MATHEMATICAL MODELLING HOMEWORK 2

[57/57]

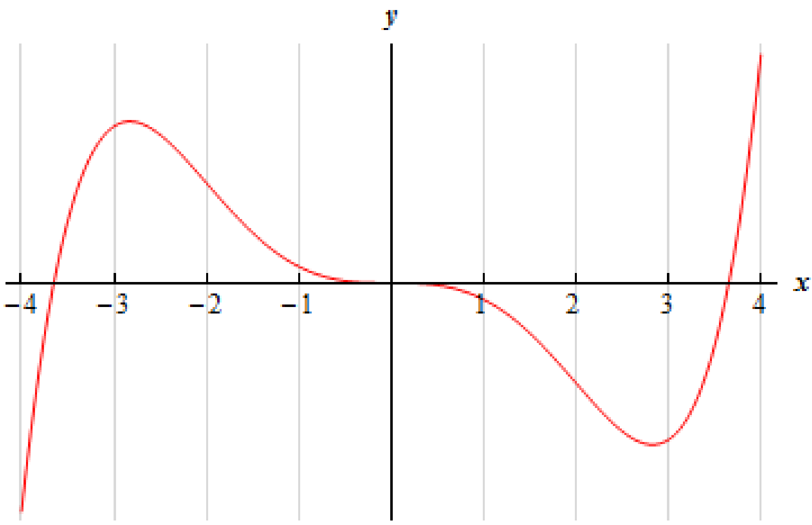
Please upload a typewritten or (legible!) scanned copy of your homework on Brightspace. Collaboration is encouraged, but please ensure you state at the start of your homework with whom you've collaborated. For full marks you must provide a justification of your answers, i.e. only a (correct) answer without any work shown will not be considered sufficient.

1) [18/18] Sometimes we want to compute the derivative of a function y that is implicitly a function of x, i.e. $y(x)$, but that we can't write directly as a function of x. For example, the circle $x^2 + y^2 = 1$. In this case, we use the chain rule, treating y as a function of x, i.e. the derivative of y^2 with respect to x is then $2y\frac{dy}{dx}$, where the derivative of y is not yet known explicitly.
   (a) [2/2] Calculate the derivative y' for the circle $x^2 + y^2 = 1$. Your answer may depend on both y and x.
   (b) [3/3] Find an equation to the tangent line of $x^2 + y^2 = 9$ at the point $(-2, \sqrt{5})$.

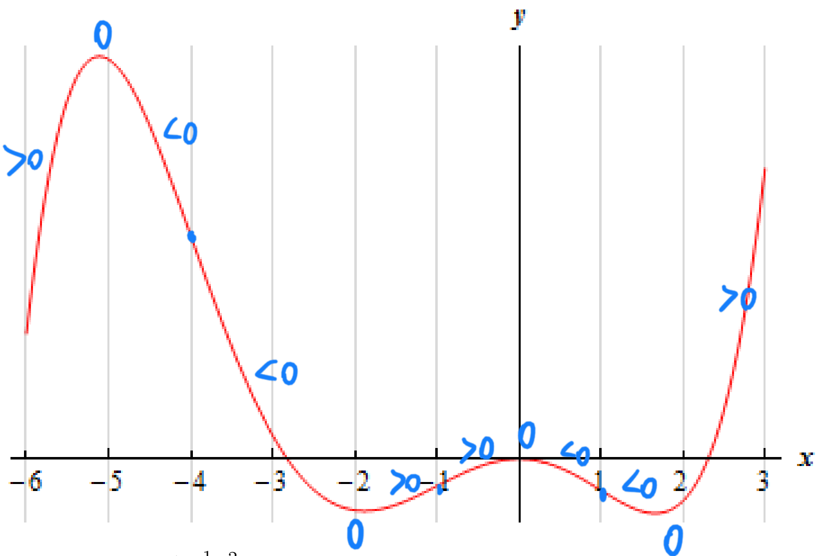
   We can use this technique of *implicit differentiation* to calculate related rates of change.
   (c) [3/3] A spherical balloon has volume $V = \frac{4}{3}\pi r^3$, where r is its radius. Suppose the balloon's volume is increasing at a rate of $\frac{dV}{dt} = 5\text{cm}^3/s$. Using implicit differentiation, find an equation for $\frac{dr}{dt}$, the rate of change of the radius of the balloon when the radius is $5cm$.
   (d) [10/10] You light a fuse on a model rocket ship, and immediately start running away at a speed of 4m/s. Three seconds after you start running, the rocket blasts straight upwards at a constant speed of 20m/s. Find the rate of change of the distance between you and your rocket i) 1 second after lighting the rocket and ii) 4 seconds after lighting it.

2) [6/6] Another method that can be used in differentiation is *logarithmic differentiation*. That is, sometimes it's easier if, given $y = f(x)$, to first take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then you can compute the derivative, using implicit differentiation to compute the derivative of $\ln(y)$ as $\frac{y'}{y}$.
   (a) 3/3] Compute the derivative of $y = x^x$ using logarithmic differentiation. You'll have to use certain properties of the logarithm!
   (b) [4/4] Compute the derivative of $y = x^{\ln(x)}$ using logarithmic differentiation.

3) [27/27]
   (a) [3/3] Determine the interval(s) on which the function shown in the graph below is concave up and concave down.



(b) [4/4] Determine the interval(s) on which the function shown in the graph below is concave up and concave down.



(c) [20/20] Let $f(x) = 3xe^{1-\frac{1}{4}x^2}$.

(i) [6/6] Find the critical point(s) of $f(x)$ and identify them as local maxima or minima, if possible.

(ii) [2/2] Determine the interval(s) on which $f(x)$ is increasing or decreasing.

(iii) [8/8] Find the interval(s) on which $f(x)$ is concave up or concave down. Find the inflection point(s) of $f(x)$.

(iv) [2/2] Find the inflection point(s) of $f(x)$.

(v) [2/2] Sketch the graph of the function, using the information from the previous parts.

4)

[6/6] A printer needs to make a poster that will have a total area of $500cm^2$ that will have $3cm$ margins on the sides and $2cm$ margins on the top and bottom for the printed area. What dimensions of the poster will give the largest printed area?