Subspaces, independence, dimension and rank

4433LALG3: Linear Algebra

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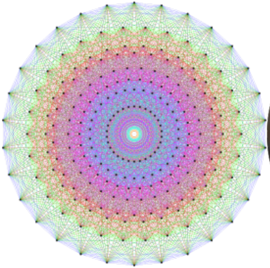

Overview

- Recap: Linear combinations and span
- Subspaces, linear independence and dimension
- Rank and nullity
- Supplementary material

References:

- Nicholson §5.1–5.2 & §5.4 (read also §1.3)
- 3Blue1Brown Ch.2 & Ch.7

Section 1

Recap: Linear combinations and span

Recap: Linear combinations and span

In this section we will quickly review some concepts that you should already know from **3Blue1Brown Ch.2**.

Definition

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v_n}$, a **linear combination** of them is any expression of the form

$$a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n,$$

where $a_1, \ldots, a_n$ are *scalars* (i.e. real numbers).

Definition

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v_n}$, the **span** of them is the set of all linear combinations of them:

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v_n}\} = \{a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \mid a_i \text{ in } \mathbb{R}\}.$$

Section 2

Subspaces, linear independence and dimension

Subspaces of $\mathbb{R}^n$

Motivation:

- If $\mathbf{x} \neq \mathbf{0}$, then $\mathrm{span}(\mathbf{x})$ is a line through the origin: we call it a *linear subspace of dimension one*.
- If $\mathbf{x}, \mathbf{y}$ are not a multiple of each other, then $\mathrm{span}\{\mathbf{x}, \mathbf{y}\}$ is a plane through the origin: we call it a *linear subspace of dimension two*.

Definition

A set U of vectors in $\mathbb{R}^n$ is called a (linear) **subspace** of $\mathbb{R}^n$ if :

- The zero vector $\mathbf{0}$ is in U,
- If $\mathbf{x} \in U$ and $\mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y} \in U$,
- If $\mathbf{x} \in U$, then $a\mathbf{x} \in U$ for any real number a.

Subspaces of $\mathbb{R}^n$

The most general example of a subspace is:

- If S is any (non-empty) collection of vectors in $\mathbb{R}^n$, then $\operatorname{span} S$ is a subspace of $\mathbb{R}^n$.

Subspaces associated to a linear transformation

Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Definition

The **kernel** of T is the linear subspace of $\mathbb{R}^n$:

$$\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}.$$

The **image** of T is the linear subspace of $\mathbb{R}^m$:

$$\operatorname{im} T = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x}\}.$$

Subspaces associated to a matrix

Suppose $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n]$ is a $m \times n$ matrix.

Definition

The **null space** of A is the linear subspace of $\mathbb{R}^n$:

$$\operatorname{null} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

The **column space** of A is the linear subspace of $\mathbb{R}^m$:

$$\operatorname{col} A = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}.$$

The standard basis

We usually work with the **standard basis** of $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Two obvious, but important, facts:

1. $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} = \mathbb{R}^n$
2. You actually need *all* these vectors to span $\mathbb{R}^n$:
   Any strictly smaller subset, $S \subset \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, will not span $\mathbb{R}^n$

Redundancy

Consider two vectors $\mathbf{x}, \mathbf{y}$ that are not one the multiple of the other.

Notice how $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\} = \operatorname{span}\{\mathbf{x}, \mathbf{y}\}$.

Adding $\mathbf{x}+\mathbf{y}$ to the set $S = \{\mathbf{x}, \mathbf{y}\}$ does **not** increase its span, because $\mathbf{x}+\mathbf{y}$ was already in $\operatorname{span} S$.

Adding $\mathbf{x}+\mathbf{y}$ to the spanning set is *redundant*.

The technical concept for redundancy is **linear dependence**:

The set $\{\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\}$ is *linearly dependent* because at least one of its elements is redundant.

Independence

Definition

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is called **linearly independent** when the following condition holds:

$$\text{If} \;\; a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_k\mathbf{x}_k = \mathbf{0}, \;\; \text{then} \;\; a_1 = a_2 = \ldots = a_k = 0.$$

Linear independence means:

- The **only** way to combine the $\mathbf{x}_i$ and get $\mathbf{0}$ is by making all coefficients $a_i = 0$.
- It is **impossible** to express a vector $\mathbf{x}_j$ as a linear combination of the other vectors.

Bases and dimension

Definition

Let U be a linear subspace of $\mathbb{R}^n$.

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is called a **basis** for U if:

- $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = U$,
- The set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is linearly independent.

In that case, we say that the **dimension** of U is equal to k.

Section 3

Rank and nullity

Rank of a linear transformation or a matrix

Definition

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The **rank** of T is the *dimension* of $\operatorname{im} T$.

Definition

Let A be a $m \times n$ matrix.

The **rank** of A is the *dimension* of $\operatorname{col} A$.

Recall:

- $\operatorname{im} T = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x}\}$
- $\operatorname{col} A = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$

Nullity of a linear transformation or a matrix

Definition

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The **nullity** of T is the *dimension* of $\ker T$.

Definition

Let A be a $m \times n$ matrix.

The **nullity** of A is the *dimension* of $\operatorname{null} A$.

Recall:

- $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$
- $\operatorname{null} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$

Some comments about the rank

Remark

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- $\operatorname{rank} T \leq m$, because $\operatorname{im} T \subseteq \mathbb{R}^m$,
- $\operatorname{rank} T \leq n$, because a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ cannot increase the dimension.

If $\operatorname{rank} T < n$, it means that T has *squished* the dimension of $\mathbb{R}^n$.

But that means that some vectors $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$ are mapped to $\mathbf{0} \in \mathbb{R}^m$.

In fact: The larger $\ker T$ the smaller $\operatorname{im} T$

The larger the nullity the smaller the rank

The rank-nullity theorem

The following theorem is fundamental.

Theorem

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

$$\dim(\ker T) + \dim(\operatorname{im} T) = n.$$

Some equivalent formulations (A is the matrix associated to T):

- $\operatorname{nullity} T + \operatorname{rank} T = n$
- $\dim(\operatorname{null} A) + \dim(\operatorname{col} A) = n$
- $\operatorname{nullity} A + \operatorname{rank} A = n$

Rank and invertibility

The rank-nullity theorem has an immediate important consequence:

Suppose A is an $n \times n$ matrix whose rank equals n.

The rank-nullity theorem says:

$$\begin{aligned} \operatorname{nullity} A + \operatorname{rank} A &= n \\ \operatorname{nullity} A + n &= n \\ \dim(\operatorname{null} A) &= 0 \end{aligned}$$

That means $\operatorname{null} A = \{\mathbf{0}\}$.

Corollary

If an $n \times n$ matrix has rank n, then it is invertible.

Section 4

Supplementary material

Independence test

Independence test: version 1

To test whether a set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is independent we proceed as follows:

- Set an equation: $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \ldots + t_k\mathbf{x}_k = \mathbf{0}$, for $t_1, \ldots, t_k$ unknown.
- Solve the equation and analyze the set of solutions:
  - If $t_1 = t_2 = \ldots = t_k = 0$ is the **only** solution, then the set is independent.
  - Else the set is not independent.

Independence test: example

Example 1:
Determine whether $S = \{(1,0,-2,5),(2,1,0,-1),(1,1,2,1)\}$ is independent.

Solution:
Set $t_1(1,0,-2,5)+t_2(2,1,0,-1)+t_3(1,1,2,1)=(0,0,0,0)$. This can be rewritten as:

$$\begin{aligned} t_1+2t_2+t_3&=0,\\ t_2+t_3&=0,\\ -2t_1+2t_3&=0,\\ 5t_1-t_2+t_3&=0. \end{aligned}$$

A short computation shows $t_1=t_2=t_3=t_4=0$ is the only possible solution.

We conclude that the set is indeed independent.

Independence test: matrix version

Notice that we can write the system of equations on the previous slide in matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & 2 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice how the columns of these matrix are the vectors in the set S (duh!)

Independence test

Independence test: version 2

To test whether a set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is independent we proceed as follows:

- Form the $n \times k$ matrix $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_k \end{bmatrix}$
- Find the solutions to $X\mathbf{t} = \mathbf{0}$ (e.g. compute the null space of X).
  - If $\operatorname{null} X = \{\mathbf{0}\}$, the set is independent.
  - Else the set is not independent.

Independence test: example

Example 1:
Determine whether $S = \{(1, 0, -2, 5), (2, 1, 0, -1), (1, 1, 2, 1)\}$ is independent.

Solution:

```
> library(matlib)

> X <- matrix(c(1,0,-2,5, 2,1,0,-1, 1,1,2,1), ncol=3); X
     [,1] [,2] [,3]
[1,]    1    2    1
[2,]    0    1    1
[3,]   -2    0    2
[4,]    5   -1    1

> gaussianElimination(X)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
[4,]    0    0    0
```

We conclude that the set is indeed independent.

Column space and row space

We have defined the *column space* of a matrix $m \times n$ matrix A: it is the subspace of $\mathbb{R}^m$ spanned by its columns.

Similarly, we have:

Definition

The **row space** of A, $\mathrm{row}\, A$, is the linear subspace of $\mathbb{R}^n$ spanned by the rows of A.

Sometimes, we talk about **row rank** and **column rank**. These are the dimensions of $\mathrm{row}\, A$ and of $\mathrm{col}\, A$, respectively. However, as the next theorem claims, they are numerically the same.

Theorem

Let A denote any $m \times n$ matrix. Then, $\dim(\mathrm{col}\, A) = \dim(\mathrm{row}\, A)$.

Therefore, we usually just talk about the rank of A.

Computation of rank

The general algorithm for computing rank uses Gaussian elimination.

Theorem

Let A denote any $m \times n$ matrix.

Suppose R is the reduced row-echelon form of A. Then $\operatorname{rank} A$ equals to the number of leading 1s in R.

Computation of the column space

Computation of $\operatorname{col} A$ uses Gaussian elimination (yet again).

By "computation of $\operatorname{col} A$" we mean: finding a basis for the subspace $\operatorname{col} A$.

Theorem

Let A denote any $m \times n$ matrix.

Suppose R is the reduced row-echelon form of A.

If the leading 1s lie in columns $j_1, j_2, \ldots, j_r$ of R, then the columns $j_1, j_2, \ldots, j_r$ <u>of A</u> form a basis for $\operatorname{col} A$.

Rank example

Example 2:

Compute the rank of $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, and find a basis for $\operatorname{col} A$.

Solution:

```
> library(matlib)
> A <- matrix(c(1,3,1, 2,6,2, 2,5,1, -1,0,2), ncol=4)
> gaussianElimination(A)
     [,1] [,2] [,3] [,4]
[1,]    1    2    0    5
[2,]    0    0    1   -3
[3,]    0    0    0    0
```

There are two leading 1s: in column one and column three.

We conclude that $\operatorname{rank} A = 2$, and that $\operatorname{col} A$ is spanned by the first and third columns of A.

Rank example

Notice that we can get extra information from the row-echelon form.

The solutions to $A\mathbf{x} = \mathbf{0}$ are the solutions to the system: $\begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ x_3 - 3x_4 = 0 \end{cases}$

The free variables are x_2, x_4. We set parameters $x_2 = s$, $x_4 = t$.

All solutions are of the form: $\begin{bmatrix} -2s - 5t \\ s \\ 3t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, for arbitrary s, t.

We use the *basic solutions* to understand the redundancy between the columns:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad -2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{a}_2 = 2\mathbf{a}_1$$

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad -5\mathbf{a}_1 + 3\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0} \quad \Rightarrow \quad \mathbf{a}_4 = 5\mathbf{a}_1 - 3\mathbf{a}_3$$

Rank example

Thus, the final conclusion is:

- $\operatorname{col} A = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_3\}$
- Columns $\mathbf{a}_2$ and $\mathbf{a}_4$ are not in the basis because they are redundant:
  - $\mathbf{a}_2 = 2\mathbf{a}_1$
  - $\mathbf{a}_4 = 5\mathbf{a}_1 - 3\mathbf{a}_3$

Rank example and the null space

Incidentally, we have also computed a basis for the null space:

$$\operatorname{null} A = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

If A has rank r, then when we consider the system $A\mathbf{x} = \mathbf{0}$ and apply Gaussian elimination we get:

- r basic variables (one for each leading 1),
- $n - r$ free variables, which give $n - r$ basic solutions.

Theorem

Let A denote an $m \times n$ matrix of rank r. A basis for $\operatorname{null} A$ can be formed by the $n - r$ basic solutions created by the Gaussian elimination algorithm.

You can read more on Nicholson §5.4 (see Theorem 5.4.2).
The concept of a *basic solution* is discussed in Nicholson §1.3.