Practice Exercises – Solutions

1. Suppose that a box contains one fair coin and one coin with a head at each side. Suppose that a coin is selected at random and that when it is tossed three times, a head is obtained three times. Determine the probability that the coin is the fair coin.

Solution:

$$\mathrm{P(fair|HHH)} = \frac{P(HHH|fair)P(fair)}{P(HHH|fair)P(fair) + P(HHH|unfair)P(unfair)} = \frac{0.5^3 * 0.5}{0.5^3 * 0.5 + 1 * 0.5}$$

2.

a. If four dice are rolled, what is the probability that each of the four numbers that appear will be different?

b. Suppose that a committee of 12 people is selected in a random manner from a group of 100 people. Determine the probability that in the chosen committee two particular people A and B will both be selected.

c. Five red and seven blue marbles are arranged in a row. We want to find the probability that both the end marbles are red.

Solution:

a. The sample space contains 6^4 elements. Number of possibilities where all numbers are different: 6*5*4*3. P(all different) = 6*5*4*3/6^4.
b. The possible ways to choose 12 out of 100 is $\binom{100}{12}$. Number of ways to select the remaining 10 members is $\binom{98}{10}$ and thus P(AB is selected) = $\frac{\binom{98}{10}}{\binom{100}{12}}$.
c. Number the marbles from 1 to 12, letting the red marbles be numbered from 1 to 5 for convenience. The sample space consists of all the possible permutations of 12 distinct objects, so the sample space contains 12! points, each of which, we will assume, is equally likely. Now we must count the number of points in which the end points are both red. We have five choices for the marble at the left end and four choices for the marble at the right end. The remaining marbles, occupying places between the ends, can be arranged in 10! ways, so P (end marbles are both red) = (5 · 4 · 10!)/12! = (5 · 4 · 10!)/(12 · 11 · 10!)=(5 · 4)/(12 · 11)= 5/33.

3. If the probability that a student A will pass a certain physics examination is 0.5, the probability that student B will pass the examination is 0.2 and the probability that both students will pass the examination is 0.1, what is the probability that at least one of these 2 students will pass the examination?

Solution:

P(at least 1 pass) =P(A union B) = P(A) + P(B) – P(A intersection B) = 0.5 + 0.2-0.1=0.6.

4. Suppose that you are walking down the street and notice that the Department of Public Health is giving a free medical test for a certain disease. The test is 90 percent reliable in the following sense: If a person has the disease, there is a probability of 0.9 that the test will give a positive response; whereas, if a person does not have the disease, there is a probability of only 0.1 that the test will give a positive response. Data indicate that your chances of having the disease are only 1 in 10,000. However, since the test costs you nothing, and is fast and harmless, you decide to stop and take the test. A few days later you learn that you had a positive response to the test. Now, what is the probability that you have the disease?

Solution:

We shall let B1 denote the event that you have the disease, and let B2 denote the event that you do not have the disease. The events B1 and B2 form a partition. Also, let A denote the event that the response to the test is positive. The event A is information we will learn that tells us something about the partition elements. Then, by Bayes' theorem,
Pr(B1|A) = Pr(A|B1) Pr(B1)
Pr(A|B1) Pr(B1) + Pr(A|B2) Pr(B2) = (0.9)(0.0001)/((0.9)(0.0001) + (0.1)(0.9999))
= 0.00090.
Thus, the conditional probability that you have the disease given the test result is approximately only 1 in 1000.

5. Let Z be a standard normal random variable and let $0 < a < 1$.
(a) Determine the value of the constant c such that $P(|Z| < c) = P(-c < Z < c) = 1 - a$.
(b) What is the value of c when a = 0.05?
For the computations you may use the Tables at the end of the Rice book.

Solution:

(a) $P(-c < Z < c) = \Phi(c) - \Phi(-c) = \Phi(c) - [1 - \Phi(c)] = 2^* \Phi(c) - 1$.
Note that $\Phi(-c) = [1 - \Phi(c)]$ because the normal pdf is symmetric.

Thus, P(-c < Z < c) = 1 - a if and only if 2 * Φ(c) - 1 = 1 - a or, equivalently, Φ(c)= 1 - a/2. Therefore, the desired value of c is given by c = Φ^{-1} (1- a/2) = $z_{\left(1-\frac{a}{2}\right)}$.

(b) $z_{0.975} = 1.96$

6. The normal distribution typically provides a good approximation to the height of an individual selected at random from a large homogeneous population of adults. Suppose the height in inches of such a randomly selected individual has mean 68 and standard deviation 2. Use this normal approximation to determine

   (a) the probability that the height of such a randomly selected individual is between 66 inches and 72 inches.

   (b) the quartiles of the distribution of height.

Solution:

(a) Let Y denote the height in inches of a randomly selected individual. Suppose Y is normally distributed with mean $\mu = 68$ and standard deviation $\sigma = 2$.

$P(66 \leq Y \leq 72) = P\left(\frac{66-68}{2} \leq \frac{Y-68}{2} \leq \frac{72-68}{2}\right) = P\left(-1 \leq \frac{Y-68}{2} \leq 2\right) = \Phi$ (2) - Φ(-1).
Since Φ (2) = .977 and Φ (-1) = 1 - Φ (1) = . 159, we get that P(66$\leq$ Y $\leq$72) = .819.

(b) The quantiles of Y are the values $y_{0.25}$, $y_{0.5}, y_{0.75}$ for which P(Y$\leq$ $y_{0.25}$)=0.25, etc
Since we need to use the Tables of the standard Normal, we will write
$P(Y \leq y_{0.25}) = P\left(\frac{Y-68}{2} \leq \frac{y_{0.25}-6}{2}\right) = P\left(Z \leq \frac{y_{0.25}-68}{2}\right) = \Phi\left(\frac{y_{0.25}-6}{2}\right)$.
Thus $\Phi\left(\frac{y_{0.25}-6}{2}\right)$=0.25 and from the tables we have $\frac{y_{0.25}-68}{2} = -0.674$ and $y_{0.25} = 66.652$.
Similarly $y_{0.5} = 68, y_{0.75} = 69.3$.

7. Suppose the time to failure of a newly installed light bulb is exponential with scale parameter p = 100 hours. Determine the probability that the time to failure is between 50 hours and 200 hours.

Solution:

The distribution function of the time to failure Y is given by F(y)=1-exp{-y/100}, y>0.

Thus, P(50 < Y < 200) = F(200) - F(50) = 0.471.

8. The average number of traffic accidents on a certain section of highway is two per week.
(a) Find the probability of no accidents on this section of highway during a 1-week period.

(b) Find the probability of at most three accidents on this section of highway during a 2-week period.

Solution:

(a) The average number of accidents per week is μ=2. Therefore, the probability of no accidents on this section of the highway during a given week is:
$P(X = 0) = \frac{2^0 e^{-2}}{0!} = e^{-2} = 0.135.$

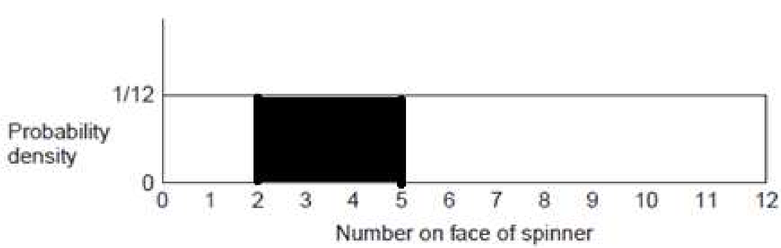
(b) During a 2-week period, the average number of accidents in tis section of the highway is 2*2=4. The probability of at most three accidents during a 2-week period is:
$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3),$
Where

$$P(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4} = 0.018$$
$$P(X = 1) = \frac{4^1 e^{-4}}{1!} = e^{-4} = 0.073$$
$$P(X = 2) = \frac{4^2 e^{-4}}{2!} = e^{-4} = 0.147$$
$$P(X = 3) = \frac{4^3 e^{-4}}{3!} = e^{-4} = 0.195$$

Thus, $P(X \leq 3) = 0.434.$

9. We will consider the example of Lecture 5 with the wheel that has a spinner attached on it. The numbers 1-12 are drawn on its circumference at equally spaced positions. Let Y the rv that gives the number at which the spinner/arrow stops whenever we spin the wheel. The support of Y is the interval [0, 12] and the pdf is given on the Figure below.

   a. What does the shaded area represent?
   b. Based on the Figure below what is P(Y=5)?
   c. Based on the Figure below compute P(5<Y<6) and P(5≤Y<6).



Solution:

   a. The shaded area represents P(2 $\leq Y \leq$ 5)=3/12.
   b. P(Y=5)=0
   c. P(5<Y<6)=P(5≤Y<6)=1/12.

10. A rv X can only assume the values 3, 4, 7, 8 and 9. It is known that P(X = 3)=1/3, P(X=4)=1/4, P(X=7)=1/6 and P(X=8)=1/6.
    a. Find P(X = 9).
    b. Compute E(X) and Var(X).

Solution:

a. The sum of the probabilities should equal 1 and thus P(X=9)=1/12.
b. E(X)=3*1/3+4*1/4+7*1/6+8*1/6+9*1/12 =5.25
c. Var(X)= (3-5.25)^2*1/3+(4-5.25)^2*1/4+(7-5.25)^2*1/6+(8-5.25)^2*1/6+(9-5.25)^2*1/12=5.02.