Exercises from the PDF

Exercise 2.1

We have an exponential distribution with an hazard rate $h(x) = 1/3$. Hence the survival function (as reported on the slides) is:

$$S(x) = \exp\Big(-\int_0^x \frac{1}{3}du\Big) = \exp\Big(-\frac{1}{3}x\Big) = e^{-1/3x}$$

As for the mean survival time, we have that, by definition of mean survival time:

$$\mu = \int_0^\infty x \cdot f(x)dx$$

Hence, since $f(x) = -\frac{d(S(x))}{dx} = \lambda \exp(\lambda x)$ then:

$$\mu = \int_0^\infty x \cdot f(x)dx = \int_0^\infty \frac{1}{3}x \cdot e^{-1/3x}dx =$$
$$\Big[-e^{-1/3x}\Big]_0^\infty + \int_0^\infty \frac{1}{3} \cdot 3e^{-1/3x}dx =$$
$$0 + \Big[-3e^{-1/3x}\Big]_0^\infty = +3 \quad \Big(=\frac{1}{\lambda}\Big)$$

Another way to find the mean survival time is to use the following relation that can be found on the slides:

$$mrl(x) = \frac{1}{\lambda}$$

Hence

$$\mu = mrl(0) = \frac{1}{\lambda}$$

In general, for an exponential distribution:

$$\mu = \frac{1}{\lambda}$$

Exercise 2.1

We have a Weibull distribution with $\gamma = 1.1$ and $\lambda = 0.05$. Hence the survival function (as reported on the slides) is:

$$S(x) = \exp(-\lambda x^\gamma) = \exp(-0.05x^{1.1})$$

and the hazard function is:

$$h(x) = \lambda\gamma x^{\gamma-1} = 0.05 \cdot 1.1x^{0.1}.$$

It follows that hazard and survival functions at 1 day and 1 week are:

$$S(1) = \exp(-0.05)$$
$$S(7) = \exp(-0.05 \cdot 7^{1.1})$$
$$h(1) = 0.05 \cdot 1.1$$
$$h(7) = 0.05 \cdot 1.1 \cdot 7^{0.1}$$

Exercises from Klein and Moeschberger

Exercise 2.1

We have an exponential distribution with an hazard rate $h(x) = 1/1000$.

(a) $mu = 1/\lambda = 1000$ hours

(b) We need to solve $S(x_{0.5}) = 0.5$. This corresponds to $e^{\lambda x_{0.5}} = 0.5$.
Taking the logarithm on both sides we get $-\lambda x_{0.5} = \ln(1/2)$.
So $x_{0.5} = \ln(2)/\lambda = \ln(2)1000$ hours

(c) $S(2000) = \exp(-2000/1000) = e^{-2}$

Exercise 2.2

We have a Weibull distribution with $\alpha = 2$ and $\lambda = 1/1000$.

(a) $S(x) = \exp(-\lambda x^{\alpha}) = \exp(-1/1000x^2)$. So:
$S(30) = \exp(-1/1000 \cdot 30^2)$
$S(45) = \exp(-1/1000 \cdot 45^2)$
$S(60) = \exp(-1/1000 \cdot 60^2)$

(b) From the slides we know that the mean of a Weibull distribution is

$$\frac{\Gamma(1 + 1/\alpha)}{\lambda^{1/\alpha}} = \frac{\Gamma(3/2)}{1/1000^{1/2}} = \frac{1}{2}\pi 10\sqrt{10}$$

(c) $h(x) = \lambda \alpha x^{\alpha-1} = 1/500x$. So:
$h(30) = 30/500$
$h(45) = 45/500$
$h(60) = 60/500$

(d) We need to solve $S(x_{0.5}) = 0.5$. This corresponds to $e^{\lambda x_{0.5}^{\alpha}} = 0.5$.
Taking the logarithm on both sides we get $-\lambda x_{0.5}^{\alpha} = \ln(1/2)$.
So $x_{0.5} = \sqrt[\alpha]{\ln(2)/\lambda} = \sqrt{\ln(2)1000}$

Exercise 2.6

We have a Gompertz distribution with $\theta = 0.01 = 1/100$ and $\alpha = 0.25 = 1/4$.

(a) $S(x) = \exp\left(\frac{\theta}{\alpha}(1 - e^{ax})\right) = \exp\left(\frac{4}{100}(1 - e^{1/4x})\right)$. So:
$S(12) = \exp\left(\frac{1}{25}(1 - e^{12/4})\right) = \exp\left(\frac{1}{25}(1 - e^{3})\right)$

(b) $1 - S(6) = 1 - \exp\left(\frac{1}{25}(1 - e^{6/4})\right) = 1 - \exp\left(\frac{1}{25}(1 - e^{3/2})\right)$

(c) We need to solve $S(x_{0.5}) = 0.5$. This corresponds to $\exp\left(\frac{\theta}{\alpha}(1 - e^{ax_{0.5}})\right) = 0.5$.
Taking the logarithm on both sides we get $\frac{\theta}{\alpha}(1 - e^{ax_{0.5}}) = \ln(1/2)$.
It follows that $e^{ax_{0.5}} = \frac{\alpha}{\theta}\ln(2)$. Taking again the logarithm: $x_0.5 = \frac{1}{a}\ln\left(\frac{\alpha}{\theta}\ln(2)\right)$

Exercise 2.7

We have a Gamma distribution with $\beta = 3$ and $\lambda = 0.2$.

(a) $S(x) = 1 - I(\lambda x, \beta) = 1 - I(0.2x, 3)$. So:
$S(18) = 1 - I(18/5, 3)$

(b) $1 - S(12) = 1 - I(12/5, 3)$

(c) $\mu = \beta/\lambda = 3/0.2 = 3 \cdot 5 = 15$ months

Exercise 2.9

Time to relapse is given by $Y = \ln(X) = 2 + 0.5Z + 2W$. So here X is a continuous outcome (time to relapse) and it depends on the two variables Z and W.

(a) For Treatment A we have $\ln(X) = 2.5 + 2W$. Hence the time to relapse depends on the value on W and, vice versa, given a time to relapse we can find a value of W and a survival probability.
At one year we have:

$$\begin{aligned} S(12) =& P(X > 12) = P(Y > \ln(12)) = P(2.5 + 2W > \ln(12)) = \\ & P\Big(W > \frac{\ln(12) - 2.5}{2}\Big) = 1 - \Phi\Big(\frac{\ln(12) - 2.5}{2}\Big) \end{aligned}$$

At two years we have:

$$\begin{aligned} S(24) =& P(X > 24) = P(Y > \ln(24)) = P(2.5 + 2W > \ln(24)) = \\ & P\Big(W > \frac{\ln(24) - 2.5}{2}\Big) = 1 - \Phi\Big(\frac{\ln(24) - 2.5}{2}\Big) \end{aligned}$$

At five years we have:

$$\begin{aligned} S(24) =& P(X > 60) = P(Y > \ln(60)) = P(2.5 + 2W > \ln(60)) = \\ & P\Big(W > \frac{\ln(60) - 2.5}{2}\Big) = 1 - \Phi\Big(\frac{\ln(60) - 2.5}{2}\Big) \end{aligned}$$

For treatment B you just need to repeat the same computations with $Z = 0$.

(b) Same as point (a) but this time $P\Big(W > \frac{\ln(24)-2.5}{2}\Big)$ will mean finding the p-value of a logistic distribution instead of a normal distribution.

Exercise 2.11

This is a distribution where at the beginning the hazard rate is 0, and then it follows the hazard rate of a Weibull distribution.

(a)

$$h(x) = \begin{cases} 0 & \text{if } x < \phi \\ \alpha\lambda(x - \phi)^{\alpha-1} & \text{if } x \geq \phi \end{cases}$$

(b) We can just take the mean and median survival time of a regular Weibull distribution and add ϕ. So:

$$\mu = \frac{\Gamma(1+1/\alpha)}{\lambda^{1/\alpha}} + \phi$$

$$x_{0.5} = \sqrt[\alpha]{\ln(2)/\lambda} + \phi$$

Exercise 2.12

(a) If

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

then

$$F(x) = \begin{cases} \frac{1}{\theta}x & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

so

$$S(x) = \begin{cases} 1 - \frac{1}{\theta}x & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

(b) From the previous point it follows that:

$$h(x) = \begin{cases} \frac{1}{\theta} \cdot \frac{1}{(1-x/\theta)} = \frac{1}{\theta - x} & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

(c) For $0 \leq x \leq \theta$

$$\begin{aligned} mrl(x) = &\frac{\int_x^\infty S(t)dt}{S(x)} = \frac{\int_x^\theta S(t)dt}{S(x)} = \\ &\frac{\left[x - \frac{1}{2\theta}x^2\right]_0^\theta}{1 - x/\theta} = \frac{x - \frac{1}{2\theta}x^2 + \frac{1}{2}\theta}{1 - x/\theta} \end{aligned}$$

For $x > \theta$, $mrl(x) = 0$.

Exercise 2.17

We know that $mrl(x) = x + 10$.

(a) $\mu = mrl(0) = 10)$

(b) Using the relation at page 35 of the book:

$$h(x) = \frac{\frac{d}{dx}mrl(x) + 1}{mrl(x)} = \frac{1+1}{x+10} = \frac{2}{x+10}$$

(c) Using the relation at page 35 of the book:

$$S(x) = \frac{mrl(0)}{mrl(x)} \exp\Big(-\int_0^x \frac{du}{mrl(u)}\Big) = \frac{10}{x+10} \exp\Big(-\int_0^x \frac{du}{u+10}\Big) =$$
$$\frac{10}{x+10} \exp\Big(-\Big[\ln(x+10)\Big]_0^x\Big) = \frac{10}{x+10} \exp\Big(-\ln\Big(\frac{x+10}{10}\Big)\Big) = \Big(\frac{10}{x+10}\Big)^2$$

Exercise 2.18

This is the same distribution as Ex 2.12 for $\theta = 100$. So

(a)

$$S(x) = \begin{cases} 1 - \frac{1}{100}x & \text{for } 0 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

So $S(25) = 1 - 25/100 = 3/4$; $S(50) = 1 - 50/100 = 1/2$; $S(75) = 1 - 75/100 = 3/4$.

(b)

$$mrl(25) = \frac{25 - \frac{1}{200}25^2 + 50}{1 - 1/4}$$
$$mrl(50) = \frac{50 - \frac{1}{200}50^2 + 50}{1 - 1/2}$$
$$mrl(75) = \frac{75 - \frac{1}{200}75^2 + 50}{1 - 3/4}$$

(c) Let's start by the median residual lifetime at 25 days.
We need to solve $S(x_{0.5})/S(25) = 0.5$. This corresponds to $\frac{1 - x_{0.5}/100}{3/4} = 0.5$.
Hence we get $x_{0.5} = 100(1 - \frac{3}{8})$.
The other two cases are analogous.