Advances in Continuum Mechanics
48

Franz Chouly
Patrick Hild
Yves Renard

Finite Element Approximation of Contact and Friction in Elasticity

Birkhäuser

Birkhäuser

Advances in Mechanics and Mathematics

Advances in Continuum Mechanics

Volume 48

This series includes collective thematic books and monographs on advanced research topics in continuum mechanics. It is a unique forum for reference reviews discussing critically foundational issues in classical field theories under the light of up-to-date problems raised by the offspring of condensed matter physics and industrial needs, including those concerning the mechanics of microstructures and the relevant multi-field (or phase-field) representations. The series aims also to show new mathematical methods in continuum mechanics and/or recent developments of existing powerful practice. Also, the series pays attention to applications with concrete conceptual content. The series editors select editors of books on specific themes, which may split into sequential volumes.

Franz Chouly • Patrick Hild • Yves Renard

Finite Element Approximation of Contact and Friction in Elasticity

Birkhäuser

Franz Chouly
University of Burgundy
Dijon, France

Patrick Hild
Paul Sabatier University
Toulouse, France

Yves Renard
National Institute of Applied Sciences
of Lyon
Lyon, France

ISSN 1571-8689 ISSN 1876-9896 (electronic)
Advances in Mechanics and Mathematics
ISSN 2524-4639 ISSN 2524-4647 (electronic)
Advances in Continuum Mechanics
ISBN 978-3-031-31422-3 ISBN 978-3-031-31423-0 (eBook)
https://doi.org/10.1007/978-3-031-31423-0

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The first mathematical analysis of some finite element methods for contact problems were published in the late 1970s, a few years after the mathematical theory of the Finite Element Method (FEM) has emerged. Many achievements come from this period, but many difficulties have been identified from then, either to prove important mathematical properties of some existing approximation methods, or to design new ones, attractive for implementation or numerical solution of complex problems, such as multi-body contact or large-strain contact.

As a result, it has always remained an active field, particularly in the last two decades, where some issues have been better understood, and new ideas from the applied mathematics and computational mechanics community have been applied successfully to contact and friction.

This book was intended to summarize some of the last advances in numerical methods for contact and friction and in their analysis, in terms of existence and uniqueness of discrete solutions, and convergence properties of the approximation. A large part of this book is focused on the Signorini problem, of frictionless unilateral contact in small strain. Indeed, this setting has an elegant mathematical structure, as a variational inequality of the first kind, and its properties are well understood since the pioneering works of G. Fichera, J.-L. Lions, and G. Stampacchia in the 1960s. Therefore, its structure makes it amenable for numerical analysis and allows to underline clearly where specific problems arise, especially in the convergence analysis. By the way, these problems have been solved only very recently. The last chapters of this book are dedicated to more complex and applications-oriented problems, such as frictional contact, multi-body contact, and large-strain (self-)contact. For all of these, the present mathematical theory remains largely open, but, at least, we tried to summarize the existing issues and solutions.

We hope this presentation will be helpful for researchers in the field, or will be incentive for researchers in numerical analysis, scientific computing or computational mechanics to get interested in this topic.

Dijon, France	Franz Chouly
Toulouse, France	Patrick Hild
Lyon, France	Yves Renard
January 2023	

Acknowledgments

First of all, we thank our collaborators that participated directly to some of the works described in this book: Faker Ben Belgacem, Stéphane Bordas, Karol Cascavita, Guillaume Drouet, Alexandre Ern, Mathieu Fabre, Miguel Fernández, Riad Hassani, Norbert Heuer, Ioan Ionescu, Patrick Laborde, Vanessa Lleras, Qingyuan Hu, Rabii Mlika, Nicolas Pignet, and Konstantinos Poulios.

We thank also Nabile Boussaïd, Alexandre Ern, Yohan Payan, and Mircea Sofonea for their support and advices related to the edition of this book. We thank Stéphane Bordas for having proofread some chapters and for his suggestions of improvements. We thank Rodolfo Araya, Hao Huang, Roland Masson, and Julien Yves Rolland for some feedback and advices.

This book has been partly written during F.C.'s stay at the Center for Research in Mathematical Engineering (CI^{2}MA) and F.C. thanks all the members of the CI^{2}MA for their kindness, as well as the CMM, the CNRS, and the University of Burgundy for having made this possible.

Finally, we thank people in Springer, particularly Chris Elder and Kritheka Elango, for their support related to the edition and publication of this book.

Contents

Acronyms

AL	Augmented Lagrangian
BEM	Boundary Element Method
DG	Discontinuous Galerkin
DPG	Discontinuous Petrov Galerkin
FEM	Finite Element Method
HDG	Hybrid Discontinuous Galerkin
HHO	Hybrid High Order
IGA	IsoGeometric Analysis
KKT	Karush–Kuhn–Tucker
LAC	Local Average Contact
VEM	Virtual Element Method
XFEM	eXtended Finite Element Method

Notations

As usual, the Kronecker symbol is denoted by δ_{ij} and defined as:

$$\delta_{ij} := \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

We note vectors in $\mathbb{R}^d$ ($d \geq 1$) using bold notation. A vector $\mathbf{x} \in \mathbb{R}^d$ is written

$$\mathbf{x} = (x_1, \ldots, x_d).$$

The canonical basis of $\mathbb{R}^d$ is denoted by $(\mathbf{e}_i)_{i=1,\ldots,d}$, with

$$\mathbf{e}_i := (0, \ldots, \underbrace{1}_{i-\text{th position}}, \ldots, 0),$$

such that $\mathbf{x} \in \mathbb{R}^d$ can also be written

$$\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e}_i .$$

The scalar (dot) product between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ is denoted by $\mathbf{x} \cdot \mathbf{y}$ and defined as:

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{d} x_i y_i \in \mathbb{R}.$$

The euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^d$ is denoted by $|\mathbf{x}|$:

$$|\mathbf{x}| := (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(x_1^2 + \ldots + x_d^2\right)^{\frac{1}{2}} \in \mathbb{R}.$$

We recall that, with the above conventions, $(\mathbf{e}_i)_{i=1,\ldots,d}$ is an orthonormal basis of $\mathbb{R}^d$:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad |\mathbf{e}_i|_2 = 1,$$

for $i, j = 1, \ldots, d$. Moreover, for $\mathbf{x} \in \mathbb{R}^d$, there holds

$$\mathbf{x} \cdot \mathbf{e}_i = \left(\sum_{j=1}^{d} x_j \mathbf{e}_j \right) \cdot \mathbf{e}_i = \sum_{j=1}^{d} x_j \left(\mathbf{e}_j \cdot \mathbf{e}_i \right) = \sum_{j=1}^{d} x_j \delta_{ij} = x_i,$$

for $i = 1, \ldots, d$.

The notation $\otimes$ denotes the tensor product between either two vector spaces or two vectors. The space of second-order tensors built from $\mathbb{R}^d$ is denoted by $\mathbb{R}^d \otimes \mathbb{R}^d$, and we denote by $(\mathbf{e}_i \otimes \mathbf{e}_j)_{i,j=1,\ldots,d}$ its canonical basis, built from the canonical basis $(\mathbf{e}_i)_{i=1,\ldots,d}$ of $\mathbb{R}^d$. The tensor product between two vectors $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^d$ is denoted by $\mathbf{x} \otimes \mathbf{y} \in \mathbb{R}^d \otimes \mathbb{R}^d$ and there holds:

$$\mathbf{x} \otimes \mathbf{y} = \left(\sum_{i=1}^{d} x_i \mathbf{e}_i \right) \otimes \left(\sum_{j=1}^{d} y_j \mathbf{e}_j \right) = \sum_{i=1}^{d} \sum_{j=1}^{d} x_i y_j \mathbf{e}_i \otimes \mathbf{e}_j.$$

For a second-order tensor $\boldsymbol{\tau} \in \mathbb{R}^d \otimes \mathbb{R}^d$, we denote by τ_{ij} its scalar components in the canonical basis: $\tau_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, d$. As a result, we write

$$\boldsymbol{\tau} = \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_{ij} \mathbf{e}_i \otimes \mathbf{e}_j,$$

and we note that in practice, we often manipulate the tensor via the matrix of its components:

$$(\tau_{ij})_{i,j=1,\ldots,d}.$$

The double dot product, or double contraction product, between two tensors $\boldsymbol{\tau} \in \mathbb{R}^d \otimes \mathbb{R}^d$ and $\boldsymbol{\kappa} \in \mathbb{R}^d \otimes \mathbb{R}^d$ is denoted by $\boldsymbol{\tau} : \boldsymbol{\kappa}$ and can be computed in the canonical basis as:

$$\boldsymbol{\tau} : \boldsymbol{\kappa} := \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_{ij} \kappa_{ij} \in \mathbb{R}.$$

Note that

$$\mathbf{e}_i \otimes \mathbf{e}_j : \mathbf{e}_k \otimes \mathbf{e}_l = \delta_{ij} \delta_{kl},$$

for $i, j, k, l = 1, \ldots, d$. We use the symbol $^{\mathrm{T}}$ to denote the transpose of a tensor. So, for $\boldsymbol{\tau} \in \mathbb{R}^d \otimes \mathbb{R}^d$, we define

$$\boldsymbol{\tau}^{\mathrm{T}} := \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_{ji} \mathbf{e}_i \otimes \mathbf{e}_j,$$

the tensor whose components in the canonical basis are $(\tau_{ji})_{i,j=1,\ldots,d}$.

The product (contraction) between a second-order tensor $\boldsymbol{\tau}$ and a vector $\mathbf{v}$ is denoted simply

$$\boldsymbol{\tau}\mathbf{v} := \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \tau_{ij} v_j \right) \mathbf{e}_i,$$

instead of $\tau \cdot \mathbf{v}$ so that it matches with the matrix / vector product (when the tensors are expressed in the canonical basis).

Scalars can be viewed as zero order tensors. Tensors of arbitrary order $k \in \mathbb{N}$ can be defined in the same way as tensors of order 2, using the properties of the tensor product $\otimes$. For introductory material about tensors, the interested reader can refer for instance to [46, Chapter 2], [69, Chapter 2], or [237, Chapter 2]. Other classical references on the subject are [2, 49, 197].

The letter C stands for a generic positive constant whose value may change at each occurence. This constant will be generally independent of some specified quantities, particularly the mesh size.

Part I
Basic Concepts

Chapter 1
Introduction

For a wide range of systems in structural mechanics, it is crucial to take into account contact and friction between rigid or elastic bodies. Among numerous applications, let us mention foundations in civil engineering, metal forming processes, crash tests of cars, and design of car tires (see, e.g., [266]). Contact and friction conditions are usually formulated with a set of inequalities and nonlinear equations on the boundary of each body, with corresponding unknowns that are displacements, velocities, and surface stresses. Basically, contact conditions allow to enforce nonpenetration on the whole candidate contact surface, and the actual contact surface is not known in advance. A friction law may be taken into account additionally, and various models exist that correspond to different surface properties, the most popular one being Coulomb friction (see, e.g., [181] and the references therein). Frictional contact problems can be formulated weakly within the framework of variational or quasi-variational inequalities (see, e.g., [114, 128, 172, 181]). Those are the very basis of most existing Finite Element Methods (FEMs), which are discretization methods commonly used in engineering to compute approximate solutions, see, e.g., [139, 147, 153, 154, 181, 190, 264, 266].

This book describes in detail the mathematical theory that allows to formulate some typical contact and friction problems and to analyze their approximation using variational methods of Ritz–Galerkin type, such as the Finite Element Method. It also aims at providing a state of the art about the new techniques of numerical approximation, and about some improvements in their mathematical analysis that have been made in the last two decades.

1.1 An Overview of the Content

This book contains eleven chapters, including this one, and one appendix. It starts from the basic notions that allow to formulate and study frictionless contact. Then, it

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_1

introduces some finite element approximations and their numerical analysis. It ends with some extensions to frictional contact and multibody contact in large strain. Below we describe briefly the content of each chapter and how they are related.

Chapter 2 presents the fundamental notions to define functions that are candidate solutions to contact and friction problems. It starts with the basic notions about distributions and then introduces fractional order Sobolev spaces. Since contact and friction conditions occur on the boundary of a domain, we review the notion of Lipschitz boundary and of trace space on the boundary. The trace theorem and the lifting theorem play indeed a key role in the analysis of approximation methods for contact and friction. We end the chapter with an adaptation of the Deny–Lions lemma to the fractional setting, which allows later on to obtain optimal error estimates for various Sobolev regularities.

Chapter 3 describes the Signorini problem in small strain elasticity, which is among the simplest problems that involve frictionless unilateral contact. It has been widely studied in numerical analysis since it is simple enough to enter into a nice functional setting, and it already incorporates the most important difficulties since contact conditions on the boundary are given by a set of inequalities and nonlinear equations. We formulate it as a constrained minimization problem and recover its weak form by writing the first order optimality condition which is a variational inequality of the first kind. Then, we derive the strong form and recover the Signorini contact conditions as the Karush–Kuhn–Tucker (KKT) conditions of optimality for the constrained minimization problem. We also study the well-posedness of the weak formulation and, thanks to Stampacchia's theorem, prove that it admits one unique solution. This chapter ends with some considerations about the regularity of the solution, and more specifically on an assumption about the topology of the solution on the contact boundary, which has been at the center of many studies of numerical analysis.

Chapter 4 recalls some basic notions about Lagrange finite elements, and related interpolation results, that are needed for the next chapters, where the numerical analysis of various methods are carried out. We introduce Lagrange finite elements of arbitrary order on simplicial meshes and state the most important interpolation results, taking into account the functional setting associated with fractional order Sobolev spaces.

In Chap. 5, we go to the core of the topic and introduce some families of discrete variational inequalities that approximate the Signorini problem, based on Lagrange finite elements. We then derive optimal a priori error estimates in the natural norm for these direct finite element approximations. We make use of Falk's lemma to obtain a first abstract error estimate. This lemma is the extension of the well-known Cea's lemma, targeted for variational inequalities of the first kind. Then, we show that a specific term is difficult to bound and has been object of intensive works for decades. We take advantage of a recent work to show how it can be bounded optimally. To this aim, we need to dissociate the two-dimensional and three-dimensional cases and detail separately some extra difficulties that occur in the three-dimensional setting.

It is not usual to rely upon the discrete variational inequalities for practical implementation, and, for commodity reasons, people generally prefer to reformulate the variational inequality prior to finite element approximations. Many possibilities have been found for this purpose and may still be found in the future. The most usual ones are based either on duality arguments or on penalization/regularization. In the first case, one obtains, for instance, mixed or augmented Lagrangian formulations. In the other case, one can obtain penalized contact. In Chap. 6, we provide an alternative to these two paths, based on Nitsche's method. Nitsche's method is now well known for essential boundary conditions but has been adapted to the Signorini problem just a few years ago. It is closely related to the Barbosa and Hughes stabilized mixed form but remains primal and does not involve any additional unknown, conversely to mixed or augmented Lagrangian formulations. It does not regularize contact conditions and thus differs also from the penalty techniques. Notably it remains consistent. We show that this method leads also to optimal a priori error bounds, but with mathematical arguments quite different to those of Chap. 5, and more related in some sense to Strang lemmas for nonconforming finite element methods. Furthermore, Nitsche's technique is easy to implement when combined with a semi-smooth Newton solver, and we provide some information on this topic at the end. We provide among others the tangent weak form needed at each Newton iteration.

Chapter 7 is about Lagrange multipliers and the augmented Lagrangian. These are well-known techniques for minimization problems under constraints and have been successfully applied to contact problems for some decades. We review how these methods can be derived from duality arguments and study their mathematical properties. As shown with some recent works, optimal convergence in the natural norm can be recovered for these approximations. Particular emphasis is also placed on the link between the proximal augmented Lagrangian and Nitsche's method.

Chapter 8 is an introduction to friction, with a simple model: Tresca friction. Tresca friction is a simplification of Coulomb friction, where the threshold is supposed to be known. It has limited interest in terms of modelling but is particularly useful when fixed point arguments are involved for Coulomb friction. So, Tresca friction is a preliminary step before tackling the subject of Coulomb friction, which is much more difficult. We present first the Tresca model, which can be recast as a well-posed variational inequality of the second kind. Despite this friendly mathematical structure, some special issues arise for numerical approximation, notably to obtain optimal a priori error estimates. We try to explain why and how it can be circumvented, at least for some special discretization techniques based on Nitsche or penalty.

Chapter 9 deals with Coulomb friction. Unilateral contact with Coulomb friction can be recast as a quasi-variational inequality, which is difficult to study. We sum up known results about existence and (non-)uniqueness of solutions, depending on the value of the friction coefficient. This is still today an open problem. Then, we present finite element methods for Coulomb friction based on mixed and Nitsche formulations.

Chapter 10 introduces a setting more general and more relevant to applications than the Signorini problem detailed in Chap. 3, for frictionless contact, and Chap. 8, for frictional contact. It involves two bodies, still in small deformations, with less restrictive assumptions on the geometry of their contact boundary. It is thought mostly as a preparatory chapter to the last chapter.

Last Chap. 11 deals with contact in large strain and for possibly multiple bodies and/or self-contact. It is among the most difficult problems, but it is also among the most relevant problems for practical and industrial applications. Of course, in this setting, existence, uniqueness, and convergence properties are very difficult or even impossible to obtain with current mathematical techniques. At least the mathematical theory we have at our disposal helps to build some numerical methods that can be more robust or accurate. We detail particularly the difficulties associated with the gap function and its derivative. Indeed the gap function needs in this case to be defined from the (unknown) deformed configuration and becomes unknown. We provide some finite element formulations as well based on Nitsche or augmented Lagrangian technologies, with emphasis on an unbiased Nitsche method designed to handle self-contact. We end this section with numerical results that illustrate the performance of the presented methods in such complex situations.

1.2 Prerequisites

This book assumes the reader has a basic knowledge in the mathematical theory of the Finite Element Method for elliptic partial differential equations and also some notions of continuum mechanics, at least some knowledge of the small strain elasticity framework. It does not assume that the reader has some familiarity with contact or friction problems, neither with the formalism of variational inequalities.

Though convex analysis is a very powerful tool for the problems under consideration in this book, we did not assume that the reader has a deep knowledge around this topic. We tried to mention the useful notions at an introductory level and hope it will make the reader curious to know more about this field of mathematical analysis.

1.3 An Informal Presentation of the Methods

Many numerical methods can be designed for solving contact and friction. To provide an explicit formulation for most of them, we can focus on the emblematic Signorini problem, which is among the simplest contact problems in elasticity, but that already presents some important difficulties from the numerical approximation viewpoint.

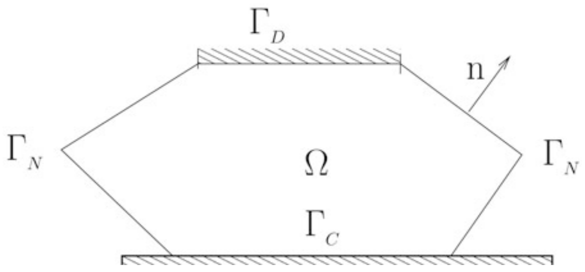

Fig. 1.1 An elastic body that occupies the domain Ω, and Γ_C is the contact boundary with nonpenetration into the rigid support at its bottom

1.3.1 The Signorini Problem

The Signorini problem consists in finding the final configuration of an elastic body after deformation, in the small strain framework, when this body is subjected to prescribed bulk and boundary forces, and possibly in frictionless contact with a plane rigid surface: see Fig. 1.1. This problem is presented in detail in Chap. 3.

For the Signorini problem, the unknown is the displacement $\mathbf{u}$, a vector valued function defined in the domain $\overline{\Omega}$. This domain represents the reference position of the elastic body. The boundary conditions (Signorini conditions) on the potential contact surface Γ_C are

$$u_{\mathbf{n}} \leq 0, \qquad \sigma_{\mathbf{n}} \leq 0, \qquad \sigma_{\mathbf{n}}\, u_{\mathbf{n}} = 0, \tag{1.1}$$

where $u_{\mathbf{n}}$ represents the normal displacement to the contact boundary, and $\sigma_{\mathbf{n}}$ stands for the normal Cauchy stress, also called the contact pressure. For regular enough solutions, these conditions are satisfied punctually, i.e., at every point on the contact boundary Γ_C, otherwise they should be understood in a weaker sense.

First of all note that, conversely to standard (Dirichlet or Neumann) boundary conditions, Signorini conditions are nonlinear and made of two inequalities plus a product equal to zero. These are a special case of Karush–Kuhn–Tucker (KKT) optimality conditions for minimization problems under inequality constraints. Furthermore, remark the following specificity of the Signorini problem: the effective contact surface (corresponding to the subset of Γ_C where $u_{\mathbf{n}} = 0$) is not known in advance and is determined only after the problem has been solved. This is a free boundary problem. The corresponding terminology to describe such a situation is *unilateral contact*, in opposition to bilateral contact problems where the effective contact zone is supposed to be known in advance, and for which the corresponding model incorporates simply a (linear) generalized Dirichlet boundary condition $u_{\mathbf{n}} = 0$. In the sequel, we use the following terminology to design the subset of Γ_C that corresponds to the condition $u_{\mathbf{n}} = 0$, by saying this is the region where contact is activated, or effective. In the complementary region corresponding to $u_{\mathbf{n}} < 0$, the contact is not activated.

Roughly speaking, the first condition $u_{\mathbf{n}} \leq 0$ prevents the penetration of the elastic body into the rigid support, and the second condition $\sigma_{\mathbf{n}} \leq 0$ means that the contact forces can solely be repulsive (i.e., without adhesion phenomena). The third and last condition $\sigma_{\mathbf{n}}\, u_{\mathbf{n}} = 0$ states that on the region where contact is not activated ($u_{\mathbf{n}} < 0$) the contact pressure $\sigma_{\mathbf{n}}$ must be equal to zero. There can be a region of Γ_C where both $\sigma_{\mathbf{n}}$ and $u_{\mathbf{n}}$ are zero. This is possible but not frequent, and this is called grazing contact. Notably, since this last condition $\sigma_{\mathbf{n}}\, u_{\mathbf{n}} = 0$ forces the contact pressure $\sigma_{\mathbf{n}}$ to vanish where contact is not activated ($u_{\mathbf{n}} < 0$), it prevents unphysical situations of negative contact pressure combined with no contact activation.

Suppose, for the sake of simplicity, that the body Ω is just subjected to bulk forces $\mathbf{f}$, and there are no boundary forces. The corresponding weak form associated with the Signorini problem can be obtained from the elasticity equations (local equilibrium) combined with the Green formula:

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}} - u_{\mathbf{n}}), \tag{1.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the small strain tensor, and $\mathbf{v}$ is an arbitrary test function (virtual displacement). Denote by $\mathbf{K}$ the set of admissible displacements that satisfy the nonpenetration condition, i.e., for $\mathbf{v} \in \mathbf{K}$, there holds $v_{\mathbf{n}} \leq 0$. We observe that $\mathbf{K}$ is a convex cone, and Signorini conditions (1.1) imply that $\mathbf{u} \in \mathbf{K}$. Now, for $\mathbf{v} \in \mathbf{K}$, still from Signorini conditions (1.1), there holds

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}} - u_{\mathbf{n}}) = \int_{\Gamma_C} \underbrace{\sigma_{\mathbf{n}}(\mathbf{u})}_{\leq 0} \underbrace{v_{\mathbf{n}}}_{\leq 0} - \int_{\Gamma_C} \underbrace{\sigma_{\mathbf{n}}(\mathbf{u}) u_{\mathbf{n}}}_{=0} \geq 0.$$

This inequality combined with the Green formula (1.2) yields

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}), \tag{1.3}$$

for every test function (admissible virtual displacement) $\mathbf{v} \in \mathbf{K}$. The above weak form (1.3) is a *variational inequality* and this reformulation allows us to prove that the Signorini problem has a sound mathematical structure. Indeed we detail in Chap. 3 the precise conditions of equivalence between the different (strong and weak) formulations and establish that the Signorini problem is a well-posed problem in the sense it admits one unique solution. Moreover, we show in Chap. 3 how this problem can be derived using energy minimization arguments, and this point is also useful for the derivation of some numerical methods.

A direct finite element approximation of the variational inequality (1.3) can be carried out, and its main interest is theoretical: it allows to study the convergence properties of the finite element approximations, and the difficulties that arise in the mathematical analysis of the corresponding discrete variational inequality are also present for other methods. This point is the object of Chap. 5.

Nevertheless, the formulation (1.3) is not very practical for implementation into standard finite element libraries, which generally do not allow to handle easily inequality constraints, and people traditionally prefer to reformulate the contact conditions in a different way for this purpose, notably to circumvent the cumbersome inequalities.

1.3.2 Some Numerical Approximations for Signorini Contact

Among the simplest strategies to approximate Signorini Problem (1.3), one consists in penalizing the penetration of the elastic body into the rigid support, in other terms to substitute to the Signorini conditions (1.1) the following one:

$$\sigma_{\mathbf{n}} = -\frac{1}{\varepsilon}[u_{\mathbf{n}}]_{+}, \tag{1.4}$$

where $\varepsilon > 0$ is the penalty parameter, and $[\cdot]_{+}$ denotes the positive part of a function. As a result, the above condition allows a small amount of penetration between the two bodies, but that is expected to be smaller when ε gets smaller. Indeed from (1.4), one deduces that there is no reaction when there is no penetration and that the reaction forces are proportional to the amount of the penetration and to the inverse of ε. Condition (1.4) also ensures that the condition $\sigma_{\mathbf{n}} \leq 0$ in (1.1) is enforced (at least at the continuous level). Suppose once again, and for the sake of simplicity, that the body Ω is just subjected to bulk forces $\mathbf{f}$, and there are no boundary forces. To derive a weak formulation for penalized contact, we start as in (1.2) from elasticity equations (local equilibrium) combined with the Green formula and get

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) v_{\mathbf{n}} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \tag{1.5}$$

with $\mathbf{v}$ still an arbitrary test function (virtual displacement). We inject (1.4) into the term associated with the contact boundary Γ_C and discretize all the fields, and we deduce a first discrete approximation for the Signorini problem, which reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that} \\ \displaystyle\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\varepsilon} \left[u^h_{\mathbf{n}} \right]_+ v^h_{\mathbf{n}} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}^h, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

In the above formulation, $\mathbf{u}^h$ is the (unknown) discrete displacement and $\mathbf{V}^h$ a finite dimensional vector space (built using the Finite Element Method, for instance). As explained in Chap. 7, this formulation can be alternatively obtained from an Augmented Lagrangian formalism combined with an Uzawa solving procedure and corresponds to the problem solved at the first iteration of the Uzawa solver. The penalty formulation for contact consists then in solving a nonlinear equation, but with no more inequality constraint. This makes it attractive for implementation. Its main issue consists in choosing appropriately the penalty parameter ε, which needs to be as small as possible to mimic the nonpenetration condition, but not too small in order to ensure the convergence of any iterative solver for the nonlinear problem.

As a result, a second possibility to incorporate Signorini conditions (1.1) is to reformulate them as follows:

$$\sigma_{\mathbf{n}} = -\left[\gamma u_{\mathbf{n}} - \sigma_{\mathbf{n}}\right]_+ , \tag{1.6}$$

for any positive function γ that lives on the contact boundary Γ_C. This is the starting point for both Nitsche and Augmented Lagrangian formulations. As we show in Chap. 6, the above condition (1.6) is equivalent to Signorini conditions (1.1) and is not an approximation, conversely to the penalized formulation (1.4). Starting once again from (1.5) and using the above condition (1.6) in the contact boundary term, we get the following Nitsche weak form:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that} \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) + \int_{\Gamma_C} [\gamma u^h_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u}^h)]_+ v^h_{\mathbf{n}} = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

Remark that the above formulation is not symmetric, but this is not really an issue and a symmetric formulation can be derived as well. A detailed mathematical study of Nitsche's formulations is made in Chap. 6.

A third possibility consists in introducing a Lagrange multiplier as a new unknown for the contact pressure: $\lambda = \sigma_{\mathbf{n}}(\mathbf{u})$. Then, we can rewrite the Signorini contact conditions (1.6) as

$$\lambda = -\left[\gamma u_{\mathbf{n}} - \lambda\right]_+ , \tag{1.7}$$

still with γ a positive function that is defined on the contact boundary Γ_C. Now, from the above contact condition (1.7), we get an Augmented Lagrangian formulation

$$\begin{cases} \text{Find } (\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times W^h \text{ such that} \\ \displaystyle\int_\Omega \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) + \int_{\Gamma_C} \left[\gamma u^h_{\mathbf{n}} - \lambda^h\right]_+ v^h_{\mathbf{n}} = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \displaystyle -\int_{\Gamma_C} \frac{1}{\gamma}\left(\left[\gamma u^h_{\mathbf{n}} - \lambda^h\right]_+ + \lambda^h\right)\mu^h = 0, \quad \forall \mu^h \in W^h, \end{cases} \tag{1.8}$$

where λ^h is a discrete approximation of the contact pressure and W^h is a finite dimensional vector space of functions defined on Γ_C. The second equation in (1.8) enforces (1.7) weakly. The first equation in (1.8) comes directly from (1.5) once again, where we have first substituted λ with $\sigma_{\mathbf{n}}$ and then used (1.7). Augmented Lagrangian formulations, as well as mixed methods, are object of Chap. 7.

As for the penalty method, Nitsche and Augmented Lagrangian formulations do not involve any more inequality constraints, and corresponding weak formulations are nonlinear variational equalities, that can be solved with any solver for nonlinear nonsmooth problems. Moreover, the function γ still involves a user-defined numerical parameter γ_0 (Nitsche's parameter or augmentation parameter), but, conversely to the parameter ε in the penalty formulation, this parameter γ_0 does not need to be very small or very large to mimic the contact condition.

A popular method to solve nonlinear weak forms as those described above is the semi-smooth Newton method. The tangent weak form that is needed in this case for each Newton iteration is obtained without difficulty, whatever the formulation is (penalty, Nitsche, Augmented Lagrangian). Indeed, to get the contribution of the nonlinear contact term in the tangent system, it suffices to apply the following formula for the positive part operator:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x]_+ = H_s(x),$$

where H_s is the Heaviside function. Of course, the positive part function is not differentiable at point 0 and admits left and right derivatives. In such a situation, one makes an arbitrary choice between the two possibilities. The resulting iterative method converges provided the initial guess and the numerical parameter are well chosen. At this point, it remains only to apply a standard finite element assembly procedure and a solver for the linear system at each Newton iteration. There are now many finite element libraries that allow an easy implementation of such methods: AceFEM, FEniCS, scikit-fem and GetFEM, to mention only a few ones.

1.4 How to Use This Book

For a reader who wants to learn about the field of numerical approximation of contact, this book can be studied more or less linearly from this chapter to the end. The first part, made of four chapters, contains all the basic notions to prepare the understanding of the further chapters, focused on numerical methods and their analysis. The chapters on Sobolev spaces and on finite elements can eventually be skipped by the reader already familiar with these topics.

The second part is the core of this book, with the three chapters that present various possibilities to discretize contact. Chapters 5 and 6 on discrete variational inequalities and Nitsche, respectively, can be studied independently, whereas Chap. 7 on mixed methods relies on some results presented in Chap. 5.

The third and last part focuses on frictional contact and multi-body contact. Each chapter can be studied rather independently if the reader is familiar with these topics. However, Chap. 8 on Tresca friction is in some sense preparatory to the next chapter (Chap. 9) on Coulomb friction. In the same spirit, Chap. 10 for multi-body contact in small strain introduces at an elementary level some notions for the last chapter (Chap. 11) on (self-)contact in large strain.

A specialized course inspired from the present monograph could be based on a combination of Chaps. 3 and 5, adorned possibly with other ones according to the interest of the reader and of the audience.

Chapter 2
Sobolev Spaces

Contact and friction problems, as well as many other problems involving elliptic partial differential equations and some specific boundary conditions, are formulated and studied appropriately using fractional order Sobolev spaces. This functional setting allows to provide some meaningful reformulations of the original mathematical models (in strong form) and also to study their properties in terms of existence, uniqueness of solutions, as well as their approximability. We start this chapter with some definitions and basic results about functions, distributions, and distributional derivatives. Then, we introduce the fundamental notions about fractional order Sobolev spaces and Lipschitz domains. This allows further to provide some details about trace operators, which play a particularly important role in the next chapters, since they allow to define appropriately the restriction of regular distributions on the boundary of a Lipschitz domain. As well, lifting operators are studied, since they allow, the other way round, to extend to a whole domain a function defined only on a boundary. Moreover, these operators allow to derive Green formulas, which is the basic tool to go from the strong form to the weak form, and conversely. We end the chapter with a presentation of polynomial approximation theory in fractional order Sobolev spaces, from which we will derive later on interpolation estimates for finite element spaces.

We start in Sect. 2.1 by recalling basic facts about functions and distributions. Then, Sect. 2.2 is devoted to fractional order Sobolev spaces. Special attention is dedicated to Lipschitz domains in Sect. 2.3. Results about trace and lifting operators for Lipschitz domains are presented in Sect. 2.4, which is followed by Sect. 2.5 related to Green formulas. Results of polynomial approximation theory in fractional order Sobolev spaces are provided in Sect. 2.6. Some comments and bibliography are finally provided in Sect. 2.7.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_2

2.1 Distributions

Before going into more complicated results, we start by recalling the definitions and properties of some function spaces and of distributions, which are at the very basis of the theory of partial differential equations. We tried to do it as shortly and simply as it was possible, and to recall only what is strictly necessary for the sequel of this book. The reader interested in going further into this topic can find some references at the end of the chapter.

We first present some notations related to continuous and differentiable functions, in Sect. 2.1.1 and then introduce the space of test functions in Sect. 2.1.2. Distributions and the fundamental notion of distributional derivative are presented in Sect. 2.1.3. We end in Sect. 2.1.4 with the notion of regular distributions, since they play a key role in the forthcoming theory.

2.1.1 Continuous and Differentiable Functions

For $d \geq 1$, let ω be a Lebesgue-measurable subset of $\mathbb{R}^d$, with non-empty interior. The notation $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ stands for a generic point in ω. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ be a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d \in \mathbb{N}$. For a regular enough function $\psi : \omega \to \mathbb{R}$, the partial derivative of ψ with respect to the multi-index α is:

$$D^\alpha \psi := \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.$$

The gradient of ψ is a vectorial field on ω with values in $\mathbb{R}^d$ defined as follows:

$$\nabla \psi := \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}, \ldots, \frac{\partial \psi}{\partial x_d} \right).$$

For a regular enough vector-valued function $\boldsymbol{\psi} : \omega \to \mathbb{R}^d$, using the notation $\boldsymbol{\psi} := (\psi_1, \psi_2, \ldots, \psi_d)$ for its components, we define its divergence as

$$\operatorname{div} \boldsymbol{\psi} := \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} + \ldots + \frac{\partial \psi_d}{\partial x_d}.$$

The set of continuous real-valued functions in ω is denoted by $\mathscr{C}(\omega)$. In the same way, the set of real-valued functions in ω that admit continuous Fréchet-derivatives up to order k ($k \geq 0$) is denoted by $\mathscr{C}^k(\omega)$ ($\mathscr{C}^0(\omega) = \mathscr{C}(\omega)$). The set of real-valued functions in ω that are infinitely Fréchet-differentiable is noted $\mathscr{C}^\infty(\omega)$, i.e.,

$$\mathscr{C}^\infty(\omega) = \bigcap_{k \geq 0} \mathscr{C}^k(\omega).$$

For vector-valued functions, we will use the notation $\mathscr{C}(\omega; \mathbb{R}^d)$ for continuous vector-valued functions defined in ω and that take their values in $\mathbb{R}^d$. In the same manner, we note $\mathscr{C}^k(\omega; \mathbb{R}^d)$, resp. $\mathscr{C}^\infty(\omega; \mathbb{R}^d)$, for functions with continuous derivatives up to order k, resp. infinitely differentiable.

2.1.2 Test Functions

Let now Ω be a Lebesgue-measurable, non-empty, open subset of $\mathbb{R}^d$. For any function $f : \Omega \to \mathbb{R}$, the support of f is

$$\operatorname{supp} f = \overline{\{x \in \Omega \mid f(x) \neq 0\}} \subset \mathbb{R}^d.$$

We say that f has a compact support if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} f \subset K$. Notably if f has a compact support, it vanishes in a neighborhood of $\partial\Omega$.

The set of test functions, denoted by $\mathcal{D}(\Omega)$, is the set of infinitely Fréchet-differentiable functions with a compact support, i.e.,

$$\mathcal{D}(\Omega) := \{f \in \mathscr{C}^\infty(\Omega) \mid \exists K \subset \Omega,\ K \text{ compact},\ \operatorname{supp} f \subset K\}.$$

The most basic example of test function is the following one: the function $\psi \in \mathcal{D}(\mathbb{R})$, defined explicitly as:

$$\psi(x) := \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in \mathbb{R}$. From the above function, we can build a whole family of, smooth, test functions of d variables, whose support is arbitrarily small. Take $\varepsilon > 0$, $\mathbf{x} \in \mathbb{R}^d$, and define

$$\psi_\varepsilon(\mathbf{x}) := \left(\int_{\mathbb{R}^d} \psi\left(\frac{|\mathbf{y}|}{\varepsilon}\right) \mathrm{d}\mathbf{y}\right)^{-1} \psi\left(\frac{|\mathbf{x}|}{\varepsilon}\right).$$

This allows to have a family of test functions $(\psi_\varepsilon)_{\varepsilon>0}$ with many interesting properties. Indeed, there holds:

$$\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d), \quad \operatorname{supp} \psi_\varepsilon = \overline{\mathcal{B}(\mathbf{0}, \varepsilon)}, \quad \psi_\varepsilon \geq 0, \quad \int_{\mathbb{R}^d} \psi_\varepsilon(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1,$$

where $\mathcal{B}(\mathbf{0}, \varepsilon)$ is the open ball of center $\mathbf{0} \in \mathbb{R}^d$ and radius $\varepsilon > 0$.

Many other test functions can be built, and, in supplement, we provide another useful family of test functions, which correspond to smooth characteristic functions associated with open balls (see, e.g., [239]):

Proposition 2.1 *For every open ball $\mathcal{B}(\mathbf{x}_0; R)$ of center $\mathbf{x}_0 \in \mathbb{R}^d$, and of radius $R > 0$, for every $\varepsilon > 0$, there exists $\varphi_{\mathcal{B}} \in \mathcal{D}(\mathbb{R}^d)$ such that:*

$$\varphi_{\mathcal{B}}(\mathbf{x}) = 1, \qquad \forall \mathbf{x} \in \mathcal{B}(\mathbf{x}_0; R),$$

and

$$0 \leq \varphi_{\mathcal{B}} \leq 1, \qquad supp\ \varphi_{\mathcal{B}} \subset \overline{\mathcal{B}(\mathbf{x}_0; R+\varepsilon)},$$

where $\mathcal{B}(\mathbf{x}_0; R+\varepsilon)$ is the open ball in $\mathbb{R}^d$ of center $\mathbf{x}_0$ and radius $R+\varepsilon$.

Proof We introduce first a smooth step function, defined, for $x \in \mathbb{R}$, as:

$$h(x) = \frac{1}{\displaystyle\int_{\mathbb{R}} \psi(2r-1)\mathrm{d}r} \int_{-\infty}^{x} \psi(2r-1)\mathrm{d}r.$$

We check that it has the following properties: $h \in \mathscr{C}^\infty(\mathbb{R})$, $0 \leq h \leq 1$ and $h' \geq 0$. Moreover, there holds:

$$h(x) = 0 \text{ for } x \leq 0, \qquad h(x) = 1 \text{ for } x \geq 1.$$

Thereafter we define, for $\mathbf{x} \in \mathbb{R}^d$:

$$\varphi_{\mathcal{B}}(\mathbf{x}) = h\left(\frac{R+\varepsilon-|\mathbf{x}-\mathbf{x}_0|}{\varepsilon}\right)$$

and we check it has the required properties. □

Moreover, let us note that if φ is a test function, all its derivatives are test functions as well. Notably, they have all compact support: take a point $\mathbf{x} \in \Omega$ that does not belong to the support of φ. Then, not only there holds $\varphi(\mathbf{x}) = 0$ but there exists an open ball of center $\mathbf{x}$ on which φ is equal to zero identically. This implies that $D^\alpha\varphi = 0$ on this open ball for any multi-index α. It results that $\mathbf{x}$ does not belong to the support of $D^\alpha\varphi$.

We endow $\mathcal{D}(\Omega)$ with the following convergence property:

Definition 2.1 A sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ tends to 0 in $\mathcal{D}(\Omega)$ if:

1. There exists a compact set $K \subset \Omega$ such that, for all $n \in \mathbb{N}$:

$$\text{supp}\ \varphi_n \subset K.$$

2. For any multi-index α, the function $D^\alpha \varphi_n$ tends to 0 uniformly in K:

$$\max_{\mathbf{x}\in\Omega} |D^\alpha \varphi_n(\mathbf{x})| \xrightarrow{n\to+\infty} 0.$$

We will use the following notation for this convergence:

$$\varphi_n \xrightarrow{\mathcal{D}(\Omega)} 0.$$

For $\varphi \in \mathcal{D}(\Omega)$, we say that $\varphi_n \xrightarrow{\mathcal{D}(\Omega)} \varphi$ if and only if $(\varphi_n - \varphi) \xrightarrow{\mathcal{D}(\Omega)} 0$. □

Finally, the vector space of vector-valued test functions defined in Ω and with values in $\mathbb{R}^d$ is denoted by $\mathcal{D}(\Omega; \mathbb{R}^d)$.

2.1.3 Distributions and Distributional Derivatives

Let us now recall the notion of distribution:

Definition 2.2 A distribution T is a continuous linear form on $\mathcal{D}(\Omega)$, which means that, for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ of test functions such that

$$\varphi_n \xrightarrow{\mathcal{D}(\Omega)} 0,$$

there holds:

$$\langle T, \varphi_n \rangle \xrightarrow{n\to+\infty} 0,$$

where $\langle T, \varphi \rangle$ denotes the duality pairing between $\mathcal{D}(\Omega)$ and its topological dual, noted $\mathcal{D}'(\Omega)$. The vector space of distributions is thus denoted $\mathcal{D}'(\Omega)$. □

One of the main interests of distributions is that they always admit derivatives of any order, if we define the derivative in the following sense:

Definition 2.3 Let $T \in \mathcal{D}'(\Omega)$ be a distribution and α be a multi-index. The partial derivative of T with respect to the multi-index α is denoted $D^\alpha T \in \mathcal{D}'(\Omega)$ and defined by

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\Omega)$. □

One readily checks that this definition makes sense and that $D^\alpha T$ belongs indeed to $\mathcal{D}'(\Omega)$. This definition is inspired by integration by parts. The partial derivative of a distribution T is called distributional derivative.

We can define also vector-valued distributions as follows: $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{D}'(\Omega; \mathbb{R}^d)$ is a vector-valued distribution if each of its components T_i, $i = 1, \cdots, d$, is a (scalar) distribution ($T_i \in \mathcal{D}'(\Omega)$), and there holds, for every vector-valued test function $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_d) \in \mathcal{D}(\Omega; \mathbb{R}^d)$:

$$\langle \mathbf{T}, \boldsymbol{\varphi} \rangle = \sum_{i=1}^{d} \langle T_i, \varphi_i \rangle.$$

2.1.4 Regular Distributions and Dirac Distribution

We now examine some classical examples of distributions, more specifically we recall that a large class of functions can be viewed as distributions. This is the class of locally integrable functions. The vector space of locally integrable functions on Ω is defined as follows:

$$L^1_{\mathrm{loc}}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \int_K |f(\mathbf{x})| \mathrm{d}\mathbf{x} < +\infty,\ \forall K \subset \Omega,\ K \text{ compact} \right\}.$$

To any function $f \in L^1_{\mathrm{loc}}(\Omega)$, we can associate a distribution $T_f \in \mathcal{D}'(\Omega)$ using the following definition:

$$\langle T_f, \varphi \rangle = \int_\Omega f(\mathbf{x}) \varphi(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{2.1}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Since $f \in L^1_{\mathrm{loc}}(\Omega)$, and φ is continuous and has compact support, the above definition is meaningful.

For locally integrable functions, there holds the following variational lemma, whose importance is considerable in applied analysis:

Lemma 2.1 *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$\int_\Omega f(\mathbf{x}) \varphi(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then, $f = 0$ almost everywhere in Ω. □

Proof We proceed using regularization by convolution, as in [207, Theorem 3.7, p.64]. Let $K \subset \Omega$ be a compact set, and ω an open bounded set such that

$$K \subset \omega \subset \overline{\omega} \subset \Omega.$$

Let us define $\tilde{f}$ as follows, for $\mathbf{x} \in \mathbb{R}^d$:

$$\tilde{f}(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Since ω is an open set that contains the compact set K, there exists $\varepsilon_0 > 0$, such that, for every $0 < \varepsilon < \varepsilon_0$ and for every $\mathbf{x} \in K$, $\mathcal{B}(\mathbf{x}, \varepsilon) \subset \omega$.

Now, let us recall the family of test functions $(\psi_\varepsilon)_{\varepsilon>0}$ introduced above in Sect. 2.1.2. The above result means that, for $\mathbf{x} \in K$, and for $0 < \varepsilon < \varepsilon_0$, $\psi_\varepsilon(\mathbf{x} - \cdot) \in \mathcal{D}(\omega)$, where

$$\psi_\varepsilon(\mathbf{x} - \cdot) : \mathbf{y} \mapsto \psi_\varepsilon(\mathbf{x} - \mathbf{y}).$$

Let us compute the convolution between $\tilde{f}$ and ψ_ε, for all $\mathbf{x} \in K$:

$$\begin{aligned}\left(\tilde{f} * \psi_\varepsilon\right)(\mathbf{x}) &= \int_{\mathbb{R}^d} \tilde{f}(\mathbf{y})\psi_\varepsilon(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y} = \int_\omega f(\mathbf{y})\psi_\varepsilon(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_\Omega f(\mathbf{y})\psi_\varepsilon(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y},\end{aligned}$$

where we used first the definition of $\tilde{f}$, and then that $\psi_\varepsilon(\mathbf{x} - \cdot)$ vanishes outside of ω. Then, since $\psi_\varepsilon(\mathbf{x} - \cdot) \in \mathcal{D}(\omega) \subset \mathcal{D}(\Omega)$, we can apply the assumption of the Lemma, and get:

$$\left(\tilde{f} * \psi_\varepsilon\right)(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in K.$$

On the other hand (see [207, Theorem 3.4]), there holds:

$$\int_{\mathbb{R}^d} \left|(\tilde{f} * \psi_\varepsilon)(\mathbf{x}) - \tilde{f}(\mathbf{x})\right| \mathrm{d}\mathbf{x} \xrightarrow{\varepsilon \to 0} 0.$$

This implies that $\tilde{f} * \psi_\varepsilon$ tends to $\tilde{f}$ almost everywhere in $\mathbb{R}^d$. As a result, there holds $\tilde{f} = 0$ almost everywhere in $\mathbb{R}^d$, and particularly $f = 0$ almost everywhere in K. Since K was chosen arbitrarily in Ω, this means that, finally, $f = 0$ almost everywhere in Ω. □

Remark 2.1 Another way to prove this result is to make use of the Lebesgue differentiation Theorem, as in [239, Proposition 1.1]. □

This lemma means in some sense that the space of test functions is rich enough to characterize any locally integrable function. The first important consequence of the above lemma is that the application

$$L^1_{\mathrm{loc}}(\Omega) \ni f \mapsto T_f \in \mathcal{D}'(\Omega)$$

is injective, so that $L^1_{\mathrm{loc}}(\Omega)$ can be identified with a subspace of $\mathcal{D}'(\Omega)$. This particular subspace is called the subspace of regular distributions. As it is usual when dealing with distributions, we will use the notation f instead of T_f for the regular distribution associated with a locally integrable function f.

We end this paragraph with some important facts. First of all, though a lot of usual functions are locally integrable, this notion does not allow to associate a distribution to some specific usual functions. For instance, functions with strong singularities such as $1/x$, when $d = 1$, do not belong to $L^1_{\mathrm{loc}}(\Omega)$ if $0 \in \Omega$.

Also, $L^1_{\mathrm{loc}}(\Omega)$ is a proper subspace of $\mathcal{D}'(\Omega)$, since some distributions cannot be associated with some "usual" locally integrable functions. This is the case of the Dirac distribution (or Dirac mass) at point $\mathbf{x} \in \Omega$, which is defined for any $\varphi \in \mathcal{D}(\Omega)$ by

$$\langle \delta_{\mathbf{x}}, \varphi \rangle = \varphi(\mathbf{x}).$$

One readily checks that $\delta_{\mathbf{x}} \in \mathcal{D}'(\Omega)$ and it can be shown (see, e.g., [239, Chapter 1]) that $\delta_{\mathbf{x}}$ cannot be represented by any $f \in L^1_{\mathrm{loc}}(\Omega)$.

2.1.5 *Distributional Gradient and Divergence*

Thanks to the notion of distributional derivative, we can define the (distributional) gradient and divergence operators, for any distribution. Let $T \in \mathcal{D}'(\Omega)$, the distributional gradient of T is defined, in cartesian coordinates, as follows

$$\nabla T := \left(\frac{\partial T}{\partial x_1}, \frac{\partial T}{\partial x_2}, \dots, \frac{\partial T}{\partial x_d} \right) \in \mathcal{D}'(\Omega; \mathbb{R}^d),$$

where each partial derivative is taken in the sense of distributions. The definition of the distributional derivative implies the following rule: we take $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_d) \in \mathcal{D}(\Omega; \mathbb{R}^d)$ and compute

$$\langle \nabla T, \boldsymbol{\varphi} \rangle = \sum_{i=1}^{d} \left\langle \frac{\partial T}{\partial x_i}, \varphi_i \right\rangle = -\sum_{i=1}^{d} \left\langle T, \frac{\partial \varphi_i}{\partial x_i} \right\rangle = -\left\langle T, \sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial x_i} \right\rangle = -\left\langle T, \operatorname{div} \boldsymbol{\varphi} \right\rangle.$$

In the same manner, for a vector-valued distribution $\mathbf{T} \in \mathcal{D}'(\Omega; \mathbb{R}^d)$, with the notation $\mathbf{T} = (T_1, \dots, T_d)$, its distributional divergence is

$$\operatorname{div} \mathbf{T} := \frac{\partial T_1}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \dots + \frac{\partial T_d}{\partial x_d},$$

where all the derivatives are in the distributional sense. We can derive a similar rule as above, using the definition of the distributional derivatives. Indeed, if we take $\varphi \in \mathcal{D}(\Omega)$, there holds:

$$\langle \operatorname{div} \mathbf{T}, \varphi \rangle = \left\langle \left(\sum_{i=1}^{d} \frac{\partial T_i}{\partial x_i} \right), \varphi \right\rangle = \sum_{i=1}^{d} \left\langle \frac{\partial T_i}{\partial x_i}, \varphi \right\rangle = - \sum_{i=1}^{d} \left\langle T_i, \frac{\partial \varphi}{\partial x_i} \right\rangle.$$

As a result, we obtain

$$\langle \operatorname{div} \mathbf{T}, \varphi \rangle = -\langle \mathbf{T}, \nabla \varphi \rangle.$$

We provide finally an important result, whose proof is rather long and difficult [239, Theorem 1.1]. This result allows to characterize distributions with vanishing gradient, which are in fact (regular) constant distributions:

Theorem 2.1 *Let Ω be an open and connected set in $\mathbb{R}^d$. If $T \in \mathcal{D}'(\Omega)$ satisfies $\nabla T = \mathbf{0}$, then, there exists $c \in \mathbb{R}$ such that $T = c$.* □

2.1.6 Level Sets of Locally Integrable Functions

We end this section by providing a definition for the level sets of locally integrable functions. Indeed, this notion is obvious for continuous functions, but it is no more the case for functions that belong only to $L^1_{\mathrm{loc}}(\Omega)$, since there in fact equivalence classes of functions. We follow here the presentation from [193] and provide the definition:

Definition 2.4 Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $c \in \mathbb{R}$, the level set of f associated with c is the set

$$E_c(f) := \left\{ \mathbf{x} \in \Omega \;\middle|\; 0 = \lim_{\varepsilon \to 0} \frac{1}{|\mathcal{B}(\mathbf{x}; \varepsilon)|} \int_{\mathcal{B}(\mathbf{x}; \varepsilon)} |f(\mathbf{y}) - c| \mathrm{d}\mathbf{y} \right\},$$

where $\mathcal{B}(\mathbf{x}; \varepsilon)$ is the ball of center $\mathbf{x}$ and radius ε, and $|\mathcal{B}(\mathbf{x}; \varepsilon)|$ denotes its measure. □

We check that this definition is indeed meaningful for any $f \in L^1_{\mathrm{loc}}(\Omega)$, and that it coincides with the usual definition whenever f is continuous. We have moreover the following property, which is a direct consequence of the Lebesgue differentiation Theorem [193, Proposition 3.3]:

Proposition 2.2 *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $c \in \mathbb{R}$, then, $f = c$ almost everywhere in $E_c(f)$ and $f \neq c$ almost everywhere in $\Omega \setminus E_c(f)$.* □

2.2 Fractional Order Sobolev Spaces

In this section we provide main definitions and properties of fractional order Sobolev spaces, following mostly the book of W. McLean [207] as well as some class notes of N. Heuer [161]. Here Ω is still a Lebesgue-measurable, non-empty, open subset of $\mathbb{R}^d$. We start with recalling some results about the Lebesgue space and the Fourier transform, then, we introduce the notion of fractional order Sobolev space, using the Fourier transform. We provide another usual definition using the Sobolev-Slobodeckij semi-norm. We end by stating some usual properties.

2.2.1 Square-Integrable Functions

First, $L^2(\Omega)$ is the Lebesgue space of real-valued square-integrable functions defined on Ω:

$$L^2(\Omega) := \left\{ v : \Omega \to \mathbb{R} \,\middle|\, \int_\Omega |v(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} < +\infty \right\}$$

endowed with the scalar product

$$(v, w)_\Omega := \int_\Omega v(\mathbf{x}) w(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

for $v, w \in L^2(\Omega)$. The associated norm is

$$\|v\|_{0,\Omega} := \left(\int_\Omega |v(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \right)^{1/2}.$$

Moreover, the notation $L^2(\Omega; \mathbb{R}^d)$ stands for the space of vector-valued functions defined in Ω that are square-integrable. We keep the same notations for the scalar product and the norms as in the real case.

With the above scalar product, $L^2(\Omega)$ is a separable Hilbert space. For the proof, one can refer, for instance, to the corresponding chapter of H. Brezis about L^p spaces: see [52, Theorem IV.8] (Fischer-Riesz Theorem) and [52, Theorem IV.13]. As we will see, this property will be transmitted to the Sobolev spaces defined below.

In complement, we will sometimes need to use the other L^p spaces. For $p \in [1; +\infty)$ they are defined as

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \,\middle|\, \int_\Omega |v(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} < +\infty \right\}$$

and the corresponding norm is denoted by

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} .$$

For $p = +\infty$, we set

$$L^\infty(\Omega) := \left\{ v : \Omega \to \mathbb{R} \,\middle|\, \|v\|_{L^\infty(\Omega)} < +\infty \right\},$$

where

$$\|v\|_{L^\infty(\Omega)} := \inf \{C \in [0; +\infty] \mid |v(\mathbf{x})| \leq C \text{ for almost every } \mathbf{x} \in \Omega\} .$$

We recall that the above L^p spaces are Banach spaces [52, Chapter IV].

2.2.2 Sobolev Spaces of Fractional Order

To capture finely the regularity of the solutions of some partial differential equations, and to characterize precisely, later on, trace spaces (i.e., spaces associated with the value of functions on the boundary of a domain), we need to introduce the family of fractional order Sobolev spaces defined on Ω (see, for instance, [161, 207]).

A convenient definition relies on the Fourier transform. For a function $v \in L^1(\mathbb{R}^d)$, its Fourier transform is defined as follows:

$$\hat{v}(\boldsymbol{\xi}) := \mathcal{F}v(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} v(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

for every $\boldsymbol{\xi} \in \mathbb{R}^d$. Similarly, if $\hat{v} \in L^1(\mathbb{R}^d)$, the inverse (conjugate) Fourier transform is given by

$$v(\mathbf{x}) = \bar{\mathcal{F}}\hat{v}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \hat{v}(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}$$

for every $\mathbf{x} \in \mathbb{R}^d$. Moreover, using the density of $L^1(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, we can extend the domain of $\mathcal{F}$ to $L^2(\mathbb{R}^d)$ and prove that

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a unitary isomorphism (Plancherel's Theorem [207, Theorem 3.12]). This implies the identity:

$$\|\hat{v}\|_{0,\mathbb{R}^d} = \|v\|_{0,\mathbb{R}^d}, \quad \forall v \in L^2(\mathbb{R}^d), \tag{2.2}$$

as well as the inversion formula

$$\bar{\mathcal{F}}\mathcal{F}v = v, \quad \forall v \in L^2(\mathbb{R}^d). \tag{2.3}$$

For any real $s \geq 0$, and for functions in $\mathbb{R}^d$, we introduce the family of fractional order Sobolev spaces:

$$H^s(\mathbb{R}^d) := \left\{ v \in L^2(\mathbb{R}^d) \;\middle|\; (1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{v} \in L^2(\mathbb{R}^d) \right\},$$

endowed with the following scalar product

$$(v, w)_{s,\mathbb{R}^d} := \left((1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{v}, (1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{w} \right)_{\mathbb{R}^d}$$

for $v, w \in H^s(\mathbb{R}^d)$ and thus the corresponding norm

$$\|v\|_{s,\mathbb{R}^d} := \|(1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{v}\|_{0,\mathbb{R}^d}$$

for $v \in H^s(\mathbb{R}^d)$.

Endowed with the above norm, fractional order Sobolev spaces are complete, as stated below (for the proof, see, e.g., [239, Proposition 13.19]).

Proposition 2.3 *For any $s \geq 0$, $(H^s(\mathbb{R}^d), (\cdot, \cdot)_{s,\mathbb{R}^d})$ is a Hilbert space.* □

We are given now a whole scale of spaces that can capture more precisely the regularity of a specified function. For a domain $\Omega \subset \mathbb{R}^d$, we simply define

$$H^s(\Omega) := \left\{ v \in L^2(\Omega) \;\middle|\; \exists V \in H^s(\mathbb{R}^d),\ v = V|_\Omega \right\},$$

with the corresponding norm

$$\|v\|_{s,\Omega} := \inf_{V|_\Omega = v} \|V\|_{s,\mathbb{R}^d}.$$

For any real $s \geq 0$, we define the subspace $H^s_0(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$, for the norm $\|\cdot\|_{s,\Omega}$. As well, for vector-valued functions, we denote by $H^s(\Omega; \mathbb{R}^d)$, $s \geq 0$, the Sobolev space of vectorial fields of dimension d in Ω. We keep the same notation for the Sobolev norm $\|\cdot\|_{s,\Omega}$, as in the real case.

Remark 2.2 Another possibility to build the above family of spaces is to use the Slobodeckij semi-norm (see later on), or also hilbertian interpolation (see, e.g., [160, 161, 207] for more details). □

2.2.3 The Slobodeckij Semi-norm

In a first place, let us recall the usual definition of Sobolev spaces, when their order is a natural number. Let us start with the following consideration. Let $K \subset \Omega$ be a compact set. For $v \in L^2(\Omega)$, we use Cauchy–Schwarz inequality to bound:

$$\begin{aligned}\int_K |v(\mathbf{x})|\mathrm{d}\mathbf{x} &\leq \left(\int_K 1^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \left(\int_K |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} = |K|^{\frac{1}{2}} \left(\int_K |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \\ &\leq |K|^{\frac{1}{2}} \|v\|_{0,\Omega}.\end{aligned}$$

This means that $v \in L^1_{\mathrm{loc}}(\Omega)$. As a consequence, any function in $L^2(\Omega)$ can be considered as a regular distribution using formula (2.1), which implies that any function in $L^2(\Omega)$ is infinitely differentiable in the sense of distributions. This simple fact allows then to introduce the following definition:

For every exponent $m \in \mathbb{N}$ the Sobolev space of order m on Ω is defined as

$$W^m(\Omega) := \left\{ v \in L^2(\Omega) \,\middle|\, D^\alpha v \in L^2(\Omega), |\alpha| \leq m \right\},$$

where $D^\alpha v$ is the distributional derivative of v respectively to the multi-index α. We adopt the usual convention: $W^0(\Omega) := L^2(\Omega)$. We introduce on $W^m(\Omega)$ the following scalar product

$$(v, w)_{W^m(\Omega)} := \sum_{|\alpha| \leq m} (D^\alpha v, D^\alpha w)_\Omega$$

for $v, w \in W^m(\Omega)$ with the corresponding norm

$$\|v\|_{W^m(\Omega)} := \Big(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{0,\Omega}^2 \Big)^{\frac{1}{2}}$$

for $v \in W^m(\Omega)$.

Now, to define in another way fractional order Sobolev spaces, we introduce the Slobodeckij (or Gagliardo, or Aronszajn) semi-norm, as follows. Let θ be in $(0, 1)$, for a real-valued function v defined on Ω we set:

$$|v|_{\theta,\Omega} := \left(\int_\Omega \int_\Omega \frac{(v(\mathbf{x}) - v(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^{d+2\theta}} \,\mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}}.$$

For any $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we set $s = m + \theta$, $m \in \mathbb{N}$, $\theta \in (0, 1)$, and the fractional order Sobolev space $W^s(\Omega)$ is

$$W^s(\Omega) := \left\{ v \in W^m(\Omega) \;\middle|\; |D^\alpha v|_{\theta,\Omega} < +\infty, |\alpha| = m \right\}.$$

This space is endowed with the norm

$$\|v\|_{W^s(\Omega)} := \left(\|v\|_{m,\Omega}^2 + \sum_{|\alpha|=m} |D^\alpha v|_{\theta,\Omega}^2 \right)^{\frac{1}{2}}.$$

The following result ensures that the above definition is equivalent to the first one using Fourier transform, in the case $\Omega = \mathbb{R}^d$ (see, e.g., [207, Theorem 3.16]):

Proposition 2.4 *For any $s \geq 0$ we have*

$$W^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$$

and the norms $\|\cdot\|_{s,\mathbb{R}^d}$ and $\|\cdot\|_{W^s(\mathbb{R}^d)}$ are equivalent. □

In the sequel, we will need also, sometimes, fractional order Sobolev spaces that are based on L^p spaces. So first, for $p \in [1; +\infty]$, and for every exponent $m \in \mathbb{N}$, we set

$$W^{m,p}(\Omega) := \left\{ v \in L^p(\Omega) \;\middle|\; D^\alpha v \in L^p(\Omega), |\alpha| \leq m \right\},$$

which is a Banach space when endowed with the corresponding norm

$$\|v\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

for $v \in W^{m,p}(\Omega)$. Then, let θ be in $(0, 1)$, for a real-valued function v defined on Ω, we set:

$$|v|_{\theta,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|v(\mathbf{x}) - v(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{d+p\theta}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \right)^{\frac{1}{p}},$$

and for any $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we set $s = m + \theta$, $m \in \mathbb{N}$, $\theta \in (0, 1)$, and the fractional order Sobolev space $W^{s,p}(\Omega)$ is now

$$W^{s,p}(\Omega) := \left\{ v \in W^{m,p}(\Omega) \;\middle|\; |D^\alpha v|_{\theta,p,\Omega} < +\infty, |\alpha| = m \right\}.$$

This space is also a Banach space when endowed with the norm

$$\|v\|_{W^{s,p}(\Omega)} := \left(\|v\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} |D^\alpha v|_{\theta,p,\Omega}^p \right)^{\frac{1}{p}}.$$

Remark that, of course, $W^{s,2}(\Omega) = W^s(\Omega)$ with identical norms.

We will need furthermore the space of vector-valued square-integrable functions that admit a square-integrable distributional divergence

$$\mathbf{H}(\mathrm{div}\,;\Omega) := \left\{ \boldsymbol{\psi} \in L^2(\Omega;\mathbb{R}^d) \;\middle|\; \mathrm{div}\,\boldsymbol{\psi} \in L^2(\Omega) \right\}$$

and in this case the divergence of $\boldsymbol{\psi}$ verifies the following relationship:

$$\int_\Omega (\mathrm{div}\,\boldsymbol{\psi}(\mathbf{x}))\,\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_\Omega \boldsymbol{\psi}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

The space $\mathbf{H}(\mathrm{div}\,;\Omega)$ is endowed with the norm

$$\|\boldsymbol{\psi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)} := \left(\|\boldsymbol{\psi}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{\psi}\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

2.2.4 Some Useful Properties

First, we give a useful density result (see [207, Lemma 3.24] for the proof):

Proposition 2.5 *For any $s \geq 0$, the set $\mathcal{D}(\mathbb{R}^d)$ is dense into $H^s(\mathbb{R}^d)$.* □

Then, we provide a compactness result in fractional order Sobolev spaces that generalizes the Rellich-Kondrachov theorem (compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$). For its proof, see, for instance, the book of W. McLean [207, Theorem 3.27]:

Theorem 2.2 *Let Ω be an open, bounded, Lebesgue-measurable, and non-empty set of $\mathbb{R}^d$, and let s and t be two reals with $0 \leq s < t$. Then, $H^t(\Omega)$ is compactly embedded into $H^s(\Omega)$.* □

Another important result at this stage concerns the Sobolev embedding Theorem, that states precisely which Sobolev regularity is needed for a function to be at least continuous (more precisely Hölder). Notably, this depends on the dimension d, and, for instance, in two or three dimensions, functions that belong to $H^1(\mathbb{R}^d)$ are not necessarily continuous. This has important implications, especially because, in this situation, we cannot define straightforwardly their value punctually. Let us state the following theorem, whose detailed proof can be found in [207, Theorem 3.26]:

Theorem 2.3 *Let $d/2 < s < d/2 + 1$ and take $v \in H^s(\mathbb{R}^d)$, then, v is almost everywhere equal to a Hölder continuous function. More precisely, there holds:*

$$|v(\mathbf{x})| \leq C\|v\|_{s,\mathbb{R}^d}$$

and

$$|v(\mathbf{x}) - v(\mathbf{y})| \leq C\|v\|_{s,\mathbb{R}^d}|\mathbf{x} - \mathbf{y}|^{s-\frac{d}{2}}$$

for almost every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. □

As a result, in two dimensions, to ensure continuity, a Sobolev regularity strictly above 1 is necessary, and in three dimensions, this Sobolev regularity needs to be strictly above $\frac{3}{2}$. Remark also that the above theorem extends directly to functions in $H^s(\Omega)$, where Ω is a domain in $\mathbb{R}^d$, as a consequence of the definition of $H^s(\Omega)$ we introduced in Sect. 2.2.2.

Last but not least, an important, and difficult, result, which will play a strong role in proving some optimal error estimates, is about how the gradient of a function in $H^1(\Omega)$ vanishes on level sets. We quote the result below and refer to [193, Theorem 3.2] for the detailed proof:

Theorem 2.4 *Let* Ω *be an open, Lebesgue-measurable, and non-empty set of* $\mathbb{R}^d$, *and take* $v \in H^1(\Omega)$, *then, for any value* $c \in \mathbb{R}$, $\nabla v = 0$ *almost everywhere on* $E_c(v)$. □

2.3 Lipschitz Domains

To be specific about what happens on the boundary, and to give a meaning to values of functions or their derivative on the boundary, we need to add some geometric assumptions to our domain Ω. We will follow the most common assumption in the literature about partial differential equations and their approximation and consider Lipschitz domains.

As it will be seen, this choice has some strong advantages, since it is not too restrictive and includes a large class of domains in two or three dimensions, particularly a large class of polygonal or polyhedral domains. Indeed it is very convenient to describe in a first presentation all the numerical methods for polygonal or polyhedral domains, since in this case there is no error to take into account in the analysis for the approximation of the boundary. This is what will be often done in the next chapters, but not for the description of the continuous problems, so as to remain somehow general. Furthermore, there is a wide range of literature about this class of domains, or that make use of the Lipschitz assumption. Finally, we will see also that this formalism can be combined in a quite satisfying manner with the formalism of Sobolev spaces seen before.

We start with providing a precise definition and some characteristics of what a Lipschitz domain is. We focus on the notion of bi-Lipschitz homeomorphism, which is convenient both to describe a Lipschitz boundary, and to handle Sobolev spaces. Then, we state a particularly helpful result of partition of unity that will serve to build Sobolev spaces on the boundary of a domain. Finally, we provide more properties of Sobolev spaces on Lipschitz domains, and another characterization using the Slobodeckij semi-norm.

2.3.1 A First Definition Using Lipschitz Hypographs

Broadly speaking, for Ω to be a Lipschitz domain, its boundary needs to be described locally as the graph of a Lipschitz function. To state this more precisely, we follow the presentation provided in [207, 239].

Let us first take the case where the domain is simply

$$\widetilde{\Omega}_f := \{(\widetilde{\mathbf{x}}', \widetilde{x}_d) \in \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \mathbb{R} \mid \widetilde{x}_d < f(\widetilde{\mathbf{x}}')\} \subset \mathbb{R}^d,$$

where $\mathcal{B}_{d-1}(\mathbf{0}, 1)$ is the open unit ball of center $\mathbf{0}$ in $\mathbb{R}^{d-1}$ and $f : \mathcal{B}_{d-1}(\mathbf{0}, 1) \to \mathbb{R}$ is a Lipschitz function, i.e., there exists $L > 0$ such that

$$|f(\widetilde{\mathbf{x}}') - f(\widetilde{\mathbf{y}}')| \leq L|\widetilde{\mathbf{x}}' - \widetilde{\mathbf{y}}'|$$

for all $\widetilde{\mathbf{x}}', \widetilde{\mathbf{y}}' \in \mathcal{B}_{d-1}(\mathbf{0}, 1)$. Such a domain is called a Lipschitz hypograph. We also define interior and exterior neighborhoods of $\widetilde{\Omega}_f$, as well as its boundary: we introduce a parameter $\eta > 0$, that represents a thickness, and set:

$$\begin{aligned}
\widetilde{\Omega}_f^{\eta,+} &:= \{(\widetilde{\mathbf{x}}', \widetilde{x}_d) \in \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \mathbb{R} \mid f(\widetilde{\mathbf{x}}') < \widetilde{x}_d < f(\widetilde{\mathbf{x}}') + \eta\}, \\
\widetilde{\Omega}_f^{\eta,-} &:= \{(\widetilde{\mathbf{x}}', \widetilde{x}_d) \in \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \mathbb{R} \mid f(\widetilde{\mathbf{x}}') - \eta < \widetilde{x}_d < f(\widetilde{\mathbf{x}}')\}, \\
\widetilde{\Gamma}_f &:= \{(\widetilde{\mathbf{x}}', \widetilde{x}_d) \in \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \mathbb{R} \mid f(\widetilde{\mathbf{x}}') = \widetilde{x}_d\}.
\end{aligned}$$

Then, we are ready to define a (strong) Lipschitz domain. For the sake of simplicity, we suppose that Ω is bounded, but this assumption can be omitted if necessary (see [207]):

Definition 2.5 Let Ω be an open, measurable, non-empty and bounded subset of $\mathbb{R}^d$, whose boundary is denoted $\Gamma := \partial\Omega$. The domain Ω is called a (strong) Lipschitz domain if it satisfies the following property: for every point $\mathbf{x} \in \Gamma$, there exists a Lipschitz function f, a parameter $\eta > 0$, an invertible orthogonal transformation $\mathbf{T}$ (i.e., a rigid motion composed with a scaling) such that

$$\begin{aligned}
\mathbf{T}(\widetilde{\Omega}_f^{\eta,+}) &\subset (\overline{\Omega})^c, \\
\mathbf{T}(\widetilde{\Omega}_f^{\eta,-}) &\subset \Omega, \\
\mathbf{T}(\widetilde{\Gamma}_f) &\subset \Gamma.
\end{aligned}$$

Note that in the above definition, the function f, the parameter η and the transformation $\mathbf{T}$ depend on $\mathbf{x}$ (we do not mention it explicitly to alleviate the notation). Since $\overline{\Omega}$ is compact, this definition implies that Ω can be recovered by a finite collection of sets that are, locally, Lipschitz hypographs (see Fig. 2.1).

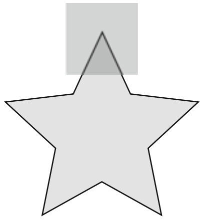

Fig. 2.1 A (strong) Lipschitz domain: the star Ω (in light gray), with boundary Γ in black. Each portion of the boundary defines locally a Lipschitz hypograph (here in dark gray, for instance)

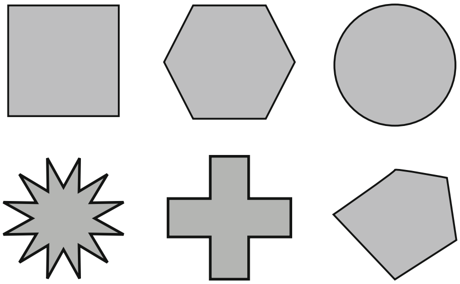

Fig. 2.2 All the domains on the right Ω (gray) in $\mathbb{R}^2$ are Lipschitz domains (their boundary Γ, in black, is locally the graph of a Lipschitz function). These domains can be polygons, or not, and can be convex or not

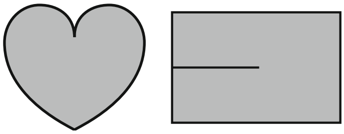

Fig. 2.3 The two above domains Ω (gray) in $\mathbb{R}^2$ are not Lipschitz domains. Unfortunately, the nice heart on the left is not Lipschitz, and this is the case of all the domains with cusps. Neither the crack domain on the right is Lipschitz

2.3.2 Examples and Counterexamples of Lipschitz Domains

First, all bounded domains with very smooth boundaries, such as circles, ellipses, etc., are Lipschitz domains. More importantly, in two dimensions, all polygons with interior angles strictly lower than 2π are Lipschitz. In three dimensions also, a large variety of polyhedra are Lipschitz. Remark that domains can be Lipschitz though they are not convex, or not connected. See Fig. 2.2 for some illustrations.

However, domains with cracks or tips are not Lipschitz, as well as domains with a cusp. There is indeed an impossibility near some specific points (the cusp, the crack tip) to represent them locally as the hypograph of a Lipschitz function. This is unfortunate for contact mechanics, in which domain with cracks are frequent and represent many relevant situations in engineering. As a result, they are object of a specific study, which is out the scope of this book. Moreover, in three dimensions, the so-called double-brick is not Lipschitz. See Fig. 2.3 for some illustrations. Some other illustrations of examples and counterexamples can be found, in, e.g., [141, 207, 214, 239].

Remark 2.3 In fact, it can be proven that every domain Ω which is both convex and bounded is Lipschitz, see, e.g., [101, Lemma 2.3]. □

2.3.3 Bi-Lipschitz Homeomorphisms

As we presented earlier, fractional order Sobolev spaces are easier to define and handle in the whole space or in a half-space, where it is possible to make use of the Fourier transform. As a result, for an arbitrary domain, it should be very convenient to introduce maps to a half-space. This is why we provide another characterization of Lipschitz domains. For this purpose, we introduce the following notion:

Definition 2.6 Let $\mho$ be an open subset of $\mathbb{R}^d$. The application $\mathbf{F} : \mho \to \Omega_{\mathbf{F}} := \mathbf{F}(\mho) \subset \mathbb{R}^d$ is a bi-Lipschitz homeomorphism if $\mathbf{F}$ is invertible from $\mho$ to $\Omega_{\mathbf{F}}$ and if both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are Lipschitz. This means there exists positive constants c and C such that, for every $\mathbf{x}$ and $\mathbf{y}$ in $\mho$, there holds

$$c\,|\mathbf{x}-\mathbf{y}| \le |\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})| \le C\,|\mathbf{x}-\mathbf{y}|.$$

□

We state here an important property of Lipschitz transformations, which is a difficult result, and that will be very helpful in the sequel. In fact, it is a fundamental tool that will help to prove some further results, such as the trace theorem, by going back to fractional order Sobolev spaces defined in $\mathbb{R}^d$. This property is the following: Lipschitz transformations preserve H^s spaces, for $0 \le s \le 1$. We state this fact precisely below:

Theorem 2.5 *Let $s \in [0,1]$. Let $\mho$ be an open set in $\mathbb{R}^d$ and $\mathbf{F} : \mho \to \Omega_{\mathbf{F}}$ a bi-Lipschitz homeomorphism. Then, there holds*

$$v \in H^s(\Omega_{\mathbf{F}}) \quad \textit{if and only if} \quad v \circ \mathbf{F} \in H^s(\mho)$$

and moreover there exists $c, C > 0$ such that:

$$c\|v\|_{s,\Omega_{\mathbf{F}}} \le \|v\circ\mathbf{F}\|_{s,\mho} \le C\|v\|_{s,\Omega_{\mathbf{F}}},$$

for any function $v \in H^s(\Omega_{\mathbf{F}})$. □

Proof In the case $s = 0$ (i.e., $v \in L^2(\Omega_{\mathbf{F}})$), the statement is just a consequence of the change of variable formula in the integral (due to Rademacher's theorem, the Jacobian of $\mathbf{F}$ is well-defined and bounded from below almost everywhere). The proof for $s = 1$ is rather technical and can be found in [239, Corollary 3.1]. For the intermediate cases $s \in (0,1)$, we use Hilbertian interpolation [207, Theorem B.8]. □

Remark that the second statement of the Theorem tells us that the mapping $H^s(\Omega_F) \mapsto H^s(\mho)$ is bounded. The theorem also motivates in some sense the introduction of the notion of Lipschitz domain, which is a (quite) practical framework to work with partial differential equations (or even integral equations on the boundary) and related fractional order Sobolev spaces. Particularly, they allow to prove some results on simple domains (rectangles or the whole space) and then map them for a general Lipschitz domain.

2.3.4 Lipschitz Domains as a Collection of Lipschitz Mappings

Let us go back to the characterization of a Lipschitz domain. We now take $\mho := \mathcal{B}_{d-1}(\mathbf{0}, 1) \times (-1, 1)$ as a reference cylinder in $\mathbb{R}^d$ and define then the following subdomains of $\mho$:

$$\begin{aligned} \mho^+ &:= \mathcal{B}_{d-1}(\mathbf{0}, 1) \times (0, 1), \\ \mho^- &:= \mathcal{B}_{d-1}(\mathbf{0}, 1) \times (-1, 0), \\ \Sigma &:= \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \{0\}. \end{aligned}$$

Take now $\mathbf{x} \in \Gamma$, and corresponding $(f, \eta, \mathbf{T})$ associated with above Definition 2.5. We introduce the mapping $\widetilde{\mathbf{F}} : \mho \to \mathcal{B}_{d-1}(\mathbf{0}, 1) \times \mathbb{R}$ defined by

$$\widetilde{\mathbf{F}}(\widetilde{\mathbf{x}}', \widetilde{x}_d) = (\widetilde{\mathbf{x}}', f(\widetilde{\mathbf{x}}') + \eta\, \widetilde{x}_d).$$

It is by construction a bi-Lipschitz homeomorphism, built such that it realizes exactly the following mappings:

$$\widetilde{\mathbf{F}}(\mho^+) = \widetilde{\Omega}_f^{\eta,+}, \quad \widetilde{\mathbf{F}}(\mho^-) = \widetilde{\Omega}_f^{\eta,-}, \quad \widetilde{\mathbf{F}}(\Sigma) = \widetilde{\Gamma}_f.$$

Then, take $\mathbf{F} := \mathbf{T} \circ \widetilde{\mathbf{F}} : \mho \to \mathbb{R}^d$: it is a bi-Lipschitz homeomorphism that verifies

$$\mathbf{F}(\mho^+) \subset (\overline{\Omega})^c, \quad \mathbf{F}(\mho^-) \subset \Omega, \quad \mathbf{F}(\Sigma) \subset \Gamma.$$

So to each point $\mathbf{x}$ on the boundary Γ, there exists an application $\mathbf{F}$ that maps $\mho$ to a set $\mathbf{F}(\mho)$ that covers Γ. Since Γ is compact, we can extract a finite covering of Γ by such sets. We state this property as follows:

Proposition 2.6 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, then there exists a finite collection of charts $(\mathbf{F}_j)$ and open sets (Ω_j), $j = 1, \ldots, J$, with*

$$\mathbf{F}_j : \mho \to \Omega_j := \mathbf{F}_j(\mho) \subset \mathbb{R}^d,$$

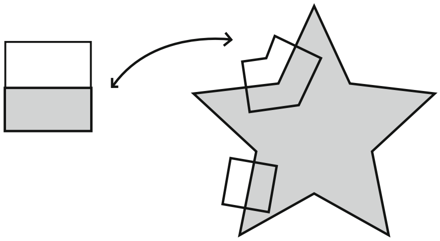

Fig. 2.4 Covering of a Lipschitz domain Ω (in gray, on the right). The boundary Γ of Ω is such that it can be covered by a finite union of sets Ω_j that are images of $\mho$ (on the left) by a bi-Lipschitz homeomorphism. Moreover, this covering preserves the partition of $\mho$ into $\mho^-$ (gray), Σ (black) and $\mho^+$ (white)

that are bi-Lipschitz homeomorphisms (for $j = 1, \dots, J$) and that verify the following relationships:

$$\begin{aligned} \Omega_j^+ &:= \mathbf{F}_j(\mho^+) = \Omega_j \cap (\overline{\Omega})^c, \\ \Omega_j^- &:= \mathbf{F}_j(\mho^-) = \Omega_j \cap \Omega, \\ \Gamma_j &:= \mathbf{F}_j(\Sigma) = \Omega_j \cap \Gamma, \quad \forall j = 1, \dots, J. \end{aligned}$$

Moreover, there holds

$$\bigcup_{j=1}^{J} \Gamma_j = \Gamma.$$

The situation described in Proposition 2.6 above is illustrated Fig. 2.4.

Finally, note that, thanks to Rademacher's theorem, a surface measure can be associated with a Lipschitz boundary. We will denote this measure by $\mathrm{d}s(\mathbf{x})$, for $\mathbf{x} \in \Gamma$ ($\mathbf{x}$ will be sometimes omitted when it is not necessary). Moreover, an outward unit normal vector can be defined almost everywhere. We will denote it by $\mathbf{n}(\mathbf{x})$, for $\mathbf{x} \in \Gamma$ such that this vector exists. Remark that, for instance, for a polygonal domain, the set of points where the normal is not defined is non-empty, but finite. We will see in next chapters it has some consequences for contact problems. Detailed construction of this surface measure and this normal vector can be found, in, e.g., [207] or [239].

2.3.5 Partition of Unity for a Lipschitz Boundary

We provide here a very useful result of partition of unity, whose detailed proof can be found, in, e.g., [239, Section 3.4]:

Proposition 2.7 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, associated with the covering $(\mathbf{F}_j, \Omega_j)_{j=1,\ldots,J}$ described in Proposition 2.6, then there exists a partition of unity, i.e., a finite collection of smooth functions $(\varphi_j)_{j=0,\ldots,J}$ ($\varphi_j \in \mathcal{D}(\mathbb{R}^d)$, $j = 0, \ldots, J$), with the following properties:*

$$\operatorname{supp}\varphi_0 \subset \Omega, \quad \operatorname{supp}\varphi_j \subset \Omega_j \ (j = 1, \ldots, J),$$

and, in a neighborhood of Ω:

$$\sum_{j=0}^{J} \varphi_j = 1.$$

This property plays a key role in the definition of Sobolev spaces on the boundary, as well as in the construction of the trace and lifting operators, as we will see below.

2.3.6 Sobolev Spaces on the Boundary

Let Ω be a Lipschitz domain in $\mathbb{R}^d$, with boundary denoted by $\Gamma := \partial\Omega$. To define a Sobolev space on the boundary Γ, the idea is to go back ("pullback") to the "whole" subspace $\mathbb{R}^{d-1}$ (the straight line in two dimensions), where it is easy to define a fractional order Sobolev space.

To this purpose, take the covering $(\mathbf{F}_j, \Omega_j)_{j=1,\ldots,J}$ and the corresponding partition of unity $(\varphi_j)_{j=0,\ldots,J}$ from the above Proposition 2.7. Then, consider a real $s \in [0, 1]$, that represents the Sobolev regularity (for spaces on Lipschitz boundaries, it is not always possible to consider other positive values). For a function $w \in L^2(\Gamma)$, let us write

$$w = w \sum_{j=0}^{J} \varphi_j = \sum_{j=0}^{J} (w\varphi_j).$$

Remark that each term of the sum above, $(w\varphi_j)$, is equal to 0 out of Γ_j, since the support of φ_j is contained in Ω_j. We then use the chart $\mathbf{F}_j$ to map $(w\varphi_j)$ back to Σ. Since $\varphi_j \in \mathcal{D}(\Omega_j)$, we can extend the corresponding function, $(w\varphi_j) \circ \mathbf{F}_j$, from Σ to the whole subspace $\mathbb{R}^{d-1}$. We note this extension as follows, for $\widetilde{\mathbf{x}}' \in \mathbb{R}^{d-1}$:

$$\widetilde{(w\varphi_j) \circ \mathbf{F}_j}(\widetilde{\mathbf{x}}') := \begin{cases} (w\varphi_j) \circ \mathbf{F}_j(\widetilde{\mathbf{x}}'), & \text{if } \mathbf{x}' \in \Sigma, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Note indeed that $(w\varphi_j) \circ \mathbf{F}_j$ is equal to zero in a neighborhood of $\partial\Sigma$ (the Lipschitz map $\mathbf{F}_j$ preserves the fact that $w\varphi_j$ has compact support). As a result, we can associate a fractional Sobolev norm to the above function. This motivates the introduction of the following norm:

$$\|w\|_{s,\Gamma} := \left(\sum_{j=1}^{J} \|\widetilde{(w\varphi_j)} \circ \mathbf{F}_j\|_{s,\mathbb{R}^{d-1}} \right) \tag{2.4}$$

for $w \in L^2(\Gamma)$. Then, we define

$$H^s(\Gamma) := \{w \in L^2(\Gamma) \mid \|w\|_{s,\Gamma} < +\infty\}$$

and $(\cdot,\cdot)_{s,\Gamma}$ the corresponding scalar product defined from the norm thanks to polarity identities.

One can check that this defines a norm (and a scalar product), and that this definition does not depend on the choice of the partition of unity, in the sense that the norms are equivalent for two distinct partitions of unity. One can check also that $H^s(\Gamma)$ is a separable Hilbert space.

As before, we denote $H^s(\Gamma;\mathbb{R}^d)$ the Sobolev space of vector-valued functions on the boundary and keep the same notation $\|\cdot\|_{s,\Gamma}$ as in the scalar case.

2.3.7 Sobolev Spaces on a Part of the Boundary

Since, in the next chapters, we will deal with boundary conditions defined only on a part of the boundary, we need to introduce appropriate Sobolev spaces. Following [239], we consider an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, with boundary $\Gamma := \partial\Omega$. Let us take Γ_1 a relatively open subset in Γ, of positive Lebesgue measure ($|\Gamma_1| > 0$), and define Γ_2 the complementary set in Γ:

$$\Gamma_2 := \Gamma \setminus \overline{\Gamma_1}.$$

First, we simply define, for $s \in [0;1]$

$$H^s(\Gamma_1) := \{w|_{\Gamma_1} \mid w \in H^s(\Gamma)\},$$

where $H^s(\Gamma)$ has been defined previously. Note that the restriction operation $w|_{\Gamma_1}$ is meaningful, since $w \in L^2(\Gamma)$ ($H^s(\Gamma) \subset L^2(\Gamma)$) and $|\Gamma_1| > 0$. The space $H^s(\Gamma_1)$ is a separable Hilbert space when endowed with the norm

$$\|w_1\|_{s,\Gamma_1} := \inf\{\|w\|_{s,\Gamma} \mid w \in H^s(\Gamma),\ w_1 = w|_{\Gamma_1}\}, \tag{2.5}$$

for $w_1 \in H^s(\Gamma_1)$. We also introduce the Lions-Magenes space

$$H^s_{00}(\Gamma_1) := \{w|_{\Gamma_1} \mid w \in H^s(\Gamma),\ w|_{\Gamma_2} = 0\} = \{w_1 \in H^s(\Gamma_1) \mid \widetilde{w_1} \in H^s(\Gamma)\},$$

where $\widetilde{w_1}$ denotes the extension of w_1 by 0 outside of Γ_1:

$$\widetilde{w_1}|_{\Gamma_1} = w_1, \quad \widetilde{w_1}|_{\Gamma_2} = 0.$$

The space $H^s_{00}(\Gamma_1)$ is endowed with the norm

$$\|w_1\|_{H^s_{00}(\Gamma_1)} := \|\widetilde{w_1}\|_{s,\Gamma}.$$

There holds

$$H^s_{00}(\Gamma_1) \subset H^s(\Gamma_1) \subset L^2(\Gamma_1)$$

with bounded injections. Moreover, the injections $H^s_{00}(\Gamma_1) \subset L^2(\Gamma_1)$ and $H^s(\Gamma_1) \subset L^2(\Gamma_1)$ are dense.

As usual, we denote $H^s(\Gamma_1; \mathbb{R}^d)$ the Sobolev space of vector-valued functions on Γ_1, and keep the same notation $\|\cdot\|_{s,\Gamma_1}$ as in the scalar case (the same convention holds also for the Lions-Magenes space).

2.3.8 Other Density Theorems

We will need in the sequel the following density result. Let us take an open set $\Omega \subset \mathbb{R}^d$ and first define the set of functions:

$$\mathcal{D}(\overline{\Omega}) := \{v|_\Omega \mid v \in \mathcal{D}(\mathbb{R}^d)\}.$$

Then, there holds (see [207, Theorem 3.29]):

Proposition 2.8 *For all $s \geq 0$ and $\Omega \subset \mathbb{R}^d$ a Lipschitz domain, the set $\mathcal{D}(\overline{\Omega})$ is dense in $H^s(\Omega)$.* □

Similarly, there also holds, for vector-valued functions (see [137, Theorem 2.4]):

Proposition 2.9 *For $\Omega \subset \mathbb{R}^d$ a Lipschitz domain, the set $\mathcal{D}(\overline{\Omega}; \mathbb{R}^d)$ is dense in* $\mathbf{H}(\mathrm{div}\,; \Omega)$. □

2.3.9 The Slobodeckij Semi-norm, Once Again

For Lipschitz domains, there is an equivalence between Sobolev spaces defined using Fourier transform and Sobolev spaces defined using the Slobodeckij semi-norm. We state the precise result below (see, e.g., [207, Theorem 3.30]) that extends previous Proposition 2.4:

Proposition 2.10 *For any $s \geq 0$ and $\Omega \subset \mathbb{R}^d$ a Lipschitz domain we have*

$$W^s(\Omega) = H^s(\Omega)$$

and the norms $\|\cdot\|_{s,\Omega}$ *and* $\|\cdot\|_{W^s(\Omega)}$ *are equivalent.* □

A similar result holds for Sobolev spaces defined on the boundary $\Gamma := \partial\Omega$, or on a part of the boundary, as stated below:

Proposition 2.11 *Let* $\theta \in (0, 1)$ *and* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain of boundary* $\Gamma := \partial\Omega$. *Let* Γ_1 *be a relatively open subset in* Γ, *of positive Lebesgue measure. For* $w \in H^\theta(\Gamma_1)$, *let us define the semi-norm*

$$|w|_{\theta,\Gamma_1} := \left(\int_{\Gamma_1}\int_{\Gamma_1} \frac{(w(\mathbf{x}) - w(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^{d-1+2\theta}}\, \mathrm{d}s(\mathbf{x})\, \mathrm{d}s(\mathbf{y})\right)^{\frac{1}{2}}$$

and the norm

$$\|w\|_{W^\theta(\Gamma_1)} := \left(\|w\|_{0,\Gamma_1}^2 + |w|_{\theta,\Gamma_1}^2\right)^{\frac{1}{2}}.$$

Then, the norms $\|\cdot\|_{\theta,\Gamma_1}$, *defined in* (2.5), *and* $\|\cdot\|_{W^\theta(\Gamma_1)}$, *defined above, are equivalent.* □

Based on these equivalences, we introduce some useful notations for semi-norms of functions defined in Sobolev spaces. For $m \in \mathbb{N}$, and $v \in H^m(\Omega)$, we first define the semi-norm

$$|v|_{m,\Omega} := \left(\sum_{|\alpha|=m} \|D^\alpha v\|_{0,\Omega}^2\right)^{\frac{1}{2}}.$$

Then, for any $s \in \mathbb{R}_+ \setminus \mathbb{N}$, still we set $s = m + \theta$, $m \in \mathbb{N}$, $\theta \in (0, 1)$, and, for $v \in H^s(\Omega)$, we define the semi-norm

$$|v|_{s,\Omega} := \left(\sum_{|\alpha|=m} |D^\alpha v|_{\theta,\Omega}^2\right)^{\frac{1}{2}}.$$

Last but not least, we will need the following result for the convergence analysis of a mixed finite element method for Coulomb friction, in Chap. 9:

Proposition 2.12 *Let* $d \geq 1$ *and* Ω *be either equal to* $\mathbb{R}^d$ *or an open bounded Lipschitz domain in* $\mathbb{R}^d$. *Let* $s > d/2$. *Then,* $H^s(\Omega)$ *is a Banach (multiplicative) algebra: the application*

$$H^s(\Omega) \times H^s(\Omega) \ni (v, w) \mapsto vw \in H^s(\Omega)$$

is a well-defined, continuous bilinear map. Particularly, there exists $C > 0$ *such that, for* $v, w \in H^s(\Omega)$

$$\|vw\|_{s,\Omega} \leq C\|v\|_{s,\Omega}\|w\|_{s,\Omega}.$$

Proof This result is a consequence of the Sobolev embedding Theorem 2.3 and can be obtained as a special case of [141, Theorem 1.4.4.2] (see also [270, Théorème 2]). □

2.3.10 Domains with Smoother Boundary

To end this section, we need to mention conventions for domains with smoother boundary, which will be sometimes useful in next chapters. Following W. McLean [207, Chapter 3], for Ω an open non-empty set in $\mathbb{R}^d$, and for $k \in \mathbb{N}$, we define a $\mathscr{C}^k$ domain when Definition 2.5 applies, but with the functions f being $\mathscr{C}^k$ instead of Lipschitz. Also, for $\mu \in (0; 1]$, we define a $\mathscr{C}^{k,\mu}$ domain, if, moreover, the derivatives of f of order k are Hölder-continuous with exponent μ: with this convention, a Lipschitz domain is a $\mathscr{C}^{0,1}$ domain.

2.4 Trace and Lifting Operators

As we have seen concerning Sobolev embeddings, for a function that belongs to $H^1(\Omega)$, where Ω is an open set of $\mathbb{R}^d$, it is not obvious to define its value (trace) on the boundary Γ since this function is not necessarily continuous.

Moreover, we need to characterize precisely the Sobolev space of traces on Γ, for functions that belong to a Sobolev space of given regularity $H^s(\Omega)$, and see precisely how to map functions from the whole domain Ω to the boundary (trace), and conversely (lifting). The corresponding results will be very useful for the numerical analysis of the forthcoming boundary value problems we will study in the next chapters.

For technical reasons, it is much more convenient to analyze what happens for the trace on a hyperplane, of a function that belongs to the whole domain. This allows to work with the Fourier transform which is behind the definition of fractional order Sobolev spaces. In a second step, we state the main theorem for a Lipschitz domain, using all the technology provided in the previous section about Lipschitz domains and mappings. The same results hold to define a lifting operator that allows to extend a function from the boundary to the whole domain.

2.4.1 The Trace on a Hyperplane

We reproduce here the proof in [161] and [207]. For a smooth function $v \in \mathcal{D}(\mathbb{R}^d)$, we define the trace function on the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ as

$$\Upsilon v : \mathbf{x}' \mapsto v(\mathbf{x}', 0)$$

for $\mathbf{x}' \in \mathbb{R}^{d-1}$. The notation Υ stands here for the trace operator that, for the moment, maps linearly $\mathcal{D}(\mathbb{R}^d)$ into $\mathcal{D}(\mathbb{R}^{d-1})$. We can state the following result:

Proposition 2.13 *Let $s > \frac{1}{2}$, there is a unique extension of the trace operator Υ to a bounded linear operator*

$$\Upsilon : H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1}).$$

Proof We take $v \in \mathcal{D}(\mathbb{R}^d)$ and will conclude using density (Proposition 2.8). First, let us make use of the Fourier inversion formula, for $\mathbf{x}' \in \mathbb{R}^{d-1}$:

$$(\Upsilon v)(\mathbf{x}') = v(\mathbf{x}', 0) = \bar{\mathcal{F}}(\hat{v})(\mathbf{x}', 0) = \left[\int_{\mathbb{R}^d} e^{2i\pi \boldsymbol{\xi}\cdot\mathbf{x}} \hat{v}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\right]_{\mathbf{x}=(\mathbf{x}',0)}.$$

We use the notation $\boldsymbol{\xi} = (\boldsymbol{\xi}', \xi_d) \in \mathbb{R}^d$, with $\boldsymbol{\xi}' \in \mathbb{R}^{d-1}$ and $\xi_d \in \mathbb{R}$. We apply Fubini's Theorem and get

$$(\Upsilon v)(\mathbf{x}') = \int_{\mathbb{R}^d} e^{2i\pi \boldsymbol{\xi}'\cdot\mathbf{x}'} \hat{v}(\boldsymbol{\xi}', \xi_d)\, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^{d-1}} e^{2i\pi \boldsymbol{\xi}'\cdot\mathbf{x}'} \left(\int_{\mathbb{R}} \hat{v}(\boldsymbol{\xi}', \xi_d)\, \mathrm{d}\xi_d\right) \mathrm{d}\boldsymbol{\xi}'.$$

This allows to obtain the Fourier transform of Υv, which is:

$$\widehat{\Upsilon v}(\boldsymbol{\xi}') = \int_{\mathbb{R}} \hat{v}(\boldsymbol{\xi}', \xi_d)\, \mathrm{d}\xi_d.$$

We introduce the corresponding term associated with the H^s-norm in the Fourier domain:

$$\widehat{\Upsilon v}(\boldsymbol{\xi}') = \int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^{-\frac{s}{2}} (1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{v}(\boldsymbol{\xi})\, \mathrm{d}\xi_d.$$

Then, we apply the Cauchy–Schwarz inequality:

$$|\widehat{\Upsilon v}(\boldsymbol{\xi}')|^2 \le \left(\int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^{-s}\, \mathrm{d}\xi_d\right) \left(\int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^{s} |\hat{v}(\boldsymbol{\xi})|^2\, \mathrm{d}\xi_d\right). \qquad (2.6)$$

Let us introduce the following notation for the first term at the right of the above inequality:

$$\begin{aligned} C_s(\boldsymbol{\xi}') &:= \int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^{-s}\, \mathrm{d}\xi_d \\ &= \int_{\mathbb{R}} \frac{1}{(1 + |\boldsymbol{\xi}'|^2 + |\xi_d|^2)^s}\, \mathrm{d}\xi_d. \end{aligned}$$

To evaluate this term, we use the substitution $\xi_d = (1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}} t$, for $t \in \mathbb{R}$. Note that this is here that we lose the Sobolev regularity of $\frac{1}{2}$. This yields:

$$C_s(\boldsymbol{\xi}') = \int_{\mathbb{R}} \frac{1}{((1 + |\boldsymbol{\xi}'|^2)(1 + t^2))^s} (1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}} \mathrm{d}t$$
$$= \frac{1}{(1 + |\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}}} \underbrace{\int_{\mathbb{R}} \frac{1}{(1 + t^2)^s} \mathrm{d}t}_{C_s} .$$

This last integral C_s is finite provided that $s > \frac{1}{2}$. Using this last equality in Eq. (2.6) we get:

$$(1 + |\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}} |\widehat{\Upsilon v}(\boldsymbol{\xi}')|^2 \le C_s \left(\int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^s |\hat{v}(\boldsymbol{\xi})|^2 \, \mathrm{d}\xi_d \right).$$

We perform integration with respect to $\boldsymbol{\xi}'$:

$$\int_{\mathbb{R}^{d-1}} (1 + |\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}} |\widehat{\Upsilon v}(\boldsymbol{\xi}')|^2 \, \mathrm{d}\boldsymbol{\xi}' \le C_s \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} (1 + |\boldsymbol{\xi}|^2)^s |\hat{v}(\boldsymbol{\xi})|^2 \, \mathrm{d}\xi_d \right) \mathrm{d}\boldsymbol{\xi}'.$$

Finally, we apply Fubini's Theorem:

$$\|\Upsilon v\|^2_{s-\frac{1}{2}, \mathbb{R}^{d-1}} \le C_s \|v\|^2_{s, \mathbb{R}^d}.$$

This ends the proof. □

Remark finally that the trace constant C_s depends on the Sobolev regularity s.

2.4.2 The Trace on a Lipschitz Boundary

With the material presented previously and the Trace Theorem on a hyperplane, we are ready to prove the main theorem of this section, which is the Trace Theorem for a Lipschitz domain (as in [207, Theorem 3.37]). Let us go.

Theorem 2.6 *Let* $\frac{1}{2} < s \le 1$ *and* Ω *be a Lipschitz domain in* $\mathbb{R}^d$. *Denote* $\Gamma := \partial\Omega$ *its boundary. The trace mapping* $\Upsilon : v \mapsto v|_\Gamma$, *well-defined in* $\mathcal{D}(\overline{\Omega})$, *can be uniquely extended by density to the linear bounded operator*

$$\Upsilon : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma).$$

Particularly, there holds the following trace inequality*:*

$$\|\Upsilon v\|_{s-\frac{1}{2},\Gamma} \le C\|v\|_{s,\Omega} \tag{2.7}$$

for any $v \in H^s(\Omega)$, where $C > 0$ does not depend on v, but of the Sobolev regularity s (and of Ω). □

Proof First, we need the covering $(\mathbf{F}_j, \Omega_j)_{j=1,\ldots,J}$ as well as the corresponding partition of unity $(\varphi_j)_{j=0,\ldots,J}$ from Proposition 2.7. Then, let us take $v \in \mathcal{D}(\overline{\Omega})$, and $1/2 < s \le 1$. Since $v|_\Gamma \in \mathscr{C}^\infty(\Gamma)$, there holds also $v|_\Gamma \in H^{s-\frac{1}{2}}(\Gamma)$. Recall the definition of the Sobolev norm on Γ:

$$\|v|_\Gamma\|_{s-\frac{1}{2},\Gamma} = \left(\sum_{j=1}^{J} \|\widetilde{(v|_\Gamma\varphi_j)} \circ \mathbf{F}_j\|_{s-\frac{1}{2},\mathbb{R}^{d-1}}\right).$$

For $j = 1, \ldots, J$ Proposition 2.13 above allows to bound each term of the sum as follows:

$$\|\widetilde{(v|_\Gamma\varphi_j)} \circ \mathbf{F}_j\|_{s-\frac{1}{2},\mathbb{R}^{d-1}} \le C_s\|\widetilde{(v\varphi_j)} \circ \mathbf{F}_j\|_{s,\mathbb{R}^d} = C_s\|(v\varphi_j) \circ \mathbf{F}_j\|_{s,\mho}.$$

The last equality just comes from the definition of the Sobolev norm on a domain, since $(v\varphi_j) \circ \mathbf{F}_j$ vanishes near the boundary $\partial\mho$. Then, we use Theorem 2.5 to go back to the domain Ω, with preservation of the H^s-Sobolev norm:

$$\|(v\varphi_j) \circ \mathbf{F}_j\|_{s,\mho} \le C_j\|v\varphi_j\|_{s,\Omega_j}$$

with $C_j > 0$ a constant. This yields:

$$\|\widetilde{(v|_\Gamma\varphi_j)} \circ \mathbf{F}_j\|_{s-\frac{1}{2},\mathbb{R}^{d-1}} \le C_s C_j\|v\varphi_j\|_{s,\Omega_j}.$$

There remains to sum from $j = 1$ to J and to use an equivalence of norms for the partition of unity (coming from [207, Theorem 3.20]):

$$\|v|_\Gamma\|_{s-\frac{1}{2},\Gamma} \le C_s\left(\sum_{j=1}^{J} C_j\|v\varphi_j\|_{s,\Omega_j}\right) \le C\|v\|_{s,\Omega}.$$

The proof is concluded by density. □

Remark that, contrarily to what happened for a hyperplane, we are limited here in Sobolev regularity, with $\frac{1}{2} < s \le 1$. This limitation comes from the assumption of a Lipschitz domain. We can overcome this limitation by assuming that the boundary is smoother. We state here the precise result, which will be helpful in the sequel (see, still, [207, Theorem 3.37], for the detailed proof).

Theorem 2.7 *Let $k \geq 1$, $\frac{1}{2} < s \leq k$ and Ω be a $\mathscr{C}^{k-1,1}$ domain in $\mathbb{R}^d$. Denote $\Gamma := \partial\Omega$ its boundary. The trace mapping $\Upsilon : v \mapsto v|_\Gamma$, well-defined in $\mathcal{D}(\overline{\Omega})$, can be uniquely extended by density to the linear bounded operator*

$$\Upsilon : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma),$$

with the trace inequality*:*

$$\|\Upsilon v\|_{s-\frac{1}{2},\Gamma} \leq C\|v\|_{s,\Omega} \tag{2.8}$$

for any $v \in H^s(\Omega)$. □

For a polygon or a polyhedron, the above result is not useful, but, if we restrict the trace to an edge or face, we can expect to overcome the regularity restriction of Theorem 2.6 for functions with a Sobolev regularity larger than 1, as stated below (see Bernardi et al. [38, Corollary 4.3, p.21] for a more general statement and the proof).

Theorem 2.8 *Let $\frac{1}{2} < s$ and Ω be a Lipschitz domain in $\mathbb{R}^d$, with polygonal ($d = 2$) or polyhedral ($d = 3$) boundary $\Gamma := \partial\Omega$, divided into J edges ($d = 2$) or (polygonal) faces ($d = 3$) $\Gamma_1, \ldots, \Gamma_J$. The trace mapping*

$$\Upsilon : v \mapsto (v|_{\Gamma_1}, \ldots, v|_{\Gamma_J}),$$

well-defined in $\mathcal{D}(\overline{\Omega})$, can be uniquely extended by density to the linear bounded operator

$$\Upsilon : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma_1) \times \cdots \times H^{s-\frac{1}{2}}(\Gamma_J).$$

Particularly, there holds the following trace inequality*:*

$$\|\Upsilon v|_{\Gamma_j}\|_{s-\frac{1}{2},\Gamma_j} \leq C\|v\|_{s,\Omega} \tag{2.9}$$

for $j = 1, \ldots, J$ and any $v \in H^s(\Omega)$, where $C > 0$ does not depend on v, but of the Sobolev regularity s (and of Ω). □

Above theorem will be particularly useful to derive optimal error estimates for contact, when the contact boundary is an edge or a planar face.

2.4.3 The Lifting Operator

Here we state a result corresponding to the existence of a (nonunique) bounded lifting operator.

Theorem 2.9 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ of boundary $\Gamma := \partial\Omega$. Let $\frac{1}{2} < s \leq 1$. There exists $C > 0$ such that for all $w \in H^{s-\frac{1}{2}}(\Gamma)$, there is (at least) one function v_w in $H^s(\Omega)$ that verifies*

$$\Upsilon v_w = w, \quad \|v_w\|_{s,\Omega} \leq C\|w\|_{s-\frac{1}{2},\Gamma}. \tag{2.10}$$

Such a function v_w is called a lifting (or an extension) of w into $H^s(\Omega)$. □

Proof The proof is an adaptation of [207, Lemma 3.36, Theorem 3.37], that we simplify (the statement in the book of W. Mc Lean is more general). First, we consider the case of a hyperplane, and smooth functions, as for the first version of the trace theorem (Proposition 2.13). Therefore, we want to build a lifting operator:

$$\mathcal{L} : \mathcal{D}(\mathbb{R}^{d-1}) \to \mathscr{C}^\infty(\mathbb{R}^d)$$

that satisfies:

$$\mathcal{L}v(\mathbf{x}', 0) = v(\mathbf{x}') \tag{2.11}$$

for $v \in \mathcal{D}(\mathbb{R}^{d-1})$, and where we still use the notation $\mathbf{x} = (\mathbf{x}', x_d) \in \mathbb{R}^d$, with $\mathbf{x}' \in \mathbb{R}^{d-1}$. This operator can be built explicitly thanks to the Fourier transform.

For this purpose, we use Proposition 2.1 and take $\varphi_1 \in \mathcal{D}(\mathbb{R}^{d-1})$ a smooth test function such that

$$\varphi_1(\mathbf{x}') = 1, \text{ when } |\mathbf{x}'| \leq 1.$$

Let us denote now

$$\boldsymbol{\psi}(\boldsymbol{\xi}', x_d) := \varphi_1\left[(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}} x_d\right],$$

for $\boldsymbol{\xi}' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. We can use the inverse Fourier transform to define the lifting of v, as:

$$\mathcal{L}v(\mathbf{x}', x_d) = \int_{\mathbb{R}^{d-1}} \widehat{v}(\boldsymbol{\xi}')\boldsymbol{\psi}(\boldsymbol{\xi}', x_d)e^{2\pi i\mathbf{x}'\cdot\boldsymbol{\xi}'}\mathrm{d}\boldsymbol{\xi}'.$$

Moreover, using the property $\varphi_1(0) = 1$, there holds:

$$\mathcal{L}v(\mathbf{x}', 0) = \int_{\mathbb{R}^{d-1}} \widehat{v}(\boldsymbol{\xi}')\boldsymbol{\psi}(\boldsymbol{\xi}', 0)e^{2\pi i\mathbf{x}'\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}' = \int_{\mathbb{R}^{d-1}} \widehat{v}(\boldsymbol{\xi})e^{2\pi i\mathbf{x}'\cdot\boldsymbol{\xi}'}\mathrm{d}\boldsymbol{\xi}' = v(\mathbf{x}'),$$

which is the required property (2.11). Now, we compute the Fourier transform of the lifting:

$$
\begin{aligned}
&\widehat{\mathcal{L}v}(\boldsymbol{\xi}', \xi_d)\\
&= \int_{\mathbb{R}^d} \mathcal{L}v(\mathbf{x}', x_d) e^{-2\pi i(\mathbf{x}'\cdot\boldsymbol{\xi}' + x_d\xi_d)} \mathrm{d}\mathbf{x}'\mathrm{d}x_d\\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \mathcal{L}v(\mathbf{x}', x_d) e^{-2\pi i\mathbf{x}'\cdot\boldsymbol{\xi}'} \mathrm{d}\mathbf{x}' \right) e^{-2\pi i x_d\xi_d} \mathrm{d}x_d\\
&= \int_{\mathbb{R}} \widehat{v}(\boldsymbol{\xi}') \boldsymbol{\psi}(\boldsymbol{\xi}', x_d) e^{-2\pi i x_d\xi_d} \mathrm{d}x_d\\
&= \widehat{v}(\boldsymbol{\xi}') \int_{\mathbb{R}} \boldsymbol{\psi}(\boldsymbol{\xi}', x_d) e^{-2\pi i x_d\xi_d} \mathrm{d}x_d.
\end{aligned}
$$

Above we used Fubini's Theorem, the definition of the lifting and the properties of the Fourier transform and its inverse. Then, we apply the change of variable $y = (1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}} x_d$ to reformulate

$$
\begin{aligned}
&\int_{\mathbb{R}} \boldsymbol{\psi}(\boldsymbol{\xi}', x_d) e^{-2\pi i x_d\xi_d} \mathrm{d}x_d\\
&= \int_{\mathbb{R}} \varphi_1(y) e^{-2\pi i(1+|\boldsymbol{\xi}'|^2)^{-\frac{1}{2}} y\xi_d} (1 + |\boldsymbol{\xi}'|^2)^{-\frac{1}{2}} \mathrm{d}y\\
&= \frac{1}{(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \int_{\mathbb{R}} \varphi_1(y) e^{-2\pi i(1+|\boldsymbol{\xi}'|^2)^{-\frac{1}{2}} \xi_d y} \mathrm{d}y\\
&= \frac{1}{(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \widehat{\varphi_1} \left[\frac{\xi_d}{(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \right].
\end{aligned}
$$

So we get finally:

$$
\widehat{\mathcal{L}v}(\boldsymbol{\xi}', \xi_d) = \frac{1}{(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \widehat{\varphi_1} \left[\frac{\xi_d}{(1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \right] \widehat{v}(\boldsymbol{\xi}'). \tag{2.12}
$$

Finally, we use the same techniques as for the trace operator (Proposition 2.13) to show the inequality in Sobolev norms in (2.10). We detail this below. First, we use the definition of the norm:

$$
\|\mathcal{L}v\|_{s,\mathbb{R}^d}^2 = \int_{\mathbb{R}^d} (1 + |\boldsymbol{\xi}'|^2 + \xi_d^2)^s \widehat{\mathcal{L}v}^2(\boldsymbol{\xi}', \xi_d) \, \mathrm{d}\boldsymbol{\xi}'\mathrm{d}\xi_d.
$$

We use the above expression (2.12) for the Fourier transform of the lifting operator, the change of variable $\xi_d = (1 + |\boldsymbol{\xi}'|^2)^{\frac{1}{2}} t$, the relationship

$$
(1 + |\boldsymbol{\xi}'|^2 + \xi_d^2)^s = (1 + |\boldsymbol{\xi}'|^2)^s (1 + t^2)^s
$$

and we obtain:

$$
\begin{aligned}
&\|\mathcal{L}v\|^2_{s,\mathbb{R}^d}\\
&= \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}'|^2+\xi_d^2)^s \left(\frac{1}{(1+|\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \widehat{\varphi_1} \left[\frac{\xi_d}{(1+|\boldsymbol{\xi}'|^2)^{\frac{1}{2}}} \right] \widehat{v}(\boldsymbol{\xi}') \right)^2 \, \mathrm{d}\boldsymbol{\xi}' \mathrm{d}\xi_d\\
&= \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}} (1+t^2)^s \left(\widehat{\varphi_1}\,(t)\, \widehat{v}(\boldsymbol{\xi}') \right)^2 \, \mathrm{d}\boldsymbol{\xi}' \mathrm{d}t.
\end{aligned}
$$

We apply Fubini's Theorem:

$$
\|\mathcal{L}v\|^2_{s,\mathbb{R}^d} = \int_{\mathbb{R}^{d-1}} (1+|\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}} \widehat{v}(\boldsymbol{\xi}')^2 \left(\int_{\mathbb{R}} (1+t^2)^s \widehat{\varphi_1}^2\,(t)\, \mathrm{d}t \right) \mathrm{d}\boldsymbol{\xi}'.
$$

Then, we set

$$
C := \int_{\mathbb{R}} (1+t^2)^s \widehat{\varphi_1}^2\,(t)\, \mathrm{d}t < +\infty
$$

and get

$$
\|\mathcal{L}v\|^2_{s,\mathbb{R}^2} = C \int_{\mathbb{R}^{d-1}} (1+|\boldsymbol{\xi}'|^2)^{s-\frac{1}{2}} \widehat{v}(\boldsymbol{\xi}')^2 \, \mathrm{d}\boldsymbol{\xi}' = C\|v\|^2_{s-\frac{1}{2},\mathbb{R}}.
$$

Finally, we need first the density result of Proposition 2.8, from which we deduce that the application $\mathcal{L}$ is in fact a bounded application from $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$ to $H^s(\mathbb{R}^d)$, for $s \geq 1/2$. Then, for an arbitrary Lipschitz domain Ω, and $s \leq 1$, we need the partition of unity result of Proposition 2.7. Indeed, from this result and proceeding the same way as in Theorem 2.6 for the trace operator, we get the statement of the Theorem for a Lipschitz domain Ω. □

As in the proof of the above theorem, we can introduce a special notation for the lifting operator:

$$
\mathcal{L} : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\Omega)
$$

which is a linear bounded operator, such that, for any $w \in H^{s-\frac{1}{2}}(\Gamma)$, there holds $\mathcal{L}w = v_w$ with v_w that satisfies (2.10). The same can be done for functions in $H^{s-\frac{1}{2}}(\Gamma_1)$ where Γ_1 is a relatively open subset of Γ of positive Lebesgue measure. A similar notation will be used in the next chapter for discrete lifting operators.

2.4.4 First Consequences of Trace and Lifting Theorems

In this section we recall that Γ is the boundary of the Lipschitz domain Ω and take $\frac{1}{2} < s \leq 1$. We provide first an important, and useful result, to characterize the kernel of the trace operator Υ:

$$H_0^s(\Omega) = \{v \in H^s(\Omega) \mid \Upsilon v = 0\},$$

as stated in, e.g., [207, Theorem 3.40]. From the above results, and the Lifting Theorem 2.9, it also appears that the trace operator Υ is surjective. As a result, the range of Υ is $H^{s-\frac{1}{2}}(\Gamma)$:

$$H^{s-\frac{1}{2}}(\Gamma) = \{\Upsilon v \mid v \in H^s(\Omega)\}.$$

We can even go a little bit further, and, as a consequence of the trace and the lifting theorems above, we can define another norm for functions on the boundary that are traces of H^s-functions. For $w \in H^{s-\frac{1}{2}}(\Gamma)$, the set

$$\{\|v\|_{s,\Omega} \mid v \in H^s(\Omega),\, \Upsilon v = w\}$$

is non-empty, because of Theorem 2.9, and bounded below by 0, so it admits an infimum. Thus we can define the quantity

$$\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma} := \inf\{\|v\|_{s,\Omega} \mid v \in H^s(\Omega),\, \Upsilon v = w\}.$$

It defines a norm. Particularly if $\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma} = 0$, it means that there exists a sequence (v_n) in $H^s(\Omega)$, with $\Upsilon v_n = w$ for any $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \|v_n\|_{s,\Omega} = 0.$$

Then, using the Trace Theorem 2.6, we get:

$$\|w\|_{s-\frac{1}{2},\Gamma} = \|\Upsilon v_n\|_{s-\frac{1}{2},\Gamma} \leq C_s \|v_n\|_{s,\Omega}.$$

With $n \to +\infty$, we obtain $\|w\|_{s-\frac{1}{2},\Gamma} = 0$ and thus $w = 0$. Moreover, a consequence of the above trace and lifting theorems is that this new norm $\|\cdot\|_{s-\frac{1}{2},\Upsilon,\Gamma}$ is equivalent to the Sobolev norm $\|\cdot\|_{s-\frac{1}{2},\Gamma}$, as shown below:

Proposition 2.14 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ of boundary $\Gamma := \partial\Omega$. Let $\frac{1}{2} < s \leq 1$. The two norms $\|\cdot\|_{s-\frac{1}{2},\Upsilon,\Gamma}$ and $\|\cdot\|_{s-\frac{1}{2},\Gamma}$ are equivalent on $H^{s-\frac{1}{2}}(\Gamma)$.*

Proof Let $w \in H^{s-\frac{1}{2}}(\Gamma)$. By definition of the infimum, there exists $v \in H^s(\Omega)$, with $\Upsilon v = w$, such that

$$\|v\|_{s,\Omega} \leq 2\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma}.$$

With the Trace Theorem 2.6, we deduce

$$\|w\|_{s-\frac{1}{2},\Gamma} = \|\Upsilon v\|_{s-\frac{1}{2},\Gamma} \leq C_s\|v\|_{s,\Omega} \leq 2C_s\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma}.$$

Then, the Lifting Theorem 2.9 ensures there exists v_w in $H^s(\Omega)$ that verifies

$$\Upsilon v_w = w, \quad \|v_w\|_{s,\Omega} \leq C_s\|w\|_{s-\frac{1}{2},\Gamma}.$$

Then, by definition of the norm:

$$\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma} \leq \|v_w\|_{s,\Omega} \leq C_s\|w\|_{s-\frac{1}{2},\Gamma}.$$

Since all the constants do not depend on w, this proves the result. □

The above result will be especially useful in the case $s = 1$. The same way we can introduce a new norm for Sobolev spaces on a part of the boundary. For Γ_1 a relatively open subset in Γ, of positive Lebesgue measure, we define

$$\|w\|_{s-\frac{1}{2},\Upsilon,\Gamma_1} := \inf\{\|v\|_{s,\Omega} \mid v \in H^s(\Omega), \Upsilon v|_{\Gamma_1} = w\}$$

for any $w \in H^{s-\frac{1}{2}}(\Gamma_1)$ and this is an equivalent norm to $\|\cdot\|_{s-\frac{1}{2},\Gamma_1}$ on $H^{s-\frac{1}{2}}(\Gamma_1)$.

2.5 Green Formulas

To go from strong forms to weak forms, and conversely, we need Green formulas (see [118, Lemma B.56, Corollary B.57]). We introduce first some notations and preliminary results, then state the Green formulas, and end this section with Green formulas on a part of the boundary.

2.5.1 Preliminaries

We need to introduce $H^{-\frac{1}{2}}(\Gamma)$ that we define as the topological dual of $H^{\frac{1}{2}}(\Gamma)$. , We denote by $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing in $H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$ and denote by $\|\cdot\|_{-\frac{1}{2},\Gamma}$ the dual (operator) norm defined as

$$\|\varphi\|_{-\frac{1}{2},\Gamma} := \sup_{w\in H^{\frac{1}{2}}(\Gamma)} \frac{\langle\varphi, w\rangle_\Gamma}{\|w\|_{\frac{1}{2},\Gamma}},$$

for $\varphi \in H^{-\frac{1}{2}}(\Gamma)$. Also, the normal derivative of a regular enough function v on the boundary $\partial\Omega$ of a Lipschitz domain Ω is denoted as:

$$\partial_{\mathbf{n}} v := \nabla v \cdot \mathbf{n}.$$

Let us now recall the divergence theorem (see, e.g., [207, Theorem 3.34] for the proof):

Proposition 2.15 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, of boundary $\Gamma := \partial\Omega$, and $\boldsymbol{\psi} \in \mathscr{C}^1(\overline{\Omega}; \mathbb{R}^d)$ a vector field. Then, there holds*

$$\int_\Omega \operatorname{div}(\boldsymbol{\psi})(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \boldsymbol{\psi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\,\mathrm{d}s(\mathbf{x}). \tag{2.13}$$

The following intermediate result is a consequence of the divergence theorem and of the lifting theorem (Theorem 2.9) (see [137, Theorem 2.5]):

Proposition 2.16 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, of boundary $\Gamma := \partial\Omega$, then the mapping $\Upsilon_{\mathbf{n}} : \boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} \cdot \mathbf{n}$, defined for $\mathcal{D}(\overline{\Omega}; \mathbb{R}^d)$, can be extended uniquely into a linear bounded operator*

$$\Upsilon_{\mathbf{n}} : \mathbf{H}(\operatorname{div}; \Omega) \to H^{-\frac{1}{2}}(\Gamma).$$

Proof We pick at first $\boldsymbol{\varphi}$ and ψ as test functions, respectively, in $\mathcal{D}(\overline{\Omega}; \mathbb{R}^d)$ and $\mathcal{D}(\overline{\Omega})$ and then, we apply the divergence formula:

$$\int_\Omega \operatorname{div}(\psi\boldsymbol{\varphi})(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \psi(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x}).$$

Then, we use the identity $\operatorname{div}(\psi\boldsymbol{\varphi}) = \nabla\psi \cdot \boldsymbol{\varphi} + \psi \operatorname{div}(\boldsymbol{\varphi})$ and get:

$$\int_\Omega \nabla\psi(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega \psi(\mathbf{x}) \operatorname{div}(\boldsymbol{\varphi})(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \psi(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x}). \tag{2.14}$$

By density, and using the trace theorem, the above identity (2.14) also holds for $\psi \in H^1(\Omega)$. Indeed, let $\psi \in H^1(\Omega)$ and (ψ_n) a sequence of smooth functions that converges to ψ in the H^1-norm. We use the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\left| \int_\Omega \nabla\psi(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_\Omega \nabla\psi_n(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \\
&= \left| \int_\Omega (\nabla\psi(\mathbf{x}) - \nabla\psi_n(\mathbf{x})) \cdot \boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \\
&\leq \|\nabla(\psi - \psi_n)\|_{0,\Omega} \|\boldsymbol{\varphi}\|_{0,\Omega} \\
&\leq \|\psi - \psi_n\|_{1,\Omega} \|\boldsymbol{\varphi}\|_{0,\Omega},
\end{aligned}$$

which tends to 0 when $n \to +\infty$. The same can be done for the second term. For the third term, we use once again Cauchy–Schwarz and the trace theorem ($\|\psi_n - \psi\|_{\frac{1}{2},\Gamma} \leq C\|\psi_n - \psi\|_{1,\Omega}$):

$$\left|\int_\Gamma \psi_n(\mathbf{x})(\boldsymbol{\varphi}\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) - \int_\Gamma \psi(\mathbf{x})(\boldsymbol{\varphi}\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x})\right|$$
$$\leq C\|\psi_n - \psi\|_{1,\Omega}\|\boldsymbol{\varphi}\cdot\mathbf{n}\|_{0,\Gamma}.$$

As a result, for $\boldsymbol{\varphi}$ smooth, and any ψ in $H^1(\Omega)$, we bound, using (2.14) and applying two times the Cauchy–Schwarz inequality:

$$\left|\int_\Gamma \Upsilon\psi(\mathbf{x})(\boldsymbol{\varphi}\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x})\right| \leq \left|\int_\Omega \nabla\psi(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| + \left|\int_\Omega \psi(\mathbf{x})\,\mathrm{div}\,(\boldsymbol{\varphi})(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|$$
$$\leq \|\psi\|_{1,\Omega}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)}.$$

Now, take $w \in H^{\frac{1}{2}}(\Gamma)$ arbitrary. Using Theorem 2.9, there exists $v_w \in H^1(\Omega)$ such that $\Upsilon v_w = w$, and

$$\|v_w\|_{1,\Omega} \leq C\|w\|_{\frac{1}{2},\Gamma}.$$

Using these last two inequalities, let us bound, for $\boldsymbol{\varphi}$ smooth:

$$\left|\int_\Gamma w(\mathbf{x})(\boldsymbol{\varphi}\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x})\right| = \left|\int_\Gamma \Upsilon v_w(\mathbf{x})(\boldsymbol{\varphi}\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x})\right|$$
$$\leq \|v_w\|_{1,\Omega}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)}$$
$$\leq C\|w\|_{\frac{1}{2},\Gamma}\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)}.$$

So, by definition of the dual norm:

$$\|\boldsymbol{\varphi}\cdot\mathbf{n}\|_{-\frac{1}{2},\Omega} \leq C\|\boldsymbol{\varphi}\|_{H(\mathrm{div}\,;\Omega)}.$$

And we obtain the final result by density, using Proposition 2.9. □

2.5.2 Green Formulas

Proposition 2.16 allows to state a first Green formula:

Proposition 2.17 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, of boundary $\Gamma := \partial\Omega$. For every functions $\psi \in H^1(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}(\mathrm{div}\,;\Omega)$, there holds $(\boldsymbol{\varphi}\cdot\mathbf{n}) \in H^{-\frac{1}{2}}(\Gamma)$ as well as:*

$$-\int_\Omega \mathrm{div}\,(\boldsymbol{\varphi})(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{\varphi}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} - \langle\boldsymbol{\varphi}\cdot\mathbf{n}, \Upsilon\psi\rangle_\Gamma.$$

Proof Take $\boldsymbol{\varphi} \in \mathbf{H}(\mathrm{div}\,;\Omega)$, from previous Proposition 2.16, there holds $(\boldsymbol{\varphi}\cdot\mathbf{n}) \in H^{-\frac{1}{2}}(\Gamma)$, and from Proposition 2.9, there exists a sequence $(\boldsymbol{\varphi}_n)$ of functions in $\mathcal{D}(\overline{\Omega};\mathbb{R}^d)$ that tends to $\boldsymbol{\varphi}$ in $\mathbf{H}(\mathrm{div}\,;\Omega)$. Take ψ in $H^1(\Omega)$. From (2.14), there holds:

$$\int_\Omega \nabla\psi(\mathbf{x})\cdot\boldsymbol{\varphi}_n(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega \psi(\mathbf{x})\,\mathrm{div}\,(\boldsymbol{\varphi}_n)(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Gamma \Upsilon\psi(\mathbf{x})(\boldsymbol{\varphi}_n\cdot\mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x}).$$

We take the limit when $n \to +\infty$ and obtain the required result. Note that, for the boundary term, we can use the results of above Proposition 2.9 to get the bound:

$$|\langle\boldsymbol{\varphi}\cdot\mathbf{n} - \boldsymbol{\varphi}_n\cdot\mathbf{n}, \Upsilon\psi\rangle_\Gamma| \le C\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_n\|_{H(\mathrm{div}\,;\Omega)}\|\psi\|_{1,\Omega}.$$

□

We state below another, useful, Green formula (see, e.g., [118, Corollary B.59]), which is a direct consequence of the formula above:

Proposition 2.18 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, of boundary $\Gamma := \partial\Omega$. For every functions $v \in H^1(\Omega)$ and $u \in H^1(\Omega)$ such that* $\mathrm{div}\,(\nabla u) \in L^2(\Omega)$*, there holds $\partial_\mathbf{n} u \in H^{-\frac{1}{2}}(\Gamma)$ as well as:*

$$-\int_\Omega \mathrm{div}\,(\nabla u)(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega \nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})\,\mathrm{d}\mathbf{x} - \langle\partial_\mathbf{n} u, \Upsilon v\rangle_\Gamma.$$

Proof We apply Proposition 2.17 with the choice:

$$\boldsymbol{\varphi} = \nabla u, \quad \psi = v.$$

□

2.5.3 *Green Formulas on a Part of the Boundary*

Since boundary conditions are not always defined in the same manner on the whole boundary of a domain (i.e., when mixed boundary conditions are applied), it is also useful to have Green formulas adapted to these situations. We adopt the same setting as in Sect. 2.3.7, and we consider an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, with boundary $\Gamma := \partial\Omega$, Γ_1 a relatively open subset in Γ, of positive Lebesgue measure ($|\Gamma_1| > 0$). We still define Γ_2 the complementary set in Γ: $\Gamma_2 := \Gamma \setminus \overline{\Gamma_1}$. We define as follow the subset of functions with vanishing trace on Γ_2:

$$V_1 := \{v \in H^1(\Omega) \mid \Upsilon v|_{\Gamma_2} = 0\},$$

where $\Upsilon v|_{\Gamma_2} = 0$ means $\Upsilon v = 0$ almost everywhere on Γ_2. Thanks to the Trace Theorem 2.6, the definition of V_1 is meaningful and V_1 is a closed subspace of $H^1(\Omega)$. For the sake of simplicity, we denote the Lions-Magenes space associated with Γ_1 as follows here:

$$W_1 := H_{00}^{\frac{1}{2}}(\Gamma_1),$$

with norm $\|\cdot\|_{W_1} := \|\cdot\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)}$. We denote by W_1' the dual space of W_1, with the operator norm. Remark that for any $v \in V_1$, there holds $\Upsilon v|_{\Gamma_1} \in W_1$.

The Green formula we will present later on is based on the following observation, that we can modify above Proposition 2.16 to give a weak meaning for the normal trace on Γ_1 of a function in $\mathbf{H}(\mathrm{div}\,;\Omega)$:

Proposition 2.19 *Let Ω, Γ, Γ_1, and W_1 be defined with the assumptions at the beginning of this Sect. 2.5.3. Then, the mapping $\Upsilon_{\mathbf{n}} : \boldsymbol{\varphi} \mapsto (\boldsymbol{\varphi} \cdot \mathbf{n})|_{\Gamma_1}$, defined for $\mathcal{D}(\overline{\Omega}; \mathbb{R}^d)$, can be extended uniquely into a linear bounded operator*

$$\Upsilon_{\mathbf{n}} : \mathbf{H}(\mathrm{div}\,;\Omega) \to W_1'.$$

Proof We proceed almost exactly as in Proposition 2.16. We pick at first $\boldsymbol{\varphi}$ and ψ, respectively, in $\mathcal{D}(\overline{\Omega}; \mathbb{R}^d)$ and in $H^1(\Omega)$. There still holds:

$$\left| \int_\Gamma \Upsilon \psi(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) \right| \le \|\psi\|_{1,\Omega} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)}.$$

Now, take $w \in W_1$ and $\widetilde{w} \in H^{\frac{1}{2}}(\Gamma)$ its extension by 0 out of Γ_1. Using Theorem 2.9, there exists $v_{\widetilde{w}} \in H^1(\Omega)$ such that $\Upsilon v_{\widetilde{w}} = \widetilde{w}$, and

$$\|v_{\widetilde{w}}\|_{1,\Omega} \le C \|\widetilde{w}\|_{\frac{1}{2},\Gamma}.$$

Moreover, there holds, by definition, $\|w\|_{W_1} = \|\widetilde{w}\|_{\frac{1}{2},\Gamma}$ so

$$\|v_{\widetilde{w}}\|_{1,\Omega} \le C \|w\|_{W_1}.$$

Using these above inequalities, let us bound, for $\boldsymbol{\varphi}$ smooth:

$$\begin{aligned}
\left| \int_{\Gamma_1} w(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) \right| &= \left| \int_\Gamma \widetilde{w}(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) \right| \\
&= \left| \int_\Gamma (\Upsilon v_{\widetilde{w}})(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\, \mathrm{d}s(\mathbf{x}) \right| \\
&\le \|v_{\widetilde{w}}\|_{1,\Omega} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)} \\
&\le C \|w\|_{W_1} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div}\,;\Omega)}.
\end{aligned}$$

So, by definition of the dual norm:

$$\|\boldsymbol{\varphi} \cdot \mathbf{n}\|_{W_1'} \leq C \|\boldsymbol{\varphi}\|_{H(\mathrm{div}\,;\Omega)}.$$

And we obtain the final result by density (Proposition 2.9). □

We can state now our variant of the Green formula adapted for a part of the boundary.

Proposition 2.20 *Let* Ω*,* Γ*,* Γ_1*, and* W_1 *be defined with the assumptions at the beginning of this Sect.* 2.5.3*. Then, for every functions* $\psi \in V_1$ *and* $\boldsymbol{\varphi} \in \mathbf{H}(\mathrm{div}\,;\Omega)$*, there holds* $(\boldsymbol{\varphi} \cdot \mathbf{n}) \in W_1'$ *as well as:*

$$-\int_\Omega \mathrm{div}\,(\boldsymbol{\varphi})(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{\varphi}(\mathbf{x}) \cdot \nabla\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} - \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \Upsilon\psi|_{\Gamma_1}\rangle_{W_1', W_1}.$$

Proof The fundamental observation is that, for $\psi \in V_1$, there holds $\Upsilon\psi|_{\Gamma_1} \in W_1$. Then, we proceed exactly as in the proof of Proposition 2.17, starting from a smooth test function $\boldsymbol{\varphi} \in \mathcal{D}(\overline{\Omega};\mathbb{R}^d)(\subset \mathbf{H}(\mathrm{div}\,;\Omega))$, $\psi \in V_1$ and the formula

$$\int_\Omega \nabla\psi(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega \psi(\mathbf{x})\,\mathrm{div}\,(\boldsymbol{\varphi})(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Gamma_1} \Upsilon\psi(\mathbf{x})(\boldsymbol{\varphi} \cdot \mathbf{n})(\mathbf{x})\,\mathrm{d}s(\mathbf{x}),$$

which is (2.14), where we took into account $\psi|_{\Gamma_2} = 0$. We end up using Proposition 2.19, instead of Proposition 2.16. □

2.6 Polynomial Approximation in Fractional Sobolev Spaces

We provide here some tools of functional analysis that allow to derive the interpolation error estimates in fractional Sobolev norms presented in Chap. 4, notably an extension to the fractional setting of the Deny–Lions lemma. Indeed most of the textbooks about finite element theory derive the interpolation error only for Sobolev spaces with integer exponents (see, for instance, Ph. G. Ciarlet [87], P.A. Raviart and J.M. Thomas [231], A. Quarteroni and A. Valli [230], S. Brenner and R. Scott [50], A. Ern and J.L. Guermond [118, 120]). The fractional setting is generally addressed only in some specialized research articles: the most famous one is the paper of T. Dupont and R. Scott [112], but there are also other references, sometimes much more recent, such as, for instance [119].

Here we follow one of the most usual paths, inspired from A. Quarteroni and A. Valli, or A. Ern and J.L. Guermond. To adapt the results to the fractional setting, we mostly need a fractional Poincaré-Friedrichs inequality, that we present first. For other usual presentations of these results, for integer exponents, see otherwise [50, 87, 231]. For another presentation that addresses fractional exponents, see [112].

2.6.1 A Fractional Poincaré-Friedrichs Inequality

We begin by providing, with its proof, a fractional Poincaré-Friedrichs inequality. We follow the paper from B. Faermann [125, Lemma 3.4] (see also N. Heuer [160]). This result can be stated as follows:

Lemma 2.2 *Let $\omega \subset \mathbb{R}^d$ ($d \geq 1$), an open, bounded, Lipschitz domain. Let $s \in (0, 1]$. There exists a constant $C > 0$, such that for any $v \in H^s(\omega)$:*

$$\|v\|_{0,\omega} \leq C \left(|v|^2_{s,\omega} + \left(\int_\omega v \right)^2 \right)^{\frac{1}{2}}. \tag{2.15}$$

The constant $C > 0$ depends only on d, on the shape of ω and on the Sobolev index s. □

Proof The proof is usual when $s = 1$ [120, Lemma 3.30]. Let us suppose $0 < s < 1$ and take $v \in H^s(\omega)$. We use the notation

$$I_v = \int_\omega v$$

and the identity

$$(v(\mathbf{x}) - v(\mathbf{y}))^2 = v(\mathbf{x})^2 + v(\mathbf{y})^2 - 2v(\mathbf{x})v(\mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in \omega$, to compute the double integral

$$\begin{aligned}
&\int_\omega \int_\omega (v(\mathbf{x}) - v(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \int_\omega \int_\omega v(\mathbf{x})^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \int_\omega \int_\omega v(\mathbf{y})^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} - 2 \int_\omega \int_\omega v(\mathbf{x})v(\mathbf{y}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= 2|\omega| \int_\omega v(\mathbf{x})^2 \mathrm{d}\mathbf{x} - 2 \left(\int_\omega v(\mathbf{x}) \mathrm{d}\mathbf{x} \right) \left(\int_\omega v(\mathbf{y}) \mathrm{d}\mathbf{y} \right) \\
&= 2|\omega| \int_\omega v(\mathbf{x})^2 \mathrm{d}\mathbf{x} - 2I_v^2,
\end{aligned}$$

where we used the Fubini-Tonelli theorem. We then use the identity

$$1 = \frac{|\mathbf{x} - \mathbf{y}|^{d+2s}}{|\mathbf{x} - \mathbf{y}|^{d+2s}}$$

to start to recover the Aronszajn-Gagliardo-Slobodeckij semi-norm

$$2|\omega|\int_\omega v(\mathbf{x})^2\mathrm{d}\mathbf{x}-2I_v^2=\int_\omega\int_\omega\frac{|v(\mathbf{x})-v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}|\mathbf{x}-\mathbf{y}|^{d+2s}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

Since ω is bounded, there exists $C_\omega>0$ (that is, in fact, the diameter of ω), such that, for every $\mathbf{x},\mathbf{y}\in\omega$:

$$|\mathbf{x}-\mathbf{y}|^{d+2s}\le C_\omega^{d+2s}.$$

Using the above inequality, we get:

$$2|\omega|\int_\omega v(\mathbf{x})^2\mathrm{d}\mathbf{x}-2I_v^2\le C_\omega^{d+2s}\int_\omega\int_\omega\frac{|v(\mathbf{x})-v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s}}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

This can be rewritten:

$$2|\omega|\|v\|_{0,\omega}^2\le 2I_v^2+C_\omega^{d+2s}|v|_{s,\omega}^2,$$

which ends the proof. □

2.6.2 Fractional Deny–Lions Lemma

First, we adapt the Deny–Lions lemma for the fractional setting and provide a result similar to T. Dupont and R. Scott [112]. We just derive it differently, using the Poincaré-Friedrichs inequality (2.15), and following otherwise the path of the proof in the book of A. Quarteroni and A. Valli [230]. We denote as follows the vector space of d-variate real polynomials of maximal degree $k\ge 1$, viewed as functions in a domain $\omega\subset\mathbb{R}^d$:

$$\mathbb{P}_k(\omega)$$
$$:=\left\{p:\omega\to\mathbb{R}\;\middle|\;p(x_1,\ldots,x_d)=\sum_{0\le i_1+i_2+\ldots+i_d\le k}\alpha_{i_1i_2\ldots i_d}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d},\quad\alpha_{i_1i_2\ldots i_d}\in\mathbb{R}\right\}.$$

Lemma 2.3 *Let $\omega\subset\mathbb{R}^d$ be an open, bounded, Lipschitz domain. Let $k\ge 0$ and $s\in(0;1]$, then, there exists $C>0$ such that, for every $v\in H^{k+s}(\omega)$:*

$$\inf_{p\in\mathbb{P}_k(\omega)}\|v+p\|_{k+s,\omega}\le C|v|_{k+s,\omega}. \tag{2.16}$$

Proof Let us take $k\ge 0$ and $s\in(0;1]$ (for the proof when $s=1$, see, for instance, [230, Proposition 3.4.4, p.88]). First, we prove that there exists $C>0$ such that, for every $v\in H^{k+s}(\omega)$:

$$\|v\|_{k+s,\omega}^2 \leq C \left(|v|_{k+s,\omega}^2 + \sum_{|\alpha| \leq k} \left(\int_\omega D^\alpha v \right)^2 \right). \tag{2.17}$$

We proceed as usual by contradiction (see as well the Peetre–Tartar Lemma 3.5 later on) and suppose that (2.17) does not hold. So there exists a sequence $v_n \in H^{k+s}(\omega)$ such that

$$\|v_n\|_{k+s,\omega} = 1$$

and

$$\left(|v_n|_{k+s,\omega}^2 + \sum_{|\alpha| \leq k} \left(\int_\omega D^\alpha v_n \right)^2 \right) \leq \frac{1}{n^2}.$$

Since $s > 0$ and ω is bounded, the injection $H^{k+s}(\omega) \hookrightarrow H^k(\omega)$ is compact, thanks to Rellich's Theorem 2.2. Thus, there is a subsequence, still denoted (v_n), that converges strongly in $H^k(\omega)$. Moreover, because of the above inequality, (v_n) is also a Cauchy sequence in $H^{k+s}(\omega)$. Therefore there exists a limit function $v^\star \in H^{k+s}(\omega)$, which satisfies additionally

$$\|v^\star\|_{k+s,\omega} = 1, \qquad |D^\alpha v^\star|_{s,\omega} = 0 \ (|\alpha| = k), \qquad \int_\omega D^\alpha v^\star = 0 \ (|\alpha| \leq k).$$

For each multi-index α such that $|\alpha| = k$, we can use then the fractional Poincaré-Friedrichs inequality (2.15) to deduce that

$$\|D^\alpha v^\star\|_{0,\omega} = 0,$$

and thus $D^\alpha v^\star = 0$ almost everywhere on ω. If $k = 0$, then we deduce directly that $v^\star = 0$ and obtain a contradiction with $\|v^\star\|_{k+s,\omega} = 1$. If $k \geq 1$, then we deduce that $v^\star \in \mathbb{P}_{k-1}(\omega)$. Then, we use $\int_\omega D^\alpha v^\star = 0$ for $|\alpha| \leq k-1$ to deduce that $v^\star = 0$, and we obtain once again a contradiction. As a result, we have proven (2.17).

Now, for $v \in H^{k+s}(\omega)$, we define $q \in \mathbb{P}_k(\omega)$ through the relationships

$$\int_\omega D^\alpha q = -\int_\omega D^\alpha v,$$

for every multi-index α such that $|\alpha| \leq k$. We check that it defines indeed a unique q. Then, we bound

$$\inf_{p \in \mathbb{P}_k(\omega)} \|v + p\|_{k+s,\omega} \leq \|v + q\|_{k+s,\omega}$$

and we use (2.17) to get

$$\|v+q\|_{k+s,\omega} \leq C \left(|v+q|^2_{k+s,\omega} + \sum_{|\alpha|\leq k} \left(\int_\omega D^\alpha (v+q) \right)^2 \right)^{\frac{1}{2}} = C\ |v+q|_{k+s,\omega}.$$

Since $q \in \mathbb{P}_k(\omega)$, there holds, for $|\alpha| = k$, $D^\alpha q = c$ (a constant), and therefore

$$\begin{aligned} |v+q|^2_{k+s,\omega} &= \sum_{|\alpha|=k} |D^\alpha v + D^\alpha q|^2_{s,\omega} = \sum_{|\alpha|=k} |D^\alpha v + c|^2_{s,\omega} = \sum_{|\alpha|=k} |D^\alpha v|^2_{s,\omega} \\ &= |v|^2_{k+s,\omega}. \end{aligned}$$

Combining the two previous inequalities and this last identity, we obtain (2.16). □

2.7 Further Comments

The presentation of this chapter has been mostly inspired by the books of W. McLean [207] and F.J. Sayas et al. [239], that include recent and self-contained presentations on Sobolev spaces and related topics. Another recent reference about fractional order Sobolev spaces is the review article of E. Di Nezza et al. [104]. Indeed recent increased interest on the topic of fractional order operators, such as the fractional Laplacian, has motivated a renewal of the study of these spaces. The same happened before in relation with the study of some specific boundary integral operators. Of course, there are some more classical references on each of these topics, which we provide thereafter. For a historical perspective, one can refer to the book L. Tartar about Sobolev spaces [255].

2.7.1 Distributions and Sobolev Spaces

The theory of distributions is attributed to L. Schwartz, who spent the years 1945–1950 in elaborating this theory. He was awarded the Fields medal in 1950 due to this breakthrough. His book [241] remains a classical reference. It has been motivated to generalize the concepts of function and derivative because of some issues related to the solutions of some partial differential equations and to the Fourier transform, among others. Between the important contributions that led to the theory of distributions were pioneering works of J. Fourier (1822), G.R. Kirchhoff (1882), and O. Heaviside (1898), which made use of distributions before the rigorous theory was achieved. There were also contributions of S. Bochner (1932) and then of S. Sobolev (1935) toward its formalization. Particularly S. Sobolev is considered at the origin of the notion of weak derivative, that is a special case of the distributional derivative. For an historical view on distributions, one can refer to the book of J. Lutzen [203], or also [241, 255].

Introductive material to distributions can be found in the book of C. Gasquet and P. Witomski [134], with an orientation toward signal processing. For presentations more oriented to partial differential equations, one can refer, for instance, to the corresponding chapter of R. Dautray and J.-L. Lions [100], or to the corresponding chapters of L. Tartar [255]. There are many books dedicated exclusively to Sobolev spaces and among classical references we can quote R.A. Adams [3] or V.G. Maz'ja [205]. For classical references that combine Sobolev spaces and the study of elliptic partial differential equations, one can refer to J. Nečas [214], R. Dautray and J.L. Lions [100], L.C. Evans [123], H. Brezis [52] or G. Allaire [7]. A classical reference, and also a pioneering work, in the theory of fractional order Sobolev spaces, is the book of J.L. Lions and E. Magenes [199], that is very related to interpolation spaces (see also related works of J.L. Lions and J. Peetre). Classical references about interpolation spaces are [37, 258].

The reader interested in further topics about fractional order Sobolev spaces can refer to the survey [104] already mentioned, and references therein. Also, let us mention the articles of N. Heuer [159, 160] and T. Tran [257], that provide more insight about the equivalence relationship between various Sobolev norms (particularly the Sobolev-Slobodeckij norm and norms obtained using interpolation spaces). Finally, various other properties can be found in the lecture notes of C. Bernardi, M. Dauge and Y. Maday [38], the monographs of J. Chazarain and A. Piriou [70] and P. Grisvard [141].

2.7.2 Lipschitz Boundaries, Traces and Green Formulas

For more information about Lipschitz boundaries, one can refer to the books of J. Nečas [214] or of P. Grisvard [141]. There are also many additional information in [120]. For trace spaces, a detailed presentation of some important results can be found in P. Grisvard [141]. A characterization of the trace spaces for Sobolev regularity $s = 1$ and $W^{1,p}$ spaces, $p \geq 1$ has been published in 1957 by E. Gagliardo [133]. Particularly for $p = 1$ the trace space is simply $L^1(\Gamma)$ and for other values of p we recover the fractional spaces with the Sobolev-Slobodeckij semi-norm. Other early references about trace theorems are [4, 214, 261]. Concerning Green formulas, an earlier presentation can be found in the book of N. Kikuchi and J.T. Oden [181], particularly well-suited for contact and friction problems and elasticity. For alternative presentations and complements, see also J.L. Lions and E. Magenes [199], P. Grisvard [141], V. Girault and P.A. Raviart [137], L. Tartar [255].

2.7.3 Deny–Lions Lemma

For the Deny–Lions lemma, one can refer to the original paper of J. Deny and J.L. Lions [103], and also it can be found in almost every book about the mathematical theory of finite elements when the Sobolev exponent is an integer. Our presentation here has been inspired from A. Quarteroni and A. Valli [230]. For fractional exponents, see T. Dupont and L.R. Scott [112].

2.7.4 Differential Operators

We focus in this book on three-dimensional elasticity, and for this reason we provided definitions of differential operators for a flat metric. For the reader interested in thin structures like shells or rods, some notions of differential geometry are needed and one can refer, for instance, to [69] or [184] and references therein.

Chapter 3
Signorini's Problem

In this chapter, we present the Signorini problem in small strain elasticity, as a first step before going into its numerical approximation and into more complex contact and friction problems. Indeed, Signorini conditions are the simplest conditions that allow to model appropriately frictionless contact between an elastic body and a rigid support, and they are formulated in terms of inequalities and a nonlinear complementarity condition. The Signorini problem can be recast weakly as a variational inequality of the first kind, for which well-posedness can be established thanks to Stampacchia's theorem.

First we describe the Signorini problem as a minimization problem under constraints of nonpenetration. This allows to recover the weak form as a first order optimality condition (variational inequality) of this minimization problem. Then, the strong form is recovered and strong–weak equivalence is established. As a next step, we detail the proof of well-posedness, and we show that this problem is well-posed in the Hadamard sense: it admits one unique solution, which is controlled by the data [118]. This makes this setting particularly suitable for numerical analysis, and this will allow us, in the next chapters, to establish well-posedness and a priori error bounds for various types of finite element approximations. We end this chapter with a few considerations about the regularity of the solution and its behavior on the contact boundary. This is an important issue since many results of numerical analysis in the last decades have been obtained under specific assumptions on the behavior of the exact solution. However, we will see in the next chapters that these assumptions are not really necessary and optimal a priori error bounds can be derived without them.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_3

3.1 Presentation

We first describe the geometrical aspects of the Signorini problem with emphasis on the assumptions made on the boundaries, and then we recall the small strain elasticity formalism. Finally, we formulate the Signorini problem as a constrained minimization problem.

3.1.1 The Domain and Its Boundaries

We consider an elastic body whose reference configuration is represented by the domain Ω in $\mathbb{R}^d$ with $d = 2$ or $d = 3$, where Ω is an open, bounded, and connected set (see Fig. 3.1 when $d = 2$). The domain Ω is supposed Lipschitz, so that we will be able to apply all the tools we presented in Chap. 2. Moreover, it discards domains such as cracks, for which singularities appear at some locations, at the crack tip, for instance, and for which the solution has low Sobolev regularity [141, 142]. Moreover, the boundary $\Gamma := \partial\Omega$ is supposed polygonal (for $d = 2$) or polyhedral (for $d = 3$), mostly to simplify the presentation in the next chapters and to avoid technicalities in the numerical analysis when the mesh is nonconforming to the boundary. The unit outward normal vector on Γ is denoted by $\mathbf{n}$. We divide the boundary into three non-overlapping parts Γ_D, Γ_N, and Γ_C:

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$$

with

$$\Gamma_D \cap \Gamma_N = \emptyset, \quad \Gamma_D \cap \Gamma_C = \emptyset, \quad \Gamma_C \cap \Gamma_N = \emptyset.$$

The sets Γ_D, Γ_N, and Γ_C are supposed open relatively to the whole boundary Γ. On each part, we apply different boundary conditions: on Γ_D a homogeneous Dirichlet boundary condition, which means here that the body is clamped on this part, on Γ_N a Neumann boundary condition, which means we impose a known surface force density, and Γ_C is the contact boundary. The contact boundary is supposed to be

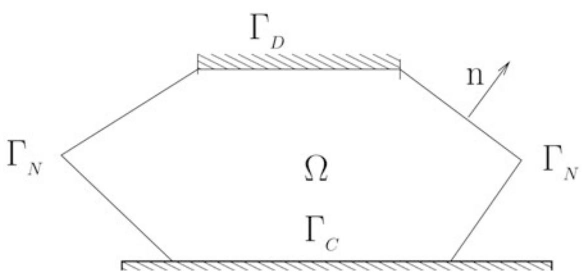

Fig. 3.1 Elastic body that occupies the domain Ω. The boundary $\partial\Omega$ is divided into three non-overlapping parts: Γ_D (the body is clamped), Γ_N (tractions are imposed), and Γ_C (contact boundary with nonpenetration into the rigid support)

a straight line segment when $d = 2$ or a plane polygon when $d = 3$, to simplify. We suppose that $|\Gamma_C| > 0$, where $|\cdot|$ denotes here the Lebesgue measure of a set of dimension $\mathbb{R}^{d-1}$. In its initial stage, the body is in contact on Γ_C with a rigid foundation and we suppose that the final contact zone after deformation is included into Γ_C. A fundamental point at this stage is the following: this final contact zone is an unknown of the problem and is determined accordingly to the material properties of the elastic body and to the density of (volume and surface) loads. The common terminology to describe this situation is *unilateral contact*, which is different from *bilateral contact*, in which this contact zone is supposed to be known exactly and is a datum of the problem. Bilateral contact is easier to handle, at least in the frictionless case.

Remark 3.1 As we should see later on, in Chap. 10, this setting can be extended to account for more general situations, and some assumptions can be alleviated. Particularly, for practical computations, the contact boundary does not need to be a straight line or a plane polygon, and an initial gap between the elastic solid and a rigid support can be included. See, for instance, the Chapter 2 in the book of N. Kikuchi and J.T. Oden [181] for such a more general description. Moreover, the setting we describe here serves as a paradigm to inspire the *master–slave* strategy to handle the contact between two elastic bodies (see Chap. 10). □

3.1.2 Small Strain Elasticity

We suppose from now on that the reader has a basic knowledge about the mathematical formulation and the mathematical theory for elastic problems, at least in the small strain regime, see otherwise [143, Chapter X] or [204, Chapter 4], especially. Other classical references about the general theory of elasticity are [88, 223, 259]. We will just recall here a very few basic facts. So we consider that the elastic body undergoes a static motion: nothing is time-dependent and we just look for an equilibrium configuration once the static loads are applied. We also make the usual small strain assumptions, which means, roughly speaking, that the overall amplitude of the motion (displacement and deformation) is small enough to make some simplifications from the finite strain theory.

As a result, we apply some loads on the elastic body $\overline{\Omega}$. We suppose that it is subjected to volume force density $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$ and to surface loads $\mathbf{F} \in L^2(\Gamma_N; \mathbb{R}^d)$. We are looking for the displacement of the body, which is a vector field

$$\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d,$$

which allows to deduce the new position of each material point $\mathbf{x} \in \overline{\Omega}$ once the equilibrium configuration is reached. More precisely, at equilibrium, the new position of $\mathbf{x}$ is $\mathbf{x} + \mathbf{u}(\mathbf{x})$. The *deformation* of the body is quantified through the small strain tensor, denoted by $\boldsymbol{\varepsilon}(\mathbf{u})$, and related to the displacement by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right) \in \mathbb{R}^d \otimes \mathbb{R}^d,$$

where $\nabla\mathbf{u} \in \mathbb{R}^d \otimes \mathbb{R}^d$ is the gradient of the displacement $\mathbf{u}$ and $\nabla\mathbf{u}^{\mathrm{T}}$ its transpose. If we denote by $(\mathbf{e}_i)_{i=1,\ldots,d}$ the canonical basis of $\mathbb{R}^d$, the above tensors can be written as follows in the canonical basis of $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$\nabla\mathbf{u} = \sum_{i=1}^{d}\sum_{j=1}^{d} \underbrace{\frac{\partial u_i}{\partial x_j}}_{=:(\nabla\mathbf{u})_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \nabla\mathbf{u}^{\mathrm{T}} = \sum_{i=1}^{d}\sum_{j=1}^{d} \underbrace{\frac{\partial u_j}{\partial x_i}}_{=:(\nabla\mathbf{u}^{\mathrm{T}})_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j.$$

We note $(\varepsilon_{ij})_{i,j=1,\ldots,d}$ the scalar components of the small strain tensor $\boldsymbol{\varepsilon}$ in the canonical basis of $\mathbb{R}^d \otimes \mathbb{R}^d$.

The internal and surface forces in the body are represented through the *Cauchy stress tensor*, denoted by $\boldsymbol{\sigma}$ which is a tensor field

$$\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

We will note as well $(\sigma_{ij})_{i,j=1,\ldots,d}$ the scalar components in the canonical basis of $\mathbb{R}^d \otimes \mathbb{R}^d$. The Cauchy stress tensor is symmetric $\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$, as a consequence of the balance of momentum [88, Theorem 2.3.1]. Its existence is ensured by Cauchy's theorem (see, e.g., [88, Theorem 2.3.1] or [143, Chapter V]). The quantitative relation between strain and stress depends on the intrinsic material properties of the elastic body. It is usually given by a constitutive relationship between the small strain tensor and the Cauchy stress tensor, which is written as follows:

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{u}), \tag{3.1}$$

where : is the double contraction (or double dot) product and $\mathbf{A}$ is a fourth order tensor called the *elasticity tensor*. The quantities $A_{ijkl} \in \mathbb{R}$, $1 \le i, j, k, l \le d$, are its scalar components in the canonical basis $(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)_{1 \le i,j,k,l \le d}$ of $\mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$. As a result, written in the canonical basis, the above relationship (3.1) means

$$\sigma_{ij} = \sum_{k=1}^{d}\sum_{l=1}^{d} A_{ijkl}\varepsilon_{kl}$$

for $i, j = 1, \ldots, d$. The elasticity tensor $\mathbf{A}$ does not depend on the unknown displacement $\mathbf{u}$. It is assumed to be completely determined, once and for all, from the microstructure of the elastic material. As a result, it may be spatially dependent, and we may note $\mathbf{A}(\mathbf{x})$ its value at the material point $\mathbf{x} \in \overline{\Omega}$, when needed. Thanks to the relationship (3.1), the Cauchy stress $\boldsymbol{\sigma}$ becomes a function of the displacement $\mathbf{u}$, which can remain the only unknown of the whole problem.

The elasticity tensor **A** has the following three properties [114]:

- It is symmetric in the sense

$$A_{ijkl} = A_{klij} = A_{jikl}, \quad \forall\, 1 \le i, j, k, l \le d. \tag{3.2}$$

- It is uniformly elliptic: there exists $\alpha_A > 0$ such that, for almost every $\mathbf{x} \in \Omega$ and for every symmetric second order tensor $\mathbf{s} = (s_{ij})_{i,j=1,\ldots,d}$,

$$\mathbf{s} : \mathbf{A} : \mathbf{s} = \sum_{i,j,k,l=1}^{d} A_{ijkl}(\mathbf{x}) s_{ij} s_{kl} \ge \alpha_A \sum_{i,j=1}^{d} s_{ij}^2 = \alpha_A\, \mathbf{s} : \mathbf{s}. \tag{3.3}$$

- It is uniformly bounded: there exists $C_A > 0$ such that, for every $1 \le i, j, k, l \le d$, and for almost every $\mathbf{x}$ in Ω,

$$|A_{ijkl}(\mathbf{x})| \le C_A. \tag{3.4}$$

Notably the above properties will be helpful to ensure that the Signorini problem is well-posed (see Sect. 3.3 later on).

Some of the symmetry properties (3.2) of the elasticity tensor **A** are inherited from the symmetry property of the small strain tensor $\boldsymbol{\varepsilon}$ and of the stress tensor $\boldsymbol{\sigma}$ (minor symmetries). The other symmetry properties of **A** are related to the existence of a stored energy function. Then, **A** can be written as the second order derivative of this stored energy, with respect to $\boldsymbol{\varepsilon}$ (see, e.g., [204, Proposition 3.5] or [173, Section 6.6]). These symmetry properties are then a consequence of Schwarz's (or Clairaut's) theorem and are called major symmetries. It results from these symmetry properties that, when $d = 3$, **A** is determined by only 21 coefficients, in general (instead of 81). Nevertheless, most of the usual elastic constitutive laws involve even less scalar parameters. For instance, the most common law, which is the Hooke law for isotropic materials, makes use of only 2 parameters, as detailed in the remark below.

Remark 3.2 When the elastic body is supposed compressible, homogeneous, and isotropic, the constitutive relationship (3.1) becomes

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda(\operatorname{div} \mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \tag{3.5}$$

where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ are the Lamé coefficients, and **I** is the metric tensor:

$$\mathbf{I} = \sum_{i=1}^{d} \sum_{j=1}^{d} \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j.$$

The above constitutive relationship is known as the Hooke law (see, for instance, [204, Proposition 3.12]). From (3.5), we check that the symmetry property (3.2) and the uniform boundedness property (3.4) are satisfied. Moreover, for reasons related to thermodynamic stability, the Lamé coefficients are supposed to satisfy:

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu > 0.$$

Particularly, if there exists $c > 0$ such that $\mu > c$ and $\lambda + \frac{2}{3}\mu > c$, this allows to satisfy the above condition and also the condition (3.3) of uniform ellipticity. The Lamé coefficients are generally expressed as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

where $E > 0$ is the Young modulus and $-1 < \nu < 1/2$ is the Poisson ratio. The values for the above quantities can be obtained from experiments and for a given material. □

We now introduce the vector space $\mathbf{V}$ of displacements that satisfy Dirichlet boundary conditions:

$$\mathbf{V} := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \;\middle|\; \Upsilon \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

Thanks to the assumptions on the boundary and to the Trace Theorem 2.6, this space is well-defined: indeed, since $\Upsilon \mathbf{v} \in H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$, the trace of $\mathbf{v}$ is a well-defined function almost everywhere on Γ, and thus we can take its restriction on the Dirichlet boundary Γ_D. Moreover, a direct application of the Trace Theorem 2.6 combined with the variational Lemma 2.1 ensures that it is a closed subspace of $H^1(\Omega; \mathbb{R}^d)$, so this is also a Hilbert space for the norm $\|\cdot\|_{1,\Omega}$.

Let us define the bilinear form $a(\cdot,\cdot)$, and the linear form $L(\cdot)$, that represent, respectively, the virtual work of internal forces, and of external (volume and surface) forces, as follows:

$$a(\mathbf{v}, \mathbf{w}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w}), \qquad L(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{F} \cdot \mathbf{v},$$

for any $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{V}$ and for $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ the double contraction of stress and deformation tensors whose expression in terms of components on the canonical basis is

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} := \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \varepsilon_{ij}.$$

Thanks to the assumption on the elasticity tensor $\mathbf{A}$, the bilinear form $a(\cdot,\cdot)$ is well-defined, symmetric, and continuous on $\mathbf{V}$. Indeed, we use the constitutive relationship (3.1), the Cauchy–Schwarz inequality, and the uniform boundedness property of $\mathbf{A}$ to get

$$|a(\mathbf{v},\mathbf{w})| \leq \|\mathbf{A}\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,\Omega}\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{0,\Omega} \leq C\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,\Omega}\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{0,\Omega} \leq C\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}, \tag{3.6}$$

for any $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{V}$ and where $C > 0$ is a constant that depends solely on C_A and d.

In the same fashion, and since $\mathbf{f} \in L^2(\Omega;\mathbb{R}^d)$, $\mathbf{F} \in L^2(\Gamma_N;\mathbb{R}^d)$, we check that the linear form $L(\cdot)$ is also well-defined and continuous on $\mathbf{V}$. Indeed, by application of the Cauchy–Schwarz inequality, there holds

$$|L(\mathbf{v})| \leq \|\mathbf{f}\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Gamma_N}\|\mathbf{v}\|_{0,\Gamma_N}.$$

Now, with the Trace Theorem 2.6, we bound

$$\|\mathbf{v}\|_{0,\Gamma_N} \leq \|\mathbf{v}\|_{0,\Gamma} \leq \|\mathbf{v}\|_{\frac{1}{2},\Gamma} \leq C\|\mathbf{v}\|_{1,\Omega}.$$

We also use the obvious bound $\|\mathbf{v}\|_{0,\Omega} \leq \|\mathbf{v}\|_{1,\Omega}$ and get

$$|L(\mathbf{v})| \leq C\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Gamma_N}\right)\|\mathbf{v}\|_{1,\Omega} \tag{3.7}$$

for any $\mathbf{v} \in \mathbf{V}$ and where $C > 0$ depends solely on the trace constant.

Finally, the total mechanical energy associated with the elastic body is given by the functional (see, e.g., [143, Chapter X. Theorem of Work and Energy] or [181, Chapter 5]):

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}), \tag{3.8}$$

which is well-defined and finite for any displacement field $\mathbf{v} \in \mathbf{V}$. The functional $\mathcal{J}(\cdot)$ takes into account the material properties of the elastic structure (constitutive and equilibrium equations), the volume load (source term), the surface load (Neumann, or natural, boundary condition), and the vector space $\mathbf{V}$ incorporates the Dirichlet (essential) boundary condition. Before we minimize $\mathcal{J}(\cdot)$ to recover the equilibrium position of the elastic solid, we need to add to this setting the contact with the rigid support. This is what we will do next.

3.1.3 The Nonpenetration Condition

The contact between the elastic body and the rigid support is addressed by enforcing explicitly the nonpenetration condition on the contact boundary Γ_C, as an essential boundary condition. As a result, we restrict even more the set of admissible displacements. To this purpose, we need first to define the normal displacement on the contact boundary. Due to the assumptions on Γ_C, the normal unit vector $\mathbf{n}$ is a constant function on Γ_C, and we denote by n_i its components ($\mathbf{n} = \sum_{i=1}^{d} n_i \mathbf{e}_i$). Then, for any field $\mathbf{v} \in \mathbf{V}$, the Trace Theorem 2.6 ensures that $\Upsilon\mathbf{v} \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^d)$, and thus $(\Upsilon\mathbf{v})|_{\Gamma_C} \in H^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$. Therefore, we can simply define

$$v_{\mathbf{n}} := (\Upsilon\mathbf{v}) \cdot \mathbf{n} = \sum_{i=1}^{d} (\Upsilon v_i) n_i ,$$

which belongs to $H^{\frac{1}{2}}(\Gamma_C)$ (and we drop the Υ symbol to alleviate the notation). We define a partial ordering relationship "$\leq$" on $L^2(\Gamma_C)$ as follows: for every $v \in L^2(\Gamma_C)$, we set

$$(v \leq 0) \text{ if and only if } (v(\mathbf{x}) \leq 0 \text{ for almost every } \mathbf{x} \in \Gamma_C).$$

We can introduce now the set $\mathbf{K}$ of admissible displacements which satisfy the non-interpenetration on the contact zone Γ_C:

$$\mathbf{K} := \{\mathbf{v} \in \mathbf{V} \mid v_{\mathbf{n}} \leq 0 \text{ on } \Gamma_C\} . \tag{3.9}$$

This set is well-defined according to the previous considerations. Moreover, it satisfies the following properties:

Proposition 3.1 *The set* $\mathbf{K}$ *defined in* (3.9) *is a non-empty closed convex cone of* $\mathbf{V}$.

□

Proof Since $\mathbf{0} \in \mathbf{K}$, $\mathbf{K}$ is non-empty. One can check directly that

$$\theta\mathbf{v} + (1-\theta)\mathbf{w} \in \mathbf{K}, \quad \lambda\mathbf{v} \in \mathbf{K},$$

for $\theta \in [0; 1]$, $\lambda \in [0; +\infty)$, $\mathbf{v} \in \mathbf{K}$ and $\mathbf{w} \in \mathbf{K}$, so $\mathbf{K}$ is a convex cone.

To prove that $\mathbf{K}$ is closed, we take a sequence $(\mathbf{v}_i)$ of functions in $\mathbf{K}$, which converges toward a function $\mathbf{v} \in \mathbf{V}$, and we check that $\mathbf{v} \in \mathbf{K}$. So we need to check that $v_{\mathbf{n}} \leq 0$ on Γ_C. Since

$$\|\mathbf{v}_i - \mathbf{v}\|_{1,\Omega} \to 0,$$

we obtain with the Trace Theorem 2.6

$$\|\mathbf{v}_i - \mathbf{v}\|_{\frac{1}{2},\Gamma_C} \to 0,$$

and particularly $v_{i,\mathbf{n}} \to v_{\mathbf{n}}$ in $L^2(\Gamma_C)$. This implies that $v_{i,\mathbf{n}} \to v_{\mathbf{n}}$ almost everywhere on Γ_C. Let us define

$$\Gamma_C^{nc} := \{\mathbf{x} \in \Gamma_C \mid v_{i,\mathbf{n}}(\mathbf{x}) \nrightarrow v_{\mathbf{n}}(\mathbf{x})\},$$

which is of zero Lebesgue measure. For each $i \in \mathbb{N}$, since $\mathbf{v}_i \in \mathbf{K}$, there holds $v_{i,\mathbf{n}} \leq 0$ almost everywhere on Γ_C, so if we define

$$\Gamma_C^i := \{\mathbf{x} \in \Gamma_C \mid v_{i,\mathbf{n}}(\mathbf{x}) \nleq 0\},$$

this set is also of zero Lebesgue measure. As a result, the set

$$\Gamma_C^{nc} \cup \left(\bigcup_{i \in \mathbb{N}} \Gamma_C^i \right)$$

is still of zero Lebesgue measure, as a countable union of sets of zero Lebesgue measure. Moreover, for every $\mathbf{x} \in \Gamma_C$ that does not belong to this set, we have both

$$v_{i,\mathbf{n}}(\mathbf{x}) \rightarrow_{(i \rightarrow +\infty)} v_{\mathbf{n}}(\mathbf{x}), \quad (v_{i,\mathbf{n}}(\mathbf{x}) \leq 0, \ \forall i \in \mathbb{N}),$$

which implies $v_{\mathbf{n}}(\mathbf{x}) \leq 0$. As a result, we obtain $v_{\mathbf{n}}(\mathbf{x}) \leq 0$ for almost every $\mathbf{x} \in \Gamma_C$ and $\mathbf{v} \in \mathbf{K}$. This proves that $\mathbf{K}$ is closed. □

Remark 3.3 There are other possibilities to define $\mathbf{K}$, which are equivalent to the definition presented here. See [181, Section 5.2] for a discussion on this topic. □

3.1.4 A First Formulation of Signorini's Problem

Using all the previous notions, we can now state a first formulation of Signorini's problem, as a frictionless contact problem in linear elasticity, and viewed as a constrained minimization problem. It reads

$$\text{Find } \mathbf{u} \in \mathbf{K} \text{ such that } \mathcal{J}(\mathbf{u}) = \min_{\mathbf{v} \in \mathbf{K}} \mathcal{J}(\mathbf{v}). \tag{3.10}$$

It means that we search the displacement $\mathbf{u}$ that minimizes the total energy of the elastic solid, provided it remains clamped on Γ_D and it does not penetrate into the rigid support on Γ_C.

3.2 Weak Form and Contact Conditions

The weak formulation of Signorini's problem can now be recovered as the first order optimality condition associated with Problem (3.10). After this, we will be able to explicit the contact conditions on Γ_C in strong form and to prove an equivalence between strong and weak formulations.

3.2.1 The Weak Formulation for Signorini's Problem

Since $a(\cdot, \cdot)$, respectively, $L(\cdot)$, is continuous on $\mathbf{V} \times \mathbf{V}$, respectively, on $\mathbf{V}$, the quadratic functional $\mathcal{J}$ is Fréchet-differentiable at any point $\mathbf{v} \in \mathbf{V}$ and its derivative is

$$\mathcal{J}'(\mathbf{v}; \mathbf{w}) = a(\mathbf{v}, \mathbf{w}) - L(\mathbf{w}).$$

Suppose that $\mathbf{u} \in \mathbf{K}$ is a local minimum of $\mathcal{J}$, and then it satisfies the following Euler inequation [181, Theorem 3.7]:

$$\mathcal{J}'(\mathbf{u}; \mathbf{v} - \mathbf{u}) \geq \mathbf{0}, \qquad \forall\, \mathbf{v} \in \mathbf{K}. \tag{3.11}$$

We use the above expression for $\mathcal{J}'$ into the first order optimality condition (3.11), and we obtain that every solution $\mathbf{u}$ to Problem (3.10) is also solution to the problem below:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \qquad \forall\, \mathbf{v} \in \mathbf{K}. \end{cases} \tag{3.12}$$

We just obtained the weak formulation of Problem (3.10) as a variational inequality (see, e.g., [128, 153, 181]). We will see later on that, for our precise model problem, formulations (3.10) and (3.12) are in fact equivalent.

Next, we provide a useful, equivalent formulation that comes from the structure of $\mathbf{K}$, which is also a cone. So every solution $\mathbf{u}$ to Problem (3.10) is also solution to

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that} \\ a(\mathbf{u}, \mathbf{u}) = L(\mathbf{u}), & (i), \\ a(\mathbf{u}, \mathbf{v}) \geq L(\mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{K}\ (ii). \end{cases} \tag{3.13}$$

We establish this equivalence in the proposition below:

Proposition 3.2 *Weak formulations* (3.12) *and* (3.13) *are equivalent.* □

Proof If $\mathbf{u} \in \mathbf{K}$ is a solution to (3.13), we check that is also a solution to (3.12) just by using the identity $a(\mathbf{u}, \mathbf{v} - \mathbf{u}) = a(\mathbf{u}, \mathbf{v}) - a(\mathbf{u}, \mathbf{u})$ and (3.13).

Let us suppose now that $\mathbf{u} \in \mathbf{K}$ is a solution to (3.12). Note that $\mathbf{0} \in \mathbf{K}$, so setting $\mathbf{v} = \mathbf{0}$ in (3.12) yields

$$a(\mathbf{u}, -\mathbf{u}) \geq L(-\mathbf{u}).$$

We take now $\mathbf{v} = 2\mathbf{u} \in \mathbf{K}$ in (3.12)

$$a(\mathbf{u}, \mathbf{u}) \geq L(\mathbf{u}).$$

We combine the two previous inequalities and get

$$a(\mathbf{u}, \mathbf{u}) = L(\mathbf{u}),$$

which is (3.13)-(i). For $\mathbf{u} \in \mathbf{K}$ and each $\mathbf{v} \in \mathbf{K}$, the weak form (3.12) reads

$$a(\mathbf{u}, \mathbf{v}) \geq a(\mathbf{u}, \mathbf{u}) + L(\mathbf{v}) - L(\mathbf{u}).$$

With (3.13)-(i), we get

$$a(\mathbf{u}, \mathbf{v}) \geq L(\mathbf{v}),$$

which is (3.13)-(ii). □

3.2.2 Displacement and Stress on the Contact Boundary

We already defined in Sect. 3.1.3 the normal displacement $v_{\mathbf{n}}$ on the contact boundary, for any displacement field $\mathbf{v} \in \mathbf{V}$. We need now to introduce more notions to define the tangential displacement, as well as the normal and tangential stress fields, with corresponding spaces. So let us define the trace space restricted to the contact boundary

$$\mathbf{W}_C := \left\{ \Upsilon \mathbf{v}|_{\Gamma_C} \mid \mathbf{v} \in \mathbf{V} \right\} \tag{3.14}$$

for all the admissible displacement fields. Remark that, since Γ_C is a segment or a planar polygon, this space is well-defined and that

$$H_{00}^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d) \subset \mathbf{W}_C \subset H^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d).$$

The first inclusion comes from the definition of $H_{00}^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$. Notably, when $|\Gamma_N| = 0$ (no Neumann boundary), there holds $\mathbf{W}_C = H_{00}^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$. The second inclusion is a direct consequence of the inclusion $\mathbf{V} \subset H^1(\Omega; \mathbb{R}^d)$. We still define, as in Sect. 3.1.3, the normal component and, now, the tangential component, for $\mathbf{w} \in \mathbf{W}_C$:

$$w_{\mathbf{n}} := \mathbf{w} \cdot \mathbf{n} = \sum_{i=1}^{d} w_i n_i, \quad \mathbf{w_t} := \mathbf{w} - w_{\mathbf{n}} \mathbf{n}.$$

And if we define

$$W_C := \{\mathbf{w} \cdot \mathbf{n} \mid \mathbf{w} \in \mathbf{W}_C\} \; (= \{v_{\mathbf{n}} \mid \mathbf{v} \in \mathbf{V}\})$$

and

$$\mathbf{W}_{C,\mathbf{t}} := \{\mathbf{w} \in \mathbf{W}_C \mid \mathbf{w} \cdot \mathbf{n} = 0\},$$

we can write $\mathbf{W}_C$ as a direct sum [181, Theorem 5.6, p.88]

$$\mathbf{W}_C = W_C \mathbf{n} \oplus \mathbf{W}_{C,\mathbf{t}}.$$

This means that for every trace function $\mathbf{w} \in \mathbf{W}_C$, we can write

$$\mathbf{w} = w_{\mathbf{n}} \mathbf{n} + \mathbf{w}_{\mathbf{t}}, \quad \mathbf{w}_{\mathbf{t}} \cdot \mathbf{n} = 0.$$

Now, thanks to the Trace Theorem 2.6, each function $\mathbf{v}$ of $\mathbf{V}$ has a well-defined trace $\Upsilon \mathbf{v}|_{\Gamma_C}$ in $\mathbf{W}_C$ (see Sect. 2.5.3), denoted still by $\mathbf{v}$, for the sake of simplicity. Moreover, this trace function $\mathbf{v}$ has a normal component $v_{\mathbf{n}} \in W_C$ and a tangential component $\mathbf{v}_{\mathbf{t}} \in \mathbf{W}_{C,\mathbf{t}}$. Conversely, by definition of $\mathbf{W}_C$, to each trace function $\mathbf{w} \in \mathbf{W}_C$, we can associate a corresponding bulk function $\mathbf{v}_{\mathbf{w}} \in \mathbf{V}$, with $\Upsilon \mathbf{v}_{\mathbf{w}}|_{\Gamma_C} = \mathbf{w}$, and $\Upsilon \mathbf{v}_{\mathbf{w}}|_{\Gamma_D} = \mathbf{0}$. The same property holds, separately, for the normal and tangential components.

In the same way as we did in Chap. 2 (see Sect. 2.4.4), we endow the space $\mathbf{W}_C$ with the following norm:

$$\|\mathbf{w}\|_{\mathbf{W}_C} := \inf\{\|\mathbf{v}\|_{1,\Omega} \mid \mathbf{v} \in \mathbf{V}, \Upsilon \mathbf{v}|_{\Gamma_C} = \mathbf{w}\},$$

for $\mathbf{w} \in \mathbf{W}_C$. We define similarly

$$\|w\|_{W_C} := \inf\{\|\mathbf{v}\|_{1,\Omega} \mid \mathbf{v} \in \mathbf{V}, v_{\mathbf{n}}|_{\Gamma_C} = w\},$$

for $w \in W_C$, and

$$\|\mathbf{w}_{\mathbf{t}}\|_{\mathbf{W}_{C,\mathbf{t}}} := \inf\{\|\mathbf{v}\|_{1,\Omega} \mid \mathbf{v} \in \mathbf{V}, \mathbf{v}_{\mathbf{t}}|_{\Gamma_C} = \mathbf{w}_{\mathbf{t}}\},$$

for $\mathbf{w}_{\mathbf{t}} \in \mathbf{W}_{C,\mathbf{t}}$.

Remark 3.4 When the Dirichlet boundary Γ_D is compactly embedded into the Neumann boundary Γ_N, there holds in fact

$$\mathbf{W}_C = H^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$$

and the norm $\|\cdot\|_{\mathbf{W}_C}$ is equivalent in this case to the Sobolev–Slobodeckij norm $\|\cdot\|_{\frac{1}{2},\Gamma_C}$. The same conclusion holds for the normal and tangential trace spaces and their corresponding norms. Though this assumption about the respective positions of the boundary is not necessary in the continuous setting, as we will see below, it will always be assumed from Chap. 5 (unless specified) for the numerical analysis of the finite element methods. The reason is mostly technical: it allows us to make use of the definition of the Sobolev–Slobodeckij norm for some error estimates, which is more practical. □

For the stress field on the boundary, we need to take care that, for a displacement $\mathbf{v} \in \mathbf{V}$, the associated stress field $\boldsymbol{\sigma}(\mathbf{v})$ is only in $L^2(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d)$ and thus has no trace defined on the boundary. As a result, we will need to work solely with boundary stresses defined in a weak (distributional) sense and not as usual L^2 functions. Therefore, we introduce the dual spaces $\mathbf{W}'_C$, respectively, W'_C and $\mathbf{W}'_{C,\mathbf{t}}$, associated with $\mathbf{W}_C$, respectively, W_C and $\mathbf{W}_{C,\mathbf{t}}$, with the operator norm. More precisely, we denote by $\langle \cdot, \cdot \rangle_{\Gamma_C}$ the duality pairing between $\mathbf{W}'_C$ and $\mathbf{W}_C$ and define

$$\|\boldsymbol{\tau}\|_{\mathbf{W}'_C} := \sup_{\mathbf{w} \in \mathbf{W}_C} \frac{\langle \boldsymbol{\tau}, \mathbf{w} \rangle_{\Gamma_C}}{\|\mathbf{w}\|_{\mathbf{W}_C}}$$

for $\boldsymbol{\tau} \in \mathbf{W}'_C$. We do the same for the normal and tangent dual spaces:

$$\|\tau\|_{W'_C} := \sup_{w \in W_C} \frac{\langle \tau, w \rangle_{\Gamma_C}}{\|w\|_{W_C}}, \qquad \|\boldsymbol{\tau}_{\mathbf{t}}\|_{\mathbf{W}'_{C,\mathbf{t}}} := \sup_{\mathbf{w}_{\mathbf{t}} \in \mathbf{W}_{C,\mathbf{t}}} \frac{\langle \boldsymbol{\tau}_{\mathbf{t}}, \mathbf{w}_{\mathbf{t}} \rangle_{\Gamma_C}}{\|\mathbf{w}_{\mathbf{t}}\|_{\mathbf{W}_{C,\mathbf{t}}}}$$

for $\tau \in W'_C$ and $\boldsymbol{\tau}_{\mathbf{t}} \in \mathbf{W}'_{C,\mathbf{t}}$. In next chapters, when there is no ambiguity, the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma_C}$ will be simply denoted $\langle \cdot, \cdot \rangle$. Passing to the usual duality arguments, we can still write [181, Chapter 5]

$$\mathbf{W}'_C = W'_C \mathbf{n} \oplus \mathbf{W}'_{C,\mathbf{t}}.$$

Thus, for every density of surface forces $\boldsymbol{\tau} \in \mathbf{W}'_C$, we can write

$$\boldsymbol{\tau} = \tau_{\mathbf{n}} \mathbf{n} + \boldsymbol{\tau}_{\mathbf{t}}, \qquad \boldsymbol{\tau}_{\mathbf{t}} \cdot \mathbf{n} = 0.$$

As a result, if we denote by $\langle \cdot, \cdot \rangle_{\Gamma_C}$ the duality pairing between $\mathbf{W}'_C$ and $\mathbf{W}_C$, there holds [181, Chapter 5]

$$\langle \boldsymbol{\tau}, \mathbf{w} \rangle_{\Gamma_C} = \langle \tau_{\mathbf{n}}, w_{\mathbf{n}} \rangle_{\Gamma_C} + \langle \boldsymbol{\tau}_{\mathbf{t}}, \mathbf{w}_{\mathbf{t}} \rangle_{\Gamma_C},$$

for every $\boldsymbol{\tau} \in \mathbf{W}'_C$ and $\mathbf{w} \in \mathbf{W}_C$, and where we used also the same notation for the duality pairings between W'_C and W_C and $\mathbf{W}'_{C,\mathbf{t}}$ and $\mathbf{W}_{C,\mathbf{t}}$, to simplify.

Last but not least, we introduce also the dual cone of weakly nonpositive (normal) forces on the contact boundary:

$$\Lambda_C := \{\tau \in W'_C \mid \langle \tau, w \rangle_{\Gamma_C} \geq 0, \ \forall w \in W_C, \ w \leq 0\},$$

where the partial ordering relationship $w \leq 0$ is defined identically as in Sect. 3.1.3.

3.2.3 Green Formula in Elasticity

To derive a strong–weak equivalence, we need to extend the Green formulas (Sect. 2.5) for elasticity equations. We state this result below:

Lemma 3.1 *Let Ω be an open Lipschitz domain in $\mathbb{R}^d$ of boundary $\Gamma := \partial\Omega$. For any $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$, with $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{v}) \in L^2(\Omega; \mathbb{R}^d)$, and $\mathbf{w} \in H^1(\Omega; \mathbb{R}^d)$, there holds $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma; \mathbb{R}^d)$ and*

$$-\int_\Omega (\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{v})) \cdot \mathbf{w} = \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w}) - \langle \boldsymbol{\sigma}(\mathbf{v})\mathbf{n}, \Upsilon\mathbf{w}\rangle_\Gamma, \tag{3.15}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma; \mathbb{R}^d)$ and $H^{\frac{1}{2}}(\Gamma; \mathbb{R}^d)$. □

Proof We apply component-wise the Green formula from Chap. 2 (Proposition 2.17). For $i = 1, \ldots, d$, we define $\boldsymbol{\varphi} := \sum_{j=1}^d \sigma_{ij}(\mathbf{v})\mathbf{e}_j$ and $\psi = w_i$ and note that, with our assumptions on $\mathbf{A}$, $\mathbf{v}$, and $\mathbf{w}$, we verify $\boldsymbol{\varphi} \in \mathbf{H}(\mathrm{div}\,; \Omega)$, and $\psi \in H^1(\Omega)$. So there holds $\boldsymbol{\varphi} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$ and

$$-\int_\Omega \left(\sum_{j=1}^d (\mathrm{div}\; \sigma_{ij}(\mathbf{v}))\mathbf{e}_j\right) w_i = \int_\Omega \left(\sum_{j=1}^d \sigma_{ij}(\mathbf{v})\frac{\partial w_i}{\partial x_j}\right) - \left\langle \left(\sum_{j=1}^d \sigma_{ij}(\mathbf{v})\mathbf{e}_j\right) \cdot \mathbf{n}, \Upsilon w_i \right\rangle_\Gamma .$$

We sum the above expression for $i = 1, \ldots, d$ and get

$$-\int_\Omega (\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{v})) \cdot \mathbf{w} = \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \nabla\mathbf{w} - \langle \boldsymbol{\sigma}(\mathbf{v})\mathbf{n}, \Upsilon\mathbf{w}\rangle_\Gamma .$$

Using the symmetry property of $\boldsymbol{\sigma}(\mathbf{v})$, we note that $\boldsymbol{\sigma}(\mathbf{v}) : \nabla\mathbf{w} = \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})$ and this ends the proof. □

As in Chap. 2 we can also derive the Green formulas that apply only to a restricted part on the boundary (as in Proposition 2.20), particularly the following one:

Lemma 3.2 *Let Ω be an open Lipschitz domain in $\mathbb{R}^d$, of boundary $\Gamma := \partial\Omega$. Suppose that Γ is partitioned as in Sect. 3.1.1 and that $|\Gamma_N| = 0$. For any $\mathbf{v} \in \mathbf{V}$, with $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{v}) \in L^2(\Omega; \mathbb{R}^d)$, and $\mathbf{w} \in \mathbf{V}$, there holds $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n} \in \mathbf{W}_C'$ and*

$$-\int_\Omega (\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{v})) \cdot \mathbf{w} = \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w}) - \langle \boldsymbol{\sigma}(\mathbf{v})\mathbf{n}, \Upsilon\mathbf{w}\rangle_{\Gamma_C}, \tag{3.16}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_C}$ denotes the duality pairing between $\mathbf{W}_C'$ and $\mathbf{W}_C$. □

Proof Note first that, since $|\Gamma_N| = 0$, there holds $\mathbf{W}_C = H_{00}^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$. Then, the proof is identical as in previous Lemma 3.1, except we use Proposition 2.20 instead of Proposition 2.17. $\square$

3.2.4 Contact Conditions and Strong Form of Signorini

Let us now analyze the weak form (3.12) in order to make explicit contact (Signorini's) conditions on Γ_C in strong form, as well as other equations in the bulk and on the other boundaries. By the way, we will prove an equivalence between strong and weak problems. We start with providing an equivalent form of (3.13), which consists in a partial differential equation in the bulk and of two variational equation and inequation on the boundary.

Lemma 3.3 *Suppose that* $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$ *and* $\mathbf{F} \in L^2(\Gamma_N; \mathbb{R}^d)$. *Then, every solution* $\mathbf{u} \in \mathbf{K}$ *to Problem* (3.13) *is a solution to Problem* (3.17) *below:*

$$\begin{cases} \textit{Find } \mathbf{u} \in \mathbf{K} \textit{ such that} & & \\ & & \\ -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \quad \textit{a.e. in } \Omega, & & (i), \\ \langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon\mathbf{u}\rangle_\Gamma = \displaystyle\int_{\Gamma_N} \mathbf{F}\cdot\Upsilon\mathbf{u}, & & (ii), \\ \langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon\mathbf{v}\rangle_\Gamma \geq \displaystyle\int_{\Gamma_N} \mathbf{F}\cdot\Upsilon\mathbf{v}, & \forall \mathbf{v} \in \mathbf{K}, & (iii). \end{cases} \tag{3.17}$$

Conversely, every solution $\mathbf{u} \in \mathbf{K}$ *to Problem* (3.17) *is a solution to Problem* (3.13). $\square$

Remark 3.5 Without extra assumptions, we cannot really provide a simpler "strong" formulation, and Problem (3.17) is the basis for the numerical analysis where the regularity of the solution $\mathbf{u}$ is low (see, e.g., [31, p.1201, formulas (2.6) and (2.7)] in the case of the scalar Signorini problem). $\square$

Proof Let us suppose that $\mathbf{u} \in \mathbf{K}$ is a solution to (3.13). Take $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; \mathbb{R}^d)$ a test function, and $\epsilon = \pm 1$. Since $\epsilon\boldsymbol{\varphi} \in \mathbf{K}$, we obtain from (3.13)-$(ii)$

$$a(\mathbf{u}, \boldsymbol{\varphi}) = L(\boldsymbol{\varphi}),$$

and therefore, using the definition of the distributional divergence, the symmetry of $\boldsymbol{\sigma}(\mathbf{u})$, the property $\boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Omega; \mathbb{R}^{d\times d})$, and the compact support of $\boldsymbol{\varphi}$:

$$\langle -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\rangle = a(\mathbf{u}, \boldsymbol{\varphi}) = L(\boldsymbol{\varphi}) = \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}.$$

This means that $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Omega; \mathbb{R}^d)$ and that

$$-\,\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \quad \text{a.e. in } \Omega,$$

which is (3.17)-(i). Since $\mathbf{u} \in H^1(\Omega; \mathbb{R}^d)$ and $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Omega; \mathbb{R}^d)$, we take $\mathbf{v} \in \mathbf{K}(\subset H^1(\Omega; \mathbb{R}^d))$, and we can apply the Green formula in elasticity (Lemma 3.1) to (3.13)-(ii):

$$\int_\Omega (-\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} + \langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v}\rangle_\Gamma \geq \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{F} \cdot \mathbf{v}.$$

Using (3.17)-(i), we get

$$\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v}\rangle_\Gamma \geq \int_{\Gamma_N} \mathbf{F} \cdot \Upsilon \mathbf{v},$$

which is (3.17)-(iii). We follow the same path from (3.13)-(i) and get

$$\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{u}\rangle_\Gamma = \int_{\Gamma_N} \mathbf{F} \cdot \Upsilon \mathbf{u},$$

which is (3.17)-(ii). Conversely, suppose now that $\mathbf{u} \in \mathbf{K}$ is a solution to Problem (3.17). Take $\mathbf{v} \in \mathbf{K}$, from the condition (3.17)-(i) and the Green formula (Lemma 3.1), we obtain

$$a(\mathbf{u}, \mathbf{v}) = \langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v}\rangle_\Gamma + \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

and using (3.17)-(iii), we recover (3.13)-(ii). Still with Lemma 3.1, we also have

$$a(\mathbf{u}, \mathbf{u}) = \langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{u}\rangle_\Gamma + \int_\Omega \mathbf{f} \cdot \mathbf{u},$$

and using (3.17)-(ii), we recover (3.13)-(i). □

To go further, we can follow N. Kikuchi and J.T. Oden [181, Chapter 6, Theorem 6.3], for instance, and suppose that there are no Neumann boundary conditions. In this case, we can reformulate contact condition on the boundary as Karush–Kuhn–Tucker optimality conditions. This is what we state now.

Theorem 3.1 *Suppose that* $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$. *Suppose moreover that* $|\Gamma_N| = 0$. *Then, every solution* $\mathbf{u} \in \mathbf{K}$ *to Problem* (3.12) *is a solution to Problem* (3.18) *below:*

$$
\begin{cases}
\textit{Find } \mathbf{u} \in \mathbf{V} \textit{ such that} \\
-\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \quad \textit{a.e. in } \Omega, \ (i), \\
\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = \mathbf{0} \textit{ in } \mathbf{W}'_{C,\mathbf{t}}, \qquad\qquad (ii), \\
u_{\mathbf{n}} \le 0 \quad \textit{a.e. on } \Gamma_C, \qquad\quad (iii), \\
\sigma_{\mathbf{n}}(\mathbf{u}) \in \Lambda_C, \qquad\qquad\qquad (iv), \\
\langle \sigma_{\mathbf{n}}(\mathbf{u}), u_{\mathbf{n}} \rangle_{\Gamma_C} = 0, \qquad\qquad (v),
\end{cases}
\tag{3.18}
$$

where we used the following notation for the boundary stress: $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \sigma_{\mathbf{n}}(\mathbf{u})\mathbf{n} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})$. *Conversely, every solution* $\mathbf{u} \in \mathbf{V}$ *to Problem* (3.18) *is a solution to Problem* (3.12). □

Proof In fact, since formulations (3.12), (3.13), and (3.17) are equivalent, we will prove directly the equivalence between (3.17) and (3.18). First, take $\mathbf{u} \in \mathbf{K}$ a solution to (3.17). Then, $\mathbf{u} \in \mathbf{V}$ and conditions (i) and (iii) of (3.18) are satisfied. Therefore, we need to prove that conditions (ii), (iv), and (v) hold.

Since $|\Gamma_N| = 0$, we are in the case $\mathbf{W}_C = H_{00}^{\frac{1}{2}}(\Gamma_C; \mathbb{R}^d)$, which allows to use Lemma 3.2 instead of Lemma 3.1, so we can rewrite conditions (ii) and (iii) of (3.17) as

$$
\begin{cases}
\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{u} \rangle_{\Gamma_C} = 0, \qquad\qquad (i), \\
\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v} \rangle_{\Gamma_C} \ge 0, \quad \forall \mathbf{v} \in \mathbf{K}, \ (ii).
\end{cases}
\tag{3.19}
$$

We now use the conventions from Sect. 3.2.2. Take any normal trace function $w_{\mathbf{n}} \in W_C$, with $w_{\mathbf{n}} \le 0$, and any tangential trace function $\mathbf{w_t} \in \mathbf{W}_{C,\mathbf{t}}$, on the contact boundary. Then set $\mathbf{w} = w_{\mathbf{n}}\mathbf{n} + \mathbf{w_t} \in \mathbf{W}_C$: there exists a lifting (Proposition 2.9) $\mathbf{v_w} \in \mathbf{K}$ such that $\Upsilon \mathbf{v_w}|_{\Gamma_C} = \mathbf{w}$, and $\Upsilon \mathbf{v_w}|_{\Gamma_D} = \mathbf{0}$. From (3.19)-$(ii)$, with the choice $\mathbf{v} = \mathbf{v_w}$ and using the decomposition into normal and tangential components from Sect. 3.2.2, we get

$$
\langle \sigma_{\mathbf{n}}(\mathbf{u}), w_{\mathbf{n}} \rangle_{\Gamma_C} + \langle \sigma_{\mathbf{t}}(\mathbf{u}), \mathbf{w_t} \rangle_{\Gamma_C} \ge 0. \tag{3.20}
$$

Choose now $w_{\mathbf{n}} = 0$, $\mathbf{w_t} \in \mathbf{W}_{C,\mathbf{t}}$ arbitrary, and once again, using $\epsilon = \pm 1$ and with $\epsilon \mathbf{w_t}$ in (3.20), we obtain

$$
\langle \sigma_{\mathbf{t}}(\mathbf{u}), \mathbf{w_t} \rangle_{\Gamma_C} = 0.
$$

This is the condition (ii). Thus the inequation (3.20) simplifies further into

$$\langle \sigma_{\mathbf{n}}(\mathbf{u}), w_{\mathbf{n}} \rangle_{\Gamma_C} \geq 0, \tag{3.21}$$

for every $w_{\mathbf{n}} \in W_C$, with $w_{\mathbf{n}} \leq 0$. This means that $\sigma_{\mathbf{n}}(\mathbf{u}) \in \Lambda_C$, and we get condition (iv). Finally, we go back to Eq. (3.19)-(i). We use the decomposition into normal and tangential components first:

$$\langle \sigma_{\mathbf{n}}(\mathbf{u}), u_{\mathbf{n}} \rangle_{\Gamma_C} + \langle \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}), \mathbf{u}_{\mathbf{t}} \rangle_{\Gamma_C} = 0.$$

Then, we use that $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = 0$ and obtain (v).

Reciprocally, let us suppose now that $\mathbf{u} \in \mathbf{V}$ satisfies (3.18). Let us show it is also a solution to (3.17). First, since $\mathbf{u} \in \mathbf{V}$ and using condition (iii), we get that $\mathbf{u} \in \mathbf{K}$. Condition (i) of (3.17) is condition (i) of (3.18). There remains to prove conditions (ii) and (iii) of (3.17). Since $|\Gamma_N| = 0$, these are equivalent to conditions (3.19). So take $\mathbf{v} \in \mathbf{K}$, and apply once again the decomposition into normal and tangential components:

$$\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v} \rangle_{\Gamma_C} = \langle \sigma_{\mathbf{n}}(\mathbf{u}), v_{\mathbf{n}} \rangle_{\Gamma_C} + \langle \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}), \mathbf{v}_{\mathbf{t}} \rangle_{\Gamma_C}.$$

With condition (3.18)-(ii), this simplifies as

$$\langle \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}, \Upsilon \mathbf{v} \rangle_{\Gamma_C} = \langle \sigma_{\mathbf{n}}(\mathbf{u}), v_{\mathbf{n}} \rangle_{\Gamma_C}.$$

From (3.18)-(iv), we deduce that this term is non-negative, which is (ii) of (3.19). The condition (i) of (3.19) follows similarly, from conditions (3.18) (ii) and (v).

□

Another interesting situation is when the solution $\mathbf{u}$ to Problem (3.12) is regular enough, so that the boundary stress becomes a measurable function, and duality pairings on the boundary become simply boundary integrals. This situation will often occur in the numerical analysis of the next chapters. In this situation, the assumption $|\Gamma_N| = 0$ becomes superfluous. We can state this result below. This result provides in fact the strong form of the Signorini problem, as it can be found in many textbooks and research papers.

Theorem 3.2 *Let us suppose that* $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$ *and* $\mathbf{F} \in L^2(\Gamma_N; \mathbb{R}^d)$*, and that* $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$*, with* $s > \frac{3}{2}$*. Then, the solution* $\mathbf{u}$ *to Problem* (3.12) *is a strong solution of Problem* (3.22)–(3.23) *below, i.e., of small strain elasticity equations*

$$\begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad & \textit{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{u}) \qquad & \textit{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \textit{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \qquad & \textit{on } \Gamma_N, \end{aligned} \tag{3.22}$$

combined with contact conditions on Γ_C*:*

$$\begin{aligned} u_{\mathbf{n}} \leq &\ 0, \quad (i) \\ \sigma_{\mathbf{n}}(\mathbf{u}) \leq &\ 0, \quad (ii) \\ \sigma_{\mathbf{n}}(\mathbf{u})\, u_{\mathbf{n}} = &\ 0, \quad (iii) \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = &\ \mathbf{0}. \quad (iv) \end{aligned} \tag{3.23}$$

This means that (3.22)–(3.23) *are satisfied almost everywhere in the domain* Ω *or on the appropriate parts of the boundary* Γ*. Conversely, if* $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$*,* $s > \frac{3}{2}$ *is a solution to Problem* (3.22)–(3.23)*, then it is solution to the weak formulation* (3.12). □

Proof We can proceed exactly as in the proof of previous Theorem 3.1. Since $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$, $s > \frac{3}{2}$, the boundary stress verifies $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \in H^{s-\frac{3}{2}}(\Omega; \mathbb{R}^d) \subset L^2(\Omega; \mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle_\Gamma$ can be interpreted simply as a boundary integral. This allows first to recover the Neumann boundary condition $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F}$ almost everywhere on Γ_N. This allows also to recover conditions (3.23). Let us just emphasize that the complementarity condition (3.18)-(v) becomes now

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) u_{\mathbf{n}} = 0,$$

and since, now, the product function $\sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}}$ is non-negative almost everywhere on Γ_C, this means that

$$\sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} = 0$$

almost everywhere on Γ_C. □

Remark 3.6 In the above contact conditions (3.23), the condition (i) comes directly from the fact that $\mathbf{u}$ belongs to $\mathbf{K}$ and means that there is no penetration into the rigid support. Condition (ii) means that the normal reaction of the rigid support is oriented inward the elastic body (contact forces can only be repulsive). The complementarity condition (iii) can be interpreted as follows: if, at any point, the elastic body is detached from the rigid support ($u_{\mathbf{n}} < 0$), then necessarily $\sigma_{\mathbf{n}}(\mathbf{u}) = 0$ (there cannot be a contact force); moreover if there is a negative contact force $\sigma_{\mathbf{n}}(\mathbf{u}) < 0$, then necessarily the elastic body must stick to the support ($u_{\mathbf{n}} = 0$). Last condition (iv) means that there is no friction. □

Remark 3.7 There exists a scalar counterpart to the above Signorini problem (3.22)–(3.23), which has a very similar mathematical structure, but that involves only a scalar unknown. This scalar Signorini problem, in strong form, reads
Find $u : \Omega \to \mathbb{R}$ solution to

$$\begin{aligned} \Delta u + f &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} u &= F && \text{on } \Gamma_N, \end{aligned} \tag{3.24}$$

and with the following contact conditions on Γ_C:

$$\begin{aligned} u \leq \;& 0, \quad (i) \\ \partial_{\mathbf{n}} u \leq \;& 0, \quad (ii) \\ u\, \partial_{\mathbf{n}} u = \;& 0, \quad (iii), \end{aligned} \tag{3.25}$$

with the notations $\Delta u = \operatorname{div}(\nabla u)$, $\partial_{\mathbf{n}} u := \nabla u \cdot \mathbf{n}$ and with data $f \in L^2(\Omega)$, and $F \in L^2(\Gamma_N)$. When $d = 2$, this problem can be interpreted as follows: we find the vertical displacement u of a thin membrane that occupies the domain Ω, subjected to forces f and F and that can come into contact with an obstacle on Γ_C. In the same manner as for the Signorini problem in small strain elasticity, this problem admits a weak form as a variational inequality of the first kind and can be recast as a minimization problem [138]. □

3.3 Well-posedness

The aim of this section is to prove that the Signorini problem in weak form (3.12) is well-posed, i.e., that it admits a unique solution $\mathbf{u}$ in $\mathbf{K}$, with Lipschitz continuity of the solution $\mathbf{u}$ with respect to the data $\mathbf{f}$ and $\mathbf{F}$. To this purpose, we will first establish the ellipticity of the bilinear form $a(\cdot, \cdot)$ on $\mathbf{V}$, and the well-posedness result follows as an application of Stampacchia's theorem.

3.3.1 Ellipticity of the Bilinear Form

First, we need the following lemma that allows to characterize the kernel of the small deformation tensor $\boldsymbol{\varepsilon}$, which is in fact constituted of infinitesimal rigid body motions (see, e.g., [121, Lemma 42.7] or [181, Lemma 6.1, p.113]):

Lemma 3.4 *Let Ω be an open, bounded, connected domain of $\mathbb{R}^d$. For any $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$, the equivalence below holds:*

$$\Big(\mathbf{v}(\mathbf{x}) = \mathbf{M}\mathbf{x} + \mathbf{b} \textit{ for a.e. } \mathbf{x} \in \Omega\,, \textit{ with } \mathbf{b} \in \mathbb{R}^d, \mathbf{M} \in \mathbb{R}^{d\times d}, \mathbf{M} = -\mathbf{M}^{\mathrm{T}}\Big)$$
$$\Leftrightarrow (\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0} \quad \textit{a.e. on } \Omega)\,.$$

Then, we need a fundamental and delicate result from small strain elasticity theory, which is the second Korn inequality (see, e.g., [181, Theorem 5.13] or [207, Theorem 10.2]):

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded Lipschitz domain. Then, there exists $c > 0$ such that*

$$c\,\|\mathbf{v}\|_{1,\Omega} \leq \left(\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,\Omega}^2 + \|\mathbf{v}\|_{0,\Omega}^2\right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in H^1(\Omega;\mathbb{R}^d). \tag{3.26}$$

And last but not least, we recall here a very useful result to prove the ellipticity of a bilinear form. This is the Peetre–Tartar Lemma, such as formulated and proven in the monograph of A. Ern and J.L. Guermond [118, Lemma A.38, p.469]:

Lemma 3.5 *Let X, Y, and Z be three Banach spaces. Let A be a continuous linear and injective operator from X to Y and T be a continuous linear and compact operator from X to Z. Suppose that there exists $C > 0$ such that*

$$\forall x \in X,\ \|x\|_X \leq C\left(\|A(x)\|_Y + \|T(x)\|_Z\right). \tag{3.27}$$

Then, there exists $c > 0$ such that

$$\forall x \in X,\ c\|x\|_X \leq \|A(x)\|_Y. \tag{3.28}$$

Proof We proceed by contradiction. Let us suppose that Eq. (3.28) does not hold, and then there exists a sequence (x_n) in X such that

$$\|x_n\|_X = 1, \quad \forall n \in \mathbb{N}, \tag{3.29}$$

and $\|A(x_n)\|_Y$ that tends to 0 when n goes to $+\infty$.

Since T is compact, there is a subsequence (x_{n_k}) such that $T(x_{n_k})$ converges in Z. Let us apply (3.27)

$$\|x_{n_k} - x_{n_l}\|_X \leq C\left(\|A(x_{n_k} - x_{n_l})\|_Y + \|T(x_{n_k}) - T(x_{n_l})\|_Z\right),$$

for $k, l \in \mathbb{N}$. From this, we ensure that (x_{n_k}) is a Cauchy sequence in X, Banach space, and then it converges toward a limit, which we denote $x \in X$. Observe that $\|x\|_X = 1$ because of (3.29).

Furthermore, since A is continuous, there holds $Ax = 0$, and since A is injective, this means that $x = 0$. This contradicts $\|x\|_X = 1$. □

Now, we can combine the above lemmas and theorem to state the following result, which ensures the $\mathbf{V}$-ellipticity of $a(\cdot,\cdot)$:

Proposition 3.3 *Let Ω be an open bounded connected Lipschitz set of $\mathbb{R}^d$, with $\Gamma := \partial\Omega$. Let $\Gamma_D \subset \Gamma$ such that $|\Gamma_D| > 0$, and define $\mathbf{V}$ as*

$$\mathbf{V} := \left\{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d) \mid \Upsilon\mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\right\}.$$

Let $a(\cdot,\cdot)$ be the bilinear form defined as

$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{w}),$$

where the elasticity tensor $\mathbf{A}$ *verifies assumptions* (3.2)*,* (3.3)*, and* (3.4)*, of symmetry, uniform ellipticity, and uniform boundedness, respectively. Then, the bilinear form $a(\cdot,\cdot)$ is continuous on* $\mathbf{V}$ *and* $\mathbf{V}$*-elliptic, i.e., there exists* $\alpha > 0$ *such that, for all* $\mathbf{v} \in \mathbf{V}$*,*

$$a(\mathbf{v},\mathbf{v}) \geq \alpha \; \|\mathbf{v}\|^2_{1,\Omega}. \tag{3.30}$$

Proof The continuity of $a(\cdot,\cdot)$ has been established previously in Sect. 3.1.2, see (3.6). Let us prove now that $a(\cdot,\cdot)$ is $\mathbf{V}$-elliptic. First, observe that, due to the uniform ellipticity of $\mathbf{A}$ (property (3.3)),

$$\begin{aligned} a(\mathbf{v},\mathbf{v}) &= \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{v}), \\ &\geq \alpha_A \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) \\ &= \alpha_A \; \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{0,\Omega}. \end{aligned} \tag{3.31}$$

Then, let us apply the Peetre–Tartar Lemma 3.5. We make the choice: $X = \mathbf{V}$, $Y = L^2(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d)$, and $Z = L^2(\Omega; \mathbb{R}^d)$. The operator A is defined as

$$A : \begin{cases} X \to \quad Y \\ \mathbf{v} \mapsto \boldsymbol{\varepsilon}(\mathbf{v}). \end{cases}$$

We check that the operator A is linear and continuous. It is injective as a consequence of Lemma 3.4 and of the property $\mathbf{v} = \mathbf{0}$ on Γ_D for any $\mathbf{v} \in \mathbf{V}$. The operator T is chosen as the canonical injection from $\mathbf{V}$ into $L^2(\Omega; \mathbb{R}^d)$. We check that it is linear and continuous. Moreover, the operator T is compact due to Rellich's theorem 2.2. Korn's second inequality (3.26) (Theorem 3.3) provides exactly in this case the condition (3.27) of Peetre–Tartar Lemma 3.5. Then, we apply the lemma and get

$$c\|\mathbf{v}\|_{1,\Omega} \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,\Omega},$$

with $c > 0$. From the above inequality and (3.31), we get

$$a(\mathbf{v},\mathbf{v}) \geq c^2\alpha_A \; \|\mathbf{v}\|^2_{1,\Omega}, \tag{3.32}$$

which is (3.30) (with $\alpha = c^2\alpha_A$). □

Remark 3.8 Note the importance of the Dirichlet boundary condition in the previous result. Though, the situation where $|\Gamma_D| = 0$ can be of interest, and rigid body motions can be suppressed provided adequate conditions on the loads and the geometry of the contact boundary. This setting has been studied in [153], see, for instance, [153, Theorem 6.3]. □

3.3.2 The Well-posedness Result

Before stating the well-posedness result, we recall Stampacchia's theorem, which is a cornerstone in the theory of variational inequalities. Its proof can be found in the original paper of J.-L. Lions and G. Stampacchia [200] or in the book of H. Brezis [52, Theorem 5.6].

Theorem 3.4 *Let V be a Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Let K be a non-empty closed convex subset of V. Let $L(\cdot)$ be a continuous linear form on V. Let $a(\cdot,\cdot)$ be a continuous bilinear form on V. Suppose that $a(\cdot,\cdot)$ is V-elliptic: there exists $\alpha > 0$ such that, for any $v \in V$,*

$$a(v, v) \geq \alpha \|v\|^2.$$

Then, the variational inequality: find $u \in K$ such that

$$a(u, v - u) \geq L(v - u), \quad \forall v \in K,$$

admits one unique solution. Moreover, if $a(\cdot,\cdot)$ is symmetric, then the solution u to the above variational inequality is additionally the unique minimizer on K of the quadratic functional

$$\mathcal{J} : K \ni v \mapsto \frac{1}{2}a(v, v) - L(v) \in \mathbb{R}.$$

Remark 3.9 In the case where a symmetric bilinear form $a(\cdot,\cdot)$ is involved, Stampacchia's theorem is a reformulation of the result of existence and uniqueness of the projection onto a closed convex set in a Hilbert space. For a possibly non-symmetric bilinear form, the proof in [52] is based on Banach's fixed point theorem [52, Theorem 5.7]. □

We now collect all the previous results and state the main theorem of this section:

Theorem 3.5 *Let us make the same assumptions as in Proposition 3.3. Let us suppose also that $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{F} \in L^2(\Gamma_N; \mathbb{R}^d)$. Then, Problem (3.12) admits one unique solution $\mathbf{u} \in \mathbf{K}$. This solution is also the unique minimizer of $\mathcal{J}(\cdot)$ on the cone $\mathbf{K}$ (so Problems (3.12) and (3.10) are in fact equivalent).*

Moreover, the solution to (3.12) *is Lipschitz continuous with respect to the data. Indeed, for* $i = 1, 2$ *and some source terms* $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ *and* $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$*, denote by* $\mathbf{u}_i \in \mathbf{K}$ *the corresponding solution to* (3.12)*. Then there holds*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \le \frac{C}{\alpha}(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

where $C > 0$*, and* $\alpha > 0$ *is the* $\mathbf{V}$*-ellipticity constant associated with* $a(\cdot, \cdot)$*.* □

Proof The existence and uniqueness of a solution to Problem (3.12) is a consequence of Proposition 3.3 and Stampacchia's theorem 3.4. Stampacchia's theorem also ensures the equivalence between Problems (3.10) and (3.12).

Let us prove now that the solution to (3.12) is Lipschitz continuous with respect to the data. For $i = 1, 2$, we denote by L_i the linear form associated with the source terms $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$, and $\mathbf{u}_i \in \mathbf{K}$ is the corresponding solution to (3.12). We take successively $\mathbf{v} = \mathbf{u}_2$ and $\mathbf{v} = \mathbf{u}_1$ in the variational inequality (3.12) and get

$$a(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) \ge L_1(\mathbf{u}_2 - \mathbf{u}_1),$$

$$a(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \ge L_2(\mathbf{u}_1 - \mathbf{u}_2).$$

We add the two inequalities and get

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) \ge (L_1 - L_2)(\mathbf{u}_2 - \mathbf{u}_1)$$

or equivalently

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \le (L_1 - L_2)(\mathbf{u}_1 - \mathbf{u}_2).$$

We make use of the $\mathbf{V}$-ellipticity of $a(\cdot, \cdot)$ as well as the continuity of $(L_1 - L_2)(\cdot)$:

$$\alpha\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega}^2 \le a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \le \|L_1 - L_2\|\,\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega}.$$

This implies

$$\alpha\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \le \|L_1 - L_2\|.$$

With the definition of $(L_1 - L_2)(\cdot)$, there holds (see (3.7) in Sect. 3.1.2)

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \le \frac{C}{\alpha}(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

with $C > 0$. This ends the proof. □

Remark 3.10 In fact, from Proposition 3.3 and the definition of $\mathcal{J}'$, there holds, for $\mathbf{v}, \mathbf{w} \in \mathbf{K}$,

$$\begin{aligned}&\mathcal{J}'(\mathbf{v}; \mathbf{v}-\mathbf{w}) - \mathcal{J}'(\mathbf{w}; \mathbf{v}-\mathbf{w})\\ &= (a(\mathbf{v}, \mathbf{v}-\mathbf{w}) - L(\mathbf{v}-\mathbf{w})) - (a(\mathbf{w}, \mathbf{v}-\mathbf{w}) - L(\mathbf{v}-\mathbf{w}))\\ &= a(\mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w})\\ &\geq \alpha \|\mathbf{v}-\mathbf{w}\|_{1,\Omega}^2.\end{aligned}$$

This ensures that $\mathcal{J}(\cdot)$ is strongly convex. A continuous strongly convex function has a unique global minimizer, and this is another way to recover the well-posedness of Problem (3.10) [7]. □

Remark 3.11 A result similar to Theorem 3.5 holds for the scalar Signorini problem (3.24)–(3.25), as a consequence of Stampacchia's theorem 3.4, see, e.g., [138, Theorem 4.1]. □

3.4 Regularity

We first provide some information about what is expected from the regularity of the solution and then summarize what is known about the behavior of the solution on the contact boundary.

3.4.1 Global Regularity of the Solution

Unilateral contact problems are limited in regularity due to the Signorini conditions (3.23) on the contact boundary Γ_C: singularities arise at contact–noncontact transitions, i.e., between connected subsets of Γ_C where $u_{\mathbf{n}} = 0$ (contact) and $u_{\mathbf{n}} < 0$ (noncontact). Additionally, other singularities can be present, which we find usually in linear elasticity problems, and that are caused, for instance, by the data, by reentrent corners of the polygonal/polyhedral domain and by Dirichlet–Neumann transitions on other parts of the boundary (see, e.g., [141, 142]).

Some papers study precisely these regularity issues, such as the seminal paper of M. Moussaoui and K. Khodja [213] that concerns the two-dimensional scalar Signorini problem (3.24)–(3.25), with the Laplace equation in $\mathbb{R}^2$. From such studies, it is generally admitted that Sobolev regularity $H^{\frac{5}{2}}(\Omega; \mathbb{R}^d)$ cannot be passed beyond.

This regularity limitation has some important consequences at the numerical level. It explains why low-order finite elements, such as linear or quadratic Lagrange finite elements, are usually preferred, when a uniform mesh refinement is considered. In particular the use of quadratic finite element methods can be of interest for the most regular solutions lying in $H^s(\Omega; \mathbb{R}^d)$, $2 < s < 5/2$, and they are studied in, e.g., [30, 166, 227, 265]. The same occurs for more recent high-order discretization techniques, such as Hybrid High-Order (HHO) methods, see, for instance, [66, 75].

3.4.2 Binding/Nonbinding Transitions on the Contact Set

A widespread conjecture in the contact community is the following: in two dimensions, the set of binding/nonbinding transitions on the contact region is finite. This means that both the subsets

$$\{\mathbf{x} \in \Gamma_C \mid u_{\mathbf{n}}(\mathbf{x}) = 0\}, \quad \overline{\{\mathbf{x} \in \Gamma_C \mid u_{\mathbf{n}}(\mathbf{x}) < 0\}}$$

are each one a union of a finite number of closed intervals. This conjecture played indeed an important historical role in the numerical analysis of finite element methods for the Signorini Problem (3.12). It has been frequently used as an extra assumption to improve the rates of a priori error estimates associated with various finite element discretizations (see the Further Comments of Chap. 5 and the related references). In 2020, some progress has been made by T. Apel and S. Nicaise [14] for the scalar Signorini problem (3.24)–(3.25) in a polygonal domain. In this situation, the authors prove that this assumption of finiteness is true. Nevertheless, for the Signorini problem in small strain elasticity, this remains an open issue. Moreover, this assumption is not straightforward to extend to the three-dimensional case: see [264, Assumption 4.4].

As it will be detailed in the next chapters, such kind of assumption is no longer necessary provided that some appropriate arguments are used for the numerical analysis or that special discretization techniques are used to incorporate Signorini conditions (3.23).

3.5 Further Comments

Many books present contact or friction problems in a continuous setting: G. Duvaut and J.L. Lions [114], W. Han and M. Sofonea [147], or C. Eck et al. [115], to quote a few ones. See also the presentation of J. Haslinger, I. Hlaváček and J. Nečas in the Handbook of Numerical Analysis [153]. The presentation in this chapter has been inspired by the book of N. Kikuchi and J.T. Oden [181].

3.5.1 About Signorini Contact

Signorini problem has its origin in some works of A. Signorini [245, 246]. It was the first mathematical model of contact with such degree of generality and with a contact region that was not known in advance. The first mathematical analysis for the Signorini problem has been carried out by G. Fichera in 1963 [128]. G. Fichera proved the well-posedness thanks to a minimization argument in abstract (Banach) spaces.

These two pioneering works inspired G. Stampacchia and J.L. Lions to develop the general theory of variational inequalities. It started in 1964 with a first article by G. Stampacchia, concerned with existence results for abstract variational inequalities, possibly nonsymmetric [249]. It was followed by other articles by G. Stampacchia and J.L. Lions in 1967 [200] and by H. Brezis in 1968 [51]. The two earliest books about the mathematical theory of variational inequalities were published by G. Duvaut and J.L. Lions in 1972 [114] and D. Kinderlehrer and G. Stampacchia in 1980 [183].

3.5.2 About the Second Korn Inequality

The second Korn inequality is difficult to establish because, fundamentally, it is nothing but obvious to bound the whole gradient of the deformation using only the components of its symmetric part.

In $H_0^1(\Omega; \mathbb{R}^d)$, a well-known trick consists in taking advantage of the vanishing trace of the displacement to bound its H^1-semi-norm by the L^2-norm of the small strain tensor (see, e.g., [121, Theorem 42.9]). As a result, one obtains the first Korn inequality. This trick does not apply any longer when the displacement is supposed to be only in $H^1(\Omega; \mathbb{R}^d)$ and does not vanish on the boundary.

The Korn inequalities have been stated originally at the beginning of last century by Korn [185–187] and proved thoroughly later on by many authors. The interested reader can refer, for instance, to the monographs of Ph. G. Ciarlet [88] and of N. Kikuchi and J.T. Oden [181] for references to different classical proofs that have been published. Theorem 3.3, for a bounded Lipschitz domain, is an adaptation from the book of N. Kikuchi and J.T. Oden, where a detailed proof can be found. This proof is based on Calderon–Zygmund inequalities for singular integral equations. Notably, with this technique, a general result in Banach $W^{s,p}$-spaces is obtained. In the Hilbertian setting, another well-known path to prove the second Korn inequality is to rely on a delicate result of distributions/Sobolev theory, whose proof is not easy: see, for instance, [114, Theorem 3.2] or [207, Lemma 4.8]. An alternative elementary proof, especially for two-dimensional domains with polygonal boundary, has been provided by J.A. Nitsche and is based on the design of extension operators that preserve the small strain tensor [220].

Chapter 4
Lagrange Finite Elements and Interpolation

The aim of this chapter is to give a brief introduction to finite element spaces, and introduce some useful interpolation estimates in fractional order Sobolev spaces which will serve later on to establish corresponding *a priori* error estimates. For simplicity, we limit the presentation to Lagrange finite elements on triangular or tetrahedral meshes. For a more detailed description of the finite element method and its variants, one can consult the numerous works on the subject such as [50, 87, 89, 118, 230, 231]. Notably, the fact that only Lagrange elements on simplices are presented here does not prevent the results given in this book to have obvious extensions for many other types of elements, such as quadrangular or hexahedral elements, isoparametric elements, etc.

In this chapter Ω is a Lipschitz domain in $\mathbb{R}^d$ ($d = 2, 3$). For simplicity, the boundary $\Gamma := \partial\Omega$ is assumed to be polygonal (for $d = 2$) or polyhedral (for $d = 3$). As in Chap. 2, we denote as follows the vector space of d-variate real polynomials of maximal degree $k \geq 1$, viewed as functions in a domain $\mathcal{O} \subset \mathbb{R}^d$:

$$\mathbb{P}_k(\mathcal{O}) := \left\{ p : \mathcal{O} \to \mathbb{R} \;\middle|\; p(x_1, \dots, x_d) = \sum_{0 \leq i_1 + i_2 + \dots + i_d \leq k} \alpha_{i_1 i_2 \dots i_d}\, x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}, \right.$$
$$\left. \alpha_{i_1 i_2 \dots i_d} \in \mathbb{R} \right\}.$$

4.1 Lagrange Finite Elements on Simplices

We recall the most basic notions about the construction of Lagrange finite element spaces, on simplicial meshes. We first recall some notations about simplicial meshes, then define Lagrange finite element spaces, and show they are H^1-conforming. We end this section with extra useful notions and definitions.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_4

4.1.1 Simplicial Meshes

We take a family $(\mathcal{T}^h)_{h>0}$ of simplicial meshes of the domain Ω. For a given h, each geometrical element $T \in \mathcal{T}^h$ is a simplex of dimension d, a triangle for $d = 2$ and a tetrahedron for $d = 3$. It is supposed to be a closed subset of $\mathbb{R}^d$: $\overline{T} = T$ (and the same assumption holds for the edges and faces). We define the mesh size h precisely as follows:

$$h := \max_{T \in \mathcal{T}^h} h_T,$$

where h_T is the diameter of the simplex T. The mesh is supposed to be regular in Ciarlet's sense: there exists $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathcal{T}^h, \tag{4.1}$$

where ρ_T is the radius of the largest ball contained in the simplex T. Additionally, for any pair (T, T') in $\mathcal{T}^h \times \mathcal{T}^h$, the intersection $T \cap T'$ should be either the empty set, a vertex, an entire edge or an entire face of the two simplices. This prevents, for instance, a vertex of one element to lie in the middle of an edge of another element.

The edges of the triangles (when $d = 2$) or the faces of the tetrahedra (when $d = 3$) will be called facets. The notation E will be used for a generic facet in $\mathcal{T}^h$ and the notation h_E stands for the diameter of the facet E.

We suppose finally that the mesh resolves the boundary Γ, which means precisely that, for each simplex, each of its facets that intersects Γ is contained completely in one and only one of the facets of Γ. Finally, the mesh has to be a partition of Ω in the sense that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}^h} T.$$

See Fig. 4.1 for an example of valid mesh.

4.1.2 Lagrange Finite Elements

The Lagrange finite element space of degree (or order) $k \geq 1$ is (see, e.g., [50, 87, 89, 118, 231]):

$$X_k^h := \left\{ v^h \in \mathscr{C}^0(\overline{\Omega}) \;\middle|\; v^h|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}^h \right\}. \tag{4.2}$$

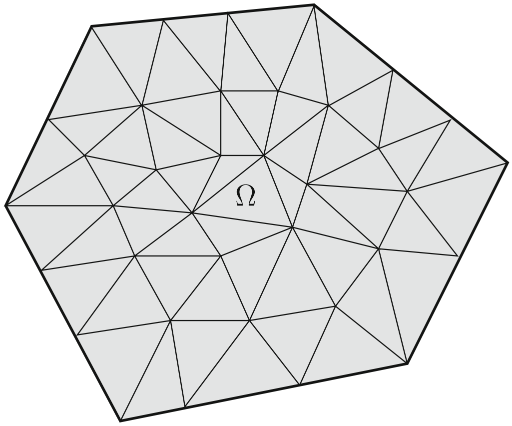

Fig. 4.1 Example of a valid triangular mesh of a polygonal domain Ω

We may omit the subscript k when there is no ambiguity and note simply X^h instead of X_k^h. It means that each function in X^h is a continuous, piecewise polynomial function, and its restriction to each simplex is a polynomial of order k. Therefore $(X^h)_{h>0}$ is a family of finite dimensional vector spaces indexed by h, built from the above family $(\mathcal{T}^h)_{h>0}$ of simplicial meshes of the domain Ω.

Similarly, for vector-valued functions, we define the corresponding Lagrange finite element space as follows:

$$\mathbf{X}_k^h := \left\{ \mathbf{v}^h \in \mathscr{C}^0(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h|_T \in \mathbb{P}_k(T; \mathbb{R}^d), \forall\, T \in \mathcal{T}^h \right\} = (X_k^h)^d. \tag{4.3}$$

Once again, we may drop the subscript k when there is no ambiguity. This definition will be particularly useful when we will deal with boundary value problems in elasticity.

4.1.3 Conformity

The space X^h defined above is H^1-conformal in the following sense:

Proposition 4.1 *Let Ω be a polygonal domain in $\mathbb{R}^2$ or a polyhedral domain in $\mathbb{R}^3$ and let X^h be the Lagrange finite element space defined in* (4.2)*. There holds*

$$X^h \subset H^1(\Omega).$$

Proof Remark first that any function $v^h \in X^h$ is a continuous, piecewise polynomial function, so it belongs to $L^2(\Omega)$ and is a regular distribution. Let us compute its distributional gradient, ∇v^h. We take $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; \mathbb{R}^d)$ a test function and compute:

$$\langle \nabla v^h, \boldsymbol{\varphi} \rangle = -\langle v^h, \operatorname{div} \boldsymbol{\varphi} \rangle = -\int_\Omega v^h (\operatorname{div} \boldsymbol{\varphi})$$
$$= -\sum_{T \in \mathcal{T}^h} \int_T v^h (\operatorname{div} \boldsymbol{\varphi}),$$

where we used the definition of the distributional gradient, the fact that v^h is a regular distribution, and where we split the integral into a sum over the simplices. Since in each simplex T, v^h and $\boldsymbol{\varphi}$ are smooth functions, we can apply the Green formula:

$$-\int_T v^h (\operatorname{div} \boldsymbol{\varphi}) = \int_T \nabla v^h \cdot \boldsymbol{\varphi} - \int_{\partial T} v^h (\boldsymbol{\varphi} \cdot \mathbf{n}_T),$$

where $\mathbf{n}_T$ is the outward unit normal vector to ∂T. When we sum once again on all the simplices, the boundary terms cancel, because $\boldsymbol{\varphi}$ is equal to 0 near Γ, and because it is smooth, so that, for each interior edge (or face), the two contributions from adjacent simplices are of opposite sign. Therefore

$$\langle \nabla v^h, \boldsymbol{\varphi} \rangle = \sum_{T \in \mathcal{T}^h} \int_T \nabla v^h \cdot \boldsymbol{\varphi}$$

which means that ∇v^h is a regular distribution: a piecewise polynomial function, discontinuous at the interfaces between simplices, that belongs to $L^2(\Omega)$. We proved that $v^h \in H^1(\Omega)$. □

Remark 4.1 The above proposition implies that the space $\mathbf{X}^h$ is also a subset of $H^1(\Omega; \mathbb{R}^d)$. □

4.1.4 Basis of Shape Functions

We denote by $N \in \mathbb{N}$ the dimension of the vector space X^h, and by $\varphi_1, \ldots, \varphi_N$ the nodal basis of shape functions of X^h. The corresponding (FE) nodes (or degrees of freedom) are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_N$, so that there holds

$$\varphi_i(\mathbf{a}_j) = \delta_{ij}$$

where $1 \le i, j \le N$ and δ_{ij} is Kronecker's symbol.

Take care that the above nodes correspond to the degrees of freedom of Lagrange finite elements: for $k = 1$ they coincide with the vertices of the geometrical mesh, but this is no more the case for $k \ge 2$, since some nodes lie on the edges (or faces) and, for $k \ge 3$, in the interior of the simplices (see [89] and Fig. 4.2).

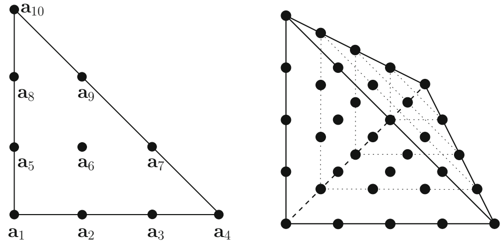

Fig. 4.2 Examples of standard Lagrange lattices for a triangular cubic element and a fourth degree tetrahedral element

We may sometimes split the set of (FE) nodes into two subsets: first $\mathrm{I} := 1, \ldots, N_{\text{int}}$ for the internal nodes, which means that:

$$\mathbf{a}_i \in \Omega, \quad \forall i \in \mathrm{I}.$$

Then, we define $\mathrm{B} := N_{\text{int}} + 1, \ldots, N$ the complementary set of nodes, which belong to the boundary Γ:

$$\mathbf{a}_i \in \Gamma, \quad \forall i \in \mathrm{B}.$$

Note we have chosen to index the nodes such that the first N_{int} nodes be internal nodes.

We may need to identify the (FE) nodes that lie only on a portion Γ_A of the boundary ($\Gamma_A \subset \Gamma$), and denote B_A the corresponding subset:

$$\mathbf{a}_i \in \Gamma_A, \quad \forall i \in \mathrm{B}_A.$$

In the above convention, A stands for a generic letter. Obviously we have $\mathrm{B}_A \subset \mathrm{B}$.

4.1.5 The Reference Element

For the derivation of interpolation estimates and related results, we will often map to the reference element, as in, *e.g.*, [118, 120, 230, 231]. We recall here the basic formulas related to this operation.

The reference element is denoted by $\widehat{T}$. We denote by J_T the Jacobian matrix of the linear geometric transformation from $\widehat{T}$ to a simplex T in a mesh $\mathcal{T}^h$. Then, we have the bounds [120, Lemma 11.1]:

$$|\det(J_T)| = \frac{|T|}{|\widehat{T}|}, \quad \|J_T\| \le \frac{h_T}{\rho_{\widehat{T}}}, \quad \|J_T^{-1}\| \le \frac{h_{\widehat{T}}}{\rho_T}, \tag{4.4}$$

where $\|J_T\| := \sup_{\widehat{\mathbf{x}}\neq 0}(\|J_T\widehat{\mathbf{x}}\|/\|\widehat{\mathbf{x}}\|)$ is the matrix norm associated with the usual Euclidean norm in $\mathbb{R}^d$ and $|T|$, $|\widehat{T}|$ stand for the measures of T, $\widehat{T}$.

4.2 Some Basic Interpolation Estimates

We present some interpolation estimates for Lagrange finite elements, in fractional order Sobolev spaces, that will serve to establish *a priori* error estimates in the next chapters. We start by defining the Lagrange interpolation operator, and then provide the interpolation estimates.

4.2.1 The Lagrange Interpolation Operator

The Lagrange interpolation operator, or Lagrange interpolator, associated with X^h will be denoted by $\mathcal{I}^h$. It is defined as follows, for any continuous function v in $\overline{\Omega}$:

$$\mathcal{I}^h(v) := \sum_{i=1,\ldots,N} v(\mathbf{a}_i)\varphi_i \in X^h.$$

For vector-valued functions, we define it similarly (and we use the same notation):

$$\mathcal{I}^h(\mathbf{v}) := \sum_{i=1,\ldots,N} \mathbf{v}(\mathbf{a}_i)\varphi_i \in \mathbf{X}^h.$$

4.2.2 Lagrange Interpolation on the Boundary

We denote by $\mathcal{I}^h_\Gamma$ the Lagrange interpolation operator associated with the boundary Γ: it is defined for any $w \in \mathscr{C}(\overline{\Gamma})$ by

$$\mathcal{I}^h_\Gamma(w) := \sum_{i\in\mathrm{B}} w(\mathbf{a}_i)\,\Upsilon\varphi_i,$$

where Υ is the trace operator. We will need later on the following useful property (see [118]):

Proposition 4.2 *The trace operator and the Lagrange interpolator are commuting, in the following sense:*

$$\Upsilon(\mathcal{I}^h v) = \mathcal{I}^h_\Gamma(\Upsilon v), \quad \forall v \in \mathscr{C}(\overline{\Omega}) \cap H^1(\Omega). \tag{4.5}$$

Proof Take $v \in \mathscr{C}(\overline{\Omega}) \cap H^1(\Omega)$, then we verify:

$$\begin{aligned}\Upsilon(\mathcal{I}^h v) &= \Upsilon\left(\sum_{i=1,\dots,N} v(\mathbf{a}_i)\varphi_i\right) = \sum_{i=1,\dots,N} v(\mathbf{a}_i)\,\Upsilon\varphi_i \\ &= \sum_{i\in \mathrm{B}}(\Upsilon v)(\mathbf{a}_i)\,\Upsilon\varphi_i = \mathcal{I}^h_\Gamma(\Upsilon v),\end{aligned}$$

where we used the linearity of the trace operator Υ and the fact that a nodal basis function φ_i vanishes on Γ if its corresponding node $\mathbf{a}_i$ is an interior node of the domain Ω. □

For Γ_A a subset of the boundary ($\Gamma_A \subset \Gamma$) we denote by $\mathcal{I}^h_{\Gamma_A}$ the corresponding Lagrange interpolation operator, defined for any $w \in \mathscr{C}(\overline{\Gamma_A})$ by

$$\mathcal{I}^h_{\Gamma_A}(w) := \sum_{i\in \mathrm{B}_A} w(\mathbf{a}_i)\,\Upsilon\varphi_i,$$

where Υ is the trace operator.

4.2.3 Estimates for Lagrange Interpolation

Here we state first the basic local error estimates for Lagrange interpolation, in the fractional setting. We follow the presentation of A. Quarteroni and A. Valli [230], with the appropriate adaptations.

Theorem 4.1 *Let k be the degree of the finite element space X^h_k and $d/2 < s \le 1 + k$. Let $u \in H^s(\Omega)$ where $\Omega \subset \mathbb{R}^d$. The following interpolation estimate holds*

$$|u - \mathcal{I}^h u|_{m,T} \le C h_T^{s-m} |u|_{s,T}, \tag{4.6}$$

for every $m \in \mathbb{N}$, $0 \le m < s$, and every simplex $T \in \mathcal{T}^h$. The positive constant C is independent of T and of h. □

Proof Using the scaling relationships and the regularity of the mesh, we deduce from a basic calculus

$$|u - \mathcal{I}^h u|_{m,T} \le C h_T^{-m} |\det J_T|^{\frac{1}{2}} |\hat{u} - \widehat{\mathcal{I}}^h \hat{u}|_{m,\hat{T}},$$

where $\widehat{\mathcal{I}}^h$ is the Lagrange interpolation operator in the reference coordinates and $\hat{u}(\hat{x}) = u(J_T(\hat{x}))$. Now, we consider the map

$$\begin{aligned}\mathscr{F} : H^s(\widehat{T}) &\to H^m(\widehat{T}),\\ \hat{u} &\mapsto \hat{u} - \widehat{\mathcal{I}}^h \hat{u},\end{aligned}$$

which is well-defined. Indeed, thanks to Theorem 2.3 and because of the assumption $s > \frac{d}{2}$, $\hat{u}$ is continuous on T and its Lagrange interpolation makes sense. Still from Theorem 2.3, there holds

$$|\hat{u}(\widehat{\mathbf{x}})| \leq C\|\hat{u}\|_{s,\widehat{T}}$$

for every $\mathbf{x} \in \widehat{T}$. Since $\widehat{\mathcal{I}}^h \hat{u}$ involves only nodal values of $\hat{u}$, it implies that, for $0 \leq m < s$

$$\|\widehat{\mathcal{I}}^h \hat{u}\|_{m,\widehat{T}} \leq C\|\hat{u}\|_{s,\widehat{T}}.$$

The above bound, combined with the triangular inequality, ensures then that $\mathscr{F}$ is continuous. Using the property $\mathscr{F}(\hat{p}) = 0$ for all $\hat{p} \in \mathbb{P}_k(\widehat{T})$, and then the continuity of $\mathscr{F}$, we can write

$$\begin{aligned}|\hat{u} - \widehat{\mathcal{I}}^h \hat{u}|_{m,\widehat{T}} &= |\mathscr{F}(\hat{u} + \hat{p})|_{m,\widehat{T}} \ \ \forall \hat{p} \in \mathbb{P}_k(\widehat{T}),\\ &\leq \|\mathscr{F}\|_{\mathscr{L}(H^s(\widehat{T}),H^m(\widehat{T}))}\|\hat{u} + \hat{p}\|_{s,\widehat{T}} \ \ \forall \hat{p} \in \mathbb{P}_k(\widehat{T})\\ &\leq C \inf_{\hat{p}\in\mathbb{P}_k(\widehat{T})} \|\hat{u} + \hat{p}\|_{s,\widehat{T}}.\end{aligned}$$

Define now

$$\underline{s} = \max\{n \in \mathbb{N} \mid n < s\},$$

and use the relationship

$$\underline{s} < s \leq 1 + k,$$

to infer that $\underline{s} \leq k$. Because of the inclusion $\mathbb{P}_{\underline{s}} \subset \mathbb{P}_k$ we get

$$\inf_{\hat{p}\in\mathbb{P}_k(\widehat{T})} \|\hat{u} + \hat{p}\|_{s,\widehat{T}} \leq \inf_{\hat{p}\in\mathbb{P}_{\underline{s}}(\widehat{T})} \|\hat{u} + \hat{p}\|_{s,\widehat{T}}.$$

Moreover, since $0 < s - \underline{s} \leq 1$, the assumption of the fractional Deny–Lions lemma is verified, see Lemma 2.3. Let us apply this Lemma and obtain

$$\inf_{\hat{p}\in\mathbb{P}_{\underline{s}}(\widehat{T})} \|\hat{u} + \hat{p}\|_{s,\widehat{T}} \leq C|\hat{u}|_{s,\widehat{T}}.$$

We combine the above relationships and get finally

$$|\hat{u} - \widehat{\mathcal{I}}^h \hat{u}|_{m,\widehat{T}} \leq C|\hat{u}|_{s,\widehat{T}}.$$

Now, proceeding to the reverse change of variable, and still using the regularity of the mesh and the expression of $|\hat{u}|_{s,\widehat{T}}$ (given also, for instance, in [112]) it comes

$$\begin{aligned}|u - \mathcal{I}^h u|_{m,T} &\leq Ch_T^{-m}|\det J_T|^{\frac{1}{2}}|\hat{u}|_{s,\widehat{T}}\\ &\leq Ch_T^{-m}h_T^s|u|_{s,T}\\ &= Ch_T^{s-m}|u|_{s,T}.\end{aligned}$$

This ends the proof. □

Global interpolation estimates on the whole domain Ω are obtained through simple summation of the local estimate (4.6) over all the simplices of the mesh. The non-local nature of the Sobolev-Slobodeckij semi-norm makes no difficulty in this situation. Indeed, using only the definition we get

$$\sum_{T\in\mathcal{T}^h} |u|_{s,T}^2 \leq |u|_{s,\Omega}^2.$$

Moreover, a similar result holds for the interpolation of the trace on the boundary, as stated below.

Corollary 4.1 *Let k be the degree of the finite element space X_k^h and $d/2 < s \leq 1/2 + k$. Let $u \in H^s(\Omega)$ where $\Omega \subset \mathbb{R}^d$. The following interpolation estimate holds*

$$|\Upsilon u - \mathcal{I}_\Gamma^h(\Upsilon u)|_{m,E} \leq Ch_E^{s-\frac{1}{2}-m}|\Upsilon u|_{s-\frac{1}{2},E}, \tag{4.7}$$

for every $m \in \mathbb{N}$, $0 \leq m < s - 1/2$, and every facet E in $\mathcal{T}^h$. The positive constant C is independent of E and of h. □

Proof For $u \in H^s(\Omega)$, its trace Υu on each facet Γ_j $(j = 1, \ldots, J)$ of the boundary is well-defined and belongs to $H^{s-\frac{1}{2}}(\Gamma_j)$, see Theorem 2.8. Then, we apply the previous result, Theorem 4.1, in dimension $d - 1$, for the function Υu (note that a facet E is still a simplex in dimension $d - 1$). □

4.3 Other Useful Results

We provide now some other general results useful for the analysis of some finite element methods. These are either additional approximation results or discrete inequalities.

4.3.1 Interpolation Estimate for the Gradient on the Boundary

The aim here is to provide a useful interpolation estimate for the gradient on the boundary, which plays a key role in the analysis of Nitsche's method for contact problems. We reproduce, with minor adaptations, the proof in [82].

Theorem 4.2 *Let k be the degree of the finite element space X_k^h and $3/2 < s \leq 1+k$. Let $u \in H^s(\Omega)$ where $\Omega \subset \mathbb{R}^d$. There holds*

$$\|\nabla(u - \mathcal{I}^h u)\|_{0,\Gamma\cap T} \leq C h_T^{s-\frac{3}{2}} |u|_{s,T}, \tag{4.8}$$

for every simplex T in the mesh $\mathcal{T}^h$ which shares one facet with the boundary Γ. The positive constant C in (4.8) is independent of T and of h. □

Proof Using the scaling relationships and the regularity of the mesh, we deduce from a basic calculus

$$\|\nabla(u - \mathcal{I}^h u)\|_{0,\Gamma\cap T} \leq C h_T^{\frac{d-3}{2}} \|\widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u})\|_{0,\hat{\Gamma}},$$

where $\hat{\Gamma}$ is the corresponding facet on the reference element, $\widehat{\nabla}$, $\widehat{\mathcal{I}}^h$ are the gradient and the Lagrange interpolation operator in the reference coordinates, respectively, and $\hat{u}(\hat{\mathbf{x}}) = u(J_T(\hat{\mathbf{x}}))$. Now, we consider the map

$$\begin{aligned} \mathscr{F} : H^s(\hat{T}) &\to L^2(\hat{\Gamma}; \mathbb{R}^d), \\ \hat{u} &\mapsto \Upsilon \widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u}). \end{aligned}$$

From the Trace Theorem 2.8 and since $s > 3/2$, we deduce that

$$\|\widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u})\|_{0,\hat{\Gamma}} \leq \|\widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u})\|_{s-\frac{3}{2},\hat{\Gamma}} \leq C \|\widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u})\|_{s-1,\hat{T}}$$

and with a similar argument as in Theorem 4.1 we conclude that $\mathscr{F}$ is continuous. Using the property $\mathscr{F}(\hat{p}) = 0$ for all $\hat{p} \in \mathbb{P}_k(\hat{T})$, we can write

$$\begin{aligned} \|\widehat{\nabla}(\hat{u} - \widehat{\mathcal{I}}^h \hat{u})\|_{0,\hat{\Gamma}} &= \|\mathscr{F}(\hat{u} + \hat{p})\|_{0,\hat{\Gamma}} \quad \forall \hat{p} \in \mathbb{P}_k(\hat{T}), \\ &\leq \|\mathscr{F}\|_{\mathscr{L}(H^s(\hat{T}), L^2(\hat{\Gamma}, \mathbb{R}^d))} \|\hat{u} + \hat{p}\|_{s,\hat{T}} \quad \forall \hat{p} \in \mathbb{P}_k(\hat{T}), \\ &\leq C \inf_{\hat{p} \in \mathbb{P}_k(\hat{T})} \|\hat{u} + \hat{p}\|_{\tau,\hat{T}} \\ &\leq C |\hat{u}|_{\tau,\hat{T}}. \end{aligned}$$

The last estimate is the application of the extension to fractional order spaces of Deny–Lions lemma, see Lemma 2.3, and the proof of Theorem 4.1. Now, proceeding to the reverse change of variable, and still using the regularity of the mesh and the expression of $|\hat{u}|_{s,\hat{T}}$ given also, for instance, in [112] it comes

$$
\begin{aligned}
\|\nabla(u - \mathcal{I}^h u)\|_{0,\Gamma\cap T} &\leq C h_T^{\frac{d-3}{2}} |\hat{u}|_{s,\hat{T}} \\
&\leq C h_T^{\frac{d-3}{2}} h_T^{-\frac{d-2s}{2}} |u|_{s,T} \\
&= C h_T^{s-\frac{3}{2}} |u|_{s,T}.
\end{aligned}
$$

□

Remark 4.2 This result is non-standard and cannot be established as a direct consequence of standard Lagrange interpolation estimates (the gradient and the Lagrange interpolator do not commute, and anyway, ∇u may not have enough Sobolev regularity to ensure continuity). Another way to prove this estimate is to make use in a specific way of the Scott-Zhang quasi-interpolator [130, 131]. □

4.3.2 Some Discrete Inverse Inequalities

Here we collect some other technical results useful for the numerical analysis of many methods. First of all, we present results for inverse inequalities that are a classical tool, see [120, Chapter 12] (or, for instance, [41, 89, 97]). In practice, they generally need to be combined with extra assumptions on the mesh $\mathcal{T}^h$. The simplest, and most popular one, is the option of quasi-uniformity. The mesh $\mathcal{T}^h$ is supposed to be quasi-uniform if there exists $c > 0$ independent of h and T, such that, for every T in $\mathcal{T}^h$, there holds:

$$ch \leq h_T.$$

Of course, this assumption is quite restrictive, but it can be alleviated if needed (see the Remark 4.3 below).
We state a first result below, which proof can be found in, e.g., [159, Lemma 4].

Proposition 4.3 *Let $\mathcal{T}^h$ be a quasi-uniform mesh and X_k^h be a Lagrange finite element space of degree $k \geq 1$, and let $0 \leq s \leq t \leq 1$. Let $d \geq 2$ and Ω be a Lipschitz polygonal or polyhedral domain, such that*

$$\partial\Omega = \cup_{j=1}^{J} \Gamma_j$$

with $J \geq 3$ and each Γ_j, $j = 1, \ldots, J$, a straight line segment ($d = 2$) or a planar polygon ($d = 3$). The following inverse inequality holds: there exists $C > 0$, such that:

$$\|v^h\|_{t,\Gamma_j} \leq C h^{s-t} \|v^h\|_{s,\Gamma_j}$$

for every $v^h \in X_k^h$ with C that depends on k, s, t, d. □

Remark 4.3 In fact, the assumption of global quasi-uniformity can be replaced, for instance, by local quasi-uniformity near the boundary [159, Lemma 4]. □

The following result will be of great help to establish well-posedness and optimal convergence of Nitsche's methods.

Lemma 4.1 *There exists $C > 0$, independent of the mesh size h, such that for all $v^h \in X_k^h$, and all $T \in \mathcal{T}^h$ such that $\partial T \cap \Gamma \neq \emptyset$:*

$$\|h_T^{\frac{1}{2}}(\Upsilon \nabla v^h)\|_{0,\partial T\cap\Gamma} \le C|v^h|_{1,T}.$$

Proof The estimate is obtained using a scaling argument (see also [256, Lemma 2.1, p.26] in the case of quasi-uniform families of meshes). Let $\hat{T}$ be the reference element and $\tau_T : \hat{T} \to T$ the corresponding affine geometric transformation. Then, for each element T having a whole facet on Γ, one has

$$\|h_T^{\frac{1}{2}}(\Upsilon\nabla v^h)\|^2_{0,\partial T\cap\Gamma} = \int_{\partial T\cap\Gamma} h_T|\nabla v^h|^2 \mathrm{d}s = \int_{\tau_T^{-1}(\partial T\cap\Gamma)} h_T|B_T^{-\mathrm{T}}\nabla v^h|^2|B_T^{-\mathrm{T}}\hat{\mathbf{n}}|J_T \mathrm{d}\hat{s},$$

where $B_T = \nabla\tau_T$, $J_T = \det(B_T)$ and $\hat{\mathbf{n}}$ is the unit outward normal vector to the facet of the reference element corresponding to $\tau_T^{-1}(\partial T \cap \Gamma)$. Note that B_T, J_T (resp. $\hat{\mathbf{n}}$) are constant over $\hat{T}$ (resp. $\tau_T^{-1}(\partial T \cap \Gamma)$) since the geometric transformation is affine. This implies that $B_T^{-\mathrm{T}}\nabla v^h \in \mathbb{P}_{k-1}(\hat{T};\mathbb{R}^d)$, and because all norms are equivalent in the finite dimensional vector space $\mathbb{P}_{k-1}(\hat{T};\mathbb{R}^d)$, one concludes there exists $D > 0$ such that

$$\|B_T^{-\mathrm{T}}\Upsilon(\nabla v^h)\|^2_{0,\tau_T^{-1}(\partial T\cap\Gamma)} \le D\|B_T^{-\mathrm{T}}\nabla v^h\|^2_{0,\hat{T}}, \quad \forall v^h \in X_k^h.$$

Moreover, due to the regularity of the family of meshes in Ciarlet's sense, there exists a constant $\beta > 0$ (depending only on the reference element) such that $|B_T^{-\mathrm{T}}\hat{\mathbf{n}}| \le \dfrac{\sigma\beta}{h_T}$, where σ is the constant in (4.1). Therefore there holds

$$\|h_T^{\frac{1}{2}}(\Upsilon\nabla v^h)\|^2_{0,\partial T\cap\Gamma} \le \sigma\beta\int_{\tau_T^{-1}(\partial T\cap\Gamma)} |B_T^{-\mathrm{T}}\nabla v^h|^2 J_T \mathrm{d}\hat{s},$$

and finally

$$\|h_T^{\frac{1}{2}}(\Upsilon\nabla v^h)\|^2_{0,\partial T\cap\Gamma} \le D\sigma\beta\int_{\hat{T}} |B^{-T}\nabla v^h|^2 J_T \mathrm{d}\hat{\mathbf{x}} = D\sigma\beta|v^h|^2_{1,T},$$

which is the result of the lemma. □

Remark 4.4 A proof of the above Lemma 4.1 can also be found in the book of V. Thomée for quasi-uniform meshes [256], and in the book of A. Ern and J.-L. Guermond [120], without restriction of quasi-uniformity. It is notable that this result holds whatever is the polynomial degree k of the finite element space. Of course, the discrete trace inverse constant C depends on this polynomial degree k and this dependency has been studied in [263], where expressions of C as a function of d and k are given for simplicial meshes and tensor-product meshes. $\square$

4.3.3 Discrete Liftings

We present in this section a result for liftings based on finite element spaces, which mimic the properties of the continuous liftings of Theorem 2.9. For this purpose we introduce a discrete trace space on a part $\Gamma_1 \subset \partial\Omega$ of the boundary:

$$W^h(\Gamma_1) := \left\{ \psi^h \in \mathscr{C}(\overline{\Gamma_1}) \;\middle|\; \exists v^h \in X^h \text{ such that } \Upsilon v^h = \psi^h \text{ on } \Gamma_1 \right\}.$$

The following lemma concerns the existence of a discrete bounded lifting from Γ_1 to Ω (for the proof, see, for instance, [44] or [109]):

Lemma 4.2 *Suppose that the mesh $\mathcal{T}^h$ resolves Γ_1 and that its restriction to Γ_1 is quasi-uniform. There exists $\mathcal{L}^h : W^h(\Gamma_1) \to X^h$ and $C > 0$, such that:*

$$\Upsilon \mathcal{L}^h(\psi^h)|_{\Gamma_1} = \psi^h, \quad \|\mathcal{L}^h(\psi^h)\|_{1,\Omega} \leq C\|\psi^h\|_{\frac{1}{2},\Gamma_1}, \tag{4.9}$$

for all $\psi^h \in W^h(\Gamma_1)$. $\square$

Of course, this result can be extended directly to finite element spaces of vector-valued functions. Note that discrete liftings are sometimes called discrete extension operators. They can be obtained by combining results for continuous liftings (such as Theorem 2.9) with a local regularization operator (as, e.g., [39, 242]).

4.3.4 Properties of Projection Operators

For the analysis of mixed methods, results on the stability and approximation properties of L^2-projections will be useful. Let $\Gamma_1 \subset \partial\Omega$ be once again a portion of the boundary, resolved by the mesh $\mathcal{T}^h$.

Let us introduce then, π^h, the global L^2-projection operator onto $W^h(\Gamma_1)$, the discrete trace space defined in the previous subsection. This projection operator is defined as follows for any function $\varphi \in L^2(\Gamma_1)$:

$$\pi^h\varphi \in W^h(\Gamma_1), \qquad \int_{\Gamma_1} (\pi^h\varphi - \varphi)\,\mu^h = 0, \qquad \forall\,\mu^h \in W^h(\Gamma_1). \tag{4.10}$$

First, we provide a stability property for this projection operator.

Proposition 4.4 *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $\Gamma_1 \subset \partial\Omega$, $\mathcal{T}^h$ be a quasi-uniform mesh, and X^h be the Lagrange finite element space of functions of degree k, with $W^h(\Gamma_1)$ the corresponding trace space on Γ_1. Let π_h be the corresponding*

L^2-projection operator onto $W^h(\Gamma_1)$, as defined above in (4.10). For all $s \in [0, 1]$ and all $\varphi \in H^s(\Gamma_1)$, we have the stability estimate:

$$\|\pi^h\varphi\|_{s,\Gamma_1} \leq C\|\varphi\|_{s,\Gamma_1}, \tag{4.11}$$

where $C > 0$ does not depend on v and h but depends on s. □

Proof From its definition, the operator π^h is stable in the $L^2(\Gamma_1)$-norm:

$$\|\pi^h\varphi\|_{0,\Gamma_1} \leq \|\varphi\|_{0,\Gamma_1}.$$

Now, let us consider $\varphi \in H^1(\Gamma_1)$ and define $\varphi^h := \pi_1^h\varphi \in W^h(\Gamma_1)$ where π_1^h is the projection operator from $H^1(\Gamma_1)$ onto $W^h(\Gamma_1)$, for the $H^1(\Gamma_1)$-norm, see [41]. There holds $\pi^h(\varphi^h) = \varphi^h$ and we use this property and the triangle inequality, to bound:

$$\|\pi^h\varphi\|_{1,\Gamma_1} \leq \|\pi^h(\varphi - \varphi^h)\|_{1,\Gamma_1} + \|\varphi^h\|_{1,\Gamma_1}.$$

Since the mesh is quasi-uniform, we can use an inverse inequality (Proposition 4.3) as follows:

$$\|\pi^h(\varphi - \varphi^h)\|_{1,\Gamma_1} \leq Ch^{-1}\|\pi^h(\varphi - \varphi^h)\|_{0,\Gamma_1} \leq Ch^{-1}\|\varphi - \varphi^h\|_{0,\Gamma_1},$$

where we used the L^2-stability of π^h. Using the approximation properties of π_1^h [41, Lemma 2.4] we obtain:

$$\|\pi^h(\varphi - \varphi^h)\|_{1,\Gamma_1} \leq C\|\varphi\|_{1,\Gamma_1}.$$

Then, by definition of π_1^h, there exists a constant C such that, for every $\varphi \in H^1(\Gamma_1)$:

$$\|\pi_1^h\varphi\|_{1,\Gamma_1} \leq C\|\varphi\|_{1,\Gamma_1}.$$

As a result we get

$$\|\pi^h\varphi\|_{1,\Gamma_1} \leq C\|\varphi\|_{1,\Gamma_1},$$

and we proved that the the operator π^h is stable in the $H^1(\Gamma_1)$ norm. Using an argument of Hilbertian interpolation, of index $0 \leq s \leq 1$ between $L^2(\Gamma_1)$ and $H^1(\Gamma_1)$, we conclude there exists a constant C independent of h such that:

$$\|\pi^h\varphi\|_{s,\Gamma_1} \leq C\|\varphi\|_{s,\Gamma_1}, \quad \forall\varphi \in H^s(\Gamma_1). \tag{4.12}$$

□

Remark 4.5 In the above proof, the quasi-uniformity of the mesh allowed to use an inverse inequality. However, note that the same stability property of π^h can be established with mesh assumptions less restrictive than global quasi-uniformity. For a discussion on this topic, see [120, Remark 22.23]. □

Approximation properties of the L^2-projection operator will also be useful for mixed methods, and are recalled in the statement below (for the proof, see, e.g., [41, 89, 97]).

Proposition 4.5 *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $\Gamma_1 \subset \partial\Omega$, $\mathcal{T}^h$ be a regular mesh, and X^h be the Lagrange finite element space of functions of degree k, with $W^h(\Gamma_1)$ the corresponding trace space on Γ_1. Let π_h be the corresponding L^2-projection operator onto $W^h(\Gamma_1)$, as defined above in (4.10). For all $r \in [0, k+1]$ and all $\varphi \in H^r(\Gamma_1)$, we have the approximation estimate:*

$$h^{-\frac{1}{2}} \|\varphi - \pi^h\varphi\|_{-\frac{1}{2},\Gamma_1} + \|\varphi - \pi^h\varphi\|_{0,\Gamma_1} \leq C\, h^r\, \|\varphi\|_{r,\Gamma_1}, \tag{4.13}$$

where $C > 0$ does not depend on φ and h. □

4.4 Further Comments

First, we provide some additional information about the finite element method, particularly the Lagrange finite elements on simplices. Also, since all the techniques presented in the next chapters are in fact applicable to all kind of variational (Galerkin) approximation methods, we provide some extra information about alternative variational techniques to the FEM, with emphasis on recent research.

4.4.1 FEM and Lagrange FEM

The general theory about Lagrange finite element spaces on simplices, their construction and their interpolation properties can be found in many textbooks about the finite element method. Notably, the construction of the Lagrange finite element spaces for an arbitrary dimension d and an arbitrary degree k is detailed carefully in the monographs of P.A. Raviart and J.M. Thomas [231], G. Allaire [7], as well as A. Ern and J.L. Guermond [118, 120]. See also the early paper of R.A. Nicolaides [218] that focuses specifically on this topic, as well as previous papers of M. Zlamal [269], for $d = 2$ and quadratic finite elements, and P.G. Ciarlet and C. Wagschal [90], still for quadratic finite elements and d arbitrary.

More generally, many textbooks exist about the mathematical theory of the finite element method. For instance, just to mention a few ones (!): [7, 42, 50, 87, 89, 118, 120–122, 129, 180, 230, 231, 250, 254].

We limited the presentation within this book to Lagrange FEM on simplices, but this is mostly to avoid technicalities for the presentation and the analysis of the methods, and to keep a unified framework as simple as possible. However, many methods for contact have been proposed and studied for many types of finite elements, such as, for instance, finite elements on quadrilaterals or Crouzeix–Raviart finite elements. Moreover, when the elasticity equations are formulated in terms of boundary integral equations, the Boundary Element Method (BEM) (see, for instance, [251]) can also be an alternative to the FEM.

4.4.2 Other Approximation Methods

Many variational approximation methods, alternative to standard finite elements, have been designed in the last decades, motivated either by theoretical considerations or practical purposes. Since most of the techniques to incorporate contact conditions such as Signorini conditions are not limited to finite elements, but are also relevant for other methods, we review them below, with corresponding references for the reader who wants to know more about the ideas behind, their mathematical properties and implementations issues (the list below may be far from being exhaustive).

Discontinuous Galerkin (DG) methods are now a well-established field of methods that have emerged in 1970–1980: see, for instance, the pioneering works [16, 196] and the recent monograph [106]. DG methods are known for their flexibility, in terms of approximation order or relatively to the mesh. In fact, they accept even general polytopal meshes, and not only simplicial meshes: see, e.g., [202] and references therein. Motivated by polytopal meshes, many methods have emerged recently as alternatives to DG, such as Hybrid Discontinuous Galerkin (HDG) [93], Hybrid High-Order (HHO) methods [91, 93, 105, 194] or the Virtual Element Method (VEM) [29, 194]. Based on some ideas of DG, Discontinuous Petrov Galerkin (DPG) methods have been object of many research in the last decade [102].

Another popular class of methods, motivated by the link between computer aided design and numerical simulation, is IsoGeometric Analysis (IGA) [96, 216]. Notably, for these methods, the degrees of freedom are control points of B-spline functions and are not nodes on the boundary [26].

Last but not least, let us mention unfitted finite elements or geometrically nonconforming finite elements, where the mesh boundary and the domain boundary do not match, as it occurs in fictitious domain methods, the eXtended Finite Element Method (XFEM) or the cut Finite Element Method (cutFEM): see, e.g., [47, 58, 140, 155, 211, 217, 226]. These methods are particularly important since they are motivated, among other, for domains with cracks, crack propagation, etc, and other applications in which contact and friction play a major role.

Part II
Numerical Approximation for Signorini

Chapter 5
Finite Elements for Signorini

This chapter is devoted to the finite element approximation of the Signorini problem (3.12) described in previous Chap. 3. We start with the simplest possible setting just to point out the main difficulty that appears in the convergence analysis of many methods. We then present discretizations based on a direct approximation of the variational inequality (3.12), with different alternatives to build discrete convex cones that approximate $\mathbf{K}$. After this, we focus on the derivation of optimal a priori error estimates in the natural H^1-norm related to the approximate solution.

5.1 Preliminaries

We consider Signorini's problem (3.12) with the same assumptions on the domain, the boundary, the source terms, and the elasticity tensor as stated in Chap. 3, e.g.,

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \qquad \forall\, \mathbf{v} \in \mathbf{K}, \end{cases} \tag{5.1}$$

where

$$\mathbf{K} := \{\mathbf{v} \in \mathbf{V} \mid v_{\mathbf{n}} \leq 0 \text{ on } \Gamma_C\},$$

and $\mathbf{V} := \left\{\mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \;\middle|\; \Upsilon \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\right\}$. Let us emphasize that we suppose that the Dirichlet boundary Γ_D is of positive Lebesgue measure and compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$. We start by introducing the finite element spaces considered in the rest of this chapter and then present a very simple discretization of Problem (5.1) using linear finite elements, followed by a preliminary, suboptimal, convergence analysis.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_5

5.1.1 Finite Element Spaces

We recall here some notations of Chap. 4. We choose standard continuous Lagrange finite elements of degree k, with $k = 1$ or $k = 2$, i.e.,

$$\mathbf{V}_k^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h_{|T} \in \mathbb{P}_k(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\},$$

where $\mathcal{T}^h$ is a simplicial mesh of the domain Ω and where $\mathbb{P}_k(T; \mathbb{R}^d)$ stands for the space of all vector-valued polynomials on T, of degree lower than or equal to k in the d variables. We suppose that the mesh $\mathcal{T}^h$ resolves the contact boundary Γ_C, which means that all the edges ($d = 2$) or faces ($d = 3$) that intersect Γ_C are included in $\overline{\Gamma_C}$. We suppose also that it resolves the Neumann boundary Γ_N and the Dirichlet boundary Γ_D.

5.1.2 A Discrete Variational Inequality

We define the discrete convex cone using linear finite elements

$$\mathbf{K}^h := \mathbf{K} \cap \mathbf{V}_1^h$$

and introduce the discrete problem:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ such that} \\ a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \geq L(\mathbf{v}^h - \mathbf{u}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{K}^h. \end{cases} \tag{5.2}$$

As a result, we discretize the variational inequality (5.1) in a rather direct manner, using a finite dimensional subset of the convex cone $\mathbf{K}$ of admissible displacements. As for the continuous formulation (5.1), Problem (5.2) is equivalent to find the minimizer on $\mathbf{K}^h$ of the functional

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}),$$

defined in (3.8) for any displacement field $\mathbf{v} \in \mathbf{V}$. Formulation (5.2) is a conforming method, since $\mathbf{K}^h \subset \mathbf{K}$, and also a consistent method, since the solution $\mathbf{u}$ to Problem (5.1) verifies

$$a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}) \geq L(\mathbf{v}^h - \mathbf{u}), \qquad \forall\, \mathbf{v}^h \in \mathbf{K}^h.$$

5.1.3 A Preliminary Error Estimate

The set of admissible displacements $\mathbf{K}^h$ is a non-empty closed convex set. Moreover, $a(\cdot,\cdot)$ is still continuous and elliptic on $\mathbf{V}^h$, as it is on $\mathbf{V}$ (see Proposition 3.3). The same occurs for $L(\cdot)$ which is continuous on $\mathbf{V}^h$. We can apply once again Stampacchia's theorem 3.4, to ensure that Problem (5.2) is well-posed: it admits one unique solution $\mathbf{u}^h \in \mathbf{K}^h$. We now derive an abstract error estimate, in the spirit of Falk's lemma [126]:

Proposition 5.1 *Suppose that the solution* $\mathbf{u} \in \mathbf{K}$ *to Problem* (5.1) *belongs to* $H^s(\Omega;\mathbb{R}^d)$ *with* $s > 3/2$. *Then the solution* $\mathbf{u}^h$ *to Problem* (5.2) *satisfies the a priori error estimate:*

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} \leq C \inf_{\mathbf{v}^h\in\mathbf{K}^h}\left(\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + \left(\int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v^h_{\mathbf{n}}-u_{\mathbf{n}})\right)^{\frac{1}{2}}\right), \tag{5.3}$$

with $C > 0$ *independent from* h *and* $\mathbf{u}$. □

Proof Let $\mathbf{v}^h \in \mathbf{K}^h$. We use the ellipticity of $a(\cdot,\cdot)$, combined with the Cauchy–Schwarz inequality:

$$\begin{aligned}\alpha\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} &\leq a(\mathbf{u}-\mathbf{u}^h,\mathbf{u}-\mathbf{u}^h)\\ &= a(\mathbf{u}-\mathbf{u}^h,(\mathbf{u}-\mathbf{v}^h)+(\mathbf{v}^h-\mathbf{u}^h))\\ &\leq C\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + a(\mathbf{u}-\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h),\end{aligned}$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot,\cdot)$. Then we use the Young inequality:

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} \leq \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|^2_{1,\Omega} + a(\mathbf{u},\mathbf{v}^h-\mathbf{u}^h) - a(\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h). \tag{5.4}$$

Since $\mathbf{v}^h$ belongs to $\mathbf{K}^h$, and $\mathbf{u}^h$ is the solution to (5.2):

$$-a(\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h) \leq -L(\mathbf{v}^h-\mathbf{u}^h).$$

Now, since $\mathbf{u}$ is a also solution to the elasticity equations (3.22), using the Green formula (3.15) and taking into account Dirichlet, Neumann, and no friction conditions (3.23), we get

$$a(\mathbf{u},\mathbf{v}^h-\mathbf{u}^h) = L(\mathbf{v}^h-\mathbf{u}^h) + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v^h_{\mathbf{n}}-u^h_{\mathbf{n}}).$$

Remark that above we wrote the boundary term as an integral term. This has been made possible thanks to the regularity assumption $\mathbf{u} \in H^s(\Omega;\mathbb{R}^d)$ with $s > \frac{3}{2}$ and using the Trace Theorem 2.6, from which we obtain $\sigma_{\mathbf{n}}(\mathbf{u}) \in L^2(\Gamma_C)$. We combine the two previous relationships with the estimate (5.4):

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h).$$

There remains to transform the last term, as follows:

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h) = \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}) + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}}^h.$$

We use the contact conditions (3.23) and the property $u_{\mathbf{n}}^h \leq 0$ (since $\mathbf{u}^h \in \mathbf{K}^h$):

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} = 0, \quad -\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}}^h \leq 0.$$

We obtain finally

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}),$$

which gives the desired error bound (5.3). □

Therefore we can obtain genuinely a first estimation of the convergence rate for the finite element approximation (5.2). Indeed, we make the choice $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$, where $\mathcal{I}^h$ is the Lagrange interpolation operator introduced in Chap. 4. Due to the regularity assumption $\mathbf{u} \in H^s(\Omega;\mathbb{R}^d)$, $s > \frac{3}{2}$, and the Sobolev embedding theorem (Theorem 2.3), $\mathbf{u}$ is a continuous function, and its Lagrange interpolation can be defined. We verify $\mathcal{I}^h\mathbf{u} \in \mathbf{K}^h$ since $\mathbf{u} \in \mathbf{K}$ and since $\mathcal{I}^h$ preserves the positivity for $\mathbb{P}_1$ Lagrange finite elements. As a result, there holds

$$(\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} \leq 0.$$

Now, let us bound the two right terms in the estimate (5.3). The first term corresponds to the interpolation estimate in the H^1-norm within the domain Ω and is classical. We just apply Theorem 4.1:

$$\|\mathbf{u}-\mathcal{I}^h\mathbf{u}\|_{1,\Omega} \leq Ch^{s-1}|\mathbf{u}|_{s,\Omega} \tag{5.5}$$

for $\frac{3}{2} < s \leq 2$. For the second term of (5.3), which is the boundary term that comes from the contact condition on Γ_C, the simplest bound is obtained thanks to Cauchy–Schwarz inequality and then from the interpolation estimate on the trace (Corollary 4.1):

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})((\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} - u_{\mathbf{n}}) \le \|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,\Gamma_C} \, \|(\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} - u_{\mathbf{n}}\|_{0,\Gamma_C}$$

$$\le C\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,\Gamma_C} \, h^{s-\frac{1}{2}} \|u_{\mathbf{n}}\|_{s-\frac{1}{2},\Gamma_C}$$

where we used the property (4.5). We take the square root and get

$$\left(\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})((\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} - u_{\mathbf{n}})\right)^{\frac{1}{2}} \le Ch^{\frac{s}{2}-\frac{1}{4}} \|\sigma_{\mathbf{n}}(\mathbf{u})\|^{\frac{1}{2}}_{0,\Gamma_C} \|u_{\mathbf{n}}\|^{\frac{1}{2}}_{s-\frac{1}{2},\Gamma_C}.$$

We use the obvious bound $\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,\Gamma_C} \le \|\sigma_{\mathbf{n}}(\mathbf{u})\|_{s-\frac{3}{2},\Gamma_C}$, and we apply two times the Trace Theorem 2.8:

$$\left(\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})((\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} - u_{\mathbf{n}})\right)^{\frac{1}{2}} \le Ch^{\frac{s}{2}-\frac{1}{4}} \|\mathbf{u}\|_{s,\Omega}.$$

Finally, we combine the above result with (5.5) and obtain

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \le Ch^{\frac{s}{2}-\frac{1}{4}} \|\mathbf{u}\|_{s,\Omega}, \tag{5.6}$$

since $(s-1) > (\frac{s}{2} - \frac{1}{4})$.

The above result is suboptimal, since, for instance, for a Sobolev regularity $s = 2$, it provides only a convergence rate $\mathcal{O}(h^{\frac{3}{4}})$, instead of the expected rate $\mathcal{O}(h)$ that comes from the interpolation estimate and that we find usually for elliptic problems with standard boundary conditions (such as small strain elasticity with mixed Neumann/Dirichlet boundary conditions).

This is the starting point of a long story that started in year 1970 with, at first, such suboptimal bounds. Much effort has been dedicated since then to improve these first results (see the comments and bibliography at the end of this chapter). One possibility to overcome this issue is to assume, in two dimensions, that the number of binding/nonbinding transitions on Γ_C is finite, so that the region where $u_{\mathbf{n}} < 0$ is made of a finite number of open intervals, and the same for the region where $\sigma_{\mathbf{n}} < 0$. Though this assumption may seem reasonable at first glance, it has no sound mathematical justification. It has been proved valid only recently [14] and solely for a scalar problem in two dimensions (see Chap. 3). We will see later on, that, in fact, this assumption is superfluous, and with some effort, optimal bounds in the H^1-norm can be recovered. Before providing details about this result, we will first introduce some variants of discretized Problem (5.2), obtained considering different usual discrete convex cones, for linear and quadratic finite elements, in two dimensions and in three dimensions.

Another issue concerns the numerical implementation of the discretization (5.2): solving the discrete variational inequality is not the easiest option in practice and not at all the favorite option of practitioners. It is usual to reformulate first contact conditions (3.23) in order to get a problem that can be solved easily with standard numerical techniques. One possibility consists in transforming it into a nonlinear variational equation, but with no more inequality constraints. Another possibility consists in transferring all the inequality constraints from the bulk to the boundary. These techniques will be detailed in the next chapters. Particularly, we will show that, for the convergence analysis of some of them, the same difficulty as above is recovered. For some other techniques, which are based on a reformulation of the Signorini conditions (3.23), this difficulty does not appear any longer.

5.2 Finite Element Approximation with Various Cones

Before improving the first analysis made in the previous section, let us introduce different variants for the discrete problem (5.2), which are common in the contact literature. This will allow us to be slightly more general. So we first describe in detail the construction of various convex cones that mimic $\mathbf{K}$, the convex cone of admissible displacements that enforce the nonpenetration condition $u_{\mathbf{n}} \leq 0$, and present the corresponding discrete variational inequalities. These discrete cones are built differently for two-dimensional problems ($d = 2$) and three-dimensional problems ($d = 3$). Also, depending on the order of the simplicial Lagrange finite elements, there are different possibilities to approximate the nonpenetration condition.

5.2.1 The Convex Cones for Linear Lagrange Finite Elements

We first focus on linear Lagrange finite elements ($k = 1$). The simplest discrete set of admissible displacements satisfying the nonpenetration conditions on the contact zone is, as introduced previously in Sect. 5.1.2,

$$\mathbf{K}_1^h := \mathbf{V}_1^h \cap \mathbf{K}.$$

Remark here that we change slightly the notation and that subscript 1 will allow us to differentiate this discrete cone from other ones that will come later on. The above discrete cone can be redefined more explicitly as

$$\mathbf{K}_1^h = \left\{ \mathbf{v}^h \in \mathbf{V}_1^h \;\middle|\; v_{\mathbf{n}}^h \leq 0 \quad \text{on } \Gamma_C \right\},$$

which means that the nonpenetration condition is imposed strongly for each node that belongs to the contact boundary, since the mesh $\mathcal{T}^h$ resolves the contact boundary Γ_C.

Alternatively we can take into account the discrete nonpenetration in average on any contact facet $T \cap \Gamma_C$:

$$\overline{\mathbf{K}_1^h} := \left\{ \mathbf{v}^h \in \mathbf{V}_1^h \,\middle|\, \int_{\partial T \cap \Gamma_C} v_{\mathbf{n}}^h \leq 0,\ \forall T \in \mathcal{T}^h \right\}.$$

The following inclusion relation holds for the continuous and discrete convex cones:

- $\mathbf{K}_1^h \subset \mathbf{K}$, by definition
- $\overline{\mathbf{K}_1^h} \not\subset \mathbf{K}$ since the definition allows the discrete normal displacement $v_{\mathbf{n}}^h$ to have positive values on some portions of the facets
- $\mathbf{K}_1^h \subset \overline{\mathbf{K}_1^h}$ as a straightforward consequence of their respective definitions.

5.2.2 The Convex Cones for Quadratic Lagrange Finite Elements

In the following we denote by $\mathbf{a}_i$, $0 \leq i \leq I$, the vertices of the triangulation or tetrahedralization located in $\overline{\Gamma_C}$ and by $\mathbf{m}_j$, $0 \leq j \leq J$, the midpoints of the contact elements when $d = 2$ (i.e., the midpoints of the segments in $\overline{\Gamma_C}$). When $d = 3$, the $\mathbf{m}_j$ $(0 \leq j \leq J)$ are the midpoints of the contact element edges (i.e., the midpoints of the edges of the triangles in $\overline{\Gamma_C}$). The first discrete set of admissible displacements satisfies the nonpenetration conditions both at the vertices and at the midpoints:

$$\mathbf{K}_2^h := \left\{ \mathbf{v}^h \in \mathbf{V}_2^h \,\middle|\, v_{\mathbf{n}}^h(\mathbf{a}_i) \leq 0,\ 0 \leq i \leq I,\ \ v_{\mathbf{n}}^h(\mathbf{m}_j) \leq 0,\ 0 \leq j \leq J \right\}.$$

The second cone involves an average nonpenetration condition on any contact facet:

$$\overline{\mathbf{K}_2^h} := \left\{ \mathbf{v}^h \in \mathbf{V}_2^h \,\middle|\, \int_{\partial T \cap \Gamma_C} v_{\mathbf{n}}^h \leq 0,\ \forall T \in \mathcal{T}^h \right\}.$$

The third cone is a combination, specific to the quadratic case, of both previous cases:

$$\widehat{\mathbf{K}_2^h} := \left\{ \mathbf{v}^h \in \mathbf{V}_2^h \,\middle|\, \int_{\partial T \cap \Gamma_C} v_{\mathbf{n}}^h \leq 0,\ \forall T \in \mathcal{T}^h,\ \ v_{\mathbf{n}}^h(\mathbf{a}_i) \leq 0,\ 0 \leq i \leq I \right\}.$$

In the case $d = 3$, we are interested in a fourth convex cone:

$$\widetilde{\mathbf{K}_2^h} := \left\{ \mathbf{v}^h \in \mathbf{V}_2^h \,\middle|\, v_{\mathbf{n}}^h(\mathbf{m}_j) \leq 0,\ 0 \leq j \leq J \right\}.$$

In fact, the similar definition when $d = 2$ does not lead to interesting convergence properties.

Note that neither of the previous convex cones is a subset of $\mathbf{K}$.

Proposition 5.2 *The following inclusion relationships hold for the discrete convex cones:*

$$\mathbf{K}_2^h \subset \widehat{\mathbf{K}_2^h} \subset \overline{\mathbf{K}_2^h}, \quad \mathbf{K}_2^h \subset \widetilde{\mathbf{K}_2^h} \subset \overline{\mathbf{K}_2^h}.$$

Proof First, the inclusions $\widehat{\mathbf{K}_2^h} \subset \overline{\mathbf{K}_2^h}$ and $\mathbf{K}_2^h \subset \widetilde{\mathbf{K}_2^h}$ are straightforward. In the case $d = 2$, the inclusion $\mathbf{K}_2^h \subset \widehat{\mathbf{K}_2^h}$ is a consequence of the Simpson quadrature rule along a contact edge.

When $d = 3$, the inclusions $\mathbf{K}_2^h \subset \widehat{\mathbf{K}_2^h}$ and $\widetilde{\mathbf{K}_2^h} \subset \overline{\mathbf{K}_2^h}$ come from the quadrature of order two on the triangle (see, e.g., [89, 118]): for any $\mathbf{v}^h \in \mathbf{V}_2^h$, there holds

$$\int_{\partial T \cap \Gamma_C} v_{\mathbf{n}}^h = \frac{|\partial T \cap \Gamma_C|}{3} \sum_{j=1}^{3} v_{\mathbf{n}}^h(\mathbf{m}_j),$$

where $|\partial T \cap \Gamma_C|$ stands for the surface of $\partial T \cap \Gamma_C$ and $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}_3$ represent the three midpoints of the edges. From the previous quadrature, we see that the integral on $\partial T \cap \Gamma_C$ of the three basis functions at the vertices vanishes. □

5.2.3 The Discrete Problems

When $\mathbf{K}^h$ is one of the (eleven) previous convex cones (five when $d = 2$ and six when $d = 3$), the discrete variational inequality issued from (5.1) is

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ satisfying} \\ a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \geq L(\mathbf{v}^h - \mathbf{u}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{K}^h. \end{cases} \tag{5.7}$$

According to Stampacchia's theorem 3.4, Problem (5.7) admits a unique solution, which is also the unique minimizer on $\mathbf{K}^h$ of the functional

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}),$$

defined in (3.8) for any displacement field $\mathbf{v} \in \mathbf{V}$.

5.2.4 An Abstract Lemma

To establish the a priori error bounds, we need an abstract tool in the spirit of Falk's lemma [126, 153, 240]. It is recalled hereafter with its proof. It extends the preliminary result of Proposition 5.1, in order to take into account nonconforming approximations ($\mathbf{K}^h \not\subset \mathbf{K}$).

Lemma 5.1 *Suppose that the solution* $\mathbf{u}$ *to Problem (5.1) belongs to* $H^s(\Omega;\mathbb{R}^d)$ *with* $s > 3/2$. *Let* $\mathbf{u}^h$ *be the solution to the discrete problem (5.7). Then*

$$c\,\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} \le \inf_{\mathbf{v}^h\in\mathbf{K}^h}\left(\|\mathbf{u}-\mathbf{v}^h\|^2_{1,\Omega} + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v^h-u)_{\mathbf{n}}\right) + \inf_{\mathbf{v}\in\mathbf{K}}\int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v-u^h)_{\mathbf{n}}, \tag{5.8}$$

where c *is a positive constant which only depends on the continuity and the ellipticity constants of* $a(\cdot,\cdot)$. □

Proof We start exactly as in the proof of Proposition 5.1 until we obtain

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} \le \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|^2_{1,\Omega} + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v^h-u)_{\mathbf{n}} + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} - \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})u^h_{\mathbf{n}}.$$

From the contact conditions (3.23), we know that $\sigma_{\mathbf{n}}(\mathbf{u}) \le 0$. Therefore, for any $\mathbf{v}\in\mathbf{K}$, we deduce the property

$$-\int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})u^h_{\mathbf{n}} \le \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v-u^h)_{\mathbf{n}}.$$

This property, combined with the contact condition $\sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} = 0$, gives

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} \le \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|^2_{1,\Omega} + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v^h-u)_{\mathbf{n}} + \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})(v-u^h)_{\mathbf{n}}.$$

We obtain finally the desired result (5.8) by taking both infimums. □

Remark 5.1 The additional term in (5.8), comparatively to (5.3), measures the error due to the nonconformity of the discrete cone $\mathbf{K}^h$. Of course, this term vanishes when $\mathbf{K}^h \subset \mathbf{K}$ and we recover then (5.3). □

We are ready to derive optimal error bounds for the above problem. For the presentation, it will be simpler to start with the two-dimensional case. As we will see later on, the three-dimensional case follows the same path but introduces some extra technical issues.

5.3 Error Analysis in the Two-Dimensional Case

The following two theorems yield optimal convergence rates in the two-dimensional case when considering either linear or quadratic finite elements. Both theorems only use the standard Sobolev regularity assumption of the solution $\mathbf{u}$ to the Signorini problem (5.1). We first state a result for linear Lagrange finite elements.

Theorem 5.1 *Let us suppose that $d = 2$ and $k = 1$. Set $\mathbf{K}^h = \mathbf{K}_1^h$ or $\mathbf{K}^h = \overline{\mathbf{K}_1^h}$. Let $\mathbf{u}$ and $\mathbf{u}^h$ be the solutions to Problem (5.1) and to Problem (5.7), respectively. Assume that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$ with $3/2 < s \leq 2$. Then, there exists a constant $C > 0$ independent of h and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq Ch^{s-1}\|\mathbf{u}\|_{s,\Omega}. \tag{5.9}$$

Then, we state a similar result for quadratic Lagrange finite elements:

Theorem 5.2 *Let us suppose that $d = 2$ and $k = 2$. Set $\mathbf{K}^h = \mathbf{K}_2^h$, $\mathbf{K}^h = \widehat{\mathbf{K}_2^h}$ or $\mathbf{K}^h = \overline{\mathbf{K}_2^h}$. Let $\mathbf{u}$ and $\mathbf{u}^h$ be the solutions to Problem (5.1) and to Problem (5.7), respectively. Assume that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$ with $3/2 < s \leq 5/2$. Then, there exists a constant $C > 0$ independent of h and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq Ch^{s-1}\|\mathbf{u}\|_{s,\Omega}. \tag{5.10}$$

5.3.1 Some Local Estimates for the Contact Stress and the Normal Displacement

This section contains preliminary results that allow to recover optimal error estimates, without any superfluous assumption on the contact set. We start with a simple estimate on the interpolation error that involves the L^1-norm of the gradient.

Proposition 5.3 *Let $d = 2$, then for every edge E in the mesh $\mathcal{T}^h$, and for every function $w \in H^s(E)$, with $s > 1$, there holds*

$$\|w - \mathcal{I}^h w\|_{0,E} \leq Ch_E^{1/2}\|w'\|_{L^1(E)}, \tag{5.11}$$

where the constant $C > 0$ is independent of E and h, $\mathcal{I}^h$ is the Lagrange interpolation operator mapping onto $\mathbf{V}_k^h$, h_E is the length of the edge E, and w' is the derivative of w along E. □

Proof Without loss of generality, we can set $E := (0, h_E)$. Let us take $w \in H^s(E)$. Since $s > 1$, and from Theorem 2.3, w is a continuous function, and its Lagrange interpolant $\mathcal{I}^h w$ is well-defined. Moreover, its distributional derivative belongs to

$L^2(E)$ and thus to $L^1(E)$. Set $\varphi := w - \mathcal{I}^h w$. Note that $\varphi(0) = \varphi(h_E) = 0$, so for any $t \in [0, h_E]$, we have

$$\varphi(t) = \int_0^t \varphi'(\xi)\mathrm{d}\xi,$$

and then

$$|\varphi(t)| \leq \int_0^t |\varphi'(\xi)|\mathrm{d}\xi \leq \|\varphi'\|_{L^1(E)}.$$

So we first bound:

$$\|w - \mathcal{I}^h w\|_{0,E}^2 = \int_0^{h_E} \varphi(t)^2 \mathrm{d}t \leq h_E \|\varphi'\|_{L^1(E)}^2.$$

It remains to estimate $\|\varphi'\|_{L^1(E)}$. When $k = 1$, since the derivative of the Lagrange interpolant corresponds to the mean value of the derivative, we use the relationship

$$(\mathcal{I}^h w)' = \frac{w(h_E) - w(0)}{h_E} = \frac{1}{h_E}\int_E w'(\xi)\mathrm{d}\xi,$$

which implies

$$\|(\mathcal{I}^h w)'\|_{L^1(E)} \leq \|w'\|_{L^1(E)}.$$

So

$$\|\varphi'\|_{L^1(E)} = \|w' - (\mathcal{I}^h w)'\|_{L^1(E)} \leq \|w'\|_{L^1(E)} + \|(\mathcal{I}^h w)'\|_{L^1(E)} \leq 2\|w'\|_{L^1(E)},$$

where we used the triangular inequality. When $k = 2$, the same conclusion holds (then $\|(\mathcal{I}^h w)'\|_{L^1(E)} \leq \frac{5}{4}\|w'\|_{L^1(E)}$, and we omit the details). So (5.11) holds. $\square$

Remark 5.2 The above proposition can also be recovered using more general error estimates for $W^{s,p}$-spaces (see, e.g., [89, Theorem 16.1]) or the interpolation estimate in Theorem 4.1 and then an inverse inequality for L^p-norms [120, Example 12.4].

For $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$, $3/2 < s \leq 2$, the solution to Problem (5.1), and using the Trace Theorem 2.8, we deduce that $u_{\mathbf{n}} \in H^{s-1/2}(\Gamma_C)$. Also, we have $\sigma_{\mathbf{n}}(\mathbf{u}) \in H^{s-3/2}(\Gamma_C)$ and $u'_{\mathbf{n}} \in H^{s-3/2}(\Gamma_C)$, where $u'_{\mathbf{n}}$ denotes the derivative of $u_{\mathbf{n}}$ along Γ_C. Let $T \in \mathcal{T}^h$ with $\partial T \cap \Gamma_C \neq \emptyset$. To simplify, we use the notation $E := \partial T \cap \Gamma_C$ to denote the corresponding contact edge of the triangle T, and h_E denotes the length of E.

We define Z_C and Z_{NC} as the contact and the noncontact sets in E, respectively, i.e.,

$$Z_C := \{\mathbf{x} \in E \mid u_{\mathbf{n}}(\mathbf{x}) = 0\},$$
$$Z_{NC} := \{\mathbf{x} \in E \mid u_{\mathbf{n}}(\mathbf{x}) < 0\}.$$

Since $u_{\mathbf{n}}$ belongs to $H^{s-1/2}(\Gamma_C)$ when $3/2 < s \le 2$, the Sobolev embeddings (see Theorem 2.3) ensure that $u_{\mathbf{n}} \in \mathscr{C}(\overline{\Gamma_C})$. As a result, the sets Z_C and Z_{NC} are measurable, as inverse images of Borel sets by a continuous function. We denote by $|Z_C|$ and $|Z_{NC}|$ their respective Lebesgue measures in $\mathbb{R}$, so $|Z_C| + |Z_{NC}| = h_E$.

Thereafter, we provide the gold lemma that states some optimal estimates in the L^1-norm and the L^2-norm, for the contact stress $\sigma_{\mathbf{n}}$ and the gradient of the normal displacement $u'_{\mathbf{n}}$, on the contact boundary Γ_C. This lemma will be used later on to derive optimal error estimates for the finite element approximations of the variational inequality (5.1), notably in the proof of Theorem 5.1. Note that it concerns Sobolev regularities lower than $5/2$.

Lemma 5.2 *Let $d = 2$. Set $3/2 < s < 5/2$. Let h_E be the length of the contact edge $E := T \cap \Gamma_C$, with $|Z_C|$ and $|Z_{NC}|$ that stand for the measures in $\mathbb{R}$ of the contact and noncontact sets Z_C and Z_{NC} in E, respectively. Assume that $|Z_C| > 0$ and $|Z_{NC}| > 0$. The following L^1 and L^2-estimates hold for $\sigma_{\mathbf{n}}(\mathbf{u})$ and $u'_{\mathbf{n}}$:*

$$\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E} \le \frac{1}{|Z_{NC}|^{1/2}} h_E^{s-1} |\sigma_{\mathbf{n}}(\mathbf{u})|_{s-3/2,E}, \tag{5.12}$$

$$\|u'_{\mathbf{n}}\|_{L^1(E)} \le \frac{|Z_{NC}|^{1/2}}{|Z_C|^{1/2}} h_E^{s-1} |u'_{\mathbf{n}}|_{s-3/2,E}. \tag{5.13}$$

□

Proof We begin with estimate (5.12). Since $\mathbf{u}$ is a solution to (5.1), with enough regularity ($\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$ with $s > 3/2$), the unilateral contact conditions (3.23) hold almost everywhere on the contact boundary (see Theorem 3.2) so we deduce that $\sigma_{\mathbf{n}}(\mathbf{u}) = 0$ almost everywhere on Z_{NC}. Therefore, if we set $\sigma_{\mathbf{n}} := \sigma_{\mathbf{n}}(\mathbf{u})$, to alleviate the notations, we can use this relationship combined with Fubini's theorem and start to transform the L^2-norm of the contact stress as follows:

$$\begin{aligned}
\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E}^2 &= \int_{Z_C} \sigma_{\mathbf{n}}(s)^2 \,\mathrm{d}s \\
&= |Z_{NC}|^{-1} \left(\int_{Z_{NC}} \mathrm{d}t \right) \left(\int_{Z_C} \sigma_{\mathbf{n}}(s)^2 \,\mathrm{d}s \right) \\
&= |Z_{NC}|^{-1} \int_{Z_C} \int_{Z_{NC}} \sigma_{\mathbf{n}}(s)^2 \,\mathrm{d}t \,\mathrm{d}s \\
&= |Z_{NC}|^{-1} \int_{Z_C} \int_{Z_{NC}} (\sigma_{\mathbf{n}}(s) - \sigma_{\mathbf{n}}(t))^2 \,\mathrm{d}t \,\mathrm{d}s.
\end{aligned} \tag{5.14}$$

We set $\nu := s-3/2$, and the assumption $3/2 < s < 5/2$ translates into $0 < \nu < 1$. It just remains now to introduce in (5.14) the terms related to the Sobolev–Slobodeckij semi-norm and to bound them appropriately as follows:

$$\begin{aligned}
\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E}^2 &= |Z_{NC}|^{-1} \int_{Z_C} \int_{Z_{NC}} \frac{(\sigma_{\mathbf{n}}(s) - \sigma_{\mathbf{n}}(t))^2}{|s-t|^{1+2\nu}} |s-t|^{1+2\nu} \,\mathrm{d}t\,\mathrm{d}s \\
&\le |Z_{NC}|^{-1} \sup_{(s,t)\in Z_C \times Z_{NC}} \left(|s-t|^{1+2\nu}\right) \\
&\quad \times \int_{Z_C} \int_{Z_{NC}} \frac{(\sigma_{\mathbf{n}}(s) - \sigma_{\mathbf{n}}(t))^2}{|s-t|^{1+2\nu}} \,\mathrm{d}t\,\mathrm{d}s \\
&\le |Z_{NC}|^{-1} h_E^{1+2\nu} |\sigma_{\mathbf{n}}|_{\nu,E}^2 \\
&= |Z_{NC}|^{-1} h_E^{2(s-1)} |\sigma_{\mathbf{n}}|_{s-3/2,E}^2,
\end{aligned} \tag{5.15}$$

which proves (5.12).

The remaining estimate (5.13) deals with $u'_{\mathbf{n}}$. We need here the non-trivial result of Theorem 2.4. Indeed, since Z_C is the set of level 0 of the continuous function $u_{\mathbf{n}}$, it results that $u'_{\mathbf{n}} = 0$ almost everywhere on Z_C. So the estimate (5.13) is proved following the same path as for (5.12) by changing $\sigma_{\mathbf{n}}$ with $u'_{\mathbf{n}}$ and inverting the sets Z_C and Z_{NC}. We just need to start with a Cauchy–Schwarz inequality:

$$\begin{aligned}
\|u'_{\mathbf{n}}\|_{L^1(E)}^2 &= \left(\int_{Z_{NC}} |u'_{\mathbf{n}}(s)| \,\mathrm{d}s\right)^2 \\
&\le \left(\int_{Z_{NC}} \mathrm{d}s\right) \left(\int_{Z_{NC}} u'_{\mathbf{n}}(s)^2 \,\mathrm{d}s\right) \\
&= |Z_{NC}||Z_C|^{-1} \left(\int_{Z_C} \mathrm{d}t\right) \left(\int_{Z_{NC}} u'_{\mathbf{n}}(s)^2 \,\mathrm{d}s\right) \\
&= |Z_{NC}||Z_C|^{-1} \int_{Z_{NC}} \int_{Z_C} u'_{\mathbf{n}}(s)^2 \,\mathrm{d}t\,\mathrm{d}s \\
&= |Z_{NC}||Z_C|^{-1} \int_{Z_{NC}} \int_{Z_C} (u'_{\mathbf{n}}(s) - u'_{\mathbf{n}}(t))^2 \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.16}$$

From the above bound (5.16) and doing exactly the same as previously in (5.15), we obtain the estimate (5.13). □

Based on this preliminary material, we detail now the proof of the two main theorems. We start with the proof for linear Lagrange finite elements.

5.3.2 Error Analysis for Linear Finite Elements

We start from estimate (5.8) and take $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$, where $\mathcal{I}^h$ is the Lagrange interpolation operator mapping onto $\mathbf{V}_1^h$, as introduced in Chap. 4. Since $\mathbf{u} \in \mathbf{K}$ and since linear interpolation on each edge preserves the sign, we deduce that

$$\mathcal{I}^h\mathbf{u} \in \mathbf{K}_1^h \subset \overline{\mathbf{K}_1^h}.$$

Moreover, from Theorem 4.1, we get

$$\|\mathbf{u} - \mathcal{I}^h\mathbf{u}\|_{1,\Omega} \le Ch^{s-1}\|\mathbf{u}\|_{s,\Omega}$$

for any $1 < s \le 2$.

5.3.2.1 Conforming Approximation

First, we deal with the case $\mathbf{K}^h = \mathbf{K}_1^h$ in which the second infimum in (5.8) disappears, as $\mathbf{K}_1^h \subset \mathbf{K}$. To prove the theorem in this case, it remains then to estimate the term

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}},$$

for $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$, $3/2 < s \le 2$. In the forthcoming proof, we will estimate

$$\int_{E} \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}} \tag{5.17}$$

for each contact edge E in the mesh $\mathcal{T}^h$. Recall that E is partitioned into a contact set Z_C and a noncontact set Z_{NC}. If either $|Z_C|$ or $|Z_{NC}|$ equals zero, then it is easy to see that the integral term (5.17) vanishes. Indeed, if $|Z_C| = 0$, from conditions (3.23), we deduce that $\sigma_{\mathbf{n}}(\mathbf{u}) = 0$ almost everywhere on E. Otherwise, if $|Z_{NC}| = 0$, this means that $u_{\mathbf{n}} = 0$ almost everywhere on E, and so does its Lagrange interpolant $\mathcal{I}^h u_{\mathbf{n}}$.

So we suppose that $|Z_C| > 0$ and $|Z_{NC}| > 0$ in the following estimation of (5.17). We next obtain two estimates of the same error term (5.17): a first one depending on $|Z_{NC}|$ and a second one depending on $|Z_C|$.

Estimate of (5.17) depending on Z_{NC} We use the Cauchy–Schwarz inequality, estimate (5.12) in Lemma 5.2, and the standard error estimate on $\mathcal{I}^h$ (Corollary 4.1), so

$$\begin{aligned}\int_E \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}} &\le \|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E}\|(\mathcal{I}^h u - u)_{\mathbf{n}}\|_{0,E}\\ &\le C\ \frac{h_E^{s-1}}{|Z_{NC}|^{1/2}}|\sigma_{\mathbf{n}}(\mathbf{u})|_{s-3/2,E}\ h_E^{s-1/2}|u'_{\mathbf{n}}|_{s-3/2,E}\\ &\le C\ \frac{h_E^{2s-3/2}}{|Z_{NC}|^{1/2}}\left(|\sigma_{\mathbf{n}}(\mathbf{u})|^2_{s-3/2,E} + |u'_{\mathbf{n}}|^2_{s-3/2,E}\right). \quad (5.18)\end{aligned}$$

Estimate of (5.17) depending on Z_C This estimate is obtained in a different way. We now use the error estimate (5.11) on $\mathcal{I}^h$ based on a L^1-norm, together with bounds (5.12) and (5.13) in Lemma 5.2:

$$\begin{aligned}\int_E \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}} &\le \|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E}\|(\mathcal{I}^h u - u)_{\mathbf{n}}\|_{0,E}\\ &\le C\|\sigma_{\mathbf{n}}(\mathbf{u})\|_{0,E} h_E^{1/2}\|u'_{\mathbf{n}}\|_{L^1(E)}\\ &\le C\frac{1}{|Z_{NC}|^{1/2}} h_E^{s-1}|\sigma_{\mathbf{n}}(\mathbf{u})|_{s-3/2,E}\\ &\quad \times h_E^{1/2}\frac{|Z_{NC}|^{1/2}}{|Z_C|^{1/2}} h_E^{s-1}|u'_{\mathbf{n}}|_{s-3/2,E}\\ &\le C\frac{h_E^{2s-3/2}}{|Z_C|^{1/2}}\left(|\sigma_{\mathbf{n}}(\mathbf{u})|^2_{s-3/2,E} + |u'_{\mathbf{n}}|^2_{s-3/2,E}\right). \quad (5.19)\end{aligned}$$

We conclude by noting that either $|Z_{NC}|$ or $|Z_C|$ is greater than $h_E/2$ and by choosing the convenient estimate (5.18) or (5.19):

$$\int_E \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}} \le C h_E^{2(s-1)}\left(|\sigma_{\mathbf{n}}(\mathbf{u})|^2_{s-3/2,E} + |u'_{\mathbf{n}}|^2_{s-3/2,E}\right).$$

By summation, using the shape regularity of the mesh and the Trace Theorem 2.8, we get

$$\begin{aligned}\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h u - u)_{\mathbf{n}} &\le C h^{2(s-1)}\left(|\sigma_{\mathbf{n}}(\mathbf{u})|^2_{s-3/2,\Gamma_C} + |u'_{\mathbf{n}}|^2_{s-3/2,\Gamma_C}\right)\\ &\le C h^{2(s-1)}\|\mathbf{u}\|^2_{s,\Omega},\end{aligned}$$

from which (5.9) follows when $\mathbf{K}^h = \mathbf{K}^h_1$.

5.3.2.2 Nonconforming Approximation

We now consider the case $\mathbf{K}^h = \overline{\mathbf{K}_1^h}$. As previously we can choose $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$ since $\mathcal{I}^h\mathbf{u} \in \overline{\mathbf{K}_1^h}$. The first infimum in (5.8) therefore satisfies the same optimal bound as in the case $\mathbf{K}^h = \mathbf{K}_1^h$. The second infimum in (5.8) is handled by choosing $\mathbf{v} = \mathbf{0}$. To prove the theorem, it remains then to estimate the term

$$-\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u^h_{\mathbf{n}}.$$

This term can be bounded optimally as well: see [110] for the detailed proof.

5.3.3 *Error Analysis for Quadratic Finite Elements*

The proof is split into two parts, the first one dealing with $3/2 < s < 5/2$ and the second one when $s = 5/2$.

5.3.3.1 Sobolev Regularity $3/2 < s < 5/2$

Since the following inclusions hold:

$$\mathbf{K}_2^h \subset \widehat{\mathbf{K}_2^h} \subset \overline{\mathbf{K}_2^h}, \tag{5.20}$$

we only have to prove the optimal approximation error bound when considering $\mathbf{K}_2^h$ and the optimal consistency error bound when considering $\overline{\mathbf{K}_2^h}$.

- Approximation error (when $\mathbf{K}^h = \mathbf{K}_2^h$):
  We choose $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$, where $\mathcal{I}^h$ is the Lagrange interpolation operator mapping onto $\mathbf{V}_2^h$, so that $\mathbf{v}^h \in \mathbf{K}_2^h$. Combining the standard error estimate on $\mathcal{I}^h$ (see Theorem 4.1) and Lemma 5.2 (which holds for $3/2 < s < 5/2$) in the same way as for linear Lagrange finite elements (i.e., when $\mathbf{K}^h = \mathbf{K}_1^h$) gives us the optimal approximation bound.
- Consistency error (when $\mathbf{K}^h = \overline{\mathbf{K}_2^h}$):
  We choose (again) $\mathbf{v} = \mathbf{0}$ and proceed in the same way as for $\mathbf{K}^h = \overline{\mathbf{K}_1^h}$, see [110] for the details. □

As a result, (5.10) holds when $3/2 < s < 5/2$ for $\mathbf{K}^h = \mathbf{K}_2^h$, $\mathbf{K}^h = \widehat{\mathbf{K}_2^h}$ or $\mathbf{K}^h = \overline{\mathbf{K}_2^h}$.

5.3.3.2 Sobolev Regularity $s = 5/2$

In this case, we need to prove (see (5.10)) that

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq Ch^{3/2}\|\mathbf{u}\|_{5/2,\Omega}. \tag{5.21}$$

In this situation, the proof is not that difficult and takes advantage of the continuity of the functions $\sigma_{\mathbf{n}}$ and $u'_{\mathbf{n}}$ on the contact boundary. See once again [110] for the detailed proof.

5.4 Error Analysis in the Three-Dimensional Case

Before giving the error estimates, we first explain that the two-dimensional proof could not be extended straightforwardly to the three-dimensional case although the key Lemma 5.2 still holds when $d = 3$. To simplify, we first consider the case $d = 3$ for linear Lagrange finite elements. In order to extend “straightforwardly” the optimal estimate (5.9), we would need that all the intermediary estimates used in the proof of Theorem 5.1 remain true in the three-dimensional case, and in particular

$$\|u_{\mathbf{n}} - \mathcal{I}^h u_{\mathbf{n}}\|_{0,E} \leq C\|\nabla u_{\mathbf{n}}\|_{L^1(E)}. \tag{5.22}$$

Actually, we cannot prove (5.22). This inequality is not satisfied when $d = 3$ due to the fact that the functions of $W^{1,1}(\Gamma_C)$ are not necessarily continuous when Γ_C is a subset of $\mathbb{R}^2$. So, we could not extend straightforwardly the optimal error bounds to the three-dimensional case. We next explain how we circumvent such difficulties.

5.4.1 Extreme Points and New Discrete Convex Cones

To circumvent (5.22), the basic idea is to change the Lagrange interpolation operator with a quasi-interpolation operator adapted to rough functions (see, e.g., [92, 162, 242, 253]). We also need some positivity preserving properties and more than first order accuracy (which requires, roughly speaking, that affine functions are locally reproduced). Since the functions we consider do not necessarily vanish on the boundary of Γ_C, there is an impossibility result at the extreme points of $\overline{\Gamma_C}$ (see [221]) which forces us to slightly change the convex sets $\mathbf{K}_1^h$ and $\overline{\mathbf{K}_1^h}$ on the triangles containing an extreme point of $\overline{\Gamma_C}$. We first recall the definition of extreme points, see [221]:

Definition 5.1 A point $\mathbf{e} \in \partial\Gamma_C$ is an extreme point of $\overline{\Gamma_C}$ if there exists an affine function a_e such that

$$a_e(\mathbf{e}) = 0 \quad \text{and} \quad a_e(\mathbf{x}) > 0, \forall \mathbf{x} \in \overline{\Gamma_C},\ \mathbf{x} \neq \mathbf{e}.$$

In other words, $\mathbf{e} \in \partial\Gamma_C$ is an extreme point of $\overline{\Gamma_C}$ if it does not lie in any open segment joining two points of $\overline{\Gamma_C}$. Therefore, a square contains 4 extreme points (the 4 corners), and an L-shaped domain contains 5 extreme points (the reentrant corner is not an extreme point).

Let N_e be the set of extreme points of $\overline{\Gamma_C}$. If $\mathbf{e} \in N_e$, let $\Delta_e \subset \overline{\Gamma_C}$ be the union of triangles (i.e., the patch) having $\mathbf{e}$ as vertex and set

$$E_e := \bigcup_{\mathbf{e} \in N_e} \Delta_e.$$

So we define

$$\mathbf{K}_1^{h,e} := \left\{ \mathbf{v}^h \in \mathbf{V}_1^h \;\middle|\; v_{\mathbf{n}}^h \leq 0 \quad \text{on } \Gamma_C \setminus E_e, \quad \int_{\Delta_e} v_{\mathbf{n}}^h \leq 0, \forall \mathbf{e} \in N_e \right\}, \tag{5.23}$$

$$\begin{aligned} \overline{\mathbf{K}_1^{h,e}} := \Bigg\{ \mathbf{v}^h \in \mathbf{V}_1^h \;\Bigg|\; & \int_{\partial T \cap \Gamma_C} v_{\mathbf{n}}^h \leq 0, \forall T \in \mathcal{T}^h, \partial T \cap \Gamma_C \subset \Gamma_C \setminus E_e, \\ & \int_{\Delta_e} v_{\mathbf{n}}^h \leq 0, \forall \mathbf{e} \in N_e \Bigg\}. \end{aligned} \tag{5.24}$$

□

We remark that neither of these two convex cones belongs to $\mathbf{K}$. We have $\mathbf{K}_1^{h,e} \subset \overline{\mathbf{K}_1^{h,e}}$, $\mathbf{K}_1^h \subset \mathbf{K}_1^{h,e}$, and $\overline{\mathbf{K}_1^h} \subset \overline{\mathbf{K}_1^{h,e}}$, and if any extreme point of $\overline{\Gamma_C}$ belongs to only one contact triangle, then $\overline{\mathbf{K}_1^h} = \overline{\mathbf{K}_1^{h,e}}$.

5.4.2 Main Results

We state the main results below. To prove them, we will use inverse inequalities on Γ_C and we suppose that the trace mesh on Γ_C is quasiuniform of characteristic diameter $h_C \leq h$.

Theorem 5.3 *Let $d = 3$ and $k = 1$. Set $\mathbf{K}^h = \mathbf{K}_1^{h,e}$ or $\mathbf{K}^h = \overline{\mathbf{K}_1^{h,e}}$. Let $\mathbf{u}$ and $\mathbf{u}^h$ be the solutions to Problems (5.1) and (5.7), respectively. Assume that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^3)$ with $3/2 < s \leq 2$. Then, there exists a constant $C > 0$ independent of h and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq C h^{s-1} \|\mathbf{u}\|_{s,\Omega}. \tag{5.25}$$

Before proving Theorem 5.3, we give the convergence result in the quadratic case. As for linear Lagrange finite elements, see (5.23) and (5.24), we can define the modified convex sets $\mathbf{K}_2^{h,e}$, $\widehat{\mathbf{K}_2^{h,e}}$, $\widetilde{\mathbf{K}_2^{h,e}}$, $\overline{\mathbf{K}_2^{h,e}}$ by keeping the same definitions as for $\mathbf{K}_2^h$, $\widehat{\mathbf{K}_2^h}$, $\widetilde{\mathbf{K}_2^h}$, $\overline{\mathbf{K}_2^h}$ (see Sect. 5.2.2) on the triangles of $\Gamma_C \setminus E_e$ (i.e., except on the patches Δ_e, where $\mathbf{e}$ is an extreme point of $\overline{\Gamma_C}$). On the patches, the nonpenetration condition becomes as in the linear case $\int_{\Delta_e} v_{\mathbf{n}}^h \le 0$.

Theorem 5.4 *Let $d = 3$ and $k = 2$. Set $\mathbf{K}^h = \mathbf{K}_2^{h,e}$ or $\mathbf{K}^h = \widehat{\mathbf{K}_2^{h,e}}$ or $\mathbf{K}^h = \widetilde{\mathbf{K}_2^{h,e}}$ or $\mathbf{K}^h = \overline{\mathbf{K}_2^{h,e}}$. Let $\mathbf{u}$ and $\mathbf{u}^h$ be the solutions to Problems (5.1) and (5.7), respectively. Assume that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^3)$ with $3/2 < s \le 5/2$. Then, there exists a constant $C > 0$ independent of h and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \le C h^{s-1} \|\mathbf{u}\|_{s,\Omega}. \tag{5.26}$$

5.4.3 A Quasi-Interpolation Operator

The following proof of Theorem 5.3 requires a specific quasi-interpolation operator whose construction and properties are given below. Let W_1^h be the normal trace space of $\mathbf{V}_1^h$ on Γ_C. The quasi-interpolation operator

$$\mathcal{J}_1^h : W^{1,1}(\Gamma_C) \to W_1^h$$

is defined as follows for a function v: for interior nodes $\mathbf{a}$ in Γ_C, we choose the definition of the Chen–Nochetto operator which is positivity preserving and also preserves local affine functions:

$$(\mathcal{J}_1^h v)(\mathbf{a}) = \frac{1}{|B|} \int_B v,$$

where B is the largest open ball of center $\mathbf{a}$ such that B is contained in the union of the elements containing $\mathbf{a}$ (see [71]) and $|B|$ stands for its measure. Next we consider the nodes on the boundary of $\overline{\Gamma_C}$. For the boundary nodes $\mathbf{a}$ in $\overline{\Gamma_C} \cap \overline{\Gamma_D}$, we set $(\mathcal{J}_1^h v)(\mathbf{a}) = 0$. For the other boundary nodes $\mathbf{a}$ which are not extreme points, we set (see [221])

$$(\mathcal{J}_1^h v)(\mathbf{a}) = \frac{1}{|L|} \int_L v,$$

where L is a (small) line segment of length $|L| = \alpha h$ (α is fixed), symmetrically placed around $\mathbf{a}$, and included in $\overline{\Gamma_C}$. Such a definition is both positivity and affine functions preserving. Finally, we have to define $\mathcal{J}_1^h v$ at the remaining extreme nodes. So we consider an extreme node $\mathbf{e}$ of $\overline{\Gamma_C}$ and the unions of triangles (i.e., patch) on $\overline{\Gamma_C}$ having $\mathbf{e}$ as vertex. Still we denote this patch by Δ_e. On Δ_e, we require that the average of v is preserved: find $\mathcal{J}_1^h v \in \mathbb{P}_1(\Delta_e)$ such that

$$\int_{\Delta_e} v - \mathcal{J}_1^h v = 0.$$

This definition both:

- Leads to a unique value of $(\mathcal{J}_1^h v)(\mathbf{e})$
- Preserves locally the affine functions

However, note that the value $(\mathcal{J}_1^h v)(\mathbf{e})$ is not necessarily nonpositive.

5.4.4 Error Analysis for Linear Finite Elements

As in the two-dimensional case, it suffices to prove the approximation error when $\mathbf{K}^h = \mathbf{K}_1^{h,e}$ and the consistency error when $\mathbf{K}^h = \overline{\mathbf{K}_1^{h,e}}$.

- Approximation error (when $\mathbf{K}^h = \mathbf{K}_1^{h,e}$):
  We choose

$$\mathbf{v}^h = \mathcal{I}^h \mathbf{u} + \mathcal{L}^h(\mathcal{J}_1^h u_{\mathbf{n}} - \mathcal{I}^h u_{\mathbf{n}}),$$

where $\mathcal{L}^h$ is a discrete (vectorial) lifting from W_1^h into $\mathbf{V}_1^h$, obtained from Lemma 4.2.

From the construction of $\mathcal{J}_1^h$, we deduce that $\mathbf{v}^h \in \mathbf{K}_1^{h,e}$. Let $v \in W^{1,p}(\Gamma_C)$ with $p \geq 1$: using a scaled trace inequality, it can be proven that for any node $\mathbf{a}$ on $\overline{\Gamma_C}$ which is not extreme we have

$$|(\mathcal{J}_1^h v)(\mathbf{a})| \leq C(h^{-2/p}\|v\|_{L^p(\Delta_{\mathbf{a}})} + h^{1-2/p}\|\nabla v\|_{L^p(\Delta_{\mathbf{a}})}),$$

where $\Delta_{\mathbf{a}}$ is the patch surrounding $\mathbf{a}$. If the node $\mathbf{a}$ is extreme, we have the same kind of estimate as before where $\Delta_{\mathbf{a}}$ has to be changed with the extended patch surrounding $\Delta_{\mathbf{a}}$ which we denote again by $\Delta_{\mathbf{a}}$ to simplify. So we have on any triangle $\partial T \cap \Gamma_C$ the stability estimate:

$$\|\mathcal{J}_1^h v\|_{0,\partial T\cap\Gamma_C} \leq C(h^{1-2/p}\|v\|_{L^p(\Delta_{\partial T\cap\Gamma_C})} + h^{2-2/p}\|\nabla v\|_{L^p(\Delta_{\partial T\cap\Gamma_C})}),$$

where $\Delta_{\partial T\cap\Gamma_C}$ is the patch surrounding $\partial T \cap \Gamma_C$. Choosing $p = 1$ and using the property that $\mathcal{J}_1^h$ preserves locally the constant functions (note that the triangles containing a node in $\overline{\Gamma_C} \cap \overline{\Gamma_D}$ are handled as in [71]) together with Corollary 4.2.3 in [268] we have

$$\|u_{\mathbf{n}} - \mathcal{J}_1^h u_{\mathbf{n}}\|_{0,\partial T\cap\Gamma_C} \le C\|\nabla u_{\mathbf{n}}\|_{L^1(\Delta_{\partial T\cap\Gamma_C})}, \tag{5.27}$$

which was the estimate we could not obtain for the Lagrange interpolation operator (see the previous discussion). Besides using the above stability estimate on $\mathcal{J}_1^h$ with $p = 2$ together with the property that $\mathcal{J}_1^h$ preserves locally the affine functions implies that $\mathcal{J}_1^h$ satisfies the same approximation properties as the linear Lagrange interpolation operator. Using the continuity of the extension operator and an inverse inequality gives

$$\begin{aligned}\|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} &\le \|\mathbf{u} - \mathcal{I}^h\mathbf{u}\|_{1,\Omega} + C\|\mathcal{J}_1^h u_{\mathbf{n}} - \mathcal{I}^h u_{\mathbf{n}}\|_{1/2,\Gamma_C}\\ &\le \|\mathbf{u} - \mathcal{I}^h\mathbf{u}\|_{1,\Omega} + Ch^{-1/2}(\|u_{\mathbf{n}} - \mathcal{I}^h u_{\mathbf{n}}\|_{0,\Gamma_C}\\ &\quad + \|u_{\mathbf{n}} - \mathcal{J}_1^h u_{\mathbf{n}}\|_{0,\Gamma_C})\\ &\le Ch^{s-1}\|\mathbf{u}\|_{s,\Omega} \end{aligned} \tag{5.28}$$

□

for any $1 < s \le 2$. Combining these estimates with a direct adaptation of Lemma 5.2 (for $d = 3$ and using $\Delta_{\partial T\cap\Gamma_C}$ instead of $\partial T \cap \Gamma_C$ in the lemma) gives us the optimal approximation bound:

$$\int_{\partial T\cap\Gamma_C} \sigma_{\mathbf{n}}(\mathcal{J}_1^h u - u)_{\mathbf{n}} \le Ch_T^{2(s-1)}\left(|\sigma_{\mathbf{n}}|^2_{s-3/2,\Delta_{\partial T\cap\Gamma_C}} + |\nabla u_{\mathbf{n}}|^2_{s-3/2,\Delta_{\partial T\cap\Gamma_C}}\right).$$

Hence, by summation,

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(v^h - u)_{\mathbf{n}} = \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathcal{J}_1^h u - u)_{\mathbf{n}} \le Ch^{2(s-1)}\|\mathbf{u}\|^2_{s,\Omega}, \tag{5.29}$$

which ends the proof of the approximation error.

- Consistency error (when $\mathbf{K}^h = \overline{\mathbf{K}_1^{h,e}}$):

  The optimal error estimate can also be obtained in this case. See [110] for the detailed proof.

This ends the proof of the theorem.

5.4.5 Error Analysis for Quadratic Finite Elements

As in the two-dimensional case, the proof is split into two parts: the first one when $3/2 < s < 5/2$ and the second one dealing with $s = 5/2$.

5.4.5.1 Sobolev Regularity $3/2 < s < 5/2$

From the inclusions

$$\mathbf{K}_2^{h,e} \subset \widehat{\mathbf{K}_2^{h,e}} \subset \overline{\mathbf{K}_2^{h,e}} \text{ and } \mathbf{K}_2^{h,e} \subset \widetilde{\mathbf{K}_2^{h,e}} \subset \overline{\mathbf{K}_2^{h,e}},$$

we only have to prove the approximation error bound when considering $\mathbf{K}_2^{h,e}$ and the consistency error bound when considering $\overline{\mathbf{K}_2^{h,e}}$.

- Approximation error (when $\mathbf{K}^h = \mathbf{K}_2^{h,e}$):
  We choose

$$\mathbf{v}^h = \mathcal{I}^h \mathbf{u} + \mathcal{L}^h(\mathcal{J}_1^h u_{\mathbf{n}} - \mathcal{I}^h u_{\mathbf{n}}),$$

  where $\mathcal{L}^h$ is an extension operator from W_2^h into $\mathbf{V}_2^h$ (W_2^h is the normal trace space of $\mathbf{V}_2^h$ on Γ_C) and $\mathcal{I}^h$ is the Lagrange interpolation operator mapping onto $\mathbf{V}_2^h$. Note that we use again the piecewise affine quasi-interpolation operator $\mathcal{J}_1^h$: this choice is sufficient for our estimates since we only use this operator on $\overline{\Gamma_C}$ where $u_{\mathbf{n}}$ is no more regular than H^2 (since $s < 5/2$). Of course, $v_{\mathbf{n}}^h$ is piecewise linear and equals $\mathcal{J}_1^h u_{\mathbf{n}}$ on Γ_C. From the definitions of $\mathbf{K}_1^{h,e}$ and $\mathbf{K}_2^{h,e}$, it is easy to check that $\mathbf{v}^h \in \mathbf{K}_2^{h,e}$. Estimate (5.28) still holds when $3/2 < s \leq 5/2$ since $\|u_{\mathbf{n}} - \mathcal{J}_1^h u_{\mathbf{n}}\|_{0,\Gamma_C} \leq Ch^{s-1/2}\|u_{\mathbf{n}}\|_{s-1/2,\Gamma_C} \leq Ch^{s-1/2}\|\mathbf{u}\|_{s,\Omega}$. Estimate (5.29) is handled exactly as in Theorem 5.3.
- The consistency error (when $\mathbf{K}^h = \overline{\mathbf{K}_2^{h,e}}$) is estimated as in Theorem 5.3 (see [110]).

5.4.5.2 Sobolev Regularity $s = 5/2$

Obviously $\sigma_{\mathbf{n}}$ and $\nabla u_{\mathbf{n}}$ are not continuous on Γ_C contrary to the two-dimensional case. A deeper insight into the proofs of the previous theorems shows us that the only result which is missing to complete the proof is an extension of Lemma 5.2 when $d = 3$ and $s = 5/2$. This case is detailed in [110].

5.5 Further Comments

For the Signorini problem discretized with standard FEM (5.7) or other FEM (such as mixed FEM), it has been quite challenging to establish optimal convergence in the case the solution $\mathbf{u}$ belongs to $H^s(\Omega)$ $(3/2 < s \leq 2)$. As a matter of fact, the first fully optimal result, without extra assumptions, for the standard FEM has been achieved only recently, in 2015, by G. Drouet and P. Hild: see [110]. The analysis in this chapter is inspired from this reference.

5.5.1 First Results of Numerical Approximation

Among the earliest papers about the numerical analysis of variational inequalities, we can quote an article of U. Mosco [212], published in 1967, and that followed closely the pioneering paper of J.L. Lions and G. Stampacchia, also from 1967 [200]. This paper analyzes the approximation of variational inequalities in an abstract setting, without explicit convergence rates.

The first articles that presented a priori error bounds in the $H^1(\Omega)$-norm were published in the 70s, after the mathematical theory of the FEM has been built and especially after the first results of polynomial interpolation in Sobolev spaces have been found and applied to the numerical approximation of partial differential equations. An important step has been first made by R.S. Falk in 1974, with, particularly, abstract estimates that generalize Cea's lemma (for variational equations) to the framework of variational inequalities [126].

After, in 1977 and 1978, a series of works were published that focused on the Signorini problem, by J. Haslinger [151], by F. Scarpini and M.A. Vivaldi in 1977 [240], and by F. Brezzi et al. in 1977 and 1978 [55, 56]. These first analyses were sub-optimal with a convergence in $\mathcal{O}(h^{\frac{s}{2}-\frac{1}{4}})$, as presented at the beginning of this chapter, and [55, 56] obtain a convergence in $\mathcal{O}(h)$ using additional regularities and assumptions. Some of the aforementioned results are presented in [153].

In the monograph of R. Glowinski, a mathematical study of a finite element method for the scalar Signorini problem is carried out, and a convergence result, without a rate, is proven [138, Theorem 4.3]. For various penalty techniques, some first studies were carried out by N. Kikuchi and J.T. Oden, see, for instance, [182] and [222], or their monograph [181].

5.5.2 Towards Optimal Rates (Higher Sobolev Regularities)

In the 2000s, with additional assumptions on the finiteness of transition points between contact and non-contact, optimality has been recovered. A first contribution has been made by F. Ben Belgacem in 2000, when $3/2 < s < 2$, see [31]. Then, for $s = 2$, a contribution from C. Hueber and B. Wohlmuth has followed in 2005, see [176]. Since then, many papers tried to improve these analysis, particularly to improve the convergence rates without the assumption on the biding/nonbiding transitions on the contact region. We refer to, e.g., [110, 170, 176, 264] for more detailed reviews on a priori error estimates for contact problems in elasticity.

5.5.3 The Case of Lower Sobolev Regularities

For lower Sobolev regularities, i.e., when the solution $\mathbf{u}$ to Problem (5.1) belongs to $H^s(\Omega)$ with $1 < s \leq \frac{3}{2}$, optimal error estimates have also been established by F. Ben

Belgacem in 2000, see [31]. When only H^1-regularity is assumed, a convergence result (without a rate) can also be proven (see, e.g., [138, Theorem 4.3, p.61] in the scalar case).

5.5.4 Errors in the L^2-Norm and Aubin–Nitsche

The obtention of optimal error estimates in the L^2-norm is still an ongoing research issue, since the usual Aubin–Nitsche duality trick cannot be applied directly to the Signorini setting. Some first results can be found in [65, 85, 86, 95, 251].

5.5.5 Related Results and Other Approximation Methods

For nearly incompressible elasticity, locking may deteriorate the convergence properties of the method, see, for instance, the recent review paper [5] and the references therein. This issue has been studied in the case of contact, especially in [36] (see also [181] or [264]).

For the Boundary Element Method (BEM) applied to contact, some error analysis can be found, for instance, in [72] or [136]. For some H^1-error estimates for the Crouzeix–Raviart nonconforming finite elements, see [175]. Numerical approximation with the discontinuous Petrov Galerkin (DPG) method has been studied, for instance, in [132]. For IsoGeometric Analysis (IGA), optimal convergence rates without extra assumptions on the behavior of the solution on the contact boundary have been obtained in [13], based on the results of [110].

For the lowest order Virtual Element Method (VEM), optimal error estimates have been obtained very recently in [262], for a solution of regularity $H^2(\Omega)$ and with the assumption that the trace of the displacement on the contact boundary is in $H^2(\Gamma_C)$.

Chapter 6
Nitsche's Method

As we have seen in the previous Chap. 5, it may not be the best idea to approximate directly the variational inequality for practical solving. As a result, many reformulations of Problem (3.12) have been studied since the 1970s, which aim at making easier the implementation and numerical resolution for contact, especially for problems more difficult and general than the Signorini problem. These reformulations have been mostly inspired from techniques to handle inequality constraints in optimization problems and techniques to take into account essential boundary conditions.

As a result, the most common techniques are based either on penalization or regularization of the Signorini conditions (3.18), or on duality arguments that lead to mixed formulations and involve Lagrange multipliers. For essential boundary conditions, reformulations with penalization or Lagrange multiplier have been proposed and studied in the early 1970s in two seminal papers of I. Babuska [18, 19]. These techniques have been since then applied to contact problems [181]. In the same period, another technique has been suggested by J.A. Nitsche [219]. This technique differs from penalty since it remains consistent. Furthermore, it remains also a primal method, with no duality arguments and no Lagrange multipliers. Nevertheless, it has been applied only recently to contact problems [78].

We present here how Nitsche's method can be adapted for the Signorini problem (3.12). First, we derive the discrete formulation, and then we detail its numerical analysis. The weak problem will be proven to be well-posed provided that the Nitsche parameter is large enough. Unlike the direct approximation of the variational inequality presented in Chap. 5, an optimal convergence result in the H^1-norm can be obtained rather directly, since no cumbersome term appears in the abstract error analysis. We will end up this section with some information about the numerical implementation of the method.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_6

6.1 A First Derivation of Nitsche's Method for Signorini Problem

First, we present a direct way to obtain the simplest variant of Nitsche's method for Signorini problem, which we will call incomplete. For this purpose, we need to first recall some elementary results about the positive part of a scalar quantity, or of a scalar function.

6.1.1 *The Positive Part Operator*

Let us first introduce the notation $[\cdot]_+$ for the positive part of a scalar quantity $a \in \mathbb{R}$, which is also the projection onto the half-line $[0; +\infty)$. It can be defined as

$$[a]_+ := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and it verifies as well

$$[a]_+ = \frac{1}{2}(a + |a|), \quad [a]_+ = \max(0, a).$$

In the sequel, we will make an extensive use of the following properties:

$$a \le [a]_+ , \quad a\,[a]_+ = [a]_+^2 , \quad [a]_+^2 \le a^2, \quad \forall a \in \mathbb{R}. \tag{6.1}$$

From the above properties, we deduce the monotonicity property below, which is a classical property of any projection onto a convex set:

$$\begin{aligned} \left([a]_+ - [b]_+\right)(a - b) &= a\,[a]_+ + b\,[b]_+ - b\,[a]_+ - a\,[b]_+ \\ &\ge [a]_+^2 + [b]_+^2 - 2\,[a]_+\,[b]_+ \\ &= \left([a]_+ - [b]_+\right)^2 \ge 0. \end{aligned} \tag{6.2}$$

Moreover, we extend the definition of the positive part to any function v that belongs to $L^2(\Gamma_C)$, and the corresponding function $[v]_+$ is still a function in $L^2(\Gamma_C)$. For such functions, similar properties as (6.1) and (6.2) hold almost everywhere [181, Section 3.3]. The Heaviside function will be denoted by $H_s(\cdot)$. It is defined as:

$$H_s(a) := \begin{cases} 1 & \text{if } a > 0, \\ \frac{1}{2} & \text{if } a = 0, \\ 0 & \text{if } a < 0, \end{cases}$$

for $a \in \mathbb{R}$. Note that the value in 0 is conventional (it is strictly speaking multivaluated in 0). Later on, we will make use of the properties below:

$$H_s(a)\,[a]_+ = [a]_+\,, \qquad \frac{\mathrm{d}}{\mathrm{d}a}\,[a]_+ = H_s(a), \quad \forall a \in \mathbb{R}. \tag{6.3}$$

Above, the derivative $\frac{\mathrm{d}}{\mathrm{d}a}\,[a]_+$ at point $a = 0$ should be understood in a weak sense.

6.1.2 A Reformulation of the Signorini Conditions

The derivation of a Nitsche-based method comes from the observation that the contact conditions (3.23) can be reformulated as follows:

Proposition 6.1 *Let $\gamma_N > 0$ be a positive integrable function on Γ_C. Let us suppose that $\mathbf{u}$ is regular enough on Ω such that* (3.23) (i)–(iii) *hold in $L^2(\Gamma_C)$. Contact conditions* (3.23) (i)–(iii) *on Γ_C are then equivalent to:*

$$\sigma_{\mathbf{n}}(\mathbf{u}) = -\,[\gamma_N u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})]_+\,. \tag{6.4}$$

Proof A detailed proof can be found in [78], but we provide here another, shorter, proof. Take $\mathbf{u}$ a vector field on Ω such that (3.23) (*i*)–(*iii*) hold in $L^2(\Omega)$, and thus almost everywhere on the contact boundary. Remark first that conditions (3.23) (*i*)–(*iii*) are equivalent to

$$0 = \max(u_{\mathbf{n}}, \sigma_{\mathbf{n}}(\mathbf{u})).$$

Then, let us write

$$-\,\sigma_{\mathbf{n}}(\mathbf{u}) = -\sigma_{\mathbf{n}}(\mathbf{u}) + 0 = -\sigma_{\mathbf{n}}(\mathbf{u}) + \max(u_{\mathbf{n}}, \sigma_{\mathbf{n}}(\mathbf{u})),$$

which means that

$$\begin{aligned}-\,\sigma_{\mathbf{n}}(\mathbf{u}) &= \max(-\sigma_{\mathbf{n}}(\mathbf{u}) + u_{\mathbf{n}}, -\sigma_{\mathbf{n}}(\mathbf{u}) + \sigma_{\mathbf{n}}(\mathbf{u}))\\ &= \max(-\sigma_{\mathbf{n}}(\mathbf{u}) + u_{\mathbf{n}}, 0) = [u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})]_+\,.\end{aligned}$$

Conversely, using the same previous calculations, we check that if $-\sigma_{\mathbf{n}}(\mathbf{u}) = [u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})]_+$ holds, the conditions (3.23) (i)–(iii) are satisfied.

As a result, to prove the equivalence between (3.23) (i)–(iii) and (6.4), there simply remains to notice that (3.23) (i)–(iii) are kept unchanged if we replace $\mathbf{u_n}$ by $\gamma_N \mathbf{u_n}$, where γ_N is an arbitrary positive integrable function. □

A reformulation such as (6.4) is mentioned, for instance, in [6] and is also a basis for augmented Lagrangian techniques (see next chapter). Observe that for $\mathbf{u} \in H^s(\Omega)$, with $s > \frac{3}{2}$, the assumption of Proposition 6.1 is satisfied, because of the Trace Theorem 2.6. In practice, we will take γ_N as a piecewise constant function on the contact boundary. Formulation (6.4) transforms Karuhn-Kush-Tucker type conditions involving inequalities into a single equality involving a nonsmooth function (the positive part). This is its main interest, because it facilitates the weak treatment of Signorini conditions. Also, since Proposition 6.1 states an equivalence relationship, it leads to consistent numerical approximations, contrary to regularizations of the contact conditions that introduce a consistency error [79, 181].

6.1.3 An Incomplete Nitsche Formulation

Let us suppose that $\mathbf{u}$, the solution to Signorini problem (3.12), is regular enough so that the following steps make sense, for instance, take $\mathbf{u} \in H^s(\Omega)$, with $s > \frac{3}{2}$:

1. Apply first the Green formula in elasticity (Lemma 3.1) and get, for any $\mathbf{v} \in \mathbf{V}$:

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) v_{\mathbf{n}} = L(\mathbf{v}).$$

2. Then, use the reformulation (6.4) of Signorini condition proved above:

$$a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_C} [\gamma_N u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})]_+ \, v_{\mathbf{n}} = L(\mathbf{v}).$$

And the first Nitsche formulation is obtained. Of course, this formulation may have no clear meaning in a continuous setting. Indeed it is not obvious to find a functional setting such that: (1) the integral on the contact boundary is properly defined (for instance, $\mathbf{u} \in \mathbf{V}$ is not enough), and (2) the corresponding weak form is well-posed. So we formulate it directly in a discrete setting coming from Lagrange finite elements, for instance, and carry out the numerical analysis in the spirit of nonconforming methods.

For this purpose, we proceed as in the previous Chap. 5 and consider the notions introduced in Chap. 4. We build our finite element space using standard continuous Lagrange finite elements of degree k, with $k = 1$ or $k = 2$, i.e.,:

$$\mathbf{V}_k^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \,\middle|\, \mathbf{v}^h_{|T} \in \mathbb{P}_k(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\},$$

where $\mathcal{T}^h$ is a simplicial mesh of the domain Ω, and where $\mathbb{P}_k(T; \mathbb{R}^d)$ stands for the space of all vector-valued polynomials on T, of degree lower or equal than k in the d variables. We will in general omit the subscript k and write simply $\mathbf{V}^h (= \mathbf{V}_k^h)$. We suppose once again that the mesh $\mathcal{T}^h$ resolves the contact boundary Γ_C, which means that all the edges ($d = 2$) or faces ($d = 3$) that intersect Γ_C are included in $\overline{\Gamma_C}$. We suppose also that it resolves the Neumann boundary Γ_N and the

Dirichlet boundary Γ_D. Moreover, to carry out the convergence analysis, and obtain error estimates, we suppose as in the previous chapter (Chap. 5) that the Dirichlet boundary Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$.

Then, we take γ_N as a discrete function on Γ_C, piecewise constant, whose expression when restricted to a boundary face or edge is:

$$\gamma_N|_{T\cap\Gamma_C} := \frac{\gamma_0}{h_T},$$

with T a simplex that shares at least one of its face/edge with the boundary Γ_C. The scalar value $\gamma_0 > 0$ is the Nitsche parameter. We will see in the next part of this chapter that it needs to be chosen large enough, in order to preserve well-posedness and optimal accuracy. Let us define also the linear operator

$$\mathrm{P}^{\mathbf{n}}_{1,N} : \begin{array}{rcl} \mathbf{V}^h & \rightarrow & L^2(\Gamma_C) \\ \mathbf{v}^h & \mapsto & \gamma_N v^h_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{v}^h). \end{array}$$

The incomplete Nitsche's method reads:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ v^h_{\mathbf{n}} = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{6.5}$$

Before we study this method, let us get slightly more general and see how a whole family of methods, with notable properties for some of them, can be obtained.

6.2 Nitsche Discrete Formulations and Variants

Following [82] (see also [77] and [174]), we introduce a family of variants for the above Nitsche's method, that is indexed by a parameter that we denote θ. Then, we focus on the special case of the symmetric variant, which can be obtained from the derivation of an energy functional, as in the original paper of J.A. Nitsche [219].

6.2.1 A Family of Methods

Take $\theta \in \mathbb{R}$. Let us consider again $\mathbf{u}$, the solution to Signorini problem (3.12), still regular enough so that the following steps make sense (for instance, $\mathbf{u} \in H^s(\Omega)$, with $s > \frac{3}{2}$). Still from the Green formula in elasticity (Lemma 3.1) and for any $\mathbf{v} \in \mathbf{V}$, we start with

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) v_{\mathbf{n}} = L(\mathbf{v}).$$

Now, let us split the boundary integral term using the following rewriting associated with the test function:

$$v_{\mathbf{n}} = \frac{1}{\gamma_N}\left(\gamma_N v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v})\right) + \frac{\theta}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{v}).$$

We get:

$$a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_C}\frac{\theta}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u})\sigma_{\mathbf{n}}(\mathbf{v}) - \int_{\Gamma_C}\frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u})\left(\gamma_N v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v})\right) = L(\mathbf{v}).$$

And we finally apply the reformulation (6.4) of Signorini conditions proved above:

$$a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_C}\frac{\theta}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u})\sigma_{\mathbf{n}}(\mathbf{v}) + \int_{\Gamma_C}\frac{1}{\gamma_N}\left[\gamma_N u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})\right]_+\left(\gamma_N v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v})\right) = L(\mathbf{v}).$$

As we did for the incomplete formulation, let us now write the discrete counterpart of the above weak form. To simplify the notations, we introduce another linear operator, which depends on θ:

$$\mathrm{P}^{\mathbf{n}}_{\theta,N} : \begin{array}{rcl}\mathbf{V}^h & \to & L^2(\Gamma_C)\\ \mathbf{v}^h & \mapsto & \gamma_N v^h_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v}^h),\end{array}$$

and the bilinear form

$$A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) := a(\mathbf{u}^h,\mathbf{v}^h) - \int_{\Gamma_C}\frac{\theta}{\gamma_N}\,\sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h).$$

The (general) Nitsche method now reads:

$$\begin{cases}\text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:}\\ A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) + \displaystyle\int_{\Gamma_C}\frac{1}{\gamma_N}\,[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+\mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.\end{cases} \tag{6.6}$$

Of course, with the choice $\theta = 0$ we recover the incomplete variant (6.5). Remark that it involves a reduced quantity of terms, which makes it easier to implement and to extend to contact problems involving nonlinear elasticity. In addition, this nonsymmetric variant $\theta = 0$ performs better in the sense it requires less Newton iterations to converge, for a wider range of the Nitsche parameter, than the variant $\theta = 1$, see [233].

The symmetric case of [78] is recovered when $\theta = 1$. When $\theta \neq 1$ positivity of the contact term in the Nitsche variational formulation is generally lost. Another notable value is $\theta = -1$ that corresponds to a skew-symmetric variant. In this case, the well-posedness of the discrete formulation and the optimal convergence are preserved irrespectively of the value of the Nitsche parameter $\gamma_0 > 0$. Note that for other boundary conditions, such as non-homogeneous Dirichlet, the symmetric variant ($\theta = 1$) as originally proposed by Nitsche [219] is the most widespread, since it preserves symmetry, and allows efficient solvers for linear systems with a symmetric matrix. However, some nonsymmetric variants have been reconsidered recently, due to some remarkable robustness properties (see, e.g., [45, 57] and the review [74]). In the context of discontinuous Galerkin methods, such nonsymmetric variants are well-known as well (see, e.g., [106, Section 5.3.1, p.199]).

6.2.2 The Symmetric Nitsche's Method

Let us focus now on the particular case $\theta = 1$, which corresponds to a symmetric variant. As in [219], it can be obtained alternatively as the first order optimality condition associated with the discrete functional

$$\mathcal{J}_N(\mathbf{v}^h) := \mathcal{J}(\mathbf{v}^h) - \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{v}^h)^2 + \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N}\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h)\right]_+^2,$$

well-defined for $\mathbf{v}^h \in \mathbf{V}^h$, and with the functional

$$\mathcal{J}(\mathbf{v}^h) = \frac{1}{2}a(\mathbf{v}^h, \mathbf{v}^h) - L(\mathbf{v}^h)$$

introduced previously in (3.8).

The first order optimality condition associated with the minimization of $\mathcal{J}_N$ simply reads:

$$\mathcal{J}'_N(\mathbf{u}^h; \mathbf{v}^h) = 0,$$

for any $\mathbf{v}^h$ in $\mathbf{V}^h$. Let us now make explicit the expression of the derivative. We use properties (6.3) and get the relationship

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[x\right]_+^2 = H_s(x)\left[x\right]_+ = \left[x\right]_+,$$

for $x \in \mathbb{R}$. As a result, we can derive also the boundary term that involves the positive part and get

$$\begin{aligned}\mathcal{J}'_N(\mathbf{u}^h; \mathbf{v}^h) = {} & a(\mathbf{u}^h, \mathbf{v}^h) - L(\mathbf{v}^h) - \int_{\Gamma_C} \frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h) \\ & + \int_{\Gamma_C} \frac{1}{\gamma_N}\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+ \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h).\end{aligned}$$

Finally, we obtain the following discrete weak form

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{1\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \end{cases} \tag{6.7}$$

which is exactly (6.6) when $\theta = 1$ and is the symmetric Nitsche form.

6.2.3 *Link with Barbosa and Hughes Stabilization*

As in [78], we emphasize here the link between Nitsche's method (6.6) and the stabilized mixed finite element methods (which do not require inf-sup conditions) introduced in [177]. We define the auxiliary variable

$$\lambda^h := -[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+,$$

which is in fact the L^2-projection of the quantity $-\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)$ on the convex cone:

$$L^2_-(\Gamma_C) := \{\mu \in L^2(\Gamma_C) \mid \mu \leq 0 \text{ a.e. on } \Gamma_C\}.$$

This means in particular that $\lambda^h \in L^2_-(\Gamma_C)$ verifies the following inequality for all $\mu \in L^2_-(\Gamma_C)$:

$$\int_{\Gamma_C} (\mu - \lambda^h)(-\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) - \lambda^h) \leq 0, \tag{6.8}$$

which is the characterization of the projection onto a closed convex set [52]. We then use the definition of the operator $\mathrm{P}^{\mathbf{n}}_{1,N}$ to obtain:

$$\begin{aligned} &\int_{\Gamma_C} (\mu - \lambda^h)(-\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) - \lambda^h) \\ &= \int_{\Gamma_C} (\mu - \lambda^h)(-\gamma_N u^h_{\mathbf{n}} + \sigma_{\mathbf{n}}(\mathbf{u}^h) - \lambda^h) \\ &= -\int_{\Gamma_C} \gamma_N \left((\mu - \lambda^h) u^h_{\mathbf{n}} + \gamma_N^{-1} (\mu - \lambda^h)(\lambda^h - \sigma_{\mathbf{n}}(\mathbf{u}^h)) \right). \end{aligned}$$

Since γ_N is a positive piecewise constant function on Γ_C, we have shown that the inequality (6.8) is in fact equivalent to:

$$\int_{\Gamma_C} (\mu - \lambda^h) u^h_{\mathbf{n}} + \int_{\Gamma_C} \gamma_N^{-1} (\mu - \lambda^h)(\lambda^h - \sigma_{\mathbf{n}}(\mathbf{u}^h)) \geq 0. \tag{6.9}$$

The next step is to rewrite the left hand side of the Nitsche-based method (6.6) as follows:

$$
\begin{aligned}
& A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) \\
&= a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \theta\gamma_N^{-1}\, \sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h) - \int_{\Gamma_C} \lambda^h (v^h_{\mathbf{n}} - \theta\gamma_N^{-1}\, \sigma_{\mathbf{n}}(\mathbf{v}^h)) \\
&= a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \lambda^h v^h_{\mathbf{n}} + \int_{\Gamma_C} \theta\gamma_N^{-1}(\lambda^h - \sigma_{\mathbf{n}}(\mathbf{u}^h))\sigma_{\mathbf{n}}(\mathbf{v}^h).
\end{aligned}
$$

We combine this last result with the inequality (6.9) to obtain an equivalent formulation of Nitsche method (6.6):

$$
\begin{cases}
\text{Find } (\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times L^2_-(\Gamma_C) \text{ such that:} \\
a(\mathbf{u}^h, \mathbf{v}^h) - \displaystyle\int_{\Gamma_C} \lambda^h v^h_{\mathbf{n}} + \int_{\Gamma_C} \theta\gamma_N^{-1}(\lambda^h - \sigma_{\mathbf{n}}(\mathbf{u}^h))\, \sigma_{\mathbf{n}}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\
\displaystyle\int_{\Gamma_C} (\mu - \lambda^h) u^h_{\mathbf{n}} + \int_{\Gamma_C} \gamma_N^{-1}(\mu - \lambda^h)(\lambda^h - \sigma_{\mathbf{n}}(\mathbf{u}^h)) \geq 0, \quad \forall\, \mu \in L^2_-(\Gamma_C).
\end{cases}
$$

This implies that Nitsche method can be regarded as a mixed method with a stabilization term (see [25]). This makes it close to the stabilized method proposed and analyzed in [169], the difference being that in our case, the space for the Lagrange multiplier λ^h is $L^2_-(\Gamma_C)$. A similar analogy has already been noticed in [252] in the case of elliptic problems with Dirichlet boundary conditions. Note as well that the inverse of Nitsche parameter γ_0^{-1} can be interpreted as a stabilization parameter.

6.3 Consistency, Well-posedness and Optimal Error Estimates

This section is devoted to the numerical analysis of Nitsche's method (6.6). First, we state the consistency of the method, which is its fundamental property. Then, since it is not a direct, conforming, approximation of the variational inequality, we need another argument to establish the well-posedness of the discrete formulation than the Stampacchia Theorem. This aspect is also detailed. Finally, we state the optimal convergence of the method in the H^1-norm.

The statements are provided for all the variants in θ but some proofs here are restricted to the case of the symmetric variant $\theta = 1$ (see [82] for the complete proofs in the general case).

6.3.1 Consistency

Nitsche's method (6.6) is consistent, as stated below:

Lemma 6.1 *Suppose that the solution* **u** *to* (3.12) *belongs to* $H^s(\Omega;\mathbb{R}^d)$*, with* $s > 3/2$*, then* **u** *satisfies*

$$A_{\theta\gamma}(\mathbf{u},\mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})\right]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Proof Since $\mathbf{u} \in H^s(\Omega;\mathbb{R}^d)$ and $s > 3/2$, we deduce from Theorem 2.6:

$$\sigma_{\mathbf{n}}(\mathbf{u}) \in H^{s-3/2}(\Gamma_C) \subset L^2(\Gamma_C).$$

Therefore, $\mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{u}) \in L^2(\Gamma_C)$ and $A_{\theta\gamma}(\mathbf{u},\mathbf{v}^h)$ are correctly defined. From the definition of $\mathrm{P}^{\mathbf{n}}_{\theta,N}(\cdot)$ and $A_{\theta\gamma}(\cdot,\cdot)$ and the reformulation (6.4) of the contact conditions we get:

$$\begin{aligned}
&A_{\theta\gamma}(\mathbf{u},\mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})\right]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) \\
&= a(\mathbf{u},\mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{u})\sigma_{\mathbf{n}}(\mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{u})(\theta\sigma_{\mathbf{n}}(\mathbf{v}^h) - \gamma_N v^h_{\mathbf{n}}) \\
&= a(\mathbf{u},\mathbf{v}^h) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) v^h_{\mathbf{n}}.
\end{aligned}$$

From Lemma 3.1, **u** satisfies also:

$$a(\mathbf{u},\mathbf{v}^h) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u}) v^h_{\mathbf{n}} = L(\mathbf{v}^h),$$

from which we can conclude the proof. □

6.3.2 Well-posedness

We provide here a proof of existence and uniqueness for the solution to the discrete problem (6.6). The idea is to introduce a nonlinear operator associated with (6.6) and prove that this operator is one-to-one. This allows to take advantage of the existing results of functional analysis for the characterization of bijective operators in Banach or Hilbert spaces. Here we make use of a theorem published by H. Brezis [51], which guarantees that monotone hemicontinuous operators are bijective. In the particular case $\theta = 1$ well-posedness can be established alternatively using arguments to characterize the minimum of a continuous (strongly) convex functional. The advantage of the method detailed here is that it can be easily extended for all the variants in θ, which are not obtained through minimization arguments.

Theorem 6.1 *Let us suppose that γ_0 is large enough ($\gamma_0 > 0$ when $\theta = -1$). Then, the Problem (6.6) admits one unique solution $\mathbf{u}^h$ in $\mathbf{V}^h$.*

Moreover, the solution to (6.6) is Lipschitz continuous with respect to the data. Indeed, for $i = 1, 2$, and some source terms $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$, denote by $\mathbf{u}_i^h \in \mathbf{V}^h$ the corresponding discrete solution to (6.6). Then, there holds

$$\|\mathbf{u}_1^h - \mathbf{u}_2^h\|_{1,\Omega} \leq C(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

where $C > 0$ does not depend on h and does not depend on γ_0 provided that it is large enough. □

Proof We suppose that $\theta = 1$. The general case $\theta \neq 1$ is treated similarly (see [82] for the details). Using the Riesz representation theorem, let us define the nonlinear operator

$$\mathbf{B}^h : \mathbf{V}^h \to \mathbf{V}^h$$

through:

$$(\mathbf{B}^h\mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} := A_{1\gamma}(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h)\right]_+ \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h),$$

for $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$. Note that Problem (6.6) is well-posed if and only if $\mathbf{B}^h$ is a one-to-one operator. Let $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$ with the property (6.2), the ellipticity of $a(\cdot,\cdot)$ (Proposition 3.3) and a straightforward application of the discrete trace inequality of Lemma 4.1 in Chap. 4, there holds:

$$\begin{aligned}
&(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\
&= A_{1\gamma}(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h)\right]_+ \right. \\
&\qquad \left. - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)\right]_+\right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{w}^h) \\
&\geq A_{1\gamma}(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) \\
&\geq \left(\alpha - \frac{C}{\gamma_0}\right) \|\mathbf{v}^h - \mathbf{w}^h\|^2_{1,\Omega}.
\end{aligned}$$

As a result, for γ_0 large enough,

$$(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \geq C\|\mathbf{v}^h - \mathbf{w}^h\|^2_{1,\Omega}, \tag{6.10}$$

with $C > 0$ independent of h and γ_0.

Let us show then that $\mathbf{B}^h$ is hemicontinuous. Since $\mathbf{V}^h$ is a vector space, it is sufficient to show that

$$[0, 1] \ni t \mapsto \varphi(t) = (\mathbf{B}^h(\mathbf{v}^h - t\mathbf{w}^h), \mathbf{w}^h)_{1,\Omega} \in \mathbb{R}$$

is a continuous function, for $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$. Take $s, t \in [0, 1]$ and bound:

$$\begin{aligned}
&|\varphi(t) - \varphi(s)| \\
&= |(\mathbf{B}^h(\mathbf{v}^h - t\mathbf{w}^h) - \mathbf{B}^h(\mathbf{v}^h - s\mathbf{w}^h), \mathbf{w}^h)_{1,\Omega}| \\
&= \Bigg| A_{1\gamma}((s-t)\mathbf{w}^h, \mathbf{w}^h) \\
&\quad + \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - t\mathbf{w}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - s\mathbf{w}^h) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h) \Bigg| \\
&\leq |s - t|\, A_{1\gamma}(\mathbf{w}^h, \mathbf{w}^h) \\
&\quad + \int_{\Gamma_C} \frac{1}{\gamma_N} \left| \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - t\mathbf{w}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - s\mathbf{w}^h) \right]_+ \right| |\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)|.
\end{aligned}$$

Using $|\, [a]_+ - [b]_+ \,| \leq |a - b|$, for $a, b \in \mathbb{R}$, and the linearity of $\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)$, we then note that:

$$\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma_N} \left| \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - t\mathbf{w}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - s\mathbf{w}^h) \right]_+ \right| |\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)| \\
&\leq \int_{\Gamma_C} \frac{1}{\gamma_N} \left| \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - t\mathbf{w}^h) - \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - s\mathbf{w}^h) \right| |\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)| \\
&= \int_{\Gamma_C} \frac{1}{\gamma_N} |(s-t)\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)||\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h)| \\
&= |s - t| \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h) \right)^2.
\end{aligned}$$

We deduce then

$$|\varphi(t) - \varphi(s)| \leq |s - t| \left(A_{1\gamma}(\mathbf{w}^h, \mathbf{w}^h) + \left\| \gamma_N^{-\frac{1}{2}} \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{w}^h) \right\|^2_{0,\Gamma_C} \right),$$

which means that φ is Lipschitz, and thus $\mathbf{B}^h$ is hemicontinuous. Since $\mathbf{B}^h$ satisfies also (6.10), let us apply then the Corollary 15 (p.126) of [51] and we conclude that $\mathbf{B}^h$ is one-to-one.

Let us prove now that the solution to (6.6) is Lipschitz continuous with respect to the data. For $i = 1, 2$, we denote by L_i the linear form associated with the source terms $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$, and $\mathbf{u}_i^h \in \mathbf{V}^h$ is the corresponding solution to (6.6). Thus $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ are the unique solutions to

$$(\mathbf{B}^h\mathbf{u}_1^h, \mathbf{v}^h)_{1,\Omega} = L_1(\mathbf{v}^h), \quad (\mathbf{B}^h\mathbf{u}_2^h, \mathbf{v}^h)_{1,\Omega} = L_2(\mathbf{v}^h),$$

respectively, for $\mathbf{v}^h \in \mathbf{V}^h$. We take $\mathbf{v}^h = \mathbf{u}_1^h - \mathbf{u}_2^h$ in the above relationships and get

$$(L_1 - L_2)(\mathbf{u}_1^h - \mathbf{u}_2^h) = (\mathbf{B}^h\mathbf{u}_1^h - \mathbf{B}^h\mathbf{u}_2^h, \mathbf{u}_1^h - \mathbf{u}_2^h)_{1,\Omega}.$$

Then, the property (6.10) yields

$$(L_1 - L_2)(\mathbf{u}_1^h - \mathbf{u}_2^h) \geq C\|\mathbf{u}_1^h - \mathbf{u}_2^h\|_{1,\Omega}^2.$$

With (3.7) we get finally:

$$\|\mathbf{u}_1^h - \mathbf{u}_2^h\|_{1,\Omega} \leq C(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

with $C > 0$. This ends the proof. □

Remark 6.1 The operator $\mathbf{B}^h$ introduced in the above proof is actually the Gâteaux-derivative of the functional $\mathcal{J}_N(\cdot)$ introduced in Sect. 6.2.2:

$$(\mathbf{B}^h\mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} = \mathcal{J}_N{}'(\mathbf{v}^h; \mathbf{w}^h).$$

Note that $\mathcal{J}_N(\cdot)$ is continuous on $\mathbf{V}^h$. The property (6.10) means that $\mathcal{J}_N(\cdot)$ is strongly convex as well. So we can once again apply a result from convex optimization to get existence and uniqueness of the discrete solution $\mathbf{u}^h \in \mathbf{V}^h$ to Problem (6.7), that is the unique global minimum of $\mathcal{J}_N$ on $\mathbf{V}^h$. □

Remark 6.2 When the condition γ_0 large is not satisfied, existence or uniqueness of the discrete solution can be lost: see [82] for some explicit examples. □

6.3.3 An Abstract a priori Error Estimate

Let us now prove that the method converges when the mesh size h vanishes and establish the convergence rate of the method in the H^1-norm. First, we derive an abstract error estimate. You will notice that, conversely to the previous chapter (Chap. 5), the boundary term associated with contact, which was difficult to bound, does not appear any longer.

Theorem 6.2 *Suppose that the solution* $\mathbf{u}$ *to Problem* (3.12) *belongs to* $H^s(\Omega; \mathbb{R}^d)$*, with* $s > 3/2$*. Suppose also that the Nitsche parameter* γ_0 *is large enough. Then, the unique solution* $\mathbf{u}^h$ *to Problem* (6.6) *satisfies:*

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} + \left\|\gamma_N^{-\frac{1}{2}}\left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+\right)\right\|_{0,\Gamma_C}$$
$$\leq C \inf_{\mathbf{v}^h\in\mathbf{V}^h}\left(\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + \left\|\gamma_N^{\frac{1}{2}}(u_{\mathbf{n}}-v^h_{\mathbf{n}})\right\|_{0,\Gamma_C} + \left\|\gamma_N^{-\frac{1}{2}}\sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h)\right\|_{0,\Gamma_C}\right), \tag{6.11}$$

where the constant $C > 0$ depends on θ but does not depend on γ_0, h and $\mathbf{u}$. *When $\theta = -1$, the above result still holds for $\gamma_0 > 0$ arbitrarily small, but with $C > 0$ that depends on γ_0.* □

Proof As we did previously for the well-posedness, we detail the proof in the symmetric case ($\theta = 1$). For the proof in the general case $\theta \in \mathbb{R}$, see [82]. Take $\mathbf{v}^h \in \mathbf{V}^h$. We use first the ellipticity and the continuity of $a(\cdot,\cdot)$ (Proposition 3.3) combined with a Young inequality:

$$\begin{aligned}
\alpha\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} &\leq a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h)\\
&= a(\mathbf{u}-\mathbf{u}^h, (\mathbf{u}-\mathbf{v}^h) + (\mathbf{v}^h-\mathbf{u}^h))\\
&\leq C\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h-\mathbf{u}^h)\\
&\leq \frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|^2_{1,\Omega} + \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|^2_{1,\Omega} + a(\mathbf{u}, \mathbf{v}^h-\mathbf{u}^h)\\
&\quad - a(\mathbf{u}^h, \mathbf{v}^h-\mathbf{u}^h),
\end{aligned} \tag{6.12}$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot,\cdot)$. Remark above that the first term can go to the left hand side, the second term is an interpolation error, and the last two terms contain both the discretization error and some other interpolation error terms. We need to continue and to handle these terms.

Since $\mathbf{u}$ and $\mathbf{u}^h$ are respective solutions to (3.12) and (6.6), with $\mathbf{u}$ regular enough ($s > 3/2$), and since Nitsche's method is consistent (Lemma 6.1), we can write:

$$\begin{aligned}
&a(\mathbf{u}, \mathbf{v}^h-\mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h-\mathbf{u}^h)\\
&= \int_{\Gamma_C}\sigma_{\mathbf{n}}(\mathbf{u})\left(v^h_{\mathbf{n}}-u^h_{\mathbf{n}}\right) - \int_{\Gamma_C}\frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h)\\
&\quad + \int_{\Gamma_C}\frac{1}{\gamma_N}\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h-\mathbf{u}^h)\\
&= \int_{\Gamma_C}\frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u})\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h-\mathbf{u}^h) + \int_{\Gamma_C}\frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u})\sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h)\\
&\quad - \int_{\Gamma_C}\frac{1}{\gamma_N}\sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h) + \int_{\Gamma_C}\frac{1}{\gamma_N}\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h-\mathbf{u}^h).
\end{aligned}$$

We continue:

$$
\begin{aligned}
& a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \\
&= \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{u}^h) \\
&\quad - \int_{\Gamma_C} \frac{1}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{u}^h - \mathbf{u}) \sigma_{\mathbf{n}}(\mathbf{v}^h - \mathbf{u}^h) \\
&= \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{u}) \\
&\quad + \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u} - \mathbf{u}^h) \\
&\quad - \int_{\Gamma_C} \frac{1}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{u}^h - \mathbf{u}) \sigma_{\mathbf{n}}(\mathbf{v}^h - \mathbf{u}^h).
\end{aligned} \tag{6.13}
$$

Above, at the third line, we used the definition of $\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)$, and to get the last equality, we simply have split the term $\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{u}^h)$. We bound the first term of (6.13) using the reformulation (6.4), and then Cauchy-Schwarz and Young inequalities:

$$
\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{u}) \\
&\le \frac{1}{2} \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|^2_{0,\Gamma_C} + \frac{1}{2} \left\| \gamma_N^{-\frac{1}{2}} \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h - \mathbf{u}) \right\|^2_{0,\Gamma_C} \\
&\le \frac{1}{2} \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|^2_{0,\Gamma_C} \\
&\quad + \left\| \gamma_N^{\frac{1}{2}} (u_{\mathbf{n}} - v^h_{\mathbf{n}}) \right\|^2_{0,\Gamma_C} + \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u} - \mathbf{v}^h) \right\|^2_{0,\Gamma_C}.
\end{aligned} \tag{6.14}
$$

For the second term of (6.13) we use (6.4) combined with (6.2):

$$
\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma_N} \left(\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ - \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \right) \mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u} - \mathbf{u}^h) \\
&\le - \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|^2_{0,\Gamma_C}.
\end{aligned} \tag{6.15}
$$

For the third term (6.13), we use two times the discrete trace inequality (Lemma 4.1), and finally Young inequality:

$$
\begin{aligned}
&\int_{\Gamma_C} \frac{1}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h) \qquad (6.16)\\
&\le \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{u}^h) \right\|_{0,\Gamma_C} \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h) \right\|_{0,\Gamma_C}\\
&\le C\gamma_0^{-\frac{1}{2}} \|\mathbf{v}^h-\mathbf{u}^h\|_{1,\Omega} \left(\left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C} + \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{v}^h-\mathbf{u}^h) \right\|_{0,\Gamma_C} \right)\\
&\le C \left(\frac{1}{\gamma_0} \|\mathbf{v}^h-\mathbf{u}^h\|_{1,\Omega}^2 + \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C}^2 \right)\\
&\le C \left(\frac{1}{\gamma_0} \|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 + \frac{1}{\gamma_0} \|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C}^2 \right).
\end{aligned}
$$

Now, we combine previous estimates (6.13)–(6.16) with (6.12):

$$
\begin{aligned}
&\left(\frac{\alpha}{2}-\frac{C}{\gamma_0}\right) \|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 + \frac{1}{2} \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}_{1,N}^{\mathbf{n}}(\mathbf{u}^h) \right]_+ \right) \right\|_{0,\Gamma_C}^2\\
&\le \left(\frac{C^2}{2\alpha}+\frac{C}{\gamma_0}\right) \|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \left\| \gamma_N^{\frac{1}{2}} (u_{\mathbf{n}} - v_{\mathbf{n}}^h) \right\|_{0,\Gamma_C}^2\\
&\quad + (C+1) \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C}^2,
\end{aligned}
$$

where C is the constant in (6.16). Choosing γ_0 large enough, we obtain (6.11). □

Remark 6.3 Note that, in the error bound (6.11), not only the error on the H^1-norm of the displacement $\mathbf{u}$ on Ω is controlled by interpolation terms but also the contact error appears and can be controlled as well.

6.3.4 Optimal a priori Error Estimate

The optimal convergence of Nitsche's method, for a regular enough solution $\mathbf{u}$, follows from the estimate (6.11) and interpolation estimates of Chap. 4, as stated below. Remark that conversely to the previous chapter (Chap. 5) where we tried to approximate directly the variational inequality, here there is no necessity to distinguish between linear and quadratic finite elements, and between problems in two or three dimensions.

Theorem 6.3 *Suppose that the solution* $\mathbf{u}$ *to Problem* (3.12) *belongs to* $H^s(\Omega; \mathbb{R}^d)$, *with* $3/2 < s \le 1 + k$, *where* $k \ge 1$ *denotes the degree of the Lagrange finite elements* $\mathbb{P}_k$. *Suppose that* γ_0 *is large enough* ($\gamma_0 > 0$ *for* $\theta = -1$). *Then, the solution* $\mathbf{u}^h$ *to Problem* (6.6) *satisfies the* a priori *error estimate:*

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} + \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|_{0,\Gamma_C} \leq Ch^{s-1}|\mathbf{u}|_{s,\Omega}, \tag{6.17}$$

with the constant $C > 0$ that does not depend on h and $\mathbf{u}$ but depends on θ and γ_0. □

Proof We need to bound the right terms in estimate (6.11) and we take $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$, where $\mathcal{I}^h$ stands for the Lagrange interpolation operator mapping onto $\mathbf{V}^h$. First, we use Theorem 4.1 and get:

$$\|\mathbf{u}-\mathcal{I}^h\mathbf{u}\|_{1,\Omega} \leq Ch^{s-1}|\mathbf{u}|_{s,\Omega}, \tag{6.18}$$

with $1 < s \leq 1+k$.

For the second term, we use Corollary 4.1 as well as the Trace Theorem 2.8:

$$\left\| \gamma_N^{\frac{1}{2}}(u_{\mathbf{n}} - (\mathcal{I}^h\mathbf{u})_{\mathbf{n}}) \right\|_{0,\Gamma_C} \leq Ch^{s-1}\,|u_{\mathbf{n}}|_{s-\frac{1}{2},\Gamma_C} \leq Ch^{s-1}|\mathbf{u}|_{s,\Omega}. \tag{6.19}$$

Finally, we use Lemma 4.2 for the third term:

$$\left\| \gamma_N^{-\frac{1}{2}}\sigma_{\mathbf{n}}(\mathbf{u}-\mathcal{I}^h\mathbf{u}) \right\|_{0,\Gamma_C} \leq Ch^{s-1}|\mathbf{u}|_{s,\Omega}. \tag{6.20}$$

It remains to inject (6.18)–(6.20) into (6.11) to obtain the final estimate (6.17). □

6.4 Implementation

Observe that Problem (6.6) involves no more constraints, so that no specific method to handle inequality or equality constraints needs to be used. It is nonlinear, and there are various possibilities to deal with the nonlinearity. One of them can be to use a semi-smooth Newton method [99, 171, 178, 188, 233, 260, 264]. It has some advantages in terms of simplicity, genericity, and convergence speed. We will see later on that, with such a method, more complex problems can be treated in the same manner, notably friction.

Roughly speaking, the idea behind the semi-smooth Newton is the same as for standard Newton techniques to compute the zero of functions. The only difference comes from the points where the function is not derivable, and in this case, one chooses arbitrarily one possible direction. In the case of frictionless contact, these situations are rare and correspond to grazing contact.

In modern finite element libraries such as GetFEM [234], scikit-fem [271] or FEniCS [8], there are functionalities such as a specific language for weak form and automatic differentiation. In such an environment, programming Nitsche's method (6.6) consists mostly in rewriting formula (6.6) in the appropriate syntax.

To implement Nitsche's method in other finite element libraries, combined with semi-smooth Newton, one needs the tangent weak form. Using (6.3) we can write it explicitly as:

$$\begin{cases} \text{Find } \delta\mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\delta\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N} H_s(\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h))\mathrm{P}^{\mathbf{n}}_{1,N}(\delta\mathbf{u}^h)\mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) \\ = -R(\mathbf{u}^h; \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \end{cases} \tag{6.21}$$

where the residual associated with the approximation $\mathbf{u}^h$ reads:

$$R(\mathbf{u}^h; \mathbf{v}^h) := A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) - L(\mathbf{v}^h).$$

Then, one updates $\mathbf{u}^h$ by adding it to $\delta\mathbf{u}^h$. Remark that the above system is symmetric for $\theta = 1$, unsymmetric otherwise, which may influence in practice the choice for θ, or for the solver associated with the linear system. From this system, standard quadrature rules can be applied, despite the nonsmooth terms coming from the Heaviside function and the positive part operator (other strategies can be also implemented, to gain in accuracy). Using a standard assembly procedure and any method to solve the corresponding linear system, one finds the solution. The value of γ_0 should be chosen with some care, especially for the symmetric variant, to preserve well-posedness from one side, and to allow convergence of the semi-smooth Newton method on the other side. See, e.g., [233] for a discussion on this topic. For a complete implementation of the method in GetFEM, you can have a look at the documentation.[1] For some numerical examples, see, e.g., [82, 233].

6.5 Further Comments

We provide here some further facts and references related to Nitsche's method applied to contact and friction.

6.5.1 About Nitsche's Method

Before 2010, Nitsche's method has hardly been considered to discretize contact and friction conditions, despite the fact that it has gained popularity for other boundary

[1] http://getfem.org/userdoc/model_Nitsche.html.

conditions. The Nitsche method has been originally proposed in 1971 by J.A. Nitsche [219]. Most of the applications of Nitsche's method during the last two decades involved linear conditions on the boundary of a domain or at the interface between sub-domains: see, e.g., [74, 252] for the Dirichlet problem, [28] for domain decomposition with non-matching meshes and [149] for a global review.

6.5.2 The First Application to Bilateral Contact

In some works published in 2004 and 2006 [148, 158], it has been adapted for bilateral (persistent) contact, which still corresponds to linear boundary conditions on the contact zone. We remark furthermore that an algorithm for unilateral contact which makes use of Nitsche's method in its original form is presented and implemented in [148], and an extension to large strain bilateral contact has been performed in [267].

6.5.3 Nitsche for Unilateral Contact

In 2013 and 2015, in [78, 82], the Nitsche-based FEM as presented in this chapter was proposed and analyzed for Signorini's problem. In the numerical analysis, optimal convergence rates in the $H^1(\Omega)$-norm have been obtained without any assumption on the behavior of the solution on the contact boundary. From these papers, various extensions to other contact and friction problems have been designed. We mention these works in the next chapters.

6.5.4 Symmetry, Skew-symmetry, Etc.

Note that for other boundary conditions, such as non-homogeneous Dirichlet, the symmetric variant ($\theta = 1$) as originally proposed by Nitsche [219] is the most widespread, since it preserves symmetry, and allows efficient solvers for linear systems with a symmetric matrix. However, some nonsymmetric variants have been reconsidered recently, due to some remarkable robustness properties (see, e.g., [45, 57]). In the context of discontinuous Galerkin methods, such nonsymmetric variants are well-known as well (see, e.g., [106, Section 5.3.1, p.199]).

A penalty-free Nitsche's method has been designed and studied in [60] for scalar Signorini's problem, that is an extension of the method studied in [57] for the Dirichlet problem. It is combined with a nonconforming discretization based on Crouzeix-Raviart finite elements. Stability and optimal convergence rate in $H^1(\Omega)$-norm are established.

6.5.5 Lower Sobolev Regularity

Remark that the a priori error bounds presented in this chapter rely strongly on the assumption that the Sobolev regularity of the solution is large enough. Precisely, we need that the unknown $\mathbf{u}$ belongs to $H^s(\Omega; \mathbb{R}^d)$, with $s > 3/2$, not only to apply some standard estimates for Lagrange interpolation but also, above all, to take into account the consistency and incorporate the Signorini conditions in the L^2-sense.

Some recent works try to overcome this issue and to analyze Nitsche's method for lower Sobolev regularities ($s \geq 1$), see, for instance, [144, 145].

6.5.6 Link with Stabilized Methods and the Augmented Lagrangian

In [150] a least-square stabilized Augmented Lagrangian method, inspired by Nitsche's method, is described for unilateral contact. It shares some common features with Nitsche's method and allows increased flexibility on the discretization of the contact pressure. This has been followed recently by some papers, first [59, 62, 64] and then [63], that explore further the link between Nitsche and the Augmented Lagrangian, for the contact problem, the obstacle problem and interface problems with adhesive contact. We discuss briefly this topic within the next chapter.

6.5.7 Other Discretization Methods

In [124] Nitsche's method is combined with a cut-FEM / fictitious domain discretization, in the small deformations framework. In [174] a Nitsche method with IsoGeometric Analysis for contact has been designed and numerically studied. In [66] and [75], some Nitsche-HHO methods have been proposed and analyzed.

Chapter 7
Mixed Methods

Mixed methods were among the first ones to be formulated for contact, along with penalty methods: see, e.g., the pioneering works of J. Haslinger in 1977 [151], or of F. Brezzi, W. W. Hager and P.A. Raviart in 1978 [56], see also the monographs [153, 181] and references therein. Indeed, such formulations, that involve a Lagrangian, are very natural for constrained problems. Notably, for contact and friction problems, they lead to weak formulations where the inequality constraints appear separately and involve only variables that live on the contact boundary. This aspect is attractive for implementation purposes.

Mixed methods involve Lagrange multipliers, that are new unknowns, and for the Signorini problem, stand for the normal Cauchy stress (contact pressure) on the contact boundary. When approximated with finite elements, the well-posedness of the mixed discrete formulation relies on a discrete inf-sup compatibility condition. This means that some choices for the discretization of the primal variable (the displacement) and for the dual variable (the Lagrange multipliers) are allowed, while others are prohibited, to avoid either spurious oscillations and/or poorer convergence rates. Various possibilities exist to satisfy the discrete inf-sup compatibility conditions, at least in the case of the simple Signorini problem, but proving it is never a simple formality. For more complex contact problems, this issue remains even less obvious. Otherwise, it is possible to use a stabilized formulation to circumvent this compatibility condition.

Augmented Lagrangian formulations have been used widely for solving contact problems, since the pioneering work of P. Alart and A. Curnier in 1988 [6]. It has been, up to now, mostly thought as a trick to transform the inequality constraints, still present in the mixed formulations, into nonlinear equality constraints, that can then be solved with a semi-smooth Newton technique or an Uzawa solver. Recently it has been reconsidered as a discretization method, because of its close relationship with the Nitsche method (see Chap. 6). Still a discrete inf-sup condition, or a stabilization, is needed for Augmented Lagrangian methods.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_7

In this chapter, we present continuous and discrete mixed formulations of the Signorini problem as well as other formulations coming from Augmented Lagrangian techniques. First, we deal with the continuous mixed formulation, its derivation and its well-posedness. Particularly, the trace theory exposed in Chap. 2 allows to recover the inf-sup condition at the continuous level. Then, we detail a simple mixed finite element method, its well-posedness and its convergence properties. We take advantage of the recent results exposed in Chap. 5 to prove an optimal a priori error estimate in the $H^1(\Omega)$-norm. Then, we present briefly some other mixed methods. The first one incorporates a residual-least square stabilization term of Barbosa-Hughes type, in order to bypass the discrete inf-sup condition and to allow any type of finite element pairs. The two other mixed methods, which are the Mortar method and the Local Average Contact (LAC) method, are motivated by multi-body contact with non-matching meshes.

The rest of this chapter is devoted to the Augmented Lagrangian. Particularly, we show how this formulation can be recovered either from duality arguments for inequality constraints or derived formally from Nitsche formulations. We end this chapter with some hints about implementation and iterative solvers for Augmented Lagrangian formulation. The penalty method is finally recovered as a byproduct of the Uzawa algorithm.

7.1 Duality Principle and Mixed Weak Form of Signorini

Lagrange multipliers allow to transform a constrained minimization problem into another problem where constraints involve only the multipliers. This new formulation can be easier to solve numerically. First, let us detail the general procedure from which a formulation with Lagrange multipliers can be obtained from the variational inequality for the Signorini Problem (3.12).

7.1.1 Obtention of a Lagrangian

We still use the setting of Chap. 3 for the Signorini problem. Let us first recall that we denoted by

$$\mathbf{V} = \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \;\middle|\; \Upsilon \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}$$

the space of admissible displacements and by

$$\mathbf{W}_C = \left\{ \Upsilon \mathbf{v}|_{\Gamma_C} \mid \mathbf{v} \in \mathbf{V} \right\}, \quad W_C = \mathbf{W}_C \cdot \mathbf{n},$$

the trace and the normal trace, respectively, of admissible displacements on the contact boundary (see (3.14) in Chap. 3). We denoted the dual cone of weakly nonpositive forces on the contact boundary by:

$$\Lambda_C = \{\tau \in W_C' \mid \langle \tau, w \rangle_{\Gamma_C} \geq 0,\ \forall w \in W_C,\ w \leq 0\}.$$

The notation $\langle \cdot, \cdot \rangle_{\Gamma_C}$ stands for the duality pairing on $W_C' \times W_C$ and we will even note it $\langle \cdot, \cdot \rangle$ to simplify and when there is no ambiguity. Now, we define the convex cone

$$W_{C,-} := \{w \in W_C \mid w \leq 0 \text{ on } \Gamma_C\}$$

of normal trace functions in W_C that are (almost everywhere) nonpositive on the contact boundary. Let $I_-(\cdot)$ be the indicator function of $W_{C,-}$, defined as:

$$I_-(w) := \begin{cases} 0 & \text{if } w \in W_{C,-}, \\ +\infty & \text{otherwise,} \end{cases}$$

for $w \in W_C$. We verify that it is a convex function. We recall the expression of the energy functional,

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})$$

for $\mathbf{v} \in \mathbf{V}$, associated with the Signorini problem and introduced previously in (3.8). The solution $\mathbf{u}$ to the Signorini problem (3.10) is equivalently the unique minimizer on $\mathbf{V}$ of the functional $\widetilde{\mathcal{J}}(\cdot)$ defined as

$$\widetilde{\mathcal{J}}(\mathbf{v}) := \mathcal{J}(\mathbf{v}) + I_-(v_{\mathbf{n}}),$$

for $\mathbf{v} \in \mathbf{V}$. The duality principle consists in introducing an extra variable $w \in W_C$ and to set

$$\mathcal{W}(\mathbf{v}, w) := \mathcal{J}(\mathbf{v}) + I_-(v_{\mathbf{n}} - w), \tag{7.1}$$

such that, first, there holds

$$\mathcal{W}(\mathbf{v}, 0) = \widetilde{\mathcal{J}}(\mathbf{v})$$

for $\mathbf{v} \in \mathbf{V}$, and, also, such that, for every $\mathbf{v} \in \mathbf{V}$ the partial function

$$w \mapsto \mathcal{W}(\mathbf{v}, w)$$

is convex. Therefore, the solution $\mathbf{u}$ to the Signorini problem (3.10) is the unique minimizer of $\mathcal{W}(\cdot, 0)$ on $\mathbf{V}$. The next step now consists in writing the corresponding

Lagrangian functional, and this can be done in a systematic manner by applying the duality theory. We provide the details in the lemma below.

Lemma 7.1 *The Lagrangian obtained by application of the Fenchel-Legendre conjugate to the functional $\mathcal{W}$, defined in (7.1), has the following expression:*

$$\mathscr{L}(\mathbf{v}, \lambda) = \mathcal{J}(\mathbf{v}) - \langle \lambda, v_{\mathbf{n}} \rangle_{\Gamma_C} - I_{\Lambda_C}(\lambda), \tag{7.2}$$

for $\mathbf{v} \in \mathbf{V}$ and $\lambda \in W_C'$, where $I_{\Lambda_C}(\cdot)$ is the indicator function of Λ_C, defined, for $\lambda \in W_C'$, as:

$$I_{\Lambda_C}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in \Lambda_C, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof We apply the Fenchel-Legendre conjugate to $\mathcal{W}(\mathbf{v}, \cdot)$, see, e.g., [117, 236], from which we can write the following Lagrangian:

$$\mathscr{L}(\mathbf{v}, \lambda) = - \sup_{w \in W_C} \left(\langle \lambda, w \rangle - \mathcal{W}(\mathbf{v}, w) \right),$$

for $\mathbf{v} \in \mathbf{V}$ and $\lambda \in W_C'$. The above expression of this Lagrangian can be transformed as follows:

$$\begin{aligned} \mathscr{L}(\mathbf{v}, \lambda) &= - \sup_{w \in W_C} \left(\langle \lambda, w \rangle - \mathcal{J}(\mathbf{v}) - I_-(v_{\mathbf{n}} - w) \right) \\ &= \mathcal{J}(\mathbf{v}) - \sup_{w \in W_C} \left(\langle \lambda, w \rangle - I_-(v_{\mathbf{n}} - w) \right). \end{aligned} \tag{7.3}$$

Then, we compute separately:

$$\begin{aligned} &\sup_{w \in W_C} \left(\langle \lambda, w \rangle - I_-(v_{\mathbf{n}} - w) \right) \\ &= \sup_{\zeta \in W_C} \left(\langle \lambda, v_{\mathbf{n}} - \zeta \rangle - I_-(\zeta) \right) \\ &= \langle \lambda, v_{\mathbf{n}} \rangle + \sup_{\zeta \in W_C} \left(\langle -\lambda, \zeta \rangle - I_-(\zeta) \right) \end{aligned} \tag{7.4}$$

with the change of variable $\zeta = v_{\mathbf{n}} - w$ ($w = v_{\mathbf{n}} - \zeta$). If we remember the definition of the indicator I_-, we get:

$$\sup_{\zeta \in W_C} \left(\langle -\lambda, \zeta \rangle - I_-(\zeta) \right) = \sup_{\zeta \in W_{C,-}} \langle -\lambda, \zeta \rangle.$$

Now, remark that if $\lambda \in \Lambda_C$, there holds, by definition:

$$\sup_{\zeta \in W_{C,-}} \langle -\lambda, \zeta \rangle = 0.$$

Otherwise, if $\lambda \notin \Lambda_C$, this means there exists $\zeta^\star \in W_{C,-}$, with $\langle \lambda, \zeta^\star \rangle < 0$, so that, for any real number $\alpha > 0$:

$$\langle -\lambda, \alpha\zeta^\star \rangle = \alpha \langle -\lambda, \zeta^\star \rangle$$

from which we deduce this time

$$\sup_{\zeta \in W_{C,-}} \langle -\lambda, \zeta \rangle = +\infty.$$

We can summarize the two above considerations as:

$$\sup_{\zeta \in W_{C,-}} \langle -\lambda, \zeta \rangle = I_{\Lambda_C}(\lambda),$$

where $I_{\Lambda_C}(\cdot)$ is the indicator function of Λ_C. From (7.3)-(7.4) combined with the above result, we deduce finally the following expression of the Lagrangian:

$$\mathscr{L}(\mathbf{v}, \lambda) = \mathcal{J}(\mathbf{v}) - \langle \lambda, v_{\mathbf{n}} \rangle - I_{\Lambda_C}(\lambda).$$

This is (7.2). □

Observe that, in the above Lagrangian, the inequality constraints involve no more the displacement, but are now associated with the Lagrange multiplier, thanks to the indicator function $I_{\Lambda_C}(\cdot)$.

7.1.2 *Mixed Problem as a Saddle-Point*

We know (see, e.g., [117, 153]) that the solution $\mathbf{u}$ to (3.10) is obtained through computing the saddle-point of $\mathscr{L}(\cdot, \cdot)$, in other terms through finding a pair

$$(\mathbf{u}, \lambda) \in \mathbf{V} \times W'_C$$

that satisfies

$$\mathscr{L}(\mathbf{u}, \mu) \leq \mathscr{L}(\mathbf{u}, \lambda) \leq \mathscr{L}(\mathbf{v}, \lambda), \qquad \forall \mathbf{v} \in \mathbf{V},\ \forall \mu \in W'_C.$$

Moreover, this saddle-point can be characterized as the solution to the first order optimality system computed by (sub)differentiation of the Lagrangian. By this way, we can get a mixed form for the Signorini problem. We detail this point below. Before we introduce the notation

$$b(\mathbf{v}, \lambda) := \langle \lambda, v_{\mathbf{n}} \rangle_{\Gamma_C}$$

for the bilinear form on $\mathbf{V} \times W_C'$ associated with the duality pairing, on the contact boundary, between the multiplier λ and the normal trace $v_\mathbf{n}$ of the primal variable $\mathbf{v}$.

Proposition 7.1 *Any pair* $(\mathbf{u}, \lambda) \in \mathbf{V} \times W_C'$ *that is a saddle-point of* $\mathscr{L}(\cdot, \cdot)$ *is solution to the following mixed problem:*

$$\begin{cases} \text{Find } (\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda_C \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \lambda) = L(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, \mu - \lambda) \geq 0, \qquad \forall \mu \in \Lambda_C. \end{cases} \tag{7.5}$$

Proof Let us denote by $(\mathbf{u}, \lambda) \in \mathbf{V} \times W_C'$ the saddle-point of $\mathscr{L}(\cdot, \cdot)$. The derivation in the $\mathbf{v}$ variable makes no difficulty and we get the first equation of (7.5):

$$0 = \mathscr{L}_\mathbf{v}'(\mathbf{u}, \lambda; \mathbf{v}) = \mathcal{J}'(\mathbf{u}; \mathbf{v}) - \langle \lambda, v_\mathbf{n} \rangle = a(\mathbf{u}, \mathbf{v}) - \langle \lambda, v_\mathbf{n} \rangle - L(\mathbf{v}),$$

where $\mathscr{L}_\mathbf{v}'$ denotes the partial derivative with respect to the first variable $\mathbf{v}$. Since the Lagrangian $\mathscr{L}(\cdot, \cdot)$ involves an indicator function in λ, which is not differentiable, we can write the following necessary condition of optimality [236, Theorem 8.12]:

$$0 \in \partial \mathscr{L}_\lambda'(\mathbf{u}, \lambda; \cdot),$$

where $\partial \mathscr{L}_\lambda'(\mathbf{u}, \lambda; \cdot)$ is the subdifferential of $\mathscr{L}(\cdot, \cdot)$, considered as a function of λ. This condition is rewritten:

$$0 \in \{u_\mathbf{n}\} + \partial I_{\Lambda_C}'(\lambda).$$

There remains to compute the subdifferential $\partial I_{\Lambda_C}'(\cdot)$ of the indicator function $I_{\Lambda_C}(\cdot)$. For this purpose, take $\lambda_0 \in \Lambda_C$ (Λ_C is the domain of I_{Λ_C}). Take any $w \in \partial I_{\Lambda_C}'(\lambda_0)$ that is a subgradient of I_{Λ_C} at point λ_0. By definition, it verifies, for all $\lambda \in W_C'$:

$$I_{\Lambda_C}(\lambda) \geq I_{\Lambda_C}(\lambda_0) + \langle \lambda - \lambda_0, w \rangle,$$

with alleviated notations (we identify the bidual of W_C as W_C itself). Since $\lambda_0 \in \Lambda_C$, the above equation means simply:

$$I_{\Lambda_C}(\lambda) \geq \langle \lambda - \lambda_0, w \rangle.$$

If $\lambda \notin \Lambda_C$, then $I_{\Lambda_C}(\lambda) = +\infty$ and the above equation is always satisfied. In the case $\lambda \in \Lambda_C$, then $I_{\Lambda_C}(\lambda) = 0$, from which we get:

$$0 \geq \langle \lambda - \lambda_0, w \rangle.$$

Conversely, if w satisfies the above relationship for $\lambda \in \Lambda_C$, it is a subgradient of I_{Λ_C}. As a result, we deduce

$$\partial I'_{\Lambda_C}(\lambda_0) = N_{\Lambda_C}(\lambda_0) := \{w \in W_C \mid \langle \lambda_0, w\rangle \geq \langle \lambda, w\rangle, \forall \lambda \in \Lambda_C\}.$$

This is in fact the normal cone $N_{\Lambda_C}(\lambda_0)$ to Λ_C (see once again [236]). As a result, the second optimality condition reads

$$0 \in \{u_{\mathbf{n}}\} + N_{\Lambda_C}(\lambda),$$

or, more directly:

$$-u_{\mathbf{n}} \in N_{\Lambda_C}(\lambda).$$

It is a variational inequality on the boundary, and can be rewritten

$$0 \geq \langle \lambda - \mu, u_{\mathbf{n}}\rangle$$

for all $\mu \in \Lambda_C$. Last but not least, the optimality condition for the Lagrangian implies $\lambda \in \Lambda_C$. Combining the two optimality conditions, we obtain the mixed formulation (7.5). □

Remark 7.1 Under appropriate assumptions and using similar arguments as in Chap. 3, we can establish the equivalence between the mixed formulation (7.5) for the Signorini problem, and the strong form (3.22)–(3.23). This means that if $\mathbf{u} \in \mathbf{V}$ is solution to (3.22)–(3.23) then $(\mathbf{u}, \sigma_{\mathbf{n}}(\mathbf{u})) \in \mathbf{V} \times \Lambda_C$ solves (7.5), and, conversely, for $(\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda_C$ that solves (7.5), $\mathbf{u}$ solves (3.22)–(3.23) and $\lambda = \sigma_{\mathbf{n}}(\mathbf{u})$. □

7.1.3 An inf-sup Condition

As a consequence of the lifting theorem, the pair of spaces related to the above mixed form satisfies an inf-sup condition, or Ladyzhenskaya-Babuska-Brezzi (LBB) condition (see, e.g., [53, 54, 118] for the general theory about LBB conditions). This is stated below:

Proposition 7.2 *Let us suppose that Ω is a Lipschitz domain in $\mathbb{R}^d$. Assume that the contact boundary Γ_C is of positive Lebesgue measure. Then, the following inf-sup condition holds: there exists $\beta > 0$ such that*

$$\inf_{\mu \in (H^{\frac{1}{2}}(\Gamma_C))'} \sup_{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d)} \frac{b(\mathbf{v}, \mu)}{\|\mu\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{v}\|_{1,\Omega}} \geq \beta. \tag{7.6}$$

Proof Let $\mu \in (H^{\frac{1}{2}}(\Gamma_C))'$. By definition, the (dual) norm of μ reads:

$$\|\mu\|_{-\frac{1}{2},\Gamma_C} = \sup_{w \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \mu, w \rangle}{\|w\|_{\frac{1}{2},\Gamma_C}}.$$

From the Lifting Theorem 2.9, there exists $C > 0$, such that, for any $w \in H^{\frac{1}{2}}(\Gamma_C)$, we can build $\mathbf{v}_w \in H^1(\Omega; \mathbb{R}^d)$ that verifies both $v_{w,\mathbf{n}} = w$ on Γ_C and

$$\|\mathbf{v}_w\|_{1,\Omega} \leq C\|w\|_{\frac{1}{2},\Gamma_C}.$$

As a result, we can write

$$\langle \mu, w \rangle = \langle \mu, v_{w,\mathbf{n}} \rangle$$

and we bound

$$\|\mu\|_{-\frac{1}{2},\Gamma_C} = \sup_{w \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \mu, v_{w,\mathbf{n}} \rangle}{\|w\|_{\frac{1}{2},\Gamma_C}} \leq C \sup_{w \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \mu, v_{w,\mathbf{n}} \rangle}{\|\mathbf{v}_w\|_{1,\Omega}} \leq C \sup_{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d)} \frac{\langle \mu, v_{\mathbf{n}} \rangle}{\|\mathbf{v}\|_{1,\Omega}}.$$

This is (7.6), with $\beta = 1/C$: the infsup constant can be taken as the inverse of the continuity constant of the lifting operator. □

Remark 7.2 In the case where the Dirichlet boundary Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$, there holds $W_C' = (H^{\frac{1}{2}}(\Gamma_C))'$ and, following the same path as in the above proof, we can obtain another inf-sup condition:

$$\inf_{\mu \in W_C'} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, \mu)}{\|\mu\|_{W_C'} \|\mathbf{v}\|_{1,\Omega}} \geq \beta. \tag{7.7}$$

In fact, the above inf-sup condition (7.7) can be obtained even when the above topological condition on Γ_D and Γ_C is relaxed, if one uses the definition of the norm on W_C provided in Chap. 3. Nevertheless, in the sequel, since we need to use the Sobolev-Slobodeckij norm for the numerical analysis, we will always make the assumption that Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$ and use the condition (7.6).

Remark 7.3 For dual spaces on the boundary, we prefer to keep notations such as $(H^{\frac{1}{2}}(\Gamma_C))'$. For more details about the Sobolev spaces associated with a dual norm, other notations and their properties, see, for instance, [207, 239] and references therein. □

At this point, we reached our goal: the inequality constraints are now only on the boundary and associated with the Lagrange multiplier, conversely to the variational inequality in primal form (3.12), where the whole set of solutions is defined with an inequality. Note that the second equation is a way to impose weakly the contact conditions and is equivalent to the three last, KKT, conditions in (3.18).

The existence and uniqueness of a pair $(\mathbf{u}, \lambda)$ solution to (7.5) can be proven independently as well using results about saddle-points of Lagrangian formulations, see, e.g., [153, Theorem 3.11].

7.2 A Mixed Finite Element Method

We still consider the same setting as presented in Chap. 4 for the finite element discretization, as we did in previous chapters for the primal FEM (Chap. 5) and for Nitsche-FEM (Chap. 6). To simplify the presentation and the analysis, we limit ourselves to a two-dimensional problem and then set $d = 2$. The results could be extended to the three-dimensional case using the technicalities developed in Chap. 5, Sect. 5.4.

As in Chap. 3 we suppose that the domain Ω is a polygon and that Γ_C is a straight line segment for the sake of simplicity. Moreover, we assume afterwards that Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$ (see Remark 3.4 as well as Remark 7.2 above).

With the subdomain Ω, we then associate a regular family of triangulations $\mathcal{T}^h$ made of elements denoted T such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}^h} \overline{T}.$$

The index h is given by $h = \max_{T \in \mathcal{T}^h} h_T$ where h_T denotes the diameter of the triangle T. We suppose that the mono-dimensional traces of triangulations $\mathcal{T}^h$ on Γ_C are quasi-uniform. We make use of the Lagrange finite element spaces of order one to discretize the displacement, and use the same notations as in previous chapters:

$$\mathbf{V}^h := \mathbf{X}_1^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^2) \;\middle|\; \mathbf{v}^h|_T \in \mathbb{P}_1(T; \mathbb{R}^2),\ \forall T \in \mathcal{T}^h, \quad \mathbf{v}^h|_{\Gamma_D} = 0 \right\},$$

where $\mathbf{X}_1^h$ has been introduced in Chap. 4. Next, we define the space W^h of continuous functions which are piecewise of degree 1 on the trace mesh on Γ_C.

$$W^h := \left\{ \psi^h \in \mathscr{C}(\overline{\Gamma_C}) \;\middle|\; \exists \mathbf{v}^h \in \mathbf{V}^h \text{ such that } \mathbf{v}^h \cdot \mathbf{n} = \psi^h \text{ on } \Gamma_C \right\}.$$

7.2.1 *The Mixed Method*

Let us now approximate the closed convex cone Λ_C by a subset of W^h. The most important point is the way we translate at the discrete level the nonpositivity condition. We introduce the set M^h where nonpositivity holds in a weak sense

$$M^h := \left\{ \mu^h \in W^h \;\middle|\; b(\mathbf{v}^h, \mu^h) \geq 0, \ \ \forall \mathbf{v}^h \in \mathbf{V}^h \text{ s.t. } v_{\mathbf{n}}^h \leq 0 \text{ on } \Gamma_C \right\}.$$

Next, we define the positive polar cone $M^{h,*}$ of M^h:

$$M^{h,*} = \left\{ \mu^h \in W^h \;\middle|\; \int_{\Gamma_C} \mu^h \psi^h \geq 0,\ \forall \psi^h \in M^h \right\}.$$

We then choose a discretized mixed formulation which uses M^h as an approximation of Λ_C. The discrete problem is: find $\mathbf{u}^h \in \mathbf{V}^h$ and $\lambda^h \in M^h$ satisfying

$$\begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \lambda^h) = L(\mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mathbf{u}^h, \mu^h - \lambda^h) \geq 0, & \forall\, \mu^h \in M^h. \end{cases} \tag{7.8}$$

7.2.2 Well-Posedness

From the relation

$$\left\{ \mu^h \in W^h \;\middle|\; b(\mathbf{v}^h, \mu^h) = 0, \quad \forall \mathbf{v}^h \in \mathbf{V}^h \right\} = \{0\} \tag{7.9}$$

and the $\mathbf{V}^h$-ellipticity of $a(\cdot, \cdot)$, we get the following proposition.

Proposition 7.3 *Problem (7.8) admits a unique solution* $(\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times M^h$. □

Remark 7.4 It can be easily checked that the compatibility relation (7.9) implies the existence of a constant β^h such that

$$\inf_{\mu^h \in W^h} \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, \mu^h)}{\|\mu^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{v}^h\|_{1,\Omega}} \geq \beta^h > 0.$$

In fact, the constant β^h does not depend on h (see Proposition 7.5 hereafter). □

7.2.3 An Equivalent Discrete Variational Inequality

The next result gives the link between the mixed problem (7.8) and a discretized variational inequality issued from (3.12).

Proposition 7.4 *Let* $(\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times M^h$ *be the solution to (7.8). Then,* $\mathbf{u}^h$ *is also solution to the variational inequality:*

$$\begin{aligned} &\text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ such that} \\ &a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \geq L(\mathbf{v}^h - \mathbf{u}^h), \quad \forall \mathbf{v}^h \in \mathbf{K}^h, \end{aligned} \tag{7.10}$$

where

$$\mathbf{K}^h = \left\{\mathbf{v}^h \in \mathbf{V}^h \;\middle|\; v^h_{\mathbf{n}} \le 0 \text{ on } \Gamma_C\right\}$$

the discrete convex cone where non-interpenetration is imposed. □

Proof Taking first $\mu^h = 0$, and then $\mu^h = 2\lambda^h$ in formulation (7.8) leads to

$$b(\mathbf{u}^h, \lambda^h) = 0$$

and also to

$$b(\mathbf{u}^h, \mu^h) = \int_{\Gamma_C} \mu^h u^h_{\mathbf{n}} \ge 0, \quad \forall \mu^h \in M^h.$$

The latter inequality implies by polarity that $u^h_{\mathbf{n}} \in M^{h,*}$. Since M^h is the polar cone of the nonpositive functions of W^h, we deduce that $M^{h,*}$ stands for the bipolar cone of the nonpositive functions of W^h: so, it coincides with this same convex set of the nonpositive functions of W^h. This implies that $\mathbf{u}^h \in \mathbf{K}^h$. Consequently both (7.8) and $b(\mathbf{u}^h, \lambda^h) = 0$ lead to

$$a(\mathbf{u}^h, \mathbf{u}^h) = L(\mathbf{u}^h) \tag{7.11}$$

and for any $\mathbf{v}^h \in \mathbf{K}^h$, we get

$$a(\mathbf{u}^h, \mathbf{v}^h) - L(\mathbf{v}^h) = b(\mathbf{v}^h, \lambda^h) = \int_{\Gamma_C} \lambda^h v^h_{\mathbf{n}} \ge 0, \tag{7.12}$$

owing to $\lambda^h \in M^h$.

Putting together (7.11) and (7.12) implies that $\mathbf{u}^h$ is solution to the variational inequality (7.10), which admits a unique solution according to Stampacchia's Theorem 3.4. □

Remark 7.5 We recover exactly the discrete variational inequality described in Chap. 5 for the choice $\mathbf{K}^h = \mathbf{K}^h_1$. □

7.2.4 A Discrete inf-sup Condition

We are now interested in obtaining a uniform inf-sup condition for $b(\cdot, \cdot)$ over $\mathbf{V}^h \times W^h$. The result is given in the following proposition. The proof relies essentially on the stability of the L^2-projection projection operator in the $H^{\frac{1}{2}}(\Gamma_C)$-norm (see Proposition 4.4 in Chap. 4).

Proposition 7.5 *Suppose that $\overline{\Gamma_D}$ is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$ and that the mesh on the contact boundary Γ_C is quasi-uniform. Then, the following discrete inf-sup condition holds*

$$\inf_{\mu^h \in W^h} \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, \mu^h)}{\|\mu^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{v}^h\|_{1,\Omega}} \geq \beta > 0, \tag{7.13}$$

where β does not depend on h. □

Proof Let us introduce π^h, the global L^2 projection operator on W^h, defined in (4.10). We recall that, for a quasi-uniform mesh, it is stable in the $H^{\frac{1}{2}}$-norm, see (4.12): there exists a constant C independent of h such that:

$$\|\pi^h \varphi\|_{\frac{1}{2},\Gamma_C} \leq C\|\varphi\|_{\frac{1}{2},\Gamma_C}, \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma_C). \tag{7.14}$$

Let us show that the inf-sup condition (7.13) is a consequence of (4.12). Let $\mu^h \in W^h$. By definition of the dual norm and since we deal with Hilbert spaces, the maximum is attained so there exists a function $\psi \in H^{\frac{1}{2}}(\Gamma_C)$ with $\|\psi\|_{\frac{1}{2},\Gamma_C} = 1$, and such that

$$\langle \mu^h, \psi \rangle_{\Gamma_C} = \int_{\Gamma_C} \mu^h \psi = \|\mu^h\|_{-\frac{1}{2},\Gamma_C}. \tag{7.15}$$

Using the definition (4.10) of π^h, we get:

$$\int_{\Gamma_C} \mu^h \psi = \int_{\Gamma_C} \mu^h \pi^h \psi. \tag{7.16}$$

Let us consider now a discrete lifting $\mathcal{L}^h$ from W_h to $\mathbf{V}_h$ that satisfies (see (4.9) in Chap. 4):

$$\mathcal{L}^h(\pi_h \psi) = \pi_h \psi \mathbf{n} \ \text{ on } \ \Gamma_C \qquad \text{and} \qquad \|\mathcal{L}^h(\pi_h \psi)\|_{1,\Omega} \leq C\|\pi_h \psi\|_{\frac{1}{2},\Gamma_C}.$$

We set

$$\mathbf{w}^h := \mathcal{L}^h(\pi^h \psi) \in \mathbf{V}^h,$$

and get from the above and (7.14):

$$\|\mathbf{w}^h\|_{1,\Omega} \leq C\|\pi^h \psi\|_{\frac{1}{2},\Gamma_C} \leq C\|\psi\|_{\frac{1}{2},\Gamma_C} \leq C,$$

where, finally, we took into account the equality $\|\psi\|_{\frac{1}{2},\Gamma_C} = 1$. Moreover, from (7.15) and (7.16) the function $\mathbf{w}_h$ verifies

$$\int_{\Gamma_C} \mu^h w_{\mathbf{n}}^h = \|\mu^h\|_{-\frac{1}{2},\Gamma_C}.$$

We conclude that there exists C independent of h (and of μ_h) such that

$$C\|\mu_h\|_{-\frac{1}{2},\Gamma_C} \leq \frac{\|\mu_h\|_{-\frac{1}{2},\Gamma_C}}{\|\mathbf{w}_h\|_{1,\Omega}} = \frac{\int_{\Gamma_C} \mu^h w_{\mathbf{n}}^h}{\|\mathbf{w}^h\|_{1,\Omega}} = \frac{b(\mu^h, \mathbf{w}^h)}{\|\mathbf{w}^h\|_{1,\Omega}} \leq \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mu^h, \mathbf{v}^h)}{\|\mathbf{v}^h\|_{1,\Omega}}.$$

This is the discrete inf-sup condition (7.13). □

Remark 7.6 The discrete inf-sup condition (7.13) remains valid for $d = 3$. □

Remark 7.7 Note that a slightly different proof can be written in which the continuous inf-sup condition is used instead of the property that the maximum is attained in the definition of the dual norm (see [111]). □

Remark 7.8 The choice of the mixed continuous and discrete formulations (7.5) and (7.8) will allow us to obtain more information compared to the variational inequality approach (3.12) and (7.10). In particular, the forthcoming error estimates of the mixed method hold also for the variational inequality problem according to Proposition 7.4. □

7.3 An *a priori* Error Estimate for the Mixed Formulation

Now, we analyze the convergence of the mixed finite element method (7.8).

7.3.1 An Abstract Lemma

As in previous chapters, and as usual in the analysis of finite element methods, we start with an abstract error estimate.

Proposition 7.6 *Let* $(\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda_C$ *be the solution to (7.5) and let* $(\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times M^h$ *be the solution to (7.8). Then, there exists a positive constant* C *independent of* h *satisfying:*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \\ &\leq C \left\{ \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + \inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C} \right. \\ &\left. + \Big(\max(-b(\mathbf{u}, \lambda^h), 0)\Big)^{\frac{1}{2}} + \Big(\max(-b(\mathbf{u}^h, \lambda), 0)\Big)^{\frac{1}{2}} \right\}. \end{aligned} \tag{7.17}$$

□

Proof The proof is divided into three parts. First, an upper bound of $\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}$ will be obtained in (7.18). Then, the inf-sup condition (7.13) will lead to an upper bound of $\|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C}$ in (7.19). Both estimates will allow us to get the required estimate (7.17).

(i) Let $\mathbf{v}^h \in \mathbf{V}^h$. According to (7.5) and (7.8), we have

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h) &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \\ &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + b(\mathbf{v}^h - \mathbf{u}^h, \lambda) - b(\mathbf{v}^h - \mathbf{u}^h, \lambda^h) \\ &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + b(\mathbf{v}^h - \mathbf{u}, \lambda - \lambda^h) + b(\mathbf{u} - \mathbf{u}^h, \lambda - \lambda^h). \end{aligned}$$

Besides, the inequality of formulation (7.5) implies $b(\mathbf{u}, \lambda) = 0$. Similarly, formulation (7.8) leads to $b(\mathbf{u}^h, \lambda^h) = 0$. Therefore

$$a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h) = a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) - b(\mathbf{u} - \mathbf{v}^h, \lambda - \lambda^h) - b(\mathbf{u}^h, \lambda) - b(\mathbf{u}, \lambda^h).$$

Denoting by α the ellipticity constant of $a(\cdot, \cdot)$ on $\mathbf{V}$, by C_a the continuity constant of $a(\cdot, \cdot)$ on $\mathbf{V}$ and using the Trace Theorem 2.6, we obtain

$$\begin{aligned} \alpha \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \le C_a \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + C\|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} \\ -b(\mathbf{u}^h, \lambda) - b(\mathbf{u}, \lambda^h). \qquad (7.18) \end{aligned}$$

(ii) Now, let us consider problem (7.5). The inclusion $\mathbf{V}^h \subset \mathbf{V}$ implies

$$a(\mathbf{u}, \mathbf{v}^h) - b(\mathbf{v}^h, \lambda) = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

The latter equality together with (7.8) yields

$$a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \lambda - \lambda^h) = 0, \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Inserting $\mu^h \in W^h$, using the continuity of $a(\cdot, \cdot)$ as well as the Trace Theorem 2.6 gives

$$\begin{aligned} b(\mathbf{v}^h, \lambda^h - \mu^h) &= -a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \lambda - \mu^h) \\ &\le C_a \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{v}^h\|_{1,\Omega} + C\|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{v}^h\|_{1,\Omega}, \\ &\qquad \forall\, \mu^h \in W^h, \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{aligned}$$

This estimate and condition (7.13) allow us to write

$$\beta \|\lambda^h - \mu^h\|_{-\frac{1}{2},\Gamma_C} \le \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, \lambda^h - \mu^h)}{\|\mathbf{v}^h\|_{1,\Omega}} \le C_a \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + C\|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C},$$

for any $\mu^h \in W^h$. Since

$$\|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \leq \|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C} + \|\mu^h - \lambda^h\|_{-\frac{1}{2},\Gamma_C}, \quad \forall \mu^h \in W^h,$$

we finally come to the conclusion that there exists $C > 0$ such that

$$\|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \leq C\Big(\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C}\Big). \tag{7.19}$$

(iii) Putting together (7.19) and (7.18), we obtain for any $\mathbf{v}^h \in \mathbf{V}^h$:

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \leq C \Big\{ \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}\|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + \inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}$$
$$-b(\mathbf{u}^h, \lambda) - b(\mathbf{u}, \lambda^h)\Big\}.$$

Using estimate $ab \leq \varepsilon a^2 + \dfrac{1}{4\varepsilon} b^2$ (with $\varepsilon > 0$) from Young's inequality leads to the bound

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \leq C \Big\{ \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 + \inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{-\frac{1}{2},\Gamma_C}^2$$
$$-b(\mathbf{u}^h, \lambda) - b(\mathbf{u}, \lambda^h)\Big\}.$$

Taking the square root of this inequality which is then combined with (7.19) terminates the proof of (7.17). $\square$

As a consequence the convergence error $\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C}$ can be divided into four quantities. The first two parts are the classical approximation terms measuring the "quality" of the space $\mathbf{V}^h$ approximating $\mathbf{V}$ and of W^h approximating $H^{-\frac{1}{2}}(\Gamma_C)$. The term $\big(\max(-b(\mathbf{u}^h, \lambda), 0)\big)^{\frac{1}{2}}$ takes into account the interpenetration of the bodies (i.e., when $u_{\mathbf{n}}^h > 0$) which is possible for other discretizations than the one considered here (we will see thereafter that this term disappears). Finally, the term $\big(\max(-b(\mathbf{u}, \lambda^h), 0)\big)^{\frac{1}{2}}$ measures the possible positivity of λ^h.

7.3.2 An Optimal Error Estimate

Now, our purpose is to prove error estimates under H^s regularity hypotheses on the displacements (with $\frac{3}{2} < s \leq 2$). The optimal *a priori* error estimate for the mixed form (7.8) is stated below.

Theorem 7.1 *Let* $3/2 < s \leq 2$. *Suppose that the solution* $(\mathbf{u}, \lambda)$ *to (7.5) satisfies the regularity assumption* $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$. *Let* $(\mathbf{u}^h, \lambda^h)$ *be the solution to (7.8). Then*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \leq C(\mathbf{u})h^{s-1}, \tag{7.20}$$

where the constant $C(\mathbf{u})$ *depends linearly on* $\|\mathbf{u}\|_{s,\Omega}$.

Proof Hereafter, the notation $C(\mathbf{u})$ represents a generic constant which depends linearly on $\|\mathbf{u}\|_{s,\Omega}$. Let us choose $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$ and $\mu^h = \pi^h\lambda$ in (7.17). Using standard approximation properties (4.6) and (4.13) yields

$$\begin{aligned}&\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C}\\ &\leq C\Big\{C(\mathbf{u})h^{s-1} + \big(\max(-b(\mathbf{u}, \lambda^h), 0)\big)^{\frac{1}{2}} + \big(\max(-b(\mathbf{u}^h, \lambda), 0)\big)^{\frac{1}{2}}\Big\}.\end{aligned} \tag{7.21}$$

It remains then to estimate both terms $b(\mathbf{u}, \lambda^h)$ and $b(\mathbf{u}^h, \lambda)$.
Step 1. Estimation of $-b(\mathbf{u}, \lambda^h)$.
Let $\mathcal{I}^h_{\Gamma_C}$ be the Lagrange interpolation operator on the contact boundary. We write

$$\begin{aligned}-b(\mathbf{u}, \lambda^h) &= -\int_{\Gamma_C} \lambda^h u_{\mathbf{n}}\\ &= -\int_{\Gamma_C} \lambda^h (u_{\mathbf{n}} - \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}}) - \int_{\Gamma_C} \lambda^h\, \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}}.\end{aligned}$$

Observe that $\mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}} \leq 0$ on Γ_C. From $\lambda^h \in M_h$ and (4.7), we deduce

$$\begin{aligned}-b(\mathbf{u}, \lambda^h) &\leq -\int_{\Gamma_C} \lambda^h (u_{\mathbf{n}} - \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}})\\ &\leq -\int_{\Gamma_C} \lambda (u_{\mathbf{n}} - \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}}) + \|\lambda^h - \lambda\|_{-\frac{1}{2},\Gamma_C} \|u_{\mathbf{n}} - \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}}\|_{\frac{1}{2},\Gamma_C}\\ &\leq -\int_{\Gamma_C} \lambda (u_{\mathbf{n}} - \mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}}) + Ch^{s-1}\|u_{\mathbf{n}}\|_{s-\frac{1}{2},\Gamma_C} \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C}.\end{aligned} \tag{7.22}$$

The remaining integral term is estimated as in the proof of Theorem 5.1 in Chap. 5 (see also [110]).

$$\int_{\Gamma_C} \lambda (\mathcal{I}^h_{\Gamma_C} u_{\mathbf{n}} - u_{\mathbf{n}}) \leq (C(\mathbf{u}))^2 h^{2(s-1)}. \tag{7.23}$$

Putting together (7.22) and (7.23) and using the Trace Theorem 2.8 gives

$$-b(\mathbf{u}, \lambda^h) \leq C(\mathbf{u}) \left(h^{s-1} \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} + C(\mathbf{u}) h^{2(s-1)} \right). \quad (7.24)$$

Step 2. Estimation of $-b(\mathbf{u}^h, \lambda)$.

The term is nonpositive. Indeed there holds both $\lambda \leq 0$ (see conditions (3.23) in Chap. 3) and also $u^h_{\mathbf{n}} \leq 0$ (see Proposition 7.4).
Step 3. End of the proof.

Let us insert result (7.24) into (7.21) and use estimate $ab \leq \varepsilon a^2 + \dfrac{1}{4\varepsilon} b^2$, with $\varepsilon > 0$ chosen appropriately. The convergence result (7.20) of the theorem is then obtained.

□

7.4 Other Mixed Methods

Of course, the mixed finite element method presented in the previous section is not the only possible choice when starting from the continuous mixed formulation (7.5) of Signorini contact. In fact, there are potentially as many choices, at least, as there are possibilities to approximate the primal space and the dual cone by objects in finite dimension spaces. Nevertheless, each particular choice may lead to different properties in terms of numerical stability and approximability, with some impact on the convergence rates. It may also have different implications in terms of implementation and extension to complex problems, that may involve friction or contact between various elastic bodies, for instance.

In this section we present briefly some alternative mixed methods that have been proposed in the last years. We focus on their formulation, the main ideas and results, without going much into the details of the proofs as we did for the previous mixed discretization. For more insight into these methods, the interested reader may refer to the associated bibliographical references.

Notably, as it has been mentioned already, one important point in the design of mixed discretization methods consists in ensuring they satisfy a uniform discrete inf-sup condition. For this purpose, one needs to design carefully the bulk and boundary finite element spaces. As a result, we start by presenting in Sect. 7.4.1 a stabilized mixed method that allows to circumvent this difficulty. The idea is to add some extra terms to the discrete mixed form that enforce stability whatever the choice of the underlying pair of spaces is. We already mentioned this formulation in Chap. 6 because it is closely related to Nitsche's method.

In a second part, in Sect. 7.4.2, we present some mixed discretizations that are helpful for the implementation of contact between two elastic bodies with nonmatching meshes. We present them in the simplified situation of Signorini problem, but we will see in Chap. 10 how to adapt them for multi-body contact.

Conversely to the previous mixed method, we will account here for problems either in two dimensions ($d = 2$) or in three dimensions ($d = 3$).

7.4.1 A Stabilized Mixed Method

Let us present first the stabilized mixed technique that has been studied in [169]. Let $\mathbf{V}^h$ be the Lagrange finite element space of order $k \geq 1$ to discretize the displacement:

$$\mathbf{V}^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h|_T \in \mathbb{P}_k(T; \mathbb{R}^d),\ \forall T \in \mathcal{T}^h, \quad \mathbf{v}^h|_{\Gamma_D} = 0 \right\}$$

where $\mathbf{X}_k^h$ has been introduced in Chap. 4. This time, we define the space W^H as a space of functions which are piecewise polynomials of degree $l \geq 0$ on any boundary mesh on Γ_C, and continuous when $l \geq 1$. Note that this boundary mesh does not need to be the trace mesh of $\mathcal{T}^h$, and $H > 0$ stands for the boundary mesh size. Let $M^H \subset W^H$ be any convex cone that mimics Λ_C and that incorporates the nonpositivity condition on the multiplier (various choices are provided in [169]). Let us define a boundary function $\zeta_{BT} : \Gamma_C \to \mathbb{R}$, as follows: for any simplex T that intersects the contact boundary, we set

$$\zeta_{BH}|_{\partial T \cap \Gamma_C} := \zeta_0 h_T,$$

where $\zeta_0 > 0$ is a given (small) stabilization parameter and h_T is the diameter of T. The stabilized mixed method reads as follows:

$$\begin{cases} \text{Find } (\mathbf{u}^h, \lambda^H) \in \mathbf{V}^h \times M^H \text{ such that:} \\[2mm] a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \lambda^H) + \displaystyle\int_{\Gamma_C} \zeta_{BH}(\lambda^H - \sigma_{\mathbf{n}}(\mathbf{u}^h))\, \sigma_{\mathbf{n}}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\[2mm] b(\mathbf{u}^h, \mu^H - \lambda^H) + \displaystyle\int_{\Gamma_C} \zeta_{BH}(\mu^H - \lambda^H)(\lambda^H - \sigma_{\mathbf{n}}(\mathbf{u}^h)) \geq 0, \quad \forall\, \mu^H \in M^H. \end{cases} \tag{7.25}$$

The above method still approximates the mixed form of the Signorini problem. Yet, and conversely to the previous one (7.8), some (small) terms are added to recover stability and optimal accuracy whatever the choice of the finite element pair $\mathbf{V}^h/W^H$ is. This pair does not need to satisfy a discrete inf-sup condition anymore. Of course, if a uniform inf-sup condition holds for $\mathbf{V}^h/W^H$, the stabilization terms are not necessary anymore, and we can even take $\zeta_0 = 0$. In this situation, the extra terms vanish and we recover a standard mixed formulation such as (7.8).

This method extends the stabilization techniques originally suggested by H. Barbosa and T.J.R. Hughes in a series of papers in the early nineties, see [24] and [23, 25]. It can be obtained as the saddle-point of the following modified Lagrangian:

$$\mathscr{L}_\zeta(\mathbf{v}^h, \lambda^H) = \mathcal{J}(\mathbf{v}^h) - b(\mathbf{v}^h, \lambda^H) - I_{M^H}(\lambda^H) - \frac{1}{2}\int_{\Gamma_C} \zeta_{BH} \left(\lambda^H - \sigma_{\mathbf{n}}(\mathbf{v}^h) \right)^2,$$

where is $I_{M^H}(\cdot)$ is the indicator function of M^H [169].

The result below has been first proven in [169].

Proposition 7.7 *Suppose that $\zeta_0 > 0$ is small enough, then the stabilized mixed method (7.25) is well-posed: it admits one unique solution $(\mathbf{u}^h, \lambda^H) \in \mathbf{V}^h \times M^H$.*

Proof First, as in Theorem 6.1, we use the ellipticity of $a(\cdot,\cdot)$ (Proposition 3.3) and the discrete trace inequality of Lemma 4.1 in Chap. 4 to bound:

$$a(\mathbf{v}^h, \mathbf{v}^h) - \int_{\Gamma_C} \zeta_{BH}(\sigma_{\mathbf{n}}(\mathbf{v}^h))^2 \geq c\|\mathbf{v}^h\|^2_{1,\Omega}, \tag{7.26}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$ and for ζ_0 small enough. Above the constant $c > 0$ is then independent of h, of ζ_0, and of $\mathbf{v}^h$.

Then, from the relationship (7.26), we deduce that, for all $\mu^H \in M^H$, $\mathscr{L}_\zeta(\cdot, \mu^H)$ is strictly convex on $\mathbf{V}^h$ and that

$$\lim_{\|\mathbf{v}^h\|_{1,\Omega} \to +\infty} \mathscr{L}_\zeta(\mathbf{v}^h, \mu^H) = +\infty.$$

Moreover, the sets $\mathbf{V}^h$ and M^H are non-empty closed convex sets and the application $\mathscr{L}_\zeta(\cdot,\cdot)$ is continuous on $\mathbf{V}^h \times W^H$. We check also that, for all $\mathbf{v}^h \in \mathbf{V}^h$, $\mathscr{L}_\zeta(\mathbf{v}^h, \cdot)$ is strictly concave on M^H and that

$$\lim_{\|\mu^H\|_{0,\Gamma_C} \to +\infty} \mathscr{L}_\zeta(\mathbf{v}^h, \mu^H) = -\infty.$$

From the above, existence and uniqueness of a saddle-point to $\mathscr{L}_\zeta(\cdot,\cdot)$ comes from, e.g., [153, Theorem 3.7,Theorem 3.9]. □

The convergence analysis of the stabilized mixed method (7.25) has been carried out in [169], for $d = 2$, $k = 1$, $l = 0, 1$ and various choices of M^H. From this reference and the results of Chap. 5 we can deduce the result below.

Theorem 7.2 *We set $d = 2$, $k = 1$, $l = 0, 1$ and consider the choices of [169] for M^H. Let $(\mathbf{u}, \lambda = \sigma_{\mathbf{n}})$ and $(\mathbf{u}^h, \lambda^h)$ be the solutions to the continuous Problem (7.5) and to the stabilized mixed discrete Problem (7.25), respectively. Assume that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$ with $3/2 < s \leq 2$, and that $\zeta_0 > 0$ is small enough. Then, there exists a constant $C > 0$ independent of h and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{-\frac{1}{2},\Gamma_C} \leq Ch^{s-1}\|\mathbf{u}\|_{s,\Omega}.$$

Proof The proof follows the same lines as in [169], except that we use the results of Chap. 5 to obtain optimal bounds on the contact terms. □

As we have seen in the previous Chap. 6, the stabilized mixed formulation (7.25) is closely related to Nitsche formulation, and the stabilization parameter ζ_0 acts identically as the inverse of Nitsche parameter γ_0.

7.4.2 Mortar and LAC Methods

In many situations, contact problems involve at least two elastic bodies that come into contact. For practical purposes or when complex geometries are considered, the bodies are independently meshed so that the contact elements and contact nodes do not fit together on the contact zone. The usual terminology to refer to this configuration is "nonmatching meshes". Remark that, in this situation, the first mixed discretization presented in the previous section is not straightforward to extend. As a result, it is useful to design some mixed methods that can be adapted rather easily to the case of contact between two bodies with, possibly, nonmatching meshes. We present two of these methods here: the Mortar method and the LAC method. The simple setting of the Signorini problem allows to understand their main ideas, though, of course, the real interest of these methods concerns situations where two elastic bodies are in contact (and not only one elastic body with a perfectly rigid support). This mathematical model for multi-body contact will be presented in detail in Chap. 10, and then we will describe how to extend the methods below to this more involved setting.

Still we consider in this part the two-dimensional ($d = 2$) and the three-dimensional ($d = 3$) cases, and take $\mathbf{V}^h$ the Lagrange finite element space of order $k \geq 1$ to discretize the displacement:

$$\mathbf{V}^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h|_T \in \mathbb{P}_k(T; \mathbb{R}^d),\ \forall T \in \mathcal{T}^h, \quad \mathbf{v}^h|_{\Gamma_D} = 0 \right\},$$

where $\mathbf{X}_k^h$ has been introduced in Chap. 4.

We extend to this more general setting the previous definition in Sect. 7.2 for the space W^h of continuous functions which are piecewise of degree k on the trace mesh on Γ_C:

$$W^h := \left\{ \psi^h \in \mathscr{C}(\overline{\Gamma_C}; \mathbb{R}^{d-1}) \;\middle|\; \exists \mathbf{v}^h \in \mathbf{V}^h \text{ such that } \mathbf{v}^h \cdot \mathbf{n} = \psi^h \text{ on } \Gamma_C \right\}.$$

Remark that it is the range of $\mathbf{V}^h$ by the normal component trace operator on Γ_C. As in Chap. 4, we denote by π^h is the $L^2(\Gamma_C)$-projection operator on the space W^h (see the definition (4.10)).

7.4.2.1 Formulations

Let us describe first the Mortar method [33–35]. The discrete set of admissible displacements satisfying the mortar contact conditions on the contact zone is given by

$$\mathbf{K}_{\text{Mortar}}^h := \left\{ \mathbf{v}^h \in \mathbf{V}^h \;\middle|\; \pi^h v_{\mathbf{n}}^h \leq 0 \text{ on } \Gamma_C \right\}. \tag{7.27}$$

Of course, in the case of only one meshed body, the projection operator reduces to the identity operator and it can be removed. But in the interesting case of (e.g., two) nonmatching meshes on the contact area, the field $v_{\mathbf{n}}^h$ has to be changed with the relative normal displacement on the common contact area and the projection operator produces a (new) relative normal displacement which is of finite element type on one of both meshes. Note that this treatment then involves a nonlocal operator on the whole contact interface.

As a remedy to this issue, and for easier implementation into industrial codes, the Local Average Contact (LAC) method has been designed later on [1, 111]. For the LAC method, the discrete set of admissible displacements satisfying the average non-interpenetration conditions on the contact zone is given by

$$\mathbf{K}_{\text{LAC}}^h := \left\{ \mathbf{v}^h \in \mathbf{V}^h \,\middle|\, \int_{T^m \cap \Gamma_C} v_{\mathbf{n}}^h \le 0, \;\; \forall T^m \in \mathcal{T}^M \right\}. \tag{7.28}$$

When $d = 2$ and $k = 1$, $\mathcal{T}^M$ is a one-dimensional macro-mesh constituted by macro-segments T^m comprising two adjacent segments of the trace mesh on Γ_C. When $d = 2$ and $k = 2$, $\mathcal{T}^M$ is simply the trace mesh on Γ_C inherited by one of the meshes coming from a body. The only requirement (when $k = 1, 2$ and $d = 2, 3$) is that any element of $\mathcal{T}^M$ admits an internal degree of freedom (of course when $d = 3$, $\mathcal{T}^M$ is a two-dimensional trace mesh on Γ_C) which we call hereafter the internal degree of freedom hypothesis.

The discrete variational inequality issued from (3.12) is

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ satisfying:} \\ a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \ge L(\mathbf{v}^h - \mathbf{u}^h), \;\; \forall\, \mathbf{v}^h \in \mathbf{K}^h, \end{cases} \tag{7.29}$$

where either $\mathbf{K}^h = \mathbf{K}_{\text{LAC}}^h$ or $\mathbf{K}^h = \mathbf{K}_{\text{Mortar}}^h$. According to Stampacchia's Theorem 3.4, Problem (7.29) admits a unique solution.

Now, we provide mixed formulations for both LAC and Mortar approaches. In the LAC method, we choose piecewise constant nonpositive Lagrange multipliers on the macro-mesh $\mathcal{T}^M$ on Γ_C, i.e., in the convex cone M_{LAC}^h:

$$M_{\text{LAC}}^h := \left\{ \mu^h \in W_{\text{LAC}}^h \,\middle|\, \mu^h \le 0 \;\text{ on }\; \Gamma_C \right\}$$

where

$$W_{\text{LAC}}^h = \left\{ \mu^h \in L^2(\Gamma_C) \,\middle|\, \mu^h|_{T^m} \in \mathbb{P}_0(T^m), \forall T^m \in \mathcal{T}^M \right\}.$$

For the Mortar method we choose continuous piecewise of degree k multipliers defined on the trace mesh of Γ_C. These multipliers are weakly nonpositive in the integral sense:

$$M^h_{\text{Mortar}} = \left\{ \mu^h \in W^h \;\middle|\; \int_{\Gamma_C} \mu^h \psi^h \le 0,\ \forall \psi^h \in W^h,\ \psi^h \ge 0 \right\}.$$

The mixed formulations for both LAC and Mortar method are as follows:

Proposition 7.8 *Let us introduce the mixed problem: find* $\mathbf{u}^h \in \mathbf{V}^h$ *and* $\lambda^h \in M^h$ *such that*

$$\begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \lambda^h) = L(\mathbf{v}^h),\ \forall \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mathbf{u}^h, \mu^h - \lambda^h) \ge 0,\ \forall \mu^h \in M^h, \end{cases} \tag{7.30}$$

where $M^h = M^h_{\text{LAC}}$ *or* $M^h = M^h_{\text{Mortar}}$. *The Problem (7.29) and the Problem* (7.30) *are both well-posed and equivalent, i.e., the solution* $\mathbf{u}^h$ *of (7.29) coincides with the first component of the solution of (7.30).*

Proof For the LAC method, see [111]. For the mortar method, the general result, for $k = 1, 2$ and $d = 2, 3$, is obtained as in [166] where the case $k = d = 2$ is treated. □

7.4.2.2 The inf-sup Condition

For the LAC method, the internal degree of freedom hypothesis allows to prove the previous proposition and it ensures that the corresponding mixed method using piecewise constant Lagrange multipliers on the macro-mesh T^M verifies the inf-sup condition: there is a constant β^h such that

$$\inf_{\mu^h \in W^h_{\text{LAC}}} \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, \mu^h)}{\|\mu^h\|_{\frac{1}{2},\Gamma_C} \, \|\mathbf{v}^h\|_{1,\Omega}} \ge \beta^h > 0.$$

In the case of the mortar method, we have to change W^h_{LAC} with W^h and the previous inf-sup condition is straightforward.

Moreover, it is well known that the inf-sup constant β^h also arises in the error analysis of the mixed formulation (7.30). In order to get the best convergence rate we need to prove that β^h is independent of the mesh size h. In the case of the LAC method, the situation is simple: for any $k = 1, 2$ and $d = 2, 3$ there is a constant β such that:

$$\inf_{\mu^h \in W^h_{\text{LAC}}} \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, \mu^h)}{\|\mu^h\|_{\frac{1}{2},\Gamma_C} \, \|\mathbf{v}^h\|_{1,\Omega}} \ge \beta > 0, \tag{7.31}$$

see the proof in [111]. For the mortar method, the inf-sup condition is true when $d = 2$ (see, e.g., [95, 166]).

7.4.2.3 Optimal Error Estimates

The following theorem shows that we can obtain the same convergence rates for the solution to the mixed problem (7.30) than those stated in Chap. 5 for the discrete variational inequalities.

Theorem 7.3 *Consider the LAC method with* $d = 2, 3$ *and* $k = 1, 2$ *or the Mortar method with* $d = 2$ *and* $k = 1, 2$. *Let* $(\mathbf{u}, \lambda = \sigma_{\mathbf{n}}) \in \mathbf{V} \times \Lambda_C$ *and* $(\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times M^h$ *be the solutions to the continuous Problem (7.5) and to the discrete Problem (7.30), respectively. Assume that* $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$ *with* $3/2 < s \leq \min(k + 1, 5/2)$. *Then, there exists a constant* $C > 0$ *independent of* h *and* $\mathbf{u}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\lambda - \lambda^h\|_{\frac{1}{2},\Gamma_C} \leq C h^{s-1} \|\mathbf{u}\|_{s,\Omega}.$$

Proof The proof is straightforward and standard since the inf-sup condition (7.31) is verified. It only remains to bound similar terms to the ones present in the proofs of Chap. 5. □

These results could be extended to the three-dimensional case using the technicalities developed in Chap. 5, Sect. 5.4, for the three-dimensional case, and the inf-sup condition (7.31) which is valid in the three-dimensional case.

7.5 Proximal Augmented Lagrangian

For the practical numerical resolution, one may want to remove completely the inequality constraints, still present in mixed formulations such as Problem (7.5). This is one of the interests of Augmented Lagrangian formulations. Originally, the Augmented Lagrangian has been thought mostly as a numerical procedure to solve a problem in mixed form, but it can alternatively be considered as a numerical approximation technique in itself, as it has been illustrated recently in a series of papers (see, e.g., [63] and references therein).

7.5.1 Obtention of an Augmented Lagrangian

To derive an Augmented Lagrangian, we get back to the formula (7.1), which is our starting point. Then, let us observe that we can add any convex term in w provided that it vanishes when $w = 0$. Following, for instance [236], we can add an extra term that involves a norm in w:

$$\mathcal{W}_\gamma(\mathbf{v}, w) := \mathcal{W}(\mathbf{v}, w) + \frac{\gamma}{2}\|w\|^2_{0,\Gamma_C}, \tag{7.32}$$

where $\gamma > 0$ is an augmentation parameter. Remark that, since we make use of a norm in $L^2(\Gamma_C)$, this implicitly restricts the definition of the functional and we need to suppose that the multiplier is also in $L^2(\Gamma_C)$. We will make this assumption thereafter. By the way, it allows us to use a boundary integral instead of duality pairings on the boundary. The expression of the Augmented Lagrangian is provided in the lemma below, in the same way we did it before for the Lagrangian in Lemma 7.1

Lemma 7.2 *The Augmented Lagrangian obtained by application of the Fenchel-Legendre conjugate to the functional $\mathcal{W}_\gamma$, defined in (7.32), has the following expression:*

$$\mathscr{L}_\gamma(\mathbf{v}, \lambda) = \mathcal{J}(\mathbf{v}) + \int_{\Gamma_C} \frac{1}{2\gamma}\left([\gamma v_{\mathbf{n}} - \lambda]_+^2 - \lambda^2\right), \tag{7.33}$$

for $\mathbf{v} \in \mathbf{V}$ and $\lambda \in L^2(\Gamma_C)$, and where $[\cdot]_+$ is the positive part operator, defined in 6.1.1.

Proof Using once again the Fenchel-Legendre conjugate with respect to $w \in L^2(\Gamma_C)$, we get the Augmented Lagrangian

$$\mathscr{L}_\gamma(\mathbf{v}, \lambda) := -\sup_{w \in L^2(\Gamma_C)} \left(\int_{\Gamma_C} \lambda w - W_\gamma(\mathbf{v}, w)\right),$$

for $\mathbf{v} \in \mathbf{V}$ and $\lambda \in L^2(\Gamma_C)$. The above expression can be transformed as follows:

$$\begin{aligned}\mathscr{L}_\gamma(\mathbf{v}, \lambda) &= -\sup_{w \in L^2(\Gamma_C)} \left(\int_{\Gamma_C} \lambda w - \mathcal{J}(\mathbf{v}) - I_-(v_{\mathbf{n}} - w) - \frac{\gamma}{2}\|w\|^2_{0,\Gamma_C}\right)\\ &= \mathcal{J}(\mathbf{v}) - \sup_{w \in L^2(\Gamma_C)} \left(\int_{\Gamma_C} \lambda w - I_-(v_{\mathbf{n}} - w) - \frac{\gamma}{2}\|w\|^2_{0,\Gamma_C}\right).\end{aligned}$$

Then, we need to compute separately:

$$\begin{aligned}&\sup_{w \in L^2(\Gamma_C)} \left(\int_{\Gamma_C} \lambda w - I_-(v_{\mathbf{n}} - w) - \frac{\gamma}{2}\|w\|^2_{0,\Gamma_C}\right)\\ &= \sup_{\zeta \in L^2(\Gamma_C)} \left(\int_{\Gamma_C} \lambda(v_{\mathbf{n}} - \zeta) - I_-(\zeta) - \frac{\gamma}{2}\|v_{\mathbf{n}} - \zeta\|^2_{0,\Gamma_C}\right)\\ &= \int_{\Gamma_C} \lambda v_{\mathbf{n}} - \frac{\gamma}{2}\|v_{\mathbf{n}}\|^2_{0,\Gamma_C} + \sup_{\zeta \in L^2(\Gamma_C)} \left(-\int_{\Gamma_C} \lambda\zeta - I_-(\zeta) + \gamma\int_{\Gamma_C} v_{\mathbf{n}}\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C}\right)\end{aligned}$$

with the change of variable $\zeta = v_{\mathbf{n}} - w$ ($w = v_{\mathbf{n}} - \zeta$). If we remember the definition of the indicator I_-, we get:

$$\sup_{\zeta \in L^2(\Gamma_C)} \left(-\int_{\Gamma_C} \lambda\zeta - I_-(\zeta) + \gamma \int_{\Gamma_C} v_{\mathbf{n}}\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C} \right)$$
$$= \sup_{\zeta \le 0} \left(-\int_{\Gamma_C} \lambda\zeta + \gamma \int_{\Gamma_C} v_{\mathbf{n}}\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C} \right).$$

To alleviate the notations, now we set $P := \gamma v_{\mathbf{n}} - \lambda \in L^2(\Gamma_C)$. There remains to compute:

$$\sup_{\zeta \le 0} \left(\int_{\Gamma_C} P\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C} \right) = \sup_{\zeta \le 0} \left(\int_{\Gamma_C} P_+\zeta - \int_{\Gamma_C} P_-\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C} \right)$$

where we used the decomposition $P = P_+ - P_-$ into a positive part $P_+ = \max(0, P)$ and a negative part $P_- = \max(0, -P)$. To compute the upper bound, we use the decomposition of ζ into an orthogonal sum (in the sense of the L^2-scalar product):

$$\zeta = \alpha^+ P_+ + \alpha^- P_- + \zeta^\perp,$$

where $\alpha^+ \in \mathbb{R}$, $\alpha^- \in \mathbb{R}$, and where $\zeta^\perp$ is orthogonal to both P_+ and P_-. Using orthogonality properties, we compute

$$\int_{\Gamma_C} P_+\zeta - \int_{\Gamma_C} P_-\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C}$$
$$= \alpha^+\|P_+\|^2_{0,\Gamma_C} - \alpha^-\|P_-\|^2_{0,\Gamma_C} - \frac{\gamma}{2}\left((\alpha^+)^2\|P_+\|^2_{0,\Gamma_C} (\alpha^-)^2\|P_-\|^2_{0,\Gamma_C} + \|\zeta^\perp\|^2_{0,\Gamma_C} \right).$$

From the above expression we see that the maximizer of this expression lies necessarily in the region where $\zeta^\perp = 0$. Taking this into account, we need to find $\zeta = \alpha^+ P_+ + \alpha^- P_- \le 0$ that maximizes

$$\left(\alpha^+ - \frac{\gamma}{2}(\alpha^+)^2 \right) \|P_+\|^2_{0,\Gamma_C} - \left(\alpha^- + \frac{\gamma}{2}(\alpha^-)^2 \right) \|P_-\|^2_{0,\Gamma_C}.$$

The maximum of each term is obtained when $\alpha^+ = 1/\gamma > 0$ and $\alpha^- = -1/\gamma < 0$. Moreover, since we need to satisfy $\zeta \le 0$, this implies $\alpha^+ \le 0$ and $\alpha^- \le 0$. As a consequence, we get:

$$\sup_{\zeta \le 0} \int_{\Gamma_C} P\zeta - \frac{\gamma}{2}\|\zeta\|^2_{0,\Gamma_C} = \frac{1}{2\gamma}\|P_-\|^2_{0,\Gamma_C} = \frac{1}{2\gamma}\|[\lambda - \gamma v_{\mathbf{n}}]_+\|^2_{0,\Gamma_C}.$$

As a result of the previous calculations, the Augmented Lagrangian is:

$$\mathscr{L}_\gamma(\mathbf{u}, \lambda) = \mathcal{J}(\mathbf{u}) - \int_{\Gamma_C} \lambda u_{\mathbf{n}} - \int_{\Gamma_C} \frac{1}{2\gamma}\left[\lambda - \gamma u_{\mathbf{n}}\right]^2_+ + \int_{\Gamma_C} \frac{\gamma}{2} u^2_{\mathbf{n}}. \tag{7.34}$$

It can also be reformulated as

$$
\begin{aligned}
\mathscr{L}_\gamma(\mathbf{u},\lambda) &= \mathcal{J}(\mathbf{u}) - \int_{\Gamma_C} \lambda u_{\mathbf{n}} - \int_{\Gamma_C} \frac{1}{2\gamma}[\lambda - \gamma u_{\mathbf{n}}]_+^2 + \int_{\Gamma_C} \frac{\gamma}{2} u_{\mathbf{n}}^2 \\
&\quad + \int_{\Gamma_C} \frac{1}{2\gamma}\lambda^2 - \int_{\Gamma_C} \frac{1}{2\gamma}\lambda^2 \\
&= \mathcal{J}(\mathbf{u}) - \int_{\Gamma_C} \frac{1}{2\gamma}[\lambda - \gamma u_{\mathbf{n}}]_+^2 + \int_{\Gamma_C} \frac{1}{2\gamma}(\lambda - \gamma u_{\mathbf{n}})^2 - \int_{\Gamma_C} \frac{1}{2\gamma}\lambda^2 \\
&= \mathcal{J}(\mathbf{u}) + \int_{\Gamma_C} \frac{1}{2\gamma}[\gamma u_{\mathbf{n}} - \lambda]_+^2 - \int_{\Gamma_C} \frac{1}{2\gamma}\lambda^2
\end{aligned}
\tag{7.35}
$$

where we used the relationship $a^2 - a_+^2 = (-a)_+^2$. □

Remark 7.9 In fact, the reason why the Proximal Augmented Lagrangian allows to obtain an optimality condition without inequality constraints is related to Moreau-Yosida regularization [83].

7.5.2 An Augmented Mixed Method

In the same way as we did for the Lagrangian, we can derive an augmented mixed formulation by writing the first order optimality condition associated with the saddle-point of the Augmented Lagrangian $\mathscr{L}_\gamma(\cdot,\cdot)$ obtained previously in Lemma 7.2.

Proposition 7.9 *Any pair* $(\mathbf{u},\lambda) \in \mathbf{V} \times L^2(\Gamma_C)$ *that is a saddle-point of* $\mathscr{L}_\gamma(\cdot,\cdot)$*, defined in* (7.33)*, is solution to the following mixed problem:*

$$
\begin{cases}
\text{Find } (\mathbf{u},\lambda) \in \mathbf{V} \times L^2(\Gamma_C) \text{ such that} \\
a(\mathbf{u},\mathbf{v}) - \displaystyle\int_{\Gamma_C} [\gamma u_{\mathbf{n}} - \lambda]_+ v_{\mathbf{n}} = L(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \\
-\displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\left([\gamma u_{\mathbf{n}} - \lambda]_+ + \lambda\right)\mu = 0 \qquad \forall \mu \in L^2(\Gamma_C).
\end{cases}
\tag{7.36}
$$

Proof The saddle-point of the Augmented Lagrangian is solution to:

$$
0 = \mathscr{L}_{\gamma,\mathbf{v}}{}'(\mathbf{u},\lambda;\mathbf{v}) = a(\mathbf{u},\mathbf{v}) - L(\mathbf{v}) - \int_{\Gamma_C} [\gamma u_{\mathbf{n}} - \lambda]_+ v_{\mathbf{n}}, \quad \forall \mathbf{v} \in \mathbf{V},
$$

$$
0 = \mathscr{L}_{\gamma,\lambda}{}'(\mathbf{u},\lambda;\mu) = \int_{\Gamma_C} \frac{1}{\gamma}\left(-[\gamma u_{\mathbf{n}} - \lambda]_+ - \lambda\right)\mu, \quad \forall \mu \in L^2(\Gamma_C).
$$

This is (7.36). □

Note that some recent works are filling the gap between Nitsche and Augmented Lagrangian formulations in the case of contact and obstacle problems [62, 64, 150]. We will follow them and describe below another way to recover this Augmented Lagrangian formulation from Nitsche's method.

7.5.3 From Nitsche to Augmented Lagrangian Formulations

Following [59, 61, 62, 64, 150] we show here how to recover some Augmented Lagrangian formulations by starting from the incomplete Nitsche method presented in Chap. 6. Remember that in Chap. 6, we presented first the incomplete version of Nitsche, which was the weak form (6.5), and which corresponds to $\theta = 0$:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} [\gamma_N u^h_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u}^h)]_+ v^h_{\mathbf{n}} = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

Above the notation γ_N stands for a positive function related to the Nitsche parameter γ_0. Observe that in the above weak form, the contact conditions (6.4) are imposed implicitly thanks to the integral term on the boundary. Another way to do this is to introduce a Lagrange multiplier $\lambda (= \sigma_{\mathbf{n}}(\mathbf{u}))$ and to rewrite the Signorini contact conditions (6.4) as:

$$\lambda = -\left[\gamma_L u_{\mathbf{n}} - \lambda\right]_+, \tag{7.37}$$

for γ_L a positive function. In practice, we will do the same as for Nitsche method (and as for the stabilized mixed method), and define explicitly $\gamma_L : \Gamma_C \to \mathbb{R}$, as follows: for any simplex T that intersects the contact boundary, we set

$$\gamma_L|_{\partial T \cap \Gamma_C} := \frac{\gamma_0}{h_T},$$

where $\gamma_0 > 0$ is the augmentation parameter and h_T is the diameter of T. As for the stabilized mixed method, we use a given discrete space W^H, which is a space of functions that are piecewise polynomials of degree $l \geq 0$ on any boundary mesh on Γ_C, and continuous when $l \geq 1$. Once again, this boundary mesh does not need to be the trace mesh of $\mathcal{T}^h$, and $H > 0$ stands for the boundary mesh size. We take a discrete Lagrange multiplier $\lambda^H \in W^H$, we make use of the Green formula in elasticity (Lemma 3.1) and substitute λ^H to the normal stress:

$$a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \lambda^H v^h_{\mathbf{n}} = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h,$$

to enforce weakly the condition $\lambda = \sigma_{\mathbf{n}}(\mathbf{u})$, which means that λ stands indeed for the contact pressure. Concerning the contact condition (7.37), we impose it weakly as another variational equation on the Lagrange multiplier:

$$\int_{\Gamma_C} \frac{1}{\gamma_L} \left(\left[\gamma_L u^h_{\mathbf{n}} - \lambda^H\right]_+ + \lambda^H \right) \mu^H = 0, \quad \forall \mu^H \in W^H.$$

With the two above equations, we obtain a first Augmented Lagrangian approximation to the Signorini problem, as:

$$\begin{cases} \text{Find } (\mathbf{u}^h, \lambda^H) \in \mathbf{V}^h \times W^H \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) - \displaystyle\int_{\Gamma_C} \lambda^H v_{\mathbf{n}}^h = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ -\displaystyle\int_{\Gamma_C} \frac{1}{\gamma_L} \left(\left[\gamma_L u_{\mathbf{n}}^h - \lambda^H \right]_+ + \lambda^H \right) \mu^h = 0, \quad \forall \mu^H \in W^H. \end{cases} \tag{7.38}$$

Alternatively, we can use also the contact condition (7.37) to reformulate the first equation of (7.38) above:

$$a(\mathbf{u}^h, \mathbf{v}^h) + \int_{\Gamma_C} [\gamma_L u_{\mathbf{n}}^h - \lambda^H]_+ v_{\mathbf{n}}^h = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

This leads to another Augmented Lagrangian approximation to the Signorini problem:

$$\begin{cases} \text{Find } (\mathbf{u}^h, \lambda^H) \in \mathbf{V}^h \times W^H \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \left[\gamma_L u_{\mathbf{n}}^h - \lambda^H \right]_+ v_{\mathbf{n}}^h = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ -\displaystyle\int_{\Gamma_C} \frac{1}{\gamma_L} \left(\left[\gamma_L u_{\mathbf{n}}^h - \lambda^H \right]_+ + \lambda^H \right) \mu^H = 0, \quad \forall \mu^H \in W^H. \end{cases} \tag{7.39}$$

Observe that we recover here a discrete counterpart of the optimality system (7.33).

Discrete Problems (7.38) and (7.39) can be shown to be well-posed provided an appropriate choice of the augmentation parameter γ_0 and that $\mathbf{V}^h/W^H$ is a compatible pair, that satisfies an inf-sup condition (alternatively, a stabilization term can be added). Moreover, these formulations are optimally convergent in natural norms. Interestingly, the technique to obtain optimal *a priori* error bounds is close to the technique presented in Chap. 6 for Nitsche's method. The reader can refer for more details to [59, 62, 64, 150].

7.6 Implementation

We provide here some information about how mixed and Augmented Lagrangian methods can be solved in practice. There are various possibilities to implement and solve them.

7.6.1 Semi-Smooth Newton for the Augmented Lagrangian

A widespread technique is to go first from mixed formulations to Augmented Lagrangian formulations, in order to remove the constraint inequalities on the multipliers. As a counterpart, this introduces some nonlinear and nonsmooth terms in the variational formulations, as it was the case for Nitsche's method in the previous chapter. Then, in the same way we did for Nitsche's method, a semi-smooth Newton algorithm can be used to deal with the nonlinear terms.

As for the Nitsche method, in libraries such as GetFEM [234], scikit-fem [271] or FEniCS [8], programming Augmented Lagrangian formulations (7.38) or (7.39) consists mostly in rewriting the corresponding weak forms in the appropriate syntax.

For other implementation, one needs the tangent weak form. For instance, for formulation (7.39), using (6.3) we can write it explicitly as:

$$\begin{cases} \text{Find } (\delta\mathbf{u}^h, \delta\lambda^H) \in \mathbf{V}^h \times W^H \text{ such that:} \\ a(\delta\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} H_s\left(\gamma_L u^h_{\mathbf{n}} - \lambda^H\right) \delta u^h_{\mathbf{n}} v^h_{\mathbf{n}} = -R(\mathbf{u}^h, \lambda^H; \mathbf{v}^h, 0), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_L}\left(H_s\left(\gamma_L u^h_{\mathbf{n}} - \lambda^H\right)\delta\lambda^H - \delta\lambda^H\right)\mu^H = -R(\mathbf{u}^h, \lambda^H; 0, \mu^H), \quad \forall \mu^H \in W^H \end{cases} \tag{7.40}$$

where $R(\cdot;\cdot)$ is the residual associated with the approximation $(\mathbf{u}^h, \lambda^H)$.

From this system also, standard quadrature rules can be applied, despite the nonsmooth terms coming from the Heaviside function and the positive part operator (other strategies can be also implemented, to gain in accuracy). Using a standard assembly procedure and any method to solve the corresponding linear system, one finds the solution.

7.6.2 Uzawa's Algorithm

Solving Problem (7.39) with a semi-smooth Newton method means that in fact we compute simultaneously the two unknowns. the displacement and the Lagrange multiplier. A popular alternative implemented in many software for structural mechanics is based on Uzawa's method: see, for instance, [247] for contact problems. The interested reader can refer also to [43, 188, 189] for a general presentation of Uzawa's algorithm for an (Augmented) Lagrangian. In our case we can write the algorithm as follows:

0) Define an initial value λ_0^h for the multiplier and a step $r > 0$ for each iteration.
1) Solve the first equation of (7.39) with only the displacement as an unknown, and for the multiplier set $\lambda^{h,k}$:

$$\begin{cases} \text{Find } \mathbf{u}^{h,k} \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}^{h,k}, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \left[\gamma_L u_{\mathbf{n}}^{h,k} - \lambda^{h,k}\right]_+ v_{\mathbf{n}}^h = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

We obtain a displacement $\mathbf{u}^{h,k}$.
2) Update $\lambda^{h,k+1}$ with:

$$\lambda^{h,k+1} = \lambda^{h,k} - \frac{r}{\gamma_L}\left(\left[\gamma_L u_{\mathbf{n}}^{h,k} - \lambda^{h,k}\right]_+ + \lambda^{h,k}\right).$$

3) Go to Step 1 with $k \leftarrow k+1$ until a convergence criterion is satisfied.

An interesting property proven in [248] is that this Uzawa algorithm for the Augmented Lagrangian is always convergent for $\gamma_L = r$ and this convergence is as fast as $\gamma_L = r$ is large (this property is not satisfied by the Uzawa algorithm for the basic Lagrangian presented in Sect. 7.1.1, for which the convergence is only guaranteed for r small and is generally very slow).

7.6.3 A Penalty Formulation from Uzawa

Remark finally that, if we choose zero as the initial value of the multiplier at the first step of Uzawa's method, we recover a penalty formulation:

$$\begin{cases} \text{Find } \mathbf{u}^{h} \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}^{h}, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \left[\gamma_P u_{\mathbf{n}}^{h}\right]_+ v_{\mathbf{n}}^h = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

Here we changed the notation γ_L and replaced it by γ_P, which is still a positive function, that depends on the local mesh size and of a penalty parameter $\varepsilon > 0$, that is assumed small. The function γ_P usually scales as $1/\varepsilon$. This formulation has been very popular, and is described, for instance, in the book of N. Kikuchi and J.T. Oden [181] (see also [78, 108] and references therein). We provide in Chap. 8 results of well-posedness and optimal convergence of this method in case of Tresca frictional contact.

7.7 Further Comments

Mixed and Augmented Lagrangian formulations have a long history related to the numerical solution of contact problems. We provide below a few extra information and some related references.

7.7.1 About Mixed Methods

The duality theory has a long history in itself, related to minimization problems under constraints. Classical references on this topic are, for instance, the monographs of I. Ekeland and R. Temam [117] or of R.T. Rockafellar and R. J. B. Wets [236]. Its application in the context of the finite element method is from the seventies, see, for instance, the seminal papers of I. Babuska [18], F. Brezzi [53] or the monograph of F. Brezzi and M. Fortin [54]. For contact problems, among the first papers are those of J. Haslinger [151] and of F. Brezzi, W.W. Hager and P.A. Raviart [56].

7.7.2 Discrete inf-sup Conditions

Proving an inf-sup, or a LBB (Ladyzhenskaya-Babuska-Brezzi) compatibility condition between two discrete spaces has been a serious matter in the numerical analysis of mixed methods, see, for instance, the monographs of F. Brezzi and M. Fortin [54], of G. Gatica [135] or A. Ern and J.L. Guermond [118], and references therein.

For the Signorini problem, despite of some achievements, the panorama is not complete today. Let us point out that, in complement to the specific choices of spaces for the displacement and the multiplier, the results depend notably on:

- The dimension d of the Signorini problem
- The boundary conditions within the (external) neighborhood of the contact boundary Γ_C, and the relative position of the Dirichlet boundary Γ_D
- The properties of the mesh on the contact boundary (if it is quasi-uniform or not, for instance, or a related property)

For instance, it has been shown in [264] that, for $d = 2$, for Γ_C a straight segment, for a uniform mesh on the contact boundary, and if Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$, the pair $\mathbb{P}_1/\mathbb{P}_0$, with continuous linear functions for the displacement, and discontinuous piecewise constant functions for the multiplier cannot be (uniformly) inf-sup stable.

Also, for related results associated with the Dirichlet boundary condition, see, for instance, the reviews [74] or [252] and references therein.

7.7.3 About the Mixed Methods in This Chapter

The first method presented is from [95] and the convergence analysis presented here is new. We took advantage of the recent results of [110] (see Chap. 5) to provide an analysis with an optimal rate.

The Barbosa & Hughes stabilization technique has been first written for variational inequalities in [24]. The adaptation for Signorini contact presented in this chapter is from [169]. Note that this method has been extended later on to friction [201] and to contact with extended finite elements for a crack [9]. Other stabilization techniques have been studied, and, for instance, a fictitious domain method based on extended finite elements and a minimal stabilization procedure has been proposed in [10], and later on extended for a crack geometry with Tresca friction condition [11]. Error estimates for bubble stabilized finite element methods can be found in, e.g., [32].

The Mortar Finite Element Method has its origin in domain decomposition with nonmatching meshes, see the seminal papers [40, 41]. Then, it has been adapted to unilateral contact in [33–35]. Many works related to this method have been published since then, see, for instance, [264] and references therein. The Local Average Contact (LAC) paradigm is more recent [1] and has been mostly designed to be implemented in industrial codes, such as Code Aster.[1]

7.7.4 Augmented Lagrangian

The Augmented Lagrangian technique has a long story in relationship with computational mechanics, see, for instance, [139]. For contact and friction, its introduction is attributed to P. Alart and A. Curnier, in their article of 1988 [6]. As already mentioned, this technique has received much attention in the last years, see, for instance, [63] and references therein.

[1] https://code-aster.org.

Part III
Extension to Frictional Contact and Large Strain

Chapter 8
Tresca Friction

As a preliminary step before going into Coulomb friction, we focus on the numerical approximation of Tresca friction in this chapter. The Signorini problem with Tresca friction leads to a well-posed variational inequality of the second kind. As a result, this setting makes easier the mathematical analysis of some numerical approximations, as it was the case for the (frictionless) Signorini problem. However, as we will see, new difficulties appear in the derivation of *a priori* error estimates, and, for some classes of methods, it is still an open problem to derive estimates with the optimal rates. We will show that, still today, optimal convergence rates can be obtained solely for Nitsche and penalty methods. We will try to explain at least the sources of difficulties for some other techniques.

Moreover, in the next chapter, we will see that some existence results for discrete Coulomb problems can be established with a Brouwer fixed point argument. For this latter, we define a function that involves a Tresca problem. As a consequence, the results of discrete well-posedness presented here will be helpful for this purpose.

8.1 Setting

We introduce a setting very similar to the Signorini problem presented in Chap. 3, except that, now, we add a Tresca friction law on the contact boundary. As a result, we consider the deformation of an elastic body occupying, in the initial unconstrained configuration, a domain Ω in $\mathbb{R}^d$ ($d = 2, 3$). The Lipschitz boundary $\partial\Omega$ of Ω consists of Γ_D, Γ_N and Γ_C, where the measure of Γ_D does not vanish. The body Ω is clamped on Γ_D and subjected to surface traction force density $\mathbf{F}$ on Γ_N. The body force density is denoted by $\mathbf{f}$. For the sake of simplicity, in the initial configuration, the set Γ_C is a straight line segment or a planar polygon that represents the candidate contact boundary on a rigid foundation. Moreover, to carry out the convergence analysis, and obtain error estimates, we will suppose as in

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_8

Chap. 5 that the Dirichlet boundary Γ_D is compactly embedded into $\Gamma \setminus \overline{\Gamma_C}$. The contact is assumed to be frictional, and the stick, slip, and separation zones on Γ_C are not known in advance.

The contact problem with Tresca friction law consists of finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$ satisfying (8.1)–(8.3):

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{u}) \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \qquad & \text{on } \Gamma_N, \end{aligned} \tag{8.1}$$

with the notations already introduced in Chap. 3. On the contact boundary Γ_C, the three conditions representing unilateral contact are still given by

$$u_{\mathbf{n}} \le 0, \qquad \sigma_{\mathbf{n}}(\mathbf{u}) \le 0, \qquad \sigma_{\mathbf{n}}(\mathbf{u})\, u_{\mathbf{n}} = 0, \tag{8.2}$$

as in Chap. 3. The Tresca friction law is summarized by the following conditions:

$$\begin{cases} |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \le s_T, & \text{if } \mathbf{u}_{\mathbf{t}} = \mathbf{0}, \quad (i) \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = -s_T \dfrac{\mathbf{u}_{\mathbf{t}}}{|\mathbf{u}_{\mathbf{t}}|} & \text{otherwise}, \quad (ii) \end{cases} \tag{8.3}$$

where $s_T \in L^2(\Gamma_C)$, $s_T \ge 0$, is a given threshold. Note that conditions (i) and (ii) imply that $|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \le s_T$ in all cases and that if $|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| < s_T$, we must have $\mathbf{u}_{\mathbf{t}} = \mathbf{0}$. Moreover, these conditions imply that there always hold

$$|\mathbf{u}_{\mathbf{t}}|\; \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + s_T\; \mathbf{u}_{\mathbf{t}} = \mathbf{0}. \tag{8.4}$$

When we set the friction threshold s_T to 0 in (8.3), we recover the condition $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = \mathbf{0}$ and thus the frictionless case as a limit case.

Remark 8.1 The case of bilateral contact with Tresca friction can be obtained by substituting to Signorini conditions (8.2) the following one on Γ_C:

$$u_{\mathbf{n}} = 0. \tag{8.5}$$

This condition means that the elastic body is in contact with the rigid support along the whole potential contact surface Γ_C, and that sliding is allowed.

Remark 8.2 As we will see in the next chapter, the Tresca friction model can be obtained from the Coulomb friction model by making the additional assumption that the amplitude of the normal stress is known ($\mathscr{F}|\sigma_{\mathbf{n}}(\mathbf{u})| = s_T$, where $\mathscr{F}$ is the Coulomb friction coefficient) [181].

We recall from Chap. 3 that $\mathbf{V}$ is the space of admissible displacements that satisfy the Dirichlet boundary conditions and Λ_C is the convex cone of weakly nonpositive normal boundary stresses. We need to introduce a new notation for the admissible set associated to the tangential Lagrange multiplier, and we define it below, for any $s_T \in -\Lambda_C$,

$$\boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T) := \left\{ \boldsymbol{\mu} \in \mathbf{W}'_{C,\mathbf{t}} \;\middle|\; \langle \boldsymbol{\mu}, \mathbf{w}_\mathbf{t} \rangle_{\mathbf{W}'_{C,\mathbf{t}}, \mathbf{W}_{C,\mathbf{t}}} \leq \int_{\Gamma_C} s_T |\mathbf{w}_\mathbf{t}|, \forall \mathbf{w}_\mathbf{t} \in \mathbf{W}_{C,\mathbf{t}} \right\}.$$

We also define a global convex set that incorporates weakly all the conditions, for both the normal and tangential components:

$$\boldsymbol{\Lambda}_C(s_T) := \Lambda_C \times \boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T).$$

Other notations already introduced in Chap. 3 are adopted and recalled below:

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}), \qquad L(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{F} \cdot \mathbf{v},$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$. Now, we need to change the notation introduced in Chap. 7 for the bilinear term $b(\cdot, \cdot)$ on the contact boundary, in order to incorporate the tangential terms and the friction conditions:

$$b(\mathbf{v}, \boldsymbol{\mu}) := \langle \mu_\mathbf{n}, v_\mathbf{n} \rangle_{W'_C, W_C} + \langle \boldsymbol{\mu}_\mathbf{t}, \mathbf{v}_\mathbf{t} \rangle_{\mathbf{W}'_{C,\mathbf{t}}, \mathbf{W}_{C,\mathbf{t}}}$$

for any $\boldsymbol{\mu} = (\mu_\mathbf{n}, \boldsymbol{\mu}_\mathbf{t})$ in $\mathbf{W}'_C$.

The variational formulation of Problem (8.1)–(8.3) in its mixed form consists of finding a pair $(\mathbf{u}, \boldsymbol{\lambda}) = (\mathbf{u}, \lambda_\mathbf{n}, \boldsymbol{\lambda}_\mathbf{t}) \in \mathbf{V} \times \Lambda_C \times \boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T) = \mathbf{V} \times \boldsymbol{\Lambda}_C(s_T)$, which satisfies

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \boldsymbol{\lambda}) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\[2ex] \quad b(\mathbf{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \geq 0, & \forall \boldsymbol{\mu} = (\mu_\mathbf{n}, \boldsymbol{\mu}_\mathbf{t}) \in \boldsymbol{\Lambda}_C(s_T). \end{cases} \tag{8.6}$$

We can observe that if $(\mathbf{u}, \lambda_\mathbf{n}, \boldsymbol{\lambda}_\mathbf{t})$ is a solution to (8.6), then $\lambda_\mathbf{n} = \sigma_\mathbf{n}(\mathbf{u})$ and $\boldsymbol{\lambda}_\mathbf{t} = \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})$. To avoid more notation, we will skip over the regularity aspects of the functions defined on Γ_C and we write afterward integral terms instead of duality pairings.

The above mixed formulation can be derived using duality arguments, as in Chap. 7, and particularly can be obtained as the saddle-point of the following Lagrangian:

$$\mathscr{L}(\mathbf{v}, \boldsymbol{\lambda}) = \mathcal{J}(\mathbf{v}) - b(\mathbf{v}, \boldsymbol{\lambda}) - I_{\Lambda_C}(\lambda_\mathbf{n}) - I_{\boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T)}(\boldsymbol{\lambda}_\mathbf{t}), \tag{8.7}$$

where $I_{\Lambda_C}(\cdot)$, respectively $I_{\boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T)}(\cdot)$, is the indicator function of Λ_C, respectively, $\boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T)$, defined as follows:

$$I_{\Lambda_C}(\lambda) := \begin{cases} 0 & \text{if } \lambda \in \Lambda_C, \\ +\infty & \text{otherwise,} \end{cases}$$

for $\lambda \in W'_C$, and

$$I_{\boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T)}(\boldsymbol{\lambda}_\mathbf{t}) := \begin{cases} 0 & \text{if } \boldsymbol{\lambda}_\mathbf{t} \in \boldsymbol{\Lambda}_{C,\mathbf{t}}(s_T), \\ +\infty & \text{otherwise,} \end{cases}$$

for $\boldsymbol{\lambda}_\mathbf{t} \in \mathbf{W}'_{C,\mathbf{t}}$.

Another classical weak formulation of Problem (8.1)–(8.3) is an inequality problem. We recall the definition (3.9) of the closed convex cone of admissible displacement fields satisfying the nonpenetration conditions, already introduced in Chap. 3:

$$\mathbf{K} := \{\mathbf{v} \in \mathbf{V} \mid v_\mathbf{n} \leq 0 \text{ on } \Gamma_C\}.$$

We also introduce the functional:

$$j(s_T; \mathbf{v}) := \int_{\Gamma_C} s_T |\mathbf{v}_\mathbf{t}|,$$

for any $\mathbf{v} \in \mathbf{V}$. Then, Problem (8.1)–(8.3) is equivalent to find a displacement field $\mathbf{u}$ such that

$$\mathbf{u} \in \mathbf{K}, \qquad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(s_T; \mathbf{v}) - j(s_T; \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K}. \tag{8.8}$$

This is a variational inequality of the second kind, for which it is possible to state the following result:

Theorem 8.1 *Problem* (8.8) *admits a unique solution* $\mathbf{u} \in \mathbf{K}$*, that is the unique minimizer on* $\mathbf{K}$ *of the functional*

$$\mathcal{J}_{s_T} : \mathbf{V} \ni \mathbf{v} \mapsto \mathcal{J}(\mathbf{v}) + j(s_T; \mathbf{v}) \in \mathbb{R}.$$

Moreover, for $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$*, with* $s > 3/2$*, Problem (8.1)–(8.3) and Problem* (8.8) *are equivalent.*

Furthermore, the solution to (8.8) *is Lipschitz continuous with respect to the data. Indeed, for* $i = 1, 2$*, and some source terms* $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ *and* $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$*, denote by* $\mathbf{u}_i \in \mathbf{K}$ *the corresponding solution to* (8.8)*. Then, there holds*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \leq \frac{C}{\alpha}(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

where $C > 0$, and $\alpha > 0$ is the $\mathbf{V}$*-ellipticity constant associated to* $a(\cdot, \cdot)$.

Proof First, existence and uniqueness of the solution $\mathbf{u} \in \mathbf{K}$ to Problem (8.8) can be obtained from the Lions–Stampacchia theory. We can apply for instance [138, Theorem 4.1] since we can check all its assumptions, particularly that the functional $j(s_T; \cdot)$ is convex, proper, and lower semi-continuous on $\mathbf{V}$. Since $a(\cdot, \cdot)$ is symmetric, finding the solution to the variational inequality (8.8) is also equivalent to find the minimizer of the functional $\mathcal{J}_{s_T}$ on $\mathbf{K}$ [138, Lemma 4.1] (see also [181, Theorem 10.2]).

The strong–weak equivalence property is established following the same lines as in Chap. 3 (see also, for instance, [114, Section 5.4.3] or [181, Theorem 10.1]).

Let us prove now that the solution to (8.8) is Lipschitz continuous with respect to the data. For $i = 1, 2$, we denote by L_i the linear form associated to the source terms $\mathbf{f}_i \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{F}_i \in L^2(\Gamma_N; \mathbb{R}^d)$, and $\mathbf{u}_i \in \mathbf{K}$ is the corresponding solution to (8.8). We take successively $\mathbf{v} = \mathbf{u}_2$ and $\mathbf{v} = \mathbf{u}_1$ in the variational inequality (8.8) and get

$$a(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j(s_T; \mathbf{u}_2) - j(s_T; \mathbf{u}_1) \geq L_1(\mathbf{u}_2 - \mathbf{u}_1),$$

$$a(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) + j(s_T; \mathbf{u}_1) - j(s_T; \mathbf{u}_2) \geq L_2(\mathbf{u}_1 - \mathbf{u}_2).$$

We add the two inequalities and get

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) \geq (L_1 - L_2)(\mathbf{u}_2 - \mathbf{u}_1)$$

or equivalently

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq (L_1 - L_2)(\mathbf{u}_1 - \mathbf{u}_2).$$

We make use of the $\mathbf{V}$-ellipticity of $a(\cdot, \cdot)$ as well as the continuity of $(L_1 - L_2)(\cdot)$. With the definition of $(L_1 - L_2)(\cdot)$ and (3.7) (see Sect. 3.1.2), there holds

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \leq \frac{C}{\alpha}(\|\mathbf{f}_1 - \mathbf{f}_2\|_{0,\Omega} + \|\mathbf{F}_1 - \mathbf{F}_2\|_{0,\Gamma_N}),$$

with $C > 0$. This ends the proof. □

Remark 8.3 In the case of bilateral contact (condition (8.5)), the same weak formulation (8.8) holds, replacing simply the convex cone $\mathbf{K}$ by the vector space:

$$\mathbf{K}_b := \{\mathbf{v} \in \mathbf{V} \mid v_{\mathbf{n}} = 0 \text{ on } \Gamma_C\}.$$

8.2 Discrete Variational Inequality

We recall here some notations of Chap. 4. We choose standard continuous Lagrange finite elements of degree k, with $k = 1$ or $k = 2$, *i.e.*,

$$\mathbf{V}_k^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \,\middle|\, \mathbf{v}^h_{|T} \in \mathbb{P}_k(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\},$$

where $\mathcal{T}^h$ is a simplicial mesh of the domain Ω, and where $\mathbb{P}_k(T; \mathbb{R}^d)$ stands for the space of all vector-valued polynomials on T, of degree lower or equal than k in the d variables. We suppose that the mesh $\mathcal{T}^h$ resolves the contact boundary Γ_C, which means that all the edges ($d = 2$) or faces ($d = 3$) that intersect Γ_C are included in $\overline{\Gamma_C}$. We also suppose that it resolves the Neumann boundary Γ_N and the Dirichlet boundary Γ_D.

8.2.1 A Discrete Variational Inequality

We proceed as in Chap. 5, and we start with the most obvious approximation of Tresca problem (8.8). This will allow us to understand where are the difficulties to obtain a priori error estimates in the H^1-norm. We define

$$\mathbf{K}^h := \mathbf{K} \cap \mathbf{V}_1^h$$

and introduce the discrete problem:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) + j(s_T; \mathbf{v}^h) - j(s_T; \mathbf{u}^h) \geq L(\mathbf{v}^h - \mathbf{u}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{K}^h. \end{cases} \tag{8.9}$$

So we discretize the variational equality (8.8) using a finite dimensional subset of the convex cone $\mathbf{K}$ of admissible displacements. As for the continuous formulation (8.8), Problem (8.9) is equivalent to find the minimizer in $\mathbf{K}^h$ of the functional $\mathcal{J}_{s_T}(\cdot)$. Formulation (8.9) is a conforming method, since $\mathbf{K}^h \subset \mathbf{K}$, and also a consistent method since the solution $\mathbf{u}$ to Problem (8.8) verifies, for $\mathbf{v}^h \in \mathbf{K}^h$:

$$a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}) + j(s_T; \mathbf{v}^h) - j(s_T; \mathbf{u}) \geq L(\mathbf{v}^h - \mathbf{u}).$$

8.2.2 A Preliminary Convergence Result

The set of admissible displacements $\mathbf{K}^h$ is a non-empty closed convex set. Moreover, $a(\cdot, \cdot)$ is still continuous and elliptic on $\mathbf{V}^h$, as it is on $\mathbf{V}$ (see Proposition 3.3).

The same occurs for $L(\cdot)$, which is continuous on $\mathbf{V}^h$. We can apply once again Lions–Stampacchia's theory to ensure that Problem (8.9) is well-posed: it admits one unique solution $\mathbf{u}^h \in \mathbf{K}^h$. We provide below an abstract error estimate, still in the spirit of Falk's Lemma [126]:

Proposition 8.1 *Suppose that the solution* $\mathbf{u} \in \mathbf{K}$ *to Problem* (8.8) *belongs to* $H^s(\Omega; \mathbb{R}^d)$ *with* $s > 3/2$*. Then, the solution* $\mathbf{u}^h$ *to Problem* (8.9) *satisfies the* a priori *error estimate:*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq C \inf_{\mathbf{v}^h \in \mathbf{K}^h} \left(\|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + \left(\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}) \right)^{\frac{1}{2}} \right.$$
$$\left. \left(\int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}) \right)^{\frac{1}{2}} + \left(\int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}|) \right)^{\frac{1}{2}} \right), \tag{8.10}$$

with $C > 0$ *independent from* h *and* $\mathbf{u}$.

Proof Let $\mathbf{v}^h \in \mathbf{K}^h$. We use the ellipticity of $a(\cdot, \cdot)$, combined with Cauchy–Schwarz inequality:

$$\begin{aligned} \alpha \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 &\leq a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h) \\ &= a(\mathbf{u} - \mathbf{u}^h, (\mathbf{u} - \mathbf{v}^h) + (\mathbf{v}^h - \mathbf{u}^h)) \\ &\leq C\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h), \end{aligned}$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot, \cdot)$. Then, we use Young inequality:

$$\frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 + a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h). \tag{8.11}$$

Since $\mathbf{v}^h$ belongs to $\mathbf{K}^h$, and $\mathbf{u}^h$ is the solution to (8.9):

$$-a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \leq -L(\mathbf{v}^h - \mathbf{u}^h) + \int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}^h|).$$

Now, since $\mathbf{u}$ is also solution to the elasticity equations of (8.1), using the Green formula (3.15) and taking into account Dirichlet and Neumann conditions in (8.1), we get

$$a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) = L(\mathbf{v}^h - \mathbf{u}^h) + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h) + \int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}^h).$$

Remark that above we wrote the boundary term as an integral term, which has been made possible thanks to the regularity assumption $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$ with $s > \frac{3}{2}$, and using the Trace Theorem 2.6, from which we obtain $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} \in L^2(\Gamma_C; \mathbb{R}^d)$. We combine the two previous relationships with the estimate (8.11):

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h)$$
$$+ \int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}^h) + \int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}^h|).$$

We transform the contact term, as follows:

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h) = \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}}) + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}}^h.$$

We use the contact conditions (3.23) and the property $u_{\mathbf{n}}^h \leq 0$ (since $\mathbf{u}^h \in \mathbf{K}^h$):

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}} = 0, \quad -\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})u_{\mathbf{n}}^h \leq 0.$$

For the friction terms, we can split it as follows:

$$\int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}^h) + \int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}^h|)$$
$$= \int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}) + \int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{u}_{\mathbf{t}} - \mathbf{u}_{\mathbf{t}}^h)$$
$$+ \int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}|) + \int_{\Gamma_C} s_T(|\mathbf{u}_{\mathbf{t}}| - |\mathbf{u}_{\mathbf{t}}^h|).$$

Now we use the formula (8.4) for Tresca conditions

$$|\mathbf{u}_{\mathbf{t}}|\,\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + s_T\,\mathbf{u}_{\mathbf{t}} = \mathbf{0}.$$

Moreover, using (8.3), we can bound

$$-\int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot \mathbf{u}_{\mathbf{t}}^h - \int_{\Gamma_C} s_T|\mathbf{u}_{\mathbf{t}}^h|$$
$$\leq \int_{\Gamma_C} |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})||\mathbf{u}_{\mathbf{t}}^h| - \int_{\Gamma_C} s_T|\mathbf{u}_{\mathbf{t}}^h| = \int_{\Gamma_C} (|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| - s_T)|\mathbf{u}_{\mathbf{t}}^h| \leq 0.$$

We obtain finally

$$\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(v_{\mathbf{n}}^h - u_{\mathbf{n}})$$

$$\int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\mathbf{v}_{\mathbf{t}}^h - \mathbf{u}_{\mathbf{t}}) + \int_{\Gamma_C} s_T(|\mathbf{v}_{\mathbf{t}}^h| - |\mathbf{u}_{\mathbf{t}}|),$$

which gives the desired error bound (8.10). □

Therefore, we proceed as in the beginning of Chap. 5 to get a preliminary estimation of the convergence rate for the finite element approximation (8.9). Indeed, we make the choice $\mathbf{v}^h = \mathcal{I}^h \mathbf{u}$, where $\mathcal{I}^h$ is the Lagrange interpolation operator we introduced in Chap. 4. Due to the regularity assumption $\mathbf{u} \in H^s(\Omega; \mathbb{R}^d)$, $s > \frac{3}{2}$, and the Sobolev embedding theorem (Theorem 2.3), $\mathbf{u}$ is a continuous function, and its Lagrange interpolation can be defined. We verify $\mathcal{I}^h \mathbf{u} \in \mathbf{K}^h$ since $\mathbf{u} \in \mathbf{K}$ and since $\mathcal{I}^h$ preserves the positivity for $\mathbb{P}_1$ Lagrange finite elements. As a result, we verify

$$(\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} \leq 0.$$

Let us bound now the different terms on the right in the estimate (8.10). The first term corresponds to the interpolation estimate in the H^1-norm within the domain Ω and is classical. We just apply Theorem 4.1:

$$\|\mathbf{u} - \mathcal{I}^h \mathbf{u}\|_{1,\Omega} \leq C h^{s-1} \|\mathbf{u}\|_{s,\Omega} \tag{8.12}$$

for $\frac{3}{2} < s \leq 2$. Now, we focus on the first term related to Tresca friction. We proceed in the same way as we did at the very beginning of Chap. 5 and get a suboptimal bound:

$$\left(\int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot ((\mathcal{I}^h(\mathbf{u}))_{\mathbf{t}} - \mathbf{u}_{\mathbf{t}}) \right)^{\frac{1}{2}} \leq C h^{\frac{s}{2} - \frac{1}{4}} \|\mathbf{u}\|_{s,\Omega}.$$

For the second term of (8.10), which is the boundary term that comes from the contact condition on Γ_C, we do the same and get the suboptimal bound:

$$\left(\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})((\mathcal{I}^h(\mathbf{u}))_{\mathbf{n}} - u_{\mathbf{n}}) \right)^{\frac{1}{2}} \leq C h^{\frac{s}{2} - \frac{1}{4}} \|\mathbf{u}\|_{s,\Omega}.$$

The other friction term needs to be handled differently. A simple treatment consists in using $||a| - |b|| \leq |a - b|$, then the Cauchy–Schwarz inequality, and once again the interpolation estimate for the trace:

$$\begin{aligned}
&\int_{\Gamma_C} s_T(|\mathcal{I}^h \mathbf{u}_{\mathbf{t}}| - |\mathbf{u}_{\mathbf{t}}|) \\
&\leq \int_{\Gamma_C} s_T |\mathcal{I}^h \mathbf{u}_{\mathbf{t}} - \mathbf{u}_{\mathbf{t}}| \\
&\leq \|s_T\|_{0,\Gamma_C} \|\mathcal{I}^h \mathbf{u}_{\mathbf{t}} - \mathbf{u}_{\mathbf{t}}\|_{0,\Gamma_C} \leq C \|s_T\|_{0,\Gamma_C} h^{s-\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega}.
\end{aligned}$$

As a result, we obtain

$$\left(\int_{\Gamma_C} s_T(|\mathcal{I}^h \mathbf{u_t}| - |\mathbf{u_t}|)\right)^{\frac{1}{2}} \leq C h^{\frac{s}{2}-\frac{1}{4}} \|\mathbf{u}\|_{s,\Omega}^{\frac{1}{2}}.$$

Finally, we combine the above result with (8.12) and obtain

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \leq C(\mathbf{u}) h^{\frac{s}{2}-\frac{1}{4}}, \tag{8.13}$$

since $(s-1) > (\frac{s}{2} - \frac{1}{4})$, this is suboptimal. The bottleneck here is made of the two integral terms modeling friction (i.e., involving the tangential components). Indeed, conversely to the two previous ones, the techniques presented in Chap. 5 are not applicable straightforwardly, in order to get an optimal bound. As a result, for this formulation, and some others, the obtention of an optimal error bound is still an open problem in the case of Tresca friction (see the discussion and the related references at the end of this chapter). As for frictionless contact, a possibility to overcome this issue is to change the discretization, using a reformulation of the friction condition. This is what we show next.

8.3 Nitsche for Tresca Friction

This section presents a Nitsche method for the Tresca friction problem that extends what we described in Chap. 6 for the frictionless Signorini problem. The first interest of such a formulation is that it leads to optimal convergence rates in the H^1-norm. Its second interest is related to numerical implementation, as we will see later on.

8.3.1 Setting

Here, we also suppose that the mesh $\mathcal{T}^h$ is conformal to the subdivision of the boundary into Γ_D, Γ_N, and Γ_C (*i.e.*, a face of an element $T \in \mathcal{T}^h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision). We choose a standard Lagrange finite element method of degree k with $k = 1$ or $k = 2$, i.e.,

$$\mathbf{V}^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}^0(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h{}_{|T} \in \mathbb{P}_k(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

For any $\alpha \in \mathbb{R}^+$, we introduce the notation $[\cdot]_\alpha$ for the orthogonal projection onto $\mathscr{B}(\mathbf{0}, \alpha) \subset \mathbb{R}^{d-1}$, where $\mathscr{B}(\mathbf{0}, \alpha)$ is the closed ball centered at the origin $\mathbf{0}$ and of radius α. This operation can be defined analytically, for $\mathbf{x} \in \mathbb{R}^{d-1}$ by

$$[\mathbf{x}]_\alpha = \begin{cases} \mathbf{x} & \text{if } |\mathbf{x}| \le \alpha, \\ \alpha \dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise.} \end{cases}$$

We recall the following property of projections, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$:

$$(\mathbf{y} - \mathbf{x}) \cdot ([\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha) \ge |[\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha|^2. \tag{8.14}$$

As we did for Signorini conditions in Chap. 6, we reformulate Tresca conditions (8.3) as a nonlinear equality, which is the fundamental first step to derive a Nitsche method. This is the object of the following proposition. In fact, this next result has been pointed out earlier in [6].

Proposition 8.2 *Let γ be a positive function defined on Γ_C. The Tresca friction conditions* (8.3) *can be reformulated as follows:*

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t}]_{s_T} . \tag{8.15}$$

Proof We proceed in two steps:
(1) First, let us suppose that $\mathbf{u}_\mathbf{t}$ and $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})$ verify Eq. (8.3).

Consider the case $\mathbf{u}_\mathbf{t} = \mathbf{0}$; then from (8.3) (i), we have the inequality $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| \le s_T$. Due to the property of projection $[\mathbf{x}]_{s_T} = \mathbf{x}$ for $\mathbf{x} \in \mathscr{B}(\mathbf{0}, s_T)$, it results

$$[\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t}]_{s_T} = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})]_{s_T} = \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}),$$

so that (8.15) holds.

In the case $\mathbf{u}_\mathbf{t} \neq \mathbf{0}$, from (8.3) (ii), we obtain

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t} = -\left(\frac{s_T}{|\mathbf{u}_\mathbf{t}|} + \gamma\right)\mathbf{u}_\mathbf{t}.$$

It results that

$$|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t}| = \gamma |\mathbf{u}_\mathbf{t}| + s_T \ge s_T,$$

which means that its projection onto $\mathscr{B}(\mathbf{0}, s_T)$ is simply

$$[\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t}]_{s_T} = -s_T \frac{\mathbf{u}_\mathbf{t}}{|\mathbf{u}_\mathbf{t}|}.$$

Finally, using again (8.3) (ii) yields

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = -s_T \frac{\mathbf{u}_\mathbf{t}}{|\mathbf{u}_\mathbf{t}|} = [\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma \mathbf{u}_\mathbf{t}]_{s_T} ,$$

which is (8.15).
(2) Suppose now that the condition (8.15) is satisfied by $\mathbf{u_t}$ and $\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})$.

We first consider the case $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}| \leq s_T$, for which $[\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}]_{s_T} = \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}$. In this case, condition (8.15) becomes

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t},$$

so $\mathbf{u_t} = \mathbf{0}$. Moreover, in this case, since $\gamma > 0$,

$$|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| = |\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}| \leq s_T,$$

which means that (8.3) (i) is satisfied.

Consider then the case $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}| > s_T$. Then there is $\lambda \in (0, 1)$ such that $[\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}]_{s_T} = \lambda(\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t})$. Relationship (8.15) can now be written:

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \lambda(\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}),$$

or

$$(1 - \lambda)\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = -\lambda\gamma\mathbf{u_t}.$$

Thus we have

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = \frac{\gamma\lambda}{\lambda - 1}\mathbf{u_t}, \tag{8.16}$$

with the quantity $\frac{\gamma\lambda}{\lambda-1} < 0$. Since in this case $[\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{u_t}]_{s_T} \in \partial\mathscr{B}(\mathbf{0}, s_T)$ and owing to relationship (8.15), we deduce $|\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u})| = s_T$. Taking then the norm of (8.16) yields

$$s_T = \frac{\gamma\lambda}{1 - \lambda}|\mathbf{u_t}|,$$

from which we obtain finally $\mathbf{u_t} \neq \mathbf{0}$ and

$$\boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) = -s_T\frac{\mathbf{u_t}}{|\mathbf{u_t}|},$$

that is (8.3) (ii). □

From now on, we proceed exactly as in Chap. 6 (see also [73] and [76]). Let now $\theta \in \mathbb{R}$ be a fixed parameter. We take γ_N as a discrete function on Γ_C, piecewise constant, whose expression when restricted to a boundary face or edge is

$$\gamma_N|_{T\cap\Gamma_C} := \frac{\gamma_0}{h_T},$$

with T a simplex that shares at least one of its face/edge with the boundary Γ_C. The scalar value $\gamma_0 > 0$ is the Nitsche parameter. Let us also define the linear operators

$$\mathrm{P}^{\mathbf{n}}_{1,N} : \begin{array}{rcl} \mathbf{V}^h & \to & L^2(\Gamma_C) \\ \mathbf{v}^h & \mapsto & \gamma_N v^h_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{v}^h), \end{array} \qquad \text{and} \qquad \mathbf{P}^{\mathbf{t}}_{\theta,N} : \begin{array}{rcl} \mathbf{V}^h & \to & L^2(\Gamma_C; \mathbb{R}^{d-1}) \\ \mathbf{v}^h & \mapsto & \gamma_N \mathbf{v}^h_{\mathbf{t}} - \theta \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h) \end{array}.$$

Define the bilinear form

$$A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma_N}\, \sigma_{\mathbf{n}}(\mathbf{u}^h)\sigma_{\mathbf{n}}(\mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma_N}\, \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h) \cdot \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h).$$

Nitsche method for contact with Tresca friction now reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N}\, [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h)\right]_{s_T} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,N}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{8.17}$$

The role of θ is the same as described in Chap. 6 and allows to choose between different variants with different properties. Particularly, for $\theta = 1$, we recover a symmetric variant that can be derived from a discrete energy functional [76], and for $\theta = -1$, we recover a skew-symmetric variant, with well-posedness and optimal *a priori* error estimates irrespectively of the value of the Nitsche parameter.

Remark 8.4 For the symmetric variant ($\theta = 1$), Nitsche formulation (8.17) can be obtained as the first order optimality condition associated with the energy functional

$$\begin{aligned} \mathcal{J}_{N,s_T}(\mathbf{v}^h) := \mathcal{J}(\mathbf{v}^h) &- \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N} \sigma_{\mathbf{n}}(\mathbf{v}^h)^2 + \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{v}^h)\right]_+^2 \\ &- \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N} |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)|^2 + \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N} \left|\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{v}^h)\right|^2 \\ &- \frac{1}{2}\int_{\Gamma_C} \frac{1}{\gamma_N} \left|\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{v}^h) - \left[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{v}^h)\right]_{s_T}\right|^2, \end{aligned} \tag{8.18}$$

defined for $\mathbf{v}^h \in \mathbf{V}^h$, see [76, Proposition 2]. Note the expression of the additional terms that incorporate friction, in comparison to the functional for the symmetric Nitsche method (6.7) in the case of frictionless contact.

8.3.2 Well-Posedness and Error Estimates

As for frictionless unilateral contact (see Chap. 6), our Nitsche-based formulation (8.17) for the Tresca problem is consistent. It is a direct consequence of reformulations (8.15) and (6.4) followed by integration by parts.

In the following, we recall the well-posedness and the optimal convergence of the method (8.17) for unilateral contact with friction. The proofs are a direct adaptation of those in Chap. 6 (see also [73]). Moreover, these results can be adapted without difficulty in the case of bilateral contact with friction.

8.3.2.1 Well-Posedness

To show that Problem (8.17) is well-posed, we use an argument by Brezis for M-type and pseudo-monotone operators and proceed exactly as in the proof of Theorem 6.1.

Theorem 8.2 *Suppose that one of the two following assumptions hold:*

1. $\theta \neq -1$ *and* $\gamma_0 > 0$ *is large enough.*
2. $\theta = -1$ *and* $\gamma_0 > 0$.

Then Problem (8.17) *admits one unique solution* $\mathbf{u}^h$ *in* $\mathbf{V}^h$. *Moreover, the solution to* (8.17) *is Lipschitz continuous with respect to the data.*

8.3.2.2 Error Analysis

First, there holds an abstract error estimate, which extends Theorem 6.2.

Theorem 8.3 *Suppose that the solution* $\mathbf{u}$ *to Problem* (8.8) *belongs to* $H^s(\Omega; \mathbb{R}^d)$, *with* $s > 3/2$. *Suppose also that the Nitsche parameter* γ_0 *is large enough. Then the unique solution* $\mathbf{u}^h$ *to Problem* (8.17) *satisfies*

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} + \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|_{0,\Gamma_C} \\
&+ \left\| \gamma_N^{-\frac{1}{2}} \left(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \left[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h) \right]_{s_T} \right) \right\|_{0,\Gamma_C} \\
\leq\ & C \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left(\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + \left\| \gamma_N^{\frac{1}{2}} (u_{\mathbf{n}} - v^h_{\mathbf{n}}) \right\|_{0,\Gamma_C} + \left\| \gamma_N^{-\frac{1}{2}} \sigma_{\mathbf{n}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C} \right. \\
&\left. + \left\| \gamma_N^{\frac{1}{2}} (\mathbf{u}_{\mathbf{t}} - \mathbf{v}^h_{\mathbf{t}}) \right\|_{0,\Gamma_C} + \left\| \gamma_N^{-\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}-\mathbf{v}^h) \right\|_{0,\Gamma_C} \right),
\end{aligned} \tag{8.19}$$

where the constant $C > 0$ *depends on* θ *but does not depend on* γ_0*,* h*, and* $\mathbf{u}$*. When* $\theta = -1$*, the above result still holds for* $\gamma_0 > 0$ *arbitrarily small, but with* $C > 0$ *that depends on* γ_0*.*

The optimal convergence of the method is stated below, and the result is proven exactly as for Theorem 6.3.

Theorem 8.4 *Suppose that the solution* $\mathbf{u}$ *to Problem* (8.8) *belongs to* $H^s(\Omega; \mathbb{R}^d)$*, with* $3/2 < s \leq 1 + k$*, where* $k \geq 1$ *denotes the degree of the Lagrange finite elements* $\mathbb{P}_k$*. Suppose that* γ_0 *is large enough (*$\gamma_0 > 0$ *for* $\theta = -1$*). Then the solution* $\mathbf{u}^h$ *to Problem* (8.17) *satisfies the* a priori *error estimate:*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \\
&+ \left\| \gamma_N^{-\frac{1}{2}} \left(\sigma_{\mathbf{n}}(\mathbf{u}) + \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h) \right]_+ \right) \right\|_{0,\Gamma_C} + \left\| \gamma_N^{-\frac{1}{2}} \left(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \left[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h) \right]_{s_T} \right) \right\|_{0,\Gamma_C} \\
\leq\ & C h^{s-1} |\mathbf{u}|_{s,\Omega},
\end{aligned} \tag{8.20}$$

with the constant $C > 0$ *that does not depend on* h *and* $\mathbf{u}$ *but depends on* θ *and* γ_0*.*

Remark finally that the above result, which is an optimal a priori error estimate, holds in two and three dimensions, for linear and quadratic finite elements and without any extra assumption on the behavior of the exact solution $\mathbf{u}$ on the contact and friction boundary.

8.4 Mixed and Augmented Lagrangian Methods

As in Chap. 7, we can write mixed and Augmented Lagrangian formulations for Tresca friction. Particularly, we can extend the mixed method for frictionless contact, and this result will be useful when we will study Coulomb friction in the next chapter.

8.4.1 Global Setting

We still approximate Problem (8.6) with Lagrange finite elements and follow the setting of Chap. 4. We recall that $\mathbf{V}^h := \mathbf{V}^h_1$ is a space of continuous piecewise affine Lagrange finite elements, built from a simplicial mesh $\mathcal{T}^h$ of the domain Ω. As in the previous Chap. 7, in order to express the contact constraints by using Lagrange multipliers on the contact zone, we have to introduce the range of $\mathbf{V}^h$ by the trace operator on Γ_C:

$$\mathbf{W}^h := \left\{ \boldsymbol{\mu}^h \in \mathscr{C}(\overline{\Gamma_C};\mathbb{R}^d) \;\middle|\; \exists \mathbf{v}^h \in \mathbf{V}^h \text{ s.t. } \mathbf{v}^h = \boldsymbol{\mu}^h \text{ on } \Gamma_C \right\},$$

and, as in the continuous case, we split this space into trace spaces for the normal and tangential components, as

$$\mathbf{W}^h := W^h \times \mathbf{W}^h_{\mathbf{t}}, \quad W^h := \mathbf{W}^h \cdot \mathbf{n}.$$

8.4.2 A Mixed Method

Now, we define the following closed convex cones:

$$M^h_{\mathbf{n}} := \left\{ \mu^h \in W^h \;\middle|\; \int_{\Gamma_C} \mu^h \psi^h \geq 0,\ \forall \psi^h \in W^h,\ \psi^h \leq 0 \right\},$$

and, for $s^h \in -M^h_{\mathbf{n}}$:

$$\mathbf{M}^h_{\mathbf{t}}(s^h) := \left\{ \boldsymbol{\mu}^h \in \mathbf{W}^h_{\mathbf{t}} \;\middle|\; -\int_{\Gamma_C} |\boldsymbol{\mu}^h|\psi^h \leq -\int_{\Gamma_C} s^h \psi^h \ \forall \psi^h \in W^h,\ \psi^h \leq 0 \right\}.$$

Remark 8.5 One can check that the functions in $M^h_{\mathbf{n}}$ are not necessarily nonpositive. In the same way the functions in $\mathbf{M}^h_{\mathbf{t}}(s^h)$ do not satisfy $|\boldsymbol{\mu}^h| \leq s^h$ everywhere.

Let us set now $s^h_T \in -M^h_{\mathbf{n}}$ a discrete approximation of the Tresca threshold s_T. The discretized mixed formulation of the contact problem with Tresca friction (8.6) is to find $\mathbf{u}^h \in \mathbf{V}^h$ and $\boldsymbol{\lambda}^h \in \mathbf{M}^h(s^h_T) := M^h_{\mathbf{n}} \times \mathbf{M}^h_{\mathbf{t}}(s^h_T)$ satisfying:

$$\begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \boldsymbol{\lambda}^h) = L(\mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mathbf{u}^h, \boldsymbol{\mu}^h - \boldsymbol{\lambda}^h) \geq 0, & \forall\, \boldsymbol{\mu}^h \in \mathbf{M}^h(s^h_T). \end{cases} \tag{8.21}$$

The above Problem (8.21) is in fact equivalent of finding a saddle-point $(\mathbf{u}^h, \boldsymbol{\lambda}^h) \in \mathbf{V}^h \times \mathbf{M}^h(s^h_T)$ verifying

$$\mathcal{L}(\mathbf{u}^h, \boldsymbol{\mu}^h) \leq \mathcal{L}(\mathbf{u}^h, \boldsymbol{\lambda}^h) \leq \mathcal{L}(\mathbf{v}^h, \boldsymbol{\lambda}^h), \qquad \forall \mathbf{v}^h \in \mathbf{V}^h,\ \forall \boldsymbol{\mu}^h \in \mathbf{M}^h(s^h_T),$$

where the Lagrangian is

$$\mathcal{L}(\mathbf{v}^h, \boldsymbol{\mu}^h) := \frac{1}{2} a(\mathbf{v}^h, \mathbf{v}^h) - \int_{\Gamma_C} \mu^h_{\mathbf{n}} v^h_{\mathbf{n}} - \int_{\Gamma_C} \boldsymbol{\mu}^h_{\mathbf{t}} \cdot \mathbf{v}^h_{\mathbf{t}} - L(\mathbf{v}^h).$$

This Lagrangian can be obtained the same way as in Chap. 7.
The well-posedness of the mixed formulation (8.21) is stated below.

Proposition 8.3 *Problem* (8.21) *admits one unique solution.*

Proof By using standard arguments on saddle-point problems as in [153, Theorem 3.9], we deduce the existence of a saddle-point of the Lagrangian $\mathcal{L}(\cdot,\cdot)$. The strict convexity of $a(\cdot,\cdot)$ implies that the first argument $\mathbf{u}^h$ is unique. Suppose that the second argument has two solutions $\boldsymbol{\lambda}^{1h}$ and $\boldsymbol{\lambda}^{2h}$: then the equality in (8.21) implies

$$b(\mathbf{v}^h, \boldsymbol{\lambda}^{1h} - \boldsymbol{\lambda}^{2h}) = 0, \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

The definition of $\mathbf{W}^h$ allows us to choose $\mathbf{v}^h = \boldsymbol{\lambda}^{1h} - \boldsymbol{\lambda}^{2h}$ on Γ_C. From the definition of $b(\cdot,\cdot)$, we come to the conclusion that $\boldsymbol{\lambda}^{1h} - \boldsymbol{\lambda}^{2h} = \mathbf{0}$. Consequently, the second argument $\boldsymbol{\lambda}^h$ is unique and (8.21) admits a unique solution. □

The obtention of optimal error estimates in the $H^1(\Omega)$-norm, without extra assumptions on the contact/friction boundary, is still unresolved. Indeed, the optimal result presented in Chap. 7 for frictionless contact is not straightforward to extend to the frictional case, for reasons similar as those exposed for the discrete variational inequality. We will extend this mixed method in Chap. 9 to the case of Coulomb friction and derive an error estimate.

8.4.3 An Augmented Lagrangian Formulation

Finally, in the same way as in Chap. 7, we can define an augmented Lagrangian formulation, which is the following:

$$\begin{cases} \text{Find } (\mathbf{u}^h, \boldsymbol{\lambda}^h) \in \mathbf{V}^h \times \mathbf{W}^h \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \left[\gamma_L u^h_{\mathbf{n}} - \lambda^h_{\mathbf{n}}\right]_+ v^h_{\mathbf{n}} + \int_{\Gamma_C} \left[\gamma_L \mathbf{u}^h_{\mathbf{t}} - \boldsymbol{\lambda}^h_{\mathbf{t}}\right]_{S_T} \cdot \mathbf{v}^h_{\mathbf{t}} = L(\mathbf{v}^h), \\ \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ -\displaystyle\int_{\Gamma_C} \frac{1}{\gamma_L} \left(\left[\gamma_L u^h_{\mathbf{n}} - \lambda^h_{\mathbf{n}}\right]_+ + \lambda^h_{\mathbf{n}} \right) \mu^h_{\mathbf{t}} = 0, \quad \forall \mu^h_{\mathbf{t}} \in W^h, \\ -\displaystyle\int_{\Gamma_C} \frac{1}{\gamma_L} \left(\left[\gamma_L \mathbf{u}^h_{\mathbf{t}} - \boldsymbol{\lambda}^h_{\mathbf{t}}\right]_{S_T} + \boldsymbol{\lambda}^h_{\mathbf{t}} \right) \cdot \boldsymbol{\mu}^h_{\mathbf{t}} = 0, \quad \forall \boldsymbol{\mu}^h_{\mathbf{t}} \in \mathbf{W}^h_{\mathbf{t}}, \end{cases} \tag{8.22}$$

where γ_L is a discrete positive function on the contact boundary Γ_C that depends on an augmentation parameter, and where the notation $[\cdot]_{S_T}$ has already been introduced for the Nitsche method. As previously, there are at least two possibilities to find such a formulation: starting from duality arguments to get an (continuous) Augmented Lagrangian, or starting from the Nitsche formulation (8.17). Though this formulation is not really new, and one can go back to the seminal paper of P.

Alart and A. Curnier [6], the analysis of this method for Tresca friction has not been carried out to the best of our knowledge. The corresponding (continuous) augmented Lagrangian can be obtained following the same path as in Lemma 7.2 and reads, for $\mathbf{v} \in \mathbf{V}$ and $\boldsymbol{\lambda} \in L^2(\Gamma_C; \mathbb{R}^d)$:

$$\begin{aligned}
&\mathscr{L}_\gamma(\mathbf{v}, \boldsymbol{\lambda}) \\
&= \mathcal{J}(\mathbf{v}) - \int_{\Gamma_C} \lambda v_{\mathbf{n}} - \int_{\Gamma_C} \frac{1}{2\gamma}[\lambda - \gamma v_{\mathbf{n}}]_+^2 + \int_{\Gamma_C} \frac{\gamma}{2} v_{\mathbf{n}}^2 \\
&\quad - \int_{\Gamma_C} \boldsymbol{\lambda}_{\mathbf{t}} \cdot \mathbf{v}_{\mathbf{t}} - \int_{\Gamma_C} \frac{1}{2\gamma} \left|\gamma \mathbf{v}_{\mathbf{t}} - \boldsymbol{\lambda}_{\mathbf{t}} - [\gamma \mathbf{v}_{\mathbf{t}} - \boldsymbol{\lambda}_{\mathbf{t}}]_{sT}\right|^2 + \int_{\Gamma_C} \frac{\gamma}{2} |\mathbf{v}_{\mathbf{t}}|^2 . \quad (8.23)
\end{aligned}$$

8.5 Penalized Frictional Contact

We end this review of approximation methods for Tresca friction with penalty, which is among the simplest techniques. We already have seen in the previous chapter that this method can be recovered from an Uzawa solver, in the frictionless case. This is still the case for Tresca friction. Interestingly, combining the results [79] and [110], we are able now to get an optimal convergence rate for this method (the error estimates in [79] were suboptimal). We present this result in this section. It is valid in two dimensions and is limited to lowest order Lagrange finite elements $\mathbb{P}_1$.

8.5.1 *The Penalty Method*

The penalty method associated to Problem (8.8) of Signorini contact with Tresca friction can be first formulated at a continuous level and reads

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon \in \mathbf{V} \text{ such that:} \\ a(\mathbf{u}_\varepsilon, \mathbf{v}) + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} [u_{\varepsilon,\mathbf{n}}]_+ v_{\mathbf{n}} + \dfrac{1}{\varepsilon} \displaystyle\int_{\Gamma_C} [\mathbf{u}_{\varepsilon,\mathbf{t}}]_{\varepsilon sT} \cdot \mathbf{v}_{\mathbf{t}} = L(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \end{cases} \quad (8.24)$$

where $\varepsilon > 0$ is the (small) penalty parameter. In the above formulation, the exact contact conditions (8.2) are approximated by

$$\sigma_{\mathbf{n}}(\mathbf{u}_\varepsilon) = -\frac{1}{\varepsilon}[u_{\varepsilon,\mathbf{n}}]_+ .$$

In the same way, the exact friction conditions (8.3) are approximated by

$$\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}_\varepsilon) = -\frac{1}{\varepsilon}[\mathbf{u}_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T}.$$

As an important consequence of the above approximations, the resulting discrete penalized formulation is not consistent, and the penalty parameter needs to be chosen small enough to recover frictional contact condition with some accuracy.

We check that, for any positive ε, Problem (8.24) admits a unique solution. It suffices to note that for any $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{V}$ we have $([\mathbf{v_t}]_{\varepsilon s_T} - [\mathbf{w_t}]_{\varepsilon s_T})(\mathbf{v_t} - \mathbf{w_t}) \geq 0$ and $|[\mathbf{v_t}]_{\varepsilon s_T} - [\mathbf{w_t}]_{\varepsilon s_T}| \leq |\mathbf{v_t} - \mathbf{w_t}|$ on Γ_C, and then to use the result of H. Brezis on M-type and pseudo-monotone operators, as in Theorem 6.1. For the detailed proof in the frictionless case $s_T = 0$, the reader can refer to [79, Theorem 2.2] (the extension to Tresca friction makes no extra difficulty).

Now, we choose standard continuous Lagrange finite elements of degree 1:

$$\mathbf{V}^h := \mathbf{X}_1^h \cap \mathbf{V} = \left\{\mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^d) \;\middle|\; \mathbf{v}^h_{|T} \in \mathbb{P}_1(T; \mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D\right\},$$

where $\mathcal{T}^h$ is a simplicial mesh of the domain Ω, and where $\mathbb{P}_1(T; \mathbb{R}^d)$ stands for the space of all vector-valued polynomials on T, of degree lower or equal than 1 in the d variables. We suppose that the mesh $\mathcal{T}^h$ resolves the contact boundary Γ_C, which means that all the edges ($d = 2$) or faces ($d = 3$) that intersect Γ_C are included in $\overline{\Gamma_C}$. We also suppose that it resolves the Neumann boundary Γ_N and the Dirichlet boundary Γ_D.

The finite element discretized problem issued from (8.24) reads

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon^h \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}_\varepsilon^h, \mathbf{v}^h) + \dfrac{1}{\varepsilon}\displaystyle\int_{\Gamma_C} [u^h_{\varepsilon,\mathbf{n}}]_+ v^h_{\mathbf{n}} + \dfrac{1}{\varepsilon}\displaystyle\int_{\Gamma_C} [\mathbf{u}^h_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T^h} \cdot \mathbf{v}^h_{\mathbf{t}} = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{8.25}$$

Using the same argument as for the continuous problem (8.24) proves that Problem (8.25) admits a unique solution.

Remark 8.6 Setting $\gamma_P = 1/\varepsilon$, we can rewrite the above formulation as in Chap. 7:

$$\begin{cases} \text{Find } \mathbf{u}_\gamma^h \in \mathbf{V}^h \text{ such that:} \\ a(\mathbf{u}_\gamma^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} [\gamma_P u^h_{\gamma,\mathbf{n}}]_+ v^h_{\mathbf{n}} + \displaystyle\int_{\Gamma_C} [\gamma_P \mathbf{u}^h_{\gamma,\mathbf{t}}]_{s_T^h} \cdot \mathbf{v}^h_{\mathbf{t}} = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases}$$

In this case, γ_P is closer to a Nitsche or an augmentation parameter and needs to be chosen large enough. Moreover, it can be a function that depends on the local mesh size on the boundary.

8.5.2 Convergence Analysis

The convergence analysis when $\varepsilon \to 0$ is as follows (see [79, Theorem 4.1] for the proof):

Theorem 8.5 *Suppose that* $\mathbf{u} \in \mathbf{V}$, *the solution of Problem* (8.8), *belongs to* $H^s(\Omega;\mathbb{R}^2)$ *with* $s \in (3/2, 2]$. *Let* $\mathbf{u}_\varepsilon \in \mathbf{V}$ *be the solution of Problem* (8.24). *We have the a priori estimates:*

$$\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{1,\Omega} \le C\varepsilon^{s-1}\|\mathbf{u}\|_{s,\Omega},$$

$$\left\|\sigma_{\mathbf{n}}(\mathbf{u}) + \frac{1}{\varepsilon}[u_{\varepsilon,\mathbf{n}}]_+\right\|_{0,\Gamma_C} + \varepsilon^{\frac{3}{2}-s}\left\|\sigma_{\mathbf{n}}(\mathbf{u}) + \frac{1}{\varepsilon}[u_{\varepsilon,\mathbf{n}}]_+\right\|_{\frac{3}{2}-s,\Gamma_C} \le C\varepsilon^{s-\frac{3}{2}}\|\mathbf{u}\|_{s,\Omega},$$

$$\left\|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\varepsilon}[\mathbf{u}_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T}\right\|_{0,\Gamma_C} + \varepsilon^{\frac{3}{2}-s}\left\|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\varepsilon}[\mathbf{u}_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T}\right\|_{\frac{3}{2}-s,\Gamma_C} \le C\varepsilon^{s-\frac{3}{2}}\|\mathbf{u}\|_{s,\Omega},$$

with $C > 0$ *a constant, independent of* ε *and* $\mathbf{u}$.

The next theorem gives the convergence rates for the discrete penalty problem when $\varepsilon = h$.

Theorem 8.6 *Suppose that the solution* $\mathbf{u}$ *of Problem* (8.8) *belongs to* $H^s(\Omega;\mathbb{R}^2)$ *with* $s \in (3/2, 2]$. *Suppose also that the penalty parameter is chosen as* $\varepsilon = h$. *The solution* $\mathbf{u}^h_\varepsilon$ *of the discrete penalty problem* (8.25) *satisfies the following error estimates:*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h_\varepsilon\|_{1,\Omega} &+ h^{\frac{1}{2}}\left\|\sigma_{\mathbf{n}}(\mathbf{u}) + \frac{1}{\varepsilon}[u^h_{\varepsilon,\mathbf{n}}]_+\right\|_{0,\Gamma_C} + h^{2-s}\left\|\sigma_{\mathbf{n}}(\mathbf{u}) + \frac{1}{\varepsilon}[u^h_{\varepsilon,\mathbf{n}}]_+\right\|_{\frac{3}{2}-s,\Gamma_C}\\ &+ h^{\frac{1}{2}}\left\|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\varepsilon}[\mathbf{u}^h_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T}\right\|_{0,\Gamma_C} + h^{2-s}\left\|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\varepsilon}[\mathbf{u}^h_{\varepsilon,\mathbf{t}}]_{\varepsilon s_T}\right\|_{\frac{3}{2}-s,\Gamma_C}\\ &\le Ch^{s-1}\|\mathbf{u}\|_{s,\Omega}\end{aligned} \tag{8.26}$$

with $C > 0$ *a constant, independent of* h *and* $\mathbf{u}$.

Proof The result comes from [79, Theorem 4.2]. The only difference is that we use here the same argument as in Chap. 5 to estimate the integral term on Γ_C:

$$\int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})(\mathcal{I}^h\mathbf{u})_{\mathbf{n}}$$

and this allows to conclude the proof. □

Remark 8.7 The proof in [79] is done for the two-dimensional case but can be extended to the three-dimensional case. Its extension for quadratic finite elements is an open issue.

8.6 Further Comments

Further comments are first about the Tresca model in itself and the obtention of error estimates. We end this section with a few words about the implementation.

8.6.1 Tresca Friction

The Tresca friction model is among the simplest mathematical models for friction, which, notably, allows to recover stick and slip regions that are not known in advance. It has been mentioned already in the monograph of G. Duvaut and J.L. Lions [114] and has been studied for instance in [138] and [181]. Its interest for practical purposes is limited, since it relies on the assumption that the friction threshold is known, and since it allows friction forces to act even if contact is not activated ($\sigma_{\mathbf{n}} = 0$).

8.6.2 First Error Estimates for Tresca

For the Tresca problem, the obtention of optimal error estimates with the weakest additional assumptions in standard numerical approximations with FEM has revealed itself to be a challenging issue (see, e.g., [21, 264]). To our knowledge, the first estimates for this problem have been obtained by J. Haslinger and I. Hlavácek in 1982 [152]: in 2D and for a mixed $\mathbb{P}_1/\mathbb{P}_0$ FEM approximation, with the assumption that the solution has a regularity $H^s(\Omega)$, $s > 1$ (and the normal stress a regularity $L^2(\Gamma_C)$), they obtain a rate $O(h^{\min(\frac{1}{4},s-1)})$ [152, Theorem 4.1]. For a regularity $H^2(\Omega)$ and with the additional assumption that the set of transition points on the contact/friction boundary is finite, they manage to increase this rate to $O(h^{\frac{1}{2}})$ [152, Theorem 4.2] (see also [153]).

8.6.3 Further Error Estimates for Tresca

Later, still in 2D, a rate of $O(h^{\frac{3}{4}})$ for a regularity $H^2(\Omega)$ is obtained by P. Hild in 1997 [163], for a mortar $\mathbb{P}_1$ method. The same rate is given by L. Baillet and T. Sassi in 2002 for a mixed $\mathbb{P}_1/\mathbb{P}_1$ (and a special $\mathbb{P}_1/\mathbb{P}_0$) method, under the same assumption of regularity $H^2(\Omega)$ [20]. The first results in the three-dimensional case, still with a mixed $\mathbb{P}_1/\mathbb{P}_0$ method, are established by J. Haslinger and T. Sassi in 2004 [156]: a rate of $O(h^{\frac{1}{2}})$ is obtained for a regularity $H^2(\Omega)$. This rate was improved to $O(h^{\frac{3}{4}})$ with the additional assumption that the friction threshold is a constant [156, Remark 5.1]. Later, in 2006 [21], a rate of $O(h^{\frac{3}{4}})$ is still mentioned by L. Baillet and T. Sassi for a mixed $\mathbb{P}_1/\mathbb{P}_1$ FEM in the 2D case [21, Theorem 4.4], but this result is improved to the quasi-optimal rate of $O(h|\log(h)|^{\frac{1}{4}})$ with additional regularity assumptions on the tangential displacement, the tangential Lagrange multiplier, and the friction threshold, and, also, with the assumption that the number of transition points on the contact/friction zone is finite [21, Theorem 4.5].

In 2011, the rate of $O(h^{s-1})$, for a regularity $H^s(\Omega)$ $(1 \leq s \leq 2)$, is obtained by B. Wohlmuth in the 2D case for a mixed low-order FEM, under technical assumptions on the contact/friction set [264, Theorem 4.9] (this is the first optimal bound to the best of our knowledge). In the 3D case, the rate $O(h^{\min(\frac{1}{2},s-1)})$ is also given, without additional assumption [264, Theorem 4.10].

In 2013, for the penalty method, and using results from [170], the quasi-optimal rate of $O(h|\log h|^{\frac{1}{2}})$ for a regularity $H^2(\Omega)$ is established by F. Chouly and P. Hild in [79], without additional assumptions on the contact/friction set. As we have seen in this chapter, an optimal rate can be obtained now. Also, following a different path, an analysis of the penalty method for Tresca has been carried out by I. Dione [107]. The author obtained other a priori error estimates involving the penalty parameter ε and the mesh size h. Moreover, assuming an appropriate relationship between ε and h, and for ε small enough, an optimal rate can be established.

For Nitsche, in 2014 [73], it has been proven the optimal convergence of this method in the $H^1(\Omega)$-norm, of order $O(h^{s-1})$ provided the solution has a regularity $H^s(\Omega)$, $3/2 < s \leq 1 + k$ ($k = 1, 2$ is the polynomial degree of the Lagrange finite elements). To this purpose, this does not need any additional assumption on the contact/friction zone, such as an increased regularity of the tangential components (displacement and stress) or a finite number of transition points between contact and noncontact. The proof applies in two-dimensional and three-dimensional cases and for piecewise linear and quadratic finite elements. This result has been reported in this chapter.

New results with optimal rates have been published for the Hybrid High Order method in 2020 [75] and for a stabilized mixed method in 2022 [146].

8.6.4 Implementation

As in the previous chapters, semi-smooth Newton and Uzawa techniques can be used to implement the finite element formulations presented for Tresca. This issue will be treated more in detail in Chap. 10.

Chapter 9
Coulomb Friction

This chapter is about the Coulomb frictional contact model. As we have already seen for Tresca friction, when taking into account friction in addition to the contact model, there are supplementary nonlinearities that have to be taken into account. Remark there exist simplified and/or different models for friction: Tresca friction (see the previous chapter), normal compliance, smoothed Coulomb friction [181, 244], to mention only a few. Coulomb friction generates numerous mathematical difficulties that remain partly unsolved. In this chapter, we mainly consider the so-called static Coulomb friction problem introduced in [113, 114] that roughly speaking corresponds to an incremental problem in the time discretized quasi-static model. For this model, existence of solutions holds when the friction coefficient is small enough, see [115, 116] and the references quoted therein. A recently submitted paper [272] announces that existence holds for any friction coefficient. Besides, the solutions are generally nonunique when the friction coefficient is large enough, see [164, 165]. A first uniqueness result has been obtained in 2006 by Y. Renard [232] with the assumption that a "regular" solution exists and that the friction coefficient is sufficiently small. From a numerical viewpoint, it is well known that the finite element problem, associated with the continuous static Coulomb friction model, always admits a solution and that the solution is unique if the friction coefficient is small enough. Unfortunately, the denomination small depends on the discretization parameter and the bound ensuring uniqueness vanishes as the mesh is refined, see e.g., [153].

The chapter is organized as follows. In Sect. 9.1, we introduce the equations modelling the frictional unilateral contact problem between an elastic body and a rigid foundation. We write in Sect. 9.2 the problem in weak form and reformulate it as a mixed formulation where the unknowns are the displacement field in the body and the frictional contact pressures on the contact area. We also provide some results about the existence, uniqueness, and non-uniqueness of solutions. In Sect. 9.3, we choose a discretization involving continuous finite elements of degree one and continuous piecewise affine multipliers on the contact zone. We

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_9

establish an *a priori* error estimate. Thanks to the uniqueness result of [232], we obtain a bound of the error. A priori error estimates are scarce for contact with Coulomb friction, and among the existing results, a similar bound has been proven in [168], but for a different method. Finally, in Sect. 9.4, we describe how the Nitsche method described in Chap. 6 can be adapted for Coulomb friction. The analysis of this method is still ongoing, and a first result of existence of discrete solutions is mentioned.

9.1 The Frictional Contact Problem in Elasticity

We introduce a setting very similar to the Signorini problem presented in Chap. 3, except that, now, we add a friction law on the contact boundary. As well, for the sake of simplicity, we restrict ourselves to the two-dimensional case. As a result, we consider the deformation of an elastic body occupying, in the initial unconstrained configuration, a domain Ω in $\mathbb{R}^2$, where plane strain assumptions are assumed. The Lipschitz boundary $\partial\Omega$ of Ω consists of Γ_D, Γ_N and Γ_C, where the measure of Γ_D does not vanish. The body Ω is clamped on Γ_D and subjected to surface traction forces $\mathbf{F}$ on Γ_N. The body forces are denoted by $\mathbf{f}$. In the initial configuration, the set Γ_C is a straight line segment considered as the candidate contact boundary on a rigid foundation, for the sake of simplicity. The contact is assumed to be frictional, and the stick, slip, and separation zones on Γ_C are not known in advance. We denote by $\mathscr{F} \geq 0$ the given friction coefficient on Γ_C. The unit outward normal and tangent vectors of $\partial\Omega$ are $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (-n_2, n_1)$, respectively.

The contact problem with Coulomb friction law consists of finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ satisfying (9.1)–(9.6)

$$\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{9.1}$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A} : \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \tag{9.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{9.3}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{F} \quad \text{on } \Gamma_N, \tag{9.4}$$

with the notations already introduced in Chap. 5. For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on the contact boundary Γ_C, we adapt the notations previously provided in Chap. 3 to the two-dimensional setting:

$$\mathbf{v} = v_{\mathbf{n}}\mathbf{n} + v_{\mathbf{t}}\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_{\mathbf{n}}(\mathbf{v})\mathbf{n} + \sigma_{\mathbf{t}}(\mathbf{v})\mathbf{t}.$$

As a result, there holds $v_{\mathbf{t}} = \mathbf{v}_{\mathbf{t}} \cdot \mathbf{t}$ and $\sigma_{\mathbf{t}}(\mathbf{v}) = \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}) \cdot \mathbf{t}$ where the notations $\mathbf{v}_{\mathbf{t}}$ and $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v})$ have been previously introduced in Chap. 3. On the contact boundary Γ_C, the three conditions representing unilateral contact are still given by

$$u_{\mathbf{n}} \leq 0, \qquad \sigma_{\mathbf{n}}(\mathbf{u}) \leq 0, \qquad \sigma_{\mathbf{n}}(\mathbf{u})\, u_{\mathbf{n}} = 0, \tag{9.5}$$

as in Chap. 3, and the Coulomb friction law is summarized by the following conditions:

$$\begin{cases} u_{\mathbf{t}} = 0 \implies |\sigma_{\mathbf{t}}(\mathbf{u})| \leq \mathscr{F}|\sigma_{\mathbf{n}}(\mathbf{u})|, \\ u_{\mathbf{t}} \neq 0 \implies \sigma_{\mathbf{t}}(\mathbf{u}) = -\mathscr{F}|\sigma_{\mathbf{n}}(\mathbf{u})|\dfrac{u_{\mathbf{t}}}{|u_{\mathbf{t}}|}. \end{cases} \tag{9.6}$$

Remark 9.1 A problem where much interest was focused on in the recent years is the so-called quasi-static problem that is an evolution problem. It consists of finding a displacement field

$$\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^2$$

that satisfies equations and conditions (9.1)–(9.5) and in which the friction conditions are as follows on $\Gamma_C \times [0, T]$:

$$\begin{cases} \dot{u}_{\mathbf{t}} = 0 \implies |\sigma_{\mathbf{t}}(\mathbf{u})| \leq -\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u}), \\ \dot{u}_{\mathbf{t}} \neq 0 \implies \sigma_{\mathbf{t}}(\mathbf{u}) = \mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u})\dfrac{\dot{u}_{\mathbf{t}}}{|\dot{u}_{\mathbf{t}}|}, \end{cases} \tag{9.7}$$

where $\dot{u}_{\mathbf{t}}$ denotes the time derivative of the tangential displacement. Moreover, an initial condition for the displacement has to be given on $\Omega : \mathbf{u}_{|t=0} = \mathbf{u}^0$. The existence of a solution has been proved for the quasi-static problem if the friction coefficient is small enough (see [12, 235]).

9.2 Weak Formulation and Existence of Solutions

We provide two weak forms for the frictional contact problems, with a mixed formulation and a quasi-variational inequality. Then we discuss issues related to existence and uniqueness of solutions.

9.2.1 Weak Formulations

We recall from Chap. 3 that $\mathbf{V}$ is the set of admissible displacements and Λ_C the convex cone of weakly nonpositive normal boundary stresses. In agreement with the two-dimensional setting of this chapter, we introduce the following notation for tangential boundary displacements

$$W_{C,\mathbf{t}} := \{w_{\mathbf{t}} = \mathbf{w} \cdot \mathbf{t} \mid \mathbf{w} \in \mathbf{W}_C\}$$

and $W'_{C,\mathbf{t}}$ its topological dual. Note that $\mathbf{W}_{C,\mathbf{t}} = W_{C,\mathbf{t}}\mathbf{t}$. We need to introduce a new notation for the admissible set associated to the tangential Lagrange multiplier, and we define it below, for any $s_C \in \Lambda_C$:

$$\Lambda_{C,\mathbf{t}}(s_C) := \left\{ \mu \in W'_{C,\mathbf{t}} \;\middle|\; \langle \mu, w_\mathbf{t} \rangle_{W'_{C,\mathbf{t}}, W_{C,\mathbf{t}}} \leq -\langle s_C, |w_\mathbf{t}| \rangle_{W'_C, W_C}, \forall w_\mathbf{t} \in W_{C,\mathbf{t}} \right\}.$$

Note that, to alleviate the notations, we changed in this chapter the sign convention related to the threshold s_C, comparatively to the previous chapter on Tresca friction. Now this threshold is supposed weakly nonpositive, which is more convenient for Coulomb friction.

The variational formulation of Problem (9.1)–(9.6) in its mixed form consists of finding a pair $(\mathbf{u}, \boldsymbol{\lambda}) = (\mathbf{u}, \lambda_\mathbf{n}, \lambda_\mathbf{t}) \in \mathbf{V} \times \Lambda_C \times \Lambda_{C,\mathbf{t}}(\mathscr{F}\lambda_\mathbf{n}) =: \mathbf{V} \times \boldsymbol{\Lambda}_C(\mathscr{F}\lambda_\mathbf{n})$, which satisfies

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \boldsymbol{\lambda}) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\[2ex] \qquad b(\mathbf{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \geq 0, & \forall \boldsymbol{\mu} = (\mu_\mathbf{n}, \mu_\mathbf{t}) \in \boldsymbol{\Lambda}_C(\mathscr{F}\lambda_\mathbf{n}). \end{cases} \tag{9.8}$$

In (9.8), the notations previously introduced in Chaps. 3 and 7 are adopted and adapted to the two-dimensional setting:

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}), \qquad L(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{F} \cdot \mathbf{v},$$

$$b(\mathbf{v}, \boldsymbol{\mu}) := \langle \mu_\mathbf{n}, v_\mathbf{n} \rangle_{W'_C, W_C} + \langle \mu_\mathbf{t}, v_\mathbf{t} \rangle_{W'_{C,\mathbf{t}}, W_{C,\mathbf{t}}}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and $\boldsymbol{\mu} = \mu_\mathbf{n}\mathbf{n} + \mu_\mathbf{t}\mathbf{t}$ in $\mathbf{W}'_C$. We can observe that if $(\mathbf{u}, \lambda_\mathbf{n}, \lambda_\mathbf{t})$ is a solution to (9.8), then $\lambda_\mathbf{n} = \sigma_\mathbf{n}(\mathbf{u})$ and $\lambda_\mathbf{t} = \sigma_\mathbf{t}(\mathbf{u})$. To avoid more notation, we will skip over the regularity aspects of the functions defined on Γ_C, and we write afterward integral terms instead of duality pairings. Another classical weak formulation of problem (9.1)–(9.6) is an inequality problem: find a displacement field $\mathbf{u}$ such that

$$\mathbf{u} \in \mathbf{K}, \qquad a(\mathbf{u}, \mathbf{v}-\mathbf{u}) - \mathscr{F} \int_{\Gamma_C} \sigma_\mathbf{n}(\mathbf{u})(|v_\mathbf{t}| - |u_\mathbf{t}|) \geq L(\mathbf{v}-\mathbf{u}), \qquad \forall \mathbf{v} \in \mathbf{K}, \tag{9.9}$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the nonpenetration conditions, already introduced in Chap. 3.

9.2.2 *Existence, Uniqueness, and Non-uniqueness of Solutions*

The existence of a solution to (9.9) has been first proved for small friction coefficients in [215] (in two space dimensions), and the bounds ensuring existence have been improved and generalized in [179] and [116] (see also [115]). More Precisely, existence holds if

$$\mathscr{F} \leq \frac{\sqrt{3-4\nu}}{2-2\nu},$$

where $0 \leq \nu < 1/2$ denotes Poisson ratio. A recently submitted paper [272] announces that existence holds for any friction coefficient. In [164, 165], some multi-solutions of the problem (9.1)–(9.6) are exhibited for triangular or quadrangular domains. These multiple solutions involve either an infinite set of slipping solutions or two isolated (stick and separation) configurations. Note that these examples of non-uniqueness involve large friction coefficients (i.e., $\mathscr{F} > \sqrt{(1-\nu)/\nu}$) and tangential displacements with a constant sign on Γ_C. Actually, it seems that no multi-solution has been detected for an arbitrary small friction coefficient in the continuous case, although such a result exists for finite element approximations in [157], but for a variable geometry.

The forthcoming partial uniqueness result is obtained in [232]: it defines some cases where it is possible to affirm that a solution to the Coulomb friction problem is in fact the unique solution. More precisely, if a "regular" solution to the Coulomb friction problem exists (here the denomination "regular" means, roughly speaking, that the transition is smooth when the slip direction changes), and if the friction coefficient is small enough, then this solution is the only one.

We now introduce the space of multipliers M of the functions ξ defined on Γ_C such that the following norm $\|\xi\|_M$ is finite:

$$\|\xi\|_M := \sup_{v_{\mathbf{t}} \in W_{C,\mathbf{t}} \setminus \{0\}} \frac{\|\xi v_{\mathbf{t}}\|_{W_{C,\mathbf{t}}}}{\|v_{\mathbf{t}}\|_{W_{C,\mathbf{t}}}}.$$

Since Γ_C is assumed to be straight, M contains for any $\varepsilon > 0$ the space $H^{1/2+\varepsilon}(\Gamma_C)$ (see [206] for a complete discussion on the theory of multipliers in a pair of Hilbert spaces). The partial uniqueness result is given assuming that

$$\lambda_{\mathbf{t}} = \mathscr{F} \lambda_{\mathbf{n}} \xi$$

with $\xi \in M$. It is easy to see that it implies $|\xi| \leq 1$ *a.e.* on the support of $\lambda_{\mathbf{n}}$. More precisely, this implies that $\xi \in \mathrm{Dir}_{\mathbf{t}}(u_{\mathbf{t}})$ *a.e.* on the support of $\lambda_{\mathbf{n}}$, where $\mathrm{Dir}_{\mathbf{t}}(\cdot)$ is the subdifferential of the convex map $x_{\mathbf{t}} \longmapsto |x_{\mathbf{t}}|$. This means that it is possible to assume that $\xi \in \mathrm{Dir}_{\mathbf{t}}(u_{\mathbf{t}})$ *a.e.* on Γ_C.

Proposition 9.1 ([232]) *Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *be a solution to Problem (9.8) such that*

$$\lambda_{\mathbf{t}} = \mathscr{F}\lambda_{\mathbf{n}}\xi$$

with $\xi \in M$, $\xi \in Dir_{\mathbf{t}}(u_{\mathbf{t}})$ *a.e. on* Γ_C, *and*

$$\mathscr{F} < C_0\|\xi\|_M^{-1},$$

where $C_0 > 0$ *is independent of* ξ. *Then,* $(\mathbf{u}, \boldsymbol{\lambda})$ *is the unique solution to Problem (9.8).*

The case $\xi \equiv 1$ corresponds to an homogeneous sliding direction, and the previous result is complementary with the non-uniqueness results obtained in [164, 165]. The multiplier ξ has to vary from -1 to $+1$ each time the sign of the tangential displacement changes from negative to positive. The set M does not contain any multiplier having a discontinuity of the first kind. Consequently, in order to satisfy the assumptions of Proposition 9.1, the tangential displacement of the solution $\mathbf{u}$ cannot pass from a negative value to a positive value and being zero only at a single point of Γ_C. For a more precise discussion concerning the assumption $\lambda_{\mathbf{t}} = \mathscr{F}\lambda_{\mathbf{n}}\xi$, $\xi \in M$, $\xi \in \mathrm{Dir}_{\mathbf{t}}(u_{\mathbf{t}})$, and the cases where the assumption cannot be fulfilled independently of the regularity of the solution, we refer the reader to [168, Remark 2].

9.3 Mixed Finite Element Approximation

We introduce first the setting for a mixed finite element approximation to frictional contact. Then, we study the existence and uniqueness of solutions to this discrete problem. We end this section with an *a priori* error estimate.

9.3.1 Setting

We approximate Problem (9.8) with Lagrange finite elements and follow the setting of Chap. 4. We recall that $\mathbf{V}^h := \mathbf{V}_1^h$ is a space of continuous piecewise linear Lagrange finite elements, built from a simplicial mesh $\mathcal{T}^h$ of the domain Ω. The corresponding family of meshes is supposed to be regular in Ciarlet's sense. We recall that the contact area is a straight line segment to simplify. The extension to a contact area that is a broken line can be made without additional technical difficulties (see, e.g., [167]). As in the previous Chap. 7, in order to express the contact constraints by using Lagrange multipliers on the contact zone, we have to introduce the range of $\mathbf{V}^h$ by the normal trace operator on Γ_C:

$$W^h := \left\{ \mu^h \in \mathscr{C}(\overline{\Gamma_C}) \;\middle|\; \exists \mathbf{v}^h \in \mathbf{V}^h \text{ s.t. } v^h_{\mathbf{n}} = \mu^h \text{ on } \Gamma_C \right\},$$

which coincides with the range of $\mathbf{V}^h$ by the tangent trace operator on Γ_C. We also set $\mathbf{W}^h := W^h \times W^h (\subset \mathbf{W}_C)$. The choice of the space W^h allows us to define the following closed convex cones:

$$M^h_{\mathbf{n}} := \left\{ \mu^h \in W^h \;\middle|\; \int_{\Gamma_C} \mu^h \psi^h \geq 0, \;\forall \psi^h \in W^h, \psi^h \leq 0 \right\},$$

and, for $s^h_C \in M^h_{\mathbf{n}}$:

$$M^h_{\mathbf{t}}(s^h_C) := \left\{ \mu^h \in W^h \;\middle|\; \left| \int_{\Gamma_C} \mu^h \psi^h \right| \leq \int_{\Gamma_C} s^h_C \psi^h \;\forall \psi^h \in W^h, \psi^h \leq 0 \right\}.$$

Remark 9.2 One can check that the functions in $M^h_{\mathbf{n}}$ are not necessarily nonpositive. In the same way, the functions in $M^h_{\mathbf{t}}(s^h_C)$ do not satisfy $|\mu^h| \leq -s^h_C$ everywhere.

The discretized mixed formulation of the frictional contact problem is to find $\mathbf{u}^h \in \mathbf{V}^h$ and $\boldsymbol{\lambda}^h \in \mathbf{M}^h(\mathscr{F}\lambda^h_{\mathbf{n}}) := M^h_{\mathbf{n}} \times M^h_{\mathbf{t}}(\mathscr{F}\lambda^h_{\mathbf{n}})$ satisfying

$$\begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \boldsymbol{\lambda}^h) = L(\mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mathbf{u}^h, \boldsymbol{\mu}^h - \boldsymbol{\lambda}^h) \geq 0, & \forall\, \boldsymbol{\mu}^h = (\mu^h_{\mathbf{n}}, \mu^h_{\mathbf{t}}) \in \mathbf{M}^h(\mathscr{F}\lambda^h_n). \end{cases} \tag{9.10}$$

The following result proved in [94] gives explicitly the discrete frictional contact conditions.

Proposition 9.2 *Let $(\mathbf{u}^h, \boldsymbol{\lambda}^h)$ be a solution to (9.10). Suppose that $dim(W^h) = p$ and let*

$$\psi_{\mathbf{x}_i}, \; 1 \leq i \leq p$$

denote the basis functions of W^h on Γ_C. The p-by-p mass matrix $\mathcal{M} = (m_{ij})_{1 \leq i,j \leq p}$ on Γ_C is given by $m_{ij} = \int_{\Gamma_C} \psi_{\mathbf{x}_i} \psi_{\mathbf{x}_j}$. Let U_N and U_T denote the vectors of components the nodal values of $u^h_{\mathbf{n}}$ and $u^h_{\mathbf{t}}$, respectively, and let L_N and L_T denote the vectors of components the nodal values of $\lambda^h_{\mathbf{n}}$ and $\lambda^h_{\mathbf{t}}$, respectively. Then, the discrete frictional contact conditions in (9.10) are as follows; for any $1 \leq i \leq p$:

$$(\mathcal{M}L_N)_i \leq 0, \qquad (U_N)_i \leq 0, \qquad (\mathcal{M}L_N)_i (U_N)_i = 0, \tag{9.11}$$

$$|(\mathcal{M}L_T)_i| \leq -\mathscr{F}(\mathcal{M}L_N)_i, \tag{9.12}$$

$$|(\mathcal{M}L_T)_i| < -\mathscr{F}(\mathcal{M}L_N)_i \implies (U_T)_i = 0, \tag{9.13}$$

$$(\mathcal{M}L_T)_i (U_T)_i \leq 0. \tag{9.14}$$

Remark 9.3 As for frictionless contact, many other choices can be made for the dual sets of Lagrange multipliers: see for instance [168] or [264] and references therein.

9.3.2 Existence and Uniqueness

To study the well-posedness of the mixed formulation (9.10), we will make use of fixed point theorems. The fixed point mapping is built with the help of an auxiliary problem based on Tresca friction. So let $\mathscr{F} > 0$ be given. We introduce the problem of friction $\mathcal{P}(s_C^h)$ with a given threshold $\mathscr{F} s_C^h$, for an arbitrary $s_C^h \in M_{\mathbf{n}}^h$. It consists of finding $\mathbf{u}^h \in \mathbf{V}^h$ and $\boldsymbol{\lambda}^h \in \mathbf{M}^h(\mathscr{F} s_C^h) := M_{\mathbf{n}}^h \times M_{\mathbf{t}}^h(\mathscr{F} s_C^h)$ satisfying:

$$\begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, \boldsymbol{\lambda}^h) = L(\mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \quad b(\mathbf{u}^h, \boldsymbol{\mu}^h - \boldsymbol{\lambda}^h) \geq 0, & \forall\, \boldsymbol{\mu}^h = (\mu_{\mathbf{n}}^h, \mu_{\mathbf{t}}^h) \in \mathbf{M}^h(\mathscr{F} s_C^h). \end{cases} \tag{9.15}$$

Problem (9.15) is identical to the discrete mixed problem (8.21) introduced in the previous chapter about Tresca friction, with a slight change of notations ($s_T^h = -\mathscr{F} s_C^h$). It is well-posed according to Proposition 8.3. The next lemma is a straightforward consequence of the definition of problems (9.10) and (9.15).

Lemma 9.1 *The solutions of Coulomb discrete frictional contact problem (9.10) are the solutions of $\mathcal{P}(\widetilde{\lambda}_{\mathbf{n}}^h)$, where $\widetilde{\lambda}_{\mathbf{n}}^h$ is a fixed point of Φ^h. The functional Φ^h is defined as follows:*

$$\begin{aligned} \Phi^h : M_{\mathbf{n}}^h &\longrightarrow M_{\mathbf{n}}^h \\ s_C^h &\longmapsto \lambda_{\mathbf{n}}^h, \end{aligned}$$

where $(\mathbf{u}^h, \boldsymbol{\lambda}^h)$ is the solution to $\mathcal{P}(s_C^h)$.

The next theorem deals with an existence and uniqueness result for the discrete problem (9.10).

Theorem 9.1 *For any positive friction coefficient $\mathscr{F}$, Problem (9.10) admits at least a solution. Moreover, there is a constant C such that, if*

$$\mathscr{F} \leq C h^{1/2},$$

then the solution is unique.

Proof To establish existence of a fixed point of Φ^h, we use Brouwer fixed point theorem [115, 193]. First, we prove that the mapping Φ^h is continuous. Set

$$\widetilde{\mathbf{V}}^h := \left\{ \mathbf{v}^h \in \mathbf{V}^h \;\middle|\; v_{\mathbf{t}}^h = 0 \text{ on } \Gamma_C \right\}.$$

From the definition of W^h, we check that the definition of $\|\cdot\|_{-\frac{1}{2},h}$ given by

$$\|\mu\|_{-\frac{1}{2},h} = \sup_{\mathbf{v}^h \in \widetilde{\mathbf{V}}^h} \frac{\displaystyle\int_{\Gamma_C} \mu v_{\mathbf{n}}^h}{\|\mathbf{v}^h\|_{1,\Omega}}$$

is a norm on W^h. Take s_C^h and $\overline{s_C^h}$ two functions in $M_{\mathbf{n}}^h$. Let $(\mathbf{u}^h, \lambda_{\mathbf{n}}^h, \lambda_{\mathbf{t}}^h)$ and $(\overline{\mathbf{u}^h}, \overline{\lambda_{\mathbf{n}}^h}, \overline{\lambda_{\mathbf{t}}^h})$ be the solutions to $\mathcal{P}(s_C^h)$ and $\mathcal{P}(\overline{s_C^h})$, respectively. On the one hand, we get

$$a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \lambda_{\mathbf{n}}^h v_{\mathbf{n}}^h = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \widetilde{\mathbf{V}}^h,$$

$$a(\overline{\mathbf{u}^h}, \mathbf{v}^h) - \int_{\Gamma_C} \overline{\lambda_{\mathbf{n}}^h} v_{\mathbf{n}}^h = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \widetilde{\mathbf{V}}^h.$$

Subtracting the previous equalities and using the continuity of the bilinear form $a(\cdot,\cdot)$ give

$$\int_{\Gamma_C} (\lambda_{\mathbf{n}}^h - \overline{\lambda_{\mathbf{n}}^h}) v_{\mathbf{n}}^h = a(\mathbf{u}^h - \overline{\mathbf{u}^h}, \mathbf{v}^h) \le C\|\mathbf{u}^h - \overline{\mathbf{u}^h}\|_{1,\Omega}\|\mathbf{v}^h\|_{1,\Omega} \qquad \forall \mathbf{v}^h \in \widetilde{\mathbf{V}}^h.$$

Hence, we get a first estimate

$$\|\lambda_{\mathbf{n}}^h - \overline{\lambda_{\mathbf{n}}^h}\|_{-\frac{1}{2},h} \le C\|\mathbf{u}^h - \overline{\mathbf{u}^h}\|_{1,\Omega}. \tag{9.16}$$

On the other hand, we have from (9.10)

$$a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \lambda_{\mathbf{n}}^h v_{\mathbf{n}}^h - \int_{\Gamma_C} \lambda_{\mathbf{t}}^h v_{\mathbf{t}}^h = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \mathbf{V}^h, \tag{9.17}$$

$$a(\overline{\mathbf{u}^h}, \mathbf{v}^h) - \int_{\Gamma_C} \overline{\lambda_{\mathbf{n}}^h} v_{\mathbf{n}}^h - \int_{\Gamma_C} \overline{\lambda_{\mathbf{t}}^h} v_{\mathbf{t}}^h = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \mathbf{V}^h. \tag{9.18}$$

Choosing $\mathbf{v}^h = \mathbf{u}^h - \overline{\mathbf{u}^h}$ in (9.17) and $\mathbf{v}^h = \overline{\mathbf{u}^h} - \mathbf{u}^h$ in (9.18) implies by addition:

$$\begin{aligned} a(\mathbf{u}^h - \overline{\mathbf{u}^h}, \mathbf{u}^h - \overline{\mathbf{u}^h}) = &\int_{\Gamma_C} (\lambda_{\mathbf{n}}^h - \overline{\lambda_{\mathbf{n}}^h})(u_{\mathbf{n}}^h - \overline{u_{\mathbf{n}}^h}) \\ &+ \int_{\Gamma_C} (\lambda_{\mathbf{t}}^h - \overline{\lambda_{\mathbf{t}}^h})(u_{\mathbf{t}}^h - \overline{u_{\mathbf{t}}^h}). \end{aligned} \tag{9.19}$$

Let us notice that the inequality in (9.15) is equivalent to the two following conditions:

$$\int_{\Gamma_C} (\mu_{\mathbf{n}}^h - \lambda_{\mathbf{n}}^h) u_{\mathbf{n}}^h \geq 0, \qquad \forall \mu_{\mathbf{n}}^h \in M_{\mathbf{n}}^h, \tag{9.20}$$

$$\int_{\Gamma_C} (\mu_{\mathbf{t}}^h - \lambda_{\mathbf{t}}^h) u_{\mathbf{t}}^h \geq 0, \qquad \forall \mu_{\mathbf{t}}^h \in M_{\mathbf{t}}^h(\mathscr{F} s_C^h). \tag{9.21}$$

According to the definition of $M_{\mathbf{n}}^h$, we can choose $\mu_{\mathbf{n}}^h = 0$ and $\mu_{\mathbf{n}}^h = 2\lambda_{\mathbf{n}}^h$ in (9.20), which gives

$$\int_{\Gamma_C} \lambda_{\mathbf{n}}^h u_{\mathbf{n}}^h = 0 \quad \text{and} \quad \int_{\Gamma_C} \mu_{\mathbf{n}}^h u_{\mathbf{n}}^h \geq 0, \qquad \forall \mu_{\mathbf{n}}^h \in M_{\mathbf{n}}^h,$$

and from which we deduce that

$$\int_{\Gamma_C} (\lambda_{\mathbf{n}}^h - \overline{\lambda_{\mathbf{n}}^h})(u_{\mathbf{n}}^h - \overline{u_{\mathbf{n}}^h}) \leq 0.$$

Hence, (9.19) becomes

$$\|\mathbf{u}^h - \overline{\mathbf{u}^h}\|_{1,\Omega}^2 \leq C \int_{\Gamma_C} (\lambda_{\mathbf{t}}^h - \overline{\lambda_{\mathbf{t}}^h})(u_{\mathbf{t}}^h - \overline{u_{\mathbf{t}}^h}). \tag{9.22}$$

From the definition of $M_{\mathbf{t}}^h(\mathscr{F} s_C^h)$ and according to the notations of Proposition 9.2, we get

$$\begin{aligned} &- \int_{\Gamma_C} \lambda_{\mathbf{t}}^h \overline{u_{\mathbf{t}}^h} \\ &= - \sum_{i=1}^{p} (\mathcal{M}L_T)_i (\overline{U_T})_i \leq \sum_{i=1}^{p} -\mathscr{F}(\mathcal{M}L_N)_i |(\overline{U_T})_i| = -\mathscr{F} \int_{\Gamma_C} s_C^h \mathcal{I}^h \left(\left| \overline{u_{\mathbf{t}}^h} \right| \right). \end{aligned}$$

A similar expression can be obtained when integrating the term $\overline{\lambda_{\mathbf{t}}^h} u_{\mathbf{t}}^h$. Besides, from (9.21):

$$\begin{aligned} \int_{\Gamma_C} \lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h &\leq \int_{\Gamma_C} \mu_{\mathbf{t}}^h u_{\mathbf{t}}^h \\ &= \sum_{i=1}^{p} (\mathcal{M}N_T)_i (U_T)_i \quad \forall N_T \text{ such that } |(\mathcal{M}N_T)_i| \leq -\mathscr{F}(\mathcal{M}L_N)_i. \end{aligned}$$

If $(U_T)_i \geq 0$, we choose $(\mathcal{M}N_T)_i = -\mathscr{F}(\mathcal{M}L_N)_i$, and if $(U_T)_i \leq 0$, we choose $(\mathcal{M}N_T)_i = \mathscr{F}(\mathcal{M}L_N)_i$. This yields the following bound:

$$\int_{\Gamma_C} \lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h \leq \mathscr{F} \sum_{i=1}^{p} (\mathcal{M} L_N)_i |(U_T)_i| = \mathscr{F} \int_{\Gamma_C} s_C^h \mathcal{I}^h \left(\left| u_{\mathbf{t}}^h \right| \right).$$

A similar expression can be obtained when integrating the term $\overline{\lambda_{\mathbf{t}}^h}\, \overline{u_{\mathbf{t}}^h}$. Finally, (9.22) becomes

$$\|\mathbf{u}^h - \overline{\mathbf{u}^h}\|_{1,\Omega}^2 \leq C\mathscr{F} \int_{\Gamma_C} \left(s_C^h - \overline{s_C^h} \right) \mathcal{I}^h \left(\left| u_{\mathbf{t}}^h \right| - \left| \overline{u_{\mathbf{t}}^h} \right| \right)$$
$$\leq C\mathscr{F} \left\| s_C^h - \overline{s_C^h} \right\|_{0,\Gamma_C} \left\| \mathcal{I}^h \left(\left| u_{\mathbf{t}}^h \right| - \left| \overline{u_{\mathbf{t}}^h} \right| \right) \right\|_{0,\Gamma_C}. \quad (9.23)$$

The first term is bounded using an inverse inequality (see Proposition 4.3) and the discrete inf-sup condition:

$$\left\| s_C^h - \overline{s_C^h} \right\|_{0,\Gamma_C} \leq C h^{-\frac{1}{2}} \left\| s_C^h - \overline{s_C^h} \right\|_{-\frac{1}{2},\Gamma_C} \leq C h^{-\frac{1}{2}} \left\| s_C^h - \overline{s_C^h} \right\|_{-\frac{1}{2},h}. \quad (9.24)$$

The second term is bounded using elementary calculus, which gives

$$\left\| \mathcal{I}^h \left(\left| u_{\mathbf{t}}^h \right| - \left| \overline{u_{\mathbf{t}}^h} \right| \right) \right\|_{0,\Gamma_C} = \left\| \mathcal{I}^h \left(\left| \left| u_{\mathbf{t}}^h \right| - \left| \overline{u_{\mathbf{t}}^h} \right| \right| \right) \right\|_{0,\Gamma_C} \leq \left\| \mathcal{I}^h \left(\left| u_{\mathbf{t}}^h - \overline{u_{\mathbf{t}}^h} \right| \right) \right\|_{0,\Gamma_C}$$
$$\leq 2\sqrt{2} \left\| \left| u_{\mathbf{t}}^h - \overline{u_{\mathbf{t}}^h} \right| \right\|_{0,\Gamma_C}$$
$$\leq C \left\| \mathbf{u}^h - \overline{\mathbf{u}^h} \right\|_{1,\Omega}. \quad (9.25)$$

The bound with $2\sqrt{2}$ (the constant is maybe non-optimal) is obtained by bounding on any element T the quantity $\|\mathcal{I}^h(|u_{\mathbf{t}}^h - \overline{u_{\mathbf{t}}^h}|)\|_{0,T}$ and using a decomposition on the basis functions. So (9.23), (9.16), (9.24), and (9.25) imply

$$\left\| \lambda_{\mathbf{n}}^h - \overline{\lambda_{\mathbf{n}}^h} \right\|_{-\frac{1}{2},h} \leq C\mathscr{F} h^{-1/2} \left\| s_C^h - \overline{s_C^h} \right\|_{-\frac{1}{2},h}. \quad (9.26)$$

Hence, Φ^h is continuous.

Let $(\mathbf{u}^h, \lambda_{\mathbf{n}}^h, \lambda_{\mathbf{t}}^h)$ be the solution to $P(s_C^h)$. Taking $\mathbf{v}^h = \mathbf{u}^h$ in (9.15) gives

$$a(\mathbf{u}^h, \mathbf{u}^h) - \int_{\Gamma_C} \lambda_{\mathbf{n}}^h u_{\mathbf{n}}^h - \int_{\Gamma_C} \lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h = L(\mathbf{u}^h). \quad (9.27)$$

According to

$$\int_{\Gamma_C} \lambda_{\mathbf{n}}^h u_{\mathbf{n}}^h = 0 \quad \text{and} \quad \int_{\Gamma_C} \lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h \leq 0,$$

we deduce from (9.27), the $\mathbf{V}$-ellipticity of $a(\cdot,\cdot)$ and the continuity of $L(\cdot)$:

$$\|\mathbf{u}^h\|_{1,\Omega}^2 \le Ca(\mathbf{u}^h,\mathbf{u}^h) \le CL(\mathbf{u}^h) \le C\|\mathbf{u}^h\|_{1,\Omega}.$$

So we deduce that $\|\mathbf{u}^h\|_{1,\Omega}$ is bounded. In other respects,

$$a(\mathbf{u}^h,\mathbf{v}^h) - \int_{\Gamma_C} \lambda_{\mathbf{n}}^h v_{\mathbf{n}}^h = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \widetilde{\mathbf{V}}^h,$$

leads to

$$c\int_{\Gamma_C} \lambda_{\mathbf{n}}^h v_{\mathbf{n}}^h \le \|\mathbf{u}^h\|_{1,\Omega}\|\mathbf{v}^h\|_{1,\Omega} + \|\mathbf{v}^h\|_{1,\Omega}, \qquad \forall \mathbf{v}^h \in \widetilde{\mathbf{V}}^h.$$

Therefore,

$$\|\Phi^h(s_C^h)\|_{-\frac{1}{2},h} = \|\lambda_{\mathbf{n}}^h\|_{-\frac{1}{2},h} \le C\left(\|\mathbf{u}^h\|_{1,\Omega}+1\right) \le C,$$

for all $s_C^h \in M_{\mathbf{n}}^h$. This boundedness of Φ^h together with the continuity of Φ^h proves that there exists at least a solution of Coulomb discrete frictional contact problem according to Brouwer fixed point theorem. The uniqueness result follows from (9.26). □

9.3.3 An a priori Error Estimate

We prove the following *a priori* error bound for the solution to the discrete mixed problem that approximates contact with friction:

Theorem 9.2 *Let* $(\mathbf{u},\boldsymbol{\lambda})$ *be the solution to Problem (9.8) such that*

$$\lambda_{\mathbf{t}} = \mathscr{F}\lambda_{\mathbf{n}}\xi, \tag{9.28}$$

$$\xi \in H^{\frac{1}{2}+\varepsilon}(\Gamma_C),\ \varepsilon>0, \quad \xi \in Dir_{\mathbf{t}}(u_{\mathbf{t}}) \text{ a.e. on } \Gamma_C, \quad \mathscr{F} < C_1\|\xi\|^{-1}_{\frac{1}{2}+\varepsilon,\Gamma_C}, \tag{9.29}$$

with $C_1 < C_0$ *small enough. Assume that* $\mathbf{u} \in H^s(\Omega;\mathbb{R}^2)$*, with* $3/2 < s \le 2$*, and let* $(\mathbf{u}^h,\boldsymbol{\lambda}^h)$ *be a solution to the discrete problem (9.10). Then, there exists a constant* $C>0$ *independent of h and* $\mathbf{u}$ *such that*

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \le Ch^{\frac{1}{2}}\|\mathbf{u}\|_{s,\Omega}. \tag{9.30}$$

Proof Let $\mathbf{v}^h \in \mathbf{V}^h$ be an arbitrary test function. First, we use the expressions in Problem (9.8) and Problem (9.10) to compute

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h) &= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \\ &= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + \int_{\Gamma_C} \lambda_{\mathbf{n}}(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h) + \int_{\Gamma_C} \lambda_{\mathbf{t}}(v_{\mathbf{t}}^h - u_{\mathbf{t}}^h) \\ &\quad - \int_{\Gamma_C} \lambda_{\mathbf{n}}^h(v_{\mathbf{n}}^h - u_{\mathbf{n}}^h) - \int_{\Gamma_C} \lambda_{\mathbf{t}}^h(v_{\mathbf{t}}^h - u_{\mathbf{t}}^h) \\ &= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) \\ &\quad + \int_{\Gamma_C} (\lambda_{\mathbf{n}} - \lambda_{\mathbf{n}}^h)(v_{\mathbf{n}}^h - u_{\mathbf{n}}) + \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda_{\mathbf{t}}^h)(v_{\mathbf{t}}^h - u_{\mathbf{t}}) \\ &\quad + \int_{\Gamma_C} (\lambda_{\mathbf{n}} - \lambda_{\mathbf{n}}^h)(u_{\mathbf{n}} - u_{\mathbf{n}}^h) + \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda_{\mathbf{t}}^h)(u_{\mathbf{t}} - u_{\mathbf{t}}^h). \end{aligned} \tag{9.31}$$

The continuous and discrete complementary conditions in (9.5) and (9.11), respectively, imply that

$$\int_{\Gamma_C} \lambda_{\mathbf{n}} u_{\mathbf{n}} = \int_{\Gamma_C} \lambda_{\mathbf{n}}^h u_{\mathbf{n}}^h = 0.$$

Hence, (9.31) becomes

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h) &= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) \\ &\quad + \int_{\Gamma_C} (\lambda_{\mathbf{n}} - \lambda_{\mathbf{n}}^h)(v_{\mathbf{n}}^h - u_{\mathbf{n}}) + \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda_{\mathbf{t}}^h)(v_{\mathbf{t}}^h - u_{\mathbf{t}}) \\ &\quad - \int_{\Gamma_C} \left(\lambda_{\mathbf{n}} u_{\mathbf{n}}^h + \lambda_{\mathbf{n}}^h u_{\mathbf{n}}\right) + \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda_{\mathbf{t}}^h)(u_{\mathbf{t}} - u_{\mathbf{t}}^h). \end{aligned}$$

Using the ellipticity and the continuity of the bilinear form $a(\cdot, \cdot)$, we obtain

$$\begin{aligned} c\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \le\ & C\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C}\|\mathbf{u}-\mathbf{v}^h\|_{\frac{1}{2},\Gamma_C} \\ & \underbrace{- \int_{\Gamma_C} \left(\lambda_{\mathbf{n}} u_{\mathbf{n}}^h + \lambda_{\mathbf{n}}^h u_{\mathbf{n}}\right)}_{\mathcal{T}_C} + \underbrace{\int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda_{\mathbf{t}}^h)(u_{\mathbf{t}} - u_{\mathbf{t}}^h)}_{\mathcal{T}_F}. \end{aligned} \tag{9.32}$$

Besides, we consider the equilibrium equation. From (9.1)–(9.6), the Green formula (Lemma 3.1), and the inclusion $\mathbf{V}^h \subset \mathbf{V}$, we get first

$$a(\mathbf{u}, \mathbf{v}^h) = L(\mathbf{v}^h) + \int_{\Gamma_C} \boldsymbol{\lambda} \cdot \mathbf{v}^h \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Since there also holds for the discrete solution

$$a(\mathbf{u}^h, \mathbf{v}^h) = L(\mathbf{v}^h) + \int_{\Gamma_C} \boldsymbol{\lambda}^h \cdot \mathbf{v}^h \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h,$$

we deduce by subtraction that

$$a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) = \int_{\Gamma_C} (\boldsymbol{\lambda} - \boldsymbol{\lambda}^h) \cdot \mathbf{v}^h \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Consequently, for any $\mathbf{v}^h \in \mathbf{V}^h$ and any $\boldsymbol{\mu}^h \in \mathbf{W}^h$, we use once again the continuity of the bilinear form $a(\cdot,\cdot)$ combined with the Trace Theorem 2.8 and get

$$\begin{aligned}\int_{\Gamma_C} (\boldsymbol{\lambda}^h - \boldsymbol{\mu}^h) \cdot \mathbf{v}^h &= \int_{\Gamma_C} (\boldsymbol{\lambda}^h - \boldsymbol{\lambda}) \cdot \mathbf{v}^h + \int_{\Gamma_C} (\boldsymbol{\lambda} - \boldsymbol{\mu}^h) \cdot \mathbf{v}^h \\ &= a(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + \int_{\Gamma_C} (\boldsymbol{\lambda} - \boldsymbol{\mu}^h) \cdot \mathbf{v}^h \\ &\le C \left(\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\boldsymbol{\lambda} - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} \right) \|\mathbf{v}^h\|_{1,\Omega}.\end{aligned}$$

The mesh independent inf-sup condition (7.13) (see Chap. 7) implies that, for any $\boldsymbol{\mu}^h \in \mathbf{W}^h$:

$$\begin{aligned}c\|\boldsymbol{\lambda}^h - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} &\le \sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{\displaystyle\int_{\Gamma_C} (\boldsymbol{\lambda}^h - \boldsymbol{\mu}^h) \cdot \mathbf{v}^h}{\|\mathbf{v}^h\|_{1,\Omega}} \\ &\le C \left(\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\boldsymbol{\lambda} - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} \right).\end{aligned}$$

By the triangular inequality

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \le \|\boldsymbol{\lambda} - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} + \|\boldsymbol{\mu}^h - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C},$$

we come to the conclusion that

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \le C \left(\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \inf_{\boldsymbol{\mu}^h \in \mathbf{W}^h} \|\boldsymbol{\lambda} - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} \right). \tag{9.33}$$

Keeping in mind that $\mathbf{u} \in H^s(\Omega; \mathbb{R}^2)$ with $s > \frac{3}{2}$, we choose $\mathbf{v}^h = \mathcal{I}^h \mathbf{u}$, where $\mathcal{I}^h$ denotes the Lagrange interpolation operator mapping onto $\mathbf{V}^h$. From Theorem 4.1 in Chap. 4, if $0 < s - 3/2 \le 1/2$, there holds

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} \le C h^{s-1} \|\mathbf{u}\|_{s,\Omega}. \tag{9.34}$$

The Trace Theorem 2.8 ensures that $\boldsymbol{\lambda} \in L^2(\Gamma_C; \mathbb{R}^2)$. Then, we can take $\boldsymbol{\mu}^h = \pi^h \boldsymbol{\lambda}$, where π^h represents the L^2-projection operator mapping onto $\mathbf{W}^h$. We get from Theorem 4.5:

$$\inf_{\boldsymbol{\mu}^h \in \mathbf{W}^h} \|\boldsymbol{\lambda} - \boldsymbol{\mu}^h\|_{-\frac{1}{2},\Gamma_C} \leq C h^{s-1} \|\mathbf{u}\|_{s,\Omega}. \tag{9.35}$$

As a result, (9.33) becomes

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \leq C \left(\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + h^{s-1} \|\mathbf{u}\|_{s,\Omega} \right). \tag{9.36}$$

We now estimate the two terms $\mathcal{T}_C$ in (9.32) coming from the contact approximation. As in Theorem 7.1 of Chap. 7, the first term is nonpositive, since $\lambda_{\mathbf{n}} \leq 0$, and because of the relationship (9.11) in Proposition 9.2:

$$-\int_{\Gamma_C} \lambda_{\mathbf{n}} u^h_{\mathbf{n}} \leq 0. \tag{9.37}$$

In order to estimate the second term of $\mathcal{T}_C$ in (9.32) coming from the contact approximation, we proceed exactly as in the proof of Theorem 7.1 in Chap. 7, where the same term appears. We get

$$-\int_{\Gamma_C} \lambda^h_{\mathbf{n}} u_{\mathbf{n}} \leq C \left(h^{s-1} \|\lambda_{\mathbf{n}} - \lambda^h_{\mathbf{n}}\|_{-\frac{1}{2},\Gamma_C} + h^{2(s-1)} \right). \tag{9.38}$$

We now estimate the terms $\mathcal{T}_F$ corresponding to the friction approximation in (9.32). First, we add and subtract the term $\mathscr{F} \lambda^h_{\mathbf{n}} \xi$ and then use the assumption (9.28) in the statement of the theorem to write:

$$\begin{aligned} \mathcal{T}_F &= \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \lambda^h_{\mathbf{t}})(u_{\mathbf{t}} - u^h_{\mathbf{t}}) \\ &= \int_{\Gamma_C} (\lambda_{\mathbf{t}} - \mathscr{F} \lambda^h_{\mathbf{n}} \xi)(u_{\mathbf{t}} - u^h_{\mathbf{t}}) + \int_{\Gamma_C} (\mathscr{F} \lambda^h_{\mathbf{n}} \xi - \lambda^h_{\mathbf{t}})(u_{\mathbf{t}} - u^h_{\mathbf{t}}) \\ &= \int_{\Gamma_C} \mathscr{F} (\lambda_{\mathbf{n}} - \lambda^h_{\mathbf{n}}) \xi (u_{\mathbf{t}} - u^h_{\mathbf{t}}) + \int_{\Gamma_C} (\mathscr{F} \lambda^h_{\mathbf{n}} \xi - \lambda^h_{\mathbf{t}})(u_{\mathbf{t}} - u^h_{\mathbf{t}}). \end{aligned} \tag{9.39}$$

The estimate of the first integral term in (9.39) gives, after application of the Trace Theorem 2.8,

$$\int_{\Gamma_C} \mathscr{F} (\lambda_{\mathbf{n}} - \lambda^h_{\mathbf{n}}) \xi (u_{\mathbf{t}} - u^h_{\mathbf{t}}) \leq C \mathscr{F} \|\xi\|_M \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}. \tag{9.40}$$

The second integral term in (9.39) is written as follows:

$$\int_{\Gamma_C}(\mathscr{F}\lambda_{\mathbf{n}}^h\xi-\lambda_{\mathbf{t}}^h)(u_{\mathbf{t}}-u_{\mathbf{t}}^h)=\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\xi(u_{\mathbf{t}}-u_{\mathbf{t}}^h)-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}+\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h.$$

From the inequality (9.21), the definition of the discrete set $M_{\mathbf{t}}^h(\mathscr{F}\lambda_{\mathbf{n}}^h)$, and according to the notations of Proposition 9.2, we get

$$\begin{aligned}\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h &\leq \int_{\Gamma_C}\mu_{\mathbf{t}}^h u_{\mathbf{t}}^h \quad \forall \mu_{\mathbf{t}}^h\in M_{\mathbf{t}}^h(\mathscr{F}\lambda_{\mathbf{n}}^h)\\ &=\sum_{i=1}^{p}(\mathcal{M}N_T)_i(U_T)_i \quad \forall N_T \text{ such that } |(\mathcal{M}N_T)_i|\leq -\mathscr{F}(\mathcal{M}L_N)_i.\end{aligned}$$

If $(U_T)_i\geq 0$, we choose $(\mathcal{M}N_T)_i=-\mathscr{F}(\mathcal{M}L_N)_i$, and if $(U_T)_i\leq 0$, we choose $(\mathcal{M}N_T)_i=\mathscr{F}(\mathcal{M}L_N)_i$. This yields the following bound:

$$\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}^h \leq \mathscr{F}\sum_{i=1}^{p}(\mathcal{M}L_N)_i|(U_T)_i| = \mathscr{F}\int_{\Gamma_C}\lambda_{\mathbf{n}}^h\mathcal{I}^h\left(\left|u_{\mathbf{t}}^h\right|\right). \tag{9.41}$$

So we obtain

$$\int_{\Gamma_C}(\mathscr{F}\lambda_{\mathbf{n}}^h\xi-\lambda_{\mathbf{t}}^h)(u_{\mathbf{t}}-u_{\mathbf{t}}^h)\leq\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\xi(u_{\mathbf{t}}-u_{\mathbf{t}}^h)-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\mathcal{I}^h|u_{\mathbf{t}}^h|.$$

We rewrite the above estimate as follows:

$$\begin{aligned}\int_{\Gamma_C}(\mathscr{F}\lambda_{\mathbf{n}}^h\xi-\lambda_{\mathbf{t}}^h)(u_{\mathbf{t}}-u_{\mathbf{t}}^h)&\leq\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\xi(u_{\mathbf{t}}-u_{\mathbf{t}}^h)-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h u_{\mathbf{t}}+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\mathcal{I}^h|u_{\mathbf{t}}^h|\\ &\quad+\int_{\Gamma_C}\lambda_{\mathbf{t}}^h\mathcal{I}^h u_{\mathbf{t}}-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h\mathcal{I}^h u_{\mathbf{t}}\\ &\leq\int_{\Gamma_C}\lambda_{\mathbf{t}}^h(\mathcal{I}^h u_{\mathbf{t}}-u_{\mathbf{t}})+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\left(\mathcal{I}^h|u_{\mathbf{t}}^h|-\xi u_{\mathbf{t}}^h\right)\\ &\quad+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\xi u_{\mathbf{t}}-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h\mathcal{I}^h u_{\mathbf{t}}.\end{aligned}$$

We use now the relationship $\xi u_{\mathbf{t}}=|u_{\mathbf{t}}|$ from the assumption (9.29), and we obtain

$$\begin{aligned}\int_{\Gamma_C}(\mathscr{F}\lambda_{\mathbf{n}}^h\xi-\lambda_{\mathbf{t}}^h)(u_{\mathbf{t}}-u_{\mathbf{t}}^h)&\leq\int_{\Gamma_C}\lambda_{\mathbf{t}}^h(\mathcal{I}^h u_{\mathbf{t}}-u_{\mathbf{t}})+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h\left(\mathcal{I}^h|u_{\mathbf{t}}^h|-\xi u_{\mathbf{t}}^h\right)\\ &\quad+\int_{\Gamma_C}\mathscr{F}\lambda_{\mathbf{n}}^h|u_{\mathbf{t}}|-\int_{\Gamma_C}\lambda_{\mathbf{t}}^h\mathcal{I}^h u_{\mathbf{t}}.\end{aligned}$$

We finally add and subtract the term $\int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h \mathcal{I}^h |u_{\mathbf{t}}^h|$ in the second line and obtain

$$\begin{aligned}
&\int_{\Gamma_C} (\mathscr{F}\lambda_{\mathbf{n}}^h \xi - \lambda_{\mathbf{t}}^h)(u_{\mathbf{t}} - u_{\mathbf{t}}^h) \\
&\leq \underbrace{\int_{\Gamma_C} \lambda_{\mathbf{t}}^h (\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}})}_{\mathcal{T}_1} + \underbrace{\int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h (\mathcal{I}^h |u_{\mathbf{t}}^h| - \xi u_{\mathbf{t}}^h)}_{\mathcal{T}_2} \\
&\quad + \underbrace{\int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h \mathcal{I}^h |u_{\mathbf{t}}| - \lambda_{\mathbf{t}}^h \mathcal{I}^h u_{\mathbf{t}}}_{\mathcal{T}_3} + \underbrace{\int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h (|u_{\mathbf{t}}| - \mathcal{I}^h |u_{\mathbf{t}}|)}_{\mathcal{T}_4} .
\end{aligned} \tag{9.42}$$

The estimate of the first term $\mathcal{T}_1$ in (9.42) is achieved as follows. First, we use the identity $\lambda_{\mathbf{t}}^h - \lambda_{\mathbf{t}} + \lambda_{\mathbf{t}}$ to rewrite it, and this is followed by the definition of the dual norm and Cauchy–Schwarz inequality:

$$\begin{aligned}
\mathcal{T}_1 &= \int_{\Gamma_C} (\lambda_{\mathbf{t}}^h - \lambda_{\mathbf{t}})(\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}}) + \int_{\Gamma_C} \lambda_{\mathbf{t}} (\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}}) \\
&\leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}}\|_{\frac{1}{2},\Gamma_C} + \|\boldsymbol{\lambda}\|_{0,\Gamma_C} \|\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}}\|_{0,\Gamma_C} .
\end{aligned}$$

Then, we use Corollary 4.1 for the interpolation error on the boundary combined with the Trace Theorem 2.8

$$\mathcal{T}_1 \leq C \left(h^{s-1} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}} \|\boldsymbol{\lambda}\|_{0,\Gamma_C} \|\mathbf{u}\|_{s,\Omega} \right) .$$

Note that, above, we used different indices of Sobolev regularity to obtain two different powers for the interpolation error, taking advantage that the solution is regular enough ($\boldsymbol{\lambda} \in H^{s-3/2}(\Gamma_C; \mathbb{R}^2)$). We finally use $s > 3/2$, the bound $\|\boldsymbol{\lambda}\|_{0,\Gamma_C} \leq C\|\mathbf{u}\|_{s,\Omega}$, and get

$$\mathcal{T}_1 \leq C h^{\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega} \left(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} + h^{\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega} \right) . \tag{9.43}$$

The estimate of the second term $\mathcal{T}_2$ in (9.42) uses the properties (9.29), from which we deduce $|u_{\mathbf{t}}^h| - \xi u_{\mathbf{t}}^h \geq 0$. As a result of the above inequality and (9.11), we get

$$\begin{aligned}
\mathcal{T}_2 &= \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h (\mathcal{I}^h |u_{\mathbf{t}}^h| - \xi u_{\mathbf{t}}^h) - \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h \mathcal{I}^h (\xi u_{\mathbf{t}}^h) + \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h \mathcal{I}^h (\xi u_{\mathbf{t}}^h) \\
&= \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h \mathcal{I}^h (|u_{\mathbf{t}}^h| - \xi u_{\mathbf{t}}^h) + \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h (\mathcal{I}^h (\xi u_{\mathbf{t}}^h) - \xi u_{\mathbf{t}}^h)
\end{aligned}$$

$$\le \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h(\xi u_{\mathbf{t}}^h) - \xi u_{\mathbf{t}}^h)$$
$$= \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h(\xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) - \xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) + \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h(\xi u_{\mathbf{t}}) - \xi u_{\mathbf{t}})$$
$$= \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h(\xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) - \xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) + \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h|u_{\mathbf{t}}| - |u_{\mathbf{t}}|).$$

At the last line, we used once again (9.29). Remark that above, the last term is precisely the opposite of the fourth term $\mathcal{T}_4$ in (9.42). So the sum of the second and fourth terms in (9.42) is bounded as follows ($\varepsilon > 0$ is chosen small):

$$\begin{aligned}
&\mathcal{T}_2 + \mathcal{T}_4\\
&\le \int_{\Gamma_C} \mathscr{F}\lambda_{\mathbf{n}}^h(\mathcal{I}^h(\xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) - \xi(u_{\mathbf{t}}^h - u_{\mathbf{t}}))\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} h^{\frac{1}{2}+\varepsilon}\|\xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} h^{\frac{1}{2}+\varepsilon}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\|u_{\mathbf{t}}^h - u_{\mathbf{t}}\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} h^{\frac{1}{2}+\varepsilon}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\left(\|u_{\mathbf{t}}^h - \mathcal{I}^h u_{\mathbf{t}}\|_{\frac{1}{2}+\varepsilon,\Gamma_C} + \|\mathcal{I}^h u_{\mathbf{t}} - u_{\mathbf{t}}\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\right)\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} h^{\frac{1}{2}+\varepsilon}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\left(h^{-\varepsilon}\|u_{\mathbf{t}}^h - \mathcal{I}^h u_{\mathbf{t}}\|_{1/2,\Gamma_C} + h^{s-1-\varepsilon}\|u_{\mathbf{t}}\|_{s-\frac{1}{2},\Gamma_C}\right)\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} h^{\frac{1}{2}+\varepsilon}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\left(h^{-\varepsilon}\|u_{\mathbf{t}}^h - \mathcal{I}^h u_{\mathbf{t}}\|_{1/2,\Gamma_C} + h^{s-1-\varepsilon}\|\mathbf{u}\|_{s,\Omega}\right)\\
&\le C\mathscr{F}\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\left(h^{1/2}\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + h^{s-1/2}\|\mathbf{u}\|_{s,\Omega}\right).
\end{aligned}$$

We used here the fact that $H^{\frac{1}{2}+\varepsilon}(\Gamma_C)$ is a multiplicative algebra in one dimension (see Proposition 2.12). In the second line, we used Corollary 4.1 for Lagrange interpolation on the boundary. In the fifth line, we used an inverse inequality (Proposition 4.3) as well as Corollary 4.1. The sixth line is a consequence of the Trace Theorem 2.8.

Besides, using the (global) $L^2(\Gamma_C)$-projection operator π^h onto W^h and an inverse inequality (see Theorem 4.5 and Proposition 4.3), we write

$$\begin{aligned}
\|\lambda_{\mathbf{n}}^h\|_{0,\Gamma_C} &\le \|\lambda_{\mathbf{n}}^h - \pi^h\lambda_{\mathbf{n}}\|_{0,\Gamma_C} + \|\pi^h\lambda_{\mathbf{n}} - \lambda_{\mathbf{n}}\|_{0,\Gamma_C} + \|\lambda_{\mathbf{n}}\|_{0,\Gamma_C}\\
&\le C\left(h^{-1/2}\|\lambda_{\mathbf{n}}^h - \pi^h\lambda_{\mathbf{n}}\|_{-\frac{1}{2},\Gamma_C} + \|\lambda_{\mathbf{n}}\|_{0,\Gamma_C}\right)\\
&\le C\left(h^{-1/2}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} + \|\mathbf{u}\|_{s,\Omega}\right).
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\mathcal{T}_2 + \mathcal{T}_4 \le & \int_{\Gamma_C} \mathscr{F} \lambda_{\mathbf{n}}^h (\mathcal{I}^h(\xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) - \xi(u_{\mathbf{t}}^h - u_{\mathbf{t}})) \\
& \le C\mathscr{F} \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C} \left(h^{-1/2} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} + \|\mathbf{u}\|_{s,\Omega} \right) \\
& \qquad (h^{1/2}\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + h^{s-1/2}\|\mathbf{u}\|_{s,\Omega}).
\end{aligned}
$$

We develop, make use of (9.36) and then of the relationship $s > 3/2$:

$$
\begin{aligned}
& \mathcal{T}_2 + \mathcal{T}_4 \\
& \le C\mathscr{F} \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C} \Big(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + h^{s-1} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u}\|_{s,\Omega} \\
& \qquad + h^{1/2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega}^2 \Big) \\
& \le C\mathscr{F} \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C} \Big(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + h^{s-1} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u}\|_{s,\Omega} \\
& \qquad + h^{2s-2} \|\mathbf{u}\|_{s,\Omega}^2 + h^{1/2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega}^2 \Big) \\
& \le C\mathscr{F} \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C} \Big(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \\
& \qquad + h^{1/2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega}^2 \Big).
\end{aligned}
$$

We use the relationship (9.12) to conclude that the third term $\mathcal{T}_3$ in (9.42) is nonpositive, as follows:

$$
\begin{aligned}
& -\int_{\Gamma_C} \lambda_{\mathbf{t}}^h \mathcal{I}^h u_{\mathbf{t}} \\
& = \sum_{i=1}^{p} -(\mathcal{M}L_T)_i (U_T)_i \le \sum_{i=1}^{p} -\mathscr{F}(\mathcal{M}L_N)_i |(U_T)_i| = -\mathscr{F} \int_{\Gamma_C} \lambda_{\mathbf{n}}^h \mathcal{I}^h(|u_{\mathbf{t}}|).
\end{aligned}
$$

Now, when we combine the estimates (9.43) and above for $\mathcal{T}_i$, $i = 1, \ldots, 4$, we get from (9.42):

$$
\begin{aligned}
\int_{\Gamma_C} (\mathscr{F}\lambda_{\mathbf{n}}^h \xi - \lambda_{\mathbf{t}}^h)(u_{\mathbf{t}} - u_{\mathbf{t}}^h) \le & \; C h^{\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega} \left(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} + h^{\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega} \right) \\
& + C\mathscr{F} \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C} \Big(\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \\
& \qquad + h^{1/2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}} \|\mathbf{u}\|_{s,\Omega}^2 \Big).
\end{aligned}
\tag{9.44}
$$

From (9.39), the above estimate (9.44), and the previous one (9.40), we can bound the friction term

$$\begin{aligned}\mathcal{T}_F \leq{}& Ch^{\frac{1}{2}}\|\mathbf{u}\|_{s,\Omega}\left(\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C} + h^{\frac{1}{2}}\|\mathbf{u}\|_{s,\Omega}\right)\\ &+C\mathscr{F}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}\Big(\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\\ &+h^{1/2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}\|_{s,\Omega} + h^{s-\frac{1}{2}}\|\mathbf{u}\|^2_{s,\Omega}\Big)\\ &+C\mathscr{F}\|\xi\|_M\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}.\end{aligned} \tag{9.45}$$

We make use once again of (9.36), of $s > 3/2$, and we regroup the terms:

$$\begin{aligned}\mathcal{T}_C \leq{}& C\mathscr{F}(\|\xi\|_M + \|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C})\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^h\|_{-\frac{1}{2},\Gamma_C}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\\ &+Ch(1+\mathscr{F}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C})\|\mathbf{u}\|^2_{s,\Omega}\\ &+Ch^{1/2}(1+\mathscr{F}\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C})\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}\|_{s,\Omega}.\end{aligned} \tag{9.46}$$

We combine (9.32) with (9.36), (9.37), (9.38) and (9.46). We use as many times as necessary the Cauchy–Schwarz and Young inequalities and make use of the assumption (9.29) and of the bound $\|\xi\|_M \leq C\|\xi\|_{\frac{1}{2}+\varepsilon,\Gamma_C}$ for the first right term in $\mathcal{T}_C$. We get finally

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} \leq Ch^{\frac{1}{2}}\|\mathbf{u}\|_{s,\Omega}.$$

Using once again (9.36), we obtain (9.30) and this concludes the proof. □

Remark 9.4 The rate of convergence of order $1/2$ in the theorem does not depend on $\varepsilon > 0$. Actually, we are not able to obtain a better convergence rate even if ε increases. A bit more regularity than $H^{\frac{3}{2}}(\Omega)$ is needed to apply the trace theorem, when writing $\|\lambda_{\mathbf{n}}\|_{0,\Gamma_C} \leq C\|\mathbf{u}\|_{\frac{3}{2}+\varepsilon,\Omega}$ (see [141]).

Remark 9.5 Note that we do not prove that the solution to the discrete problem is unique under the assumptions of Theorem 9.1. This seems to be an open question that is actually under investigation. Note also that this possible loss of uniqueness would not be embarrassing in the a priori error analysis of Theorem 1. As a matter of fact, even if there are multiple solutions to the discrete problem, any solution would converge toward the unique solution of the continuous model.

Remark 9.6 In [168], the same convergence rate has been obtained, but with another finite element method, especially another choice of the discrete sets for the Lagrange multipliers. The error estimate for the mixed method presented in this chapter is new to the best of our knowledge.

9.4 Finite Element Approximation with Nitsche

In this section, we present, as an alternative to the mixed method detailed above, a Nitsche method for frictional contact that extends the method of Chap. 6. As for frictionless contact, this method does not need an extra unknown, and there is no issue related to the discrete inf-sup condition.

9.4.1 Preliminaries

For any $\alpha \in \mathbb{R}^+$, we recall the notation $[\cdot]_\alpha$ for the orthogonal projection onto $\mathscr{B}(\mathbf{0}, \alpha) \subset \mathbb{R}^{d-1}$, where $\mathscr{B}(\mathbf{0}, \alpha)$ is the closed ball centered at the origin $\mathbf{0}$ and of radius α. We also recall the analytical expression of this projection, for $\mathbf{x} \in \mathbb{R}^{d-1}$ by

$$[\mathbf{x}]_\alpha = \begin{cases} \mathbf{x} & \text{if } |\mathbf{x}| \le \alpha, \\ \alpha \dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise.} \end{cases}$$

As in Chap. 6 for frictionless contact, and in Chap. 8 for Tresca friction, we reformulate the Coulomb friction conditions as follows (the proof is almost identical as for the Tresca friction case, see Proposition 8.2, see also [81]):

Proposition 9.3 *Let γ be a positive function defined on Γ_C. Coulomb friction conditions (9.6) can be reformulated as follows:*

$$\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = -[\gamma \mathbf{u}_{\mathbf{t}} - \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})]_{(-\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u}))} = -[\gamma \mathbf{u}_{\mathbf{t}} - \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})]_{\mathscr{F}[\gamma u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})]_+}. \tag{9.47}$$

Both formulations in (9.47) can be used to derive a Nitsche method, but we will see the interest of the second one below. We consider the same finite element space as in Chaps. 6 and 8, of Lagrange finite elements of order k:

$$\mathbf{V}^h := \mathbf{X}_k^h \cap \mathbf{V} = \left\{ \mathbf{v}^h \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^2) \;\middle|\; \mathbf{v}^h_{|T} \in \mathbb{P}_k(T; \mathbb{R}^2), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

9.4.2 Nitsche Formulation

We consider in what follows γ_N, a positive piecewise constant function on the contact interface Γ_C that satisfies

$$\gamma_{N|T \cap \Gamma_C} = \frac{\gamma_0}{h_T}, \tag{9.48}$$

for every T that has a non-empty intersection of dimension 1 with Γ_C, and where γ_0 is a positive given constant (the Nitsche parameter). Note that the value of γ_N on element intersections has no influence. Given θ a fixed parameter, we introduce the discrete linear operator

$$\mathbf{P}^{\mathbf{t}}_{\theta,N} : \begin{array}{rcl} \mathbf{V}^h & \rightarrow & L^2(\Gamma_C;\mathbb{R}) \\ \mathbf{v}^h & \mapsto & \gamma_N \mathbf{v}^h_{\mathbf{t}} - \theta \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h) \end{array}.$$

Define as well the bilinear form:

$$A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) := a(\mathbf{u}^h,\mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma_N}\, \boldsymbol{\sigma}(\mathbf{u}^h)\mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{v}^h)\mathbf{n}.$$

The Nitsche method for Coulomb frictional contact reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N}\, [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ \mathrm{P}^{\mathbf{n}}_{\theta,N}(\mathbf{v}^h) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma_N} \left[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h)\right]_{\left(\mathscr{F}\left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)\right]_+\right)} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,N}(\mathbf{v}^h) = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{9.49}$$

Remark that this formulation is very close to the case of Tresca friction (8.17). Remark that the threshold for the projection operator related to Coulomb remains always nonnegative, which is a desirable property in terms of numerical stability. Indeed, at the discrete level, the first reformulation in (9.47) would imply the expression $-\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u}^h)$ for the threshold, but, for the discrete solution, nothing guarantees that $\sigma_{\mathbf{n}}(\mathbf{u}^h)$ remains nonpositive.

9.4.3 Existence and Uniqueness Results

In this section, we state that the discrete problem (9.49) admits solutions when γ_0 is large (here the denomination "large" depends on θ) and that the solution is unique under an additional smallness assumption on $\mathscr{F}^2\gamma_0 h^{-1}$. The following result has been proven in [81]:

Theorem 9.3 *For any value of $\theta \in \mathbb{R}$, Problem* (9.49) *admits at least one solution when γ_0 is large enough. Moreover, if the quantity $\mathscr{F}^2\gamma_0 h^{-1}$ is small enough, this solution is unique.*

Remark finally that, conversely to the case of Tresca friction, where a Nitsche reformulation facilitated in some sense the obtention of optimal a priori error estimates, there is no special improvement here in comparison to the mixed method. Especially, the same kind of condition for uniqueness as for the mixed method is recovered.

9.5 Further Comments

For introductive material about Coulomb friction, one can refer to [114], to [115], or to [181]. Issues related to the existence and uniqueness of solutions at the continuous level are detailed, for instance in [22, 115, 232].

9.5.1 Mixed Methods and Error Estimates

Many mixed methods have been derived for Coulomb friction, as well as Augmented Lagrangian formulations. In fact, most of the techniques presented in the Chap. 7 can be extended for Coulomb friction. Then, as mentioned in this chapter, existence results can be obtained by using the Brouwer fixed point theorem. To recover uniqueness of the discrete solution, a sufficient condition involves the friction coefficient and the mesh size. It is still unclear if this condition can be improved.

There are very few error estimates for finite elements and Coulomb friction, and the only results that we know are in [168] and in the present chapter, for two distinct mixed methods. A first convergence result, without convergence rate, has been obtained by J. Haslinger and I. Hlaváček in 1982 [152]. The best rates obtained are still largely suboptimal, and it seems quite challenging to improve them.

9.5.2 Nitsche Method

The mathematical study of Nitsche's method for Coulomb friction started only a few years ago, and preliminary results can be found in [15, 27, 80, 81]. In [27], a modified version of Nitsche, with local projections of contact and friction terms on the boundary faces, has been designed, and its relation with stabilized mixed methods has been studied.

There is still no result of convergence for Nitsche method applied to Coulomb friction.

9.5.3 Bifurcation Tracking

Fixed point techniques for Coulomb friction have been studied in, e.g., [189]. Continuation techniques to follow possible bifurcations in case of Coulomb friction have been studied in, e.g., [198].

Chapter 10
Contact Between Two Elastic Bodies

In this chapter, we detail the formulation for the contact between two elastic bodies, in the small deformation framework. This is mostly a preliminary step before the study of contact in large strains that involve much more technical difficulties in terms of formulation and implementation. Indeed, we focused up to now on the Signorini problem, studied first in Chap. 3 in the frictionless case. This choice was motivated by the (relative) simplicity of this problem and its sound mathematical structure. Moreover, this allowed to present various numerical approximations within a unified framework. Nevertheless, from the application viewpoint, this setting has obvious limitations, since one of the bodies in contact is supposed to be perfectly rigid, and since there are strong geometrical assumptions on the contact surfaces.

As a result, here, we describe the setting associated to two elastic bodies that are potentially in contact. We present first a biased, or master–slave setting, which mimics the Signorini problem, in the sense that the contact conditions are written solely on one of the two contact surfaces (the slave surface). This setting is the most usual one. We then describe another setting where contact conditions are written everywhere on the potential contact surfaces. This allows easier extensions for multi-body contact and self-contact. It is made possible by some specific formulations of the contact conditions (penalized and Nitsche formulations). As well, we introduce the notion of gap function that represents the initial distance between the two contact surfaces and plays a fundamental role in writing the nonpenetration condition. It is an important preliminary, since, for the extension to large strain, very specific difficulties appear due to this gap function, as we will see in the next chapter. From the mathematical viewpoint, the small strain assumption allows still to recover some well-posed variational inequalities, at least for the frictionless and Tresca frictional cases. This allows to extend most of the well-posedness and convergence results presented in the previous chapters for Signorini contact. However, this point will not be detailed here.

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_10

After this description of the setting, we will provide the formulation for some of the numerical methods already studied in the previous chapters: Nitsche's method, mixed and Augmented Lagrangian methods, and the penalty method. We will also mention some specific aspects related to implementation.

10.1 Setting

First, we describe the setting for the frictional contact between two elastic bodies, in Sect. 10.1.1, and then in Sect. 10.1.7, this problem is recasted as a (quasi-)variational inequality.

10.1.1 The General Configuration

We consider two elastic bodies whose reference configurations are respectively denoted by Ω^1 and Ω^2: two open bounded connected sets of $\mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundaries, see Fig. 10.1. The boundaries $\partial\Omega^1$ and $\partial\Omega^2$ of Ω^1 and Ω^2 are decomposed as follows: Dirichlet boundaries Γ^1_D and Γ^2_D, of nonempty interior, Neumann boundaries Γ^1_N and Γ^2_N, and contact boundaries Γ^1_C and Γ^2_C. These boundaries are supposed to be a partition without overlap of $\partial\Omega^1$ and $\partial\Omega^2$. On these two elastic bodies, we apply some bulk force densities, denoted by $\mathbf{f}^1$ and $\mathbf{f}^2$, and on the Neumann boundaries Γ^1_N and Γ^2_N, we apply boundary force densities denoted by $\mathbf{F}^1$ and $\mathbf{F}^2$, respectively.

10.1.2 Biased Contact

As it is usual in the contact mechanics literature (see, for instance, [190, Chapter 3] or [266, Chapter 6]), we first adopt a biased, or master–slave paradigm, which consists in enforcing the unilateral contact conditions solely on one of the two (potential) contact boundaries. Therefore, we need at this point to make an arbitrary choice and to set conventions:

- Γ^1_C is the slave boundary: contact conditions analogous to Signorini conditions (3.18) will be incorporated on this boundary.
- Γ^2_C is the master boundary: the contact force applied will come from action–reaction principle.

With the above choice, the body Ω^1 plays an analogous role to the elastic body in the Signorini problem, and the body Ω^2 plays a role similar to the rigid support.

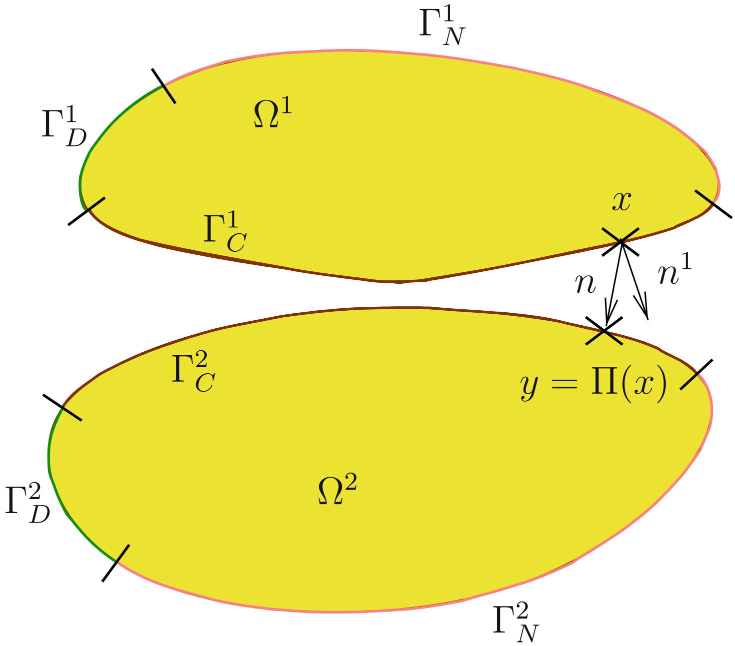

Fig. 10.1 Two elastic bodies Ω^1 and Ω^2 with their respective contact boundaries Γ^1_C and Γ^2_C

10.1.3 Contact Pairing

Now, let us describe precisely the unilateral contact condition with friction on the slave surface Γ^1_C. For any point $\mathbf{x} \in \Gamma^1_C$, we need to find the corresponding point $\mathbf{y} \in \Gamma^2_C$ that comes potentially in contact with it on the master surface. This is called contact pairing. In the framework of small strain elasticity, this pairing is defined in the reference configuration and is not updated during the deformation. Generally, it is obtained using an orthogonal projection of $\mathbf{x}$ into the master surface Γ^2_C. Nevertheless, this is not the only possible choice (see Chap. 11 for another alternative, called *raytracing*). Let

$$\begin{aligned} \Pi : \Gamma^1_C &\to \Gamma^2_C, \\ \mathbf{x} &\mapsto \mathbf{y} = \Pi(\mathbf{x}) \end{aligned}$$

be the mapping that ensures contact pairing. Once this mapping is set, we can define an external unit vector that shares the same direction with $\mathbf{y} - \mathbf{x}$, denoted by $\mathbf{n}$. Therefore, there are in fact two notable external vectors to the contact boundary at point $\mathbf{x}$ (see Fig. 10.1): the outward unit normal to the boundary of Ω^1, denoted by $\mathbf{n}^1$, and the external unit vector $\mathbf{n}$. This last one can be defined as

$$\mathbf{n} := \begin{cases} \dfrac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} & \text{if } (\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^1 > 0, \\ \mathbf{n}^1 & \text{if } (\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^1 = 0, \\ -\dfrac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} & \text{if } (\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^1 < 0. \end{cases}$$

The last two cases, for $(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^1 \leq 0$, are designed, either to account for the situation where contact is already activated in the reference configuration, or if the two domains overlap. The two unit vectors $\mathbf{n}^1$ and $\mathbf{n}$ need not to be identical, generally. The unit vector $\mathbf{n}$ is usually called the contact normal.

10.1.4 The Contact Conditions

To write the contact condition, one needs to define first what the normal stress means. Let

$$\mathbf{u}^1 : \Omega^1 \to \mathbb{R}^d$$

be the displacement associated to the first body and $\boldsymbol{\sigma}(\mathbf{u}^1)$ the corresponding Cauchy stress tensor. Then, we define

$$\sigma_{\mathbf{n}} := (\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1)\cdot\mathbf{n}, \quad \boldsymbol{\sigma}_{\mathbf{t}} := (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})(\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1),$$

the decomposition into normal and tangential components of the stress $\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1$ on the slave contact boundary. Be careful that here, as in Chap. 3, this is still $\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1$ that represents the density of boundary forces, in agreement with Cauchy's Theorem, and the contact normal vector $\mathbf{n}$ serves only for the geometric decomposition according to the contact pairing mapping Π. We introduce now

$$g_0 := (\mathbf{y}-\mathbf{x})\cdot\mathbf{n}$$

the initial gap between the two contact surfaces and

$$[\![\mathbf{u}]\!] := \mathbf{u}^1 - \mathbf{u}^2\circ\Pi, \quad [\![u_{\mathbf{n}}]\!] := [\![\mathbf{u}]\!]\cdot\mathbf{n} = (\mathbf{u}^1 - \mathbf{u}^2\circ\Pi)\cdot\mathbf{n}$$

the jumps of the displacements and of the normal displacements, where $\circ$ is the composition operator. From this point, we are ready to adapt the original definition of Signorini conditions (3.18) to this new setting:

$$[\![u_{\mathbf{n}}]\!] \leq g_0, \quad \sigma_{\mathbf{n}} \leq 0, \quad \left([\![u_{\mathbf{n}}]\!] - g_0\right)\sigma_{\mathbf{n}} = 0. \tag{10.1}$$

The above conditions (10.1) have the same meaning as Signorini conditions, and, particularly, the first condition is the nonpenetration condition. Remark how it has been reformulated in this new setting, where nonpenetration means that, locally, the sum of the normal displacements of the two bodies cannot exceed the initial gap.

10.1.5 Friction

To write the friction condition, we need a slip velocity. If we suppose a quasi-steady evolution, we can use instead an increment of tangential displacement, denoted by $\mathbf{d_t}$ (see Chaps. 8 and 9). In the books of G. Duvaut and J.L. Lions, and of N. Kikuchi and J.T. Oden [114, 181], the authors make use of the following expression:

$$\mathbf{d_t}(\mathbf{u}^1, \mathbf{u}^2) = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})[\![\mathbf{u}]\!].$$

The resulting problem has the same mathematical structure as if a temporal discretization of the slip velocity would have been taken into account. If we take into account the temporal discretization, we can write then

$$\mathbf{d_t}(\mathbf{u}^1, \mathbf{u}^2) = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \left([\![\mathbf{u}]\!] - [\![\mathbf{u}^0]\!] \right),$$

with $[\![\mathbf{u}^0]\!]$ that stands for the jump of displacements at the previous time step. The friction condition then reads

$$|\boldsymbol{\sigma}_\mathbf{t}| \leq S(\sigma_\mathbf{n}), \qquad \text{if } \mathbf{d_t} \neq \mathbf{0} \text{ then } \boldsymbol{\sigma}_\mathbf{t} = -S(\sigma_\mathbf{n}) \frac{\mathbf{d_t}}{|\mathbf{d_t}|}, \tag{10.2}$$

where $S(\sigma_\mathbf{n})$ is a given function of the contact pressure $\sigma_\mathbf{n}$. For $S(\sigma_\mathbf{n}) = s_T^1$, where s_T^1 is a given non-negative function on Γ_C^1, we recover Tresca friction (see Chap. 8). For $S(\sigma_\mathbf{n}) = -\mathscr{F} \sigma_\mathbf{n}$, with $\mathscr{F} > 0$ the friction coefficient, we recover Coulomb friction (see Chap. 9).

10.1.6 Equilibrium

Conversely to the Signorini problem described in Chap. 3, we need here to make explicit the third Newton law (action–reaction principle) that ensures the balance of forces between the two bodies on the region where contact is activated and that corresponds to $[\![u_\mathbf{n}]\!] = g_0$. Of course, the third Newton law is also implicitly ensured in the formulation of the Signorini problem: when the problem is solved, the density of forces applied from the elastic body on the surface of the rigid support is given by $-\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$ and is the opposite of the density of forces applied to the elastic body $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$.

This condition can be derived as follows. The third Newton law means that for each arbitrary subset γ_C^1 of Γ_C^1, the resulting boundary forces applied by each body Ω^1 and Ω^2 to the other one are opposite. As a result, there holds:

$$\int_{\gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1(\mathbf{x}))\mathbf{n}^1(\mathbf{x})\mathrm{d}s(\mathbf{x}) + \int_{\gamma_C^2} \boldsymbol{\sigma}(\mathbf{u}^2(\mathbf{y}))\mathbf{n}^2(\mathbf{y})\mathrm{d}s(\mathbf{y}) = \mathbf{0},$$

where $\gamma_C^2 := \Pi(\gamma_C^1)$ is the image of γ_C^1 obtained with the contact mapping Π and $\mathbf{n}^2(\mathbf{y})$ is the outward unit normal to Γ_C^2 at point $\mathbf{y} = \Pi(\mathbf{x})$. We make a change of variable on the second boundary term, in order to map it back to γ_C^1:

$$\int_{\gamma_C^2} \boldsymbol{\sigma}(\mathbf{u}^2(\mathbf{y}))\mathbf{n}^2(\mathbf{y})\mathrm{d}s(\mathbf{y}) = \int_{\gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^2(\Pi(\mathbf{x})))\mathbf{n}^2(\Pi(\mathbf{x}))\, J_\Pi(\mathbf{x})\mathrm{d}s(\mathbf{x}),$$

where J_Π is the Jacobian of Π:

$$J_\Pi := \det(\nabla\Pi)\,.$$

If we combine the above relationships, we obtain

$$\boldsymbol{\sigma}(\mathbf{u}^1(\mathbf{x}))\mathbf{n}^1(\mathbf{x}) + \boldsymbol{\sigma}(\mathbf{u}^2(\Pi(\mathbf{x})))\mathbf{n}^2(\Pi(\mathbf{x}))\, J_\Pi(\mathbf{x}) = \mathbf{0},$$

at any point $\mathbf{x} \in \Gamma_C^1$ (since γ_C^1 is arbitrary). We can reformulate the above relationship in a more compact form, as

$$\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 + \boldsymbol{\sigma}(\mathbf{u}^2 \circ \Pi)(\mathbf{n}^2 \circ \Pi)\, J_\Pi = \mathbf{0}. \tag{10.3}$$

Remark that the above Eq. (10.3) remains always true, whenever the two bodies are in effective contact or not.

There remains now to write the small strain elasticity equations in Ω^i, $i = 1, 2$:

$$\begin{cases} -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}^i) = \mathbf{f}^i & \text{in } \Omega^i, \\ \boldsymbol{\sigma}(\mathbf{u}^i) = \mathbf{A}^\mathbf{i} : \boldsymbol{\varepsilon}(\mathbf{u}^i) & \text{in } \Omega^i, \\ \mathbf{u}^i = \mathbf{0} & \text{on } \Gamma_D^i, \\ \boldsymbol{\sigma}(\mathbf{u}^i)\mathbf{n}^i = \mathbf{F}^i & \text{on } \Gamma_N^i, \end{cases} \tag{10.4}$$

where, for each index $i = 1, 2$, $\mathbf{A}^i$ is the elasticity tensor associated to the material properties of Ω^i. These two tensors $\mathbf{A}^i$, $i = 1, 2$, are supposed endowed with the same properties of symmetry (3.2), strong ellipticity (3.3), and uniform boundedness (3.4) as the elasticity tensor $\mathbf{A}$ introduced in Chap. 3 for the Signorini problem.

10.1.7 Weak Form

Following [114, 181] for instance, we can recast the contact problem as a variational inequality. For this purpose, let us introduce the space:

$$\mathbf{V} = \Big\{ \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in H^1(\Omega^1; \mathbb{R}^d) \times H^1(\Omega^2; \mathbb{R}^d) \;\Big|\; \mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_D^1 \text{ and } \mathbf{v}^2 = \mathbf{0} \text{ on } \Gamma_D^2 \Big\}$$

and the convex cone of admissible displacements

$$\mathbf{K} = \Big\{ \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in \mathbf{V} \;\Big|\; [\![v_{\mathbf{n}}]\!] - g_0 \le 0 \Big\}.$$

Let us suppose that $(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $L^2(\Omega^1; \mathbb{R}^d) \times L^2(\Omega^2; \mathbb{R}^d)$, and $(\mathbf{F}^1, \mathbf{F}^2)$ belongs to $L^2(\Gamma_N^1; \mathbb{R}^d) \times L^2(\Gamma_N^2; \mathbb{R}^d)$. We can define the following bilinear and linear forms on $\mathbf{V}$:

$$a(\mathbf{u}, \mathbf{v}) := \sum_{i=1,2} \int_{\Omega^i} \sigma(\mathbf{u}^i) : \boldsymbol{\varepsilon}(\mathbf{v}^i), \quad L(\mathbf{v}) = \sum_{i=1,2} \int_{\Omega^i} \mathbf{f}^i \cdot \mathbf{v}^i + \sum_{i=1,2} \int_{\Gamma_N^i} \mathbf{F}^i \cdot \mathbf{v}^i,$$

as well as the functional associated to the friction:

$$j(S, \mathbf{v}) := \langle S, |\mathbf{d_t}(\mathbf{v}^1, \mathbf{v}^2)| \rangle,$$

where $S(= S(\sigma_{\mathbf{n}}))$ is the friction threshold, and $\langle \cdot, \cdot \rangle$ is the duality pairing associated to the normal trace space on Γ_C^1.

First, suppose that, for simplicity, we are in the frictionless case, and we set $S(\sigma_{\mathbf{n}}) = 0$ in the friction condition (10.2). It becomes then

$$\boldsymbol{\sigma}_{\mathbf{t}} = \mathbf{0}.$$

To obtain a weak formulation, we can start from equilibrium equations (10.4) and apply the Green formula, which yields

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot (\mathbf{v}^1 - \mathbf{u}^1) - \int_{\Gamma_C^2} \boldsymbol{\sigma}(\mathbf{u}^2)\mathbf{n}^2 \cdot (\mathbf{v}^2 - \mathbf{u}^2) = L(\mathbf{v} - \mathbf{u}),$$

where $\mathbf{u} \in \mathbf{K}$ solves (10.4), and $\mathbf{v} \in \mathbf{K}$ is an arbitrary test function. Let us now map the boundary integral Γ_C^2 back to the slave boundary Γ_C^1:

$$\int_{\Gamma_C^2} \boldsymbol{\sigma}(\mathbf{u}^2)\mathbf{n}^2 \cdot (\mathbf{v}^2 - \mathbf{u}^2) = \int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^2 \circ \Pi)(\mathbf{n}^2 \circ \Pi) \cdot (\mathbf{v}^2 - \mathbf{u}^2) \circ \Pi \; J_\Pi.$$

We combine the above equation with the third Newton law (10.3) and get

$$\begin{aligned}
&\int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot (\mathbf{v}^1 - \mathbf{u}^1) + \int_{\Gamma_C^2} \boldsymbol{\sigma}(\mathbf{u}^2)\mathbf{n}^2 \cdot (\mathbf{v}^2 - \mathbf{u}^2) \\
&= \int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot (\mathbf{v}^1 - \mathbf{u}^1) + \int_{\Gamma_C^1} \underbrace{\boldsymbol{\sigma}(\mathbf{u}^2 \circ \Pi)(\mathbf{n}^2 \circ \Pi)\ J_\Pi}_{-\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1} \cdot (\mathbf{v}^2 - \mathbf{u}^2) \circ \Pi \\
&= \int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot \left[(\mathbf{v}^1 - \mathbf{u}^1) - (\mathbf{v}^2 - \mathbf{u}^2) \circ \Pi\right] \\
&= \int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot [\![\mathbf{v} - \mathbf{u}]\!].
\end{aligned}$$

We decompose this last term according to the contact normal $\mathbf{n}$ and apply the frictionless condition $\boldsymbol{\sigma}_\mathbf{t} = \mathbf{0}$:

$$\begin{aligned}
\int_{\Gamma_C^1} \boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot [\![\mathbf{v} - \mathbf{u}]\!] &= \int_{\Gamma_C^1} (\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot \mathbf{n})([\![\mathbf{v} - \mathbf{u}]\!] \cdot \mathbf{n}) \\
&\quad + \int_{\Gamma_C^1} (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}(\mathbf{u}^1)\mathbf{n}^1 \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})[\![\mathbf{v} - \mathbf{u}]\!] \\
&= \int_{\Gamma_C^1} \sigma_\mathbf{n}(\mathbf{u})[\![v_\mathbf{n} - u_\mathbf{n}]\!] + \int_{\Gamma_C^1} \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}) \cdot [\![\mathbf{v}_\mathbf{t} - \mathbf{u}_\mathbf{t}]\!] \\
&= \int_{\Gamma_C^1} \sigma_\mathbf{n}(\mathbf{u})[\![v_\mathbf{n} - u_\mathbf{n}]\!].
\end{aligned}$$

We split this last term and introduce the gap function g_0:

$$\int_{\Gamma_C^1} \sigma_\mathbf{n}(\mathbf{u})[\![v_\mathbf{n} - u_\mathbf{n}]\!] = \int_{\Gamma_C^1} \sigma_\mathbf{n}(\mathbf{u}) \left([\![v_\mathbf{n}]\!] - g_0\right) - \int_{\Gamma_C^1} \sigma_\mathbf{n}(\mathbf{u}) \left([\![u_\mathbf{n}]\!] - g_0\right).$$

Thanks to contact conditions (10.1), and since $\mathbf{v}$ belongs to $\mathbf{K}$, this last term is non-negative. As a result, combining all the previous considerations, we obtain finally the variational inequality:

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}).$$

Conversely, following the same path as we did in Chap. 3 for the Signorini problem, we can check that the solution $\mathbf{u} \in \mathbf{K}$ to the above variational inequality solves the contact problem in strong form (10.1)–(10.4).

If we proceed the same way as above, but taking friction into account, we get the following weak form associated to (10.1)–(10.4):

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(S(\sigma_{\mathbf{n}}(\mathbf{u})), \mathbf{v}) - j(S(\sigma_{\mathbf{n}}(\mathbf{u})), \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \\ \hfill \forall \mathbf{v} \in \mathbf{K}. \end{cases} \tag{10.5}$$

In the frictionless case $S = 0$, we recover a variational inequality of the first kind and Stampacchia's Theorem 3.4 allows to ensure well-posedness. Moreover, in the case of Tresca friction, a variational inequality of the second kind is recovered, and well-posedness can be obtained following the same arguments as in Chap. 8. For Coulomb friction, we obtain a quasi-variational inequality, and the existence of a solution can be established for a small friction coefficient $\mathscr{F} > 0$. The same issues for existence and uniqueness as in Chap. 9 arise.

Moreover, in the case of Tresca friction (or in the frictionless case), the solution $\mathbf{u} \in \mathbf{K}$ to Problem (10.5) is the unique minimizer on $\mathbf{K}$ of the functional

$$\mathcal{J}_{s_T^1}(\mathbf{v}) := \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + j(s_T^1, \mathbf{v}).$$

For frictionless contact ($s_T^1 = 0$), we will denote it simply $\mathcal{J}(\cdot)$, instead of $\mathcal{J}_0(\cdot)$.

It will be useful in the sequel to define the trace spaces on the contact and friction boundary, such as

$$\mathbf{W}_C^1 := \left\{ \Upsilon \mathbf{v}^1|_{\Gamma_C^1} \;\middle|\; \mathbf{v} \in \mathbf{V} \right\} \tag{10.6}$$

and, in the same way as in Chap. 3 the space W_C^1 for the normal component of the trace, and $\mathbf{W}_{C,\mathbf{t}}^1$ for the tangential component.

10.2 Finite Element Approximations

We present now various possibilities to approximate the contact problem between two elastic bodies that are extensions of the formulations proposed in the previous chapters.

For this purpose, we introduce $\mathbf{V}^h \subset \mathbf{V}$, a conforming finite element space built from two meshes of bodies Ω^1 and Ω^2. Unless specified, the trace meshes on the two contact surfaces are not supposed to be matching. The notation h stands for the mesh size on the slave body. A new gap function is obtained from this meshing, which we denote by g_0^h. This is an approximation of the exact gap function g_0.

10.2.1 Discrete Variational Inequality, Mortar and LAC

As in Chap. 5, we can discretize directly the weak formulation (10.5), as follows:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{K}^h \text{ such that} \\ a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) + j(S(\sigma_{\mathbf{n}}(\mathbf{u}^h)), \mathbf{v}^h) - j(S(\sigma_{\mathbf{n}}(\mathbf{u}^h)), \mathbf{u}^h) \geq L(\mathbf{v}^h - \mathbf{u}^h) \\ \hfill \forall \mathbf{v}^h \in \mathbf{K}^h. \end{cases} \tag{10.7}$$

Above $\mathbf{K}^h$ is a discrete convex cone built from the finite element space $\mathbf{V}^h$ and that approximates the convex cone $\mathbf{K}$. Various choices for $\mathbf{K}^h$ lead to various finite element methods, and for instance, one can try to adapt the various cones presented in Chap. 5. For matching meshes, this adaptation is more or less straightforward, but for nonmatching meshes, it is not obvious at all. For this reason, the Mortar and LAC methods presented previously in Chap. 7 for the Signorini problem can be formulated here for the contact between two bodies, to handle the nonmatching meshes, and this is why they have been designed.

To simplify we next suppose that the candidate contact area Γ_C is a straight line segment when $d = 2$ or a polygon when $d = 3$. We will suppose also that there is no gap and no friction.

For the local average contact (LAC) method, the discrete set of admissible displacements satisfying the average non-interpenetration conditions on the contact zone is given by

$$\mathbf{K}^h_{\mathrm{LAC}} = \left\{ \mathbf{v}^h \in \mathbf{V}^h \;\middle|\; \int_{T^m} [\![v^h_{\mathbf{n}}]\!] \leq 0 \;\; \forall T^m \in T^M \right\}. \tag{10.8}$$

We recall here, that, when $d = 2, k = 1$, then T^M is a one-dimensional macro-mesh constituted by macro-segments T^m comprising two adjacent segments of the trace mesh of Ω^1 on Γ_C. When $d = 2, k = 2$, then T^M is simply the trace mesh on Γ_C inherited by the mesh of Ω^1. The only requirement (when $k = 1, 2$ and $d = 2, 3$) is that any element of T^M admits an internal degree of freedom (of course when $d = 3$, T^M is a two-dimensional trace mesh on Γ_C), which we call hereafter the internal degree of freedom hypothesis.

The discrete set of admissible displacements satisfying the mortar contact conditions on the contact zone is given by

$$\mathbf{K}^h_{\mathrm{Mortar}} = \left\{ \mathbf{v}^h \in \mathbf{V}^h \;\middle|\; \pi^h [\![v^h_{\mathbf{n}}]\!] \leq 0 \;\text{ on } \Gamma_C \right\}, \tag{10.9}$$

where π^h is the $L^2(\Gamma^1_C)$-projection operator on the space $W^h \subset W^1_C$, which is the range of $\mathbf{V}^h$ by the normal component trace operator on Γ^1_C. Note that, as for the LAC method, the Mortar method uses one of both meshes to define the contact conditions, so the contact conditions are not symmetrical (we are still in biased contact).

As for the Signorini problem, there exist mixed formulations associated to LAC and Mortar methods that we will present later on.

10.2.2 Nitsche's Method

Let us now provide a Nitsche formulation. Such a formulation allows as well to handle nonmatching meshes for contact between two bodies. As in Chap. 6, the parameter θ allows to switch between different variants (symmetric, incomplete, skew-symmetric) and $\gamma_N > 0$ is a function on the slave contact boundary that scales as γ_0/h. The Nitsche formulation reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ solution to} \\ a(\mathbf{u}^h, \mathbf{v}^h) - \displaystyle\int_{\Gamma_C^1} \frac{\theta}{\gamma_N} \boldsymbol{\sigma}(\mathbf{u}^h)\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h)\mathbf{n} \\ + \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma_N} \left(\gamma_N \left([\![u_\mathbf{n}^h]\!] - g_0^h\right) - \sigma_\mathbf{n}(\mathbf{u}^h)\right)_+ \left(\gamma_N [\![v_\mathbf{n}^h]\!] - \theta\sigma_\mathbf{n}(\mathbf{v}^h)\right) \\ + \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma_N} \left(\gamma_N \mathbf{d_t}(\mathbf{u}^h) - \boldsymbol{\sigma}_\mathbf{t}(\mathbf{u}^h)\right)_{S^h(\mathbf{u}^h)} \cdot \left(\gamma_N [\![\mathbf{v}_\mathbf{t}^h]\!] - \theta\boldsymbol{\sigma}_\mathbf{t}(\mathbf{v}^h)\right) \\ = L(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{10.10}$$

In the above formulation, $S^h(\mathbf{u}^h)$ is an approximation of the friction threshold S defined previously. In the case of Tresca friction, it does not depend on the discrete solution and can be given by

$$S^h(\mathbf{u}^h) = s_T^{1,h},$$

where $s_T^{1,h}$ is a discrete approximation of s_T^1. In the case of Coulomb friction, the discrete threshold does depend on $\mathbf{u}^h$ and can be obtained for instance using the reformulation of Coulomb friction condition given in Chap. 9:

$$S^h(\mathbf{u}^h) = \mathscr{F}\left(\gamma_N \left([\![u_\mathbf{n}^h]\!] - g_0^h\right) - \sigma_\mathbf{n}(\mathbf{u}^h)\right)_+.$$

The above expression is always non-negative, which is an advantage in terms of numerical robustness.

The above formulation (10.10) can be derived from the strong form following exactly the same steps as in Chap. 6 (in the frictionless case). For the symmetric variant of Nitsche ($\theta = 1$), it can also be recovered as the first order optimality condition associated with the modified energy functional

$$\mathcal{J}_{b,N}(\mathbf{u}^h) := \mathcal{J}(\mathbf{u}^h) \\ -\frac{1}{2}\int_{\Gamma_C^1} \frac{1}{\gamma_N} \left(\sigma_\mathbf{n}(\mathbf{u}^h)\right)^2$$

$$+\frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma_N}\left(\gamma_N\left([\![u_{\mathbf{n}}^h]\!]-g_0^h\right)-\sigma_{\mathbf{n}}(\mathbf{u}^h)\right)_+^2.$$

The above modified energy corresponds to frictionless contact. For Tresca friction, it is still possible to write a modified energy functional that mimics (8.18).

10.2.3 Mixed Methods

As in Chap. 7, we can also write a mixed method for the above setting, and this method reads

$$\begin{cases}\text{Find } \mathbf{u}^h\in\mathbf{V}^h,\ \lambda_{\mathbf{n}}^h\in M_{\mathbf{n}}^h \text{ and } \boldsymbol{\lambda}_{\mathbf{t}}^h\in M_{\mathbf{t}}^h(S^h(\lambda_{\mathbf{n}}^h)) \text{ such that}\\ a(\mathbf{u}^h,\mathbf{v}^h)-\displaystyle\int_{\Gamma_C^1}\lambda_{\mathbf{n}}^h[\![v_{\mathbf{n}}^h]\!]-\int_{\Gamma_C^1}\boldsymbol{\lambda}_{\mathbf{t}}^h\cdot[\![\mathbf{v}_{\mathbf{t}}^h]\!]=L(\mathbf{v}^h) & \forall\mathbf{v}^h\in\mathbf{V}^h,\\ \displaystyle\int_{\Gamma_C^1}\left([\![u_{\mathbf{n}}^h]\!]-g_0^h\right)(\mu_{\mathbf{n}}^h-\lambda_{\mathbf{n}}^h)\geq 0 & \forall\mu_{\mathbf{n}}^h\in M_{\mathbf{n}}^h,\\ \displaystyle\int_{\Gamma_C^1}\mathbf{d}_{\mathbf{t}}(\mathbf{u}^h)\cdot(\boldsymbol{\mu}_{\mathbf{t}}^h-\boldsymbol{\lambda}_{\mathbf{t}}^h)\geq 0 & \forall\boldsymbol{\mu}_{\mathbf{t}}^h\in M_{\mathbf{t}}^h(S^h(\lambda_{\mathbf{n}}^h)),\end{cases} \tag{10.11}$$

where $\mathbf{M}_{\mathbf{n}}^h$ and $M_{\mathbf{t}}^h$ are discrete sets. Many possibilities exist to define these sets. As previously, the expression of $S^h(\lambda_{\mathbf{n}}^h)$ depends on the type of friction that is considered. For Tresca friction, we set

$$S^h(\lambda_{\mathbf{n}}^h)=s_T^{1,h},$$

where $s_T^{1,h}$ is a discrete approximation of s_T^1. In the case of Coulomb friction, we write it as in Chap. 9 and set

$$S^h(\lambda_{\mathbf{n}}^h)=\mathscr{F}\lambda_{\mathbf{n}}^h.$$

In the frictionless case and for Tresca friction, this formulation can be obtained by duality arguments, from the weak problem, and following the same steps as in Chap. 7. Notably, in the frictionless case, it can be obtained as the optimality system that characterizes the saddle-point of the following Lagrangian:

$$\mathscr{L}(\mathbf{u},\lambda_{\mathbf{n}}):=\mathcal{J}(\mathbf{u})-\int_{\Gamma_C^1}\lambda_{\mathbf{n}}u_{\mathbf{n}}-I_{\Lambda_C^1}(\lambda_{\mathbf{n}}),$$

where

$$\Lambda_C^1:=\{\tau\in(W_C^1)'\mid\langle\tau,w\rangle_{\Gamma_C^1}\geq 0,\ \forall w\in W_C^1,\ w\leq 0\}.$$

We now consider the mixed formulations for both LAC and mortar approaches, in the case of frictionless contact. In the LAC method, we choose piecewise constant nonpositive Lagrange multipliers on the macro-mesh T^M on Γ_C^1, i.e., in the convex cone:

$$M^h_{\mathrm{LAC}} = \left\{ \mu^h \in W^h_{\mathrm{LAC},1} \;\middle|\; \mu^h \leq 0 \;\text{ on }\; \Gamma_C^1 \right\},$$

where

$$W^h_{\mathrm{LAC},1} = \left\{ \mu^h \in L^2(\Gamma_C^1) \;\middle|\; \mu^h|_{T^m} \in \mathbb{P}_0(T^m), \forall T^m \in T^M \right\}.$$

For the Mortar method, we choose continuous piecewise of degree k multipliers defined on one trace mesh of Γ_C^1. These multipliers are weakly nonpositive in the integral sense:

$$M^h_{\mathrm{Mortar}} = \left\{ \mu^h \in W_1^h \;\middle|\; \int_{\Gamma_C} \mu^h \psi^h \leq 0,\; \forall \psi^h \in W_1^h,\, \psi^h \geq 0 \right\},$$

where

$$W_1^h := \left\{ \psi^h \in \mathscr{C}(\overline{\Gamma_C^1}) \;\middle|\; \exists \mathbf{v}^h \in \mathbf{V}_1^h \;\text{ such that }\; \mathbf{v}^h \cdot \mathbf{n}_1 = \psi^h \text{ on } \Gamma_C^1 \right\}.$$

10.2.4 Augmented Lagrangian

As in Chap. 7, an Augmented Lagrangian formulation for the above frictional contact problem between two elastic bodies can be obtained and reads as follows:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h,\, \lambda^h_{\mathbf{n}} \in W^h,\, \boldsymbol{\lambda}^h_{\mathbf{t}} \in \mathbf{W}^h_{\mathbf{t}} \text{ such that, for } (\mathbf{v}^h, \mu^h_{\mathbf{n}}, \boldsymbol{\mu}^h_{\mathbf{t}}) \in \mathbf{V}^h \times W^h \times \mathbf{W}^h_{\mathbf{t}} \\ a(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C^1} \left(\gamma_L \left([\![u^h_{\mathbf{n}}]\!] - g^h_0 \right) - \lambda^h_{\mathbf{n}} \right)_+ [\![v^h_{\mathbf{n}}]\!] \\ + \displaystyle\int_{\Gamma_C^1} \left(\gamma_L d_{\mathbf{t}}(\mathbf{u}^h) - \boldsymbol{\lambda}^h_t \right)_{S^h(u^h_{\mathbf{n}}, \lambda^h_{\mathbf{n}})} \cdot [\![\mathbf{v}^h_{\mathbf{t}}]\!] = L(\mathbf{v}^h) \\ - \displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma_L} \left(\left(\lambda^h_{\mathbf{n}} + \left(\gamma_L \left([\![u^h_{\mathbf{n}}]\!] - g^h_0 \right) - \lambda^h_{\mathbf{n}} \right)_+ \right) \mu^h_{\mathbf{n}} \right. \\ \left. + \left(\boldsymbol{\lambda}^h_{\mathbf{t}} + \left(\gamma_L d_{\mathbf{t}}(\mathbf{u}^h) - \boldsymbol{\lambda}^h_t \right)_{S^h(u^h_{\mathbf{n}}, \lambda^h_{\mathbf{n}})} \right) \cdot \boldsymbol{\mu}^h_{\mathbf{t}} \right) = 0. \end{cases} \tag{10.12}$$

The notation $\gamma_L > 0$ stands for a discrete positive function on the slave contact boundary that scales as γ_0/h, where here γ_0 plays the role of an augmentation parameter. Once again, $S^h(u^h_{\mathbf{n}}, \lambda^h_{\mathbf{n}})$ is an approximation of the friction threshold S

defined previously. In the case of Tresca friction, it is given by

$$S^h(u^h_{\mathbf{n}}, \lambda^h_{\mathbf{n}}) = s_T^{1,h},$$

where $s_T^{1,h}$ is an approximation of s_T^1. In the case of Coulomb friction, the discrete threshold can be

$$S^h(u^h_{\mathbf{n}}, \lambda^h_{\mathbf{n}}) = \mathscr{F}\left(\gamma_L([\![u^h_{\mathbf{n}}]\!] - g_0^h) - \lambda^h_{\mathbf{n}}\right)_+.$$

As for Nitsche, the above expression is always non-negative, which is an advantage in terms of numerical robustness.

In the frictionless case and for Tresca friction, this formulation can be obtained by duality arguments, from the weak problem and following the same steps as in Chap. 7. Notably, in the frictionless case, it can be obtained as the optimality system that characterizes the saddle-point of the following augmented Lagrangian:

$$\mathscr{L}_{\gamma_L}(\mathbf{u}, \lambda_{\mathbf{n}}) := \mathcal{J}(\mathbf{u}) + \int_{\Gamma_C} \frac{1}{2\gamma_L}\left(\left[\gamma_L\left([\![u_{\mathbf{n}}]\!] - g_0\right) - \lambda_{\mathbf{n}}\right]_+^2 - \lambda_{\mathbf{n}}^2\right).$$

10.2.5 Penalty

Finally, a simple penalty method reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that } \forall\, \mathbf{v}^h \in \mathbf{V}^h \\ a(\mathbf{u}^h, \mathbf{v}^h) \\ +\displaystyle\int_{\Gamma_C^1} \gamma_P \left([\![u^h_{\mathbf{n}}]\!] - g_0^h\right)_+ [\![v^h_{\mathbf{n}}]\!] + \int_{\Gamma_C^1} \gamma_P \left(\mathbf{d_t}(\mathbf{u}^h)\right)_{S_P^h(\mathbf{u}^h)} \cdot [\![v^h_{\mathbf{t}}]\!] = L(\mathbf{v}^h), \end{cases} \tag{10.13}$$

where $\gamma_P > 0$ is a discrete positive function and $S_P^h(\mathbf{u}^h)$ is an approximation of the friction threshold S defined previously. In the case of Tresca friction, it is given by

$$S_P^h(\mathbf{u}^h) = s_T^{1,h},$$

where $s_T^{1,h}$ is an approximation of s_T^1. In the case of Coulomb friction, the discrete threshold is

$$S_P^h(\mathbf{u}^h) := \mathscr{F}\left([\![u^h_{\mathbf{n}}]\!] - g_0^h\right)_+.$$

There are many possibilities to derive this method. As in Chap. 7, it can be considered as a byproduct of the Augmented Lagrangian formulation combined with an Uzawa solver. It can be obtained as well heuristically from Nitsche's formulation if one takes the parameter γ_N large enough and neglects the terms involving the normal Cauchy stress. Last but not least, in the frictionless case, formulation (10.13) corresponds to the first order optimality condition associated to the penalized functional:

$$\mathcal{J}_{b,P}(\mathbf{u}^h) := \mathcal{J}(\mathbf{u}^h)^2 + \int_{\Gamma_C^1} \frac{\gamma_P}{2} \left([\![u_{\mathbf{n}}^h]\!] - g_0^h \right)_+^2 .$$

10.3 Unbiased Formulation for Self- and Multi-body Contact

If the master–slave formulation consists in a natural extension of the Signorini problem that models contact between a deformable body and a rigid ground, it has no complete theoretical justification and induces detection difficulties in the case of self-contact and multi-body contact. We provide in this section an alternative unbiased formulation presented in [84] that circumvents these difficulties. We do not distinguish between a master surface and a slave one since we impose the nonpenetration and the friction conditions on both of them. Unbiased contact and friction formulations have been considered before in [238] and references therein.

10.3.1 Derivation

In order to obtain an unbiased method, we prescribe the contact condition on the two surfaces in a symmetric way. We introduce two contact mappings, one for each surface, denoted by Π^1 and Π^2, with associated Jacobians J^1 and J^2. We will suppose, to simplify, that $\Pi^2 = (\Pi^1)^{-1}$. The conditions describing contact on Γ_C^i $(i = 1, 2)$ are now

$$[\![u_{\mathbf{n}}]\!]^i - g_0 \leq 0, \quad \sigma_{\mathbf{n}}(\mathbf{u}^i) \leq 0, \quad \sigma_{\mathbf{n}}(\mathbf{u}^i)[\![u_{\mathbf{n}}]\!]^i = 0, \tag{10.14}$$

with

$$[\![u_{\mathbf{n}}]\!]^1 = u_{\mathbf{n}}^1 - u_{\mathbf{n}}^2 \circ \Pi^1$$

and

$$[\![u_{\mathbf{n}}]\!]^2 = u_{\mathbf{n}}^2 - u_{\mathbf{n}}^1 \circ \Pi^2 = -[\![u_{\mathbf{n}}]\!]^1 \circ \Pi^2.$$

Let us also define

$$[\![\mathbf{u_t}]\!]^1 = \mathbf{u_t}^1 - \mathbf{u_t}^2 \circ \Pi^1$$

and

$$[\![\mathbf{u_t}]\!]^2 = \mathbf{u_t}^2 - \mathbf{u_t}^1 \circ \Pi^2 = -[\![\mathbf{u_t}]\!]^1 \circ \Pi^2.$$

We consider the Tresca model for friction. Let us denote by $s_T^i \in L^2(\Gamma_C^i)$, $s_T^i \geq 0$, the Tresca friction threshold associated with each surface. The Tresca friction conditions on Γ_C^i, $i = 1, 2$ read

$$\begin{cases} \|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^i)\| \leq s_T^i & \text{if } [\![\mathbf{u}_{\mathbf{t}}]\!]^i = 0, \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^i) = -s_T^i \dfrac{[\![\mathbf{u}_{\mathbf{t}}]\!]^i}{\|[\![\mathbf{u}_{\mathbf{t}}]\!]^i\|} & \text{otherwise.} \end{cases} \tag{10.15}$$

Finally, we need to take into account explicitly the third Newton law between the two bodies:

$$\begin{cases} \displaystyle\int_{\gamma_C^1} \sigma_{\mathbf{n}}(\mathbf{u}^1) - \int_{\gamma_C^2} \sigma_{\mathbf{n}}(\mathbf{u}^2) = 0, \\ \displaystyle\int_{\gamma_C^1} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^1) + \int_{\gamma_C^2} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^2) = \mathbf{0}, \end{cases}$$

where γ_C^1 is an arbitrary subset of Γ_C^1 and $\gamma_C^2 = \Pi^1(\gamma_C^1)$. Mapping all terms on γ_C^1 allows to write:

$$\begin{cases} \sigma_{\mathbf{n}}(\mathbf{u}^1) - J^1 \sigma_{\mathbf{n}}(\mathbf{u}^2 \circ \Pi^1) = 0, \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^1) + J^1 \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^2 \circ \Pi^1) = \mathbf{0}, \end{cases} \quad \text{on } \Gamma_C^1. \tag{10.16}$$

Let us mention that, due to third Newton law, we need to fix the friction threshholds $\mathfrak{s}_T^1$ and $\mathfrak{s}_T^2$ such that

$$-s_T^1 \frac{[\![\mathbf{u}_{\mathbf{t}}]\!]^1}{\|[\![\mathbf{u}_{\mathbf{t}}]\!]^1\|} = \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^1) = -J^1 \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^2 \circ \Pi^1) = J^1 s_T^2 \frac{[\![\mathbf{u}_{\mathbf{t}}]\!]^2 \circ \Pi^1}{\|[\![\mathbf{u}_{\mathbf{t}}]\!]^2 \circ \Pi^1\|} = -J^1 s_T^2 \frac{[\![\mathbf{u}_{\mathbf{t}}]\!]^1}{|[\![\mathbf{u}_{\mathbf{t}}]\!]^1|}.$$

It results that the following compatibility condition on s_T^1 and s_T^2 needs to be satisfied:

$$s_T^1 = J^1 s_T^2 \circ \Pi^1. \tag{10.17}$$

10.3.2 The Unbiased Method

The reformulation of the contact and friction conditions (10.14)–(10.15) reads

$$\sigma_{\mathbf{n}}(\mathbf{u}^i) = \left[\gamma_N^i [\![u_{\mathbf{n}}]\!]^i - \sigma_{\mathbf{n}}(\mathbf{u}^i)\right]_+ ,$$

$$\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^i) = \left[\gamma_N^i [\![\mathbf{u_t}]\!]^i - \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^i)\right]_{\mathfrak{S}_T^i},$$

where γ_N^i plays the same role as γ_N on each contact surface Γ_C^i. Still using the Green formula, the different equations considered as well as the Nitsche's writing of the contact and friction conditions, we obtain the following unbiased finite element approximation:

$$\left\{\begin{array}{l} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that,} \\[1ex] A_{\theta\gamma}^{1,2}(\mathbf{u}^h, \mathbf{v}^h) + \dfrac{1}{2}\displaystyle\sum_{i=1,2}\int_{\Gamma_C^i} \dfrac{1}{\gamma_N^i}\left[\mathrm{P}_{1,\gamma^i}^{i,\mathbf{n}}(\mathbf{u}^h) - g_0\right]_+ \mathrm{P}_{\theta,\gamma^i}^{i,\mathbf{n}}(\mathbf{v}^h) \\ +\dfrac{1}{2}\displaystyle\sum_{i=1,2}\int_{\Gamma_C^i} \dfrac{1}{\gamma_N^i}\left[\mathbf{P}_{1,\gamma^i}^{i,\mathbf{t}}(\mathbf{u}^h)\right]_{\mathfrak{S}_T^i} \cdot \mathbf{P}_{\theta,\gamma^i}^{i,\mathbf{t}}(\mathbf{v}^h) = L(\mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \end{array}\right. \tag{10.18}$$

with the notations:

$$\mathrm{P}_{\theta,\gamma^i}^{i,\mathbf{n}}(\mathbf{v}) = \gamma_N^i [\![v_\mathbf{n}]\!]^i - \theta\sigma_\mathbf{n}(\mathbf{v}^i),$$

and

$$\mathbf{P}_{\theta,\gamma^i}^{i,\mathbf{t}}(\mathbf{v}) = \gamma_N^i [\![\mathbf{v_t}]\!]^i - \theta\boldsymbol{\sigma}_\mathbf{t}(\mathbf{v}^i),$$

as well as

$$A_{\theta\gamma}^{1,2}(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{2}\left(\int_{\Omega^i} \boldsymbol{\sigma}(\mathbf{u}^i) : \boldsymbol{\varepsilon}(\mathbf{v}^i) - \frac{1}{2}\int_{\Gamma_C^i} \frac{\theta}{\gamma_N^i}\boldsymbol{\sigma}(\mathbf{u}^i)\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^i)\mathbf{n}\right).$$

Note that in the above method the two contact surfaces are treated similarly. This small strain formulation allows an extended mathematical study that is impossible to perform in a large transformations framework. Indeed, all the mathematical analysis carried out for the unilateral contact problem can be adapted for this method and yields the same theoretical properties of well-posedness and convergence. This analysis as well as a complete numerical study in the small strain framework can be found in [84].

10.4 Further Comments

Many different methods have been reviewed in this chapter to take into account contact between two elastic bodies in the small deformation framework. For practical solving, the same techniques as already described in Chaps. 6 and 7 can be applied. We provide here some additional information about numerical solution methods.

10.4.1 Semi-smooth Newton Once Again

The three discrete weak forms obtained using Nitsche, the Augmented Lagrangian and penalty require an iterative solver that treats the nonlinear terms. For the Augmented Lagrangian, an Uzawa solver, as detailed in Chap. 7, can be implemented. Otherwise, remark that the three methods can be recasted as

$$B(\mathbf{x}, \mathbf{y}) + \langle \varphi \circ P(\mathbf{x}), Q(\mathbf{y}) \rangle = L(\mathbf{y}),$$

where $\mathbf{x}$ is the unknown, $\mathbf{y}$ the test function, $B(\cdot, \cdot)$ is a bilinear form, $P(\cdot)$ and $Q(\cdot)$ are affine operators on the boundary, and φ is a nonlinear function that incorporates the contact and friction conditions on the boundary. It is supposed nonsmooth, but it is generally Lipschitz continuous and nondifferentiable in the classical sense only at some specific points. Such a formulation can be solved with a semi-smooth Newton method, and the tangent problem can be expressed generally as

$$B(\delta\mathbf{x}, \mathbf{y}) + \langle \varphi'(P(\mathbf{x}); P'(\delta\mathbf{x})), Q(\mathbf{y}) \rangle = -R(\mathbf{x}; \mathbf{y}),$$

where φ' is the directional derivative of φ and $R(\cdot; \cdot)$ is the residual associated to the weak formulation.

10.4.2 Numerical Integration

Comparatively to partial differential equations with standard boundary conditions, there are specific issues in the case of contact and friction, since the integrands involve strong and weak discontinuities. Indeed, there are nonmatching meshes, and the tangent systems involve for instance the Heaviside function and the positive part operators. These issues have been addressed for instance in [127] for the Mortar method and [208, 209] for Nitsche's method.

Chapter 11
Contact and Self-contact in Large Strain

In many engineering applications, the elastic bodies in contact may undergo large transformations. In this case, a setting such as presented in Chap. 3 and in Chap. 10 is no longer valid, since it was restricted to small displacements and small strain. Taking into account large displacements and large strain, for instance in the hyperelastic framework, introduces many additional difficulties. First, one starts with losing many of the nice mathematical properties associated to the Signorini problem, notably the minimization problem does not involve anymore a quadratic functional and a convex set of admissible displacements. Therefore, it is much more challenging to study the well-posedness of the continuous problem, or to prove the convergence of the numerical methods. Moreover, difficulties appear in the formulation and numerical resolution of the contact and friction problems in this case. Notably, the gap function is no longer a datum (conversely to the setting in previous Chap. 10) but becomes an unknown: computing its derivative is challenging, and some issues of nondifferentiability, even of noncontinuity, can alter the behavior of solving procedures such as semi-smooth Newton. Fortunately, these issues have been given attention for some decades, and the existing techniques allow to compute some numerical solutions within this framework. This chapter will try to review some of them.

11.1 Setting

We consider one or many deformable bodies and suppose that they undergo a static or quasi-static deformation that can be described using hyperelasticity [46, 88, 129, 143, 204].

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0_11

11.1.1 Hyperelasticity

As it is the most common choice in computational solid mechanics, we describe their motion in the Lagrangian framework. The notation $\mathbf{X}$ is used for the Lagrangian coordinates and used as an index for the Lagrangian differential operators, so that, for instance, $\nabla_{\mathbf{X}}$ denotes the gradient with respect to the Lagrangian coordinates. Let Ω be an open bounded set in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary. It stands for the reference configuration of the elastic body/bodies. The displacement is described by the vector field

$$\mathbf{u} : \overline{\Omega} \to \mathbb{R}^d.$$

To differentiate the reference and the deformed configurations, we will make use of the notations provided in Fig. 11.1. We denote the transformation map by

$$\begin{aligned} \boldsymbol{\varphi} : \overline{\Omega} &\longrightarrow \mathbb{R}^d, \\ \mathbf{X} &\longmapsto \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}). \end{aligned}$$

We denote by Γ_D the Dirichlet boundaries for each solid: each body is clamped on Γ_D. The notation Γ_N stands for the Neumann boundary.

The remaining regions of the boundary are the contact boundaries, and we denote them by Γ_C. On these boundaries, contact and friction may possibly happen. In the case of only one elastic body, self-contact can happen on this region. If a master–slave strategy is chosen, we will divide accordingly the contact boundary between a slave surface, denoted by Γ_C^S, and a master surface, denoted by Γ_C^M. Otherwise, for unbiased contact, we will make no specific difference and apply the contact conditions on the whole contact surface. Now, contrarily to what happened for

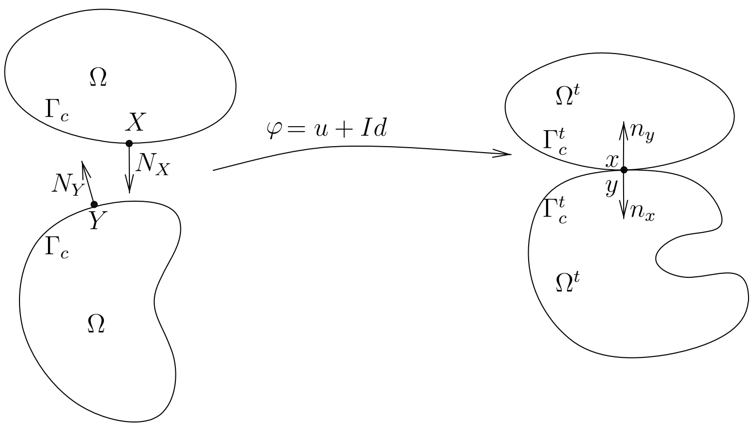

Fig. 11.1 Deformation of elastic bodies in large strain (Lagrangian framework)

the Signorini problem (Chap. 3), or for contact between two bodies in small strain (Chap. 10), formulating the nonpenetration condition is not straightforward, and, for this purpose, we need the following mapping:

$$\begin{aligned} \Pi : \Gamma_C^S &\longrightarrow \Gamma_C^M \\ \mathbf{X} &\longmapsto \mathbf{Y} = \Pi(\mathbf{X}) \end{aligned}$$

that associates material points from the slave surface Γ_C^S to their contact candidates on the master surface Γ_C^M. Contrary to the small deformation framework, this mapping is not a given datum but depends on the displacement $\mathbf{u}$. We will provide later on two ways of specifying it.

As usual, we denote by $\mathbf{F} := \nabla_{\mathbf{X}}\boldsymbol{\varphi} = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}$ the gradient of the deformation, with $J = \det(\mathbf{F})$ the corresponding Jacobian. The right Cauchy-Green tensor is denoted by $\mathbf{C}(:= \mathbf{F}^{\mathrm{T}}\mathbf{F})$, and the Green–Lagrange deformation tensor is denoted by $\mathbf{E} := (\mathbf{C} - \mathbf{I})/2$. We still use the notation $\boldsymbol{\sigma}$ for the Cauchy stress, and $\widehat{\boldsymbol{\sigma}} := J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ are, respectively, the first and the second Piola–Kirchhoff stress tensors. We recall that, for hyperelastic bodies, there exists an energy potential $\mathcal{W}$ whose value depends on the deformation through the tensors $\mathbf{E}$ or $\mathbf{C}$, see, e.g., [88, 143], and that the expression of the second Piola–Kirchhoff stress tensor is obtained thanks to

$$\mathbf{S} = \frac{\partial\mathcal{W}}{\partial\mathbf{E}}(\mathbf{E}) = 2\frac{\partial\mathcal{W}}{\partial\mathbf{C}}(\mathbf{C}).$$

To emphasize on the writing of contact and friction terms, we will use the potential energy of the whole system, $\mathcal{J}(\mathbf{u})$, which is written for instance, when gravitational forces are taken into account:

$$\mathcal{J}(\mathbf{u}) := \int_\Omega \mathcal{W}(\mathbf{E}) - \int_\Omega \rho_0\, \mathbf{g}\cdot\mathbf{u},$$

where ρ_0 is the density in the reference configuration and $\mathbf{g}$ is the gravity vector. Of course, for different problems, this potential energy may contain other terms, for instance, associated to the Neumann boundary Γ_N.

11.1.2 The Contact Mapping and the Gap Function

This section provides more details about how the contact mapping and the gap function can be defined. We introduced in the last section the contact mapping Π that links every point $\mathbf{X}$ of the slave contact surface to another point $\mathbf{Y}$ of the master contact surface. There are many ways to build this mapping, and the most common strategy relies on the orthogonal projection of $\mathbf{x} = \varphi(\mathbf{X})$ on the deformed master contact surface, as depicted in Fig. 11.2a and suggested in, e.g., [190].

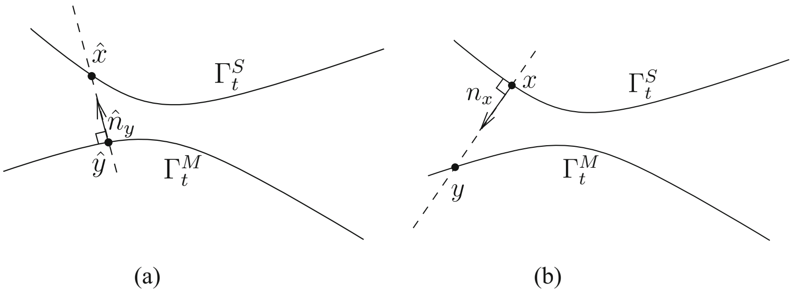

Fig. 11.2 Two possible strategies to build the contact mapping. (**a**) Contact mapping with projection. (**b**) Contact mapping with raytracing

Another strategy that can be called raytracing corresponds to Fig. 11.2b. It consists in defining $\mathbf{y}$ as the nearest intersection between the master contact surface and the straight line coming from $\mathbf{x}$ and following the normal $\mathbf{n_x}$ to the slave contact surface [228]. For both strategies, the corresponding gap functions are given by

$$\begin{aligned} g &= \mathbf{n_x} \cdot (\mathbf{y} - \mathbf{x}), \qquad &\text{(raytracing)},\\ \text{and} \quad g &= \mathbf{n_y} \cdot (\mathbf{x} - \mathbf{y}), \qquad &\text{(projection)}. \end{aligned}$$

There also holds:

$$\mathbf{y} = \mathbf{x} + g\,\mathbf{n_x}, \qquad \text{(raytracing)}, \tag{11.1}$$

$$\text{and} \quad \mathbf{y} = \mathbf{x} - g\,\mathbf{n_y}, \qquad \text{(projection)}. \tag{11.2}$$

Note that there is a certain reciprocity between these two strategies since for the raytracing one, the point $\mathbf{x}$ is most of the time the projection of the point $\mathbf{y}$ on the slave contact surface.

11.1.3 Contact and Friction Conditions

The nonpenetration condition is written as

$$g(\mathbf{u}) \geq 0.$$

The frictional contact condition may be described either with the Cauchy stress tensor or with the first Piola–Kirchhoff stress tensor. Here we choose to describe it with the Piola stress. For this purpose, we split the Piola stress at point $\mathbf{X}$ (slave boundary) into normal and tangential components:

$$\widehat{\boldsymbol{\sigma}}(\mathbf{u})\mathbf{N} = \underbrace{(\widehat{\boldsymbol{\sigma}}(\mathbf{u})\mathbf{N}\cdot\mathbf{n})}_{=\widehat{\sigma}_{\mathbf{n}}(\mathbf{u})}\mathbf{n} + \underbrace{(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\,\widehat{\boldsymbol{\sigma}}(\mathbf{u})\mathbf{N}}_{=\widehat{\boldsymbol{\sigma}}_{\mathbf{t}}(\mathbf{u})},$$

where $\mathbf{n}$ is either $\mathbf{n_x}$, for raytracing, or $-\mathbf{n_y}$, for projection, such that there always holds

$$\mathbf{y} = \mathbf{x} + g\,\mathbf{n}. \tag{11.3}$$

When contact is activated ($\widehat{\sigma}_{\mathbf{n}} < 0$), normals $\mathbf{n_x}$ and $\mathbf{n_y}$ are opposite. The quantity $\widehat{\sigma}_{\mathbf{n}}$ represents the contact pressure, in reference configuration, at point $\mathbf{X}$, and it should be nonpositive. As a result, contact conditions can be written as

$$g(\mathbf{u}) \geq 0 \tag{11.4a}$$

$$\widehat{\sigma}_{\mathbf{n}}(\mathbf{u}) \leq 0 \qquad\qquad \text{on } \Gamma_C^S. \tag{11.4b}$$

$$\widehat{\sigma}_{\mathbf{n}}(\mathbf{u})\; g(\mathbf{u}) = 0. \tag{11.4c}$$

For an arbitrary positive function γ on the slave contact boundary Γ_C^S, the above contact conditions (11.4a)–(11.4b)–(11.4c) can be rewritten alternatively

$$\widehat{\sigma}_{\mathbf{n}}(\mathbf{u}) = -[-\widehat{\sigma}_{\mathbf{n}}(\mathbf{u}) - \gamma g(\mathbf{u})]_+. \tag{11.5}$$

The above equation is obtained exactly as in Proposition 6.1 of Chap. 6. For Augmented Lagrangian formulations, it is generally preferred to use the negative part

$$[x]_- = \max(0, -x),$$

for $x \in \mathbb{R}$, notably because of the chosen convention for the gap ($g(\mathbf{u}) \geq 0$). If we reformulate contact conditions (11.4a)–(11.4b)–(11.4c) using the negative part, we get

$$\widehat{\sigma}_{\mathbf{n}}(\mathbf{u}) = -[\widehat{\sigma}_{\mathbf{n}}(\mathbf{u}) + \gamma g(\mathbf{u})]_-.$$

When there is friction, the normal and tangential stress are coupled to the relative slip velocity. This relative velocity could have been defined as

$$\mathbf{v}_r(\mathbf{X}) = \dot{\varphi}(\mathbf{X}) - \dot{\varphi}(\mathbf{Y}),$$

where $\dot{\varphi}$ is the time derivative of φ, but this definition has a drawback: this relative velocity does not respect the objectivity principle when the gap function is positive. Then, we should better use the definition

$$\mathbf{v}_r(\mathbf{X}) = \dot{\varphi}(\mathbf{X}) - \dot{\varphi}(\mathbf{Y}) + g\,\dot{\mathbf{n}}\,,$$

which is an objective quantity (see [98]) and is equal to the slip velocity when contact is activated (above $\dot{\mathbf{n}}$ denotes the time derivative of $\mathbf{n}$). These definitions are made for a time-dependent problem, and we can adapt them for a static setting thanks to a time discretization. For this purpose, let us introduce the notations φ_0 and $\mathbf{n}_0$, for the deformation and the normal at the previous time step, respectively, and $\Delta t > 0$ for the discrete time step. We should set in this case

$$\mathbf{v}_r(\mathbf{X}) = \frac{1}{\Delta t}(\varphi(\mathbf{X}) - \varphi(\mathbf{Y})) - \frac{1}{\Delta t}(\varphi_0(\mathbf{X}) - \varphi_0(\mathbf{Y})),$$

for the simple relative velocity, and

$$\begin{aligned}\mathbf{v}_r(\mathbf{X}) &= \frac{1}{\Delta t}(\varphi(\mathbf{X}) - \varphi(\mathbf{Y}) + g\,\mathbf{n}) - \frac{1}{\Delta t}(\varphi_0(\mathbf{X}) - \varphi_0(\mathbf{Y}) + \mathbf{g}\,\mathbf{n}_0) \\ &= -\frac{1}{\Delta t}(\varphi_0(\mathbf{X}) - \varphi_0(\mathbf{Y}) + g\,\mathbf{n}_0),\end{aligned}$$

for the objective relative velocity. Note that we used the relationship (11.3) to obtain the second line.

In the continuous case, this relative velocity should be tangent when the contact is persistent, but this might not be the case anymore when the problem is discretized in space. As a result, one can define the tangent increment of displacement as

$$\mathbf{d_t} = \Delta t(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{v}_r.$$

In the sequel, and for simplicity reasons, we will just set

$$\mathbf{d_t} = \Delta t\, \mathbf{v}_r.$$

We note $\mathscr{F}$ the friction coefficient and write the Coulomb friction as

$$\begin{cases} \|\widehat{\boldsymbol{\sigma}}_\mathbf{t}(\mathbf{u})\| \le -\mathscr{F}\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u}) & \text{if } \mathbf{d_t} = \mathbf{0}, \\ \widehat{\boldsymbol{\sigma}}_\mathbf{t}(\mathbf{u}) = \mathscr{F}\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u})\dfrac{\mathbf{d_t}}{\|\mathbf{d_t}\|} & \text{otherwise.} \end{cases} \tag{11.6}$$

Still for $\gamma > 0$, the above friction condition is equivalent to

$$\widehat{\boldsymbol{\sigma}}_\mathbf{t}(\mathbf{u}) = [\widehat{\boldsymbol{\sigma}}_\mathbf{t}(\mathbf{u}) - \gamma\mathbf{d_t}]_{-\mathscr{F}\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u})}, \tag{11.7}$$

where the notation $[\cdot]_{-\mathscr{F}\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u})}$ stands for the projection onto the closed ball of radius $-\mathscr{F}\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u}) \ge 0$, as in Chap. 8. The equivalence between formulations (11.6) and (11.7) can be proven exactly the same way as in Proposition 8.2.

11.2 Augmented Lagrangian Formulations

We provide now weak and finite element formulations for contact and friction in large strain. We start in this section with Augmented Lagrangian formulations that are the most widespread in this context and have a long time history. The next section will be dedicated to Nitsche method.

11.2.1 A First Augmented Lagrangian Formulation

Let us first give an Augmented Lagrangian formulation for frictionless contact. In a similar way as in the small deformations framework (see Sect. 7.5), we can define an Augmented Lagrangian as follows:

$$\mathscr{L}_\gamma(\mathbf{u}, \lambda_\mathbf{n}) := \mathscr{J}(\mathbf{u}) + \int_{\Gamma_C^S} \frac{1}{2\gamma_L}\left[[\lambda_\mathbf{n} + \gamma_L g(\mathbf{u})]_-^2 - \lambda_\mathbf{n}^2 \right],$$

for every admissible displacement $\mathbf{u}$ and Lagrange multiplier $\lambda_\mathbf{n}$. Above the Lagrange multiplier $\lambda_\mathbf{n}$ stands formally for the contact pressure $\widehat{\boldsymbol{\sigma}}_\mathbf{n}(\mathbf{u})$ on the slave surface Γ_C^S. The augmentation parameter is denoted by γ_L. As in previous chapters, it can be in fact a function on the slave contact boundary that depends locally of the mesh size. The first order optimality system associated with this Augmented Lagrangian is obtained by direct derivation and reads

$$\begin{cases} \text{Find } \mathbf{u} \text{ and } \lambda_\mathbf{n} \text{ solutions to:} \\ \mathscr{J}'(\mathbf{u}; \delta\mathbf{u}) - \displaystyle\int_{\Gamma_C^S} [\lambda_\mathbf{n} + \gamma_L g(\mathbf{u})]_- g'(\mathbf{u}; \delta\mathbf{u}) = 0, \quad \forall\, \delta\mathbf{u}, \\ -\displaystyle\int_{\Gamma_C^S} \frac{1}{\gamma_L} (\lambda_\mathbf{n} + [\lambda_\mathbf{n} + \gamma_L g(\mathbf{u})]_-)\, \delta\lambda_\mathbf{n} = 0, \quad \forall\, \delta\lambda_\mathbf{n}. \end{cases} \tag{11.8}$$

Remark already that this formulation involves the directional derivative $g'(\mathbf{u}; \delta\mathbf{u})$ in the direction $\delta\mathbf{u}$. This point is not obvious and is detailed in Sect. 11.4. For frictional contact, the augmented Lagrangian formulation reads (see, e.g., [228])

$$\begin{cases} \text{Find } \mathbf{u} \text{ and } (\lambda_\mathbf{n}, \boldsymbol{\lambda}_\mathbf{t}) \text{ solutions to:} \\ \mathscr{J}'(\mathbf{u}; \delta\mathbf{u}) - \displaystyle\int_{\Gamma_C^S} [\lambda_\mathbf{n} + \gamma_L g(\mathbf{u})]_- g'(\mathbf{u}; \delta\mathbf{u}) \\ \qquad - \displaystyle\int_{\Gamma_C^S} [\boldsymbol{\lambda}_\mathbf{t} - \gamma_L \mathbf{d}_\mathbf{t}(\mathbf{u})]_{\mathscr{F}[\lambda_\mathbf{n}+\gamma_L g(\mathbf{u})]_-} \cdot \mathbf{d}_\mathbf{t}'(\mathbf{u}; \delta\mathbf{u}) = 0, \quad \forall\, \delta\mathbf{u}, \\ -\displaystyle\int_{\Gamma_C^S} \frac{1}{\gamma_L} \left(\boldsymbol{\lambda} + [\lambda_\mathbf{n} + \gamma_L g(\mathbf{u})]_- \mathbf{n} - [\boldsymbol{\lambda}_\mathbf{t} - \gamma_L \mathbf{d}_\mathbf{t}(\mathbf{u})]_{\mathscr{F}[\lambda_\mathbf{n}+\gamma_L g(\mathbf{u})]_-}\right) \cdot \delta\boldsymbol{\lambda} = 0, \quad \forall\, \delta\boldsymbol{\lambda}, \end{cases} \tag{11.9}$$

with $\boldsymbol{\lambda} = \lambda_{\mathbf{n}}\mathbf{n} + \boldsymbol{\lambda}_{\mathbf{t}}$. We recover in the second equation of (11.9) a weak equivalent formulation of contact and friction conditions (11.4) and (11.6). However, the multipliers $\lambda_{\mathbf{n}}$ and $\boldsymbol{\lambda}_{\mathbf{t}}$ can be interpreted as boundary stresses only if g' and $\mathbf{d}'_{\mathbf{t}}$ correspond to relative displacements. We will see later on that this is indeed the case for g' when projection is chosen (see formula (11.24)), but supplementary terms are present for raytracing (see formula (11.23)). The derivative $\mathbf{d}'_{\mathbf{t}}$ reads:

$$\mathbf{d}'_t(\mathbf{u}; \delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}) + (\mathbf{F}_0 - \mathbf{F})Y'(\mathbf{u}; \delta\mathbf{u}),$$

for the simple relative velocity and

$$\mathbf{d}'_t(\mathbf{u}; \delta\mathbf{u}) = \mathbf{F}_0\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) - g'(\mathbf{u}; \delta\mathbf{u})\mathbf{n}_0,$$

for the objective relative velocity. In both cases, the quantity $\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u})$ appears, see Sect. 11.4 for detailed calculations. Moreover, this supposes a calculation of the second order directional derivative of $\mathbf{Y}$ for the tangent system, when the problem is solved with semi-smooth Newton.

11.2.2 Another Augmented Lagrangian Formulation

We can also simplify (11.9) and substitute $g'(\mathbf{u}; \delta\mathbf{u})$ with $-\mathbf{n} \cdot (\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}))$ and $\mathbf{d}'_{\mathbf{t}}(\mathbf{u}; \delta\mathbf{u})$ with $(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}))$. This modifies the interpretation of the multiplier $\boldsymbol{\lambda}$ but preserves the correct imposition of contact and friction conditions. We then obtain

$$\begin{cases} \text{Find } \mathbf{u} \text{ and } (\lambda_{\mathbf{n}}, \boldsymbol{\lambda}_{\mathbf{t}}) \text{ solutions to:} \\ \mathcal{J}'(\mathbf{u}; \delta\mathbf{u}) \\ + \displaystyle\int_{\Gamma_C^S} \left([\lambda_{\mathbf{n}} + \gamma_L g(\mathbf{u})]_- \mathbf{n} - [\boldsymbol{\lambda}_{\mathbf{t}} - \gamma_L \mathbf{d}_{\mathbf{t}}(\mathbf{u})]_{\mathscr{F}[\lambda_{\mathbf{n}} + \gamma_L g(\mathbf{u})]_-}\right) \cdot (\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y})) = 0, \\ - \displaystyle\int_{\Gamma_C^S} \frac{1}{\gamma_L} \left(\boldsymbol{\lambda} + [\lambda_{\mathbf{n}} + \gamma_L g(\mathbf{u})]_- \mathbf{n} - [\boldsymbol{\lambda}_{\mathbf{t}} - \gamma_L \mathbf{d}_{\mathbf{t}}(\mathbf{u})]_{\mathscr{F}[\lambda_{\mathbf{n}} + \gamma_L g(\mathbf{u})]_-}\right) \cdot \delta\boldsymbol{\lambda} = 0, \\ \forall\, (\delta\mathbf{u}, \delta\boldsymbol{\lambda}). \end{cases} \tag{11.10}$$

With the above modification, the symmetry of the tangent problem is lost, even if the friction coefficient $\mathscr{F}$ vanishes.

Now, let $\mathbf{V}^h$ and $\mathbf{W}^H$ be two finite element spaces for the displacement and the Lagrange multiplier, where h and H stand for mesh sizes on the bulk and on the slave contact boundary, respectively. A finite element method for contact and friction in large strain, based on the above Augmented Lagrangian, reads

$$\left\{\begin{array}{l}\text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ and } \boldsymbol{\lambda}^H \in \mathbf{W}^H \text{ solutions to:}\\ \mathcal{J}'(\mathbf{u}^h;\delta\mathbf{u}^h)\\ +\displaystyle\int_{\Gamma_C^S}\Big([\lambda_{\mathbf{n}}^H+\gamma_L g(\mathbf{u}^h)]_-\mathbf{n}-[\boldsymbol{\lambda}_{\mathbf{t}}^H-\gamma_L\mathbf{d}_{\mathbf{t}}(\mathbf{u}^h)]_{\mathscr{F}[\lambda_{\mathbf{n}}^H+\gamma_L g(\mathbf{u}^h)]_-}\Big)\\ \qquad\qquad\cdot(\delta\mathbf{u}^h(\mathbf{X})-\delta\mathbf{u}^h(\mathbf{Y}))=0,\\ \\ -\displaystyle\int_{\Gamma_C^S}\frac{1}{\gamma_L}\Big(\boldsymbol{\lambda}^H+[\lambda_{\mathbf{n}}^H+\gamma_L g(\mathbf{u}^h)]_-\mathbf{n}-[\boldsymbol{\lambda}_{\mathbf{t}}^H-\gamma_L\mathbf{d}_{\mathbf{t}}(\mathbf{u}^h)]_{\mathscr{F}[\lambda_{\mathbf{n}}^H+\gamma_L g(\mathbf{u}^h)]_-}\Big)\\ \qquad\qquad\cdot\delta\boldsymbol{\lambda}^H=0,\\ \\ \ \forall(\delta\mathbf{u}^h,\delta\boldsymbol{\lambda}^H)\in\mathbf{V}^h\times\mathbf{W}^H.\end{array}\right. \tag{11.11}$$

11.3 Unbiased and Biased Nitsche Formulations

Using Nitsche's formulation of contact conditions, an unbiased method can be derived directly, and then we will adapt this formulation for biased contact with no difficulty. So, we follow in this section [210], and, to start with, we do not distinguish between a master and slave surfaces and get back to the original notation Γ_C for the potential contact surface, in the reference configuration, of the different bodies, including the situation of self-contact. In this case, the contact mapping Π can be defined for the whole contact surface, as follows:

$$\begin{aligned}\Pi:\Gamma_C &\longrightarrow \Gamma_C\\ \mathbf{X} &\longmapsto \mathbf{Y}=\Pi(\mathbf{X}).\end{aligned}$$

As in the previous chapters, we denote by $\gamma_N > 0$ the Nitsche parameter. As in the previous chapter, it can be a function on the contact boundary that depends locally on the size of the boundary facets. Moreover, we introduce, formally, appropriate operators for large strain contact and friction:

$$\mathrm{P}^{\mathbf{n}}_{1,N} : \mathbf{v} \mapsto -\widehat{\sigma}_{\mathbf{n}}(\mathbf{v}) - \gamma_N g(\mathbf{v}), \qquad \text{and} \qquad \mathbf{P}^{\mathbf{t}}_{1,N} : \mathbf{v} \mapsto \widehat{\boldsymbol{\sigma}}_{\mathbf{t}}(\mathbf{v}) - \gamma_N \mathbf{d}_{\mathbf{t}}(\mathbf{v})\,.$$

Note that they extend naturally the previous definitions of $\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)$ and $\mathbf{P}^{\mathbf{t}}_{1,N}(\cdot)$, but that they are no more linear operators.

11.3.1 Frictionless Contact

To start with, let us focus on the case of frictionless contact. As in the small strain framework, one first possibility consists in defining a symmetric version from the derivation of a modified energy functional. The following result holds:

Proposition 11.1 *Let us define the energy functional $\mathcal{J}_{\mathrm{N}}(\cdot)$ that takes into account the body deformation as well as nonpenetration formulated in a Nitsche's manner:*

$$\mathcal{J}_{\mathrm{N}}(\mathbf{u}) := \mathcal{J}(\mathbf{u}) - \int_{\Gamma_C} \frac{1}{4\gamma_N} \widehat{\sigma}_{\mathbf{n}}^2(\mathbf{u}) + \int_{\Gamma_C} \frac{1}{4\gamma_N} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})\right]_+^2, \tag{11.12}$$

where $\gamma_N > 0$ is the Nitsche's parameter and $\widehat{\sigma}_{\mathbf{n}}$ is the normal stress in reference configuration. The first order optimality system associated to $\mathcal{J}_{\mathrm{N}}(\cdot)$ reads

$$\begin{cases} \textit{Find } \mathbf{u} \textit{ solution to:} \\ \mathcal{J}'(\mathbf{u};\delta\mathbf{u}) - \displaystyle\int_{\Gamma_C} \frac{1}{2\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u})\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u};\delta\mathbf{u}), + \int_{\Gamma_C} \frac{1}{2\gamma_N} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})]_+ {\mathrm{P}^{\mathbf{n}}_{1,N}}'(\mathbf{u};\delta\mathbf{u}) = 0 \\ \quad \forall\, \delta\mathbf{u}, \end{cases} \tag{11.13}$$

where the derivative of $\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)$ *reads*

$${\mathrm{P}^{\mathbf{n}}_{1,N}}'(\mathbf{u};\delta\mathbf{u}) = -\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u};\delta\mathbf{u}) - \gamma_N g'(\mathbf{u};\delta\mathbf{u}).$$

Proof Let us write the optimality system associated to $\mathcal{J}_{\mathrm{N}}(\cdot)$:

$$\begin{aligned} \mathcal{J}'_{\mathrm{N}}(\mathbf{u};\delta\mathbf{u}) &= \mathcal{J}'(\mathbf{u};\delta\mathbf{u}) - \int_{\Gamma_C} \frac{1}{2\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u})\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u};\delta\mathbf{u}) \\ &+ \int_{\Gamma_C} \frac{1}{4\gamma_N} \left([\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)]_+^2\right)'(\mathbf{u};\delta\mathbf{u}) = 0 \quad \forall\,\delta\mathbf{u}. \end{aligned}$$

As in Chap. 6, with properties (6.3), we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}[x]_+^2 = H_s(x)\,[x]_+ = [x]_+\,,$$

for $x \in \mathbb{R}$. With the above result and the chain rule, we obtain

$$\int_{\Gamma_C} \frac{1}{4\gamma_N} \left([\mathrm{P}^{\mathbf{n}}_{1,N}(\cdot)]_+^2\right)'(\mathbf{u};\delta\mathbf{u}) = \int_{\Gamma_C} \frac{1}{2\gamma_N} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})]_+ {\mathrm{P}^{\mathbf{n}}_{1,N}}'(\mathbf{u};\delta\mathbf{u})$$

and thus the weak form (11.13). □

Remark that, conversely to mixed, penalty, or Augmented Lagrangian formulations, the derivative of the normal stress, $\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u}^h;\delta\mathbf{u}^h)$, is involved. This is not something usual, and particularly, for the tangent system, the second order derivative of the normal stress is needed. This means that, in fact, the third order derivative of the hyperelastic potential $\mathcal{W}(\cdot)$ is needed [210]. Some useful formulas and an example for a Saint Venant–Kirchhoff material are provided in [210].

We can also derive an incomplete method, following the same ideas as in Chap. 6. The starting point is the following weak formulation:

$$\mathcal{J}'(\mathbf{u}; \delta\mathbf{u}) - \int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}} \delta u_{\mathbf{n}} = 0, \quad \forall \delta\mathbf{u}, \tag{11.14}$$

which stands for (quasi-)static equilibrium, and where contact forces have not been taken into account, yet. Then, we take into account third Newton law (action–reaction principle) that can be formulated, here, as

$$\widehat{\sigma}_{\mathbf{n}}(\mathbf{X}) + \widehat{\sigma}_{\mathbf{n}}(\mathbf{Y}) = 0,$$

where the two points $\mathbf{X}$ and $\mathbf{Y}$ on the contact boundary are related by the contact mapping: $\mathbf{Y} = \Pi(\mathbf{X})$. With the above relationship, we reformulate the boundary term as follows:

$$\begin{aligned}
&\int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}} \delta u_{\mathbf{n}} \\
&= \frac{1}{2} \int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}}(\mathbf{X}) \delta u_{\mathbf{n}}(\mathbf{X}) + \frac{1}{2} \int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}}(\mathbf{Y}) \delta u_{\mathbf{n}}(\mathbf{Y}) \\
&= \frac{1}{2} \int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}}(\mathbf{X}) \left(\delta u_{\mathbf{n}}(\mathbf{X}) - \delta u_{\mathbf{n}}(\mathbf{Y}) \right).
\end{aligned}$$

The above relationship combined with (11.14) yields

$$\mathcal{J}'(\mathbf{u}; \delta\mathbf{u}) - \frac{1}{2} \int_{\Gamma_C} \widehat{\sigma}_{\mathbf{n}}(\mathbf{X}) \left(\delta u_{\mathbf{n}}(\mathbf{X}) - \delta u_{\mathbf{n}}(\mathbf{Y}) \right) = 0, \quad \forall \delta\mathbf{u},$$

where we used the contact mapping $\mathbf{Y} = \Pi(\mathbf{X})$. There remains to take into account weakly contact conditions (11.5), and we get

$$\mathcal{J}'(\mathbf{u}; \delta\mathbf{u}) + \frac{1}{2} \int_{\Gamma_C} \left[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}) \right]_+ \left(\delta u_{\mathbf{n}}(\mathbf{X}) - \delta u_{\mathbf{n}}(\mathbf{Y}) \right) = 0, \quad \forall \delta\mathbf{u}. \tag{11.15}$$

This incomplete formulation does not involve anymore the derivative $\widehat{\sigma}'_{\mathbf{n}}$ and, for this reason, can be more attractive for implementation.

From the incomplete and the symmetric versions above, and following the same path in Chap. 6, we can also formulate a whole family of methods, indexed by a parameter $\theta \in \mathbb{R}$ that plays the same role as in Chap. 6. We introduce the following operators indexed by θ:

$$\mathrm{P}^{\mathbf{n}}_{\theta,N} : \mathbf{v} \mapsto -\theta\widehat{\boldsymbol{\sigma}}_{\mathbf{n}}(\mathbf{v}) - \gamma_N g(\mathbf{v}), \qquad \text{and} \qquad \mathbf{P}^{\mathbf{t}}_{\theta,N} : \mathbf{v} \mapsto \theta\widehat{\boldsymbol{\sigma}}_{\mathbf{t}}(\mathbf{v}) - \gamma_N \mathbf{d}_{\mathbf{t}}(\mathbf{v}) .$$

The (general) Nitsche-based formulation for large strain reads, formally,

$$\begin{cases} \text{Find } \mathbf{u} \text{ solution to:} \\ \mathcal{J}'(\mathbf{u}; \delta\mathbf{u}) - \displaystyle\int_{\Gamma_C} \frac{\theta}{2\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u})\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u}; \delta\mathbf{u}) + \int_{\Gamma_C} \frac{1}{2\gamma_N}[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u})]_+ {\mathrm{P}^{\mathbf{n}}_{\theta,N}}'(\mathbf{u}; \delta\mathbf{u}) = 0 \\ \quad \forall\, \delta\mathbf{u}. \end{cases} \tag{11.16}$$

Of course, we recover formulation (11.13) for $\theta = 1$ and formulation (11.15) for $\theta = 0$, and when the projection strategy is used for the contact mapping (see Sect. 11.4).

Let now $\mathbf{V}^h$ be a discrete (finite element) space of admissible displacements. Then, an unbiased Nitsche method for frictionless contact in large transformations reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ solution to:} \\ \mathcal{J}'(\mathbf{u}^h; \delta\mathbf{u}^h) - \displaystyle\int_{\Gamma_C} \frac{\theta}{2\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u}^h)\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u}^h; \delta\mathbf{u}^h), \\ + \displaystyle\int_{\Gamma_C} \frac{1}{2\gamma_N}[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ {\mathrm{P}^{\mathbf{n}}_{\theta,N}}'(\mathbf{u}^h; \delta\mathbf{u}^h) = 0 \quad \forall\, \delta\mathbf{u}^h \in \mathbf{V}^h. \end{cases} \tag{11.17}$$

11.3.2 Frictional Contact

In case of frictional contact, with Coulomb friction, the unbiased Nitsche formulation reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ solution to:} \\ \mathcal{J}'(\mathbf{u}^h; \delta\mathbf{u}^h) - \displaystyle\int_{\Gamma_C} \frac{\theta}{2\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u}^h)\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u}^h; \delta\mathbf{u}^h) - \int_{\Gamma_C} \frac{\theta}{2\gamma_N} \widehat{\sigma}_{\mathbf{t}}(\mathbf{u}^h) \cdot \widehat{\sigma}'_{\mathbf{t}}(\mathbf{u}^h; \delta\mathbf{u}^h) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{2\gamma_N}[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ {\mathrm{P}^{\mathbf{n}}_{\theta,N}}'(\mathbf{u}^h; \delta\mathbf{u}^h) \\ + \displaystyle\int_{\Gamma_C} \frac{1}{2\gamma_N}[\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h)]_{\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+} \cdot {\mathbf{P}^{\mathbf{t}}_{\theta,N}}'(\mathbf{u}^h; \delta\mathbf{u}^h) = 0 \quad \forall\, \delta\mathbf{u}^h \in \mathbf{V}^h. \end{cases} \tag{11.18}$$

For the incomplete variant $\theta = 0$, this formulation can be obtained following the same path as previously for the frictionless case. For the general method ($\theta \in \mathbb{R}$), see, for instance, [210] for the detailed derivation.

Remark 11.1 From the above formulations, it is easy to recover an unbiased method for penalty, as [238]. Let γ_P be the penalty parameter. An unbiased penalty method reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ solution to:} \\ \mathcal{J}'(\mathbf{u}^h; \delta\mathbf{u}^h) \\ +\dfrac{1}{2}\displaystyle\int_{\Gamma_C} \Big[-\gamma_P\, g(\mathbf{u}^h)\Big]_+ \Big(\delta u^h_{\mathbf{n}}(\mathbf{X}) - \delta u^h_{\mathbf{n}}(\mathbf{Y})\Big) \\ +\dfrac{1}{2}\displaystyle\int_{\Gamma_C} [\gamma_P \mathbf{d}_{\mathbf{t}}(\mathbf{u}^h)]_{\mathscr{F}[-\gamma_P g(\mathbf{u})]_+} \cdot \Big(\delta\mathbf{u}^h_{\mathbf{t}}(\mathbf{X}) - \delta\mathbf{u}^h_{\mathbf{t}}(\mathbf{Y})\Big) = 0 \quad \forall\, \delta\mathbf{u}^h \in \mathbf{V}^h. \end{cases} \tag{11.19}$$

Note that with mixed or Augmented Lagrangian methods, it is not obvious to do something equivalent, and solving for instance self-contact problems with these methods remains challenging.

11.3.3 A Biased Nitsche Method

Of course, for a master–slave formulation, we can also define a biased Nitsche formulation. For this purpose, we can follow the same path as previously, except that we take into account weakly the contact and friction conditions only on the slave surface. In other terms, there is no need to transform the contact boundary term as we did previously. The resulting biased Nitsche formulation, for contact with Coulomb friction, reads

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ solution to:} \\ \mathcal{J}'(\mathbf{u}^h; \delta\mathbf{u}^h) - \displaystyle\int_{\Gamma_C^S} \frac{\theta}{\gamma_N} \widehat{\sigma}_{\mathbf{n}}(\mathbf{u}^h)\widehat{\sigma}'_{\mathbf{n}}(\mathbf{u}^h; \delta\mathbf{u}^h) - \int_{\Gamma_C^S} \frac{\theta}{\gamma_N} \widehat{\sigma}_{\mathbf{t}}(\mathbf{u}^h) \cdot \widehat{\sigma}'_{\mathbf{t}}(\mathbf{u}^h; \delta\mathbf{u}^h) \\ + \displaystyle\int_{\Gamma_C^S} \frac{1}{\gamma_N} [\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+ {\mathrm{P}^{\mathbf{n}}_{\theta,N}}'(\mathbf{u}^h; \delta\mathbf{u}^h) \\ + \displaystyle\int_{\Gamma_C^S} \frac{1}{\gamma_N} [\mathbf{P}^{\mathbf{t}}_{1,N}(\mathbf{u}^h)]_{\mathscr{F}[\mathrm{P}^{\mathbf{n}}_{1,N}(\mathbf{u}^h)]_+} \cdot {\mathbf{P}^{\mathbf{t}}_{\theta,N}}'(\mathbf{u}^h; \delta\mathbf{u}^h) = 0 \quad \forall\, \delta\mathbf{u}^h \in \mathbf{V}^h. \end{cases} \tag{11.20}$$

Formally speaking, the only differences from the unbiased formulation are: 1) the surface integral is solely the slave contact surface Γ_C^S (and not the whole potential contact surface Γ_C) and 2) the factor $1/2$ that is dropped and replaced by 1.

Furthermore, because of the link between Nitsche and Augmented Lagrangian methods, we have already seen in Chap. 7, an alternative to write a biased Nitsche formulation for frictional contact in large strain consists in starting from the simplified augmented Lagrangian (11.11). Then, we substitute the multiplier $\boldsymbol{\lambda}$ with $\widehat{\boldsymbol{\sigma}}\mathbf{N}$, and we recover an incomplete variant $\theta = 0$.

11.4 Directional Derivatives of the Gap

As we have seen previously, we need the directional derivatives of various quantities such as the gap function. This is the object of the present section that details their derivation.

11.4.1 A Preliminary Result

First, we need a preliminary result about the directional derivative of the point $\mathbf{y}$ with respect to the displacement field $\mathbf{u}$. This calculation is not so straightforward, not to say a little bit awkward, since everything there is intricate: this derivative depends on the transformation map φ and also of $\mathbf{Y}$, but $\mathbf{Y}$ depends itself of the transformation map φ since it is defined indirectly from the deformed configuration (keep in mind both Figs. 11.1 and 11.2). So $\mathbf{Y}$ is not a datum and has its own directional derivative. Therefore, this calculation if the object of the lemma below:

Lemma 11.1 *Let* $\mathbf{y} = \varphi(\mathbf{Y})$ *be a point on the master contact surface in the deformed configuration. Let* $\delta\mathbf{u}$ *be an arbitrary increment of the displacement field. Suppose that* $\mathbf{y}$ *and* $\mathbf{Y}$ *are smooth enough with respect to variable* $\mathbf{u}$*. Then, there holds:*

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \mathbf{F}(\mathbf{Y}(\mathbf{u}))\,\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) + \delta\mathbf{u}(\mathbf{Y}(\mathbf{u}))\,. \tag{11.21}$$

Proof Let $\mathbf{y} = \varphi(\mathbf{Y})$, $\mathbf{u}$, $\delta\mathbf{u}$, and ε be given. We start with the relationship

$$\mathbf{y}(\mathbf{u}+\varepsilon\delta\mathbf{u}) = \mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u}) + (\mathbf{u}+\varepsilon\delta\mathbf{u})(\mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u})), \tag{11.22}$$

and we carry out Taylor expansions up to the first order in ε. First, there holds:

$$\mathbf{y}(\mathbf{u}+\varepsilon\delta\mathbf{u}) = \mathbf{y}(\mathbf{u})+\varepsilon\mathbf{y}'(\mathbf{u};\delta\mathbf{u})+\mathcal{O}(\varepsilon^2), \quad \mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u}) = \mathbf{Y}(\mathbf{u})+\varepsilon\mathbf{Y}'(\mathbf{u};\delta\mathbf{u})+\mathcal{O}(\varepsilon^2).$$

For the third term, we split it as follows:

$$\begin{aligned}
&(\mathbf{u}+\varepsilon\delta\mathbf{u})(\mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u}))\\
&= \mathbf{u}(\mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u})) + \varepsilon\delta\mathbf{u}(\mathbf{Y}(\mathbf{u}+\varepsilon\delta\mathbf{u}))\\
&= \mathbf{u}(\mathbf{Y}(\mathbf{u}) + \varepsilon\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) + \mathcal{O}(\varepsilon^2)) + \varepsilon\,(\delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) + \mathcal{O}(\varepsilon))\,.
\end{aligned}$$

Then, we apply the chain rule and get

$$\mathbf{u}(\mathbf{Y}(\mathbf{u}) + \varepsilon\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) + \mathcal{O}(\varepsilon^2)) = \mathbf{u}(\mathbf{Y}(\mathbf{u})) + \varepsilon\mathbf{u}'(\mathbf{Y}(\mathbf{u});\mathbf{Y}'(\mathbf{u};\delta\mathbf{u})) + \mathcal{O}(\varepsilon^2).$$

Moreover, there holds, by definition:

$$\mathbf{u}'(\mathbf{Y}(\mathbf{u}); \mathbf{Y}'(\mathbf{u}; \delta\mathbf{u})) = \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{Y}(\mathbf{u}))\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}).$$

Combining all the above expressions in (11.22), and then equaling all terms of first order in ε, leads to

$$\mathbf{y}'(\mathbf{u}; \delta\mathbf{u}) = \mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{Y}(\mathbf{u}))\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) + \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})).$$

We finally note that

$$\mathbf{F}(\mathbf{Y}(\mathbf{u}))\ \mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) = \mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{Y}(\mathbf{u}))\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u})$$

and we obtain (11.21). □

11.4.2 Derivatives of the Gap

Using the previous definitions of the gap function, we can now provide the expressions for the derivative. This is the object of the next proposition:

Proposition 11.2 *Suppose that the gap function g is smooth enough, and defined, either by* (11.1) *or by* (11.2)*. Then, the directional derivative of g can be expressed as*

$$g'(\mathbf{u}; \delta\mathbf{u}) = -\frac{\mathbf{n_y}}{\mathbf{n_x} \cdot \mathbf{n_y}} \cdot \big(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) + g\ \mathbf{n_x}'(\mathbf{u}; \delta\mathbf{u})\big)\ , \tag{11.23}$$

for raytracing,

$$g'(\mathbf{u}; \delta\mathbf{u}) = \mathbf{n_y} \cdot (\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})))\ , \tag{11.24}$$

for projection.

Remark that in the above expressions, we managed to get rid of the derivatives of $\mathbf{Y}$ and $\mathbf{y}$, and then these expressions depend only of computable quantities.

Proof We obtain first a useful orthogonality relationship. Indeed, we start by observing that the vector $\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u})$ is tangent to the master contact boundary Γ_C^M. It results that the vector $\mathbf{F}\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u})$ is tangent to the master contact boundary $\Gamma_C^{M,t}$, in deformed configuration, and therefore,

$$\mathbf{n_y} \cdot \mathbf{F}\ \mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) = 0\,. \tag{11.25}$$

Note also that, since $\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X})$, there holds

$$\mathbf{x}'(\mathbf{u}; \delta\mathbf{u}) = \mathbf{u}'(\mathbf{X})(\mathbf{u}; \delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}).$$

Using the above result as well as the product rule, we compute the derivatives of the expressions (11.1) and (11.2) in order to get alternative expressions for the derivative of $\mathbf{y}$:

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) + g'(\mathbf{u};\delta\mathbf{u})\,\mathbf{n_x} + g\,\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u})\,, \qquad \text{for raytracing,} \tag{11.26}$$

and

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - g'(\mathbf{u};\delta\mathbf{u})\,\mathbf{n_y} - g\,\mathbf{n_y}'(\mathbf{u};\delta\mathbf{u})\,, \qquad \text{for projection.} \tag{11.27}$$

For raytracing, it remains now to combine expressions (11.21) and (11.26):

$$\mathbf{F}(\mathbf{Y}(\mathbf{u}))\,\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) + \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) = \delta\mathbf{u}(\mathbf{X}) + g'(\mathbf{u};\delta\mathbf{u})\,\mathbf{n_x} + g\,\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u})$$

and then take the dot product with $\mathbf{n_y}$ and use (11.25):

$$\mathbf{n_y}\cdot\delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) = \mathbf{n_y}\cdot\delta\mathbf{u}(\mathbf{X}) + g'(\mathbf{u};\delta\mathbf{u})\,(\mathbf{n_y}\cdot\mathbf{n_x}) + g\,\mathbf{n}_x'(\mathbf{u};\delta\mathbf{u})\cdot\mathbf{n_y}.$$

We rearrange the terms and obtain (11.23).

We do the same for projection and combine expressions (11.21) and (11.27):

$$\mathbf{F}(\mathbf{Y}(\mathbf{u}))\,\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) + \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) = \delta\mathbf{u}(\mathbf{X}) - g'(\mathbf{u};\delta\mathbf{u})\,\mathbf{n_y} - g\,\mathbf{n_y}'(\mathbf{u};\delta\mathbf{u}).$$

Again, we take the dot product with $\mathbf{n_y}$ and use (11.25), as well as the orthogonality relationship $\mathbf{n_y}\cdot\mathbf{n_y}' = 0$:

$$\mathbf{n_y}\cdot\delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) = \mathbf{n_y}\cdot\delta\mathbf{u}(\mathbf{X}) - g'(\mathbf{u};\delta\mathbf{u}).$$

We end up with rearranging the terms and obtain (11.24). □

11.4.3 More and More Derivatives

The calculation of the derivative of the slave normal $\mathbf{n_x}$ does not add special difficulty and is detailed below.

Proposition 11.3 *Suppose that the slave surface Γ_C^S is smooth enough, as well as the displacement field* $\mathbf{u}$*. Then, the directional derivative of the normal* $\mathbf{n_x}$ *with respect to the displacement field* $\mathbf{u}$ *in a direction* $\delta\mathbf{u}$ *is given by*

$$\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}) = -(\mathbf{I} - \mathbf{n_x}\otimes\mathbf{n_x})(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}\,(\nabla_{\mathbf{X}}\delta\mathbf{u})^{\mathrm{T}}\,\mathbf{n_x}.$$

Proof The starting point is the Nanson formula [129] that maps boundary infinitesimal surfaces from the reference to the deformed configuration and that provides an expression of the normal $\mathbf{n_x}$ in the deformed configuration from the normal $\mathbf{N_X}$ in the reference configuration, using the gradient deformation tensor, as follows:

$$\mathbf{n_x}(\mathbf{u}) = \frac{(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}\mathbf{N_X}}{\|(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}\mathbf{N_X}\|}. \tag{11.28}$$

Let us do it now step by step. First, notice that $\mathbf{N_X}$ does not depend of the deformation $\mathbf{u}$, and apply the chain rule, as well as the formula for the directional derivative of the inverse of a tensor [46]:

$$(\mathbf{F}(\mathbf{u})^{-\mathrm{T}})'(\mathbf{u}; \delta\mathbf{u}) = -(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}(\mathbf{F}(\mathbf{u})^{\mathrm{T}})'(\mathbf{u}; \delta\mathbf{u})(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}.$$

From the definition of the gradient deformation tensor, we get

$$(\mathbf{F}(\mathbf{u})^{\mathrm{T}})'(\mathbf{u}; \delta\mathbf{u}) = (\nabla_{\mathbf{X}}\delta\mathbf{u})^{\mathrm{T}}.$$

Let us now set $\widetilde{\mathbf{n}}(\mathbf{u}) := (\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}\mathbf{N_X}$ so that $\mathbf{n_x}(\mathbf{u}) = \widetilde{\mathbf{n}}(\mathbf{u})/\|\widetilde{\mathbf{n}}(\mathbf{u})\|$. First, remark that combining the above calculations yields

$$\widetilde{\mathbf{n}}'(\mathbf{u}; \delta\mathbf{u}) = -(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}(\nabla_{\mathbf{X}}\delta\mathbf{u})^{\mathrm{T}}(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}\mathbf{N_X} = -(\mathbf{F}(\mathbf{u}))^{-\mathrm{T}}(\nabla_{\mathbf{X}}\delta\mathbf{u})^{\mathrm{T}}\widetilde{\mathbf{n}}(\mathbf{u}).$$

Now, from the expression (11.28), we apply the chain rule and the product rule and get, after some simplifications:

$$\mathbf{n_x}'(\mathbf{u}; \delta\mathbf{u}) = \frac{1}{\|\widetilde{\mathbf{n}}(\mathbf{u})\|}\left(\mathbf{I} - \mathbf{n_x}(\mathbf{u}) \otimes \mathbf{n_x}(\mathbf{u})\right)\widetilde{\mathbf{n}}'(\mathbf{u}; \delta\mathbf{u}).$$

There remains to combine the two above expressions, and the result follows. □

Since finite element formulations also require the expression of the derivative of $\mathbf{Y}$, we provide it in the following proposition.

Proposition 11.4 *Suppose that the gap function g is smooth enough and defined, either by (11.1) or by (11.2). Then, the directional derivative of $\mathbf{Y}$ can be expressed as*

$$\begin{aligned}
&\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) \\
&= \mathbf{F}^{-1}(\mathbf{Y}(\mathbf{u}))\left(\mathbf{I} - \frac{\mathbf{n_x} \otimes \mathbf{n_y}}{\mathbf{n_x} \cdot \mathbf{n_y}}\right)\left(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) - g(\mathbf{u})\,\mathbf{F}^{-\mathrm{T}}(\mathbf{X})\,(\nabla\delta\mathbf{u}(\mathbf{X}))^{\mathrm{T}}\,\mathbf{n_x}\right), \\
&\textit{for raytracing},
\end{aligned} \tag{11.29}$$

$$\begin{aligned}
&\mathbf{Y}'(\mathbf{u}; \delta\mathbf{u}) \\
&= \mathbf{F}^{-1}(\mathbf{Y}(\mathbf{u}))\left(\mathbf{I} - \mathbf{n_y} \otimes \mathbf{n_y}\right)\left(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})) - g(\mathbf{u})\,\mathbf{n_y}'(\mathbf{u}; \delta\mathbf{u})\right), \\
&\textit{for projection}.
\end{aligned} \tag{11.30}$$

Remark that in case of projection, the derivative $\mathbf{n_y}'(\mathbf{u};\delta\mathbf{u})$ of the normal to the master surface is needed. This is a complex issue that we will discuss just after the proof.

Proof First, from relationship (11.21), we get

$$\mathbf{Y}'(\mathbf{u};\delta\mathbf{u}) = \mathbf{F}^{-1}(\mathbf{Y}(\mathbf{u}))\ \left(\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u}))\right). \tag{11.31}$$

Let us start with the raytracing. Remember in this case that

$$\mathbf{y}(\mathbf{u}) = \mathbf{x} + g(\mathbf{u})\ \mathbf{n_x}(\mathbf{u})$$

from which we get, using the product rule:

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) + g'(\mathbf{u};\delta\mathbf{u})\mathbf{n_x}(\mathbf{u}) + g(\mathbf{u})\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}).$$

Using the expression of the derivative of the gap for raytracing, we obtain

$$\begin{aligned}\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - \Bigg(&\frac{\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\cdot(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u}))\\ &+\ g(\mathbf{u})\ \mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}))\Bigg)\ \mathbf{n_x}(\mathbf{u}) + g(\mathbf{u})\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}).\end{aligned}$$

We rewrite this as

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - \left(\frac{\mathbf{n_x}\otimes\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\ (\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})))\right) + \left(\mathbf{I} - \frac{\mathbf{n_x}\otimes\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\right) g(\mathbf{u})\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}).$$

We inject the above expression into (11.31):

$$\begin{aligned}&\mathbf{Y}'(\mathbf{u};\delta\mathbf{u})\\ &= \mathbf{F}^{-1}(\mathbf{Y}(\mathbf{u}))\ \left(\delta\mathbf{u}(\mathbf{X}) - \left(\frac{\mathbf{n_x}\otimes\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\ (\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})))\right)\right.\\ &\quad \left. + \left(\mathbf{I} - \frac{\mathbf{n_x}\otimes\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\right) g(\mathbf{u})\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u}))\right)\\ &= \mathbf{F}^{-1}(\mathbf{Y}(\mathbf{u}))\ \left(\left(\mathbf{I} - \frac{\mathbf{n_x}\otimes\mathbf{n_y}}{\mathbf{n_x}\cdot\mathbf{n_y}}\right)\left(\delta\mathbf{u}(\mathbf{X}) + g(\mathbf{u})\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u}))\right)\right).\end{aligned}$$

There just remains to replace the expression of $\mathbf{n_x}'(\mathbf{u};\delta\mathbf{u})$ with those obtained at the previous proposition, and we get (11.29).

Now, for projection, we use the formula

$$\mathbf{y}(\mathbf{u}) = \mathbf{x} - g(\mathbf{u})\ \mathbf{n_y}(\mathbf{u})$$

and apply also the product rule:

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - g'(\mathbf{u};\delta\mathbf{u})\mathbf{n}_{\mathbf{y}}(\mathbf{u}) - g(\mathbf{u})\mathbf{n}_{\mathbf{y}}{}'(\mathbf{u};\delta\mathbf{u}).$$

Using the expression of the derivative of the gap for projection, we obtain

$$\mathbf{y}'(\mathbf{u};\delta\mathbf{u}) = \delta\mathbf{u}(\mathbf{X}) - \big(\mathbf{n}_{\mathbf{y}}(\mathbf{u})\cdot(\delta\mathbf{u}(\mathbf{X}) - \delta\mathbf{u}(\mathbf{Y}(\mathbf{u})))\big)\,\mathbf{n}_{\mathbf{y}}(\mathbf{u}) - g(\mathbf{u})\mathbf{n}_{\mathbf{y}}{}'(\mathbf{u};\delta\mathbf{u}),$$

and we recover the desired formula. This ends the proof. □

11.4.4 Implications in Terms of Numerical Robustness

As a result, for projection, we will need a formula for the derivative of the normal $\mathbf{n}_{\mathbf{y}}$ to the master surface, with respect to the displacement $\mathbf{u}$. The first problem that arises is that this formula may be extremely complex, particularly because this derivative depends on the curvature of the master surface (have a look at the expression in [190] for instance). The second problem is that it can even be (mathematically) dangerous since $\mathbf{n}_{\mathbf{y}}$ is not necessarily continuous with respect to $\mathbf{u}$ in some situations, notably when projection is applied. Figure 11.3 depicts some situations when there is no more continuity. These discontinuities may lead to numerical failure when a semi-smooth Newton is applied. The situation (a) in Fig. 11.3 can be avoided thanks to $\mathscr{C}^1$ finite elements, which ensure the continuity

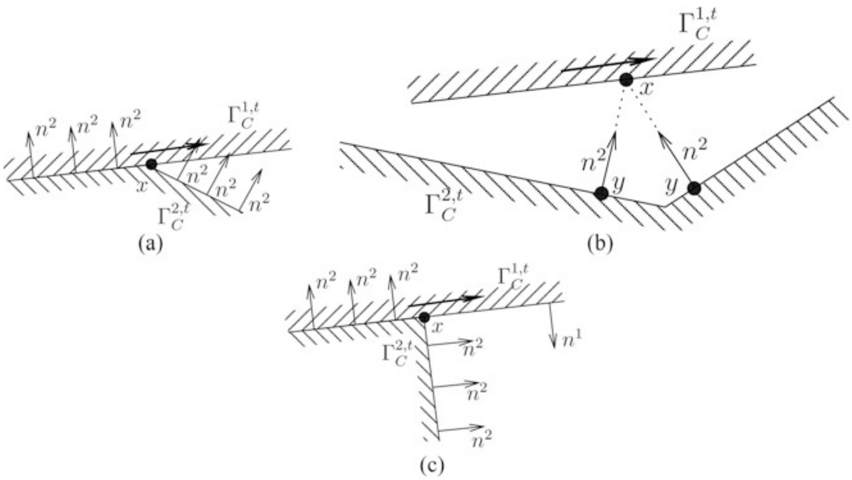

Fig. 11.3 Situations when the normal $\mathbf{n}_{\mathbf{y}}$ is discontinuous as a function of $\mathbf{x}$: (**a**) when crossing between two elements, for projection or raytracing, (**b**) in case of concavity for projection (in this case even $\mathbf{y}$ is discontinuous), (**c**) when raytracing fails (in this situation, $\mathbf{y}$ becomes undefinite)

of the normal when crossing between two adjacent elements. Nevertheless, this difficulty remains when surfaces with sharp angles are modeled. The situation (b) is harder to avoid, and the Newton algorithm may oscillate between two possible situations for $\mathbf{y}$.

11.5 Numerical Results

To end this chapter, we provide here numerical results made by using the Nitsche method and the GetFEM open source finite element library [234].

11.5.1 Elastic Half-Ring

First we report results for frictionless and frictional contact between an elastic ring undergoing large deformations and an elastic block. Both parts are assumed to exhibit neo-Hookean material behavior with Poisson ratio equal to 0.3. The elastic half-ring is assembled from outer and inner rings with the same thickness of 5 mm. The outer ring has a Young modulus E_{Rext} of 10^3 MPa, and the inner one is assumed to be 100 times stiffer. The Young modulus E_{B} is of 300 MPa for the block. The inner radius of the half-ring is equal to 90 mm. The block is 260 mm long and 50 mm high. The rectangular block is fixed at its bottom edge, while the ends of the half-ring are horizontally fixed and vertically displaced by a total distance of 70 mm in 140 steps of size 0.5 mm. Figure 11.4 shows the initial geometry and four deformed configurations at different time steps. The deformations are obtained without and with friction coefficient $\mathscr{F} = 0.5$. The coarsest mesh used in the calculations is made of 64 elements along the ring circumference and 1 element across each ring layer, while the block is discretized with 52 by 10 quadrilateral elements, in length and height directions, respectively. Parameters for Nitsche's method are: $\gamma_{0_R} = E_{\mathrm{Rext}}$, $\gamma_{0_B} = E_{\mathrm{B}}$, and $\theta = 0$.

This example allows us to test the accuracy of the Nitsche method in the case of heterogeneous materials and high friction forces. To compare the computed deformation with previous results from other methods, we measure the vertical displacement of the ring midpoint. This displacement along the load steps is plotted in Fig. 11.5, for both frictionless and frictional contact.

Figure 11.4 shows that the loaded half-ring compresses initially the elastic block on its central contact surface, as expected. At this stage, the frictional and frictionless cases are quite similar, and the central midpoint of the half-ring moves downward. This is observable on Fig. 11.5 until an amount of imposed displacement of 20–25 mm is reached. This corresponds to the first deformed configuration in Fig. 11.4. Subsequently, the tracked point is lifted progressively until 45mm of displacement. Then, in the interval between 45mm and 60mm, the lifting speed of the half-ring middle point peaks in the absence of friction, whereas it remains

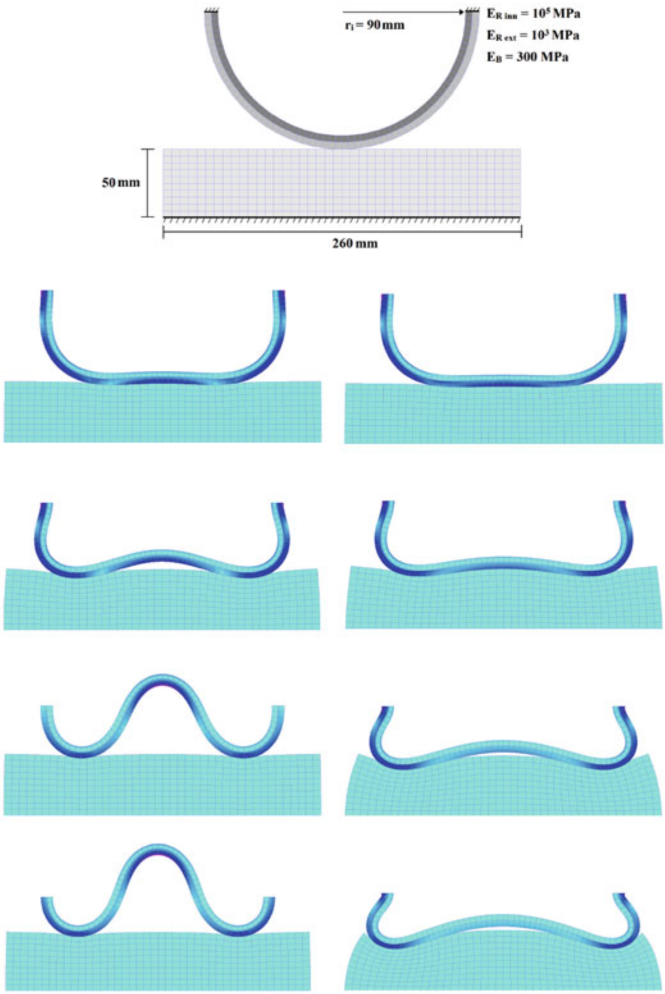

Fig. 11.4 Deformation of the elastic half-ring without friction (left) and with $\mathscr{F} = 0.5$ (right) after a loading of 25, 45, 60, and 70 mm

low in the frictional case because of extensive sliding between the ring and the block. In the remaining part of the simulation, the tracked point keeps on moving up, but with a lower speed in both cases. The results with coarse and refined meshes are very similar for the frictionless case, and also for the frictional one but only until 50 mm of displacement. In the last 20 mm of the simulation, a remarkable difference between the two approximations is observed, when considering friction. This could be due to the important sliding forces since we do not get that error in the frictionless case. Nitsche's parameter γ_0 needs to be large enough for stability and convergence, but when it is too large, the problem stiffens: some elements are

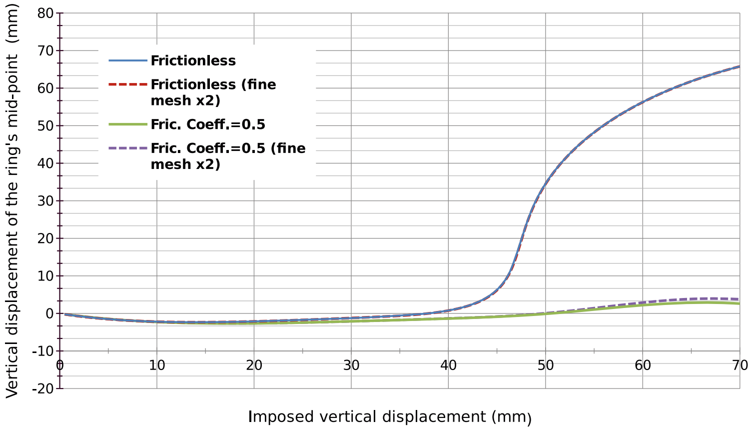

Fig. 11.5 Vertical displacement of the half-ring middle point for different mesh sizes

inverted and convergence is difficult to achieve. The optimal values of γ_0 are those near the Young modulus E. For $\gamma_0 = E$, we obtain an average of Newton's iterations of 4.45 for frictionless contact and 4.44 for frictional contact. With a fine mesh, the convergence speed is similar with a slight difference for the frictional case since the average in that case is 5.05.

11.5.2 *Crossed Tubes with Self-contact*

This second numerical example is the crossed tubes test. In this example, we simulate contact between two crossed hollow elastic cylinders. Each of the tubes has an outer diameter of 24 mm, a wall thickness equal to 0.8 mm, and a length equal to 100 mm. Neo-Hookean material behavior is considered for both tubes, with material parameters corresponding to Poisson ratio equal to 0.3 and Young moduli of $E_1 = 10^5$ MPa for the lower tube and $E_2 = 10^4$ MPa for the upper one. The tubes are forced into contact through Dirichlet conditions applied at their ends. The upper tube is displaced vertically for a total distance of 40 mm divided into 80 equal load steps. Since the enforced displacement is large, the deformations of the tubes are large, and we observe a self-contact configuration on the less rigid tube. So this test allows us to check if a numerical method is capable of taking into account self-contact. Since the geometry as well as the boundary conditions are symmetric, it is sufficient to model only one quarter of the considered structure. The actually modeled portion of each tube is colored in Fig. 11.6, and it is discretized using 16 by 24 by 2 three-dimensional elements in the length, circumferential, and radial directions, respectively.

The presented solution is based on an approximation of the geometry and of the displacement with quadratic hexahedral elements. The unbiased Nitsche method

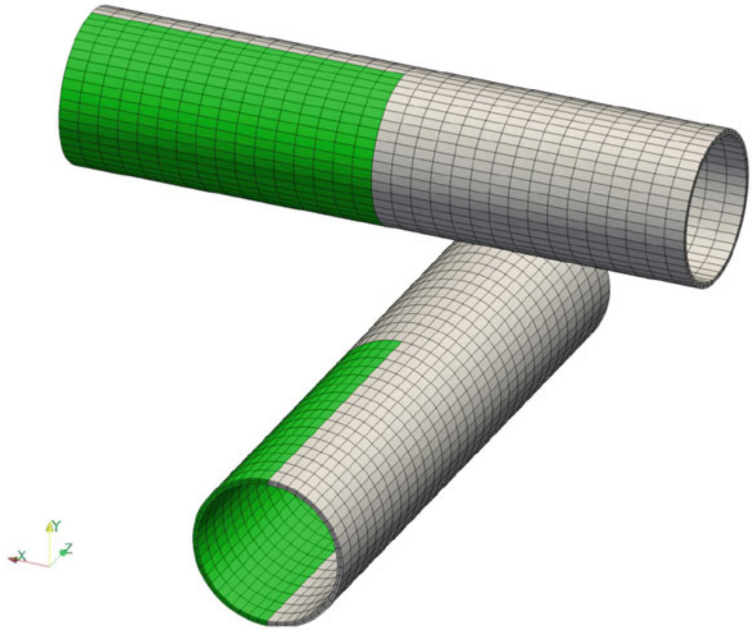

Fig. 11.6 Geometry and mesh of the crossed tubes in their initial configuration

with $\theta = 0$ is considered to deal with self-contact. The results of Figs. 11.7, 11.8, and 11.9 correspond to the frictionless case with a Nitsche's parameter $\gamma_0 = E_1$ for the lower tube and $\gamma_0 = E_2$ for the upper one. The figures depict the calculated deformed configurations for the 30th, 60th, and 80th load steps. Despite the increase of the required iterations when self-contact occurs, Newton's algorithm converges in general within a few iterations. The required iterations number for convergence increases from the 45th load step. This is due to the onset of self-contact.

11.6 Further Comments

Below we provide some extra information about theoretical and numerical issues related to contact and friction in the large strain framework.

11.6.1 Existence Results

In the case of frictionless contact in large strain, there are not many theorems that state the existence of weak solutions. Among the few existing results, let us mention

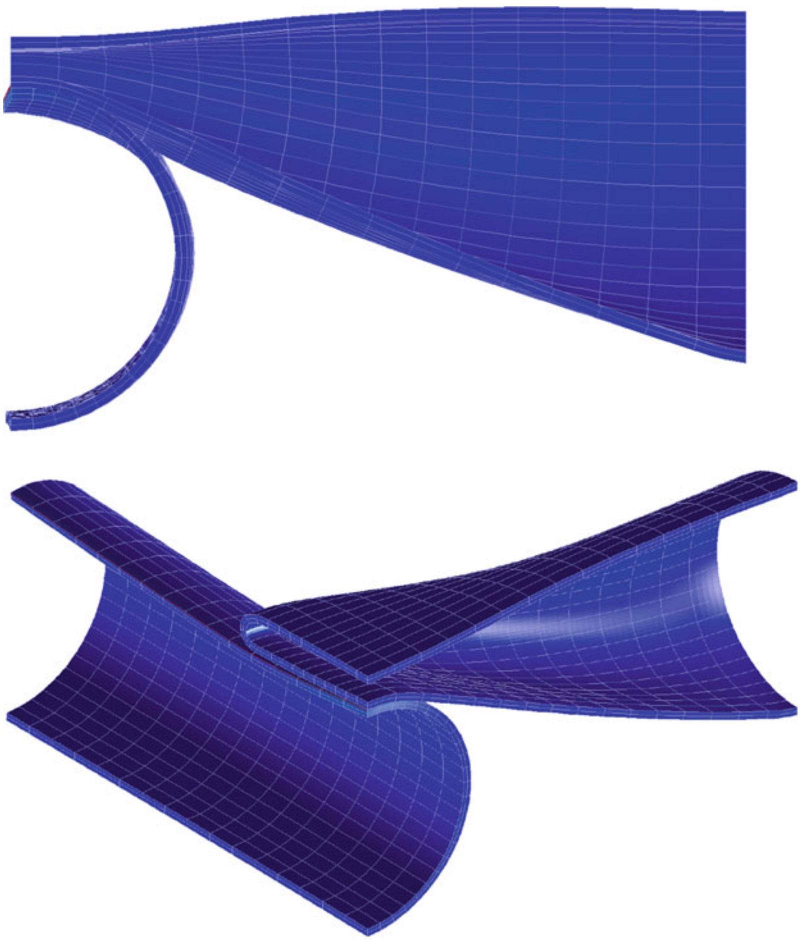

Fig. 11.7 Deformation and Von-Mises contour plot of the two crossed tubes test without friction for a loading of 20 mm

those of Ph. G. Ciarlet in 1988 [88] and O. Pantz in 2008 [224] that are restricted to hyperelasticity. For frictional contact in large strain, no existence results are known. This fact was already pointed out by Ph. G. Ciarlet, and it seems that there has been no progress since then. Similarly, for elastoplastic materials in large strain with contact, it seems no result of existence of the solution has been established yet.

Since almost no results are known for well-posedness at the continuous level and seem very hard to obtain, this makes difficult the numerical analysis in this situation. As a matter of fact, to our knowledge, there are no results of well-posedness and convergence for the discrete methods we reviewed in this chapter.

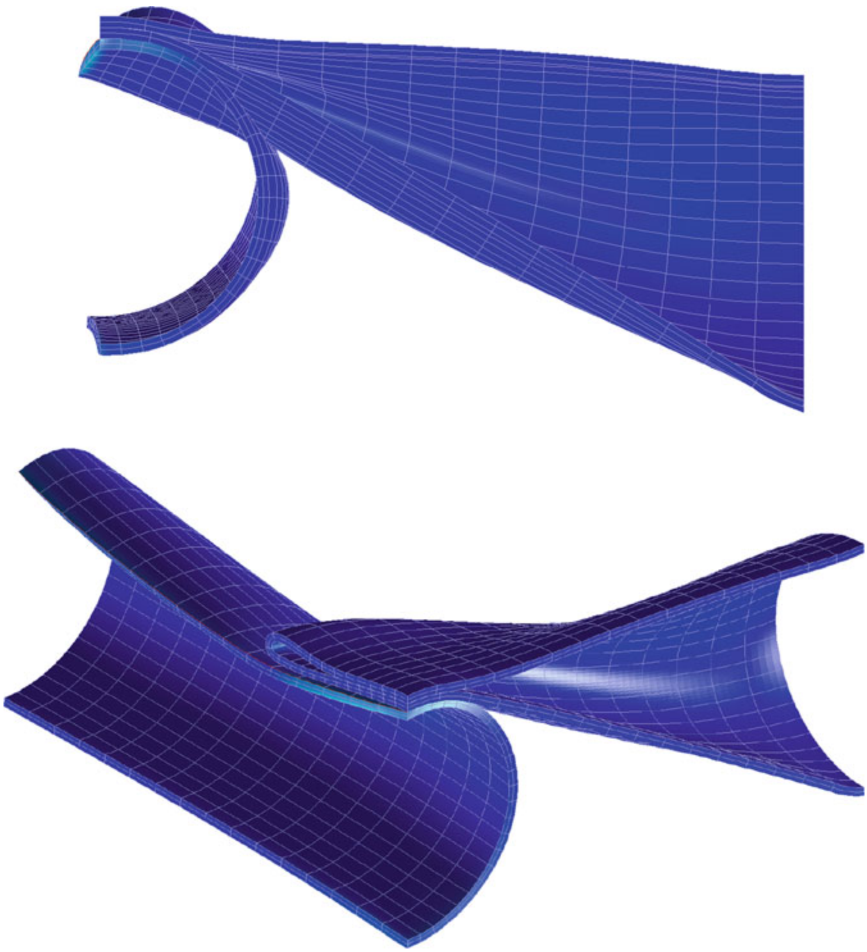

Fig. 11.8 Deformation and Von-Mises contour plot of the two crossed tubes test without friction for a loading of 30 mm

11.6.2 Numerical Methods

Among the first numerical methods for contact in large strain, let us mention the monograph of N. Kikuchi and J.T. Oden in 1988 [181, Chapter 12] that describes an incremental finite element method with penalty. Both the cases of hyperelasticity and elastoplastic deformations are treated. Let us also quote the seminal papers of J.C. Simo and T.A. Laursen in 1992 and 1993, where finite elements based on the Augmented Lagrangian are designed [191, 192, 247]. A detailed bibliography can be found in the recent articles [210, 228] that inspired the presentation of this chapter

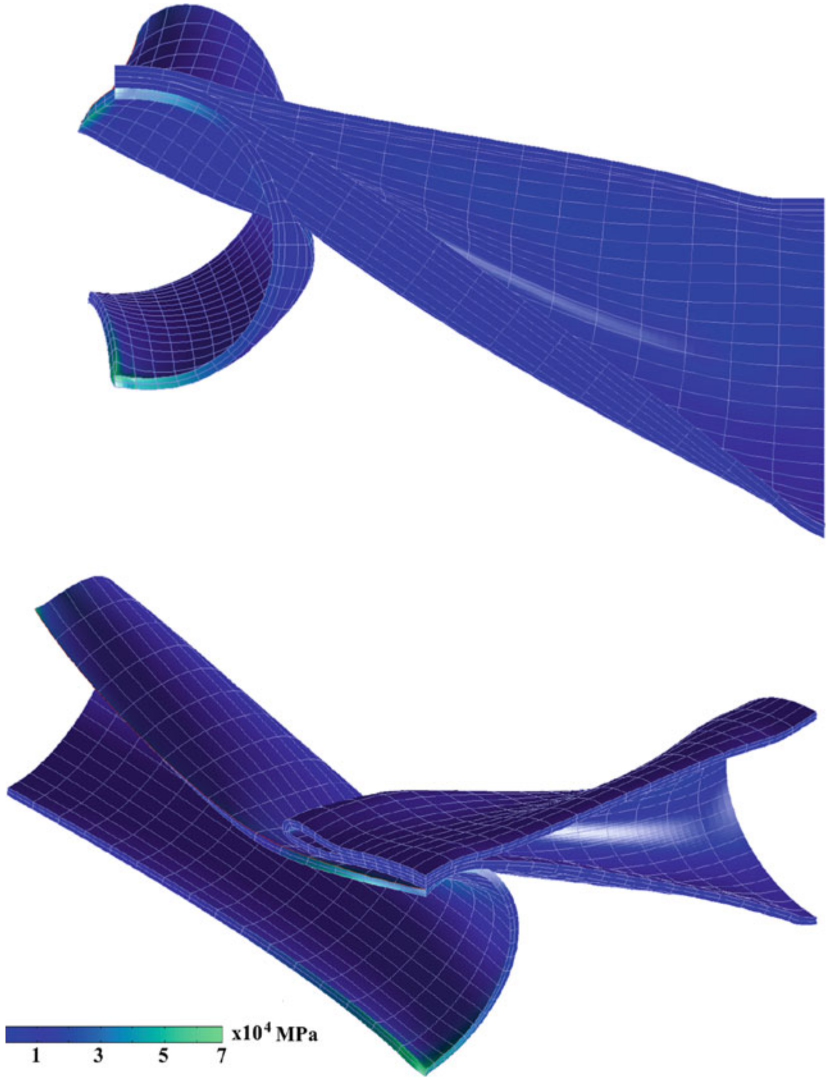

Fig. 11.9 Deformation and Von-Mises contour plot of the two crossed tubes test without friction for a loading of 40 mm

(see also the monographs of T.A. Laursen [190], of P. Wriggers [266], the article of J. Lengiewicz et al. [195], or the book chapter [83], for instance).

For unbiased contact in large strain, an original method has been designed by O. Pantz in 2011 [225], well-suited for self-contact, and applicable also for thin structures (see as well an application to fluid-structure-contact in [17]). However, this method has been designed only for frictionless contact. Later on, a penalty method for unbiased contact has been proposed in 2015 by R.A. Sauer and L. De

Lorenzis in [238]. The unbiased Nitsche method for large strain described in this chapter has been first presented in 2017 by R. Mlika et al., see [210]. Later on, in 2019, an extension to thermomechanical contact in large strain has been performed by A. Seitz et al. in [243]. In 2020, in [229], an unbiased Mortar method has been formulated by M.A. Puso and J.M. Solberg.

Appendix A
Test-Cases for Verification

For numerical experiments, especially for verification and for numerical study of the convergence rates, it is always useful to have some examples with a closed-form solution. However, for contact and friction, situations with a nontrivial closed-form solution are not frequent. In many numerical studies, the approximate solution is compared to another approximate solution, obtained for instance with a higher order discretization, a finer mesh or another method. As a result, we report in the present appendix a few test-cases where the solution or at least one component of the solution is known.

Of course, some closed-form expressions of the contact pressure are also known for Hertzian contact, which are useful to assess the performance of numerical methods. We can quote for instance from the monograph of P. Wriggers [266] the following expression for the maximal pressure, for the contact between an elastic disc of radius R and a plane rigid support:

$$p_{max} = \left(\frac{F}{\pi R} \frac{E}{(1+\nu)(1-\nu)} \right)^{\frac{1}{2}},$$

where F is the global force, E is the Young modulus and ν the Poisson ratio. Below we describe situations where the complete displacement and stress fields are known, on the entire domain. Moreover, the situations below do not involve a curved boundary, which adds another source of approximation error.

A.1 Scalar Signorini

We consider the scalar Signorini problem described in Chap. 3, in strong form (Eqs. (3.24)–(3.25)) and that we recall here, with slight modifications:

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0

$$
\begin{aligned}
\Delta u + f &= 0 && \text{in } \Omega, \\
u &= u_D && \text{on } \Gamma_D, \\
u &\leq 0, && \text{on } \Gamma_C, \\
\partial_{\mathbf{n}} u &\leq 0, && \text{on } \Gamma_C, \\
u\, \partial_{\mathbf{n}} u &= 0, && \text{on } \Gamma_C.
\end{aligned}
$$

In comparison to Chap. 3 there is no more Neumann boundary, and a non-homogeneous Dirichlet boundary condition. We take $f = 0$ and a rectangular domain

$$\Omega = (-1, 1) \times (-1, 0).$$

We define the contact boundary as

$$\Gamma_C := [-1, 1] \times \{0\}$$

and the remaining part of the boundary is Γ_D. For such a setting, we can report the closed-form solution of [66], which is, in polar coordinates:

$$u(r, \theta) = -r^{\alpha} \sin(\alpha\theta),$$

with $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r > 0$, $0 \leq \theta \leq 2\pi$, and the value

$$\alpha = \frac{11}{2}.$$

On Γ_D, we set $u_D = u$, in order to satisfy the Dirichlet boundary condition. For this solution, a transition between biding and nonbiding happens at point $(0, 0)$ on the contact boundary. This solution has been used to compute numerical convergence rates for a Nitsche Hybrid High Order (HHO) method in [66].

A.2 A Manufactured Solution for Tresca Friction

In [81] is presented a manufactured solution for contact with Tresca friction in small strain elasticity. The problem under consideration is (8.1)–(8.3), and we take $d = 2$ and

$$\Omega := (0, 1) \times (0, 1),$$

with

$$\Gamma_C := (0, 1) \times \{0\}$$

and $\Gamma_D := \partial\Omega \setminus \overline{\Gamma_C}$. We consider Hooke law with Lamé coefficients λ and μ. The source term $\mathbf{f}$ and the Dirichlet boundary condition $\mathbf{u}_D$ are such that the exact solution is given by:

$$\mathbf{u} := (u_1, u_2)$$

where

$$u_1(x, y) := \left(1 + \frac{1}{1+\lambda}\right) x\, e^{x+y}, \quad u_2(x, y) := \left(-1 + \frac{1}{1+\lambda}\right) y\, e^{x+y}.$$

The friction threshold s_T, defined on Γ_C, is given by

$$s_T(x) := \mu \left(1 + \frac{1}{1+\lambda}\right) x\, e^x,$$

for $0 \le x \le 1$. This solution allows to test the behavior in the incompressible limit $\lambda \to +\infty$.

Remark A.1 For Tresca friction, a well-known test case, without closed-form solution, has its origin in, e.g., [48]. A general methodology to obtain manufactured solutions for frictionless contact in large strain can be found, in, e.g., [68].

References

1. Abbas, M., Drouet, G., Hild, P.: The local average contact (LAC) method. Comput. Methods Appl. Mech. Eng. **339**, 488–513 (2018). https://doi.org/10.1016/j.cma.2018.05.013
2. Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications.Applied Mathematical Sciences, vol. 75, 2nd edn. Springer, New York (1988). https://doi.org/10.1007/978-1-4612-1029-0
3. Adams, R.A.: Sobolev Spaces. Pure and Applied Mathematics, vol. 65. Academic Press, New York-London (1975)
4. Agmon, S.: Lectures on Elliptic Boundary Value Problems. Van Nostrand Mathematical Studies, No. 2. D. Van Nostrand, Princeton, Toronto-London (1965)
5. Ainsworth, M., Parker, C.: Unlocking the secrets of locking: finite element analysis in planar linear elasticity. Comput. Methods Appl. Mech. Eng. **395**, 115034, 56 (2022). https://doi.org/10.1016/j.cma.2022.115034
6. Alart, P., Curnier, A.: A generalized Newton method for contact problems with friction. J Mec. Theor. Appl. **7**(1), 67–82 (1988)
7. Allaire, G.: Numerical analysis and optimization. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford (2007)
8. Alnæs, M., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M.E., Wells, G.N.: The FEniCS project version 1.5. Arch. Numer. Soft. **3**(100) (2015)
9. Amdouni, S., Hild, P., Lleras, V., Moakher, M., Renard, Y.: A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies. ESAIM Math. Model. Numer. Anal. **46**(4), 813–839 (2012). https://doi.org/10.1051/m2an/2011072
10. Amdouni, S., Moakher, M., Renard, Y.: A local projection stabilization of fictitious domain method for elliptic boundary value problems. Appl. Numer. Math. **76**, 60–75 (2014). https://doi.org/10.1016/j.apnum.2013.10.002
11. Amdouni, S., Moakher, M., Renard, Y.: A stabilized Lagrange multiplier method for the enriched finite-element approximation of Tresca contact problems of cracked elastic bodies. Comput. Methods Appl. Mech. Eng. **270**, 178–200 (2014). https://doi.org/10.1016/j.cma.2013.11.022
12. Andersson, L.E.: Existence results for quasistatic contact problems with Coulomb friction. Appl. Math. Optim. **42**(2), 169–202 (2000). https://doi.org/10.1007/s002450010009

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0

13. Antolin, P., Buffa, A., Fabre, M.: A priori error for unilateral contact problems with Lagrange multipliers and isogeometric analysis. IMA J. Numer. Anal. **39**(4), 1627–1651 (2019). https://doi.org/10.1093/imanum/dry041
14. Apel, T., Nicaise, S.: Regularity of the solution of the scalar Signorini problem in polygonal domains. Result. Math. **75**(2), 15 (2020). Id/No 75
15. Araya, R., Chouly, F.: Residual a posteriori error estimation for frictional contact with Nitsche method (2023). hal-04044900
16. Arnold, D.: An interior penalty finite element method with discontinuous elements. SIAM J. Numer. Anal. **19**, 742–760 (1982)
17. Astorino, M., Gerbeau, J.F., Pantz, O., Traoré, K.F.: Fluid-structure interaction and multi-body contact: application to aortic valves. Comput. Methods Appl. Mech. Eng. **198**(45-46), 3603–3612 (2009). https://doi.org/10.1016/j.cma.2008.09.012
18. Babuska, I.: The finite element method with Lagrangian multipliers. Numer. Math. **20**, 179–192 (1972/1973)
19. Babuska, I.: The finite element method with penalty. Math. Comp. **27**, 221–228 (1973). https://doi.org/10.2307/2005611
20. Baillet, L., Sassi, T.: Méthode d'éléments finis avec hybridisation frontière pour les problèmes de contact avec frottement. C. R. Math. Acad. Sci. Paris **334**(10), 917–922 (2002). https://doi.org/10.1016/S1631-073X(02)02356-7
21. Baillet, L., Sassi, T.: Mixed finite element methods for the Signorini problem with friction. Numer. Methods Partial Differ. Equ. **22**(6), 1489–1508 (2006). https://doi.org/10.1002/num.20147
22. Ballard, P.: Steady sliding frictional contact problems in linear elasticity. J. Elasticity **110**(1), 33–61 (2013). https://doi.org/10.1007/s10659-012-9381-6
23. Barbosa, H.J.C., Hughes, T.J.R.: Boundary Lagrange multipliers in finite element methods: error analysis in natural norms. Numer. Math. **62**(1), 1–15 (1992). https://doi.org/10.1007/BF01396217
24. Barbosa, H.J.C., Hughes, T.J.R.: Circumventing the Babu$\check{s}$ka-Brezzi condition in mixed finite element approximations of elliptic variational inequalities. Comput. Methods Appl. Mech. Eng. **97**(2), 193–210 (1992). https://doi.org/10.1016/0045-7825(92)90163-E
25. Barbosa, H.J.C., Hughes, T.J.R.: Circumventing the Babu**u**ska-Brezzi condition in mixed finite element approximations of elliptic variational inequalities. Comput. Methods Appl. Mech. Eng. **97**(2), 193–210 (1992). https://doi.org/10.1016/0045-7825(92)90163-E
26. Bazilevs, Y., Hughes, T.J.R.: Weak imposition of Dirichlet boundary conditions in fluid mechanics. Comput. Fluids **36**(1), 12–26 (2007). https://doi.org/10.1016/j.compfluid.2005.07.012
27. Beaude, L., Chouly, F., Laaziri, M., Masson, R.: Mixed and Nitsche's discretizations of Coulomb frictional contact-mechanics for mixed dimensional poromechanical models (2023). https://hal.science/hal-03949272. Hal-03949272
28. Becker, R., Hansbo, P., Stenberg, R.: A finite element method for domain decomposition with non-matching grids. M2AN Math. Model. Numer. Anal. **37**(2), 209–225 (2003)
29. Beirao da Vega, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L.D., Russo, A.: Basic principles of virtual element methods. Math. Models Methods Appl. Sci. **23**(1), 199–214 (2013). https://doi.org/10.1142/S0218202512500492
30. Belhachmi, Z., Ben Belgacem, F.: Quadratic finite element approximation of the Signorini problem. Math. Comp. **72**(241), 83–104 (2003). https://doi.org/10.1090/S0025-5718-01-01413-2
31. Ben Belgacem, F.: Numerical simulation of some variational inequalities arisen from unilateral contact problems by the finite element methods. SIAM J. Numer. Anal. **37**(4), 1198–1216 (2000). https://doi.org/10.1137/S0036142998347966
32. Ben Belgacem, F., Brenner, S.C.: Some nonstandard finite element estimates with applications to 3D Poisson and Signorini problems. Electron. Trans. Numer. Anal. **12**, 134–148 (2001)

33. Ben Belgacem, F., Hild, P., Laborde, P.: Approximation of the unilateral contact problem by the mortar finite element method. C. R. Acad. Sci. Paris Sér. I Math. **324**(1), 123–127 (1997). https://doi.org/10.1016/S0764-4442(97)80115-2
34. Ben Belgacem, F., Hild, P., Laborde, P.: The mortar finite element method for contact problems. Math. Comput. Model. **28**(4–8), 263–271 (1998). https://doi.org/10.1016/S0895-7177(98)00121-6. Recent advances in contact mechanics
35. Ben Belgacem, F., Hild, P., Laborde, P.: Extension of the mortar finite element method to a variational inequality modeling unilateral contact. Math. Models Methods Appl. Sci. **9**(2), 287–303 (1999). https://doi.org/10.1142/S0218202599000154
36. Ben Belgacem, F., Renard, Y., Slimane, L.: A mixed formulation for the Signorini problem in nearly incompressible elasticity. Appl. Numer. Math. **54**(1), 1–22 (2005). https://doi.org/10.1016/j.apnum.2004.09.036
37. Bergh, J., Löfström, J.: Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer, Berlin-New York (1976)
38. Bernardi, C., Dauge, M., Maday, Y.: Polynomials in the Sobolev World (2007). https://hal.archives-ouvertes.fr/hal-00153795. Hal-00153795
39. Bernardi, C., Girault, V.: A local regularization operator for triangular and quadrilateral finite elements. SIAM J. Numer. Anal. **35**, 1893—1916 (1998)
40. Bernardi, C., Maday, Y., Patera, A.T.: Domain decomposition by the mortar element method. In: Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters (Beaune, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 384, pp. 269–286. Kluwer Academic Publishers, Dordrecht (1993)
41. Bernardi, C., Maday, Y., Patera, A.T.: A new nonconforming approach to domain decomposition: the mortar element method. In: Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), Pitman Res. Notes Math. Ser., vol. 299, pp. 13–51. Longman Sci. Tech., Harlow (1994)
42. Bernardi, C., Maday, Y., Rapetti, F.: Discrétisations variationnelles de problèmes aux limites elliptiques, Mathématiques & Applications (Berlin), vol. 45. Springer, Berlin (2004)
43. Bertsekas, D.P.: Constrained optimization and Lagrange multiplier methods. Computer Science and Applied Mathematics. Academic Press, New York-London (1982)
44. Bjørstad, P.E., Widlund, O.B.: Iterative methods for the solution of elliptic problems on regions partitioned into substructures. SIAM J. Numer. Anal. **23**(6), 1097–1120 (1986). https://doi.org/10.1137/0723075
45. Boiveau, T., Burman, E.: A penalty-free Nitsche method for the weak imposition of boundary conditions in compressible and incompressible elasticity. IMA J. Numer. Anal. **36**(2), 770–795 (2016). https://doi.org/10.1093/imanum/drv042
46. Bonet, J., Wood, R.D.: Nonlinear continuum mechanics for finite element analysis. Cambridge University Press, Cambridge, New York (2008)
47. Bordas, S., Menk, A.: Partition of Unity Methods. Wiley, London (2023). https://books.google.cl/books?id=2ygSywAACAAJ
48. Bostan, V., Han, W.: Adaptive finite element solution of variational inequalities with application in contact problems. In: Advances in Applied Mathematics and Global Optimization, vol. 17, pp. 25–106. Springer, New York (2009). https://doi.org/10.1007/978-0-387-75714-8_3
49. Bourbaki, N.: Algebra. I. Chapters 1–3. Elements of Mathematics (Berlin). Springer, Berlin (1989)
50. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods. Texts in Applied Mathematics, vol. 15. Springer, New York (2007)
51. Brezis, H.: Équations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier (Grenoble) **18**(fasc. 1), 115–175 (1968)
52. Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York (2011)

53. Brezzi, F.: On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8**(R-2), 129–151 (1974)
54. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics, vol. 15. Springer, New York (1991)
55. Brezzi, F., Hager, W.W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities. I. Primal theory. Numer. Math. **28**(4), 431–443 (1977)
56. Brezzi, F., Hager, W.W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities. II. Mixed methods. Numer. Math. **31**(1), 1–16 (1978/1979). https://doi.org/10.1007/BF01396010
57. Burman, E.: A penalty-free nonsymmetric Nitsche-type method for the weak imposition of boundary conditions. SIAM J. Numer. Anal. **50**(4), 1959–1981 (2012). https://doi.org/10.1137/10081784X
58. Burman, E., Claus, S., Hansbo, P., Larson, M.G., Massing, A.: CutFEM: discretizing geometry and partial differential equations. Int. J. Numer. Methods Eng. **104**(7), 472–501 (2015). https://doi.org/10.1002/nme.4823
59. Burman, E., Hansbo, P.: Deriving robust unfitted finite element methods from augmented Lagrangian formulations. In: Geometrically Unfitted Finite Element Methods and Applications. Lecture Notes in Computational Science and Engineering, vol. 121, pp. 1–24. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-71431-8_1
60. Burman, E., Hansbo, P., Larson, M.G.: The penalty-free Nitsche method and nonconforming finite elements for the Signorini problem. SIAM J. Numer. Anal. **55**(6), 2523–2539 (2017). https://doi.org/10.1137/16M107846X
61. Burman, E., Hansbo, P., Larson, M.G.: Augmented Lagrangian and Galerkin least-squares methods for membrane contact. Int. J. Numer. Methods Eng. **114**(11), 1179–1191 (2018). https://doi.org/10.1002/nme.5781
62. Burman, E., Hansbo, P., Larson, M.G.: Augmented Lagrangian finite element methods for contact problems. ESAIM Math. Model. Numer. Anal. **53**(1), 173–195 (2019). https://doi.org/10.1051/m2an/2018047
63. Burman, E., Hansbo, P., Larson, M.G.: The augmented Lagrangian method as a framework for stabilised methods in computational mechanics. Arch Comput. Methods Eng. **30**, 2579–2604 (2023). https://doi.org/10.1007/s11831-022-09878-6
64. Burman, E., Hansbo, P., Larson, M.G., Stenberg, R.: Galerkin least squares finite element method for the obstacle problem. Comput. Meth. Appl. Mech. Eng. **313**, 362–374 (2017)
65. Cantin, P., Hild, P.: Error analysis of the compliance model for the Signorini problem. Calcolo **58**(3), 32, 21 (2021). https://doi.org/10.1007/s10092-021-00425-6
66. Cascavita, K.L., Chouly, F., Ern, A.: Hybrid high-order discretizations combined with Nitsche's method for Dirichlet and Signorini boundary conditions. IMA J. Numer. Anal. **40**(4), 2189–2226 (2020). https://doi.org/10.1093/imanum/drz038
67. Cazabeau, L., Lacour, C., Maday, Y.: Numerical quadratures and mortar methods. In: Computational Science for the 21st Century, pp. 119–128. John Wiley and Sons, London (1997)
68. Chamberland, E., Fortin, A., Tardieu, N.: Solutions analytiques de problèmes de contact en grandes dëformations. Mech. Ind. **11**(3-4), 271-276 (2010). https://doi.org/10.1051/meca/2010032
69. Chapelle, D., Bathe, K.J.: The Finite Element Analysis of Shells—Fundamentals, 2nd edn. Computational Fluid and Solid Mechanics. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-16408-8
70. Chazarain, J., Piriou, A.: Introduction to the Theory of Linear Partial Differential Equations. Studies in Mathematics and Its Applications, vol. 14. North-Holland Publishing, Amsterdam-New York (1982)
71. Chen, Z., Nochetto, R.: Residual type a posteriori error estimates for elliptic obstacle problems. Numer. Math. **84**, 527–548 (2000)

72. Chernov A. Maischak, M., Stephan, E.: A priori error estimates for hp penalty BEM for contact problems in elasticity. Comput. Methods Appl. Mech. Eng. **196**, 3871–3880 (2007)
73. Chouly, F.: An adaptation of Nitsche's method to the Tresca friction problem. J. Math. Anal. Appl. **411**(1), 329–339 (2014). https://doi.org/10.1016/j.jmaa.2013.09.019
74. Chouly, F.: A review on some discrete variational techniques for the approximation of essential boundary conditions (2022). https://hal.archives-ouvertes.fr/hal-03741809. Hal-03741809
75. Chouly, F., Ern, A., Pignet, N.: A hybrid high-order discretization combined with Nitsche's method for contact and Tresca friction in small strain elasticity. SIAM J. Sci. Comput. **42**(4), A2300–A2324 (2020). https://doi.org/10.1137/19M1286499
76. Chouly, F., Fabre, M., Hild, P., Mlika, R., Pousin, J., Renard, Y.: An overview of recent results on Nitsche's method for contact problems. In: Geometrically Unfitted Finite Element Methods and Applications. Lecture Notes in Computational Science and Engineering, vol. 121, pp. 93–141. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-71431-8_4
77. Chouly, F., Fabre, M., Hild, P., Pousin, J., Renard, Y.: Residual-based a posteriori error estimation for contact problems approximated by Nitsche's method. IMA J. Numer. Anal. **38**(2), 921–954 (2018). https://doi.org/10.1093/imanum/drx024
78. Chouly, F., Hild, P.: A Nitsche-based method for unilateral contact problems: numerical analysis. SIAM J. Numer. Anal. **51**(2), 1295–1307 (2013). https://doi.org/10.1137/12088344X
79. Chouly, F., Hild, P.: On convergence of the penalty method for unilateral contact problems. Appl. Numer. Math. **65**(0), 27–40 (2013). https://doi.org/10.1016/j.apnum.2012.10.003
80. Chouly, F., Hild, P., Lleras, V., Renard, Y.: Nitsche-based finite element method for contact with Coulomb friction. In: Numerical Mathematics and Advanced Applications—ENUMATH 2017. Lecture Notes in Computational Science and Engineering, vol. 126, pp. 839–847. Springer, Cham (2019)
81. Chouly, F., Hild, P., Lleras, V., Renard, Y.: Nitsche method for contact with Coulomb friction: existence results for the static and dynamic finite element formulations. J. Comput. Appl. Math. **416**, 114557 (2022). https://doi.org/10.1016/j.cam.2022.114557
82. Chouly, F., Hild, P., Renard, Y.: Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments. Math. Comp. **84**(293), 1089–1112 (2015). https://doi.org/10.1090/S0025-5718-2014-02913-X
83. Chouly, F., Hild, P., Renard, Y.: Méthodes de lagrangien et de Nitsche pour l'approximation numérique des conditions de contact avec frottement. Chapter 1 of Modélisation numérique en mécanique fortement non linéaire, pp. 8–52, 2023. Editors : Jacques Besson, Frédéric Lebon & Eric Lorentz. ISTE Editions (Collection Sciences). ISBN 978-1-78948-081-8.
84. Chouly, F., Mlika, R., Renard, Y.: An unbiased Nitsche's approximation of the frictional contact between two elastic structures. Numer. Math. **139**(3), 593–631 (2018). https://doi.org/10.1007/s00211-018-0950-x
85. Christof, C., Haubner, C.: Finite element error estimates in non-energy norms for the two-dimensional scalar Signorini problem. Numer. Math. **145**(3), 513–551 (2020). https://doi.org/10.1007/s00211-020-01117-z
86. Christof, C., Haubner, C.: Correction to: finite element error estimates in non-energy norms for the two-dimensional scalar Signorini problem. Numer. Math. **148**(2), 495 (2021). https://doi.org/10.1007/s00211-021-01204-9
87. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications, vol. 4. North-Holland Publishing, Amsterdam (1978).
88. Ciarlet, P.G.: Mathematical Elasticity. vol. I. Studies in Mathematics and Its Applications, vol. 20. North-Holland Publishing, Amsterdam (1988). Three-dimensional elasticity
89. Ciarlet, P.G.: The finite element method for elliptic problems. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, vol. II. North-Holland Publishing, Amsterdam (1991)
90. Ciarlet, P.G., Wagschal, C.: Multipoint Taylor formulas and applications to the finite element method. Numer. Math. **17**, 84–100 (1971). https://doi.org/10.1007/BF01395869

91. Cicuttin, M., Ern, A., Pignet, N.: Hybrid High-Order Methods—a Primer with Applications to Solid Mechanics. SpringerBriefs in Mathematics. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-81477-9
92. Clement, F.: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. **2**, 77–84 (1975)
93. Cockburn, B., Di Pietro, D.A., Ern, A.: Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods. ESAIM Math. Model. Numer. Anal. **50**(3), 635–650 (2016). https://doi.org/10.1051/m2an/2015051
94. Coorevits, P., Hild, P., Hjiaj, M.: A posteriori error control of finite element approximations for Coulomb's frictional contact. SIAM J. Sci. Comput. **23**(3), 976–999 (2001). https://doi.org/10.1137/S1064827500375461
95. Coorevits, P., Hild, P., Lhalouani, K., Sassi, T.: Mixed finite element methods for unilateral problems: convergence analysis and numerical studies. Math. Comp. **71**(237), 1–25 (2002). https://doi.org/10.1090/S0025-5718-01-01318-7
96. Cottrell, J.A., Hughes, T.J.R., Bazilevs, Y.: Isogeometric Analysis. Wiley, Chichester (2009). https://doi.org/10.1002/9780470749081
97. Crouzeix, M., Thomée, V.: The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. Math. Comp. **48**(178), 521–532 (1987). https://doi.org/10.2307/2007825
98. Curnier, A., He, Q.C., Klarbring, A.: Continuum mechanics modelling of large deformation contact with friction. In: Contact Mechanics, pp. 145–158. Springer, Berlin (1995)
99. Dabaghi, J., Delay, G.: A unified framework for high-order numerical discretizations of variational inequalities. Comput. Math. Appl. **92**, 62–75 (2021). https://doi.org/10.1016/j.camwa.2021.03.011
100. Dautray, R., Lions, J.L.: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 2. Springer, Berlin (1988). https://doi.org/10.1007/978-3-642-61566-5. Functional and variational methods, With the collaboration of Michel Artola, Marc Authier, Philippe Bénilan, Michel Cessenat, Jean Michel Combes, Hélène Lanchon, Bertrand Mercier, Claude Wild and Claude Zuily, Translated from the French by Ian N. Sneddon
101. Dekel, S., Leviatan, D.: Whitney estimates for convex domains with applications to multivariate piecewise polynomial approximation. Found. Comput. Math. **4**(4), 345–368 (2004). https://doi.org/10.1007/s10208-004-0096-3
102. Demkowicz, L., Gopalakrishnan, J.: Analysis of the DPG method for the Poisson equation. SIAM J. Numer. Anal. **49**(5), 1788–1809 (2011). https://doi.org/10.1137/100809799
103. Deny, J., Lions, J.L.: Les espaces du type de Beppo Levi. Ann. Inst. Fourier (Grenoble) **21**, 305–370 (1955) (1945). http://aif.cedram.org/item?id=AIF_1953__5__305_0
104. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012). https://doi.org/10.1016/j.bulsci.2011.12.004
105. Di Pietro, D.A., Droniou, J.: The Hybrid High-Order Method for Polytopal Meshes. MS&A. Modeling, Simulation and Applications, vol. 19. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-37203-3
106. Di Pietro, D.A., Ern, A.: Mathematical Aspects of Discontinuous Galerkin Methods. Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 69. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-22980-0
107. Dione, I.: Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method. J. Math. Anal. Appl. **472**(1), 266–284 (2019). https://doi.org/10.1016/j.jmaa.2018.11.023
108. Dione, I.: Optimal error estimates of the unilateral contact problem in a curved and smooth boundary domain by the penalty method. IMA J. Numer. Anal. **40**(1), 729–763 (2020). https://doi.org/10.1093/imanum/dry050
109. Domínguez, V., Sayas, F.J.: Stability of discrete liftings. C. R. Math. Acad. Sci. Paris **337**(12), 805–808 (2003). https://doi.org/10.1016/j.crma.2003.10.025
110. Drouet, G., Hild, P.: Optimal convergence for discrete variational inequalities modelling Signorini contact in 2D and 3D without additional assumptions on the unknown contact set. SIAM J. Numer. Anal. **53**(3), 1488–1507 (2015). https://doi.org/10.1137/140980697

111. Drouet, G., Hild, P.: An accurate local average contact method for nonmatching meshes. Numer. Math. **136**(2), 467–502 (2017). https://doi.org/10.1007/s00211-016-0846-6
112. Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. Math. Comp. **34**, 441–463 (1980)
113. Duvaut, G.: Problèmes unilatéraux en mécanique des milieux continus. In: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, pp. 71–77 (1971)
114. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Grundlehren der Mathematischen Wissenschaften, vol. 219. Springer, Berlin-New York (1976)
115. Eck, C., Jarušek, J., Krbec, M.: Unilateral Contact Problems. Pure and Applied Mathematics (Boca Raton), vol. 270. Chapman & Hall/CRC, Boca Raton (2005). https://doi.org/10.1201/9781420027365
116. Eck, C., Jarušek, J.: Existence results for the static contact problem with Coulomb friction. Math. Models Methods Appl. Sci. **8**(3), 445–468 (1998). https://doi.org/10.1142/S0218202598000196
117. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. Studies in Mathematics and Its Applications, vol. 1. North-Holland Publishing, Amsterdam-Oxford; American Elsevier Publishing, New York (1976)
118. Ern, A., Guermond, J.L.: Theory and Practice of Finite Elements. Applied Mathematical Sciences, vol. 159. Springer, New York (2004)
119. Ern, A., Guermond, J.L.: Finite element quasi-interpolation and best approximation. ESAIM Math. Model. Numer. Anal. **51**(4), 1367–1385 (2017). https://doi.org/10.1051/m2an/2016066
120. Ern, A., Guermond, J.L.: Finite elements I—Approximation and Interpolation. Texts in Applied Mathematics, vol. 72. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-56341-7
121. Ern, A., Guermond, J.L.: Finite elements II—Galerkin approximation, elliptic and mixed PDEs. Texts in Applied Mathematics, vol. 73. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-56923-5
122. Ern, A., Guermond, J.L.: Finite Elements III—First-Order and Time-Dependent PDEs. Texts in Applied Mathematics, vol. 74. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-57348-5
123. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19, 2nd edn. American Mathematical Society, Providence (2010). https://doi.org/10.1090/gsm/019
124. Fabre, M., Pousin, J., Renard, Y.: A fictitious domain method for frictionless contact problems in elasticity using Nitsche's method. SMAI J. Comput. Math. **2**, 19–50 (2016)
125. Faermann, B.: Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. II. The three-dimensional case. Numer. Math. **92**(3), 467–499 (2002). https://doi.org/10.1007/s002110100319
126. Falk, R.: Error estimates for the approximation of a class of variational inequalities. Math. Comput. **28**(128), 963–971 (1974)
127. Farah, P., Popp, A., Wall, W.A.: Segment-based vs. element-based integration for mortar methods in computational contact mechanics. Comput. Mech. **55**(1), 209–228 (2015)
128. Fichera, G.: Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) **7**, 91–140 (1963/1964)
129. Fortin, A., Garon, A.: Les éléments finis: de la théorie à la pratique (2020). Notes de cours, Université Laval, Québec (450 p.). https://giref.ulaval.ca/afortin/elements_finis.pdf
130. Fritz, A., Hüeber, S., Wohlmuth, B.I.: A comparison of mortar and Nitsche techniques for linear elasticity. Universität Stuttgart, IANS Preprint (8) (2003)
131. Fritz, A., Hüeber, S., Wohlmuth, B.I.: A comparison of mortar and Nitsche techniques for linear elasticity. Calcolo **41**(3), 115–137 (2004). https://doi.org/10.1007/s10092-004-0087-4
132. Führer, T., Heuer, N., Stephan, E.P.: On the DPG method for Signorini problems. IMA J. Numer. Anal. **38**(4), 1893–1926 (2018). https://doi.org/10.1093/imanum/drx048
133. Gagliardo, E.: Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili. Rend. Sem. Mat. Univ. Padova **27**, 284–305 (1957). http://www.numdam.org/item?id=RSMUP_1957__27__284_0

134. Gasquet, C., Witomski, P.: Fourier Analysis and Applications. Texts in Applied Mathematics, vol. 30. Springer, New York (1999). https://doi.org/10.1007/978-1-4612-1598-1. Filtering, numerical computation, wavelets, Translated from the French and with a preface by R. Ryan
135. Gatica, G.N.: A Simple Introduction to the Mixed Finite Element Method. SpringerBriefs in Mathematics. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-03695-3. Theory and applications
136. Gatica, G.N., Maischak, M., Stephan, E.P.: Numerical analysis of a transmission problem with Signorini contact using mixed-FEM and BEM. ESAIM Math. Model. Numer. Anal. **45**(4), 779–802 (2011). https://doi.org/10.1051/m2an/2010102
137. Girault, V., Raviart, P.A.: Finite Element Methods for Navier-Stokes Equations. Springer Series in Computational Mathematics, vol. 5. Springer, Berlin (1986). https://doi.org/10.1007/978-3-642-61623-5. Theory and algorithms
138. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer Series in Computational Physics. Springer, New York (1984). https://doi.org/10.1007/978-3-662-12613-4
139. Glowinski, R., Le Tallec, P.: Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics. SIAM Studies in Applied Mathematics, vol. 9. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1989)
140. Glowinski, R., Pan, T., Périaux, J.: A fictitious domain method for Dirichlet problem and applications. Comput. Methods Appl. Mech. Eng. **111**(3-4), 283–303 (1994)
141. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, vol. 24. Pitman (Advanced Publishing Program), Boston (1985)
142. Grisvard, P.: Problèmes aux limites dans les polygones. Mode d'emploi. EDF Bull. Direction Études Rech. Sér. C Math. Inform. (1), **3**, 21–59 (1986)
143. Gurtin, M.E.: An Introduction to Continuum Mechanics. Mathematics in Science and Engineering, vol. 158. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London (1981)
144. Gustafsson, T., Stenberg, R., Videman, J.: Nitsche's method for unilateral contact problems. Port. Math. **75**(3–4), 189–204 (2018). https://doi.org/10.4171/PM/2016
145. Gustafsson, T., Stenberg, R., Videman, J.: On Nitsche's method for elastic contact problems. SIAM J. Sci. Comput. **42**(2), B425–B446 (2020). https://doi.org/10.1137/19M1246869
146. Gustafsson, T., Videman, J.: Stabilized finite elements for Tresca friction problem. ESAIM Math. Model. Numer. Anal. **56**(4), 1307–1326 (2022). https://doi.org/10.1051/m2an/2022048
147. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. AMS/IP Studies in Advanced Mathematics, vol. 30. American Mathematical Society, Providence (2002)
148. Hansbo, A., Hansbo, P.: A finite element method for the simulation of strong and weak discontinuities in solid mechanics. Comput. Methods Appl. Mech. Eng. **193**(33-35), 3523–3540 (2004). https://doi.org/10.1016/j.cma.2003.12.041
149. Hansbo, P.: Nitsche's method for interface problems in computational mechanics. GAMM-Mitt. **28**(2), 183–206 (2005)
150. Hansbo, P., Rashid, A., Salomonsson, K.: Least-squares stabilized augmented Lagrangian multiplier method for elastic contact. Finite Elem. Anal. Des. **116**, 32–37 (2016). https://doi.org/10.1016/j.finel.2016.03.005
151. Haslinger, J.: Finite element analysis for unilateral problems with obstacles on the boundary. Appl. Math. **22**(3), 180–188 (1977)
152. Haslinger, J., Hlaváček, I.: Approximation of the Signorini problem with friction by a mixed finite element method. J. Math. Anal. Appl. **86**(1), 99–122 (1982). https://doi.org/10.1016/0022-247X(82)90257-8
153. Haslinger, J., Hlaváček, I., Nečas, J.: Numerical methods for unilateral problems in solid mechanics. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, vol. IV. North-Holland Publishing, Amsterdam (1996)

154. Haslinger, J., Miettinen, M., Panagiotopoulos, P.D.: Finite Element Method for Hemivariational Inequalities. Nonconvex Optimization and its Applications, vol. 35. Kluwer Academic Publishers, Dordrecht (1999). https://doi.org/10.1007/978-1-4757-5233-5
155. Haslinger, J., Renard, Y.: A new fictitious domain approach inspired by the extended finite element method. SIAM J. Numer. Anal. **47**(2), 1474–1499 (2009). https://doi.org/10.1137/070704435
156. Haslinger, J., Sassi, T.: Mixed finite element approximation of 3D contact problems with given friction: error analysis and numerical realization. M2AN Math. Model. Numer. Anal. **38**(3), 563–578 (2004). https://doi.org/10.1051/m2an:2004026
157. Hassani, R., Hild, P., Ionescu, I.R., Sakki, N.D.: A mixed finite element method and solution multiplicity for Coulomb frictional contact. Comput. Methods Appl. Mech. Eng. **192**(41-42), 4517–4531 (2003). https://doi.org/10.1016/S0045-7825(03)00419-5
158. Heintz, P., Hansbo, P.: Stabilized Lagrange multiplier methods for bilateral elastic contact with friction. Comput. Methods Appl. Mech. Eng. **195**(33-36), 4323–4333 (2006). https://doi.org/10.1016/j.cma.2005.09.008
159. Heuer, N.: Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen. Numer. Math. **88**(3), 485–511 (2001). https://doi.org/10.1007/s211-001-8012-7
160. Heuer, N.: On the equivalence of fractional-order Sobolev semi-norms. J. Math. Anal. Appl. **417**(2), 505–518 (2014). https://doi.org/10.1016/j.jmaa.2014.03.047
161. Heuer, N.: The boundary element method. Notes for a graduate course given at the Universidade de Santiago de Compostela, Spain (May 3–11, 2007)
162. Hilbert, S.: A mollifier useful for approximations in Sobolev spaces and some applications to approximating solutions to differential equations. Math. Comp. **27**(121), 81–89 (1973)
163. Hild, P.: Éléments finis non conformes pour un problème de contact unilatéral avec frottement. C. R. Acad. Sci. Paris Sér. I Math. **324**(6), 707–710 (1997).https://doi.org/10.1016/S0764-4442(97)86994-7
164. Hild, P.: Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity. Q. J. Mech. Appl. Math. **57**(2), 225–235 (2004). https://doi.org/10.1093/qjmam/57.2.225
165. Hild, P.: Multiple solutions of stick and separation type in the Signorini model with Coulomb friction. ZAMM Z. Angew. Math. Mech. **85**(9), 673–680 (2005). https://doi.org/10.1002/zamm.200310205
166. Hild, P., Laborde, P.: Quadratic finite element methods for unilateral contact problems. Appl. Numer. Math. **41**(3), 401–421 (2002). https://doi.org/10.1016/S0168-9274(01)00124-6
167. Hild, P., Nicaise, S.: Residual a posteriori error estimators for contact problems in elasticity. M2AN Math. Model. Numer. Anal. **41**(5), 897–923 (2007). https://doi.org/10.1051/m2an:2007045
168. Hild, P., Renard, Y.: An error estimate for the Signorini problem with Coulomb friction approximated by finite elements. SIAM J. Numer. Anal. **45**(5), 2012–2031 (2007). https://doi.org/10.1137/050645439
169. Hild, P., Renard, Y.: A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics. Numer. Math. **115**(1), 101–129 (2010). https://doi.org/10.1007/s00211-009-0273-z
170. Hild, P., Renard, Y.: An improved a priori error analysis for finite element approximations of Signorini's problem. SIAM J. Numer. Anal. **50**(5), 2400–2419 (2012). https://doi.org/10.1137/110857593
171. Hintermüller, M., Ito, K., Kunisch, K.: The primal-dual active set strategy as a semismooth Newton method. SIAM J. Optim. **13**(3), 865–888 (2003) (2002). https://doi.org/10.1137/S1052623401383558
172. Hlaváček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Applied Mathematical Sciences, vol. 66. Springer, New York (1988). https://doi.org/10.1007/978-1-4612-1048-1

173. Holzapfel, G.A.: Nonlinear Solid Mechanics. Wiley, Chichester (2000). A continuum approach for engineering
174. Hu, Q., Chouly, F., Hu, P., Cheng, G., Bordas, S.P.A.: Skew-symmetric Nitsche's formulation in isogeometric analysis: Dirichlet and symmetry conditions, patch coupling and frictionless contact. Comput. Methods Appl. Mech. Eng. **341**, 188–220 (2018). https://doi.org/10.1016/j.cma.2018.05.024
175. Hua, D., Wang, L.: The nonconforming finite element method for Signorini problem. J. Comput. Math. **25**(1), 67–80 (2007)
176. Hüeber, S., Wohlmuth, B.I.: An optimal a priori error estimate for nonlinear multibody contact problems. SIAM J. Numer. Anal. **43**(1), 156–173 (2005). https://doi.org/10.1137/S0036142903436678
177. Hughes, T.J.R., Franca, L.P.: A new finite element formulation for computational fluid dynamics. VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces. Comput. Methods Appl. Mech. Eng. **65**(1), 85–96 (1987). https://doi.org/10.1016/0045-7825(87)90184-8
178. Ito, K., Kunisch, K.: Semi-smooth Newton methods for variational inequalities of the first kind. M2AN Math. Model. Numer. Anal. **37**(1), 41–62 (2003). https://doi.org/10.1051/m2an:2003021
179. Jarušek, J.: Contact problems with bounded friction coercive case. Czechoslovak Math. J. **33**(**108**)(2), 237–261 (1983)
180. Johnson, C.: Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge University Press, Cambridge (1987)
181. Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods. SIAM Studies in Applied Mathematics, vol. 8. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1988)
182. Kikuchi, N., Song, Y.J.: Penalty-finite-element approximation of a class of unilateral problems in linear elasticity. Q. Appl. Math. **39**, 1–22 (1981)
183. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Pure and Applied Mathematics, vol. 88. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London (1980)
184. Konyukhov, A., Schweizerhof, K.: Computational Contact Mechanics. Lecture Notes in Applied and Computational Mechanics, vol. 67. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-31531-2
185. Korn, A.: Sur les équations de l'élasticité. Ann. Sci. École Norm. Sup. (3) **24**, 9–75 (1907). http://www.numdam.org/item?id=ASENS_1907_3_24__9_0
186. Korn, A.: Solution générale du problème d'équilibre dans la théorie de l'élasticité, dans le cas ou les efforts sont donnés à la surface. Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2) **10**, 165–269 (1908). http://www.numdam.org/item?id=AFST_1908_2_10__165_0
187. Korn, A.: Über die Lösung des Grundproblemes der Elastizitätstheorie. Math. Ann. **75**(4), 497–544 (1914). https://doi.org/10.1007/BF01563656
188. Kunisch, K., Stadler, G.: Generalized Newton methods for the 2D-Signorini contact problem with friction in function space. M2AN Math. Model. Numer. Anal. **39**(4), 827–854 (2005)
189. Laborde, P., Renard, Y.: Fixed point strategies for elastostatic frictional contact problems. Math. Methods Appl. Sci. **31**(4), 415–441 (2008). https://doi.org/10.1002/mma.921
190. Laursen, T.A.: Computational Contact and Impact Mechanics. Springer, Berlin (2002)
191. Laursen, T.A., Simo, J.C.: Algorithmic symmetrization of Coulomb frictional problems using augmented Lagrangians. Comput. Methods Appl. Mech. Eng. **108**(1-2), 133–146 (1993). https://doi.org/10.1016/0045-7825(93)90157-S
192. Laursen, T.A., Simo, J.C.: A continuum-based finite element formulation for the implicit solution of multibody, large deformation frictional contact problems. Int. J. Numer. Methods Eng. **36**(20), 3451–3485 (1993). https://doi.org/10.1002/nme.1620362005
193. Le Dret, H.: Nonlinear Elliptic Partial Differential Equations. Universitext. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-78390-1

194. Lemaire, S.: Bridging the hybrid high-order and virtual element methods. IMA J. Numer. Anal. **41**(1), 549–593 (2021). https://doi.org/10.1093/imanum/drz056
195. Lengiewicz, J., Korelc, J., Stupkiewicz, S.a.: Automation of finite element formulations for large deformation contact problems. Int. J. Numer. Methods Eng. **85**(10), 1252–1279 (2011). https://doi.org/10.1002/nme.3009
196. Lesaint, P., Raviart, P.A.: On a finite element method for solving the neutron transport equation. In: Mathematical Aspects of Finite Elements in Partial Differential Equations (Proceedings of Symposium Mathematics Resolution Center, Univ. Wisconsin, Madison, Wis., 1974), Publication No. 33, pp. 89–123. Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York (1974)
197. Lichnerowicz, A.: Elements of Tensor Calculus. Methuen, London; Wiley, New York (1962). Translated by J. W. Leech and D. J. Newman
198. Ligurský, T., Renard, Y.: Bifurcations in piecewise-smooth steady-state problems: abstract study and application to plane contact problems with friction. Comput. Mech. **56**(1), 39–62 (2015). https://doi.org/10.1007/s00466-015-1154-1
199. Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications. Springer, Berlin, New York (1972)
200. Lions, J.L., Stampacchia, G.: Variational Inequalities. Commun. Pure Applied Math. **XX**, 493–519 (1967)
201. Lleras, V.: A stabilized Lagrange multiplier method for the finite element approximation of frictional contact problems in elastostatics. Math. Model. Nat. Phenom. **4**(1), 163–182 (2009). https://doi.org/10.1051/mmnp/20094108
202. Lozinski, A.: A primal discontinuous Galerkin method with static condensation on very general meshes. Numer. Math. **143**(3), 583–604 (2019). https://doi.org/10.1007/s00211-019-01067-1
203. Lützen, J.: The Prehistory of the Theory of Distributions. Studies in the History of Mathematics and Physical Sciences, vol. 7. Springer, New York-Berlin (1982)
204. Marsden, J.E., Hughes, T.J.R.: Mathematical Foundations of Elasticity. Dover Publications, New York (1994). Corrected reprint of the 1983 original
205. Maz'ja, V.G.: Sobolev Spaces. Springer Series in Soviet Mathematics. Springer, Berlin (1985). https://doi.org/10.1007/978-3-662-09922-3. Translated from the Russian by T. O. Shaposhnikova
206. Maz'ya, V.G., Shaposhnikova, T.O.: Theory of Multipliers in Spaces of Differentiable Functions. Monographs and Studies in Mathematics, vol. 23. Pitman (Advanced Publishing Program), Boston (1985)
207. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
208. Mlika, R.: Nitsche method for frictional contact and self-contact: mathematical and numerical study. Theses, Institut national des sciences appliquées de Lyon (2018). https://hal.science/tel-01764646. Tel-01764646
209. Mlika, R., Chouly, F., Renard, Y.: A numerical study of quadratures for Nitsche-based contact (2022). In Preparation.
210. Mlika, R., Renard, Y., Chouly, F.: An unbiased Nitsche's formulation of large deformation frictional contact and self-contact. Comput. Methods Appl. Mech. Eng. **325**, 265–288 (2017). https://doi.org/10.1016/j.cma.2017.07.015
211. Moës, N., Béchet, E., Tourbier, M.: Imposing Dirichlet boundary conditions in the extended finite element method. Int. J. Numer. Methods Eng. **67**(12), 1641–1669 (2006). https://doi.org/10.1002/nme.1675
212. Mosco, U.: Approximation of the solutions of some variational inequalities. Ann. Scuola Norm. Sup. Pisa (3) **21**, 373–394 (1967); erratum, ibid. (3) **21**, 765 (1967)
213. Moussaoui, M., Khodja, K.: Régularité des solutions d'un problème mêlé Dirichlet-Signorini dans un domaine polygonal plan. Commun. Partial Differ. Equ. **17**(5-6), 805–826 (1992). https://doi.org/10.1080/03605309208820864

214. Nečas, J.: Les méthodes directes en théorie des équations elliptiques. Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague (1967)
215. Nečas, J., Jarušek, J., Haslinger, J.: On the solution of the variational inequality to the Signorini problem with small friction. Boll. Un. Mat. Ital. B (5) **17**(2), 796–811 (1980)
216. Nguyen, V.P., Anitescu, C., Bordas, S.P.A., Rabczuk, T.: Isogeometric analysis: an overview and computer implementation aspects. Math. Comput. Simul. **117**, 89–116 (2015). https://doi.org/10.1016/j.matcom.2015.05.008
217. Nguyen, V.P., Rabczuk, T., Bordas, S., Duflot, M.: Meshless methods: a review and computer implementation aspects. Math. Comput. Simul. **79**(3), 763–813 (2008). https://doi.org/10.1016/j.matcom.2008.01.003
218. Nicolaides, R.A.: On a class of finite elements generated by Lagrange interpolation. SIAM J. Numer. Anal. **9**, 435–445 (1972). https://doi.org/10.1137/0709039
219. Nitsche, J.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **36**, 9–15 (1971)
220. Nitsche, J.A.: On Korn's second inequality. RAIRO Anal. Numér. **15**(3), 237–248 (1981)
221. Nochetto, R., Wahlbin, L.: Positivity preserving finite element approximation. Math. Comp. **71**, 1405–1419 (2002)
222. Oden, J.T., Kim, S.J.: Interior penalty methods for finite element approximations of the Signorini problem in elastostatics. Comput. Math. Appl. **8**(1), 35–56 (1982). https://doi.org/10.1016/0898-1221(82)90038-4
223. Ogden, R.W.: Nonlinear Elastic Deformations. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood, Chichester; Halsted Press [John Wiley & Sons], New York (1984)
224. Pantz, O.: The modeling of deformable bodies with frictionless (self-)contacts. Arch. Ration. Mech. Anal. **188**(2), 183–212 (2008). https://doi.org/10.1007/s00205-007-0091-3
225. Pantz, O.: A frictionless contact algorithm for deformable bodies. ESAIM Math. Model. Numer. Anal. **45**(2), 235–254 (2011). https://doi.org/10.1051/m2an/2010041
226. Peskin, C.S.: The immersed boundary method. Acta Numer. **11**, 479–517 (2002). https://doi.org/10.1017/S0962492902000077
227. Popp, A., Wohlmuth, B.I., Gee, M.W., Wall, W.A.: Dual quadratic mortar finite element methods for 3D finite deformation contact. SIAM J. Sci. Comput. **34**(4), B421–B446 (2012). https://doi.org/10.1137/110848190
228. Poulios, K., Renard, Y.: An unconstrained integral approximation of large sliding frictional contact between deformable solids. Comput. Struct. **153**, 75–90 (2015). https://hal.archives-ouvertes.fr/hal-00937569
229. Puso, M.A., Solberg, J.M.: A dual pass mortar approach for unbiased constraints and self-contact. Comput. Methods Appl. Mech. Eng. **367**, 113092, 33 (2020). https://doi.org/10.1016/j.cma.2020.113092
230. Quarteroni, A., Valli, A.: Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics, vol. 23. Springer, Berlin (1994)
231. Raviart, P.A., Thomas, J.M.: Introduction à l'analyse numérique des équations aux dérivées partielles. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris (1983)
232. Renard, Y.: A uniqueness criterion for the Signorini problem with Coulomb friction. SIAM J. Math. Anal. **38**(2), 452–467 (2006). https://doi.org/10.1137/050635936
233. Renard, Y.: Generalized Newton's methods for the approximation and resolution of frictional contact problems in elasticity. Comput. Methods Appl. Mech. Eng. **256**, 38–55 (2012)
234. Renard, Y., Poulios, K.: GetFEM: Automated FE modeling of multiphysics problems based on a generic weak form language. ACM Trans. Math. Softw. **47**(1), 1–31 (2020)
235. Rocca, R., Cocu, M.: Existence and approximation of a solution to quasistatic Signorini problem with local friction. Int. J. Eng. Sci. **39**(11), 1233–1255 (2001). https://doi.org/10.1016/S0020-7225(00)00089-6

236. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 317. Springer, Berlin (1998). https://doi.org/10.1007/978-3-642-02431-3
237. Rudnicki, J.W.: Fundamentals of Continuum Mechanics. Wiley, London (2014)
238. Sauer, R.A., De Lorenzis, L.: An unbiased computational contact formulation for 3D friction. Int. J. Numer. Methods Eng. **101**(4), 251–280 (2015). https://doi.org/10.1002/nme.4794
239. Sayas, F.J., Brown, T.S., Hassell, M.E.: Variational Techniques for Elliptic Partial Differential Equations. CRC Press, Boca Raton (2019). https://doi.org/10.1201/9780429507069. Theoretical tools and advanced applications
240. Scarpini, F., Vivaldi, M.A.: Error estimates for the approximation of some unilateral problems. RAIRO Anal. Numér. **11**(2), 197–208 (1977)
241. Schwartz, L.: Théorie des distributions. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris (1966)
242. Scott, L.R., Zhang, S.: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp. **54**(190), 483–493 (1990). https://doi.org/10.2307/2008497
243. Seitz, A., Wall, W.A., Popp, A.: Nitsche's method for finite deformation thermomechanical contact problems. Comput. Mech. **63**(6), 1091–1110 (2019). https://doi.org/10.1007/s00466-018-1638-x
244. Shillor, M., Sofonea, M., Telega, J.J.: Models and Analysis of Quasistatic Contact: Variational Methods, vol. 655. Springer, Berlin (2004)
245. Signorini, A.: Sopra alcune questioni di elastostatica. Atti della Societa Italiana per il Progresso delle Scienze **21**(2), 143–148 (1933)
246. Signorini, A.: Questioni di elasticità non linearizzata e semilinearizzata. Rend. Mat. e Appl. (5) **18**, 95–139 (1959)
247. Simo, J.C., Laursen, T.A.: An augmented Lagrangian treatment of contact problems involving friction. Comput. Struct. **42**(1), 97–116 (1992)
248. Stadler, G.: Path-following and augmented Lagrangian methods for contact problems in linear elasticity. J. Comput. Appl. Math. **203**(2), 533–547 (2007)
249. Stampacchia, G.: Formes bilinéaires coercitives sur les ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413–4416 (1964)
250. Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Springer, New York (2008). https://doi.org/10.1007/978-0-387-68805-3. Finite and boundary elements, Translated from the 2003 German original
251. Steinbach, O., Wohlmuth, B., Wunderlich, L.: Trace and flux a priori error estimates in finite-element approximations of Signorni-type problems. IMA J. Numer. Anal. **36**(3), 1072–1095 (2016). https://doi.org/10.1093/imanum/drv039
252. Stenberg, R.: On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**(1-3), 139–148 (1995). https://doi.org/10.1016/0377-0427(95)00057-7
253. Strang, G.: Approximation in the finite element method. Numer. Math. **19**, 81—98 (1972)
254. Strang, G., Fix, G.J.: An Analysis of the Finite Element Method. Prentice-Hall Series in Automatic Computation. Prentice-Hall, Englewood Cliffs (1973)
255. Tartar, L.: An introduction to Sobolev Spaces and Interpolation Spaces. Lecture Notes of the Unione Matematica Italiana, vol. 3. Springer, Berlin; UMI, Bologna (2007)
256. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer Series in Computational Mathematics, vol. 25. Springer, Berlin (1997)
257. Tran, T.: Remarks on Sobolev norms of fractional orders. J. Math. Anal. Appl. **498**(2), 124960 (2021). https://doi.org/10.1016/j.jmaa.2021.124960
258. Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, vol. 18. North-Holland Publishing, Amsterdam-New York (1978)
259. Truesdell, C., Noll, W.: The Non-linear Field Theories of Mechanics, 3rd edn. Springer, Berlin (2004). https://doi.org/10.1007/978-3-662-10388-3. Edited and with a preface by Stuart S. Antman

260. Ulbrich, M.: Semismooth Newton methods for operator equations in function spaces. SIAM J. Optim. **13**(3), 805–842 (2003) (2002). https://doi.org/10.1137/S1052623400371569
261. Uspenskiĭ, S.V.: Imbedding theorems for the generalized classes W_p^r of Sobolev. Sibirsk. Mat. Ž. **3**, 418–445 (1962)
262. Wang, F., Reddy, B.D.: A priori error analysis of virtual element method for contact problem. Fixed Point Theory Algorithms Sci. Eng. **2022**, 10 (2022). https://doi.org/10.1186/s13663-022-00720-z
263. Warburton, T., Hesthaven, J.S.: On the constants in hp-finite element trace inverse inequalities. Comput. Methods Appl. Mech. Eng. **192**(25), 2765–2773 (2003). https://doi.org/10.1016/S0045-7825(03)00294-9
264. Wohlmuth, B.I.: Variationally consistent discretization schemes and numerical algorithms for contact problems. Acta Numer. **20**, 569–734 (2011). https://doi.org/10.1017/S0962492911000079
265. Wohlmuth, B.I., Popp, A., Gee, M.W., Wall, W.A.: An abstract framework for a priori estimates for contact problems in 3D with quadratic finite elements. Comput. Mech. **49**(6), 735–747 (2012)
266. Wriggers, P.: Computational Contact Mechanics. Wiley, London (2002)
267. Wriggers, P., Zavarise, G.: A formulation for frictionless contact problems using a weak form introduced by Nitsche. Comput. Mech. **41**, 407–420 (2008). https://doi.org/10.1007/s00466-007-0196-4
268. Ziemer, W.: Weakly Differentiable Functions. Springer, Berlin (1989)
269. Zlámal, M.: On the finite element method. Numer. Math. **12**, 394–409 (1968). https://doi.org/10.1007/BF02161362
270. Zolesio, J.L.: Multiplication dans les espaces de Besov. Proc. Roy. Soc. Edinburgh Sect. A **78**(1–2), 113–117 (1977/78). https://doi.org/10.1017/S0308210500009872
271. Gustafsson, T., Mcbain, Geordie D.: scikit-fem: A Python package for finite element assembly. J. Source Soft. **5**(52), 2369 (2020). https://hal.science/hal-02919750
272. Patrick B., Flaviana I.: Optimal existence results for the 2d elastic contact problem with Coulomb friction, arXiv 2303.16298 (2023)

Index

F. Chouly et al., *Finite Element Approximation of Contact and Friction in Elasticity*, Advances in Continuum Mechanics 48,
https://doi.org/10.1007/978-3-031-31423-0