$X \sim \exp(\lambda)$ $\hat{\lambda}_{MLE} = ?$

$f(x) = \lambda e^{-\lambda x}$; $S(x) = e^{-\lambda x}$; $F(x) = 1 - S(x)$

Likelihood: in case of complete data

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}$$

log-likelihood: $\ell(\lambda) = n \log \lambda - \lambda \sum x_i$

derivative: $\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i = 0$

$$\Rightarrow \frac{n}{\lambda} = \sum x_i \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

where $\bar{x} = \frac{\sum x_i}{n}$

likelihood for censored data:

observed data (x_i, δ_i) $i = 1..n$

$$\mathcal{L}(\lambda; x, \delta) = \prod_{i=1}^{n} \lambda^{\delta_i} e^{-\lambda x_i}$$

remember the likelihood for right censored data:

$$L(\lambda; x, \delta) = \prod_{i=1}^{n} f(x_i)^{\delta_i} S(x_i)^{1-\delta_i}$$

$$L(\lambda) = \prod_{i=1}^{n} \lambda^{\delta_i} e^{-\lambda x_i}$$

$$L(\lambda) = \prod_{i=1}^{n} \lambda^{\delta_i} e^{-\lambda x_i}$$ likelihood for right censored data

$$= \lambda^{\sum_{i}^{n} \delta_i} e^{-\lambda \sum x_i}$$

log-likelihood:

$$\ell(\lambda) = \log\left(\lambda^{\sum \delta_i} e^{-\lambda \sum x_i}\right)$$

$$= \log \lambda^{\sum \delta_i} - \lambda \sum x_i$$

$$= \left(\sum \delta_i\right) \log \lambda - \lambda \sum x_i$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\sum \delta_i}{\lambda} - \sum x_i = 0$$

$$\Rightarrow \frac{\sum \delta_i}{\lambda} = \sum x_i \Rightarrow \frac{\lambda}{\sum \delta_i} = \frac{1}{\sum x_i}$$

$$\hat{\lambda}_{MLE} = \frac{\sum \delta_i}{\sum x_i} = \frac{n^{o} \text{ failure}}{n^{o} \text{ at risk}}$$

MLE for complete data (no censoring)

$$\hat{\lambda} = \frac{n}{\sum x_i}$$

MLE for right censored data

$$\hat{\lambda} = \frac{\sum \delta_i}{\sum x_i}$$