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Sustainability and Substitutability

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Abstract Developing a quantitative science of sustainability requires bridging mathematical concepts from fields contributing to sustainability science. The concept of substitutability is central to sustainability but is defined differently by different fields. Specifically, economics tends to define substitutability as a marginal concept while fields such as ecology tend to focus on limiting behaviors. We explain how to reconcile these different views. We develop a model where investments can be made in knowledge to increase the elasticity of substitution. We explore the set of sustainable and optimal trajectories for natural capital extraction and built and knowledge capital accumulation. Investments in substitutability through knowledge stock accumulation affect the value of natural capital. Results suggest that investing in the knowledge stock, which can enhance substitutability, is critical to desirable sustainable outcomes. This result is robust even when natural capital is not managed optimally. This leads us to conclude that investments in the knowledge stock are of first order importance for sustainability.

Keywords Economics · Knowledge · Constant elasticity of substitution (CES) · Optimal control

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1 Introduction: Sustainability and Substitutability

Childhood mortality rates are declining, life expectancy is rising, and people are living longer, healthier lives.Society appears increasingly prosperous. A growing population, however increases the demand for resources, and natural resources are vanishing at an alarming rate (Millennium Ecosystem Assessment 2005). Balancing increases in human wellbeing today, while maintaining natural and other forms of capital is the crux of the sustainability problem. The problem is particularly acute for developing countries and the poor (Barbier 2011). The measurement and understanding of earth's ability to support life, and the evolution of that life, is at the core an ecological question. Efficient allocation of resources is at the core an economic question. The intersection of these questions defines sustainability. Both disciplines are full of models, many of which have been formalized mathematically. Yet, the narratives attached to these models, particularly those associated with simplifying assumptions, can stymie the merging of ecological and economic models. Bioeconomics, the application of capital theory to natural resources (Clark 2005; Wilen 1985, Fenichel et al. 2014), provides an approach to merging economic and ecological models to help forge a quantitative science of sustainability.

Sustainability is a non-declining index of something overtime (Arrow et al. 2004; Dasgupta and Maler 2000; Goodland 1995; Hanley 2000). The key question is what is non-declining? One candidate is the wellbeing in the current period, another is an index of the net present value of current and future wellbeing. Barbier (2011) refers to these measures, particularly the latter one, as the capital approach to sustainability. One could also focus on the sum value of resources or the absolute quantity of resources, which (Barbier 2011) calls the systems approach to sustainability. The choice of a useful index is interconnected with the notion of substitutability (Dasgupta and Maler 2000). Discussion of sustainability often focuses on two extreme poles: (a) strong sustainability, in which individual stocks are not at all substitutable and it is not possible to create a single index of wellbeing and (b) weak sustainability, in which all capital stocks are perfectly substitutable in production and consumption and can be combined in a straightforward matter into a single wellbeing index (Barbier 2011).[1] However, these are stylized extremes, and real sustainability solutions will likely involve some level of, but not perfect, substitution. The extent to which substitution is possible is of ultimate importance for sustainability (Dasgupta and Maler 2000; Arrow et al. 2012; Weitzman 1976; Solow 1974).[2]

The goal of "ecological sustainability" is to maintain earth's life support systems (Goodland 1995). Ecologists often pull from population biology's notion of carrying capacity, the maximum number of individuals in a population that an area can support (Daily and Ehrlich 1992), and concerns about carrying capacity date back to Malthus (1789). Carrying capacity can be thought of as productive potential of an

[1] Spatial distribution and geographic variation make the aggregation to a single index impossible and the appropriate scale and scope of aggregation is an open question (Sterner 2011).

[2] Fisher and Zhao (2002) distinguish between substitutability in production of intermediate goods and that in consumption in a conceptual model, and argue that the latter is more important for sustainability when irreversibility is present.

environment. The notion of carrying capacity is useful in the study of sustainability (Arrow et al. 1995), but the actual carrying capacity is dependent on the ability of the population to substitute resources. Indeed, the carrying capacity concept does not rule out substitution, the ecological literature is beginning to consider the role substitution in ecological production (Fox et al. 2011; Tschirhart 2000).

Prior to developing a model of sustainability that links economic and ecological concepts and incorporates substitution, it is important to bridge economic and ecological notions of substitution. Substitution is of primary interest in economics, and is well formalized. Economists focus on substitution on the margin, often implicitly assuming that substitution on the margin is feasible.[3] Ecologists are often also concerned with substitution, particularly in the application of ecological knowledge to sustainability.[4] Nevertheless, substitution is less well formalized in ecology, but use of the term substitution or substitutability in the ecological literature often implies a focus on the limiting behavior of the system. For example, Burger et al. (2012) write, "there is limited or zero scope to substitute for some resources... all known substitutes are inferior, scarcer, and more costly." However, being more costly and scarce or not doing as good a job does not imply that something is not a substitute in the economic sense. Burger et al. write no other element has the properties of copper for electronics, but copper is used in many things where there are substitutes, pennies being a high profile example (Burger et al. 2012). Another example is water. Commonly it is argued that water is essential for life, and without water we die; there are no substitutes for water. The focus in this argument is on the last drop of water, which is undoubtedly essential. However, a broom is a substitute for washing floors and driveways. Soap is a substitute for some of the water used in bathing. Alternative crops that use less water substitute for some water used in irrigation. All of these technologies can substitute for the marginal drop of water. An interdisciplinary model of sustainability should nest both of these views by allowing substitution on the margin but not in the limit, and Solow (1974) emphasizes that this is the interesting case for sustainability.[5]

Substitutability between different forms of capital in many cases is not fixed. Some investments can be made to enhance substitution opportunities. The potential to develop new substitution opportunities requires investments in ingenuity. Investments in human ingenuity are captured in models of endogenous knowledge accumulation (called technological change in the economics literature). These models play a key role in macroeconomic growth theory (Romer 1990). However, these models lock in a certain substitution pattern while considering reallocation of capital stocks to technological capital. Barbier (1999) unified these streams of research by developing an

[3] For instance, the so called *Hartwick rule* (Hartwick 1990) implies that scarcity rents from consuming non-renewable resources should not be consumed along an optimal management path, but should be converted to other forms of capital. This result implicitly assumes that natural and built capital are substitutable, at least on the margin. Hartwick's model is readily generalized to renewable natural capital. Hartwick's model is also an extension to the Ramsey–Solow model of savings and investment (Mas-Colell et al. 1995).

[4] Ecologists are also increasingly concerned with substitution opportunities associated with the ability of organisms to use resources in different ways in response to different conditions (Fox et al. 2011).

[5] Solow is concerned with intergenerational equity, but Solow's exposition implies what would likely be called sustainability today.

economic endogenous growth model constrained by natural capital. More recently, Bretschger and Smulders (2012) investigate how general research and development interacts with extraction of non-renewable resources. They argue that the elasticity of substitution, which is the percent change in input use ratio with respect to the percent change in the rate at which one input can be substituted for the other, while keeping society on a level production or consumption set (Hoy et al. 2001), is critical to sustainable development.[6] Quaas et al. (2013) show how the nature of substitutability between natural and built (human devised) capital can influence the resilience of ecological systems.

A model of sustainability must take into account some opportunity for substitution, at least on the margin, but the important questions are (i) how much improving substitution can contribute to sustainability, and (ii) how much society should invest in improving substitution opportunities.[7] In this paper we develop a dynamical systems model of substitution to link natural, built, and human knowledge capital to start addressing these questions. The contribution of this paper is to introduce a class of models that is capable of addressing dynamic concerns about substitution and sustainability – a core issue in understanding how natural capital and other forms of capital interact. We believe that explicitly modeling the dynamics of substitution opportunities is important for developing a quantitative science of sustainability. We conduct a preliminary analysis of the model's properties and layout a research agenda for the analysis of substitution and sustainability.

2 The Model Setup

We develop a model specifically to address the dynamics of substitution within in the context of sustainability. Substitution may be formally measured using the concept of elasticity of substitution. The properties of models with a changing elasticity of substitution associated with knowledge accumulation are just beginning to be addressed (Bretschger and Smulders 2012), and generally have not entered the sustainability literature. We focus on a phenomenological model of how knowledge enters the production process and analyze the implications for sustainability.

2.1 Substitution Between Natural and Built Capital

Consider a society holding three forms of capital, namely natural capital, $x(t)$, built capital, $y(t)$, and knowledge capital, $k(t)$, at time t. Natural capital assets are ecological structures. These structures change over time according to ecological relationships, albeit with human interaction (Ludwig et al. 2001; Shogren et al. 1999). Built capital assets are physical (e.g., roads and building) assets that are human devised structures,

[6] The elasticity of substitution measures the curvature of a consumption or production isocline. Expression in terms of percent reduces dimensionality. When decisions are made optimally the elasticity of substitution can be defined as optimal input levels in response to exogenous changes in their relative prices.

[7] The prior literature addresses both built and natural capital e.g., Hartwick (1990), but does not directly address changes in substitution possibilities.

support human wellbeing, and store human wealth. Knowledge capital paves the way for innovation, which increases substitution opportunities. Assume that the natural capital is renewable so that it changes over time according to

$$\dot{x} = G(x(t)) - u(t). \tag{1}$$

It is convenient to assume that $G'(x) > 0$, $G''(x) < 0$, so that ecological dynamics are described by a concave function, but this is certainly not the case for all ecological stocks (Maler et al. 2003; Carpenter et al. 1999; Brock and Starrett 2003; Dasgupta and Maler 2003). In our model, $u(t)$ could be thought of as time varying harvests, or extractions of the natural stock, which follows the observation that most often natural capital is drawn down rather than built up (Barbier 2011). Self-regeneration comes from the fact that natural systems are not closed but can harness solar energy through photosynthesis and primary productivity (Young 1991). We focus on a scalar model, but our framework could be applied to a set of natural capital stocks, where x could be viewed as a vector of natural stocks and pairs of elements in the vector x have an elasticity of substitution that relates to the services generated by the ecosystem (Finnoff and Tschirhart 2008).[8] Such measurement is an important and active area of research, but is currently in its infancy (Sanchirico and Springborn 2011; Nelson et al. 2009).

Assume that built capital accumulates according to

$$\dot{y} = v(t) - \delta y(t) \tag{2}$$

where $v(t)$ is time varying human investments in built capital, and δ is a physical depreciation rate. Physical depreciation may come from weathering, friction, and other physical processes. $y(t)$ may also be thought of as a vector, and pairs of elements would have elasticities of substitution. The form of Eq. (2) is for illustrative purposes and other functional forms are possible.

In addition to storing wealth in physical and natural capital, society may store wealth in human capital in the form of a knowledge stock, $k(t)$. Knowledge in this case refers to the ability to think critically, solve problems, and innovate. The knowledge stock enables society to respond to scarcity in innovative ways. We assume knowledge stock is accumulated according to

$$\dot{k} = z(t)m(k(t)) - \phi k(t), \tag{3}$$

where z represents investments in the knowledge stock (e.g., investment in science and education), and $m(k)$ is the marginal product of these investments in terms of new knowledge, measuring how much a unit of investment yields in terms of new knowledge capital, which may depend on the current state of knowledge. On the other hand, knowledge depreciates at rate ϕ, as old knowledge loses its value, often replaced by new knowledge. However, knowledge may suffer from a purer form of

[8] Many authors (e.g., Kellner et al. 2011; Horan et al. 2011; Brock and Xepapadeas 2002; Fenichel and Horan 2007a; Fenichel et al. 2010) have examined vectors of ecological states, but seldom have explicitly considered the substitution relationships.

depreciation. For example, knowledge may depreciate through population turnover, failure to integrate new information into education systems, or reliance on a small pool of individuals to actively engage in innovation. This pool of knowledge may include stochastic technological innovations, but is not confined to technological innovation in that knowledge capital and *ability* to innovate is rival, that is knowledge is tied to specific individuals, whereas a technological design is non-rival (Romer 1990).

Society can produce goods and services by combining extracted natural resources with built capital according to production function $F(u, y, \sigma(k))$ where $\sigma(k)$ is the elasticity of substitution between u and y, and is a function of knowledge stock k. The idea of substitution is that if society has more y or k it may generate the same level of F by extracting less from the ecosystem. Quaas et al. (2013) argue that society could persist without built capital, but one could argue that it is the ability to create built capital that separates humans from the rest of the animal kingdom. Therefore, we assume both forms of capital are essential for a society resembling modern society.

A conservation law suggests that

$$F(u(t), y(t), \sigma(k(t))) = \beta_u(u(t)) + \beta_v(v(t)) + \beta_z(z(t)) + c(t) \tag{4}$$

Instantaneous production must be allocated to resource extraction costs (which may include both direct extraction costs and pollution where the natural environment provides waste storage and removal services), investments in built capital, investments in knowledge capital, or consumption. Wealth can be stored for future use either by investing in natural capital through forgoing extraction, u, investing in built capital v, or investing in knowledge capital k.

The importance of thinking about investment in the knowledge base, k, depends in part on the model used to characterize society's production. The neoclassical economic growth literature based on the Solow–Swan model treats knowledge as another factor input: knowledge enters the model in a qualitatively similar way to other inputs, and increasing the knowledge stock shifts out production. The endogenous growth theory of Romer (1990) models knowledge as new goods. In those models knowledge has special properties that other inputs may not have, e.g., greater knowledge enables other inputs to be combined in a greater number of ways. Although the literature does not explicitly model substitutability, the possibility of a greater diversity of products from the same material inputs raises the potential of making these other inputs more substitutable.

Society produces to generate welfare. Human society obtains welfare from consumption, $c(t)$, but may also have preferences for non-consumption goods related to the stock of x either because of existence value, people placing value on resources for reasons not relating to current or future consumption (Freeman 2003), or cost savings effects (e.g., greater stocks of natural capital may reduce the cost of natural capital extraction).[9] Instantaneous welfare, a measure of how well off society is based on current consumption choices and existence or cost-savings opportunities, is $W(c, x; \eta)$ where η is the elasticity of substitution in consumption between c and x. Stable pref-

[9] It is also possible that society has preference for y, e.g., historic buildings, but this would require y to have heterogeneous vintages and is left for future exploration.

erences imply that η is constant over time.[10] A special case is when x does not directly feed into welfare or is nonessential in welfare, but u, and hence x, remain essential in production. Discounting welfare at rate ρ (Dasgupta et al. 1999), the net present value of current and future wellbeing at time τ is

$$V(x(\tau), y(\tau), k(\tau)) = \int_{\tau}^{\infty} e^{-\rho(t-\tau)} W(c(t), x(t); \eta) \mathrm{d}t \text{ s.t., (1)} - \text{(4) } x(\tau), y(\tau), k(\tau) \tag{5}$$

Model (5) allows a number of indices to be considered as sustainability measures. Following Dasgupta and Maler (2000), Arrow et al. (2004), and Arrow et al. (2012), a society is sustainable if $\mathrm{d}V/\mathrm{d}\tau \geq 0$. This definition of sustainability is general and captures many concerns, and aligns with Barbier's (2011) notion of capital sustainability. The measure $\mathrm{d}V/\mathrm{d}\tau$ is referred to as genuine investment or the adjusted net investment. This is a preferred weak sustainability index (Arrow et al. 2012). Heal (2012) reviews other approaches to measuring capital sustainability.

2.2 Special Functional Forms

We conduct a series of numerical simulation exercises to illustrate the relationship between substitutability and sustainability. Simulation of model (5) enables systems analysis that considers the time paths of $W, c, x, y, k, \mathrm{d}V/\mathrm{d}t$, which provides measures of strong or system sustainability and weak or capital sustainability. To do so, we work with specific welfare (social utility) and production functions.

A functional form for $F(\cdot)$ that allows resources to be essential in the limit but substitutable on the margin and that enables us to focus directly on the influence of knowledge on substitution is the constant elasticity of substitution (CES) specification, where constant refers to the level of input and not to the time period.[11]

$$F(u, y, \sigma(k)) = (\alpha y^{\psi} + (1-\alpha)u^{\psi})^{\frac{1}{\psi}} \text{ with } \psi = (\sigma(k)-1)/\sigma(k) \text{ and } \sigma(k) \in \mathrm{R}^{+}. \tag{6}$$

Figure 1 illustrates the CES function, which requires u and y in the limit (in the sense that as each input use approaches zero, its marginal output goes to infinity), but allows substitution on the margin. The CES specification is broadly used in economics, and has been applied in theoretical and empirical environment, energy, and health applications (e.g. Quaas et al. 2013; Kemfert 1998; Haefen and Phaneuf 2003; Fenichel 2013). When $\sigma < 1$ and declines towards zero, natural and built capital inputs are increasingly complementary, and it is more difficult to substitute between

[10] We maintain the assumption of stable preference. Changes in the knowledge stock could shift preference, but analyzing how knowledge accumulation may shift preferences is left for future work.

[11] The limit $\sigma \to 0$ implies Leontief production, which is akin to a law of the minimum approach to the production of F. The CES function can also be rescaled to create the appropriate production by multiplying by a scalar. In our analysis we set such a scalar to 1.

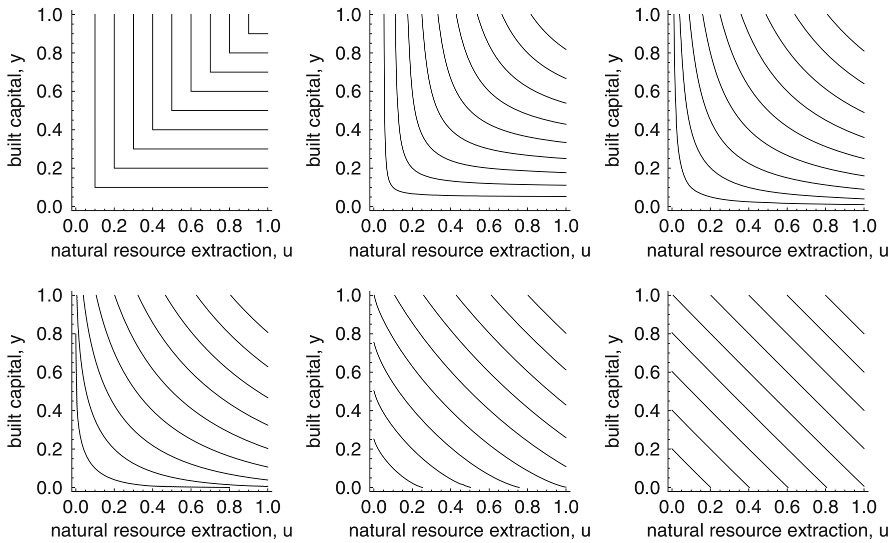

Fig. 1 The constant elasticity of substitution function where the elasticities of substitution in the *top row* are asymptotically zero, 0.5, asymptotically 1, and in the *bottom row* 1.5, 4, asymptotically infinite. The curves represent level sets of production contours

them. When $\sigma > 1$ and increasing, substitution is increasingly easy. Equation (6) only admits $\sigma = 1$ in the limit, approaching a Cobb–Douglas production function. Models of endogenous technological change, following Romer (1990), often assume Cobb–Douglas functions, e.g., they restrict $\sigma \to 1$.

To focus on substitution on the production side, we assume that instantaneous wellbeing is measured by consumption only, $W(c, x) = c$. We also assume that natural capital dynamics follow logistic growth, $G(x) = rx(1 - x)$, cost functions are convex and can be approximated by quadratic expressions, $\beta_i(i(t)) = \beta_i i(t)^2$, where β_i is a constant, $i \in \{u, v, z\}$. Finally, we assume $\sigma(k) = k$ and $m(k) = 1 \forall k$.

3 Sustainable Investment

To begin exploring the model properties we conduct simulations. Table 1 lists the parameter values used in the simulation.

Any level of built capital, y, and knowledge, k, can be maintained so long as there is enough production, F, to allocate to reinvestments and cover natural capital extraction costs. Natural capital extraction is constrained by the maximum sustained yield, given the specification. If the natural capital x is non-renewable, then constant levels of y and k cannot yield a sustainable path (Hartwick 1990). There are an infinite number of investment paths, including many constant investment paths (assuming x is renewable), that satisfy both weak and strong sustainability measures when stocks of built and knowledge capital initially are not too high. However, paths starting at low levels of capital may be relatively undesirable (Solow 1974).

Table 1 Baseline parameter values of the optimal consumption and investment path

Parameter	Definition	Value
r	Intrinsic growth rate of natural capital	0.5
δ	Depreciation rate of built capital	0.08
ϕ	Depreciation rate of knowledge capital	0.05
α	Share parameter	0.5
β_u	Twice the marginal cost of natural capital extraction	1/3
β_v	Twice the marginal cost of built capital investment	1/3
β_z	Twice the marginal cost of knowledge capital investment	1/3
ρ	Discount rate	0.02

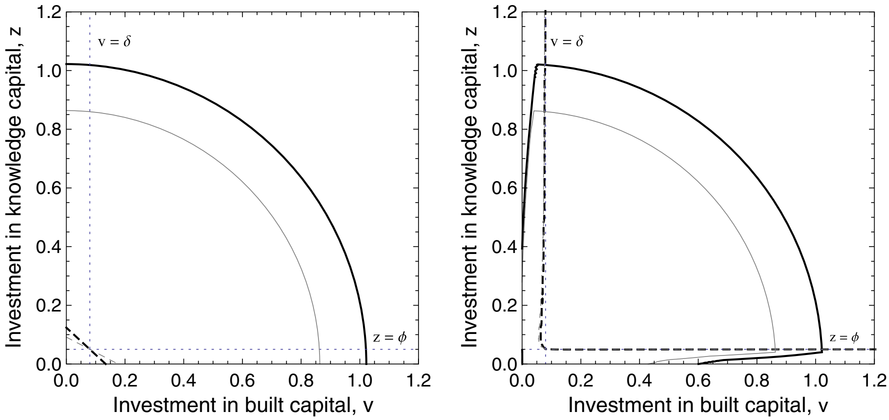

Fig. 2 Short run (*left panel*, **a**) and long run (*right panel*, **b**) constant investment strategies. *Solid curves* illustrate the consumption possibility frontier, which bounds the region where all production is allocated to consumption. The *dashed curves* show the minimum required investments to maintain $\mathrm{d}V/\mathrm{d}\tau > 0$. The *black curves* are for the base specification, and the *gray curves* are associated with 1/2 of the maximum sustainable natural capital extraction rate. The *dotted curves* show level of investment requires to maintain the initial stocks of built and knowledge capital, bounding regions of strong sustainability

We consider constant investment and consumption paths satisfying sustainability criteria for a system starting with approximately a unit of knowledge capital, a unit of built capital, and maximum sustainable yield level of natural capital (0.5 of a unit).[12] First, we consider weak or capital sustainability at time t, in which case sustainability can be measured using a single common index. For the system to be weakly sustainable, management programs must prevent negative consumption and $\mathrm{d}V/\mathrm{d}\tau \geq 0$. In Fig. 2a the black solid curve illustrates the investment possibility frontier assuming constant levels of investment and maximum sustainable yield extraction of natural

[12] The system begins with approximately a unit of knowledge capital to avoid undetermined conditions associated with $\sigma = 1$.

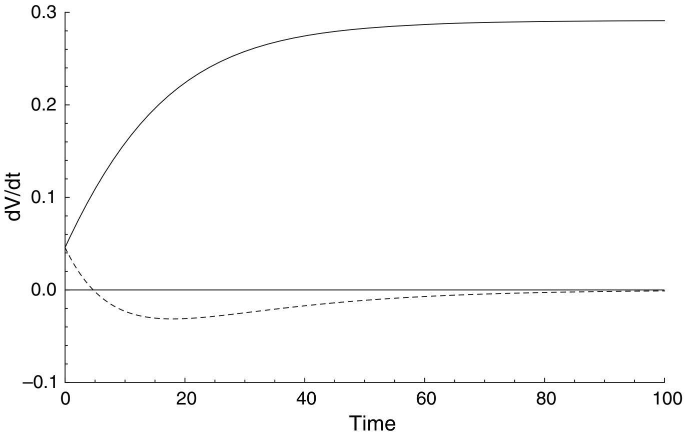

Fig. 3 Time paths for $\mathrm{d}V/\mathrm{d}\tau$ associated with constant investment strategies. The *solid curve* show a 20 % increase in z above the maintenance level for the initial stock of knowledge capital, so that knowledge capital grows overtime, and v is reduced by 20 % below the maintenance level for the initial built capital stock causing it to decline overtime. The *dashed curve* shows a 20 % decrease in z below the maintenance level for the initial knowledge stock causing it to decline over time, and v is increased by 20 % above the maintenance level for the initial built capital stock causing it to increase overtime

capital. Starting with lower levels of either knowledge or built capital causes this frontier to shift towards the origin. Only points below and to the left of this curve allow society to engage in nonnegative consumption. The black dashed curves, near the origin, (Fig. 2a) illustrate the minimum investment cutoff for investments in built and knowledge capital for $\mathrm{d}V/\mathrm{d}\tau \geq 0$ in the current period. Only points above and to the right of this curve provide sufficient investment for a society that would be considered sustainable at a given instant. The area bounded by these two curves represents investment strategies that would have a positive genuine investment (Arrow et al. 2004) or positive adjusted net investment (Heal 1998), i.e., the region of short run weak sustainability and nonnegative consumption. However, these are constant investment strategies, not time varying strategies. Therefore, it is natural to ask which subset of these strategies continues to satisfy the $\mathrm{d}V/\mathrm{d}\tau \geq 0$ condition while providing positive consumption in the long run. The long run nonnegative consumption boundary is illustrated by the solid black curve in Fig. 2b. The short and long run consumption curves only deviate for strategies with very high levels of knowledge or built capital investment and very low investment in the other form of capital. Requiring $\mathrm{d}V/\mathrm{d}\tau$ to remain nonnegative is more stringent and is illustrated by the dashed black curve in Fig. 2b. This curve shows that investment in knowledge and built capital is required over the long run for $\mathrm{d}V/\mathrm{d}\tau$ to remain nonnegative. Traditional substitution opportunities between knowledge and built capital exist above and to the right of the dashed curve, however it is not possible to maintain consumption in the long-run and maintain non-decreasing capital below and to the left of the dashed curve (Fig. 2b).

Dotted lines in Fig. 2 (identical in panels a and b), labeled $v = \delta$ and $z = \phi$, represent the minimum level of constant investment in built and knowledge capital to prevent these stocks from declining from their initial states—the minimal levels of investment for strong sustainability. We have assumed strong sustainability for natural capital through the sustained yield assumption. The simulations of constant strategies suggest investment strategies that allow declines in either capital stock in the short run can satisfy the instantaneous $\mathrm{d}V/\mathrm{d}\tau \geq 0$ sustainability criterion (Fig. 2a). However, in the long run weak sustainability and strong knowledge sustainability generally converge. Nevertheless, there is a small region where weak sustainability does not require strong sustainability in built capital as long as society invests sufficiently to grow the knowledge stock. For example, Fig. 3 shows that a 20 % increase in the knowledge capital investment relative to the maintenance level, ϕ, and a 20 % decrease in investment in built capital relative to the maintenance investment level, δ, maintains $\mathrm{d}V/\mathrm{d}\tau > 0$. The reverse is not true. As the investment level increases it becomes harder to maintain the newly achieved stock levels.

The gray curves in Fig. 2 show weakly sustainable constant investment strategies when natural capital extraction is cut in half, which reduces the overall level of V. This can be thought of as a sensitivity analysis to the initial condition of natural capital, where at some point in the past natural capital was reduced below the maximum sustain yield level and the proceeds consumed. The investment possibility frontier shifts inward, implying that society must invest less in order to guarantee nonnegative consumption. Alternatively, this can be interpreted as a society that can afford less investment. The instantaneous $\mathrm{d}V/\mathrm{d}\tau \geq 0$ boundary rotates suggesting a need for greater built capital investment relative to knowledge capital. But, the long run $\mathrm{d}V/\mathrm{d}\tau \geq 0$ boundaries hardly change, that is, the set of constant investment levels in knowledge and built capital that satisfy long-run weak sustainability criterion is mostly unaffected by the natural capital extraction level.

Once greater levels of capital stocks are achieved, greater levels of investment are needed to maintain both weak and strong sustainability (Fig. 4). However, the investment possibilities curve also shifts outward creating more opportunities for investment and consumption (Fig. 4). While capital accumulation allows increased consumption opportunities, it also creates increased investment obligations if society is to continue to satisfy weak or strong notions of sustainability. These simulations illustrate a key point about sustainability: sustainability requires a reference level. Undesirable outcomes may be sustainable in the sense of both system and capital sustainability, if low capital stock levels are used as a reference levels. This second observation is important to remember. Researchers interested in "ecological sustainability" have long been concerned with so called shifting baselines (Pauly 1995), which is closely associated with a systems approach to sustainability. Solow (1974) argues that this is a reason that a Rawlsian ethical criterion cannot be used for intertemporal allocation of wealth and resources.

3.1 Optimal Management of Natural, Built, and Knowledge Capital

The management program that maximizes $V(x, y, k)$ by choosing time paths for $u(t)$, $v(t)$, and $z(t)$ provides an alternative benchmark for considering accumulation and maintenance of wealth and capital. An optimizing path allows a consistent approach to considering time varying investment paths. The current value Hamiltonian

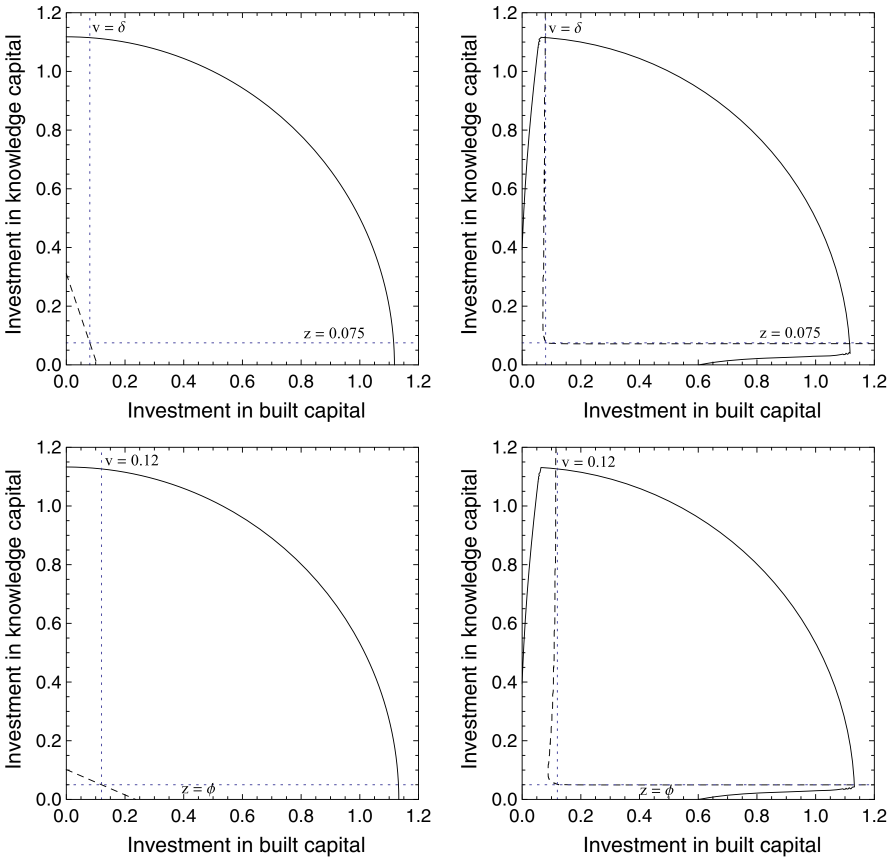

Fig. 4 Sensitivity analysis for constant investment strategies relative to Fig. 2. The *top row* represents a 50 % increase in the initial stock of knowledge capital, and the *bottom row* represents a 50 % increase in the initial stock of built capital

for this problem is

$$H = W(c, x; \eta) + \boldsymbol{\lambda}\dot{\boldsymbol{X}} \tag{7}$$

where $W(c, x; \eta) = c = F(u, y, \sigma(k)) - \beta_u u^2 - \beta_v v^2 - \beta_z z^2$ from (4), $\boldsymbol{\lambda}$ is a vector of adjoint variables (below we use superscripts to index the elements of the adjoint vector with their associated state variables), and $\dot{\boldsymbol{X}}$ is a vector of the equations of motions for the three state variables, Eqs. (1)–(3). The optimality conditions for the problem include

$$\frac{\partial H}{\partial U} = H_U + \lambda\dot{\boldsymbol{X}}_{\boldsymbol{U}} = 0 \tag{8}$$

$$\dot{\boldsymbol{\lambda}} = \rho\boldsymbol{\lambda} - H_{\boldsymbol{X}} \tag{9}$$

where subscripts represent partial derivatives and U is a vector of the three control variables, $\boldsymbol{U} = [u, v, z]$. Equation (8) implies $\boldsymbol{\lambda} = -H_U(\dot{\boldsymbol{X}}_{\boldsymbol{U}})^{-1}$. Differentiating

Table 2 Candidate interior steady-state equilibria

Variable	x	y	k	u	v	z	λ^x	λ^y	λ^k	CVH value
Candidate Equilibrium A	0.480	90.988	24.118	0.1248	7.279	1.206	0.555	4.853	0.804	26.082
Candidate Equilibrium B	0.480	10.313	1.000[a]	0.1248	0.825	0.05	4.462	0.550	0.033	0.902
"Free" natural capital Equilibrium A'	0.322	90.989	24.133	0.1092	7.279	1.207	0	4.853	0.804	26.073
"Free" natural capital Equilibrium B'	0.146	8.178	1.000[a]	0.0622	0.654	0.05	0	0.436	0.033	0.569

[a] k takes a value of 1 in the limit.
CVH value is the value of the current value Hamiltonian at equilibrium

each element in $\boldsymbol{\lambda}$ with respect to time, and setting the resulting vector equal to Eq. (9) yields a system of candidate equations for motion for the three control variables. The control variables must evolve according to these derived equations of motions as long as the derived equations of motion imply positive real values. We focus on control paths that lead to long run steady state equilibria because these satisfy the transversality conditions (Leonard and Long 1998) and provide boundary conditions for the derived control equations.[13] If these derived equations of motion for the controls imply negative or imaginary values, then the system is in a blocked interval (Arrow 1968; Fenichel and Horan 2007b) and one of the control variables must be set to zero and the problem resolved subject to the binding boundary constraint on the control variable.

The nonlinearity and dimensionality of this model makes analytical analysis intractable. Therefore, we investigate the model properties numerically. We assume that interior steady state equilibria exist and set equations of motion (1), (2), and (3) along with the derived equations of motion that satisfy the Pontryagin necessary conditions for the control variables equal to zero and numerically solve for a candidate optimal equilibrium.[14] We then verify that at the equilibrium the Hessian of the Hamiltonian, with respect to states and controls is negative semi-definite, the Mangasarian sufficiency condition (Caputo 2005). We find two interior equilibria, Equilibrium A and Equilibrium B (Table 2) that are candidates for long run optimal steady states.

Numerically solving for Equilibrium A is straight forward; however, solving for Equilibrium B is more complicated. Numerical analysis suggests that $k \rightarrow 1$ at Equilibrium B, which can only hold in the limit. Therefore, we set $z = \phi = 0.05$, which is required for $k = 1$, when $\dot{k}$. We then take the limit of the $\dot{u}$ and $\dot{v}$ as $k \rightarrow 1$, and set these modified candidate equations of motions equal to zero along with $\dot{x}$ and $\dot{y}$ to solve numerically for a candidate optimal equilibrium.

Equilibria A and B imply the same physical stock level of natural capital. Equilibrium A implies a greater level of investment in knowledge and built capital. Indeed,

[13] Equilibrium is used in the sense of a dynamical system.

[14] Numerical analysis was conducted using Mathematica 9.0 (Wolfram Research).

k is quite large suggesting a substitution pattern that approaches perfect substitution (Fig. 1 last panel). With a high elasticity of substitution, extraction of natural resource is not so important to society. The value of the current value Hamiltonian at equilibrium, the constant equivalent of consumption (Weitzman 1976; Heal 1998), is 26.0818. This equilibrium is conditionally stable. The eigenvalues of the Jacobian matrix at equilibrium are all real. Three are positive and three are negative. Using the eigenvectors associated with the negative eigenvalue with the greatest magnitude to perturb the system and simulate away from the equilibrium shows that the equilibrium can be approached from any initial level of natural capital. However, the simulation does not suggest a path across the manifolds associated with built and knowledge capital. Together, this suggests that if society has high levels of knowledge and built capital, then it would be able to choose an optimal natural capital management trajectory, without forgoing investment in built or knowledge capital in some periods. How to optimally accumulate knowledge and built capital is less clear and an important sustainability question, but it may not be optimal to control the system to equilibrium A from all initial conditions. Solving for the optimal approach paths to equilibrium A, assuming they exist, would require examining combinations of controls with either investment or built capital constrained to zero (a blocked interval).[15]

Equilibrium B has many interesting, interrelated characteristics. The value of the current value Hamiltonian at Equilibrium B is 0.9016, which is substantially less than Equilibrium A. It can only be optimal to go to Equilibrium B if a Skiba manifold exists making the transition path from some initial states to Equilibrium A excessively costly (Horan et al. 2011; Caulkins et al. 2009).[16] We linearize the system at the optimal equilibrium and numerically solve for the eigenvalues of the Jacobian matrix to characterize the stability of the equilibrium associated with the optimized system. The Jabobian has two positive real eigenvalue, two real negative eigenvalues, and two complex eigenvalues with positive real parts indicating a saddle-focus (Anishchenko et al. 2007, pp. 12–13). This suggests that for the system to be stable it must approach the equilibrium along a conditionally stable manifold faster than it travels along the manifolds that generate the unstable spiraling behavior. Moreover, it suggests a complicated adjustment path to this long-run equilibrium. It is also possible Equilibrium B is never long run optimal, and the optimal dynamics always move the system to Equilibrium A.

The elasticity of substitution associated with Equilibrium B is 1, implying Cobb–Douglas style substitution (Fig. 1, third panel in the top row) between extracted resources and built capital. This result is robust to alternative constant resource extraction and investment costs. The intuition for an equilibrium with Cobb–Douglas style substitution follows Quaas et al. (2013). As factors of production become relative complements, $\sigma = k < 1$, they become increasingly essential, and there are either

[15] An analytical solution strategy to this problem from arbitrary starting values is not immediately obvious. The problem could be solved numerically. However, given the abstract nature of the problem little new insight would be gained by such an exercise.

[16] A Skiba manifold is the multi-dimensional equivalent of a Skiba point (Brock and Starrett 2003; Skiba 1978; Wagener 2003). A Skiba point results from non-convexities in the optimized dynamic system, which create an indifference point such that a system can move along more than one trajectory to more than one long-run steady state (or stable cycle) and achieve the same maximal value.

stronger incentives for conservation or strong incentives for an optimal extinction path. However, society would prefer greater flexibility and would be willing to make investments to achieve this flexibility. Conversely, as factors of production become relative substitutes, $\sigma = k > 1$, it becomes less essential to conserve the more expensive factor. In the limit, with CES production technology, the factor is essential, and therefore a forward looking society would have an incentive to avoid depletion. This requires either a very high elasticity of substitution (e.g Equilibrium A), which may not be approachable from a region of state space, or the marginal returns to increasing substitution opportunities are diminishing. Cobb–Douglas production balances these competing effects. Finally, given the small changes associated with various levels of sustainable natural capital extraction in Fig. 2, it appears that Eq A and B are both weakly and strongly sustainable, though the paths to these equilibria may not satisfy strong or weak sustainability criteria.

3.2 Free Natural Capital

Optimal management of natural capital requires society to have institutions that allocate natural capital in a forward-looking manner. Such institutions must guide natural capital extraction in a way that accounts for the opportunity cost of current period extraction, λ^x. Historically, natural capital was relatively plentiful, and built and knowledge capital were relatively scarce. This state of the world persisted through all but the last instant of human evolution. The evolution of human society has not equipped society with institutions to deal with ecological scarcity to the extent that it has equipped society to address scarcity of other stocks of capital. Institutions seldom exist to intertemporally allocate the most critical forms of natural capital, which are often held in the public domain as common property. It is the lack of institutions to allocate natural capital intertemporally that may be the greatest barrier to sustainable management of natural capital (Barbier 2011). To contrast the optimal management of natural capital, with an equilibrium that conforms to current institutions we re-maximize (5), but *a priori* set $\lambda^x = 0$, so that natural capital is extracted as if there is zero social cost.[17]

Our solution approach is to replace $H_u = W_u(c, x; \eta) + \lambda^x \dot{x}$ with $W_u(c, x; \eta) = 0$ as a necessary condition. We differentiate $W_u(c, x; \eta)$ with respect to time and use the result along with the remaining optimality equilibrium conditions to solve for an equilibrium where investment in built and knowledge capital is optimal, but where society considers the use of natural capital to have zero opportunity cost. There are two equilibria: Equilibrium A′ and B′ (Table 2). These equilibria satisfies Mangisarian sufficiency, conditional on the *ex ante* management program. The value of the current value Hamiltonian at Equilibrium B′ is 0.5686, a 37 % reduction relative to Equilibrium B from the optimized model, and the value at A′ is 26.0734, which is nearly equivalent to the value at Equilibrium A from the optimized model.

Considering a society that optimally invests in built and knowledge capital, but not in natural capital reveals interesting features about consumption and potential

[17] There remain extraction costs, β_u.

sustainability. First, if society can accumulate substantial knowledge and built capital, then society may be able to achieve a level of sustainable consumption nearly equal to the level achieved when natural capital is managed optimally. However, if the initial conditions do not allow the attainment of such an equilibrium, which seems likely for real societies that use natural capital to build up knowledge and built capital, then such societies may be forgoing substantial consumption opportunities by not managing natural capital optimally – even if such societies ultimately settle into a sustainable trajectory. In either case, less natural capital is maintained when the natural capital is treated as free. However, more natural capital is maintained, even when natural capital goes unpriced, if there are greater substitution opportunities on the margin. This suggests the importance of considering marginal substitution opportunities for forms of natural capital even when substitutes may not exist in the limit. One may expect, however, that natural capital must be relatively scarce and costly to incentivize the development of these substitution opportunities. This may only be the case if natural capital stocks are driven to low levels so that extraction costs become large.

4 Immediate Open Questions

The numerical analysis in this paper generates more questions than it answers. There are at least five immediate extensions that fit within the context of the model developed here and many that push beyond our preliminary analysis. First, natural capital is valued for non-extraction purposes. The Millennium Ecosystem Assessment (Millennium Ecosystem Assessment 2005) discusses regulatory, cultural, and supporting services provided by natural capital left in situ. Explicitly including x in preferences or in production (e.g., flood protection services may be more like production services since built capital may be a substitute for natural capital in some cases) is likely of first order importance. Barbier (2011) emphasizes these services, and increasingly such regulating services are seen as more important than direct consumption of natural capital.

As the knowledge stock changes, preference may evolve. This is a complex question. It is unclear if a notion of fixed preference constrained by the current level of information is observationally equivalent to preferences that change as the knowledge stock changes. Nevertheless, it may be important to consider how preference may change with changes in the knowledge stock. How this can be modeled in a non-arbitrary fashion is an open question.

We have not addressed population growth in our model, but growing populations create opportunities and challenges. Most prior research either abstracts from population growth, as we have done, or includes population growth but focuses on per capita measures [see Heal 2012 for examples].

Per capita measures are good first step, but may be highly flawed, especially for analyzing democratic societies when the median individual differs substantially from the mean individual (Stiglitz et al. 2010). For example, if one were to focus on non-declining holdings of personal capital (i.e., personal wealth), then so long as the wealthiest members of society's capital holdings grew faster than those of the poor population, a society could have non-declining per capita wealth. However, policies promoting this type of “sustainable” path are likely to be politically unstable. Therefore, including population growth and distribution is an important task for a quantitative analysis of sustainability.

Incorporating distribution extends beyond ethical and political stability concerns. Distribution may matter for directly modeling capital accumulation, especially knowledge accumulation and its connection to changing substitution opportunities. In this paper we made the very simple assumption that it is possible to directly invest in substitution opportunities. However, changes in the elasticity of substitution may also come from novel but unexpected innovations. Stochastic models of $\dot{\sigma}$ need to be developed. It seems reasonable to think that how knowledge and critical thinking skills are distributed throughout the population may influence $\dot{\sigma}$, and σ may be increasing both in the total investment of knowledge and distribution of knowledge.

Fifth, our model lumps capital into three general bins. However, understanding a hierarchy of substitution is a critical area that would benefit from interdisciplinary collaboration. For example, as ecosystems are increasingly designed and invaded by new species; an important question is how can ecological elements serve as substitutes (Finnoff and Tschirhart 2008; Gozlan 2008). Ecological elements may be both substitutes in producing a given good, but also may be substitutes in consumption. For example, forests and beaches may be substitutes for providing cultural ecosystem services for relaxation. Indeed, modeling substitution in consumption and the dynamics of substitution in consumption is also important, and Fisher and Zhao (2002) suggest that substitution opportunities in consumption may be especially important for sustainability.

5 Discussion

Bridging economics and ecology is critical for developing a science of sustainability, and mathematics has been at the core of building this bridge (Clark 2005, p. 5; Polasky and Segerson 2009; Tschirhart 2009). Most prior literature has focused on feedbacks between ecological and economic systems, with considerably less work focusing on interactions between built and natural capital [Quaas et al. 2013 and Horan et al. 2008 are examples that consider interactions between built and natural capital]. Research that has considered the interaction of built and natural capital generally has not considered opportunities for changing the nature of substitution or complementarity. A separate literature in economics focus on “endogenous technical change,” which is closely related to changing substitution opportunities [Barbier 1999 provides an initial bridge between these literatures]. In this paper, we have developed a framework for investigating investments in altering substitution opportunities. Specifically, knowledge capital enters the production process by altering the elasticity of substitution.

Economists often refer to the Harwick rule, which suggests that net rents from natural resource should be invested in other forms of capital. The preliminary results presented in this paper suggest that resource rents should be reinvested in knowledge capital to enhance substitution opportunities, perhaps in a way disproportional to other forms of capital. This result is perhaps not all that surprising given the importance that substitution opportunities and technical change play in the optimal growth literature (e.g., Solow 1974). Our contribution is that we explicitly allow society to *invest* in substitution opportunities, and it appears that a society interested in accumulating wealth would invest heavily in such substitution opportunities to reduce dependence on natural resources. This seems to work in a similar manner as investing in technological change, but we have formulated the model differently from the technological change literature. Our formulation highlights the need to conceptualize substitution as a marginal property, while being keenly aware of the importance of essential capital stocks.

Initial conditions may matter for how we think about forms of capital and their relative scarcities and the institutions for allocating rights. Our preliminary investigations of optimal investment paths could not identify optimal investment paths from all initial conditions. This is because investment that increases the elasticity of substitution may create non-convexities and multiple equilibria. Non-convexities and multiple equilibria are well known in coupled ecological-economic systems (Dasgupta and Maler 2003), and knowledge and changing substitution is likely an additional source of multiple equilibria and endogenous ecological-economic thresholds (Horan et al. 2011). Such non-convexities result in non-unique prices (Starrett 1972), which begs the question can capital stocks be equally scarce but have very different absolute quantities and are sustainable strategies a convex set?

In the limit, society will not be able to substitute away from clean water, clean air, and a number of other natural resources. But, on the margin society may be able to continue to find substitutes for scarce resources. Indeed, our economic institutions help identify and allocate resources towards substitutes. However, our economic institutions have evolved in a world where built and knowledge capital were scarce, and there was little need to price natural capital and find substitutes for it. A failure to price natural capital (Barbier 2011; Heal 2012) leads to underinvestment in knowledge and innovation that can help society develop substitutes for natural capital and pursue a path towards sustainability.

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