MATHEMATICAL MODELLING HOMEWORK 2

[57/57]

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1) [18/18] Sometimes we want to compute the derivative of a function y that is implicitly a function of x, i.e. $y(x)$, but that we can't write directly as a function of x. For example, the circle $x^2 + y^2 = 1$. In this case, we use the chain rule, treating y as a function of x, i.e. the derivative of y^2 with respect to x is then $2y\frac{dy}{dx}$, where the derivative of y is not yet known explicitly.

(a) [2/2] Calculate the derivative y' for the circle $x^2 + y^2 = 1$. Your answer may depend on both y and x.

SOLUTION:

- We have $2x + 2yy' = 0$ [1 pt], or $y' = -\frac{x}{y}$. [1 pt]

(b) [3/3] Find an equation to the tangent line of $x^2 + y^2 = 9$ at the point $(-2, \sqrt{5})$.

SOLUTION:

- We still have $y' = -\frac{x}{y}$, so at the point $(-2, \sqrt{5})$ we have $y' = \frac{2}{\sqrt{5}}$ [1 pt].
- Solve for b in the equation $y = -\frac{2}{\sqrt{5}}x + b$ [1 pt], using the point as given, to find $b = \frac{9}{\sqrt{5}}$, i.e. $y = \frac{2}{\sqrt{5}}x + \frac{9}{\sqrt{5}}$ [1 pt].

We can use this technique of *implicit differentiation* to calculate related rates of change.

(c) [3/3] A spherical balloon has volume $V = \frac{4}{3}\pi r^3$, where r is its radius. Suppose the balloon's volume is increasing at a rate of $\frac{dV}{dt} = 5\text{cm}^3/s$. Using implicit differentiation, find an equation for $\frac{dr}{dt}$, the rate of change of the radius of the balloon when the radius is $5cm$.

SOLUTION:

- By implicit differentiation, we have $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ [1 pt], or $\frac{dr}{dt} = \frac{dV}{dt}/(4\pi r^2)$ [1 pt].
- Letting $V' = 5$ and $r = 5$, we obtain $r' = \frac{1}{20\pi}$. [1 pt]

(d) [10/10] You light a fuse on a model rocket ship, and immediately start running away at a speed of 4m/s. Three seconds after you start running, the rocket blasts straight upwards at a constant speed of 20m/s. Find the rate of change of the distance between you and your rocket i) 1 second after lighting the rocket and ii) 4 seconds after lighting it.

SOLUTION:

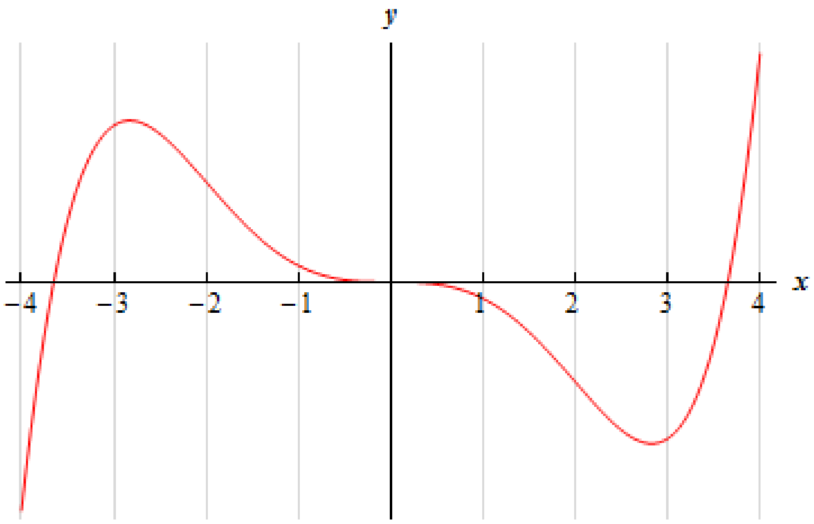
- The rocket only moves 3 seconds after you start running [1 pt], so at $t = 1$ the only movement is from yourself, i.e. $dP/dt = 4$. [1 pt]
- The rockets velocity at time t, whenever $t > 3$, is $y' = 20$ [1 pt]. It's position is $20m$ above the ground when $t = 4$ [1 pt], and your position is $16m$ [1 pt] in the x-axis with $x' = 4$ [1 pt].

- Hence z, the distance between you and the rocket, is $z = \sqrt{y^2 + x^2}$, or $z^2 = y^2 + x^2$ [1 pt]. Thus $2zz' = 2xx' + 2yy'$, [1 pt], and so $z' = \frac{xx'+yy'}{\sqrt{x^2+y^2}}$ [1 pt].
- When $x = 16$, $x' = 4$, $y = 20$, and $y' = 20$, we have $z' = 18.116$ [1 pt].

2) [6/6] Another method that can be used in differentiation is *logarithmic differentiation*. That is, sometimes it's easier if, given $y = f(x)$, to first take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then you can compute the derivative, using implicit differentiation to compute the derivative of $\ln(y)$ as $\frac{y'}{y}$.
   (a) 3/3] Compute the derivative of $y = x^x$ using logarithmic differentiation. You'll have to use certain properties of the logarithm!
   SOLUTION: [1 pts per bullet]
   - We have $\ln(y) = x\ln(x)$ by property of logarithms.
   - Thus $y'/y = \ln(x) + 1$, by taking the derivative of both sides.
   - Then $y' = y(\ln(x) + 1) = x^x(\ln(x) + 1)$.

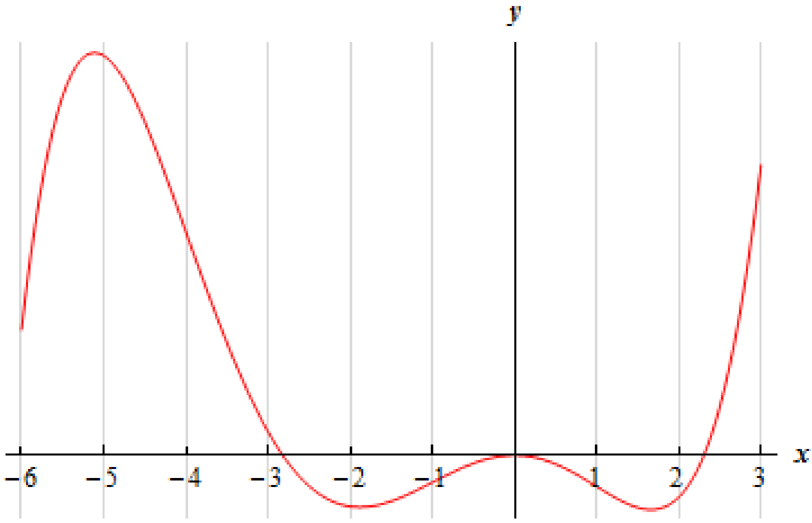
   (b) [3/3] Compute the derivative of $y = x^{\ln(x)}$ using logarithmic differentiation.
   SOLUTION: [1 pts per bullet]
   - We have $\ln(y) = \ln(x)^2$.
   - Thus $y'/y = 2\ln(x)/x$.
   - Then $y' = y2\ln(x)/x = 2\frac{\ln(x)x^{\ln(x)}}{x}$.

3) [27/27]
   (a) [3/3] Determine the interval(s) on which the function shown in the graph below is concave up and concave down.



   SOLUTION: It is concave up on $(-2, 0)$ [1 pt] and $(0, \infty)$ [1 pt]. Note that 0 should *not* be included!! If so deduce a point.
   It is concave down from $(-\infty, -2)$ [1 pt].

   (b) [4/4] Determine the interval(s) on which the function shown in the graph below is concave up and concave down.



SOLUTION: It is concave up on $(-4,-1)$ [1 pt] and on $(1,\infty)$ [1 pt]. It is concave down on $(-\infty,-4)$ [1 pt] and $(-1,1)$ [1 pt].

(c) [20/20] Let $f(x) = 3xe^{1-\frac{1}{4}x^2}$.

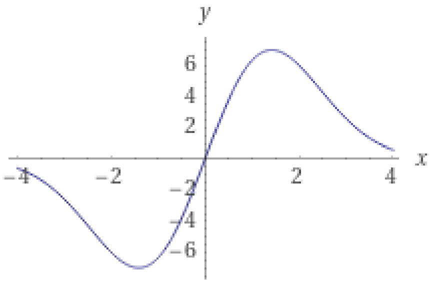
(i) [6/6] Find the critical point(s) of $f(x)$ and identify them as local maxima or minima, if possible.
SOLUTION: We have that $f'(x) = 3e^{1-\frac{1}{4}x^2} - \frac{3}{2}x^2e^{1-\frac{1}{4}x^2}$ [1 pt].
Set this equal to 0 and solve to get $x^2 = 2$, or $x = \pm\sqrt{2}$. Hence the two critical points are $(\pm\sqrt{2}, 0)$ [1 pt].
When $x = 0$, $f'(x) = 3 > 0$, and when $|x| > \sqrt{2}$, $f'(x) < 0$. Hence f is increasing on $(-\sqrt{2},\sqrt{2})$ [1 pt] and decreasing when $|x| > \sqrt{2}$ [1 pt].
Hence $-\sqrt{2}$ is a local minimum [1 pt], and $\sqrt{2}$ is a local maximum [1 pt] from the 1st derivative test (the second derivative test is also fine).

(ii) [2/2] Determine the interval(s) on which $f(x)$ is increasing or decreasing.
SOLUTION: From part (ii) we found f is increasing on $(-\sqrt{2},\sqrt{2})$ [1 pt] and decreasing when $|x| > \sqrt{2}$ [1 pt].

(iii) [8/8] Find the interval(s) on which $f(x)$ is concave up or concave down. Find the inflection point(s) of $f(x)$.
SOLUTION: We calculate $f''(x) = -\frac{3}{2}xe^{1-\frac{1}{4}x^2} - 3xe^{1-\frac{1}{4}x^2} + \frac{3}{2}x^3e^{1-\frac{1}{4}x^2}$ [1 pt].
Set this equal to 0 and rearrange to get $e^{1-\frac{1}{4}x^2}x(x^2-6) = 0$ [1 pt]. Hence either $x = 0$ [1 pt], or $x = \pm\sqrt{6}$ [1 pt].
One checks, i.e. by plugging in values, that $f(x)$ is concave up on the intervals $(-\sqrt{6},0)$ [1 pt] and $(\sqrt{6},\infty)$ [1 pt], and concave down on $(-\infty,-\sqrt{6})$ [1 pt] and $(0,\sqrt{6})$ [1 pt].

(iv) [2/2] Find the inflection point(s) of $f(x)$.
SOLUTION: Hence we conclude that $x = 0$ [1 pt] and $x = \pm\sqrt{6}$ [1 pt] are the three inflection points.

(v) [2/2] Sketch the graph of the function, using the information from the previous parts.
SOLUTION:



4)

[6/6] A printer needs to make a poster that will have a total area of $500cm^2$ that will have $3cm$ margins on the sides and $2cm$ margins on the top and bottom for the printed area. What dimensions of the poster will give the largest printed area?

SOLUTION:

i) Our two defining equations are

$$xy = 500[1pt]$$

$$A = (x-4)(y-6), [1pt]$$

where x is the width, y the vertical length, and A is the area of the printed region we want to optimise.

ii) Hence

$$A = 524 - \frac{3000}{y} - 4y, [1pt]$$

so

$$A' = \frac{3000}{y^2} - 4 = 0.[1pt]$$

iii) Hence $y = 5\sqrt{30} = 27.386$ and $x = \frac{10}{3\sqrt{30}} = 18.257$ [1 pt].

iv) Note that this indeed optimises the area, as for example $A''(y) = -\frac{6000}{y^3}$, which is always negative for y positive, which is the only realistic possibility. [1 pt]