2 Modelling road traffic

2.1 Objectives

This project consists of simulating the evolution of road traffic using methods similar to those used in hydrodynamics. A macroscopic approach will characterize road traffic in terms of density and vehicle flows.

2.2 The problem set in context

Traffic modeling is an important societal issue. A great deal of research has been carried out on this subject for several decades and new models are still on the test bench in an attempt to optimise traffic organisation and in particular urban traffic (optimisation of the length of traffic lights in cities, etc.). There are two approaches to simulating road traffic. The first microscopic approach consists in simulating the behaviour of each motorist. Such an approach makes it possible to account for the diversity of vehicles and their drivers[1] as well as to take into account the interactions between drivers (adaptation of the distance between vehicles, lane changes, etc.). However, this approach is limited in terms of calculation time, particularly when dealing with a large number of vehicles or when integrating over large time intervals or long distances. A second macrosopic approach is to look at quantities with larger distance scales. In such an approach, only "average" behaviour is considered, not individual behavior.

Road traffic can then be modelled by equations that resemble hydrodynamic problems: road traffic behaves like a fluid whose density varies over time.

[1] As far as the vehicles are concerned, we will take into account characteristics such as length, power, braking capacity, ... As far as drivers are concerned, their reaction time will be taken into account, as well as their behaviour: changing lanes, use of secondary routes, etc ...

2.3 Modelling the traffic

To proceed, we need to define the relevant quantities to write down traffic.

Physical paramters

- Traffic density, defined at every point along the route:

$$\rho(x,t) = \lim_{\Delta x \to 0} \frac{N_{\Delta x}(x,t)}{\Delta x} \tag{2.1}$$

  This is *de facto* the average number of vehicles per unit length.

- The car flow, that is, the average number of cars per unit of time:

$$q(x,t) = \lim_{\Delta t \to 0} \frac{N_{\Delta t}(x,t)}{\Delta t} \tag{2.2}$$

The average speed of motorists at x at time t, is related to the two variables just mentioned by the relationship:

$$v(x,t) = \frac{q(x,t)}{\rho(x,t)} \ . \tag{2.3}$$

Equations

Just as hydrodynamics uses the principle of conservation of mass, so here we must respect the conservation of the number of cars. This leads us to the following equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} \tag{2.4}$$

To solve this problem with two unknowns, we need to introduce a new relationship to express q as a function of ρ: we talk about **fundamental diagram**, which is equivalent to a state equation in thermodynamics.

There are more than twenty basic schemes that can be grouped into two main categories:

a) **First-order Models:**
   The flow depends only on density: $q(x,t) = q(\rho(x,t))$ = cρ(x,t) where c is not necessarily a constant, but can change with x and/or time t.

   There are several possible first-order models, here are some of them:

$$\begin{aligned} q(\rho) &= v_M \rho(x,t)(1 - \frac{\rho}{\rho_M}) && (2.5) \\ q(\rho) &= v_M \rho(x,t) e^{-\frac{\rho}{\rho_M}} && (2.6) \\ q(\rho) &= v_M \rho(x,t) e^{-\frac{1}{a}(\frac{\rho}{\rho_M})^a} && (2.7) \end{aligned}$$

   where v_M is the maximum speed and ρ_M is the maximum density. The maximum speed corresponds to the type of road considered (city, highway, freeway, ..). The maximum density corresponds to a traffic jam situation and therefore depends on the average distance between vehicles when they are stopped. The treatment is then of the type *transport* (advection).

b) **Second-order Models** :
The flow now depends on the density and its derivative:

$$q(x,t) = q(\rho, \frac{\partial \rho}{\partial x}) \ .$$

We end up with a *transport-diffusion* equation in this case. We may use the relation

$$q = c\rho - \alpha \frac{\partial \rho}{\partial x} \tag{2.8}$$

Depending on the model, α may be a constant (disturbance propagation rate) or some function of ρ.[2]

[2] For more information, please refer to page 25 of the MSc Thesis (in french) by G. Costeseque at this **url**.

To model traffic, you need to know the expression $q(rho, t)$. You can search for other, more elaborate models in the literature if you wish.

The equation can then be re-expressed as an equation at the first degree of density, *rho*, using partial derivation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial \rho}\frac{\partial \rho}{\partial x} \tag{2.9}$$

Example

As an example, we will adopt the fundamental diagram of the equation (2.5). We then obtain the partial derivation

$$\frac{\partial q}{\partial \rho} = v_M \left[1 \times (1 - \frac{\rho}{\rho_M}) + \rho \times (-\frac{1}{\rho_M}) \right] = v_M \left[1 - 2\frac{\rho}{\rho_M} \right] \ .$$

Injecting this expression into the differential equation (2.9) results in a transport-type equation for the density; we find

$$\frac{\partial \rho}{\partial t} = -v_M \left[1 - 2\frac{\rho}{\rho_M} \right] \times \frac{\partial \rho}{\partial x} \,. \tag{2.10}$$

We must then adopt a numerical scheme to integrate over time the density ρ (and also the flow, q).

2.4 *Numerical scheme*

First, we discretize the space in N intervals of length Δx as shown in figure 2.1. The scales of distances considered for the study of the problem must be large in relation to the size of a vehicle, typically

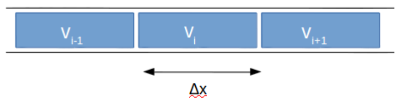

Figure 2.1: Illustration of the discretization of a stretch of road.

from a few tens to a few hundreds of meters. It's a configurable parameter.

We will consider the evolution of the system by discretizing the time in Δt intervals. The time step Δt must be less than the characteristic time it takes a vehicle to travel Δx. It is also a configurable parameter. What is your choice of parameters, expressed as the Courant-Friedrich-Levy (CFL) inequality?

We will use the **Godunov scheme** to solve for that problem. We will adopt the change of variables for a discrete representation of the continuous quantities ρ, q: $\rho \to u$, and $q \to f$ to respect conventions often encountered in literature.

Godunov integration method

First of all, initial conditions have to be set, *i.e.* know the values of the average velocity and density in each cell. Godunov's scheme allows the system to evolve over time in small steps Δt. For a given time, we determine the density values in each interval of the route:

$$u_i^n = \rho(i\Delta x, n\Delta t)$$

Use the following expression for the discrete form of the basic equation (2.4):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(f_{i+1}^n - f_i^n) \tag{2.11}$$

(Note the syntax with index for the f flow, identical to the one for the density, u). This expression to the right of the equality for the f gradient depends on the chosen scheme: we give here the form *forward in space*.

In the case of an approach to first order, according to the equation 2.5, we would obtain from (2.11) the expression :

$$u_i^{n+1} = u_i^n - v_M \times (1 - 2\frac{u_i^n}{\rho_M}) \times \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \tag{2.12}$$

Exercise

In the case of a second-order approach where we have the equation:

$$q = v_M \rho(x,t)(1 - \frac{\rho}{\rho_M}) - \alpha\frac{\partial \rho}{\partial x} \tag{2.13}$$

Show that it's possible to use the following numerical scheme:

$$u_i^{n+1} = u_i^n + \alpha \times \frac{\Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - v_M \times (1 - 2\,\frac{u_i^n}{\rho_M}) \times \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \tag{2.14}$$

Once the velocity v_M is known, along with all the integration parameters, we will be able to trace back the density u and the flow f.

Workplan

To solve the problem, it is necessary to specify the boundary conditions at the ends of the stretch of road (ρ and q at these points) and initial conditions (knowledge of ρ and v when t=0) at any point along the route.

Road map and traffic conditions

The problem can be dealt with in a number of contexts, starting with the simplest. Here are some examples of possible "routes":

- A closed car racetrack (boundary conditions: $\rho(0,t) = \rho(L,t)$)
- A stretch of motorway
- ascending / descending paths, roadwork ..

As far as boundary conditions are concerned, we can look at the following cases:

- inflow increasing over time
- with increasing inflow and congested traffic
- 'red-light' start at one end, other scenarios, ..

Results and analysis

You can illustrate the behaviour of the numerical method graphically as a function of the chosen parameters and the model under consideration. You will be able to represent the evolution of road traffic characteristics as a function of time.

Bibliography and external links:

- The Wikipedia entry for road **traffic models**
- HDR Thesis by M. Goatin (French/English) at `https://tel.archives-ouvertes.fr/file/index/docid/765410/filename/hdrGoatin.pdf`

To go further ..

- You should process different fundamental diagrams and compare them.
- You can also try to deal with non-homogeneous cases, i.e. when there are changes in maximum speed on a circuit or when the number of lanes evolves (roadwork, ..).
- Beware: non-homogeneous cases may be handled incorrectly by the Godunov scheme, so you can explore its limitations. If you wish, you can also search the bibliography for more advanced schemes.
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