

$h(x)$: hazard rate is the instantaneous rate of experiencing the event at time x given that an individual is alive at time x

$$h(x) = \lim_{\Delta x \rightarrow 0} \frac{P(x \leq X < x + \Delta x | X \geq x)}{\Delta x}$$

If Δx is very small we have:

$$P(x \leq X < x + \Delta x | X \geq x) \approx h(x) \Delta x.$$

The definition of the hazards implies that

$$h(x) = \frac{\lim_{\Delta x \rightarrow 0} P(x \leq X < x + \Delta x)}{P(X \geq x)} = \frac{f(x)}{S(x)}$$

(remember : $P(x \leq X < x + \Delta x) = f(x) \Delta x$)

$$f(x) = \frac{dF(x)}{dx} = -\frac{dS(x)}{dx} = -S'(x)$$

We can write then:

$$(*) \quad h(x) = \frac{f(x)}{S(x)} = -\frac{S'(x)}{S(x)} = -\frac{d \log \{S(x)\}}{dx}$$

From (*) by integrating both side we get

$$\begin{aligned} H(x) &= \int_0^x h(u) du = \int_0^x \frac{-S'(u)}{S(u)} du = -\log(S(u)) \Big|_0^x \\ &= -\left(\log(S(x)) - \log(\underbrace{S(0)}_{=1}) \right) \\ &= -\log\{S(x)\} \end{aligned}$$

$$H(x) = \int_0^x h(u) du = -\log\{S(x)\}$$

↓
cumulative hazard function

$$\text{Hence: } S(x) = \exp(-H(x))$$

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$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)} \Rightarrow f(x) = h(x) S(x) \\ &= h(x) \exp(-H(x)) \\ &= h(x) \exp\left(-\int_0^x h(u) du\right) \end{aligned}$$

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Exponential distribution:

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x} \quad x > 0; \quad F(x) = \int_0^x \lambda e^{-\lambda u} du \\ &= 1 - e^{-\lambda x} \\ h(x) &= \frac{f(x)}{S(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda \end{aligned}$$

$$h(x) = \lambda; \quad S(x) = 1 - F(x) = e^{-\lambda x}$$

Mean survival time:

$$\begin{aligned} \mu &= \int_0^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx = -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty} = \frac{1}{\lambda} = E^X = \int_0^{\infty} S(x) dx \end{aligned}$$

It is easier to compute the mean survival time by:

$$EX = \int_0^{\infty} S(x) dx$$

For the exponential distribution:

$$\begin{aligned} EX &= \int_0^{\infty} e^{-\lambda x} dx = -\frac{1}{\lambda} \left(e^{-\lambda x} \right) \Big|_0^{\infty} \\ &= -\frac{1}{\lambda} \end{aligned}$$

Median survival time for the exponential

$$S(x_{0.5}) = 0.5$$

$$\begin{aligned} ne(x) &= \int_0^x \lambda e^{-\lambda t} dt = 0.5 \\ -e^{-\lambda t} \Big|_0^x &= 0.5 \end{aligned}$$

$$1 - e^{-\lambda x} = 0.5 \Rightarrow e^{-\lambda x} = \frac{1}{2}$$

$$-\lambda x = \log\left(\frac{1}{2}\right) \Rightarrow \boxed{x = \log 2 / \lambda}$$

Mean residual life function:

$$\begin{aligned} mrl(x) &= E(X-x | X > x) = \frac{\int_x^{\infty} (t-x) f(t) dt}{P(X > x)} \\ &= \frac{\int_x^{\infty} (t-x) f(t) dt}{S(x)} \end{aligned}$$

Integrate by parts: (remember $f(t)dt = -dS(t)$)

$$E(X-x | X > x) \cdot S(x) = \int_x^{\infty} (t-x) \underbrace{f(t) dt}_{-dS(t)}$$

$$= -\underbrace{(t-x)S(t)}_{=0} \Big|_x^{\infty} + \int_x^{\infty} S(t) dt$$

$$= \underbrace{\frac{S(\infty) \cdot \infty}{=0}}_{=0} + \int_x^{\infty} S(t) dt$$

$$\Rightarrow E(X-x | X > x) = \frac{\int_x^{\infty} S(t) dt}{S(x)}$$

MrL for exponential distribution:

$$\begin{aligned} E(X-x | X > x) &= \frac{\int_x^{\infty} e^{-\lambda t} dt}{e^{-\lambda x}} = \frac{-\frac{1}{\lambda} e^{-\lambda t} \Big|_x^{\infty}}{e^{-\lambda x}} = \\ &= \left(\frac{1}{\lambda} \right) \end{aligned}$$

$$\text{Var}(X) = E X^2 - (E X)^2$$

find the relation between variance and survival

$$E X = \int_0^{\infty} S(t) dt$$

$$\begin{aligned} E X^2 &= \int_0^{\infty} t^2 f(t) dt = \underbrace{-t^2 S(t)}_{=0} \Big|_0^{\infty} + 2 \int_0^{\infty} t S(t) dt \\ &= 2 \int_0^{\infty} t S(t) dt \end{aligned}$$

$$\text{Var } X = 2 \int_0^{\infty} t S(t) dt - \left[\int_0^{\infty} S(t) dt \right]^2$$

$$\begin{aligned} E X &= \int_0^{\infty} t \underbrace{f(t)}_{-S(t)} dt = \underbrace{-t S(t)}_{=0} \Big|_0^{\infty} + \int_0^{\infty} S(t) dt \\ &= \int_0^{\infty} S(t) dt \end{aligned}$$

Remember that $S(\infty) = 0$, $S(0) = 1$

$$\lim_{t \rightarrow \infty} t S(t) = 0$$

Memoryless property for the exponential distribution

$$P(X > s+t | X > s) = P(X > t) \quad \forall s, t \geq 0$$

$$\begin{aligned} P(X > s+t | X > s) &= \frac{P(X > t+s, X > s)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t) \end{aligned}$$

Solution exercise car on the slides:

$$\text{average} = 10 \Rightarrow \lambda = \frac{1}{10} \quad X \sim \text{exp}(\lambda)$$

$$\begin{aligned} P(\text{remaining lifetime} > 5) &= 1 - F(5) \\ &\stackrel{!}{=} e^{-5\lambda} \\ &\stackrel{!}{=} e^{-1/2} \approx .604 \end{aligned}$$

if the life time distribution is not exponential then the relevant probability is:

$$\begin{aligned} P(X > t+5 | X > t) &= \frac{P(X > 5+t, X > t)}{P(X > t)} \\ &= \frac{P(X > 5+t)}{P(X > t)} \\ &= \frac{1 - F(t+5)}{1 - F(t)} = \frac{S(t+5)}{S(t)} \end{aligned}$$

$t =$ n° of Km that the battery had been

in use prior to the start of the trip.
If the distribution is not exponential
~~after~~ additional information is needed
(namely t), before the probability can
be calculated.