Exercises Lecture 5: Continuous random variables

1. Let us consider the following cdf for a random variable X:

$$F(X) = P(X \leq x) = \begin{cases} 0, \text{ if } x < 0 \\ 1 - \exp\{-x^2/2\}, \text{ if } x > 0. \end{cases}$$

   Which are the values of the following probabilities?

   (a) $P(X \leq 5/3)$,

   (b) $P(X > 3/2)$ and

   (c) $P(4/3 < X \leq 5/3)$.

   Solution:

   a. $P(X \leq 5/3) = P(X \leq 5/3) = 1 - \exp\{-(5/3)^2/2\} = 0.7506478$.

   b. $P(X > 3/2) = 1 - P(X \leq 3/2) = 1 - (1 - \exp\{-(3/2)^2/2\}) = 0.3246525$.

   c. $P(4/3 < X \leq 5/3) = F(5/3) - F(4/3) = (1 - \exp\{-(5/3)^2/2\}) - (1 - \exp\{-(4/3)^2/2\}) = 0.1617601$.

2. Let X be the normal random variable with $\mu = 5$ and $\sigma = 10$. Compute the following probabilities using the table of the normal distribution. You may check your answers with R.

   (a) $P(X > 10)$.

   (b) $P(-20 < X < 15)$.

   (c) What is the value of x such that $P(X > x) = 0.05$?

   Solution:

   (a) $P(X > 10) = 1 - P(X \leq 10) = 1 - F(X = 10) = 1 - F(Z = [10 - 5]/10] = 1 - F(Z = 0.5) = 0.3085$

   In R : 1-pnorm(0.5) or 1-pnorm(10,5,10).

   (b) $P(-20 < X < 15) = F(x = 15) - F(x = -20) = \ldots = 0.8351351$.

   In R: pnorm(15,5,10)-pnorm(-20,5,10).

   (c) $P(X > x) = 0.05 \Rightarrow 1 - P(X < x) = .05 \Rightarrow P(X < x) = .95 \Rightarrow x = 21.44854$.

   In R: qnorm(.95,5,10).

3. If Z follows the standard normal distribution, find

   (a) $P(Z > 2.64)$

   (b) $P(0 \leq Z < 0.87)$

   using the tables of the standard normal distribution. You may check your answers with R.

   Solution:

   (a) pnorm(2.64, lower.tail = FALSE)

   (b) pnorm(0.87) - 1/2

4. Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = 0.1$. In slides of Lecture 5, you may find the formula of its pdf and cdf.

   (a) What is the probability that the lifetime is less than 10?

   (b) What is the probability that the lifetime is between 5 and 15.

   (c) What is the value of t such that the probability that the lifetime is greater than t is 0.01?

   Solution:

   (a) For X we know $F_X(x) = 1 - \exp\{-\lambda x\}, x \geq 0$. Thus, $F_X(10) = 0.63$.

   (b) $P(5 \leq X \leq 15) = F_X(15) - F_X(5) = 0.3834$.

   (c) $P(X \geq t) = 1 - F_X(t) = 0.01$, $t = 10 \cdot ln(100)$.

5. Suppose that in a certain population, the individuals' heights are approximately normally distributed with parameters $\mu = 70$ and $\sigma = 3$inches.

   (a) What proportion of the population is over 6 ft. tall? *[Hint: 1foot = 12inches].*

   (b) What is the distribution of heights if they are expressed in centimeters? In meters? *[Hint: 1inch = 2.54cm].*

   Solution:

   (a) Let X the height in inches.

$$\begin{aligned} P(X > 72) &= P(\frac{X - 70}{3} > \frac{72 - 70}{3}) \\ &= P(Y > 0.67) = 1 - \Phi(0.67) = 1 - 0.7486 = 0.2514. \end{aligned}$$

(b) Let Y is the height in cm. Then $Y = \alpha X$. Since $X \sim N(\mu, \sigma^2)$ then $Y \sim N(\alpha\mu, \alpha^2\sigma^2)$.

6. Suppose that X is a χ^2 with 20 degrees of freedom. Answer the following questions using the table of the chi-squared distribution. You may check your answers with R.

   (a) What is the value of x_0 such that $P(X > x_0) = 0.95$?

   (b) What is the value of $P(X \leq 12.443)$?

   Solution:

   Using the tables we have: $P(X > 10.851) = 0.95$ and $P(X > 12.443) = 0.10$