Mein Titel

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ABSTRACT. – Kurze Beschreibung ...

Contents

1 TESTSection

1.1 TESTSubsection

Proposition RG-A11-16-26 (Zeitableitung der Metrik-abhängigen Volumenform). Suppose $g(t)$ is a smooth one-parameter family of metrics on a manifold M with $\frac{\partial}{\partial t} g = h$. Then the volume form $d\mu(g(t))$ evolves by

$$\frac{\partial}{\partial t} d\mu = \frac{1}{2} \operatorname{tr} h d\mu.$$

Proof RG-A11-16-27 (P: Zeitableitung der Metrik-abhängigen Volumenform). In local coordinates (x^i) the volume form can be written as $d\mu = \sqrt{\det g} dx^1 \wedge \ldots \wedge x^n$. So by (4.12) and the chain rule,

$$\begin{aligned} \frac{\partial}{\partial t} \sqrt{\det g} &= \frac{1}{2} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial t} \det g \\ &= \frac{1}{2} \frac{1}{\sqrt{\det g}} \frac{\partial \det g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial t} \\ &= \frac{1}{2} \sqrt{\det g} \left(g^{-1} \right)_{ji} h_{ij} \\ &= \frac{1}{2} g^{ij} h_{ij} \sqrt{\det g} = \frac{1}{2} \operatorname{tr} h \sqrt{\det g} \end{aligned}$$

Thus,

$$\frac{\partial}{\partial t} d\mu = \frac{\partial \sqrt{\det g}}{\partial t} dx^1 \wedge \ldots \wedge x^n = \frac{1}{2} \operatorname{tr} h d\mu$$

Theorem RG-A11-16-28 (Evolutionsformel für Riemannischen $(4,0)$ Krümmungstensor abhängig von Lösungen des Ricci-Flusses). Suppose $g(t)$ is a solution of the Ricci flow, the $(4,0)$ Riemannian tensor R evolves by

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk\ell} =& \Delta R_{ijk\ell} + 2 \left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell} \right) \\ & - g^{pq} \left(R_{pjk\ell} R_{qi} + R_{ipk\ell} R_{qj} + R_{ijkp} R_{q\ell} + R_{ijp\ell} R_{qk} \right). \end{aligned}$$

Proof RG-A11-16-29 (P: Evolutionsformel für Riemannischen $(4,0)$ Krümmungstensor abhängig von Lösungen des Ricci-Flusses). By Proposition 4.8, with $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, the time derivative of $R_{ijk\ell}$ satisfies

$$\begin{aligned} & \nabla^2_{i,\ell} R_{jk} + \nabla^2_{j,k} R_{i\ell} - \nabla^2_{i,k} R_{j\ell} - \nabla^2_{j,\ell} R_{ik} \\ &\quad = - \frac{\partial}{\partial t} R_{ijk\ell} - g^{pq} \left(R_{ijkp} R_{q\ell} + R_{ijp\ell} R_{qk} \right). \end{aligned}$$

By Proposition 4.2, with indices k and ℓ switched, the Laplacian of R satisfies

$$\begin{aligned} & \nabla^2_{i,\ell} R_{jk} - \nabla^2_{j,\ell} R_{ik} + \nabla^2_{j,k} R_{i\ell} - \nabla^2_{i,k} R_{j\ell} \\ &\quad = \Delta R_{ij\ell k} + 2 \left(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk} \right) \\ &\quad - g^{pq} \left(R_{qj\ell k} R_{pi} + R_{iq\ell k} R_{pj} \right). \end{aligned}$$

Combining these equations gives

$$\begin{aligned} -\Delta R_{ijk\ell} = \Delta R_{ij\ell k} = & - \frac{\partial}{\partial t} R_{ijk\ell} - 2 \left(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk} \right) \\ & + g^{pq} \left(R_{qj\ell k} R_{pi} + R_{iq\ell k} R_{pj} \right) \\ & - g^{pq} \left(R_{ijkp} R_{q\ell} + R_{ijp\ell} R_{qk} \right) \end{aligned}$$

Proof RG-A11-16-30 (P: Evolutionsformel für Riemannischen $(4, 0)$ Krümmungstensor abhängig von Lösungen des Ricci-Flusses). By Proposition 4.8, with $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, the time derivative of $R_{ijk\ell}$ satisfies

$$\begin{aligned}\nabla^2_{i,\ell} R_{jk} + \nabla^2_{j,k} R_{i\ell} - \nabla^2_{i,k} R_{j\ell} - \nabla^2_{j,\ell} R_{ik}\\ = -\frac{\partial}{\partial t} R_{ijk\ell} - g^{pq}\left(R_{ijkp} R_{q\ell} + R_{ijp\ell} R_{qk}\right).\end{aligned}$$

By Proposition 4.2, with indices k and ℓ switched, the Laplacian of R satisfies

$$\begin{aligned}&\nabla^2_{i,\ell} R_{jk} - \nabla^2_{j,\ell} R_{ik} + \nabla^2_{j,k} R_{i\ell} - \nabla^2_{i,k} R_{j\ell}\\ &\quad= \Delta R_{ij\ell k} + 2\left(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk}\right)\\ &\quad- g^{pq}\left(R_{qj\ell k} R_{pi} + R_{iq\ell k} R_{pj}\right).\end{aligned}$$

Combining these equations gives

$$\begin{aligned}-\Delta R_{ijk\ell} = \Delta R_{ij\ell k} = &-\frac{\partial}{\partial t} R_{ijk\ell} - 2\left(B_{ij\ell k} - B_{ijk\ell} - B_{ikj\ell} + B_{i\ell jk}\right)\\ &+ g^{pq}\left(R_{qj\ell k} R_{pi} + R_{iq\ell k} R_{pj}\right)\\ &- g^{pq}\left(R_{ijkp} R_{q\ell} + R_{ijp\ell} R_{qk}\right)\end{aligned}$$

Corollar RG-A11-16-31 (Evolution von Connection Koeffizienten unter Ricci Flow). Under the Ricci flow, the connection coefficients evolve by

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = -g^{k\ell}\left(\left(\nabla_j \operatorname{Ric}\right)(\partial_i, \partial_\ell) + \left(\nabla_i \operatorname{Ric}\right)(\partial_j, \partial_\ell) - \left(\nabla_\ell \operatorname{Ric}\right)(\partial_i, \partial_j)\right)$$

Corollar RG-A11-16-32 (Evolution der Volumenform unter Ricci Flow). Under the Ricci flow, the volume form of g evolves by

$$\frac{\partial}{\partial t} d\mu = -\operatorname{Scal} d\mu$$

Corollar RG-A11-16-33 (Evolution des Ricci-Krümmungstensors unter Ricci Flow). Under the Ricci flow,

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + \nabla^2_{ik} \mathrm{Scal} - g^{pq}\left(\nabla^2_{q,i} R_{kp} + \nabla^2_{q,k} R_{ip}\right)$$

Corollar RG-A11-16-34 (Evolution des skalaren Krümmung unter Ricci Flow). Under the Ricci flow, $\frac{\partial}{\partial t\mathrm{Scal} = 2\Delta \mathrm{Scal} - 2g^{ij} g^{pq} \nabla^2_{q,j} R_{pi} + 2|\mathrm{Ric}|^2}$

Corollar RG-A11-16-35 (Evolution des Ricci-Krümmungstensor unter Ricci Flow 2). Under the Ricci flow,

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2g^{pq} g^{rs} R_{pikr} R_{qs} - 2g^{pq} R_{ip} R_{qk}$$

Proof RG-A11-16-36 (P: Evolution des Ricci-Krümmungstensor unter Ricci Flow 2). By (4.9) the time derivative of $R_{ik} = g^{j\ell} R_{ijk\ell}$ is $\frac{\partial}{\partial t} R_{ik} = g^{j\ell} \frac{\partial}{\partial t} R_{ijk\ell} - 2g^{jp} g^{\ell q} R_{pq} R_{ijk\ell}$. Substituting the expression for $\frac{\partial}{\partial t} R_{ijk\ell}$ in Theorem 4.14 (with $g^{j\ell} \Delta R_{ijk\ell} = \Delta R_{ik}$ from (2.15)) results in

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2g^{j\ell}\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right) - g^{j\ell}g^{pq}\left(R_{pjk\ell}R_{qi} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell}\right)$$

As we find that

$$\begin{aligned} &2g^{j\ell}\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right) \\ &\quad = 2g^{j\ell}B_{ijk\ell} - 2g^{j\ell}\left(B_{i\ell jk} + B_{ij\ell k}\right) + 2g^{pr}g^{qs}R_{piqk}R_{rs} \\ &\quad = 2g^{j\ell}B_{ijk\ell} - 4g^{j\ell}B_{ij\ell k} + 2g^{pr}g^{qs}R_{piqk}R_{rs} \\ &\quad = 2g^{j\ell}\left(B_{ijk\ell} - 2B_{ij\ell k}\right) + 2g^{pr}g^{qs}R_{piqk}R_{rs} \end{aligned}$$

and

$$\begin{aligned} &g^{j\ell}g^{pq}\left(R_{pjk\ell}R_{qi} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell}\right) \\ &\quad = 2g^{pq}R_{pi}R_{qk} + g^{j\ell}g^{pq}R_{ipk\ell}R_{qj} + g^{j\ell}g^{pq}R_{ijkp}R_{q\ell} \\ &\quad = 2g^{pq}R_{pi}R_{qk} + 2g^{pr}g^{qs}R_{piqk}R_{rs}, \end{aligned}$$

it follows that

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2g^{j\ell}\left(B_{ijk\ell} - 2B_{ij\ell k}\right) + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}$$

The desired result now follows from the following claim.

Claim 4.19. For any metric g_{ij}, the tensor $B_{ijk\ell}$ satisfies the identity

$$g^{j\ell}\left(B_{ijk\ell} - 2B_{ij\ell k}\right) = 0$$

Proof of Claim.

Using the Bianchi identities,

$$\begin{aligned} g^{j\ell}B_{ijk\ell} &= g^{j\ell}g^{pr}g^{qs}R_{piqj}R_{rks\ell} \\ &= g^{j\ell}g^{pr}g^{qs}R_{pqij}R_{rsk\ell} \\ &= g^{j\ell}g^{pr}g^{qs}\left(R_{piqj} - R_{pjqi}\right)\left(R_{rks\ell} - R_{r\ell sk}\right) \\ &= 2g^{j\ell}\left(B_{ijk\ell} - B_{ij\ell k}\right) \end{aligned}$$

Corollar RG-A11-16-37 (Evolution des skalaren Krümmung unter Ricci Flow 2). Under the Ricci flow,

$$\frac{\partial}{\partial t} \text{ Scal } = \Delta \text{ Scal } + 2|\text{Ric}|^2$$

Proof RG-A11-16-38 (P: Evolution des skalaren Krümmung unter Ricci Flow 2). By (4.9) the time derivative of $R_{ik} = g^{j\ell}R_{ijk\ell}$ is $\frac{\partial}{\partial t}R_{ik} = g^{j\ell}\frac{\partial}{\partial t}R_{ijk\ell} - 2g^{jp}g^{\ell q}R_{pq}R_{ijk\ell}$. Substituting the expression for $\frac{\partial}{\partial t}R_{ijk\ell}$ in Theorem 4.14 (with $g^{j\ell}\Delta R_{ijk\ell} = \Delta R_{ik}$ from (2.15)) results in

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2g^{j\ell}\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right) - g^{j\ell}g^{pq}\left(R_{pjk\ell}R_{qi} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell}\right)$$

As we find that

$$\begin{aligned}
&2g^{j\ell}\left(B_{ijk\ell}-B_{ij\ell k}-B_{i\ell jk}+B_{ikj\ell}\right)\\
&\quad=2g^{j\ell}B_{ijk\ell}-2g^{j\ell}\left(B_{i\ell jk}+B_{ij\ell k}\right)+2g^{pr}g^{qs}R_{piqk}R_{rs}\\
&\quad=2g^{j\ell}B_{ijk\ell}-4g^{j\ell}B_{ij\ell k}+2g^{pr}g^{qs}R_{piqk}R_{rs}\\
&\quad=2g^{j\ell}\left(B_{ijk\ell}-2B_{ij\ell k}\right)+2g^{pr}g^{qs}R_{piqk}R_{rs}
\end{aligned}$$

and

$$\begin{aligned}
&g^{j\ell}g^{pq}\left(R_{pjk\ell}R_{qi}+R_{ipk\ell}R_{qj}+R_{ijp\ell}R_{qk}+R_{ijkp}R_{q\ell}\right)\\
&\quad=2g^{pq}R_{pi}R_{qk}+g^{j\ell}g^{pq}R_{ipk\ell}R_{qj}+g^{j\ell}g^{pq}R_{ijkp}R_{q\ell}\\
&\quad=2g^{pq}R_{pi}R_{qk}+2g^{pr}g^{qs}R_{piqk}R_{rs},
\end{aligned}$$

it follows that

$$\frac{\partial}{\partial t}R_{ik}=\Delta R_{ik}+2g^{j\ell}\left(B_{ijk\ell}-2B_{ij\ell k}\right)+2g^{pr}g^{qs}R_{piqk}R_{rs}-2g^{pq}R_{pi}R_{qk}$$

The desired result now follows from the following claim.

Claim 4.19. For any metric g_{ij}, the tensor $B_{ijk\ell}$ satisfies the identity

$$g^{j\ell}\left(B_{ijk\ell}-2B_{ij\ell k}\right)=0$$

Proof of Claim.

Using the Bianchi identities,

$$\begin{aligned}
g^{j\ell}B_{ijk\ell}&=g^{j\ell}g^{pr}g^{qs}R_{piqj}R_{rks\ell}\\
&=g^{j\ell}g^{pr}g^{qs}R_{pqij}R_{rsk\ell}\\
&=g^{j\ell}g^{pr}g^{qs}\left(R_{piqj}-R_{pjqi}\right)\left(R_{rks\ell}-R_{r\ell sk}\right)\\
&=2g^{j\ell}\left(B_{ijk\ell}-B_{ij\ell k}\right)
\end{aligned}$$

Note

1.2 TESTSubsection2