MATHEMATICS

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Contents

Chapter 1

Introduction

1.1 Sets

Definition 1.1.1. A *Set* is any collection of definite and distinguishable objects (*elements*).

Example 1.1.1. $A = \{1, 4, 6, 8, 20\}$
$B = \{apple, orange, cherry\}$
$N = \{0, 1, 2, 3, 4, ...\}$

Notations.

(1).Sets are usually denoted by capital letters($A, B, C, ...$). The elements of the set are usually denoted by small letters $(a, b, c, ...)$.

(2).If A is a set and a is an element of A, we write $a \in A$. (We also say that a belongs to A).

(3).If A is a set and a is not an element of A, we write $a \notin A$. (We also say that a does not belong to A).

(4).When we give a set, we generally use braces, e.g.:

(i).$S = \{a, b, c, ...\}$ where the elements are listed between braces, three dots imply that the law of formation of other elements is known.

(ii).$S = \{x \in A : p(x) \text{ is true}\}$ where x stands for a generic element of the set S and p is a property defined on the set A.

Definition 1.1.2. All sets under investigation in any application of set theory are assumed to be contained in some large fixed set called the **universal set or universe**. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies, the universal set consists of all the people in the world. We will denote the universal set by $\mathbb{U}$.

Definition 1.1.3.

i) Equal sets: We define $A = B$ if A and B have the same elements.

ii) Subset: We say that A is a *subset* of B and we write $A \subset B$ or $B \supset A$ if every element of A is also an element of B. (We also say that A is included in B or B includes A or B is a superset of A.)

iii) Proper subset: We say that A is a *proper subset* of B and we write $A \subset B$ strictly if $A \subset B$ and A $\neq B$. (There exists at least one element $b \in B$ such that $b \notin A$.)

iv) The empty set:A set which has no element is called the *empty set* which is denoted by ϕ or $\{\}$.

v) Power set of a set: Let X be a set. The set of all subsets of X is called the power set of X and is denoted by $\mathcal{P}(X)$. (That is we define $\mathcal{P}(X) = \{A : A \subset X\}$.)

vi) Disjoint sets: A and B are called disjoint sets if $A \cap B = \phi$.

Example 1.1.2.

1) $\mathbb{N} = \{0, 1, 2, 3, ...\}$ the set of all natural numbers.
2) $\mathbb{N}^{+} = \{n \in N : n > 0\}$.
3) $\mathbb{Z} = \{0, -1, +1, -2, +2, ...\}$ the set of all integer numbers.
4) $\mathbb{Q} = \{p/g : p, q \in \mathbb{Z}; q \neq 0\}$ the set of all rational numbers.
5) $\mathbb{Q}^{+} = \{r \in \mathbb{Q} : r > 0\}$
6) $\mathbb{Q}^{*} = \{x : x \notin \mathbb{Q}\}$ the set of all irrational numbers.(e.g. $\mathbb{Q}^{*} = \{\sqrt{2}, \sqrt{3}, e, \pi, ...\}$)
7) $\mathbb{R} =$ the set of real numbers.
8) $\mathbb{R}^{+} = \{x \in \mathbb{R} : x > 0\}$
9) $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}$ and $i = \sqrt{-1}\}$ the set of all complex numbers.

1.1.1 Operations Between Sets.

Union of two sets: The union of two sets A and B ,denoted by $A \cup B$, is the collection of elements which are in A or B.

$$A \cup B = \{x | x \in A \text{ or } x \in B\} \tag{1.1.1}$$

Intersection of two sets: The intersection of two sets A and B ,denoted by $A \cap B$, is the collection of elements which are in A and B.

$$A \cap B = \{x | x \in A \text{ and } x \in B\} \tag{1.1.2}$$

Difference of sets: The difference of sets A and B is defined by $A - B = \{x | x \in A \text{ and } x \notin B\}$. If $B \subset A$ then $A - B$ is called the *complement* of B with respect to (w.r.t) A

and is denoted by B^c or $\bar{B}$.

Example 1.1.3.

1) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

2)$\mathbb{Q} \cup \mathbb{Q}^* = \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{Q}^* = \phi$ so the complement of $\mathbb{Q}$ w.r.t $\mathbb{R}$ is $\mathbb{Q}^*$.

3) let $A = \{1, 2, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 6, 8\}$.
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cap B = \{2, 3\}$, $A - B = \{1, 5, 7, 9\}$ and $B - A = \{4, 6, 8\}$.

1.2 Inequality

Definition 1.2.1. Any statement on one of the forms $a < b$ and $a \leq b$ is called inequality.

Remark 1.2.1.

- The notation $a < b$ means that a is less than b (a is strictly less than b).
- The notation $a > b$ means that a is greater than b (a is strictly greater than b).
- The notation $a \leq b$ means that a is less than or equal to b (or, equivalently, not greater than b, or at most b).
- The notation $a \geq b$ means that a is greater than or equal to b (or, equivalently, not less than b, or at least b).

Order Axioms

1. For every $a, b \in \mathbb{R}$ exactly one of $a < b$, $a = b$ or $a > b$ holds.
2. For every $a, b, c \in \mathbb{R}$ satisfying $a < b$ and $b < c$ we have $a < c$.
3. For every $a, b, c \in \mathbb{R}$ satisfying $a < b$ we have $a + c < b + c$.
4. For every $a, b, c \in \mathbb{R}$ satisfying $a < b$ we have $a - c < b - c$.
5. For every $a, b, c \in \mathbb{R}$ satisfying $a < b$ and $c > 0$ we have $ac < bc$.
6. If a and b are both positive or both negative, then if $a < b$ then $\frac{1}{b} < \frac{1}{a}$.

Note 1.2.1. Many simple inequalities can be solved by adding, subtracting, multiplying or dividing both sides until you are left with the variable on its own.

But these things will change direction of the inequality:

- Multiplying or dividing both sides by a negative number.
- Swapping left and right hand sides.

Don't multiply or divide by a **variable** (unless you know it is always positive or always negative).

1.3 Absolute Value of Real Numbers

Absolute value of real numbers a is called the number itself, if $a \geq 0$, and opposite number $-a$, if $a < 0$. The absolute value of number is denoted by $|a|$.

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Geometrically $|a|$ means distance on the coordinate line of point a from point 0.

$|a - b|$ is the distance between a and b.

Theorem 1.3.1 (Properties of absolute value)**.** *If a, and b are two real numbers, then*

1. $|a| \geq 0$
2. $|a| = |-a|$
3. $|a|^2 = a^2$
4. $|a.b| = |a||b|$.
5. $|\frac{a}{b}| = \frac{|a|}{|b|}$ *if* $b \neq 0$
6. $|a + b| \leq |a| + |b|$ *(Triangle Inequality)*
7. $|a - c| \leq |a - b| + |b - c|$
8. $|a - b| \geq |a| - |b|$
9. $||a| - |b|| \leq |a - b|$
10. $\sqrt{a^2} = |a|$ *,Do not write* $\sqrt{a^2} = a$ *unless you are sure that* $a \geq 0$*.*

1.4 Interval in Real Numbers

Definition 1.4.1. Let $a, b \in \mathbb{R}$ and $a < b$,

1. The open interval which is denoted by (a, b) and defined as:

$$(a, b) = \{x|a < x < b\}.$$

2. The close interval which is denoted by $[a, b]$ and defined as:

$$[a, b] = \{x|a \leq x \leq b\}.$$

3. The half-open interval which is denoted by $[a, b)$ or $(a, b]$ and defined as; respectively:

$$[a, b) = \{x|a \leq x < b\}$$

or

$$(a, b] = \{x|a < x \leq b\}.$$

1.4.1 Absolute value and intervals

If a is any positive number

1. $|x| = a$ if and only if $x = \pm a$
2. $|x| < a$ if and only if $-a < x < a$
3. $|x| > a$ if and only if $x < -a$ or $x > a$
4. $|x| \leq a$ if and only if $-a \leq x \leq a$
5. $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$

Example 1.4.1. solve the following inequalities:
i) $2x < 3$
ii) $|3x + 2| < 4$
iii) $|3 + x| + 1 > 2$

Solution: **i)** $2x < 3 \rightarrow \; x < \frac{3}{2}$
So the set of solution for the given inequality is the set of all real numbers in which $x < \frac{3}{2}$.
ii) $|3x + 2| < 4 \; \rightarrow -4 < 3x + 2 < 4 \; \rightarrow$
This is what is called a double inequality. We must treat it as two separate inequalities. Form the left we get $-4 < 3x + 2 \; \rightarrow \; -4 - 2 < 3x \; \rightarrow \; -6 < 3x \; \rightarrow \; \frac{-6}{3} < x \; \rightarrow \; -2 < x$
From the right we get $3x + 2 < 4 \; \rightarrow \; 3x < 2 \; \rightarrow \; x < \frac{2}{3}$
We can write these solutions together as $-2 < x < \frac{2}{3}$

Example 1.4.2. Solve the following:

1. $\frac{3}{4x+1} \leq \frac{2}{3x+1}$
2. $-2 < \frac{2}{4x+3} < 5$
3. $|3x + 1| < 2|x + 5|$; where $x \neq -5$
4. $|2x + 7| - 2 = 3$
5. $x^2 + 2x + 5 > 0$
6. $(x + 1)(3x + 4) > 0$

TABLE 1.1 Types of intervals

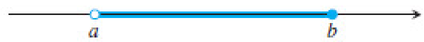
	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Figure 1.4.1: Types of intervals

Chapter 2

Functions

2.1 Functions

Basically, the concept of functions gives us a way to name the whole process of evaluating a particular expression, so we can talk about it as a whole. Informally, function is a machine or blackbox that for each input returns a corresponding output (Figure 2.1.1). You can think of a function as a cook that takes one or more ingredients and cooks them up to make a dish. depending on what you put in, you can get very different things out. Moreover, not all functions are the same. If you give one cook peanut butter, jelly, and bread, he may make a sandwich, whereas another cook may start to sculpt a volcano with the peanut butter, and use the jelly for lava after discarding the bread.

Definition 2.1.1. Let X, Y be two sets. The **Cartesian product** of two sets A and B is the set of all order pair (x, y) such that $x \in X$ and $y \in Y$, we denote it by $X \times Y$. That is,

$$X \times Y = \{(x, y) | x \in X,\, y \in Y\}.$$

If X has n elements and Y has m elements then $X \times Y$ has $n.m$ elements.

Definition 2.1.2. A relation between two sets X and Y is a subset of $X \times Y$. Symbolically, R is a relation from X to Y if and only if $R \subseteq X \times Y$.

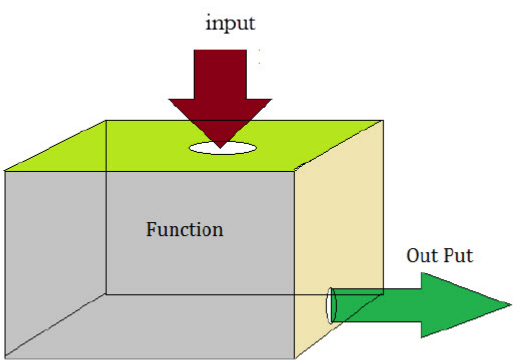


Figure 2.1.1: Function

Definition 2.1.3. Let X and Y be two non-empty sets. A **function** f as a subset of $X \times Y$ is an equation for which any $x \in X$ that can be submitted into the equation will give **exactly one** $y \in Y$ out of the equation. x and y are called independent and dependent variable respectively. In other words, The input to the function is called the independent variable and the output of the function is called the dependent variable.

Definition 2.1.4. Let f be a function from X into Y then X is called the **domain** of f (D_f) and the set $\{y \in Y | y = f(x) \text{ for some } x \in X\}$ is called the **range** of f (R_f). Y is called the **co-domain** of f.

Definition 2.1.5. Another way to visualize a function is its graph. If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is $\{(x, f(x)|x \in D)\}$.

Steps of sketching the graph of a function

1. Make a table of xy-pairs that satisfy the function rule.
2. Plot the points (x, y) whose coordinates appear in the table. Use fractions when they are convenient computationally.
3. Draw a smooth curve through the plotted points. Label the curve with its equation.

The vertical line test:

Geometrically, if a graph of a function is given in the Cartesian plane $(xy-plane)$, each vertical line has exactly one crossing point with the curve (Figure 2.1.2). From the definition above we can say, a relation from X to Y is a function iff

1. For each $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$,
2. $\forall x_1, x_2 \in X$ if $x_1 = x_2$, then we must have $f(x_1) = f(x_2)$.

Remark 2.1.1. If we say a rule $f : X \to Y$ from a set X to a set Y, is **well defined** we mean it is a function.

Example 2.1.1. which of the following relations represent a function?

2. $\{(1, 2), (2, 3), (3, 2), (3, 3), (1, 5)\}$

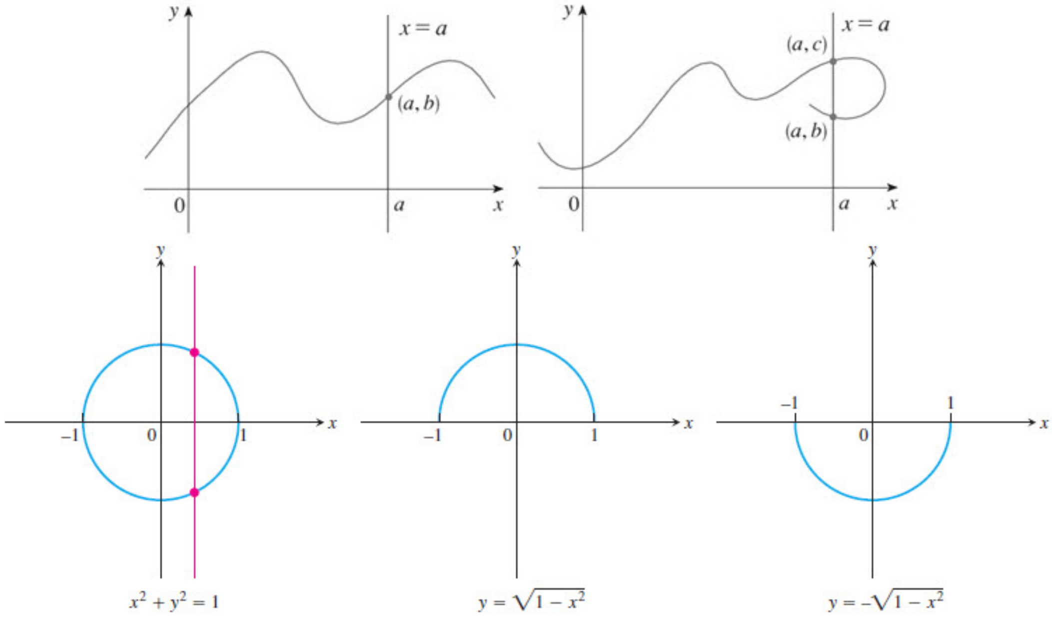


Figure 2.1.2: graph of a function in the xy-plane

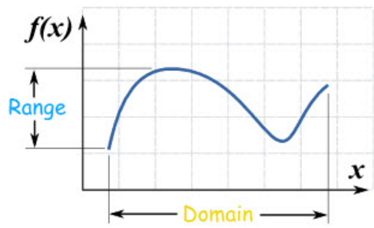


Figure 2.1.3: Domain and range

5. $f(x) = 2x - 4$
6. $y^2 = 4 - x^2$

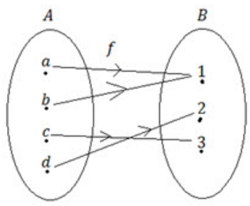
Definition 2.1.6 (Algebra of functions)**.** Let f and g be two functions then

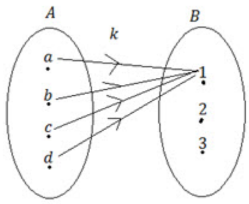
1. The sum of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

2. The difference of f and g is defined by

$$(f - g)(x) = f(x) - g(x)$$

1. 

3. 

3. The product of f and g is defined by

$$(f.g)(x) = f(x).g(x)$$

4. The quotient of f and g is defined by

$$\tfrac{f}{g}(x) = \tfrac{f(x)}{g(x)}$$

5. The scalar multiplication of a constant c and function f is defined by

$$(cf)(x) = c.f(x)$$

Remark 2.1.2. Let f and g be two functions then

$$D_{f\pm g} = D_f \cap D_g$$
$$D_{f.g} = D_f \cap D_g$$
$$D_{\frac{f}{g}} = \{D_f \cap D_g\} - \{x|\ g(x) = 0\}$$

Definition 2.1.7 (Composition of functions)**.** If $f : X \to Y$ and $g : Y \to Z$ are two functions such that $R_f \subseteq D_g$,the composite function $g \circ f : X \to Z$ (g composed with f) is defined by

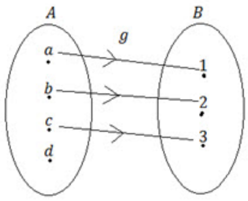
$$(g \circ f)(x) = g(f(x))$$

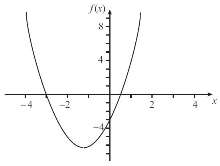
The domain of $g \circ f$ consists of the numbers x in the domain of f for which $f(x)$ lies in the domain of g,which is

$$D_{g\circ f} = \{x|x \in D_f \text{ and } f(x) \in D_g\}$$

Example 2.1.2. If $f(x) = x + 1$ and $g(x) = \sqrt{x}$, find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$.

Definition 2.1.8 (Increasing and Decreasing Functions)**.** If the graph of a function climbs or rises as you move from left to right, we say that the function is **increasing**. If the graph descends or falls as you move from left to right, the function is **decreasing**. We give formal definitions of increasing functions and decreasing functions in the coming chapters.

4. 

7. 

Example 2.1.3. Sketch the graph of the following functions.

1. $f(x) = x + 3$
2. $f(x) = x^2$
3. $f(x) = x^3$
4. $f(x) = \frac{1}{x}$

Example 2.1.4. Find domain and range of the following functions.

1. $f(x) = x^2 + 3x + 5$
2. $f(x) = \frac{x^2-5}{x^2-1}$
3. $f(x) = 4^x$
4. $f(x) = \log(x + 2)$
5. $f(x) = \sin x$
6. $f(x) = \tan(x + 3)$
7. $f(x) = \sqrt[3]{x^5}$
8. $f(x) = \sin(x) + cos(x)$

Definition 2.1.9 (Piecewise defined functions)**.**

Sometimes an equation cant be described by a single equation, and instead we have to describe it using a combination of equations. Such functions are called **piecewise defined functions**, and probably the easiest way to describe them is to look at a couple of examples.

Example 2.1.5. the function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} x^2 - 1 & \text{if } \ x \in (-\infty, 0] \\ x - 1 & \text{if } \ x \in (0, 4] \\ 3 & \text{if } x \in [4, \infty) \end{cases}$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The function g is a piecewise defined function. It is defined using three functions that we are more comfortable with: x^2-1, $x-1$ and the constant function 3. Each of these three functions is paired with an interval that appears on the right side of the same line as the function. The function f is called the absolute value function which is the most important piecewise defined function in calculus.

Example 2.1.6. Sketch the graph of f and g.

Definition 2.1.10 (One-to-one functions)**.** A one-to-one (injective) function f from set X to set Y is a function such that each x in X is related to a different y in Y. More formally, we can restate this definition as either:

$$f : X \to Y \text{ is } 1-1 \text{ provided } f(x_1) = f(x_2) \text{ implies } x_1 = x_2$$
$$\text{or}$$
$$f : X \to Y \text{ is } 1-1 \text{ provided } x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2)$$

Definition 2.1.11 (Onto functions)**.** A function $f : X \to Y$ is said to be onto (surjective) if for every y in Y, there is an x in X such that $f(x) = y$. This can be restated as: A function is onto when its image equals its range, i.e. $f(X) = Y$.

Definition 2.1.12 (Even and Odd Functions)**.** A function is

$$\text{an } \textbf{even function} \text{ of } x \text{ if } f(-x) = f(x)$$
$$\text{an } \textbf{odd function} \text{ of } x \text{ if } f(-x) = -f(x)$$

for every x in the function's domain.

Q: How can someone distinguish the graph of even and odd function?

The graphs of even and odd functions have characteristic symmetry properties.

The graph of an even function is **symmetric about the y-axis**. Since a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph . A reflection across the y-axis leaves the graph unchanged.

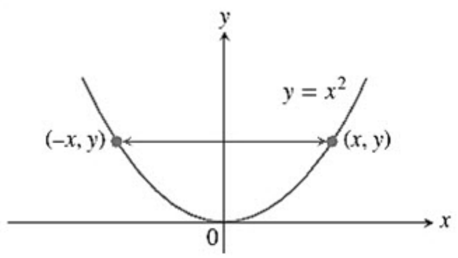


Figure 2.1.4: Even Function

The graph of an odd function is **symmetric about the origin**. Since a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph. Equivalently, a graph is symmetric about the origin if a rotation of 180 about the origin leaves the graph unchanged.

Notice that the definitions imply both x and $-x$ must be in the domain of f.

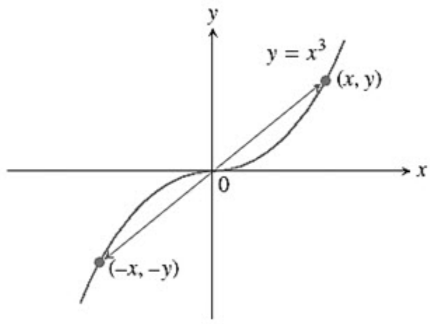


Figure 2.1.5: Odd Function

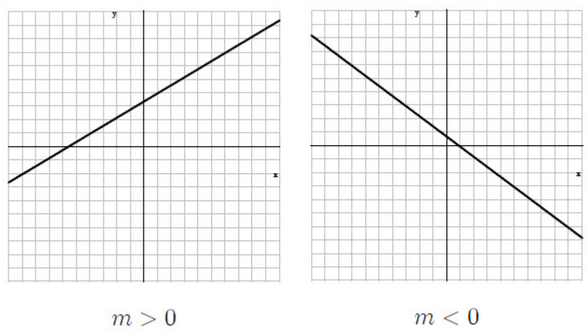


Figure 2.2.1: Linear function

Example 2.1.7. Recognizing even and odd functions

1. $f(x) = x^2 + 1$
2. $f(x) = x^2 + x$

2.2 Basic functions and their properties

1. **Linear functions** $f(x) = mx + b$ where $m \neq 0$ (Figure 2.2.1)

   case 1) if $m > 0$

   Domain: $\mathbb{R}$ & Range: $\mathbb{R}$

   One-to-one & No maximum or minimum & Strictly increasing

   Continuous on $\mathbb{R}$

   case 2) if $m < 0$

   Domain: $\mathbb{R}$ & Range: $\mathbb{R}$

   One-to-one & No maximum or minimum & Strictly decreasing

   Continuous on $\mathbb{R}$

2. **Quadratic functions** $f(x) = ax^2 + bx + c$ where $a \neq 0$

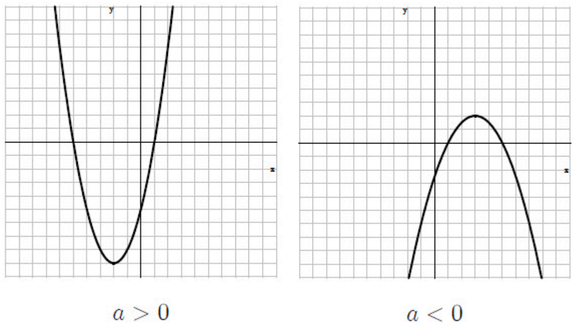


Figure 2.2.2: Quadratic function

The graph is a parabola. It opens upward if $a > 0$ and opens downward if $a < 0$ (Figure 2.2.2).

Case 1) If $a > 0$

Domain: $\mathbb{R}$

Not one-to-one & No maximum & Continuous on $\mathbb{R}$.

Case 2) If $a < 0$

Domain: $\mathbb{R}$

Not one-to-one & No minimum & Continuous on $\mathbb{R}$.

3. **Monomials** $f(x) = x^n$ (Figure 2.2.3)

   Case 1) If n is even

   Domain: $\mathbb{R}$ & Range: $(0, \infty)$

   Not one-to-one & No Maximum

   Strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$

   Continuous on $\mathbb{R}$

   Case 2) If n is odd

   Domain: $\mathbb{R}$ & Range: $\mathbb{R}$

   one-to-one & No Maximum or minimum

   Strictly increasing on $\mathbb{R}$

   Continuous on $\mathbb{R}$

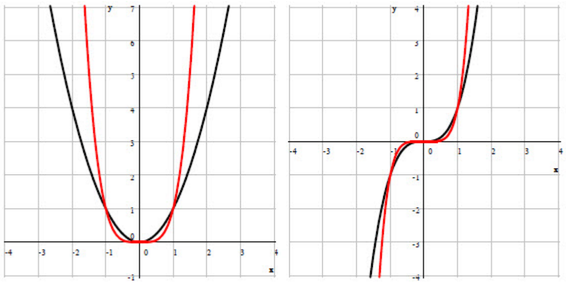
4. **Rational functions** $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ (Figure 2.2.4)

   Case 1) $f(x) = \frac{1}{x}$

   Domain: $\mathbb{R}\backslash\{0\}$ & Range: $\mathbb{R}\backslash\{0\}$

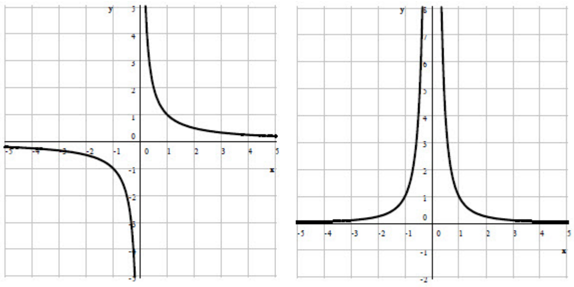
   one-to-one & no maximum or minimum

   Strictly decreasing on $(-\infty, 0)$ and on $(0, \infty)$



n is even $\qquad$ n is odd

Figure 2.2.3: Monomials



$f(x) = \dfrac{1}{x}$ $\qquad$ $g(x) = \dfrac{1}{x^2}$

Figure 2.2.4: Rational function

Not continuous at $x = 0$

Case 1) $f(x) = \frac{1}{x^2}$

Domain: $\mathbb{R}\backslash\{0\}$ & Range: $(0, \infty)$

Not one-to-one & no maximum or minimum

Strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$

Not continuous at $x = 0$

5. **Radical functions** $f(x) = \sqrt[n]{x}$ (Figure 2.2.5)

   Case 1) if n is even

   Domain: $[0, \infty)$ & Range: $[0, \infty)$

   one-to-one & no maximum

   minimum at $(0, 0)$

   Strictly increasing

   Continuous on $(0, \infty)$

   Case 1) if n is odd

   Domain: $\mathbb{R}$ & Range: $\mathbb{R}$

   one-to-one & no maximum or minimum

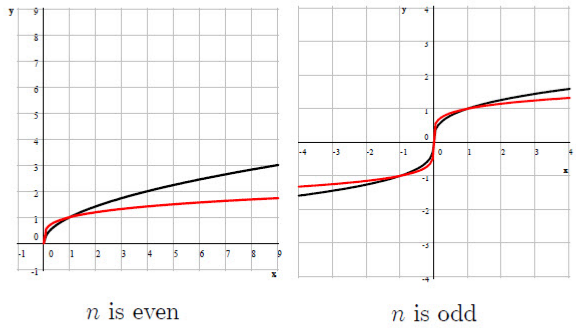


Figure 2.2.5: Radical function

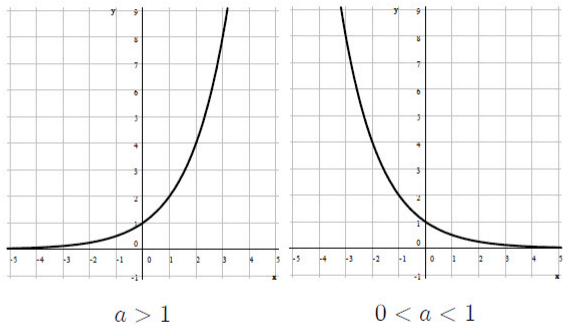


Figure 2.2.6: Exponential functions

Strictly increasing

Continuous on $\mathbb{R}$

6. **Exponential functions** $f(x) = a^x$ where $a > 0$ (Figure 2.2.6)

   Case 1) if $a > 1$

   Domain: $\mathbb{R}$ & Range: $(0, \infty)$

   No maximum or minimum

   Strictly increasing

   Continuous on $\mathbb{R}$

   Case 1) if $0 < a < 1$

   Domain: $\mathbb{R}$ & Range: $(0, \infty)$

   No maximum or minimum

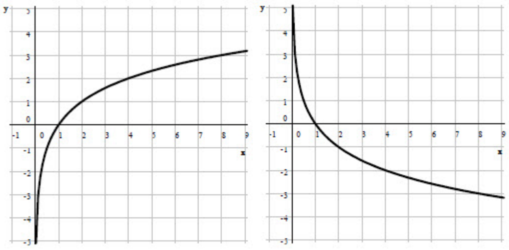
   Strictly decreasing

   Continuous on $\mathbb{R}$

7. **Logarithmic functions** $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$ (Figure 2.2.7)

   Case 1) If $a > 0$

   Domain: $(0, \infty)$ & Range: $\mathbb{R}$



$a > 1$ $\qquad$ $0 < a < 1$

Figure 2.2.7: Logarithmic functions

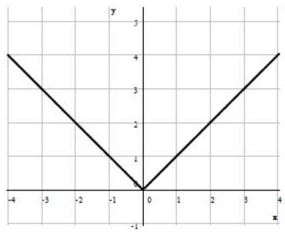


Figure 2.2.8: Absolute value functions

one-to-one

No maximum or minimum

Strictly increasing

Continuous on $(0, \infty)$

Case 2) If $0 < a < 1$

Domain: $(0, \infty)$ & Range: $\mathbb{R}$

one-to-one

No maximum or minimum

Strictly decreasing

Continuous on $(0, \infty)$

8. **Absolute value functions** $f(x) = |x|$ (Figure 2.2.8)

   Domain: $\mathbb{R}$ & Range: $[0, \infty)$

   Not one-to-one

   Minimum at $(0, 0)$

   strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$

   Continuous on $\mathbb{R}$

9. **The Greatest Integer Function** For all real numbers x, the greatest integer function returns the largest integer less than or equal to x. In other words, the greatest integer

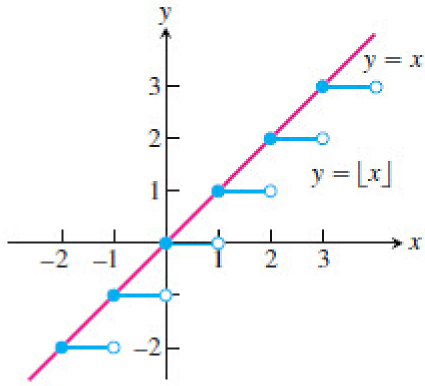


Figure 2.2.9: Greatest integer function

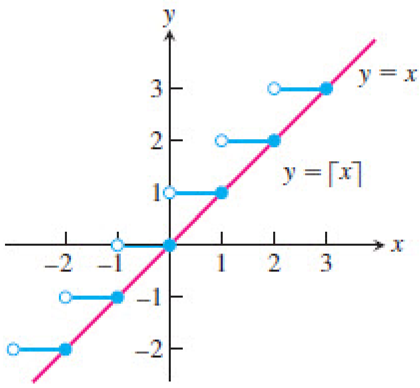


Figure 2.2.10: Least integer function

function rounds down a real number to the nearest integer and it is denoted by $\lfloor x \rfloor$ (Figure 2.2.9). For example $\lfloor 4.3 \rfloor = 4$,$\lfloor 5.7 \rfloor = 5$,$\lfloor -8.3 \rfloor = -9$.

10. **The Least Integer Function** For all real numbers x, the least integer function returns the smallest integer greater than or equal to x. In other words, the least integer function rounds up a real number to the nearest integer and it is denoted by $\lceil x \rceil$ (Figure 2.2.10). For example $\lceil 4.3 \rceil = 5$,$\lceil 5.7 \rceil = 6$,$\lceil -8.3 \rceil = -8$.

11. **Trigonometric Functions** In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations. Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by

$$\pi \quad \text{radians} = 180°$$

For example, $45°$ in radian measure is

$$45.\frac{\pi}{180} = \frac{\pi}{4} \quad \text{radian}$$

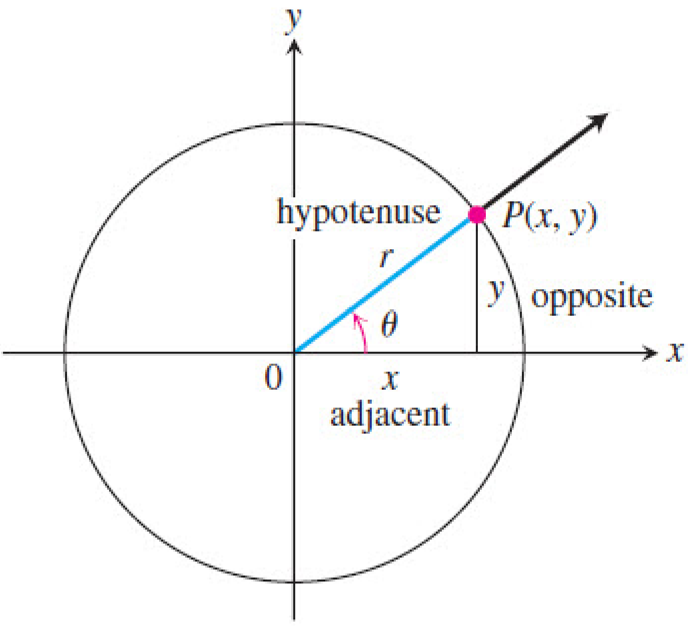


Figure 2.2.11: The new and old definitions agree for acute angles.

and $\frac{\pi}{6}$ radians is

$$\frac{\pi}{6}.\frac{180}{\pi} = 30°$$

Six basic trigonometric functions:

Sine: $\sin\theta = \frac{y}{r}$ **Cosine:** $\cos\theta = \frac{x}{r}$

Tangent: $\tan\theta = \frac{x}{y}$ **Cotangent:** $\cot\theta = \frac{y}{x}$

Secant: $\sec\theta = \frac{r}{x}$ **Cosecant:** $\csc\theta = \frac{r}{y}$

2.3 Shifting, Scaling and Reflecting a Graph of a Function

Shifting Formulas:

(Vertical Shifts)

$y = f(x) + k$ Shifts the graph of f **up** k units if $k > 0$ and shifts the graph of f **down** k units if $k < 0$.

(Horizontal Shifts)

$y = f(x + h)$ Shifts the graph of f **left** h units if $h > 0$ and shifts the graph of f **right** $|h|$ units if $h < 0$.

Degrees	**0**	**30**	**45**	**60**	**90**	**180**	**270**	**360**
θ (radians)	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		0		0

Figure 2.2.12: Value of $\sin\theta$,$\cos\theta$ and $\tan\theta$ for selected value of θ

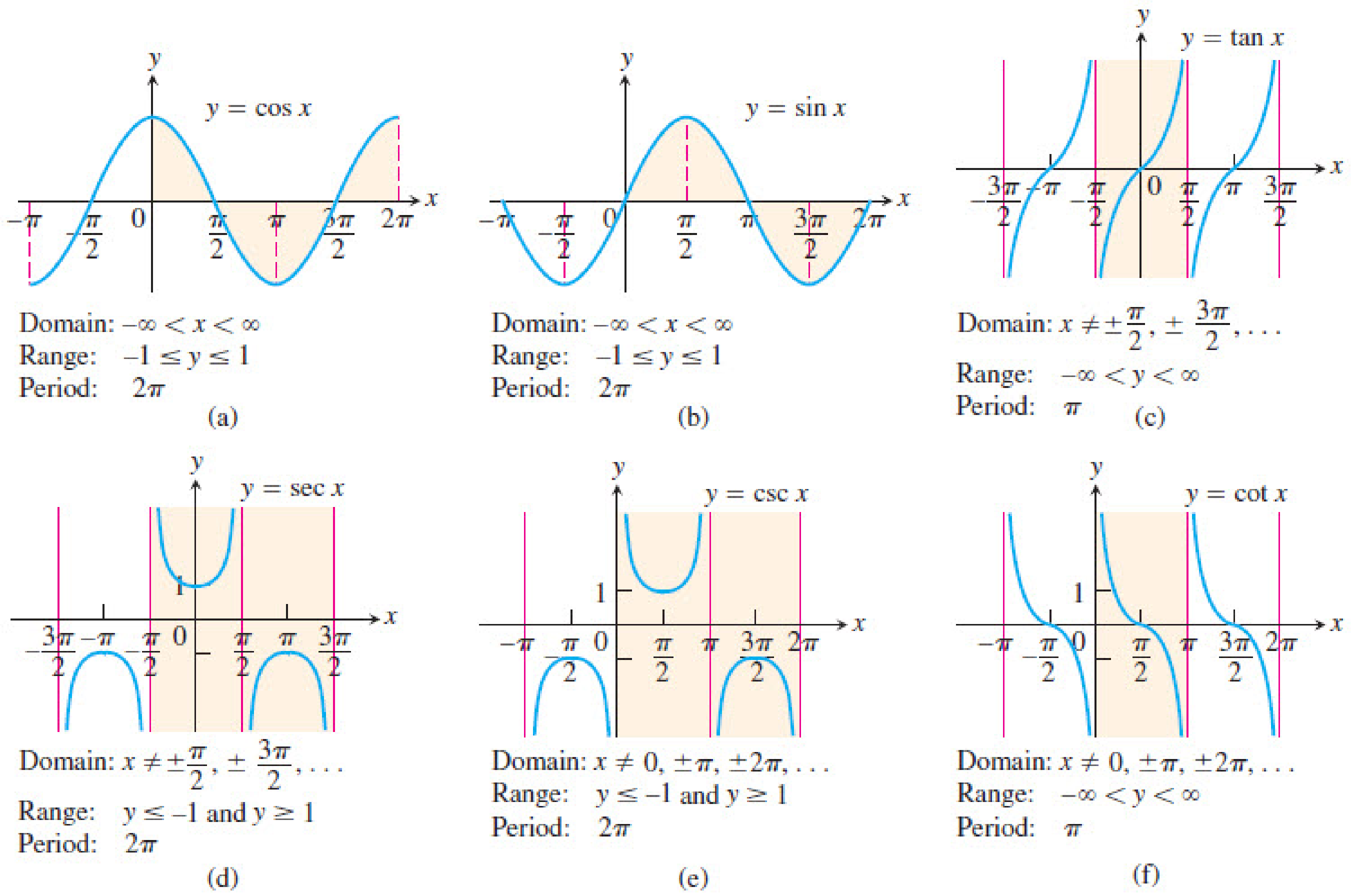


Figure 2.2.13: Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure.

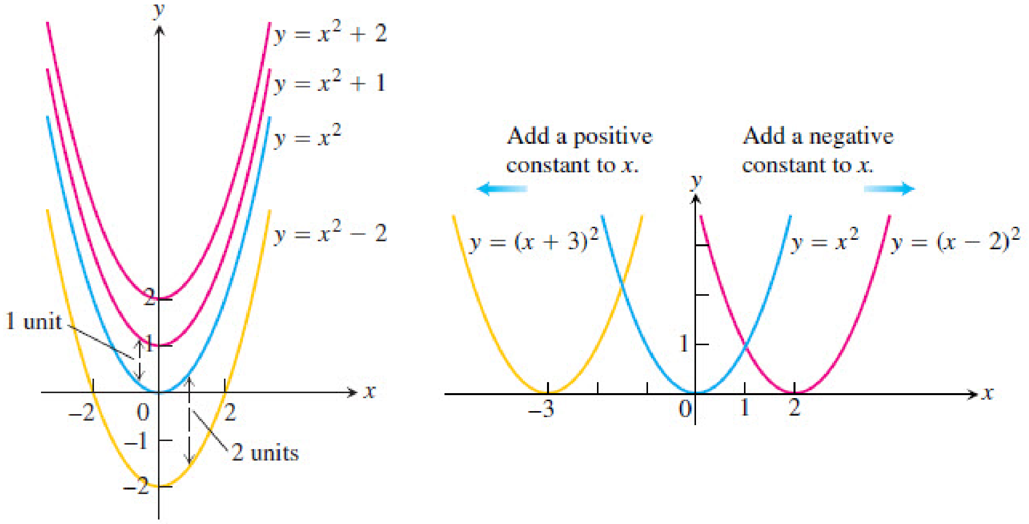


Figure 2.3.1: The function $y = x^2$ and two simple shiftings of its graph

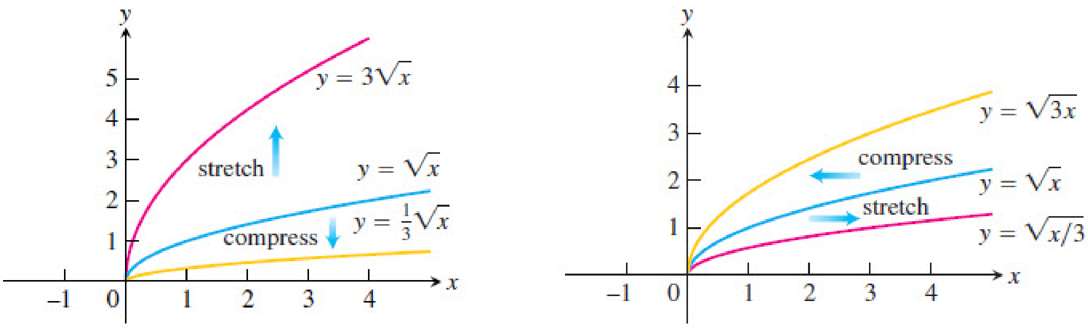


Figure 2.3.2: The function $y = \sqrt{x}$ and scaling of its graph

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function , or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Scaling and Reflecting formulas:

For $c > 1$,

- $y = cf(x)$ Stretches the graph of f vertically by a factor of c.
- $y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c.
- $y = f(cx)$ Compresses the graph of f horizontally by a factor of c.
- $y = f(\frac{1}{c}x)$ Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

- $y = cf(x)$ Reflects the graph of f across the x-axis.

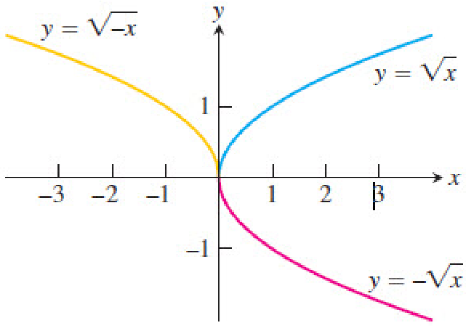


Figure 2.3.3: The function $y = \sqrt{x}$ and two reflections of its graph

- $y = f(cx)$ Reflects the graph of f across the y-axis.

Example 2.3.1. Sketch the graph of the following functions.

1. $f(x) = \frac{1}{x} + 3$
2. $f(x) = (x-2)^2$
3. $f(x) = \sin(3x) - 1$
4. $f(x) = -|2x+4|$
5. $y - 3 = -2\cos(2x)$
6. $y = \log(3-x)$
7. $y = e^{x+2} - 3$

Problem sheet 1

Q1)

Solve the following inequalities.

1. $x^2 + 3x + 4 > 0$
2. $\frac{3x+1}{x-1} < 2$
3. $(x-3)(x-2)(x-1) \leq 0$
4. $|2x+3| - |x+4| < 2$

5. $|2x+3| < 7$

6. $|x-3| < 2|x|$

7. $\frac{6-x}{4} \geq \frac{3x-4}{2}$

8. $-2 < \frac{3}{x+1} < 6$

Q2)

Find domain and range of the following functions.

1. $f(x) = x^4 - 2x^2 + 4$

2. $f(x) = -x^2 + 4$

3. $f(x) = \frac{x^2+1}{x^2-1}$

4. $f(x) = \sqrt{x^2 - 4x + 5}$

5. $f(x) = 3^{3x}$

Q3)

Recognize even and odd functions.

1. $f(x) = \frac{3x^2}{x^4+1}$

2. $f(x) = 10x + 5$

3. $f(x) = x\sqrt{x^2 - 1}$

4. $f(x) = 3x^4 + x - 5$

Q4)

Sketch the graph of the following functions.

1. $y - 4 = e^x$

2. $y = \log(-x)$

3. $y = \frac{1}{3x} - 2$

4. $y = \frac{2x}{3} + 1$

5. $y = 2|3x - 3|$

Chapter 3

Differentiation

Differentiation is the action of computing a derivative. The derivative of a function $f(x)$ of a variable x is a measure of the rate at which the value of the function changes with respect to the change of the variable. It is called the derivative of f with respect to x. If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of this graph at each point.

Definition 3.0.1. The derivative of a function f is another name for the function $f^{'}$ whose value is given by

$$f^{'} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

pivided this limit exists.

If the limit does exist, we say that f is differentiable.

The derivative is also denoted by dy/dx and other common notations are $y^{'}$,df/dx, and $D_x(y)$.

The process of finding the derivative is called differentiation. A function f is differentiable at x when the defining limit exists; and we say that f is differentiable on (an interval) (a, b) when f is differentiable at every point in (a, b).

Theorem 3.0.1. *If $f(x)$ is differentiable at a point x_0 , then its continuous at x_0.*

Differentiation Formulas

Let f be a function.

1. If f is a constant function, $f(x) = c$ for any real number c, then $f^{'} = 0$.
2. If f is a power function, $f(x) = x^n$ for any real number n, then $f^{'} = nx^{(n-1)}$.
3. If $f = g + h$ for any two functions g and h, then $f^{'} = g^{'} + h^{'}$.
4. If $f = g - h$ for any two functions g and h, then $f^{'} = g^{'} - h^{'}$.

5. If $f = ag + bh$ for any two functions g and h, and any two constants a and b, then $f^{'} = ag^{'} + bh^{'}$.
6. If $f = gh$ for any two functions g and h, then $f^{'} = gh^{'} + hg^{'}$.
7. If $f = \frac{g}{h}$ for any two functions g and h, then $f^{'} = \frac{g^{'}h - gh^{'}}{h^2}$

Example 3.0.2. Find the derivative of the following functions.

1. $f(x) = 8x^2 + \pi x^3 + \sqrt[3]{2x^2}$
2. $f(x) = (x^2 + 1)(3x^3 + 2)$
3. $f(x) = \frac{(x+1)(x^2-3)}{(x+3)x^2}$

3.0.1 Derivatives of Trigonometric Functions

The trigonometric functions sine, cosine, tangent, cotangent, cosecant, and secant are all differentiable functions on their domain and their derivative functions are:

1. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$
2. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$
3. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$
4. $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$
5. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$
6. $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

Where u is a function of x.

3.0.2 Derivatives of Inverse Trigonometric Functions (non-examinable)

The inverse trigonometric functions arcsine, *arccosine, arctangent, arccotangent, arccosecant* and *arcsecant* are all differentiable functions on their domain and their derivative functions are:

1. $\frac{d}{dx}(\sin^{-1}\frac{u}{a}) = \frac{\frac{du}{dx}}{\sqrt{a^2-u^2}}$
2. $\frac{d}{dx}(\cos^{-1}\frac{u}{a}) = \frac{-\frac{du}{dx}}{\sqrt{a^2-u^2}}$
3. $\frac{d}{dx}(\tan^{-1}\frac{u}{a}) = \frac{\frac{du}{dx}}{a^2+u^2}$
4. $\frac{d}{dx}(\cot^{-1}\frac{u}{a}) = \frac{-\frac{du}{dx}}{a^2+u^2}$
5. $\frac{d}{dx}(\sec^{-1}\frac{u}{a}) = \frac{1}{|u|}\frac{\frac{du}{dx}}{\sqrt{u^2-a^2}}$
6. $\frac{d}{dx}(\csc^{-1}\frac{u}{a}) = -\frac{1}{|u|}\frac{\frac{du}{dx}}{\sqrt{u^2-a^2}}$

Where u is a function of x and a is a constant.

3.0.3 Derivatives of Exponential Functions

1. $\frac{d}{dx}(a^u) = a^u \ln(a)\frac{du}{dx}$
2. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$
3. $\frac{d}{dx}(a^x) = a^x \ln(a)$
4. $\frac{d}{dx}(e^x) = e^x$

3.0.4 Derivatives of Logarithmic Functions

1. $\frac{d}{dx}(\log_a u) = \frac{\frac{du}{dx}}{u \ln a} = \frac{u^{'}}{u \ln a}$
2. $\frac{d}{dx}(\ln u) = \frac{\frac{du}{dx}}{u} = \frac{u^{'}}{u}$
3. $\frac{d}{dx}(log_a x) = \frac{1}{x \ln a}$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

3.0.5 Higher Order Derivatives

If f is a differentiable function, then its derivative $f^{'}$ is also a function, so $f^{'}$ may have a derivative of its own, denoted by $(f^{'})^{'} = f^{''}$.This function $f^{''}$ is called the second derivative of f Moreover, the second derivative may be differentiable, and etc.

Suppose f and $f^{'}$ are differentiable functions, then the second derivative of f is defined as$(f^{'})^{'}$ and is denoted by $f^{''}$. Further, the third derivative is defined as $(f^{''})^{'}$ and is denoted by $f^{'''}$; and the fourth derivative is defined as $(f^{'''})^{'}$ and is denoted by $f^{(4)}$, provided these functions exist. In general, if $f^{(n)}$ is differentiable, then $(f^{(n)})^{'} = f^{(n+1)}$ is the $(n+1)^{th}$derivative of f.

In Leibniz notation the first, second the third derivatives are:

$$\frac{dy}{dx}\ , \ \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2} \ \text{ and } \ \frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3} \quad .....$$

Example 3.0.3. Find the first and second derivatives of the followings.

1. $f(x) = 3x^3 + 2x^2 + x$
2. $f(x) = \frac{x^2+3x}{x+1}$
3. $f(x) = x(\cos(x^2 + 1))$
4. $f(x) = (\ln x)^2 + \log_2(\sin x + e^x)$

3.0.6 Chain Rule

With a lot of work, we can find derivatives without using the chain rule either by expanding a polynomial, by using another differentiation rule, or maybe by using a trigonometric identity. The derivative would be the same in either approach; however, the chain rule allows us to find derivatives that would otherwise be very difficult to handle. This topic gives plenty of examples of the chain rule as well an easily understandable proof of the chain rule.

If the derivatives $g^{'}(x)$ and $f^{'}(g(x))$ both exist, and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F^{'}(x)$ exists and is given by the product.

$$F^{'}(x) = f^{'}(g(x))g^{'}(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Example 3.0.4. find $f'(x)$ when $f(x) = (\frac{x^2+2}{1-x})^7$

Example 3.0.5. If $f(x) = x^3 + 1$ and $g(x) = \frac{1}{x}$ then find $(f \circ g)^{'}(x)$.

3.0.7 Implicit Differentiation

Suppose we have the following:

$$2y + 3x = 12$$

we can rewrite it as

$$y = 6 - \frac{3}{2}x$$

Now we have $y = f(x)$ and we can take the derivative

$$\frac{dy}{dx} = -\frac{3}{2}$$

Lets consider an alternative. We know that y is a function of x or, $y = y(x)$ and the derivative of y is $\frac{dy}{dx}$. If we return to our original equation, $2y+3x = 12$, we can differentiate it IMPLICITLY in the following manner

$$2\frac{dy}{dx} + 3\frac{dx}{dx} = \frac{d\ (12)}{dx}$$

where

$$\frac{dx}{dx} = 1 \quad , \quad \frac{d\ (12)}{dx} = 0$$

so

$$2\frac{dy}{dx} + 3 = 0 \rightarrow \frac{dy}{dx} = -\frac{3}{2}$$

which is what we got before!

Here there are more examples.

Example 3.0.6. Find $\frac{dy}{dx}$

1. $x^2y + 4xy^3 = 4$
2. $y^{\frac{1}{2}} - 2x^2 + 5y = 15$
3. $\sin(x + y) = y^2 \cos x$

3.1 Applications of Derivative

In the previous section we focused almost exclusively on the computation of derivatives. In this section will focus on applications of derivatives. It is important to always remember that we didnt spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we will be looking at in this section are using derivatives to determine information about graphs of functions and optimization problems.

Definition 3.1.1 (Local Maxima and Minima)**.** $f(x_0)$ is a local maximum (or local minimum) of $f(x)$ if it is the largest (or smallest) around the small interval of x_0.

In other words, We say that f has a local maximum at x_0, if for all x in some neighbourhood of x_0 we have $f(x_0) \geq f(x)$.

f has a local minimum at x_0, if for all x in some neighbourhood of x_0 we have $f(x_0) \leq f(x)$.

We say that f has a local extremum at x_0 if it either has a local minimum or a local maximum at x_0.

Definition 3.1.2 (Absolute Maximum , Absolute Minimum)**.** Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \in \ D$$

and an absolute minimum value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \in \ D$$

Definition 3.1.3 (Increasing and Decreasing Function). A function f is called increasing on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ we have that

$$x_1 \leq x_2 \rightarrow f(x_1) \leq f(x_2)$$

A function f is called decreasing on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ we have that

$$x_1 \leq x_2 \rightarrow f(x_1) \geq f(x_2)$$

Definition 3.1.4 (Critical Point). If a point c is in the domain of f and either $f'(c) = 0$ or $f'(c)$ doesn't exist, then we call $x = c$ a **critical point** or **critical value** of f.

Theorem 3.1.1 (The First Derivative Test). *Let f be continuous on the interval (a, b) and suppose that c is the only critical point of (a, b). Suppose f is differentiable on (a, b) except possibly at c. Then:*

1. *If $f'(x)$ is negative to the left of c and positive to the right of c, then f has a local min at c.*

2. *If $f'(x)$ is positive to the left of c and negative to the right of c, then f has a local max at c.*

3. *If $f'(x)$ is the same sign to the right and left of c then c is neither a max nor a min.*

   *In cases like 3 above we call the point c a **saddle point** if $f'(c) = 0$.*

Example 3.1.1. Find the critical points of the following functions and identify their nature.

1. $f(x) = \frac{x^3}{x-5}$

2. $f(x) = \sqrt{2t - t^2}$

3. $f(x) = x\sqrt{4 - x^2}$

4. $f(x) = \begin{cases} -x^2 - 2x + 4 & if \quad x \geq 1 \\ -x^2 + 6x - 4 & if \quad x < 1 \end{cases}$

5. $f(x) = \ln(3x^2 - 6x)$

6. $f(x) = e^{\sqrt{x^2-3}}$

Theorem 3.1.2 (Second Derivative Test For Local Extreme). *Suppose $f^{"}$ is continuous on an open interval that contains $x = c$.*

1. *If $f'(c) = 0$ and $f^{"}(c) < 0$ then f has local maximum at $x = c$.*

2.

3. *If $f'(c) = 0$ and $f^{"}(c) > 0$ then f has local minimum at $x = c$.*

4.

5. If $f'(c) = 0$ and $f''(c) = 0$ then the test fails. The function f may have local maximum, a local minimum, or neither.

Theorem 3.1.3 (The Second Derivative Test For Concavity)**.** *Let $y = f(x)$ be twice-differentiable on an interval I.*

1. if $f'' > 0$ on I, the graph of f over I is concave up.

2. if $f'' < 0$ on I, the graph of f over I is concave down.

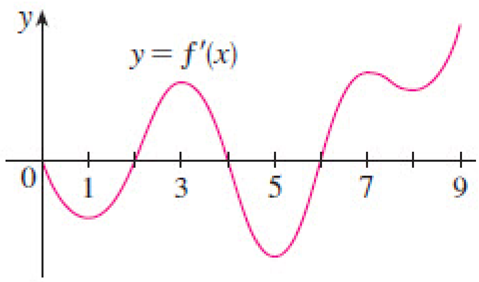
Definition 3.1.5 (point of inflection)**.** A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example 3.1.2. Identify the inflection points, local maxima and minima and Absolute maximum and absolute minimum of the following functions and identify the intervals on which the functions are concave up and concave down.

1. $f(x) = 2x^3 + 9x^2 - 24x - 10$
2. $f(x) = \frac{3}{4}(x^2 - 1)^{\frac{2}{3}}$
3. $f(x) = \sin|x| \quad -2\pi \leq x \leq 2\pi$
4. $f(x) = \frac{1-x^2}{x^3+5}$

Example 3.1.3. Below is the graph of f'.

1. On what interval f is increasing?
2. At what x-values does f have a local maximum or local minimum?
3. On what intervals f is concave up or concave down?
4. At what x-values does f have an inflection point?



Problem Sheet 2

Q1: Find y' in the following functions.

1. $y = f(u) = u^{-6} \quad u(x) = 3x^2 + 5x - 1$
2. $y = f(u) = (u^2 + u)^4 \quad u(x) = (\frac{x}{5} + \frac{1}{5x})$
3. $y = x\tan(2\sqrt{x}) + 7sin(\frac{1}{x})$
4. $y = \sin\sqrt{x}\cos^3\sqrt[3]{x}$
5. $y = x^{\sin x} + (\sin x)^x$
6. $\sqrt{y}x^3 + \log(xy) + 3x^2\sin y = 0$

Q2: a)Find all critical points and reflection points of the following functions and then Identify their nature.

b)Find the interval that the given functions are concave up or concave down.

1. $f(x) = x(x^3 - 4)^4$
2. $f(x) = |x^2 - 1|$
3. $f(x) = \frac{x}{1+x\tan x}$
4. $f(x) = x^2\sqrt{x^2 - \sin x^2}$
5. $f(x) = \cos|x|$