MFS Lecture 5

Last time: Graphs and optimisation problems

Today: Definite + indefinite integrals.

Definition: A function F is an antiderivative of a function f if F is diff. and

$$F'(x) = f(x)$$

$\forall x$ in $D(f)$.

Example: Let $f(x) = 2x$. If $F(x) = x^2$, then $F'(x) = f(x)$, so F is an antiderivative of f.

Are there others?

YES! $G(x) = x^2 + 1$ is also an antiderivative, as is $F(x) + C$ for any constant C.

Theorem: Let f be an antiderivative of f over an interval I. Then

1. for each constant C, the fcn $F(x) + C$ is also an antiderivative of f over I, and
2. if G is an antiderivative of f over I, $\exists$ a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

If you find one antiderivative, you've found them all!

Here are the common antiderivatives that we'll see a lot:

function	antiderivative
$x^n, n \neq -1$	$\frac{1}{n+1} x^{n+1} + C$
x^{-1}	$\ln(\lvert x \rvert) + C$
e^x	$e^x + C$
b^x	$\frac{1}{\ln(b)} b^x + C.$

Definition: An indefinite integral of a fcn $f(x)$, written

$$\int f(x)\,dx,$$

is a family of fcns $F(x) + C$, where $F(x)$ is an antiderivative of $f(x)$ and C is any constant.

Example. Integrate $x^{2.5} + 3^x$, i.e. solve $\int (x^{2.5} + 3^x)\,dx$.

Answer: $\int (x^{2.5}+3^x)\,dx = \frac{1}{3.5}x^{3.5} + \frac{1}{\ln(3)}3^x + C$.

Remark: Any indefinite integral $\int f(x)\,dx$ must include the constant of integration C. If you don't include it, your answer is not correct.

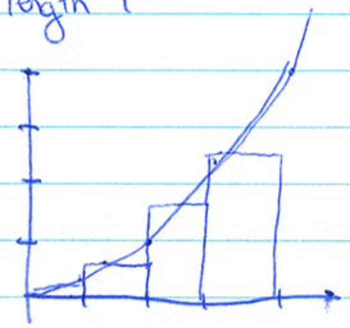
Solving indefinite integrals can be hard! We'll see some useful techniques next lecture. For now, let's look at interpreting integrals.

Areas under functions

Let $f(x)$ be a fcn. To approximate the area under f on an interval $[a,b]$, we can divide $[a,b]$ into subintervals of length Δx and pick the midpt of each subinterval; then approximate using rectangles.

Idea/theory:

Example: $f(x)=\frac{1}{4}x^2$ from 0 to 4. We can divide $[0,4]$ into 4 subintervals of length 1



We choose the midpts for each subinterval with heights $f(0.5)$, $f(1.5)$, $f(2.5)$, $f(3.5)$.

Then

area under $f \approx$ sum of rectangles

$= f(0.5)\times 1 + f(1.5)\times 1 + f(2.5)\times 1 + f(3.5)\times 1$.

How can we make this better?

Answer: use more + smaller subintervals!

i.e. divide $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$.

Choose, for each subinterval, x_i^* as the midpt of the i-th interval. Then

area under f from a to $b \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Remark: Such a sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum of f on $[a,b]$. It turns out your partition can be more general, but we don't need that level of generality.

Definition: Let $f(x)$ be a fcn defined on an interval $[a,b]$, and divide $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$. Let x_i^* be the midpt of the i-th subinterval. The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

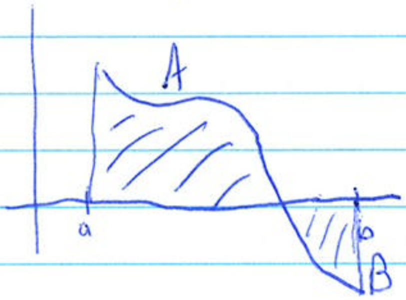
whenever this limit exists. When this limit exists, we say f is integrable on $[a,b]$

Remarks: i. How is this related to indefinite integrals? We'll see very soon.

ii. Just how we never use the formal definition of a derivative to compute derivatives, we'll never really use this definition to compute an indefinite integral.

iii. If f is positive on $[a,b]$, $\int_a^b f(x)dx$ can be interpreted as the area under f. If f is negative, this is the negative value of the area above f and below the x-axis.

In general: if f is positive on a subset $M \subset [a,b]$ and negative on $N \subset [a,b]$, $\int_a^b f(x)dx$ is the area of f above M minus the area of f under N.

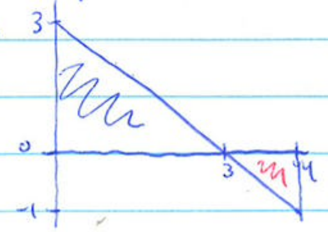


$$\int_a^b f(x)\,dx = A - B.$$

iv. We refer to $f(x)$ as the integrand, a, b as the upper/lower limits of integration.

Example: Compute $\int_0^4 (3-x)dx$.

Answer:



This is the area under the curve as shown, i.e.

$$\int_0^4 (3-x)\,dx = \frac{1}{2}(3)(3) - \frac{1}{2}(1)(1) = 4.5 - \frac{1}{2} = 4$$

Lemma: Let $f(x)$ be an integrable fcn on $[a,b]$. Then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \text{ and}$$

$$\int_a^a f(x)dx = 0.$$

Theorem: If f is cts and defined on $[a,b]$, f is integrable on $[a,b]$.

The Fundamental Theorem of Calculus

Let's look at how integrals and derivatives are related. In doing so, we'll also see how definite integrals are related to indefinite integrals.

To state the Fundamental Thm of Calculus, we'll look at a fcn of the form

$$g(x) = \int_a^x f(t)\,dt$$

where f is a cts fcn on $[a,b]$ and x is a pt in $[a,b]$. Note $g(x)$ is a function of x - it's the area under f from a to any pt x.

Ex: Consider $f(t) = t$, $a=0$. Then $g(x) = \int_0^x t\,dt$ is the area of the triangle w/ vertices $(0,0)$, $(x,0)$, and $(0,x)$, i.e. $\frac{1}{2}x^2$.

Theorem (FTC, Part I): Let $f(t)$ be a cts fcn on $[a,b]$, and set $g(x) = \int_a^x f(t)\,dt$. Then $g(x)$ is differentiable on (a,b) and $g'(x) = f(x)$.

Proof sketch: Fix x, let h be very small. Then $g(x+h) - g(x) = \int_x^{x+h} f(t)\,dt$. But if h is really small, this can be approximated by a rectangle of width h and height $f(x)$, i.e.

$$g(x+h) - g(x) \approx h f(x), \text{ or } f(x) \approx \frac{g(x+h)-g(x)}{h}$$

Let h go to 0 to get $f(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$.

Theorem (FTC, Part II): Let f be cts on an interval $[a,b]$ and let F be <u>any</u> antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof sketch: By the FTC Pt I, $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f, i.e.

$$g'(x) = f(x),$$

and so $F(x) = g(x) + C$. Hence

$$F(b) - F(a) = g(b) - g(a) = \int_a^b f(x)\,dx.$$

Remark: So to calculate a definite integral, we

1. integrate f to get an antiderivative F
2. evaluate F at the endpts + take the difference.

Ex: $\int_1^5 x^2\,dx$.

Take $F(x) = \frac{1}{3}x^3$ (we <u>can</u> add $+C$, but it'll cancel when we take the difference.

Then

$$\int_1^5 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{x=1}^{x=5}$$
$$= \tfrac{1}{3}\left[(5)^3-(1)^3\right] = \tfrac{124}{3}$$

Example: $\int_1^3 \sqrt{t}\,dt = \left[\frac{2}{3} \cancel{x} t^{3/2}\right]_{t=1}^{t=3} = \frac{2}{3}\left(3^{3/2}-1\right)$

The Net-Change Theorem: The definite integral of a rate of change is the net change.

i.e. the definite integral of velocity - $\frac{dp}{dt}$ - is the total change in position

Ex: Your velocity is $v(t)=t^2+t-9$ in $\frac{m}{s}$. How far do you travel between $t=0$ and $t=3$?

Answer: $p(3)-p(0) = \int_0^3 v(t)\,dt = \left[\frac{1}{3}t^3+\frac{1}{2}t^2-9t\right]_0^3$
$$= -\frac{27}{2} = -13.5.$$

Ex: The population of wasps in a forest changes at a rate of
$\frac{dP}{dt} = \ln(3)\,1000\,3^t$, where t is time in years.
How much does the population change between year 5 and year 7?

A: $P(7)-P(5) = \int_5^7 \frac{dP}{dt}\,dt$
$$= \ln(3)1000\int_5^7 3^t\,dt = 1000\left[3^t\right]_5^7 = 1000\left(3^7-3^5\right).$$

Improper integrals

An improper integral has one or both of
- an infinite interval of integration, i.e. $(-\infty, a]$, $[a,\infty)$, or (∞,∞)
- or f has an infinite discontinuity at some pt c in $[a,b]$.

The 1st type occurs a lot in statistics!

To do the 1st type, we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{and} \quad \int_{-\infty}^a f(x)\,dx = \lim_{t\to\infty}\int_{-t}^a f(x)\,dx.$$

Example: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1.$

If the limit(s) exist, the integral converges, otherwise it diverges.

Conversely, suppose $f(b)$ is not defined ~~from (the) right~~ at b w/ ~~$\lim_{x \to b} f(x)$~~ $\lim_{x \to b^-} f(x) = \pm\infty$.

Then we define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

If $\lim_{x \to a^+} f(x) = \pm\infty$, we set

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

Ex: $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t \to 0} 2(1 - \sqrt{t}) = 2.$