in a region around the unstable equilibrium in Example 3, then an increase in risk from σ will tend to decrease the marginal valuation of the stock, whereas prudence ($p_{ss} > 0$) will increase the marginal valuation. The overall effect of increasing σ on $p(s)$ depends on the balance of the endogenous risk and endogenous risk-aversion effects. If $2p_s + sp_{ss} > 0$ (< 0) then $\frac{\partial p}{\partial \sigma} > 0$ (< 0).

Specifically, for s above the unstable equilibrium, such that the intertemporal welfare function is risk averse in capital fluctuations ($p_s \equiv V_{ss} < 0$), it follows that

$$\frac{\partial p(s)}{\partial \sigma} \begin{cases} < 0 & \text{if } \frac{-sp_{ss}}{p_s} < 2 \\ = 0 & \text{if } \frac{-sp_{ss}}{p_s} = 2 \\ > 0 & \text{if } \frac{-sp_{ss}}{p_s} > 2 \end{cases} \tag{G.3}$$

where $\frac{-sp_{ss}}{p_s} \equiv \frac{-sV_{sss}}{V_{ss}}$ is the coefficient of relative prudence (Kimball, 1990), albeit defined with respect to the intertemporal welfare function over capital stocks, rather than over flows of consumption or wealth. It is the (negative) elasticity of the revealed intertemporal risk aversion to a change in natural capital. Positive values of the coefficient of relative prudence imply prudence ($p_{ss} \equiv V_{sss} > 0$), while negative values denote imprudence ($p_{ss} \equiv V_{sss} < 0$). In this case an increase in σ reduces the accounting price in cases of imprudence or moderate prudence, whereas relatively strong prudence is required for the accounting price to increase in σ.

For s below the unstable equilibrium stock, such that the intertemporal welfare function is risk loving in capital fluctuations ($p_s \equiv V_{ss} > 0$), it follows that

$$\frac{\partial p(s)}{\partial \sigma} \begin{cases} < 0 & \text{if } \frac{-s p_{ss}}{p_s} > 2 \\ = 0 & \text{if } \frac{-s p_{ss}}{p_s} = 2 \\ > 0 & \text{if } \frac{-s p_{ss}}{p_s} < 2 \end{cases} \tag{G.4}$$

In this case, the sign of the coefficient of relative prudence reverses, such that negative (positive) values are now consistent with prudence (imprudence). Equation G.4 shows that an increase in σ reduces the accounting price for a sufficiently high degree of imprudence, whereas a region of mild imprudence or prudence causes the accounting price to increase in σ.

The implication of these findings is that the point of no influence for σ on $p(s)$ always occurs on the concave portion of the price curve below the unstable equilibrium biomass, whereas the corresponding point above the unstable equilibrium biomass always occurs in a convex portion of the price curve.

Appendix H. Derivation of asset pricing equations with a Poisson jump

Following Reed and Heras (1992), consider the case where the resource stock is subject to a random discontinuous shock (i.e. "collapse"). For simplicity, assume that the shock can occur at most once and is, therefore, an absorbing state. In the absence of the shock, the stock grows deterministically according to $\frac{ds}{dt} = \dot{s} = \mu\left(s(t), x(s(t))\right)$. If the shock occurs, the infinite-horizon net benefit is $Z(s(t))$, so that the economic payoff may depend on the stock at the time of the arrival of collapse. Assume that the arrival time of the shock follows a Poisson process, with an instantaneous hazard rate $h(t)$. More specifically, assume that the time-varying hazard rate is dependent on time only through the state of the resource stock $h\left(s(t)\right)$, so that the probability of collapse is state dependent. This implies that the survival function, the probability of arriving at time t without the shock having occurred, is $S(t) = e^{-\int_0^t h(s(\xi))d\xi}$.

Define $M\left(s(t)\right) = W\left(s(t)\right) - \delta Z\left(s(t)\right) + Z_s\dot{s}$, suppressing the functional dependence of $W(s(t), x(s(t))$ through the economic program $x(s(t))$. The infinite horizon value function at time t, conditional on no shock having occurred up until this moment in time, can be defined as (see equation (A7) in Reed and Heras (1992)):

$$
\begin{aligned}
V\left(s(t)\right) &= \mathbb{E}_t\left[\int_t^{\infty} e^{-\delta(\tau-t)} W\left(s\left(\tau\right)\right) d\tau\right] \\
&= \int_t^{\infty} e^{-\delta(\tau-t)-\int_t^{\tau} h(s(\xi))d\xi} M\left(s(\tau)\right) d\tau + Z\left(s(t)\right) \\
&= \int_t^{\infty} \psi(t) M\left(s(\tau)\right) d\tau + Z\left(s(t)\right)
\end{aligned}
\tag{H.1}
$$

where the third equality results from redefining the modified discount factor as $\psi(t) = e^{-\delta(\tau-t)-\int_t^{\tau} h(s(\xi))d\xi}$. Note that $\frac{d\psi}{dt} = \left(\delta + h(s(t)\right)\psi(t)$. Taking the derivative of (H.1) with respect to t using Leibniz rule yields:

$$
\frac{dV}{dt} = -M\left(s\left(t\right)\right) + \left[\delta + h\left(s\left(t\right)\right)\right] \int_t^{\infty} \psi\left(t\right) M\left(s\left(\tau\right)\right) d\tau + Z_s\dot{s}
$$

which simplifies to

$$
\frac{dV}{dt} = -M\left(s\left(t\right)\right) + \left[\delta + h\left(s\left(t\right)\right)\right]\left[V\left(s\left(t\right)\right) - Z\left(s\left(t\right)\right)\right] + Z_s\dot{s}
$$

Substituting for $M\left(s\left(t\right)\right)$ and simplifying yields

$$
\frac{dV}{dt} = -W\left(s\left(t\right)\right) - h\left(s\left(t\right)\right) Z\left(s\left(t\right)\right) + \left[\delta + h\left(s\left(t\right)\right)\right] V\left(s\left(t\right)\right)
\tag{H.2}
$$

Recalling that $\frac{dV}{dt} = V_s \dot{s}$ and setting equal to (H.2) yields:

$$[\delta + h(s(t))] V(s(t)) - h(s(t)) Z(s(t)) = W(s(t)) + V_s \dot{s} \tag{H.3}$$

This expression is the stochastic Hamilton-Jacobi-Bellman equation for the Poisson shock case, analogous to (5) for the case of diffusions.

Differentiating both sides of the HJB equation (H.3) with respect to s provides the following expression for the shadow price of s, suppressing functional dependencies for the sake of clarity:

$$V_s = \frac{W_s - h_s [V(s) - Z(s)] + V_{ss} \dot{s} + h(s) Z_s}{\delta - \dot{s}_s + h(s)} \tag{H.4}$$