① a. $\begin{pmatrix}0&1&0\\1&0&0\\1&0&3\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1\\2\\-1\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$xI-A$

$Ax=\lambda x$

$(\lambda I-A)\vec{x}=\vec{0}$

b. $\begin{pmatrix}1&1&0\\-1&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$

3.3.1 a. $|\lambda I-A|=\begin{vmatrix}\lambda-1&-2\\-3&\lambda-2\end{vmatrix}=(\lambda-4)(\lambda+1)=0\Rightarrow\lambda_1=4,\lambda_2=-1$

Solve $(\lambda_1 I-A)\vec{x}=\vec{0}$, $\left(\begin{array}{cc|c}3&-2&0\\-3&2&0\end{array}\right)\xrightarrow{\frac{1}{3}R_1}\left(\begin{array}{cc|c}1&-2/3&0\\-3&2&0\end{array}\right)\xrightarrow{R_2+3R_1}\left(\begin{array}{cc|c}1&-2/3&0\\0&0&0\end{array}\right)\rightarrow$

$x_1-\frac{2}{3}x_2=0\Rightarrow\vec{x}=t\begin{pmatrix}2\\3\end{pmatrix}\Rightarrow\vec{x}_1=\begin{pmatrix}2\\3\end{pmatrix}$

Solve $(\lambda_2 I-A)\vec{x}=\vec{0}$, $\left(\begin{array}{cc|c}-2&-2&0\\-3&-3&0\end{array}\right)\xrightarrow{-\frac{1}{2}R_1}\left(\begin{array}{cc|c}1&1&0\\-3&-3&0\end{array}\right)\xrightarrow{R_2+3R_1}\left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)\rightarrow$

$x_1+x_2=0\Rightarrow\vec{x}=t\begin{pmatrix}-1\\1\end{pmatrix}\Rightarrow\vec{x}_2=\begin{pmatrix}-1\\1\end{pmatrix}$

b. $|\lambda I-A|=\begin{vmatrix}\lambda-2&4\\1&\lambda+1\end{vmatrix}=(\lambda+2)(\lambda-3)=0\Rightarrow\lambda_1=-2,\lambda_2=3$ ✓

Solve $(\lambda_1 I-A)\vec{x}=\vec{0}$, $\left(\begin{array}{cc|c}-4&4&0\\1&-1&0\end{array}\right)\xrightarrow{-\frac{1}{4}R_1}\left(\begin{array}{cc|c}1&-1&0\\1&-1&0\end{array}\right)\xrightarrow{R_2-R_1}\left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right)\rightarrow$

$x_1-x_2=0\Rightarrow\vec{x}=t\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow\vec{x}_1=\begin{pmatrix}1\\1\end{pmatrix}$ ✓

Solve $(\lambda_2 I-A)\vec{x}=\vec{0}$, $\left(\begin{array}{cc|c}1&4&0\\1&4&0\end{array}\right)\xrightarrow{R_2-R_1}\left(\begin{array}{cc|c}1&4&0\\0&0&0\end{array}\right)\rightarrow x_1+4x_2=0\Rightarrow\vec{x}=t\begin{pmatrix}-4\\1\end{pmatrix}\Rightarrow\vec{x}_2=\begin{pmatrix}-4\\1\end{pmatrix}$ ✓

c. $\lambda_1=5,\lambda_2=3,\lambda_3=2$

$\begin{cases}x_1=0\\x_3=0\end{cases}\Rightarrow\vec{x}=t\begin{pmatrix}0\\1\\0\end{pmatrix}\Rightarrow\vec{x}_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad\begin{cases}x_1-x_3=0\\x_2=0\end{cases}\Rightarrow\vec{x}=t\begin{pmatrix}1\\0\\1\end{pmatrix}\Rightarrow\vec{x}_2=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\begin{cases}x_1-0.8x_3=0\\x_2=0\end{cases}\Rightarrow\vec{x}=t\begin{pmatrix}0.8\\0\\1\end{pmatrix}\Rightarrow\vec{x}_3=\begin{pmatrix}4\\0\\5\end{pmatrix}$

d. $\lambda_1=\lambda_2=\lambda_3=2$ ✓

$x_1-x_2+3x_3=0\Rightarrow\vec{x}=t\begin{pmatrix}1\\1\\0\end{pmatrix}+s\begin{pmatrix}-3\\0\\1\end{pmatrix}\Rightarrow\vec{x}_1=\begin{pmatrix}1\\1\\0\end{pmatrix}\ \vec{x}_2=\begin{pmatrix}-3\\0\\1\end{pmatrix}$ ✓

$(x-2)(x+1)^2$

$x-2\ \overline{)\ x^3-3x-2}\quad x^2+2x+1$

x^3-2x^2

$2x^2-3x-2$

$2x^2-4x-$

$x-2$

$\begin{vmatrix}x&-1&0\\-3&x&-1\\-2&0&x\end{vmatrix}=x\begin{vmatrix}x&-1\\0&x\end{vmatrix}-(-1)\begin{vmatrix}-3&-1\\-2&x\end{vmatrix}$

$=x\,x^2+(-3x-2)=x^3-3x-2=0$

e. no eigenvalue

f. $\lambda_1=2,\lambda_2=\lambda_3=-1$ ✓

$\begin{cases}x_1-x_3=0\\x_2-2x_3=0\end{cases}\Rightarrow\vec{x}=t\begin{pmatrix}1\\2\\1\end{pmatrix}\Rightarrow\vec{x}_1=\begin{pmatrix}1\\2\\1\end{pmatrix}$ ✓

$\begin{cases}x_1+0.5x_3=0\\x_2-0.5x_3=0\end{cases}\Rightarrow\vec{x}=t\begin{pmatrix}-0.5\\0.5\\1\end{pmatrix}\Rightarrow\vec{x}_2=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ ✓

3.3.3 $A\vec{v}=\lambda\vec{v} \Rightarrow (\lambda I-A)\vec{v}=\vec{0}$, when $\lambda=0$, $|\lambda I-A|=-|A|=0 \Rightarrow |A|=0 \Rightarrow A$ not inv

3.3.19 a. $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\vec{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $A\vec{v}=\begin{pmatrix} a_{11}v_1+a_{12}v_2+a_{13}v_3 \\ a_{21}v_1+a_{22}v_2+a_{23}v_3 \\ a_{31}v_1+a_{32}v_2+a_{33}v_3 \end{pmatrix}=\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$

When $\vec{v}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A\vec{v}=\begin{pmatrix} a_{11}+a_{12}+a_{13} \\ a_{21}+a_{22}+a_{23} \\ a_{31}+a_{32}+a_{33} \end{pmatrix}=\begin{pmatrix} s \\ s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=s\vec{v}$, s is an eigenvalue.

b. $A^T=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$, $A^Tv=\begin{pmatrix} a_{11}v_1+a_{21}v_2+a_{31}v_3 \\ a_{12}v_1+a_{22}v_2+a_{32}v_3 \\ a_{13}v_1+a_{23}v_2+a_{33}v_3 \end{pmatrix}=\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$

when $\vec{v}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A^Tv=\begin{pmatrix} a_{11}+a_{21}+a_{31} \\ a_{12}+a_{22}+a_{32} \\ a_{13}+a_{23}+a_{33} \end{pmatrix}=\begin{pmatrix} s \\ s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=s\vec{v}$, s is an eigenvalue of A^T, also for A

3.3.21 a. $A\vec{x}=\lambda\vec{x} \Rightarrow A\cdot A\vec{x}=\lambda A\vec{x}=\lambda\cdot\lambda\vec{x}=\lambda^2\vec{x} \Rightarrow \lambda^2$ is an eigenvalue of A^2