Chapter Four

Matrices

Matrices and Some Operations

Definition 4.1:

An $m * n$ matrix A is a rectangular array mn real (or complex) numbers arranged in m horizontal rows and n vertical columns.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m*n}$$

The $i-th$ row of A is $[a_{i1} \;\; a_{i2} \;\; \cdots \;\; a_{in}] \quad (1 \leq i \leq m)$

The $j-th$ column is $\begin{bmatrix} a_{j1} \\ \vdots \\ a_{jm} \end{bmatrix}$

We shall say that A is m by n.

If $m = n$, we say that A is a square matrix of order n and the numbers $a_{11}, a_{22}, \cdots, a_{nn}$ from the main diagonal.

Also A can be written as $A = [a_{ij}]$, $(1 \leq i \leq m), (1 \leq j \leq n)$

Definition 4.2:

The sum of the diagonal elements of a square matrix A is called the trace of A.

Some types of Matrices

1) Zero matrix:

A matrix every element of which is zero is called a **zero matrix**.

$$A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

2) Diagonal matrix:

A square matrix $A = [a_{ij}]$, whose elements $a_{ij} = 0$ for $i \neq j$ is called **diagonal matrix**.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

3) Scalar matrix:

If in the diagonal matrix A $a_{11} = a_{22} = \cdots = a_{nn} = k$, A is called **scalar matrix**.

If $k = 1$, A is called **identity matrix**.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ is scalar matrix}$$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called identity matrix and denoted by I_3.

4) Transpose of matrix:

If $A = [a_{ij}]$ is an $m * n$ matrix then the $n * m$ matrix $A^T = [a_{ij}]^T$ where $[a_{ij}]^T = [a_{ji}]$ $(1 \le i \le m, 1 \le j \le n)$ is called **transpose** of A.

Thus the transpose of A is obtained by interchanging the rows and columns of A.

$$A = \begin{bmatrix} 3 & 6 \\ 5 & -1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 3 & 5 \\ 6 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \rightarrow B^T = [1 \ \ 3 \ \ 9]$$

5) Symmetric matrix:

A square matrix A is called **symmetric** if $A = A^T$

$$A = \begin{bmatrix} 4 & -1 & 9 \\ -1 & 7 & 2 \\ 9 & 2 & 1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 4 & -1 & 9 \\ -1 & 7 & 2 \\ 9 & 2 & 1 \end{bmatrix} \rightarrow A = A^T$$

6) Skew symmetric matrix:

A square matrix A is called **skew symmetric** matrix if $A = -A^T$

$$A = \begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & 2 \\ -9 & -2 & 0 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 0 & 1 & -9 \\ -1 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix} \rightarrow -A^T = \begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & 2 \\ -9 & -2 & 0 \end{bmatrix}$$

$\therefore A = -A^T$, then A is skew symmetric matrix.

Matrix conjugate: If $A = \begin{bmatrix} 2+3i & 1+2i \\ -i & 4 \end{bmatrix} \rightarrow \bar{A} = \begin{bmatrix} 2-3i & 1-2i \\ i & 4 \end{bmatrix}$ $\bar{A}$ is called matrix conjugate.

7) Hermitian matrix:

A square matrix $A = [a_{ij}]$ such that $A = \overline{A^T}$ is called **Hermitian matrix .**

$$A = \begin{bmatrix} 1 & 4+i \\ 4-i & 2 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 4-i \\ 4+i & 2 \end{bmatrix} \rightarrow \overline{A^T} = \begin{bmatrix} 1 & 4+i \\ 4-i & 2 \end{bmatrix}$$

$\therefore A = \overline{A^T}$ then, A is Hermitian matrix.

8) Skew Hermitian matrix:

A square matrix A is called **skew Hermitian matrix** if $A = -\overline{A^T}$

$$A = \begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$$

9) Triangular matrix:

(a) A square matrix A whose elements $a_{ij} = 0$ for $i > j$ is called **upper triangular** matrix.

$$A = \begin{bmatrix} -2 & -1 & 9 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) A square matrix A whose elements $a_{ij} = 0$ for $i < j$ is called **lower triangular** matrix.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 7 & 0 \\ 9 & 1 & 1 \end{bmatrix}$$

Definition 4.3:

i) Equality:

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if $a_{ij} = b_{ij}$ for all $1 \le i \le m\ ,\ 1 \le j \le n$.

$$A = \begin{bmatrix} -1 & 5 \\ 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} -1 & 5 \\ 0 & 3 \end{bmatrix}$$

ii) Matrix addition:

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then the sum of A and B is the matrix $C = [c_{ij}]$ defined by $c_{ij} = a_{ij} + b_{ij} \quad 1 \le i \le m\ , 1 \le j \le n$.

$$A = \begin{bmatrix} 2 & 5 & -2 \\ 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 & 2 \\ 4 & 9 & 0 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} 5 & 12 & 0 \\ 7 & 9 & 1 \end{bmatrix}$$

iii) Matrix multiplication:

If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is an $p \times n$ matrix, then the product of A and B is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj} \quad 1 \le i \le m, \qquad 1 \le j \le n$$

$$A=\begin{bmatrix}2&3\\-1&0\end{bmatrix},\quad B=\begin{bmatrix}1&3\\-2&4\end{bmatrix}$$

$$C=A*B=\begin{bmatrix}2*1+3*(-2)&2*3+3*4\\-1*1+0*2&-1*3+0*4\end{bmatrix}=\begin{bmatrix}-4&18\\-1&-3\end{bmatrix}$$

Remark 4.4:

In general $A*B\neq B*A$

iv) Scalar multiplication

If $A=[a_{ij}]$ is an $m\times n$ matrix and k is a real number, the scalar multiple of A by k is an $m\times n$ matrix $B=[b_{ij}]$ such that $b_{ij}=ka_{ij}\quad 1\leq i\leq m,\ 1\leq j\leq n$

Properties of Matrix operation

1) Matrix addition:

a) A+B=B+A

b) A+(B+C)=(A+B)+C

c) A+0=A , where 0 is the $m*n$ zero matrix.

d) For each $m*n$ matrix A , there is a unique $m*n$ matrix D s.t

A+D=0 → D=-A

∴A+(-A)=0

The matrix $-A$ is called the negative of A.

Example 4.5: If A is idempotent, show that $B=I-A$ is idempotent and that

$AB=BA=0$.

Solution:

$A^2 = A \rightarrow\ B^2 = (I - A)^2 = I^2 - 2IA + A^2 = I - A = B$

Theorem 4.6:

If A is an $n \times n$ matrix, then $A = S + K$, where S is symmetric and K is skew symmetric. Moreover, this representation is unique.

Proof:

$$Let\ S = \frac{1}{2}(A + A^T) and\ K = \frac{1}{2}(A - A^T)$$

$$S^T = (\frac{1}{2}(A + A^T))^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = S$$

So S is symmetric. Also

$$K^T = (\frac{1}{2}(A - A^T))^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -K$$

So K is skew-symmetric.

Moreover if $A = S_1 + K_1$ where S_1 is symmetric and K_1 is skew symmetric

Then $A^T = {S_1}^T + {K_1}^T = S_1 - K_1 \rightarrow\ S_1 = \frac{1}{2}(A + A^T)\ and\ S_1 = \frac{1}{2}(A - A^T)$

Assignment:

1) Prove
   a) trace $(A + B)$=trace(A)+trace(B)
   b) trace (KA)=K trace(A)
2) Show that: if A is an $n -$ square matrix then $A + \overline{A^T}$ is Hermition and $A - \overline{A^T}$ is skew-Hermition.

3) The matrix A=$\begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is Hermition.

   Is kA is Hermition if k is any real number? Or any complex number?

4) A matrix A is involuntary iff (I-A)(I+A)=0
5) If A is nilpotent of index 2. Show that $A(I \pm A)^n = A$ for any positive integer n.
6) Prove that: $(ABC)^T = C^T B^T A^T$.

2) Show that:

a) If A, B and C are of the appropriate sizes, then $A(BC) = (AB)C$

b) If A, B and C are of the appropriate sizes, then $A(B + C) = AB + AC$

c) If A, B and C are of the appropriate sizes, then $(A + B)C = AC + BC$

d) $AB = 0$ does not necessarily imply $A = 0 \; or \; B = 0$.

e) $AB = AC$ does not necessarily imply $B = C$.

3) Prove the following:

If A and B are matrices and r, s are scalars, then

a) $r(sA) = (rs)A$
b) $(r + s)A = rA + sA$
c) $r(A + B) = rA + rB$
d) $A(rB) = r(AB) = (rA)B$.

4) Show that:

If r is a scalar and A and B are matrices, then

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T$

d) $(rA)^T = rA^T$

e) $(ABC)^T = C^T B^T A^T$.

Definition 4.7: [Determinant]

Let A=$[a_{ij}]$ be an $n*n$ matrix then the **determinant** of A (denoted by $|A|$) by

$$|A| = \sum_{S} (\pm)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

Where the summation ranges over all permutations $j_1, j_2, \cdots, j_n$ of the set S={1,2,⋯, n}.
The sign is taken as + or – according to whether the permutation is even or odd.

Example 4.8:

If A=$[a_{11}]$ is an $1*1$ matrix → $|A| = a_{11}$

$A = (2) \rightarrow |A| = 2$, $A = (-3) \rightarrow |A| = -3$

Example 4.9:

If A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is an $2 * 2$ matrix then

$$|A| = \sum_{2} (\pm) a_{1j1} a_{2j2} = a_{11} a_{22} - a_{12}\, a_{21}$$

A=$\begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix} \rightarrow |A| = 1 * 6 - 3 * 5 = -9$

Example 4.10:

If A=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$|A| = \sum_{6} a_{1j1} a_{2j2} a_{3j3} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{12} a_{21}\, a_{33} - a_{11} a_{23} a_{32} - a_{13} a_{22} a_{31}$$

A=$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 0 & 4 & 6 \end{bmatrix}$

$$|A| = (1 * -1 * 6) + (3 * 3 * 0) + (5 * 2 * 4) - (3 * 2 * 6) - (1 * 3 * 4) \quad - (5 * -1 * 0)$$

$$|A| = -6 + 40 - 36 - 12 = -14$$

Also to find $|A_{3*3}|$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$-a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - \quad a_{13}a_{22}a_{31}$

A=$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 0 & 4 & 6 \end{bmatrix}$

To find $|A|$

$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 0 & 4 & 6 \end{bmatrix}\begin{matrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{matrix}$

$|A| = (1 * -1 * 6) + (3 * 3 * 0) + (5 * 2 * 4) - (5 * -1 * 0) - (1 * 3 * 4)$

$$-(3 * 2 * 6) = -6 + 40 - 12 - 36 = -14$$

Properties of Determinants

1) $|A| = |A^T|$
2) If matrix B results from matrix A by interchanging two rows (columns) of A, then $|B| = -|A|$
3) If two rows (columns) of A are equal, then $|A| = 0$
4) If every element of a row (column) of A are zero then $|A| = 0$
5) If B is obtained from A by multiplying a row (column) of A by a real number k then $|B|$ =k$|A|$
6) If B is obtained from A by adding to each element of the rth row (column) of A constant c times the corresponding element of it's sth row (column) r≠s , then $|B| = |A|$

7) If a matrix A=$[a_{ij}]$ is upper (lower) triangular, then $|A| = a_{11}a_{22}\cdots a_{nn}$

8) If a matrix A=$[a_{ij}]$ is diagonal matrix, then $|A| = a_{11}a_{22}\cdots a_{nn}$

9) $|A.B| = |A|.|B|$

10) $|\bar{A}| = \overline{|A|}$

Example 4.11:

A=$\begin{bmatrix}1 & 3\\2 & 8\end{bmatrix} \rightarrow A^T = \begin{bmatrix}1 & 2\\3 & 8\end{bmatrix}$

$$|A| = 1*8-2*3 = 2 \qquad and \qquad |A^T| = 1*8-3*2 = 2$$

Example 4.12:

Let A=$\begin{bmatrix}1 & 2 & 4\\0 & 3 & 1\\-2 & 1 & 2\end{bmatrix} \xRightarrow{R_3 \leftrightarrow R_2}$ B=$\begin{bmatrix}1 & 2 & 4\\-2 & 1 & 2\\0 & 3 & 1\end{bmatrix}$

$$|A| = (1*3*2)+(2*1*-2)+(4*0*1)-(4*3*-2)-(1*1*1)$$

$$-(2*0*2) = 6-4+24-1 = 25$$

$$|B| = (1*1*1)+(2*2*0)+(4*-2*3)-(4*1*0)-(1*2*3)$$

$$-(2*-2*1) = 1-24-6+4 = -25$$

$\therefore\ |B| = -|A|$

Example 4.13:

Let A=$\begin{bmatrix}1 & 2 & 1\\0 & 3 & 0\\2 & 4 & 2\end{bmatrix}$, since $K_1 = K_3$ then $|A| = 0$

Example 4.14:

Let A=$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 8 & 4 & 2\end{bmatrix} \rightarrow \ |A| = 0$

Example 4.15:

A=$\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$ and B=$\begin{bmatrix}6 & 3\\ 12 & 3\end{bmatrix}$ B=6A

$|A| = -1$ and $|B| = -6$

$\therefore \ |B| = 6|A|$

Example 4.16:

Let A=$\begin{bmatrix}1 & 3\\ -1 & 4\end{bmatrix}$ and B=$\begin{bmatrix}1 & 3\\ -1+2*1 & 4+2*3\end{bmatrix} = \begin{bmatrix}1 & 3\\ 1 & 10\end{bmatrix}$

$|A| = 7$ and $|B| = 7$ then $|A| = |B|$

Example 4.17:

Let A=$\begin{bmatrix}2 & 0 & 0\\ 4 & 5 & 0\\ 8 & 3 & 1\end{bmatrix} \rightarrow \ |A| = 2*5*1 = 10$

Example 4.18:

Let A=$\begin{bmatrix}2 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & 1\end{bmatrix} \rightarrow \ |A| = 2*-4*1 = -8$

Example 4.19:

Let A=$\begin{bmatrix} 2i & 3+i \\ -1+i & 2i \end{bmatrix}$ $\rightarrow$ $\bar{A} = \begin{bmatrix} -2i & 3-i \\ -1-i & -2i \end{bmatrix}$

$|A| = -2i$ then $\overline{|A|} = 2i$ and $|\bar{A}| = 2i$

$\therefore \overline{|A|} = 2i = |\bar{A}|$

Remark 4.20: If $|A| \neq 0$ then $|A^{-1}| = \frac{1}{|A|}$

Example 4.21:

If $|A| = -4$, then find:

1) $|A^2|$

2) $|A^{-1}|$

Solution:

1) $|A^2| = |A.A| = |A||A| = 16$
2) $|A^{-1}| = \frac{1}{|A|} = -\frac{1}{4} = -0.25$

Cofactor expansion and application:

Definition 4.22:

Let A=$[a_{ij}]$ be an $n * n$ matrix. Let M_{ij} be an $(n-1)*(n-1)$ sub matrix of A obtained by deleting the the ith row and jth column of A. the determinate $|M_{ij}|$ is called the minor of a_{ij}.

The factor A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j}|M_{ij}|$

Example 4.23:

Let A=$\begin{vmatrix} 2 & 1 & 3 \\ -4 & 5 & 1 \\ 3 & 2 & -4 \end{vmatrix}$

Then $|M_{12}| = \begin{vmatrix} -4 & 1 \\ 3 & -4 \end{vmatrix} = 13$, $|M_{23}| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$

$$|M_{32}| = \begin{vmatrix} 2 & 3 \\ -4 & 1 \end{vmatrix} = 14$$

Also, $A_{12} = (-1)^{1+2}|M_{12}| = -13,$ $A_{23} = (-1)^{2+3}|M_{23}| = -1$

Theorem 2.24:

Let A be an $n * n$ matrix, then

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ (Expansion of $|A|$ about the ith row)

Also

$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (Expansion of $|A|$ about the jth column)

Example 4.25:

If A=$\begin{vmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{vmatrix}$ then find $|A|$

Solution:

$$A_{11} = (-1)^{1+1}|M_{11}| = 4$$

$$A_{12} = (-1)^{1+2}|M_{12}| = 34$$

$$A_{13} = (-1)^{1+3}|M_{13}| = -31$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (3*4) + (-1*34) + (2*-31) = 12 - 96$$

$$= -84$$

Definition 4.26:

Let A be an $n*n$ matrix. The $n*n$ matrix adjA, called the **adjoint of A,** is the matrix whose i,j $-$th element is the cofactor A_{ij} of a_{ji} thus

$$adjA = \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}^T$$

Theorem 4.27:

If A=[a_{ij}] be an $n*n$ matrix, then A(adjA)=(adjA)A=$|A| * I_n$

Example 4.28:

If A=$\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$ find adjA.

Solution:

adjA=$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6$ $A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10$ $A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10$ $A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28$

$$\text{adjA} = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

$$\text{A.(adjA)} = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} = \begin{bmatrix} -94 & 0 & 0 \\ 0 & -94 & 0 \\ 0 & 0 & -94 \end{bmatrix}$$

$$= -94\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| * I_3 = \text{(adjA).A}$$

The inverse of a matrix

Definition 4.29:

An $n \times n$ matrix A is called nonsingular (or invertible) if there exists an $n \times n$ matrix B such that $AB = BA = I_n$.

The matrix B is called an inverse of A and denoted by $B = A^{-1}$, if there exists no such matrix B, then A is called singular (or non-invertable).

Example 4.30: Let

$A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}$, $B=\begin{bmatrix}6 & -2 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$

$AB=I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=BA.$

Theorem 4.31:

If A is an $n*n$ matrix and $|A| \neq 0$, then

$$A^{-1}=\frac{1}{|A|}adjA=\begin{bmatrix}\frac{A_{11}}{|A|} & \cdots & \frac{A_{n1}}{|A|}\\ \vdots & \ddots & \vdots\\ \frac{A_{1n}}{|A|} & \cdots & \frac{A_{nn}}{|A|}\end{bmatrix}$$

Proof:

A(adjA) $=$ (adjA)A$=|A|*I_n$ so if $|A| \neq 0$, then

$A\frac{1}{|A|}adjA=\frac{1}{|A|}(adjA)A=\frac{1}{|A|}|A|*I_n=I_n$

Hence $A^{-1}=\frac{1}{|A|}adjA$

Example 4.32: If A$=\begin{bmatrix}3 & -2 & 1\\ 5 & 6 & 2\\ 1 & 0 & -3\end{bmatrix}$ we have:

$|A|=-94$

$$A^{-1} = \frac{1}{|A|} adjA = \begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ \frac{-17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & \frac{-28}{94} \end{bmatrix} = \begin{bmatrix} \frac{9}{47} & \frac{3}{47} & \frac{5}{47} \\ \frac{-17}{94} & \frac{5}{47} & \frac{1}{94} \\ \frac{3}{47} & \frac{1}{47} & \frac{-14}{47} \end{bmatrix}$$

Theorem 4.33:

A matrix A is non-singular if and only if $|A| \neq 0$.

Theorem 4.34:

$adj(A.B) = adjA.adjB$

Theorem 4.35:

If a matrix A has an inverse, then the inverse is unique.

Proof:

Let B, C be two inverse for A.

$\therefore AB = BA = I_n$

$AC = CA = I_n$

$BA = AC = I_n$

$\therefore B = B * I_n = B(AC) = (BA)C = I_nC = C.$

Theorem 4.36:

a) If A is a non-singular matrix, then A^{-1} is non-singular and $(A^{-1})^{-1}=A$.

b) If A and B are non-singular matrix, then AB is non-singular and $(AB)^{-1}=B^{-1}A^{-1}$.

c) If A is a non-singular matrix, then $(A^T)^{-1}=(A^{-1})^T$.

Remark 4.37:

$(A_1A_2A_3\ldots A_n)^{-1}=(A_n^{-1}\ldots A_1^{-1})$, where $A_1, A_2, \ldots, A_n$ are non – singular matrices.

Definition 4.38:

An $m \times n$ matrix is said to be in **reduced row echelon form**, when is satisfies the following properties:

1) All rows consisting entirely of zeros (if any), are at the bottom of the matrix.
2) The first no zero entry in each row that does not consist entirely of zeros a 1, called the leading entry of its row.
3) If rows i and $i+1$ are two successive rows that do not consist entirely of zeros, then the leading entry of row $i+1$ is to the right of the leading entry of row i .
4) If a column contains a leading entry of some row, then all other entries in that column are zero.

Example 4.39:

$$A=\begin{bmatrix}1&0&0&4\\0&1&0&5\\0&0&1&2\end{bmatrix},\ B=\begin{bmatrix}1&2&0&0&2\\0&0&1&0&1\\0&0&0&1&0\end{bmatrix},\ C=\begin{bmatrix}1&0&0&3&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

are in reduced row echelon form.

Example 4.40:

$$A=\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -3\end{bmatrix},\ B=\begin{bmatrix}1 & 0 & 3 & 4\\ 0 & 2 & -2 & 5\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0 & 3 & 4\\ 0 & 1 & -2 & 5\\ 0 & 1 & 2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix},\ D=\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & -2 & 5\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

are not in reduced row echelon form.

Definition 4.41:

An elementary row operation of an $m \times n$ matrix $A = [a_{ij}]$ is one of the following operations:

a) Interchange row r and s of A , that is replace
$a_{r1}, a_{r2}, a_{r3}, \dots, a_{rn}$ by $a_{s1}, a_{s2}, a_{s3}, \dots, a_{sn}$, and $a_{s1}, \dots, a_{sn}$ by $a_{r1}, \dots, a_{rn}$.

b) Multiply row r of A by $c \neq 0$, that is replace $a_{r1}, \dots, a_{rn}$ by $ca_{r1}, \dots, ca_{rn}$.

c) Add d times row r of A to row s to A, $r \neq s$, that is ,replace
$a_{s1}, a_{s2}, \dots, a_{sn}$ by $a_{s1}+da_{r1}, a_{s2}+da_{r2}, \dots, a_{sn}+da_{rn}$.

Example 4.42:

Let $\begin{bmatrix}0 & 0 & 1 & 2\\ 2 & 3 & 0 & -2\\ 3 & 3 & 6 & -9\end{bmatrix}$ interchange rows 1 and 3 of A we obtain

B=$\begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ multiply third row of A by $\frac{1}{3}$ we obtain

C=$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$ Adding -2 times row 2 of A to row 3 of A, we obtain

D=$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$

Definition 4.43:

An $m \times n$ matrix A is said to be row equivalent to an $m \times n$ matrix B can be obtained by applying a finite sequence of elementary row operation to A.

Theorem 4.44:

Every non-zero $m \times n$ matrix is row equivalent to a unique matrix in reduced row echelon form.

Example 4.45:

$$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \xrightarrow{-2R_1+R_4} \begin{bmatrix} 1 & 1 & -5/2 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1 & 1 & -5/2 & 1 & 2\\0 & 2 & 3 & -4 & 1\\0 & 0 & 2 & 3 & 4\\0 & -2 & -1 & 7 & 3\end{bmatrix}\xrightarrow{1/2R_2}\begin{bmatrix}1 & 1 & -5/2 & 1 & 2\\0 & 1 & 3/2 & -2 & 1/2\\0 & 0 & 2 & 3 & 4\\0 & -2 & -1 & 7 & 3\end{bmatrix}$$

$$\xrightarrow{2R_2+R_4}\begin{bmatrix}1 & 1 & -5/2 & 1 & 2\\0 & 1 & 3/2 & -2 & 1/2\\0 & 0 & 2 & 3 & 4\\0 & 0 & 2 & 3 & 4\end{bmatrix}\xrightarrow{R_1-R_2}\begin{bmatrix}1 & 0 & -4 & 3 & 3/2\\0 & 1 & 3/2 & -2 & 1/2\\0 & 0 & 2 & 3 & 4\\0 & 0 & 2 & 3 & 4\end{bmatrix}$$

$$\frac{1}{2}R_3\begin{bmatrix}1 & 0 & -4 & 3 & 3/2\\0 & 1 & 3/2 & -2 & 1/2\\0 & 0 & 1 & 3/2 & 2\\0 & 0 & 2 & 3 & 4\end{bmatrix}\xrightarrow{-2R_3+R_4}\begin{bmatrix}1 & 0 & -4 & 3 & 3/2\\0 & 1 & 3/2 & -2 & 1/2\\0 & 0 & 1 & 3/2 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\xrightarrow[R_2-3/2R_3]{R_1+4R_3}\begin{bmatrix}1 & 0 & 0 & 9 & 19/2\\0 & 1 & 0 & -17/4 & -5/2\\0 & 0 & 1 & 3/2 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Remark 4.46:

1) Every matrix is row equivalent to itself, that is; $A{\sim}A$.

2) If A is row equivalent to B and B is row equivalent to A, that is; $A{\sim}\ B \rightarrow B{\sim}A$.

3) If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C, That is; $A{\sim}B \wedge B{\sim}C \rightarrow A{\sim}C$.

Practical method for finding A^{-1}

The practical method for finding A^{-1} where A is an $n \times n$ matrix, consists in forming the $n \times 2n$ matrix $(A|I_n)$ and performing elementary row operation, that transform it to $(I_n|A^{-1})$ whatever we do to a row of A, we also do to the corresponding row of I_n.

Example 4.47:

Let $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{bmatrix}$, find A^{-1}.

Solution:

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 5 & 5 & 1 & 0 & 0 & 1\end{array}\right) \xrightarrow[\frac{1}{2}R_2]{-5R_1+R_3} \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 3/2 & 0 & 1/2 & 0\\ 0 & 0 & -4 & -5 & 0 & 1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{4}R_3} \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 3/2 & 0 & 1/2 & 0\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$\xrightarrow{R_2-\frac{3}{2}R_3} \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$\xrightarrow[R_1-R_3]{R_1-R_2} \left(\begin{array}{ccc|ccc}1 & 0 & 1 & 23/8 & -1/2 & -3/8\\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$\xrightarrow{R_1-R_3} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 13/8 & -1/2 & -1/8\\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$A^{-1}=\begin{bmatrix}13/8 & -1/2 & -1/8\\ -15/8 & 1/2 & 3/8\\ 5/4 & 0 & -1/4\end{bmatrix}$$

Hence $AA^{-1}=A^{-1}A=I_3$.

Rank of a matrix

Definition 4.48:

The rank of an $m \times n$ matrix A is equal to the number of nonzero rows of final reduced row echelon form of that matrix.

Example 4.49:

Find the rank of $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 2 & 0 & 17/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$Rank of $A = 2$

Remark 4.50 Rank of I_n, (identity matrix) is n.

Q:

Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 5 \end{bmatrix}$