

TODAY: Shortest Paths Intro.

- shortest paths problem
- negative-weight cycles
- relaxation algorithm/framework
- triangle inequality
- optimal substructure

Motivation:

- Google Maps (fastest/shortest way $A \rightarrow B$)
- network routing (fastest way to route packet)

Edge-weighted graph = directed graph $G = (V, E)$
+ edge-weight function $w: E \rightarrow \mathbb{R}$

Path $p = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$
i.e. $(v_i, v_{i+1}) \in E$ for $1 \leq i < k$
- from v_1 to v_k

Weight $w(p) = w(v_1, v_2) + w(v_2, v_3) + \dots + w(v_{k-1}, v_k)$

Example:

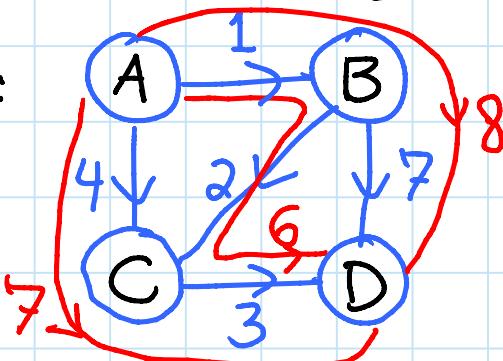


$$w(p) = 4 - 2 - 5 + 1 = -2$$

Shortest path from $u \in V$ to $v \in V$

- = a path p of minimum possible weight $w(p)$ from u to v (among all paths from u to v)
- weight of such a path is shortest-path weight $s(u,v)$

Example:



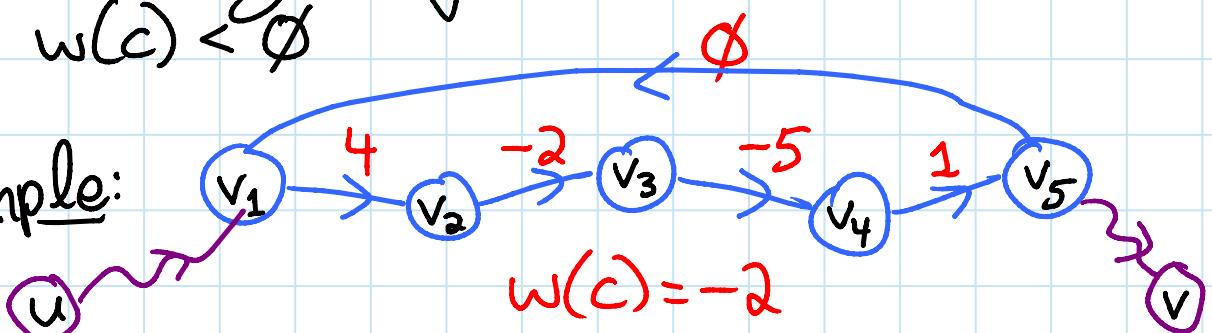
$$A \rightarrow B \rightarrow C \rightarrow D$$

$$s(A,D) = 6$$

- if no path from u to v then no shortest path either
 - define $s(u,v) = \infty$
- even if there's a path from u to v , there may not be a shortest path:
- negative-weight cycle $c = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$ has $w(c) < \emptyset$

e.g.
 $s(B,A) = \infty$

Example:



$$w(c) = -2$$

- $s(u,v) = \begin{cases} \inf \{ w(p) \mid \text{path } p \text{ from } u \text{ to } v \} \\ \infty \text{ if no path from } u \text{ to } v \\ \Rightarrow -\infty \text{ if negative-weight cycle along any path from } u \text{ to } v \end{cases}$

Brute-force algorithm:

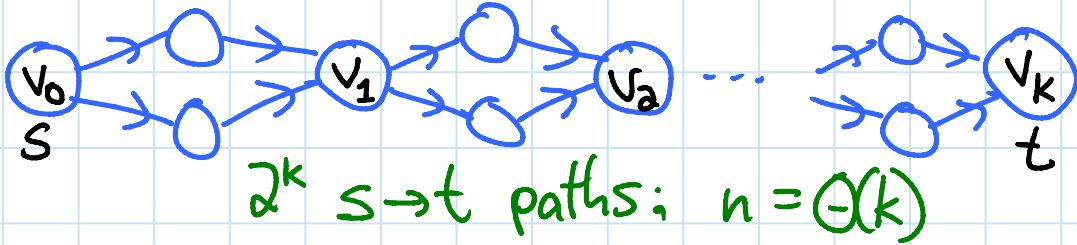
distance(s, t):

for each path p from s to t :

compute $w(p)$

return p encountered with smallest $w(p)$

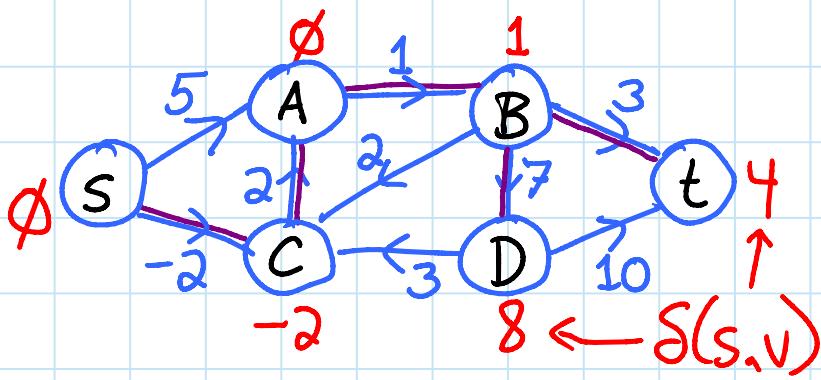
- infinitely many paths (\Rightarrow infinite time)
if $S(s, t) = -\infty$ (i.e. neg.-weight cycle on the way)
- even when finite, often exponential # paths:
 - $(n-2)!$ in complete graph
 - $2^{\Theta(n)}$ even in directed acyclic graph (DAG):



Single-Source Shortest Paths (SSSP) Problem:

- given edge-weighted directed graph $G = (V, E, w)$ & source vertex $s \in V$
 - compute $S(s, v)$ for all $v \in V$
 - turns out we can do this just as fast as computing $S(s, v)$ for a single $v \in V$
 - ideally also compute shortest-path tree containing a shortest path from s to every $v \in V$
 - represent by storing $v.\underline{\text{parent}}$ for each $v \in V$
- $\hookrightarrow \pi$ in textbook

Example:



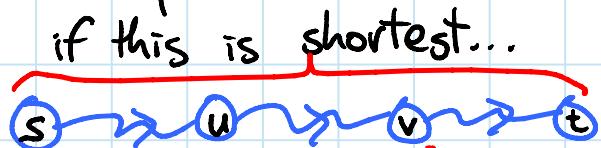
v	$v.\underline{\text{parent}}$
s	None
A	C
B	A
C	S
D	B
t	B

- why can SSSPs form a tree?

Optimal substructure: subpaths of shortest paths are shortest paths

Proof: cut & paste

- shorter $u \rightarrow v$ path would make a shorter $s \rightarrow t$ path, contradiction. \square



...then so is this

Relaxation algorithm for SSSP:

framework for most shortest-path algorithms

- maintain distance estimate $v.d$ for each $v \in V$
- goal: $v.d = s(s, v)$ for all $v \in V$
- invariant: $v.d \geq s(s, v)$ for all $v \in V$
- initialization(V):

for v in V :

$$v.d = \infty$$

$$v.parent = \text{None}$$

$$s.d = \emptyset$$

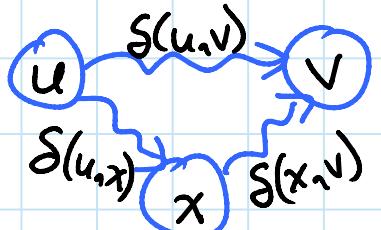
satisfies invariant

- repeatedly improve estimates until reach the goal

- "relaxation" refers to letting a solution (temporarily) violate a constraint, and trying to fix these violations
- for shortest paths, the constraint is:

Triangle inequality: $s(u, v) \leq s(u, x) + s(x, v)$
for all $u, v, x \in V$

Proof: shortest path $u \rightarrow v$
 \leq any particular path
e.g. $u \rightarrow x \rightarrow v$ \square

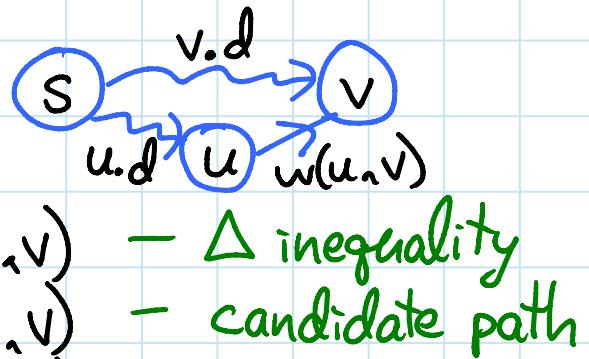


Edge relaxation:

- consider edge (u, v)

$$\begin{aligned} - S(s, v) &\leq S(s, u) + S(u, v) \\ &\leq S(s, u) + w(u, v) \end{aligned}$$

$$\Rightarrow \text{want } v.d \leq u.d + w(u, v)$$



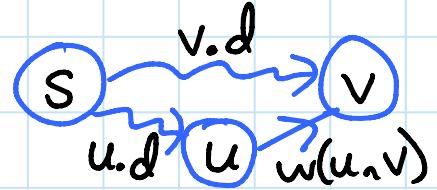
- Δ inequality
- candidate path

relax(u, v)

if $v.d > u.d + w(u, v)$

$$v.d = u.d + w(u, v)$$

$$v.\text{parent} = u$$



Safety Lemma: $\text{relax}(u, v)$ maintains invariant $x.d \geq S(s, x)$ for all $x \in V$

Proof: only $v.d$ changes, to:

$$\begin{aligned} &u.d + w(u, v) \\ &\geq S(s, u) + w(u, v) \\ &\geq S(s, u) + S(u, v) \\ &\geq S(s, v) \end{aligned}$$

□

- invariant before
- candidate path
- Δ inequality

Relaxation SSSP algorithm:

initialization(V)

while some edge (u, v) has $v.d > u.d + w(u, v)$:

pick such an edge (u, v)

$\text{relax}(u, v)$

Half Correctness: $v.d \geq S(s, v)$ for all $v \in V$

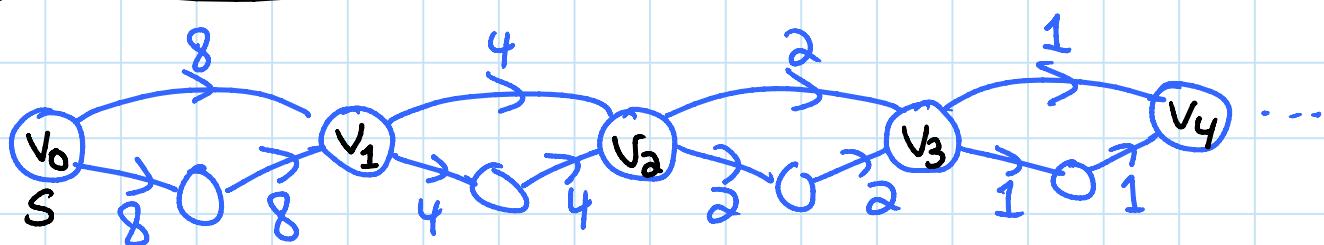
Proof: by induction on # relaxations

- base: initialization
- step: safeness lemma

Infinite relaxation: whenever negative-weight cycle reachable from source s

- relaxation continues around the cycle forever
- proof: always finite $v.d$'s, but need $-\infty$

Exponential relaxation: even in a DAG



d:	\emptyset	∞							
	8	16	20	24	26	28	29	30	29

relax upper edges
right to left

in between each:

relax lower edges
left to right

26	27	28
20	22	24
22	23	24
22	23	25
22	23	26
22	23	27
22	23	28

etc.

NEXT: Bellman-Ford algorithm finds a polynomial-length relaxation order!

Shortest Paths

Google maps from: 32 Vassar Street, Cambridge, MA to:Times Square, New York, NY Search Maps

Get Directions My Maps

32 Vassar St, Cambridge, Middlesex, Massachusetts
Times Square, New York, NY Add Destination - Show options Get Directions

Driving directions to Theater District - Times Square, New York, NY

Suggested routes

1. I-84 W 221 mi	3 hours 53 mins
2. I-90 W 209 mi	3 hours 58 mins
3. I-395 S and I-95 S 227 mi	4 hours 13 mins

This route has tolls.

A 32 Vassar St Cambridge, MA 02139

1. Head southwest on Vassar St 1.0 mi
2. Turn right at Memorial Dr 1.0 mi
3. Turn left at Western Ave 0.1 mi
4. Turn left at Soldiers Field Rd 0.2 mi
5. Take the I-90 ramp Toll road 0.3 mi
6. Keep right at the fork and merge onto I-90 W Partial toll road 52.1 mi
7. Take exit 9 to merge onto I-84 W toward US-20/Hartford/New York City Partial toll road 109 mi
8. Take exit 20 for I-684 toward NY-22/White Plains/Pawling 0.1 mi
9. Keep left at the fork and merge onto I-684 S 29.1 mi
10. Merge onto Hutchinson River Pkwy S 7.7 mi
11. Continue onto Cross County Pkwy (signs for George Washington Bridge) 4.7 mi

B 32 Vassar Street, Cambridge, MA 02139

Network Routing

http://s8.postimg.org/8kp2a7yqd/Client_routing_table.png

IPv4 Route Table

Active Routes:

Network Destination	Netmask	Gateway	Interface	Metric
0.0.0.0	0.0.0.0	192.168.3.1	192.168.3.41	281
0.0.0.0	0.0.0.0	10.8.0.1	10.8.0.2	286
10.8.0.0	255.255.255.252	On-link	10.8.0.2	286
10.8.0.2	255.255.255.255	On-link	10.8.0.2	286
10.8.0.3	255.255.255.255	On-link	10.8.0.2	286
127.0.0.0	255.0.0.0	On-link	127.0.0.1	306
127.0.0.1	255.255.255.255	On-link	127.0.0.1	306
127.255.255.255	255.255.255.255	On-link	127.0.0.1	306
169.254.0.0	255.255.0.0	On-link	169.254.112.65	276
169.254.112.65	255.255.255.255	On-link	169.254.112.65	276
169.254.255.255	255.255.255.255	On-link	169.254.112.65	276
192.168.1.0	255.255.255.0	10.8.0.1	10.8.0.2	30
192.168.3.0	255.255.255.0	On-link	192.168.3.41	281
192.168.3.41	255.255.255.255	On-link	192.168.3.41	281
192.168.3.255	255.255.255.255	On-link	192.168.3.41	281
224.0.0.0	240.0.0.0	On-link	127.0.0.1	306
224.0.0.0	240.0.0.0	On-link	169.254.112.65	276
224.0.0.0	240.0.0.0	On-link	10.8.0.2	286
224.0.0.0	240.0.0.0	On-link	192.168.3.41	281
255.255.255.255	255.255.255.255	On-link	127.0.0.1	306
255.255.255.255	255.255.255.255	On-link	169.254.112.65	276
255.255.255.255	255.255.255.255	On-link	10.8.0.2	286
255.255.255.255	255.255.255.255	On-link	192.168.3.41	281

Persistent Routes:

Network Address	Netmask	Gateway Address	Metric
0.0.0.0	0.0.0.0	10.8.0.1	Default

Shortest Paths

oracleofbacon.org/cgi-bin/movielinks

The Oracle of Bacon website interface. On the left, there's a sidebar with links: Welcome, Credits, How it Works, Contact Us, and Other stuff ». At the bottom of the sidebar, a copyright notice reads: © 1999-2010 by Patrick Reynolds. All rights reserved.

The main content area features two images: a classical statue of a head on the left and a portrait of Kevin Bacon on the right. The title "THE ORACLE OF BACON" is centered above a diagram illustrating a shortest path in the Bacon number graph.

Dr. Erik D. Demaine has a Bacon number of 3.

[Find a different link](#)

Diagram illustrating the shortest path from Dr. Erik D. Demaine to Kevin Bacon:

- Dr. Erik D. Demaine (green box) is connected to **The Man Who Saved Geometry (2009) (TV)** (purple box) with the edge label "was in".
- The Man Who Saved Geometry (2009) (TV)** (purple box) is connected to Benoît B. Mandelbrot (green box) with the edge label "with".
- Benoît B. Mandelbrot (green box) is connected to **The Revenge of the Dead Indians (1993)** (purple box) with the edge label "was in".
- The Revenge of the Dead Indians (1993)** (purple box) is connected to Dennis Hopper (green box) with the edge label "with".
- Dennis Hopper (green box) is connected to **The 2004 IFP/West Independent Spirit Awards (2004) (TV)** (purple box) with the edge label "was in".
- The 2004 IFP/West Independent Spirit Awards (2004) (TV)** (purple box) is connected to Kevin Bacon (green box) with the edge label "with".

At the bottom, there are input fields: "Kevin Bacon" and "to Dr. Erik D. Demaine", followed by "Find link" and "More options >>" buttons.

You Can't Get There From Here

Google maps from 32 Vassar Street, Cambridge, MA to:tokyo japan

Get Directions My Maps

36. Take exit 13B toward Halawa Hts. Stadium 0.3 mi

37. Merge onto I-H-201 E 4.1 mi

38. Merge onto I-H-1 E 4.1 mi

39. Take exit 23 for Punahou St toward Waikiki/Manoa 0.2 mi

40. Turn right at Punahou St 0.1 mi

41. Take the 1st right onto S Beretania St 0.1 mi

42. Take the 1st left onto Kalakaua Ave 1.9 mi

43. Kayak across the Pacific Ocean
Entering Japan 3,879 mi

44. Turn left toward 県道275号線 0.4 mi

45. Turn left toward 県道275号線 358 ft

46. Turn left toward 県道275号線 0.2 mi

47. Turn right at 県道275号線 0.1 mi

48. Turn left at 国道125号線 499 ft

49. Turn right at 県道24号線 0.6 mi

50. Turn left at 千束町（交差点） onto 国道354号線 2.0 mi

51. Turn right at 中村陸橋下（交差点） to stay on 国道354号線 1.0 mi

52. Take the ramp to 常磐自動車道
Toll road 0.3 mi

53. Keep left at the fork, follow signs for 東京 and merge onto 常磐自動車道
Toll road 23.8 mi

54. Take exit 三郷 J C T on the right toward 首都高速・銀座・渋谷 1000 mi

1000 km

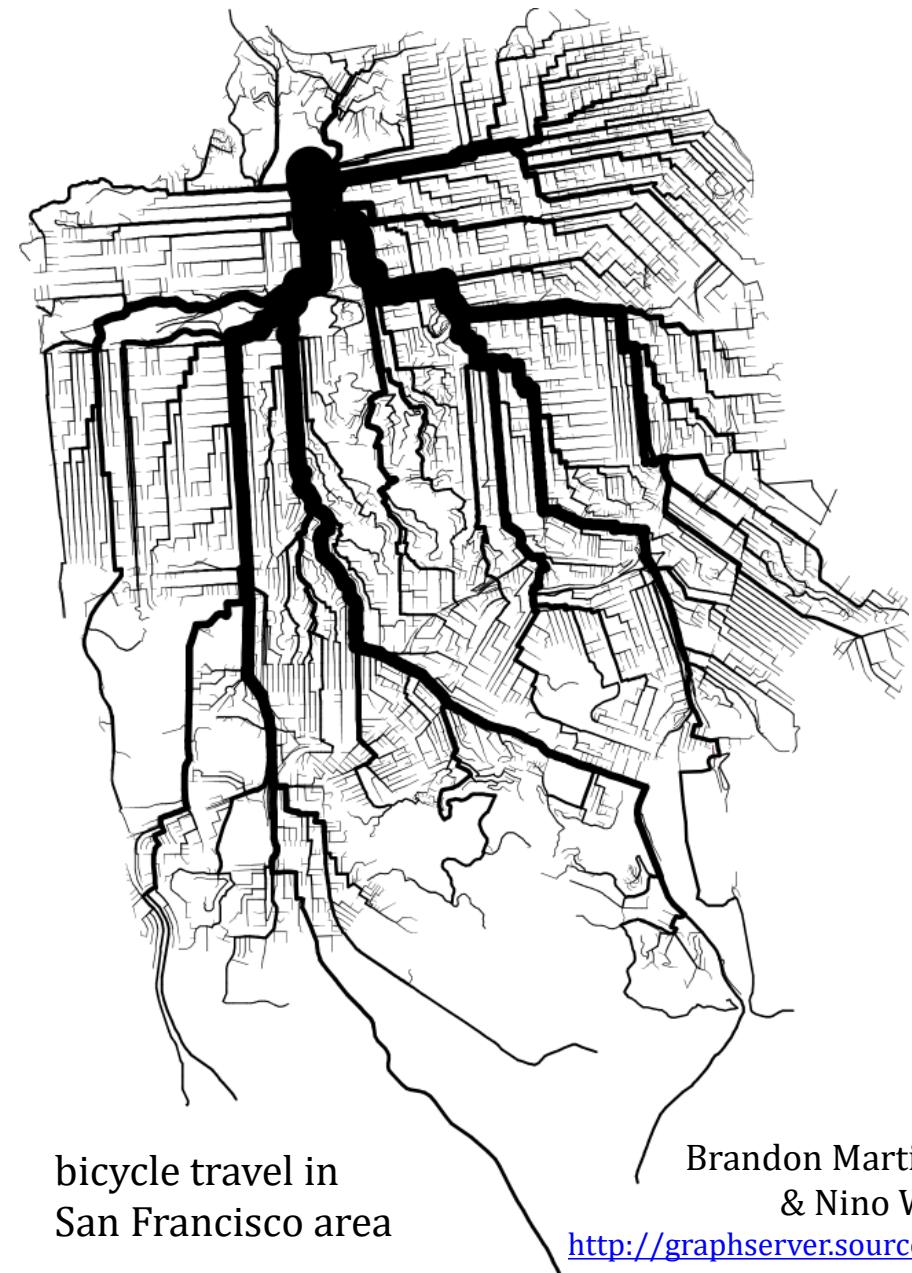
Print Send Link

Satellite

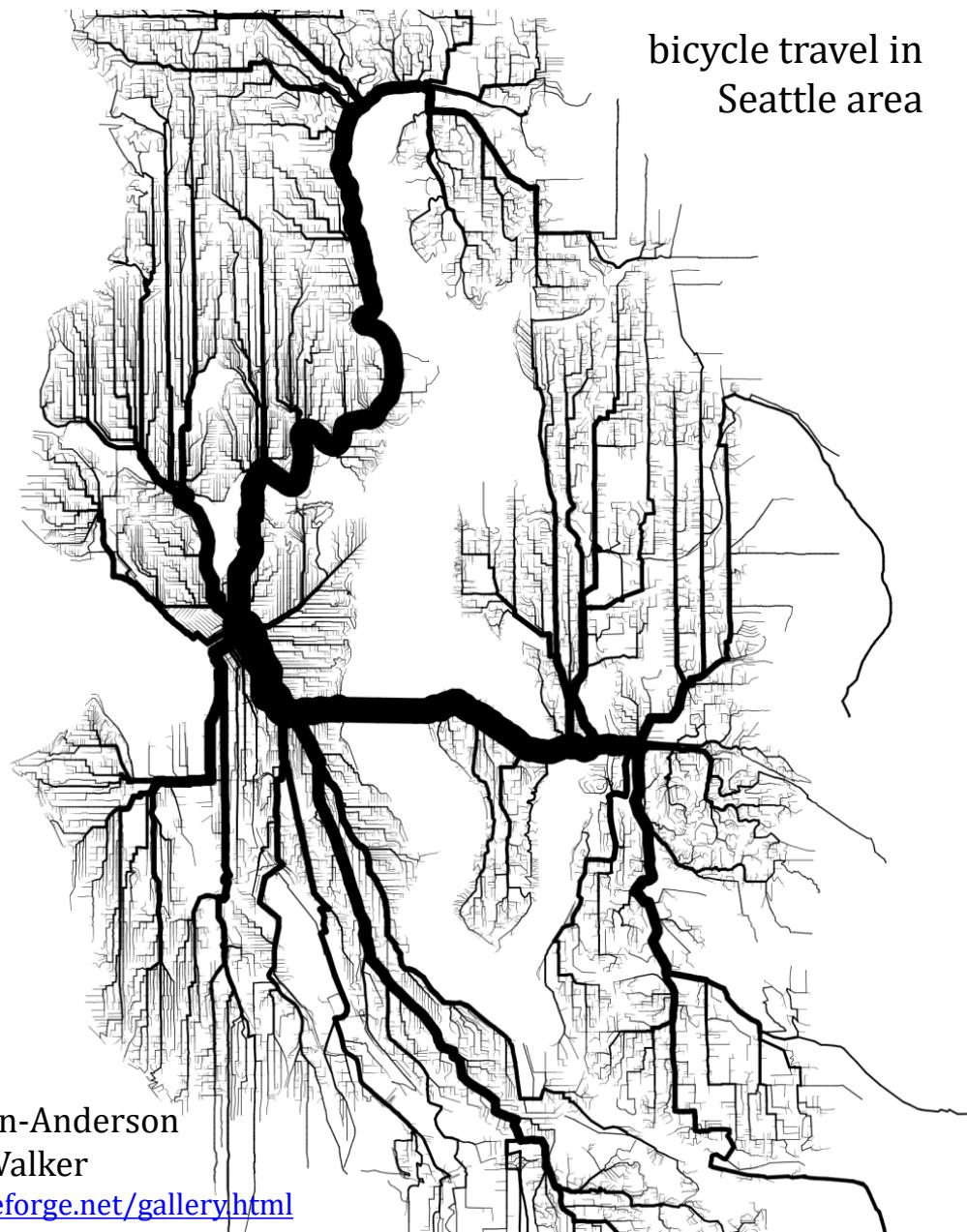
Traffic

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Shortest-Path Trees



bicycle travel in
San Francisco area



bicycle travel in
Seattle area

Brandon Martin-Anderson
& Nino Walker

<http://graphserver.sourceforge.net/gallery.html>

Relaxation

- In general, refers to letting a solution (temporarily) violate a constraint, and trying to fix these violations



Magic Geek
[http://www.youtube.com/
watch?v=Y12daEZTUYo](http://www.youtube.com/watch?v=Y12daEZTUYo)