Exercise 1. $S=\{V=mat(n,n)=[v_{ij}], i=1,\cdots,n, j=1,\cdots,n \mid v_{ij}=0 \text{ for all } i>j\}$, $A,B\in S$

$\because A+B=[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$

$a_{ij}=0$ if $i>j$

$b_{ij}=0$ if $i>j$

$\therefore a_{ij}+b_{ij}=0$ if $i>j$

$\therefore A+B\in S$ ①

$\because \alpha A=[\alpha a_{ij}]$

$a_{ij}=0$ if $i>j$

$\therefore \alpha a_{ij}=0$ if $i>j$

$\therefore \alpha A\in S$ ②

Based on ① and ②, S is a linear subspace.

2. $S=\{V=mat(n,n) \mid |V|=0\}$, suppose $A,B\in S$:

Base on $|A|=0, |B|=0$, we can't compute $|A+B|$, thus it is possible that $|A+B|\neq 0$ (i.e. $A+B\notin S$). Thus, S is not a linear subspace.

3. $S=\{X \mid Cov(X,Y)=0\}$, suppose $X_1\in S, X_2\in S$:

$Cov(X_1+X_2,Y)=cov(X_1,Y)+cov(X_2,Y)=0 \Rightarrow X_1+X_2\in S$

$cov(\alpha X_1,Y)=\alpha cov(X_1,Y)=0 \Rightarrow \alpha X_1\in S$

$\therefore$ S is a linear subspace

4. $S=\{f \mid f(0)=0\}$, $f_1, f_2\in S$:

$f_1(0)+f_2(0)=0 \Rightarrow f_1+f_2\in S$

$\alpha f_1(0)=\alpha\cdot 0=0 \Rightarrow \alpha f_1\in S$

$\therefore$ S is a linear subspace

5. $S=\{f \mid \int_0^1 f(x)dx=1\}$, $f_1, f_2\in S$:

$\int_0^1 f_1(x)+f_2(x)dx=\int_0^1 f_1(x)dx+\int_0^1 f_2(x)dx=2\neq 1 \Rightarrow f_1+f_2\notin S$

$\therefore$ S is not a linear subspace.