# Linear and Generalized Linear Models

Week 7, Lecture 1

## Logistic regression

Saskia le Cessie

Leiden University Medical Centre

## Linear regression:

- Linear model: $\boldsymbol{y} = \boldsymbol{X\beta} + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I}_n)$ and $\mathbf{X}_{n\times(k+1)}$ is the model matrix.
- Least-squares estimator: $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$.
- Variance-covariance matrix for $\mathbf{b}$ : $V(\mathbf{b}) = \sigma_\epsilon^2(\mathbf{X}'\mathbf{X})^{-1}$.
- Distribution of $\mathbf{b}$ : $\mathbf{b} \sim N_{k+1}(\boldsymbol{\beta}, \sigma_\epsilon^2(\mathbf{X}'\mathbf{X})^{-1})$

## Responses are not always continuous

- What if the response $Y$ is a 0/1 variable (dichotomous variable, binary variable) ?
  - yes / no
  - ill / healthy
  - success / failure
- How does $Y$ depend on one or more X-variables?

# Example: Predicting Rheumatoid arthritis

- Early Arthritis Clinic, Department of Rheumatology, LUMC
- Patients in Leiden area with arthritis complaints (painfull joints, swollen joints) are referred to this clinic by their GP
- Some patients are diagnosed with undifferential arthritis
- Which of these patients will get Rheumatoid Arthritis (RA) within a year ?

Aim: a model to predict the risk of RA within 1 year

Patients with high risk can be treated with new treatments( expensive, side effects)

# Notations

$Y_i$ binary response variable for person *i* : ($Y_i$ =1 , RA after 1 year, $Y_i$ = 0 no RA)

$Y_i$ follows a binomial distribution: $Y_i$ ~ binomial (1, $\pi_i$).

Several predictors $(x_{i1}, \dots, x_{ik})$ are measured

Aim: a model for $E(Y_i|x_{i1}, \dots, x_k) = P(Y_i = 1|x_{i1}, \dots, x_{ik}) = \pi_i$.

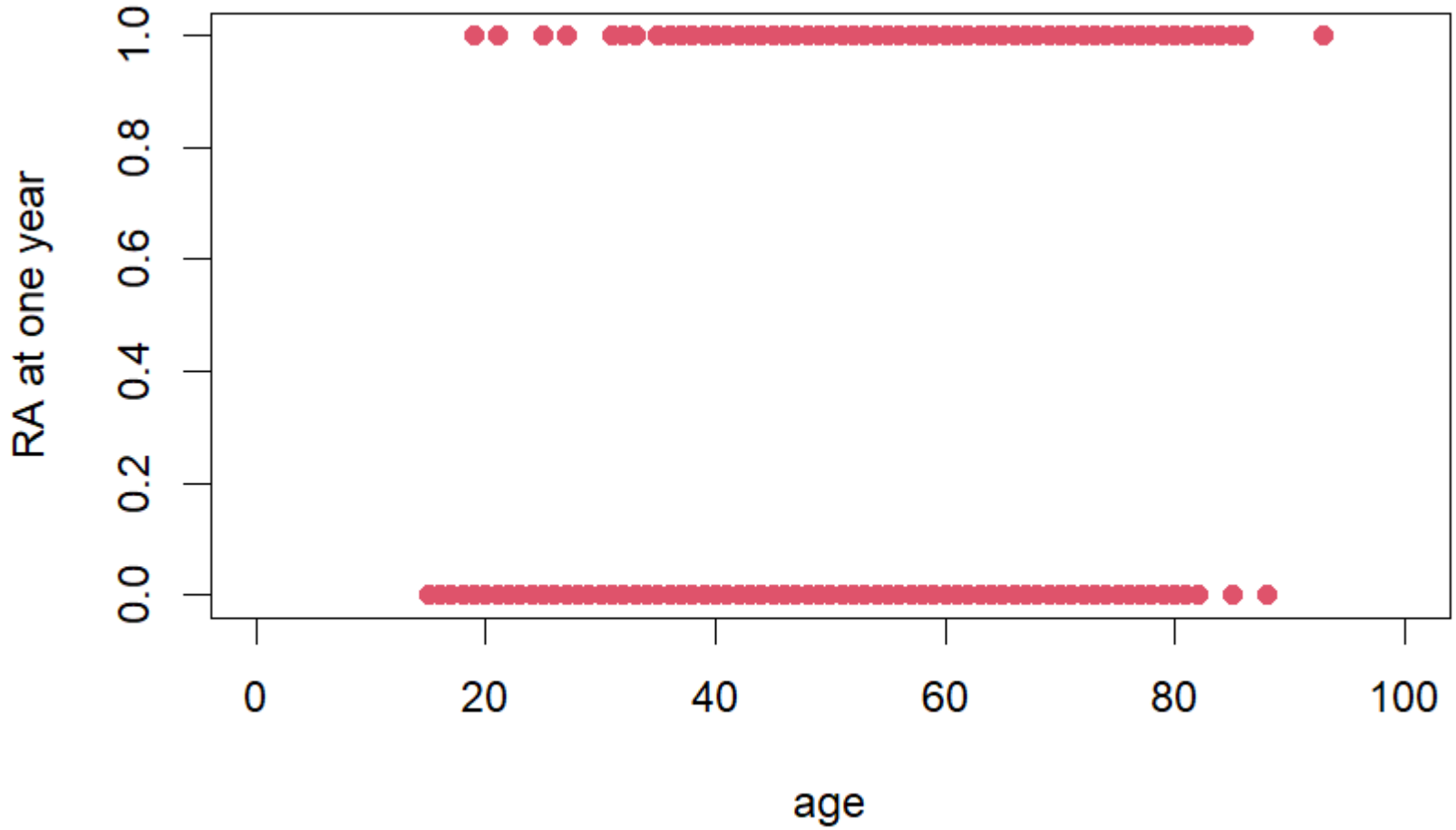
Matrix notations:

$x_i' = (1, x_{i1}, \dots, x_{ik})$

$$X = \begin{pmatrix} 1 & x_{11} & \dots & x_{1k} \\ 1 & x_{21} & & x_{2k} \\ \vdots & & \ddots & \\ 1 & x_{n1} & & x_{nk} \end{pmatrix} \qquad Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \\ Y_n \end{pmatrix}$$

We assume independent observations

# Let's start with one $x$ variable: age



# Problems with linear regression model for $\pi$

- Predicted probabilities can be greater than 1 or less than 0.
- Variance of $Y_i$ is $\pi_i(1 - \pi_i)$. However, the linear model assumes constant variance.

Therefore : develop a model directly suited for binomial data.

## Transform $\pi = P(Y = 1|x)$

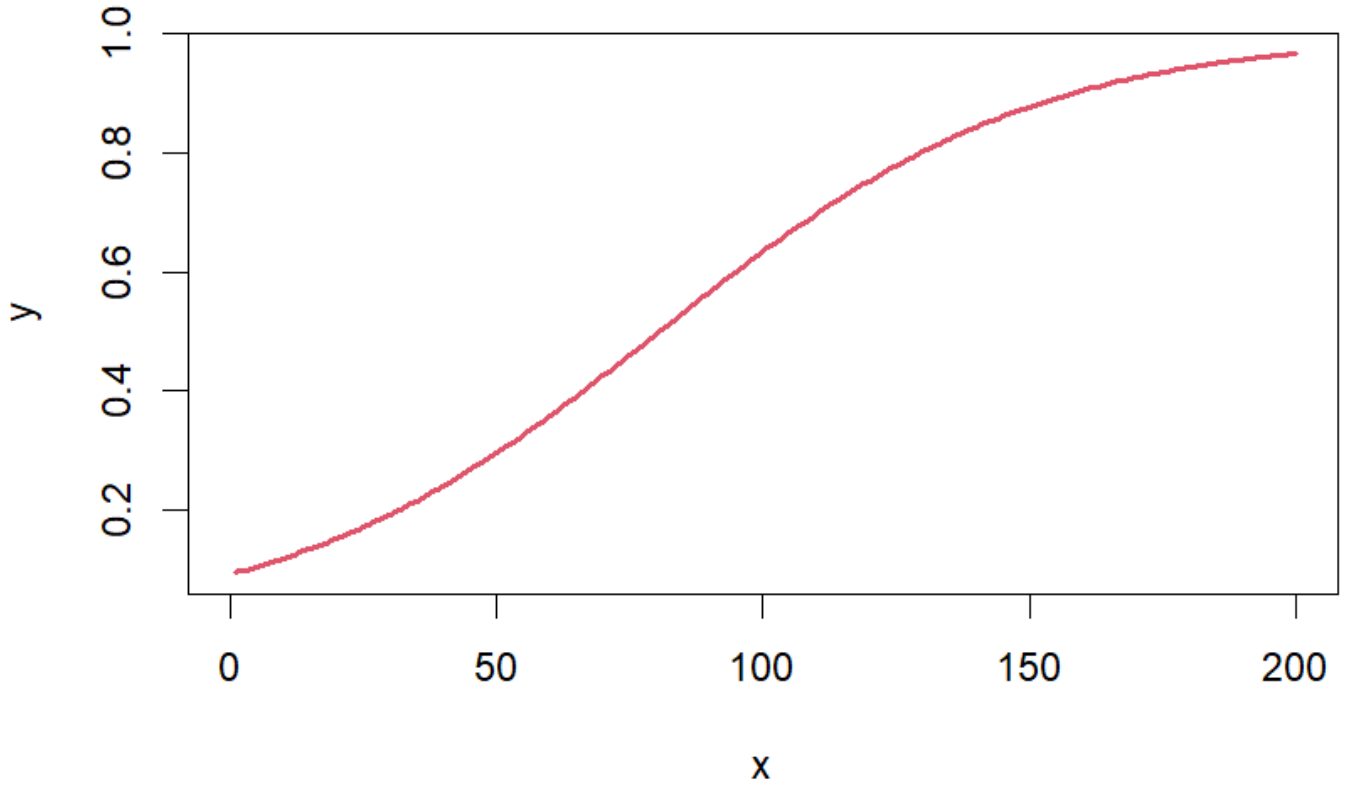
- Requirement: $0 \leq \pi \leq 1$
- Therefore $0 \leq \frac{\pi}{1-\pi} \leq \infty$ ($\frac{\pi}{1-\pi}$ is called the odds)
- And $-\infty \leq \log\left(\frac{\pi}{1-\pi}\right) \leq \infty$

→ Use a linear model for $\log\left(\frac{\pi}{1-\pi}\right)$ =logit($\pi$)

# The logistic regression model

- $\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + .. + \beta_k x_k$
- Or equivalent:
- $\pi = \frac{\exp(\beta_0+\beta_1 x_1+..+\beta_k x_k)}{1+\exp(\beta_0+\beta_1 x_1+..+\beta_k x_k)}$

(show it yourself)



## Logistic regression in R

```
> model.lr <- glm(ra1year~Age, family=binomial, data=RAdata)
> summary(model.lr)

Call:
glm(formula = ra1year ~ Age, family = binomial, data = RAdata)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.281798   0.321459  -7.098 1.26e-12 ***
Age          0.028303   0.005717   4.951 7.39e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

# The model

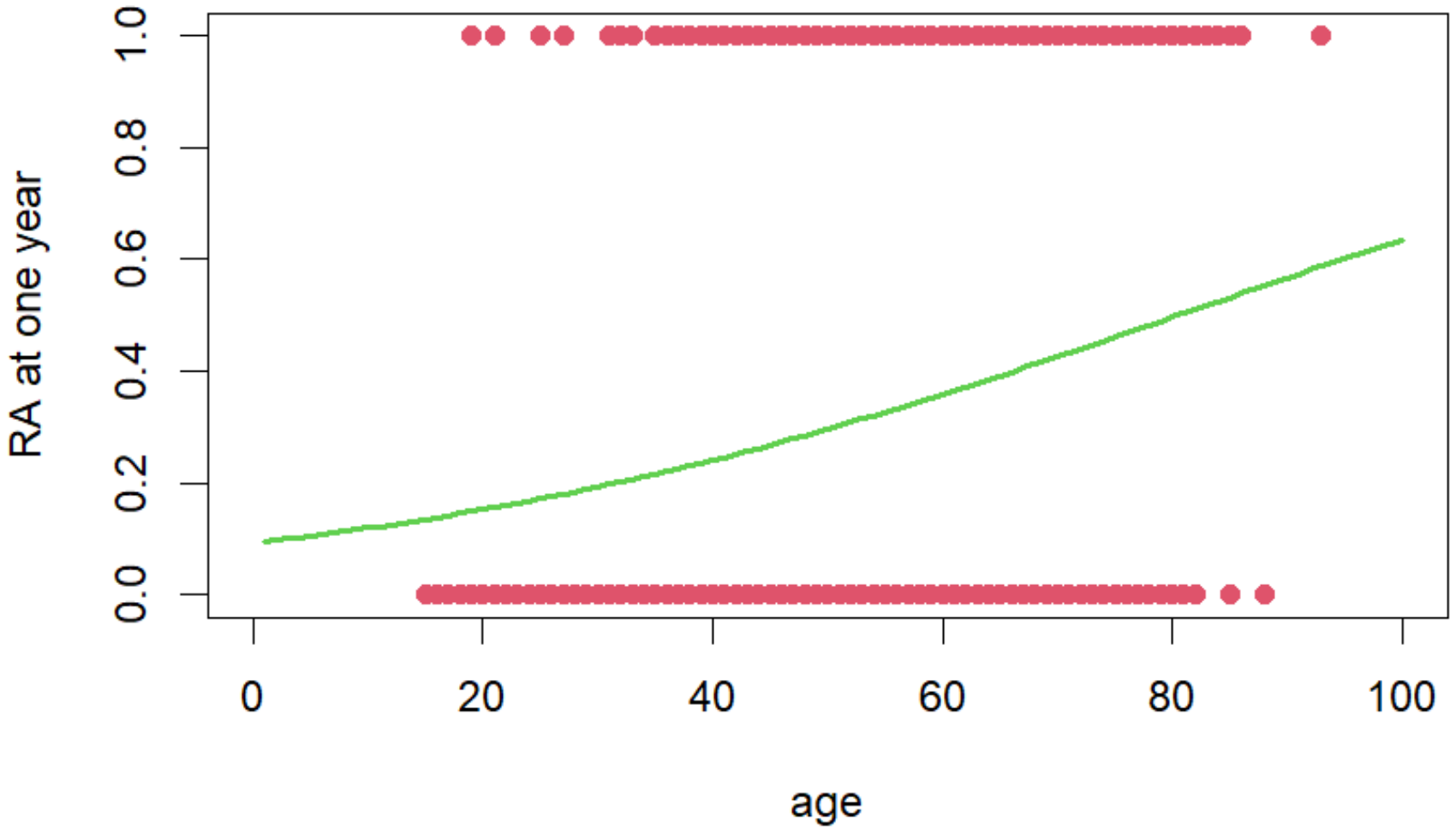
```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.281798   0.321459  -7.098 1.26e-12 ***
Age          0.028303   0.005717   4.951 7.39e-07 ***
```

Logistic model: $\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1$

Estimated: $\widehat{\log\left(\frac{\pi}{1-\pi}\right)}$ = -2.281798 + 0.028303 age

$$\hat{\pi} = P(\text{RA withing one year}) = \frac{\exp(-2.281798 + 0.028303\, age)}{1+\exp(-2.281798 + 0.028303\, age)}$$

# The logistic model



# Interpretation of regression coefficients

Interpretation coefficient for age, 0.028303:

- For two patients, with age difference of one year, the expected difference in log-odds, $\log\left(\frac{\pi}{1-\pi}\right)$ is 0.028303
- for a one year increase in age, the odds of RA becomes exp(0.028303) = 1.029 times larger.

additive

Multiplicative

*Logarithm converts multiplication and division to addition and subtraction. Exponentiation converts addition and subtraction back to multiplication and division.*

- Interpretation intercept -2.281798
- For someone of age 0, the estimated log-odds is -2.281798 and $\hat{\pi} = \frac{\exp(-2.281798)}{1+\exp(-2.281798)}$ =0.09

# Odds

INDEPENDENT BETTING

powered by gambling.com

Betting > Football

## England vs Netherlands predictions: Women's Nations League tips and odds

Our tipster offers three betting predictions for England Women's crunch Nations League encounter with Netherlands Women

Last Updated: 30th of November 2023

Joe Short • Football Writer


### England vs Netherlands betting tips:

- England to win both halves – 13/8 with Unibet
- Beth Mead to score first – 9/2 with 10Bet
- Over 2.5 goals – 8/13 with BetVictor

Most Popular

CRICKET

Big Bash League 13 predictions: Cricket betting...

FOOTBALL

# A binary risk factor: rheumafactor (RF)

| | Y | |
|---|---|---|
| X | RA | No RA |
| Rheumafactor | 84 | 56 |
| No rheumafactor | 93 | 336 |

Let $\pi_1$=P(Y=1|X=1) , $\hat{\pi}_1$ =

Let $\pi_0$=P(Y=1|X=0) , $\hat{\pi}_0$ =

Measures of association in 2 by 2 table:

- Risk difference $\pi_1$- $\pi_0$
- Risk ratio $\pi_1/\ \pi_0$
- Odds ratio $(\pi_1/(1-\pi_1))/(\ \pi_0/\ (1-\pi_0))$

# The odds ratio

$$\text{OR} = \frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}$$

OR >1 : $\pi_1 > \pi_0$

OR <1 : $\pi_1 < \pi_0$

OR = 1: $\pi_1 = \pi_0$

- If the outcome Y=1 is rare ($\pi_1$ and $\pi_0$close to 0), we have:
- $\pi_1/(1-\pi_1) \approx \pi_1$
- $\pi_0/(1-\pi_0) \approx \pi_0$
- and the odds ratio ≈ $\pi_1/\pi_0$, the relative risk

# Logistic regression with one binary predictor

```
> model.lr2 <- glm(ra1year~rfactor, family=binomial, data=RAdata)
> summary(model.lr2)

Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.2845     0.1172 -10.963  < 2e-16 ***
rfactor       1.6900     0.2085   8.104 5.33e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

- Exp(1.6900) = 5.419 → the oddsratio in the 2 by 2 table

- $e^{\beta_1}$=odds ratio

## Logistic regression with multiple x-variables

- $\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + .. + \beta_k x_k \quad = x'\boldsymbol{\beta}$

- Or equivalent:

- $\pi = \frac{\exp(\beta_0+\beta_1 x_1+..+\beta_k x_k)}{1+\exp(\beta_0+\beta_1 x_1+..+\beta_k x_k)} = \frac{\exp(x'\boldsymbol{\beta})}{1+\exp(x'\boldsymbol{\beta})}$

# Logistic regression with multiple x- variables

```
Call:
glm(formula = ra1year ~ rfactor + Age, family = binomial, data = RAdata)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.708391   0.353936  -7.652 1.98e-14 ***
rfactor      1.658540   0.212768   7.795 6.44e-15 ***
Age          0.027257   0.006126   4.450 8.61e-06 ***
```

$\log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2\, x_2$ = -2.282+1.659 rfactor+ 0.0278 age

# Logistic regression with multiple x- variables

```
Call:
glm(formula = ra1year ~ rfactor + Age, family = binomial, data = RAdata)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.708391   0.353936  -7.652 1.98e-14 ***
rfactor      1.658540   0.212768   7.795 6.44e-15 ***
Age          0.027257   0.006126   4.450 8.61e-06 ***
```

- For two patients with the same age, the odds on RA is exp(1.658540)= 5.252 times larger for those with rheumafactor present, compared to those with rheumafactor absent.

- For two patients with the same value for rfactor, differing by 1 year in age, the odds on RA is exp(0.027257)=1.0276 times larger for the older patient.

# Estimation of parameters β = ($\boldsymbol{\beta_0}$,… $\boldsymbol{\beta_p}$)’

Maximum likelihood:

- Choose values for $\beta_i$’s that are most likely to have produced the data.
- Suppose we have n observations $(x_1, y_1) \ldots (x_n, y_n)$
- What is P(data| β) = $\mathrm{P}(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n)$?

- Contribution of person *i* : $\mathrm{P}(Y_i = y_i)$
  - if $y_i$ =1 : $\pi_i$
    if $y_i$ =0 : (1- $\pi_i$)
- $\mathrm{P}(Y_i = y_i) = \pi_i^{y_i}(1\text{-}\pi_i)^{1-y_i}$

# The likelihood function

The likelihood function: $L(\beta) = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{1-y_i}$

- The larger the likelihood, the better the model fits the data.
- Select those values for b, for which L(β) attains its maximum.

- Easier to maximize the log of the likelihood
- $\log L(\beta) = \sum_{i=1}^{n} y_i \log(\pi_i) + (1 - y_i) \log(1 - \pi_i)$

- Log likelihood will take its maximum at the same values of b .

## Maximize $\log L(\boldsymbol{\beta}) = \sum_{i=1}^{n} y_i \log(\pi_i) + (1 - y_i) \log(1 - \pi_i)$

- Differentiate w.r.t. $\beta_0, \beta_1 \ldots, \beta_p$ and set the first derivatives equal to 0.

- You can show that:

$$\begin{bmatrix} \frac{\partial l(\beta)}{\partial \beta_0} \\ \frac{\partial l(\beta)}{\partial \beta_1} \\ \vdots \\ \frac{\partial l(\beta)}{\partial \beta_k} \end{bmatrix} = \begin{bmatrix} \sum_i (y_i - \pi_i) \\ \sum_i x_{i1}(y_i - \pi_i) \\ \vdots \\ \sum_i x_{ik}(y_i - \pi_i) \end{bmatrix} = X'(Y - \pi)$$

- So we need to solve : $X'(Y - \pi) = 0$
- In general, this can not be solved exactly , numerical techniques are needed.

# Standard errors of $\hat{\beta}$

- Calculate the second derivatives of the log likelihood:

- $\frac{\partial^2 \log L(\beta)}{\partial \beta \partial \beta'} = -X'V\,X$ with V $= \begin{pmatrix} \pi_1(1-\pi_1) & 0 & \cdots & 0 \\ 0 & \pi_2(1-\pi_2) & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \ddots & \pi_n(1-\pi_n) \end{pmatrix}$

- Asymptotic variance-covariance $(k+1) \times (k+1))$ matrix of MLE is then:

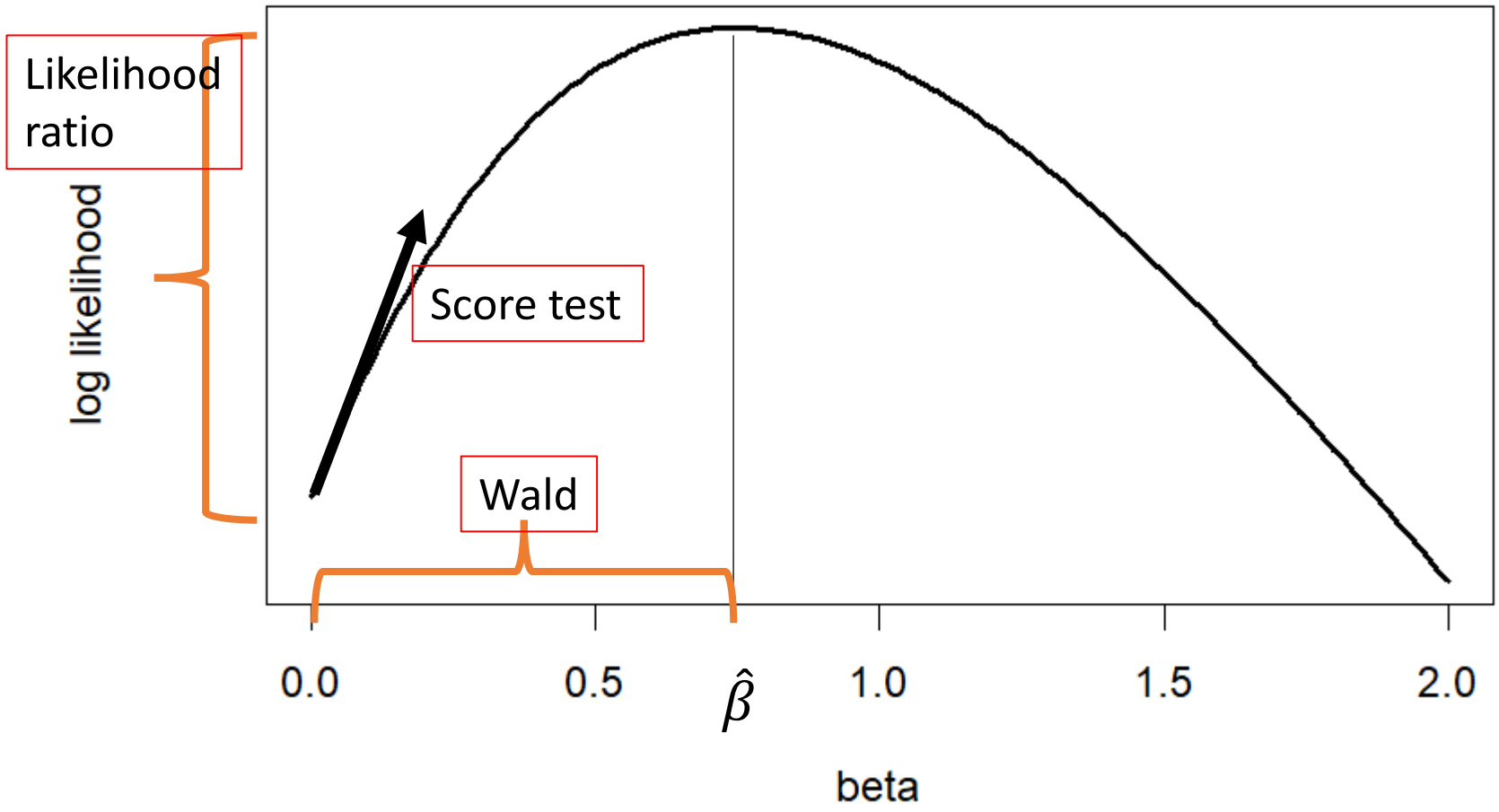
- $V(\hat{\beta}) = \left\{-E\left[\frac{\partial^2 \log L(\boldsymbol{\beta})}{\partial \beta \partial \boldsymbol{\beta}'}\right]\right\}^{-1} = (X'V\,X)^{-1}$

# Hypothesis testing for a regression coefficient

- $H_0$: $\beta_1$=0

Three different methods:

- Wald test
- Likelihood ratio test
- Score test



# Method 1: Wald test

- Consider Z = $\hat{\beta}_1 / se(\hat{\beta}_1)$. Under $H_0$ Z follows approximately a standard normal distribution. Reject $H_0$ if | Z | >$z_\alpha$
- Sometimes $Z^2$ instead of Z is used (compare $Z^2$ to a $\chi_{(1)}^2$ distribution)

- Example:

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.281798   0.321459  -7.098 1.26e-12 ***
Age          0.028303   0.005717   4.951 7.39e-07 ***
```

# Method 2: likelihood ratio test

- The (log) likelihood measures how well the model fits the data.
- Compare $\log L(\beta_0,0)$ , the log-likelihood of model with $\beta_1$=0: with $\log L(\beta_0, \beta_1)$ , the log-likelihood of complete model:
- Under $H_0$ : $2(\log L(\beta_0, \beta_1) - \log L(\beta_0,0))$ has (approximately) a $\chi_{(1)}^2$ distribution.

# Likelihood ratio tests

- Can be used more generally to compare nested models
- Can sometimes be used as goodness of fit test (to compare a model to a perfectly fitted model)
- More tomorrow

# Method 3: score test

- Use only the null model and consider how steep the likelihood function is in $\beta = 0$ is.

- The three  methods are asymptotically equivalent; for small numbers the likelihood ratio method is usually better.

# Confidence intervals

Wald method:

- 100(1-α) % confidence interval for $\beta_i$ : $(\hat{\beta}_i - z_{\alpha/2}se(\hat{\beta}_i)\ ,\ \hat{\beta}_i + z_{\alpha/2}se(\hat{\beta}_i))$
- Confidence interval for odds ratio $e^{\beta_i}$: ( $e^{\text{lower limit}}$ , $e^{\text{upper limit}}$)

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -2.281798   0.321459  -7.098 1.26e-12 ***
Age          0.028303   0.005717   4.951 7.39e-07 ***
```

95% CI for coefficient for age:

95% CI for odds ratio  for age:

# Confidence intervals-Wald method

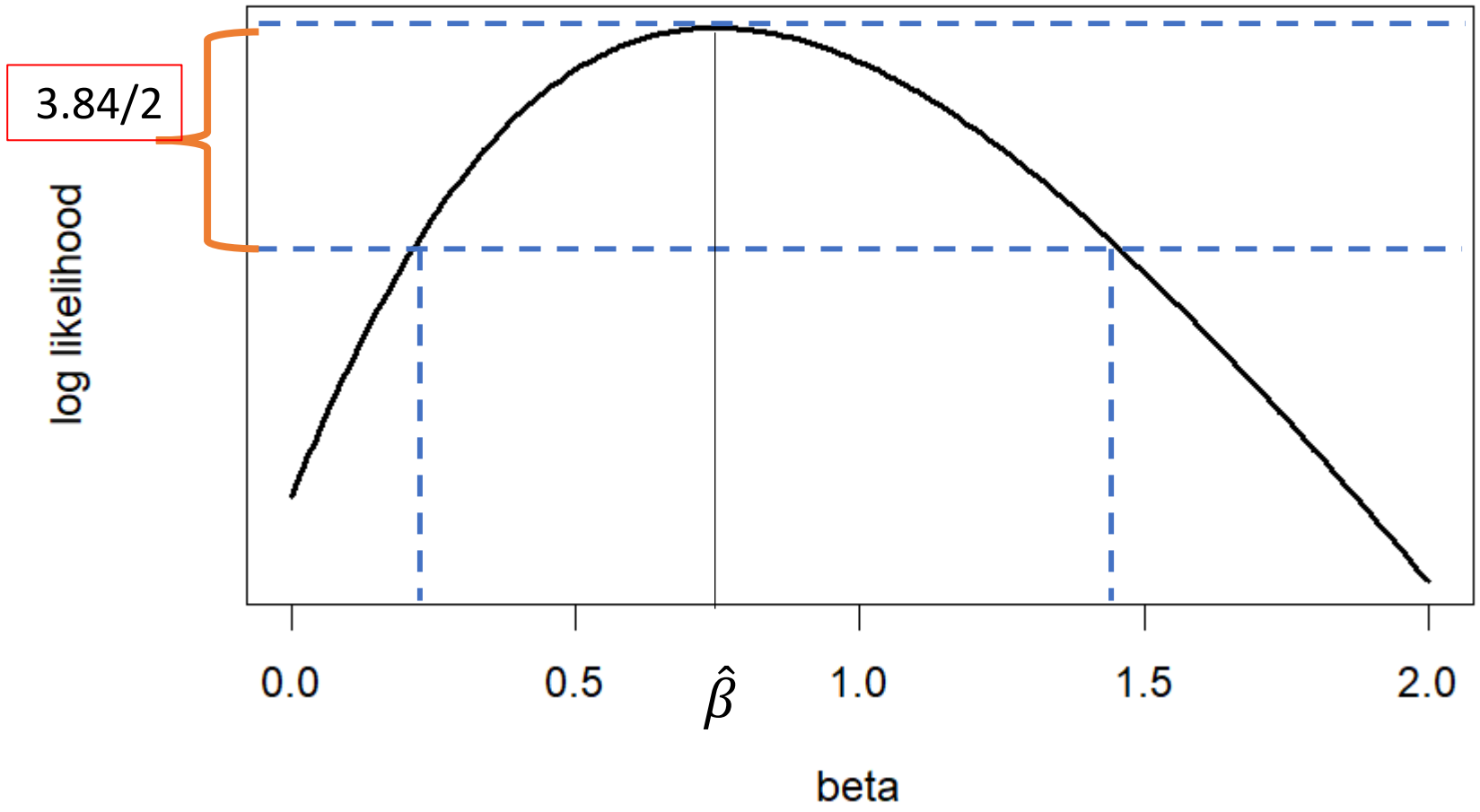
```
cbind(exp(coefficients(model.lr)), exp(confint.default(model.lr)))
                                2.5 %    97.5 %
(Intercept) 0.1021004 0.05437525 0.1917141
Age         1.0287074 1.01724528 1.0402987
```

Conclusion on relation between age and RA after 1 year based on the 95% CI?

# Confidence intervals based on the (profile) likelihood

- All values of $\beta_i$ for which the 2(profile) log likelihood is less then the critical value of the $\chi_{(1)}^2$ distribution.

- For $\alpha = 0.05$, this value is 3.84 (=$1.96^2$)



## Profile likelihood confidence intervals

```
cbind(exp(coefficients(model.lr)), exp(confint(model.lr)))
Waiting for profiling to be done...
                              2.5 %    97.5 %
(Intercept) 0.1021004 0.05354001 0.1891023
Age         1.0287074 1.01739913 1.0404893
```