Exercises Lecture 4:

1. Suppose that a random variable X has a discrete distribution with pmf:

$$f(x) = \begin{cases} c \cdot x \text{ for } x = 1, \ldots, 5 \\ 0, \text{ otherwise} \end{cases}$$

What is the value of the constant c?

Solution: It holds that $\sum_{x=1}^{5} f(x) = 1$ and thus $15c = 1$, which means that $c = 1/15$.

2. The following table shows the cdf of a discrete random variable K. Which is the corresponding pmf? Draw both the pmf and cdf.

Optional: If you want you may try it in R, as well.

k	$F(k)$
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

Solution:

k	$P(K = k)$	$F(k) = P(K \leq k)$
0	0	0
1	0.1	0.1
2	0.2	0.3
3	0.4	0.7
4	0.1	0.8
5	0.2	1.0

```
x <- 0:5
y <- c(0, 0.1, 0.2, 0.4, 0.1, 0.2)
z <- c(0, cumsum(y))

par(mfrow = c(1, 2))
plot(x, y, xlim=c(0,5), ylim=c(0,1), xlab="k", ylab="p(k)",
       main = "Frequency function", type = "p", pch = 16)
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points(x = x, y = y, type = "h")

sfun0  <- stepfun(0:5, z, f = 0)
plot(sfun0, xlim = c(-1,6), ylim = c(0, 1), main = "CDF", pch=16, ylab = "F(k)", xlab = "k")
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3. A rv X takes only the values 3, 4, 7, 8 and 9 with $P(X = 3) = 1/3, P(X = 4) = 1/4, P(X = 7) = 1/6, P(X = 8) = 1/6$. Compute the following probabilities: (a) $P(X = 9)$, (b) $F_X(5)$, (c) $P(4 \le X \le 8)$ and (d) $P(X \ge 8)$.

   Solution:

   (a) $P(X = 9) = 1 - (P(X = 3) + P(X = 4) + P(X = 7) + P(X = 8)) = 0.0833.$

   (b) $F_X(5) = P(X \le 5) = P(X = 3) + P(X = 4) = 0.5833333.$

   (c) $P(4 \le X \le 8) = P(X = 4) + P(X = 7) + P(X = 8) = 0.5833333.$

   (d) $P(X \ge 8) = P(X = 8) + P(X = 9) = 0.2499667.$

4. In a class of 50 students, the number of students n_i of each Age i is shown in the following Table:

Age	n_i
18	20
19	22
20	4
21	3
25	1

   If a student is to be selected at random from the class, what is the expected value of his age?

   Solution:

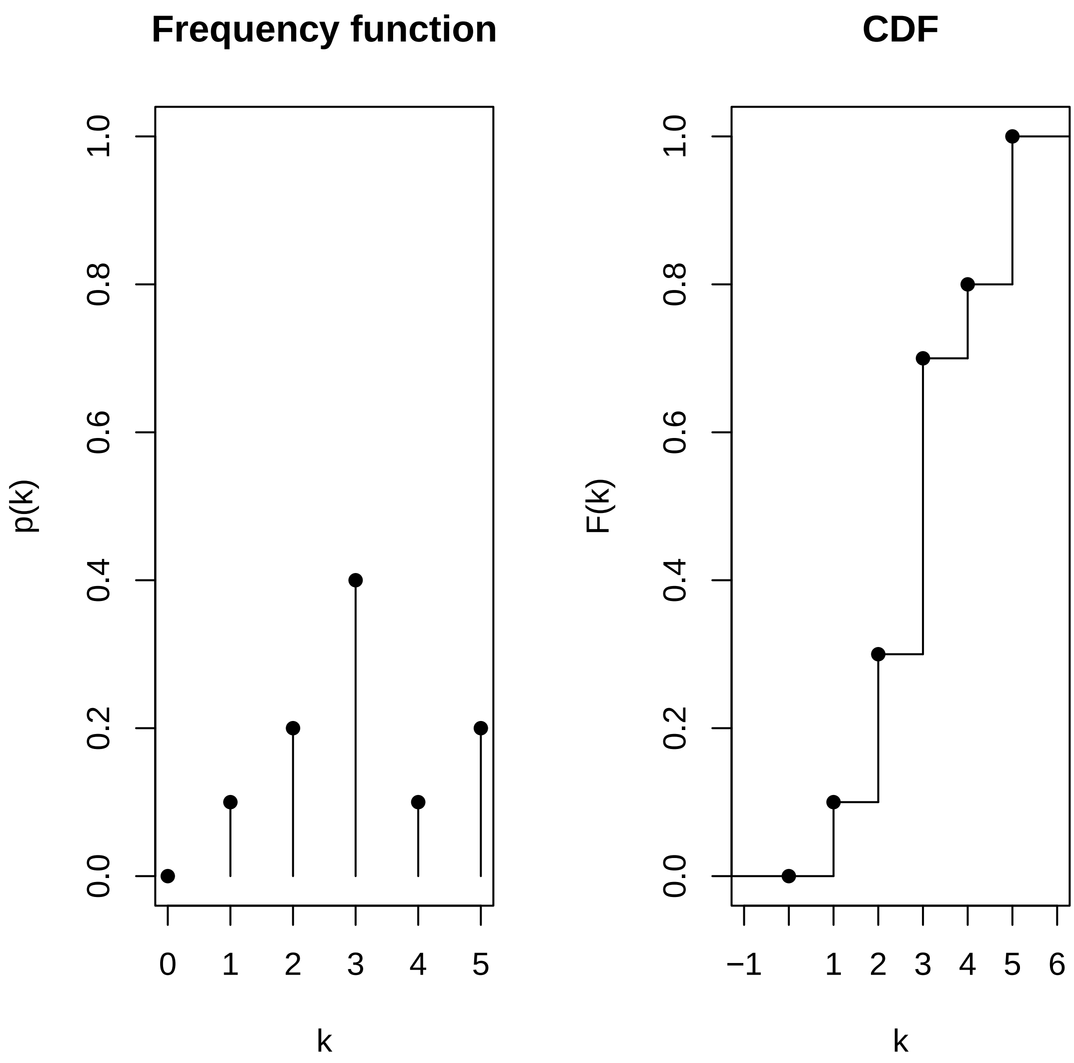
$$E(X) = 18 \cdot \frac{20}{50} + 19 \cdot \frac{22}{50} + 20 \cdot \frac{4}{50} + 21 \cdot \frac{3}{50} + 25 \cdot \frac{1}{50} = 18.92.$$

5. The following Table shows the cumulative distribution function of a discrete random variable X.

   What is the expected value of X?

   Solution: We need first to derive the frequency function.

   Then $E(X) = 0 \times 0 + 1 \times 0.1 + 2 \times 0.2 + \ldots = 3.1$


Frequency function
p(k)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
k
CDF
F(k)
0.0
0.2
0.4
0.6
0.8
1.0
−1
1
2
3
4
5
6
k

x	F(x)
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

x	0	1	2	3	4	5
f(x)	0	0.1	0.2	0.4	0.1	0.2

6. Let X be a discrete random variable that takes on values $0, 1, 2$ with probabilities $1/2, 3/8, 1/8$, respectively.

   (a) What is the value of $E(X)$?

   Solution: $E(X) = 0 \times 1/2 + 1 \times 3/8 + 2 \times 1/8 = 5/8$

   (b) Let $Y = X^2$. What is the value of $E(Y)$?

X	0	1	2
$Y = X^2$	0	1	4
$f(x)$	$1/2$	$3/8$	$1/8$

   Solution: $E(Y) = 0 \times 1/2 + 1 \times 3/8 + 4 \times 1/8 = 7/8$.

7. Let rv X with $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. What is the expected value and the variance of Z?

   Solution:

   We apply the properties of the expected value: $E(Z) = E((X - \mu)/\sigma) = (1/\sigma)E(X - \mu) = (1/\sigma)(E(X) - \mu) = 0$, and $Var(Z) = Var((X - \mu)/\sigma) = (1/\sigma^2)Var(X) = 1$.

8. Let X be a Bernoulli rv. Which is its cdf?

   Solution:

Let $X \sim Bernoulli(p)$, then the pmf is:

$$p(x) = \begin{cases} p^x(1-p)^{(1-x)} \text{ if } x = 0 \text{ or } x = 1 \\ 0, \text{ otherwise} \end{cases}$$

Thus, the cdf is:

$$F(X) = P(X \leq x) = \begin{cases} 0, \text{ if } x < 0 \\ 1-p, \text{ if } 0 \leq x < 1 \\ 1-p+p, \text{ if } x \geq 1. \end{cases}$$

9. If the probability that an individual suffers an adverse reaction from a particular drug is known to be 0.001, what is the probability that out of the 2000 individuals, (a) exactly 3 and (b) more than two individuals will suffer an adverse reaction?

   Solution: Let Y the number of individuals who suffer an adverse reaction. Then $Y \sim Bin(n = 2000, p = 0.001)$. Since n is large and p is small we can use the Poisson approximation with $\lambda = np = 2$.

   (*a*) $P(Y = 3) = \frac{2^3 \exp^{-2}}{3!} = 0.18$.

   (*b*) $P(Y > 2) = 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) = 1 - 5\exp^{-2} = 0.323$.

10. For a certain manifacturing process it is known that on average 1 in every 100 items is defective. (a) What is the probability that the first defective item found is the fifth item inspected? (b) What is the average number of items that should be sampled until the first defective is found (i.e., including the defective item)?

    Solution: Let X be the number of trials up to and including the first success. Then $X \sim Geometric(p = 0.01)$.

    (*a*) $P(X = 5) = 0.01 \cdot 0.99^4 = 0.0096$.

    (*b*) $E(X) = 1/0.01 = 100$.

11. Suppose that a rare disease has an incidence of 1 in 1000. Assuming that the members of the population are affected independently, what is the probability of k cases in a population of 10^5 for $k = 0, 1, 2$?

    Solution: Let X the number of cases in $n = 10^5$ individuals with disease probability $p = 1/1000$. Then $X \sim Bin(n = 10^5, p = 1/1000)$. Then $P(K = k) = \binom{10^5}{k}(\frac{1}{1000})^k(1 - \frac{1}{1000})^{10^5-k}$. Since n is very large and p is small, X can be approximated by a Poisson with $\lambda = n \cdot p = 100$. Then $P(K = k) = \frac{100^k}{k!}\exp^{-100}$.

12. If the probability is 0.20 that a burglar will get caught on any given job, what is the probability that he will get caught no later than on his fourth job?

Solution: Let X the number of jobs up to and including the first success. Then $X \sim Geometric(p = 0.20)$. Then $P(Y \leq 4) = P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) = 0.8^0 \cdot 0.2 + 0.8^1 \cdot 0.2 + 0.8^2 \cdot 0.2 + 0.8^3 \cdot 0.2 = 0.5904$.