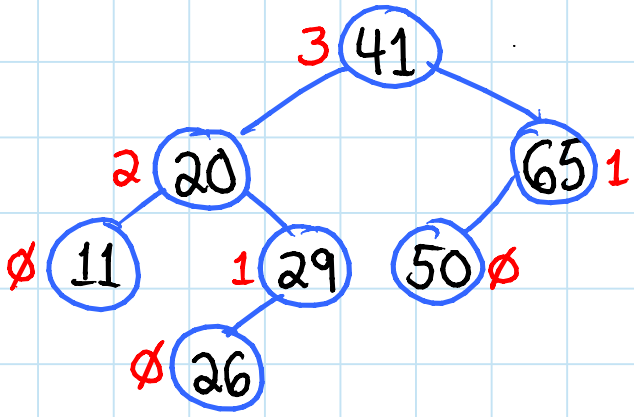
6.006	Lecture 6	Sept. 27, 2016

TODAY: Balanced BSTs

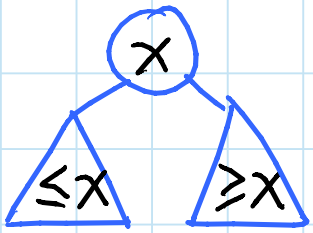
- The importance of being balanced
- AVL trees
  - definition & balance
  - rotations
  - insert, delete
- Other balanced trees
- Data structures in general

Recall: Binary Search Trees (BSTs)

- rooted binary tree
- each node has
  - key
  - left pointer
  - right pointer
  - parent pointer



- BST property:

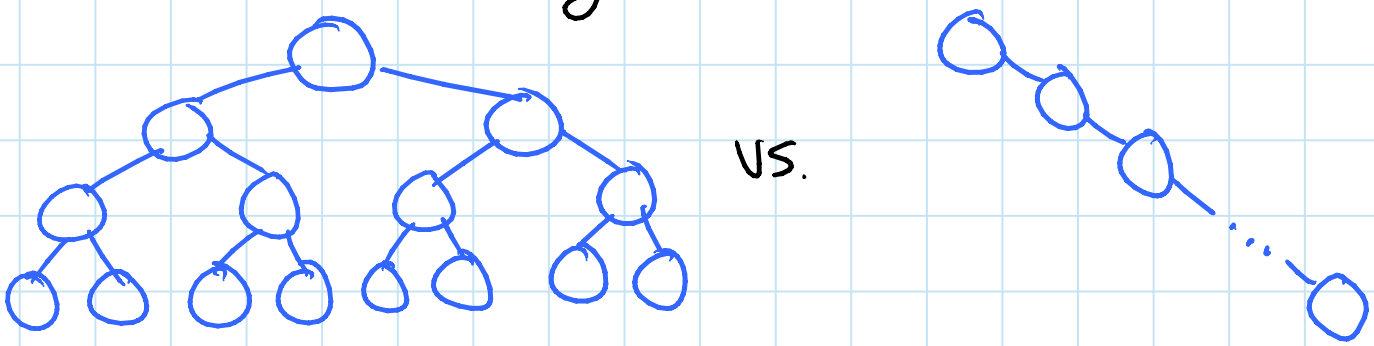


CLRS B.5

- height of node = length (# edges) of longest downward path to a leaf
- essentially, a more dynamic/flexible version of binary search

The importance of being balanced:

- BSTs support insert, delete, min, max, predecessor, successor, etc. in $O(h)$ time, where h = height of tree (= height of root)
- h is between $\lg n$ and n:

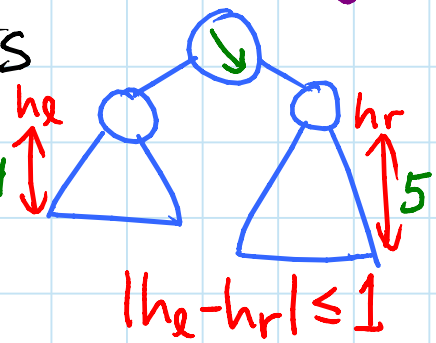


perfectly balanced path

- balanced BST maintains $h = O(\lg n)$
  $\Rightarrow$ all operations run in $O(\lg n)$ time
- today we'll see the first ever such BST data structure, and still one of the simplest: AVL trees

AVL trees: [Adel'son-Vel'skiĭ & Landis 1962]

for every node, require heights of left & right children to differ by at most ±1

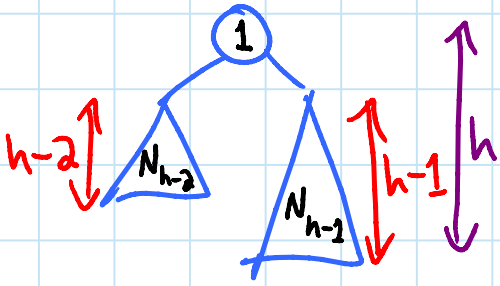


- treat nil tree as height −1
- each node stores its height (DATA STRUCTURE AUGMENTATION) (like subtree size)
  (alternatively, can just store difference in heights)

Balance: worst when every node differs by 1

- let N_h = (min.) # nodes in height-h AVL tree

$$\Rightarrow N_h = N_{h-1} + N_{h-2} + 1$$
$$> 2N_{h-2}$$
$$\Rightarrow N_h > 2^{h/2}$$
$$\Rightarrow h < 2 \lg N_h$$



Alternatively: $N_h > F_h$ (nth Fibonacci number)

- in fact $N_h = F_{h+2} - 1$ (simple induction)
- $F_h = \varphi^h/\sqrt{5}$ rounded to nearest integer
  where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ (golden ratio)

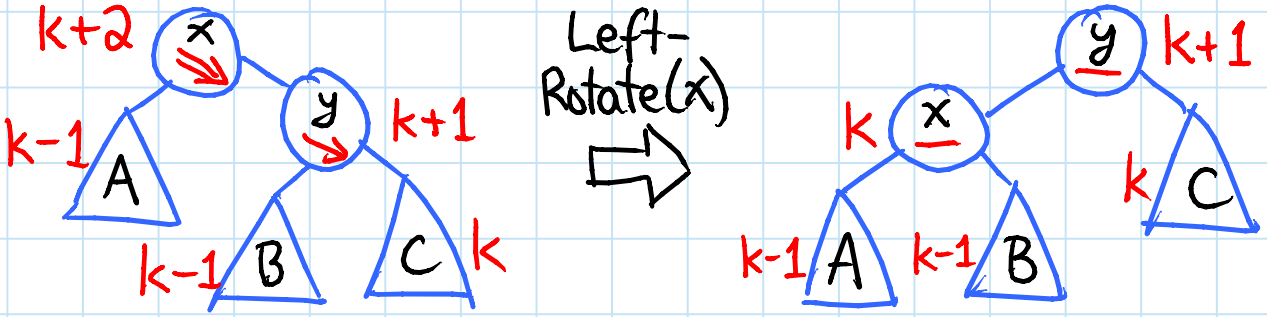
$$\Rightarrow \text{max. } h \approx \log_\varphi n \approx 1.440 \lg n$$

AVL insert:

1. insert as in simple BST [L5]
2. walk up tree, restoring AVL property (and updating heights as you go)

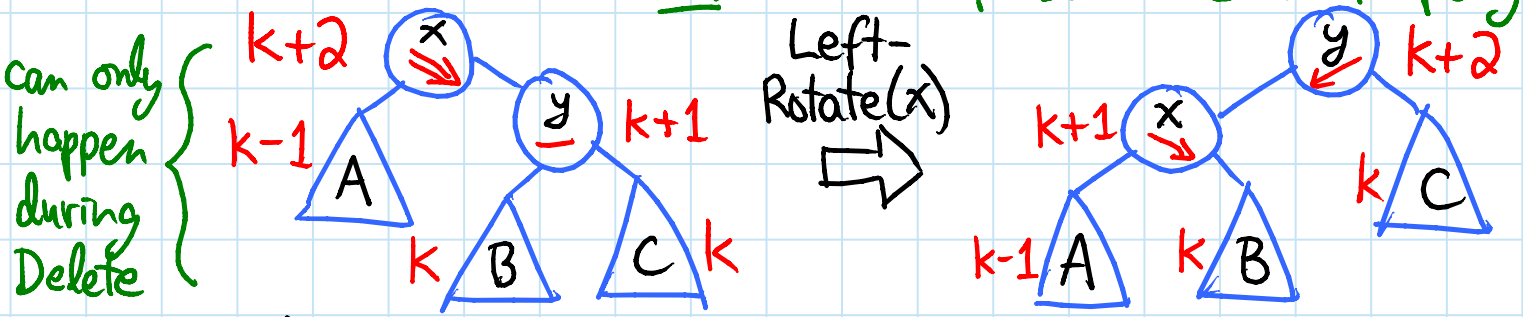
Each step:

- suppose x is lowest node violating AVL
- assume x is right-heavy (left case symmetric)
- if x's right child is right-heavy or balanced:

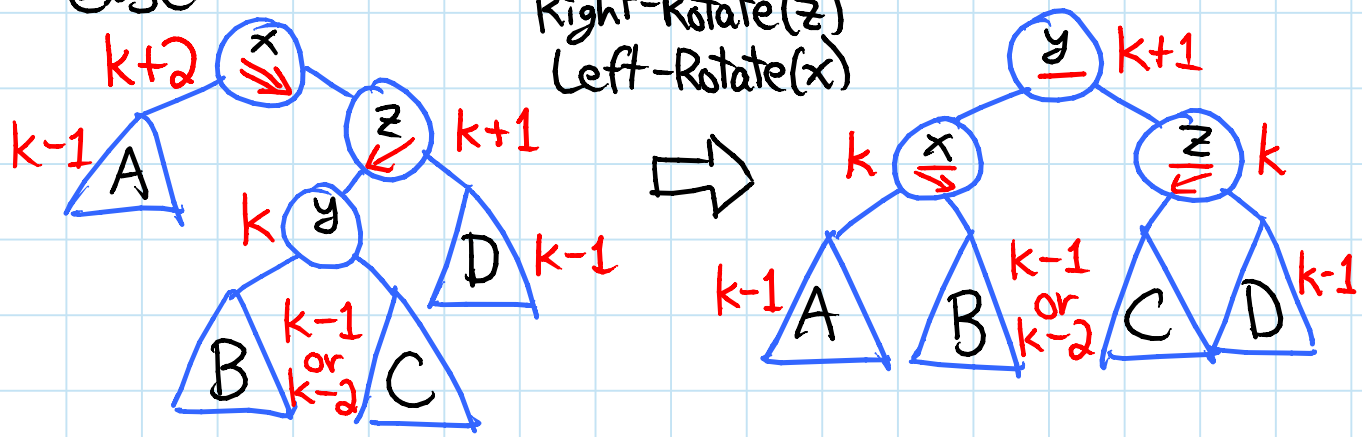


KEY: rotations preserve BST property

can only happen during Delete



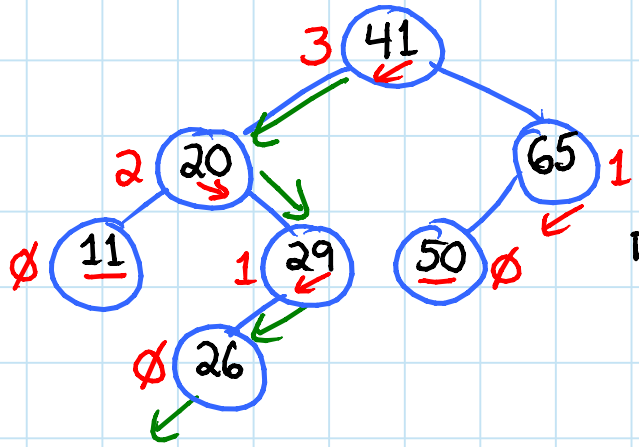
- else:



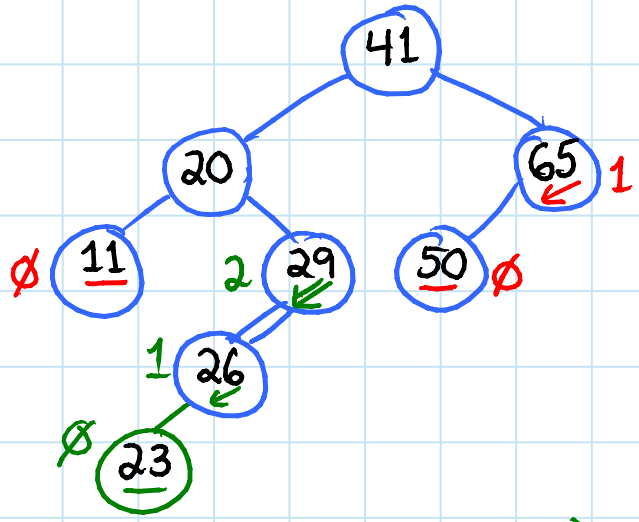
[- then continue up to x's grandparent, greatgp, ...]

Example:

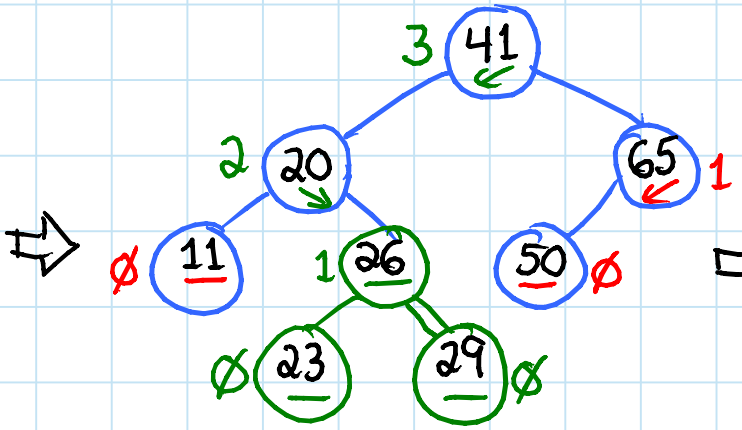
Insert(23)



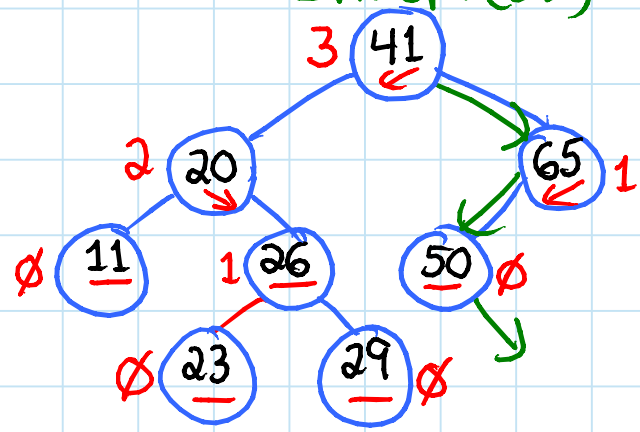
x=29: left-left case



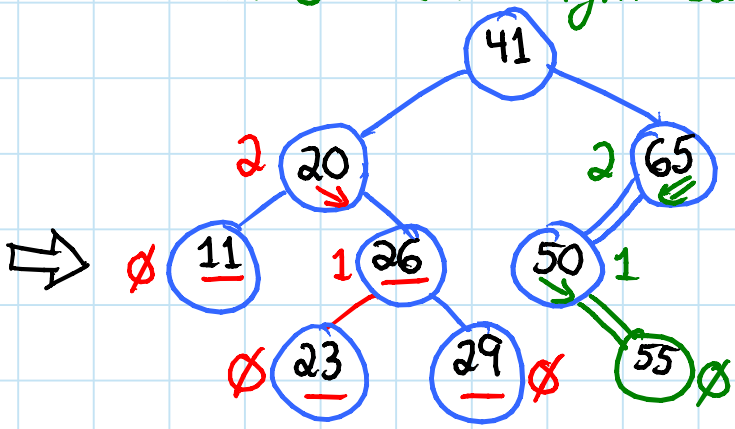
Done.



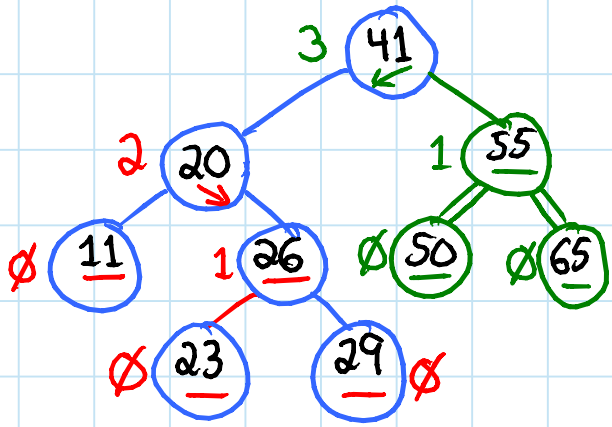
Insert(55)



x=65: left-right case



Done.



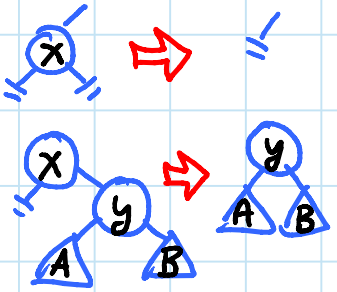
- in fact never need >1 rotation pair for an Insert:
  height of subtree after rebalancing =
  height of subtree before insert

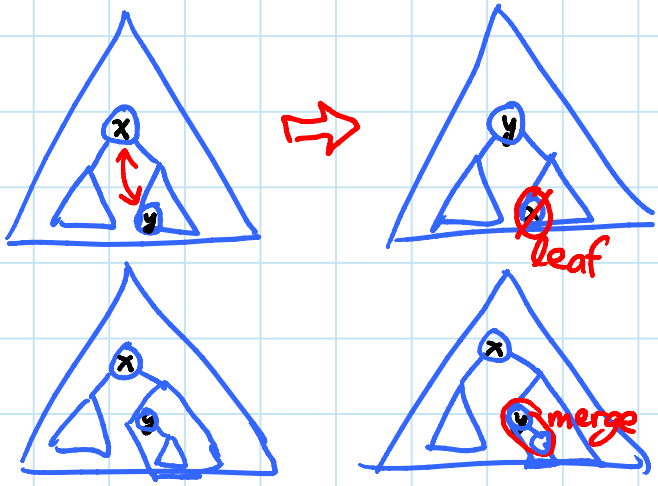
Delete is similar: (but may need $\Theta(\lg n)$ rotations)

① delete as in simple BST
- if x is a leaf: delete it
- else if x has exactly 1 child: merge x with its child
- else swap x with its successor, then apply previous case

or

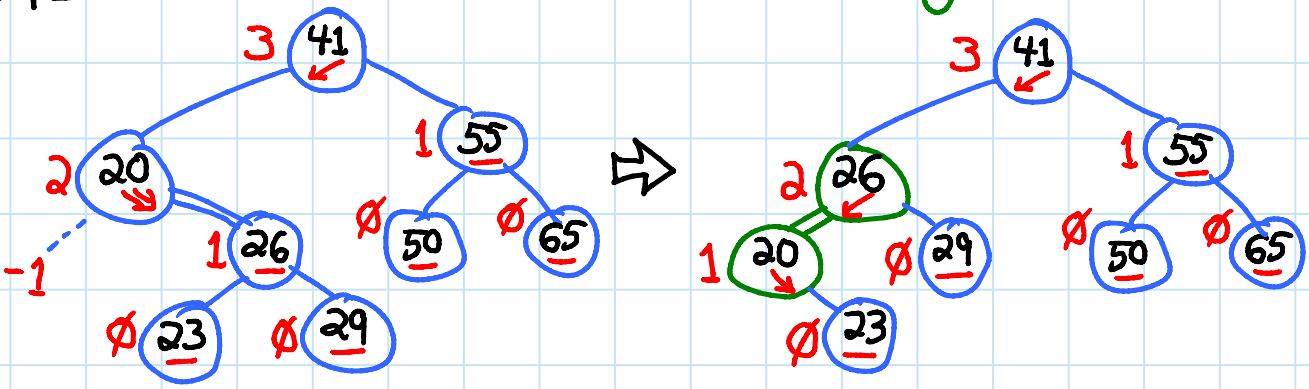
② walk up tree, restoring AVL property (& updating heights)





Example: Delete(11)

x=20: right-flat case



AVL sort:

- insert each item into AVL tree — $\Theta(n \lg n)$
- in-order traversal — $\Theta(n)$

$\Theta(n \lg n)$

Balanced search trees:

there are many!

- AVL trees — [Adel'son-Velsii & Landis 1962]
- B-trees / 2-3-4 trees — [Bayer & McCreight 1972] [CLRS 18]
- BB[α] trees — [Nievergelt & Reingold 1973]
- red-black trees — [CLRS ch. 13]
- (A) — splay trees — [Sleator & Tarjan 1985]
- (R) — skip lists — [Pugh 1989]
- (A) — scapegoat trees — [Galperin & Rivest 1993]
- (R) — treaps — [Seidel & Aragon 1996]

(R) = use random numbers to make decisions fast with high probability

(A) = "amortized": adding up costs for several operations ⇒ fast on average

e.g. splay trees are a current research topic
- see 6.854 (Advanced Algorithms) & 6.851 (Advanced Data Structures)

Big picture:

Abstract Data Type (ADT): interface spec.
vs. Data Structure (DS): algorithm for each op.

- many possible DSs for one ADT

Priority Queue ADT:	heap	AVL tree
- Q = new-empty-queue()	$\Theta(1)$	$\Theta(1)$
- Q.insert(x)	$\Theta(\lg n)$	$\Theta(\lg n)$
- x = Q.deletemin()	$\Theta(\lg n)$	$\Theta(\lg n)$
- x = Q.findmin()	$\Theta(1)$	$\Theta(\lg n)$ ↳ $\Theta(1)$

Predecessor/Successor ADT:	heap	AVL tree
- S = new-empty()	$\Theta(1)$	$\Theta(1)$
- S.insert(x)	$\Theta(\lg n)$	$\Theta(\lg n)$
- S.delete(x)	$\Theta(\lg n)$	$\Theta(\lg n)$
- y = S.predecessor(x)	$\Theta(n)$	$\Theta(\lg n)$
- y = S.successor(x)	$\Theta(n)$	$\Theta(\lg n)$