Part I: Multiple-choice questions

1. Answer: **(f)**

2. Answer: **(c)**

3. Answer: **(b)**

4. Answer: **(c)**

5. Answer: **(d)**

6. Answer: **(f)**

Part II: Short-answer questions

7. **a.** 3×3

b. Matrices cannot be added

c. Vector cannot be multiplied by these matrices

d. 1×1 (or scalar)

8. **a.** Rank is 3 because `total = body + tail`

b. $\hat{\boldsymbol{\beta}} = (X^\top X)^{-1} X^\top \mathbf{y}_{\text{ttx}}$

c. $P = X(X^\top X)^{-1} X^\top$

9. **a.** A is 4×5 and $A^\top A$ is 5×5

b. $\text{rank}(A) = 3$

c. $\{25, 16, 1, 0\}$ (0 has multiplicity 2)

d. U, V are orthogonal matrices

10. **a.** $\det(A) = 2$

b. $\text{rank}(A) = 3$

c. $\det(X) = -10$

Part III: Long-answer questions

11. The equation is solved following these steps:

$$\begin{aligned} AX^\top B + C &= I_n \\ AX^\top B &= I_n - C \\ X^\top &= A^{-1}(I_n - C)B^{-1} \\ X &= \left(A^{-1}(I_n - C)B^{-1}\right)^\top \end{aligned}$$

12. **a.** From the output we deduce that:

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 1 \\ x_3 + 3x_4 &= 2 \\ x_5 &= -1 \\ x_6 &= 0 \end{aligned}$$

The free variables are x_2 and x_4. We choose parameters $s = x_2$ and $t = x_4$, and write all variables in terms of these.

$$\begin{aligned} x_1 &= 1 + 2s - t \\ x_2 &= s \\ x_3 &= 2 - 3t \\ x_4 &= t \\ x_5 &= -1 \\ x_6 &= 0 \end{aligned}$$

The easiest choice (but others are also valid) is $s = t = 0$, which gives $\mathbf{x} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \end{bmatrix}^\top$. If you want, you can verify your answer by multiplying $A\mathbf{x}$ and checking you do get $\mathbf{b}$.

b. There are four leading-ones in the output, so $\operatorname{rank}(A) = 4$. The nullity is $6 - 4 = 2$.

c. A basis is given by those columns of A with a leading-one in the echelon form. Thus, columns number 1, 3, 5 and 6 form a basis.

d. In the output of `gaussianElimination(A, b)`, the first six columns are the reduced row-echelon form of A. From it, we solve the homogeneous system $A\mathbf{x} = \mathbf{0}$. A similar approach to the one used in part **a.**, gives:

$$\begin{aligned} x_1 &= 2s - t \\ x_2 &= s \\ x_3 &= 3t \\ x_4 &= t \\ x_5 &= 0 \\ x_6 &= 0 \end{aligned}$$

The first basic solution ($s = 1$, $t = 0$) is: $\mathbf{n}_1 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$.
The second basic solution ($s = 0$, $t = 1$) is: $\mathbf{n}_2 = \begin{bmatrix} -1 & 0 & -3 & 1 & 0 & 0 \end{bmatrix}^\top$.
You can easily check you work by verifying that $A\mathbf{n}_1 = A\mathbf{n}_2 = \mathbf{0}$.
A basis for $\operatorname{null}(A)$ is thus $\{\mathbf{n}_1, \mathbf{n}_2\}$.

13. **a.** Computing $M\mathbf{u}_1$ gives:

$$\begin{bmatrix} 9 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

This means that $\mathbf{u}_1$ is indeed an eigenvector with eigenvalue $\lambda = 10$.

b. Solving the system $(M - \lambda I_3)\mathbf{x} = \mathbf{0}$ for $\lambda = 9$ and $\lambda = 5$ gives the following (or a multiple of the following):

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{bmatrix} 1/3 \\ -4/3 \\ 1 \end{bmatrix}.$$

You can check your answer by computing $M\mathbf{u}_2$ and verifying you get $9\mathbf{u}_2$. Similar for $\mathbf{u}_3$.

c. $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$.

d. FALSE. Only a **symmetric** matrix could be diagonalized by an orthogonal matrix.

14. **a.** $\|\mathbf{a}\| = \sqrt{6}$.

b. $\mathbf{a}^\top \mathbf{b} = 3 \neq 0$, therefore they are not orthogonal.

c. $\text{proj}_{\mathbf{a}}(\mathbf{b}) = \dfrac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{3}{6} \begin{bmatrix} -1 & 1 & 2 & 0 \end{bmatrix}^\top$.

d. We see leading-ones only in columns number 1 and 2. Therefore a basis is $\{\mathbf{a}, \mathbf{b}\}$.

e. Let $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}$. The projection is $\mathbf{p} = A\boldsymbol{\alpha}$, for the vector $\boldsymbol{\alpha}$ which satisfies the normal equation: $A^\top A\boldsymbol{\alpha} = A^\top \mathbf{y}$.

$$A^\top A = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}, \quad A^\top \mathbf{y} = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Therefore, we need to solve the 2×2 system:

$$\begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

This can be done either by Gaussian elimination, or by computing $(A^\top A)^{-1}$. The (unique) solutions is $\alpha_1 = -4/3$, $\alpha_2 = 7/3$.

Finally, we get $\mathbf{p} = -\dfrac{4}{3}\mathbf{a} + \dfrac{7}{3}\mathbf{b} = \dfrac{1}{3} \begin{bmatrix} 4 \\ 3 \\ -1 \\ 7 \end{bmatrix}$.

15. **a.** A (symmetric) matrix is positive definite if all its eigenvalues are positive. Equivalently, Σ satisfies: $\mathbf{x}^\top \Sigma\, \mathbf{x} > 0$ for alll $\mathbf{x} \neq \mathbf{0}$.

b. It is enough to compute ΣQ and check what we get.

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0.6 & -0.4 & 0 \\ -0.4 & 0.6 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}.$$

This means $\Sigma Q = I_3$, and so $Q = \Sigma^{-1}$.

c. Each eigenvalue λ of Σ becomes $\lambda^{-\frac{1}{2}}$. Thus, $\widetilde{D} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

d. Because the columns are orthonormal, we can use the simplest version of the expansion theorem:

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}^\top \mathbf{p}_1)\, \mathbf{p}_1 + (\mathbf{x}^\top \mathbf{p}_2)\, \mathbf{p}_2 + (\mathbf{x}^\top \mathbf{p}_3)\, \mathbf{p}_3 \\ &= (0.7071)\mathbf{p}_1 + (8)\mathbf{p}_2 + (2.1213)\mathbf{p}_3. \end{aligned}$$