Exercises Lecture 7: Conditional Distributions

1. Let the random variables X and Y represent the number of family members in a household and the number of cars they own, respectively. The joint frequency function of the random variables X and Y is given in the following table:

	x			
y	1	2	3	4
1	0.10	0.05	0.02	0.02
2	0.05	0.20	0.05	0.02
3	0.02	0.05	0.20	0.04
4	0.02	0.02	0.04	0.10

   (a) Which is the conditional frequency function of X given $Y = 1$ and of Y given $X = 1$?

   (b) Say that we interview a household at random and we learn that it owns 2 cars. What is the expected number of members that this household has?

   (c) What is the expected family size of a household in this survey irrespective of the cars they own?

   (d) What is the value of $E(X^2 \mid Y = 2)$?

   (e) For the household, which we interviewed above, and owns 2 cars what is the variance of its members?

   Solution:

   (a) We need first the marginal: $P(Y = 1) = 0.19$.
   The conditional frequency function $P(X = x \mid Y = 1)$ is given by:
   $P(X = 1 \mid Y = 1) = \frac{P(X=1,Y=1)}{P(Y=1)} = 0.10/0.19$,
   $P(X = 2 \mid Y = 1) = 0.05/0.19$,
   $P(X = 3 \mid Y = 1) = 0.02/0.19$,
   $P(X = 4 \mid Y = 1) = 0.02/0.19$.
   For $P(Y = y \mid X = 1)$ we need first $P(X = 1) = 0.19$. Then,
   $P(Y = 1 \mid X = 1) = \frac{P(X=1,Y=1)}{P(X=1)} = 0.10/0.19$,
   $P(Y = 2 \mid X = 1) = 0.05/0.19$,
   $P(Y = 3 \mid X = 1) = 0.02/0.19$,
   $P(Y = 4 \mid X = 1) = 0.02/0.19$.

(b) We want to compute $E(X \mid Y = 2)$. So we need to derive first the conditional frequency function $P(X = x \mid Y = 2)$:
$P(X = 1 \mid Y = 2) = \frac{P(X=1,Y=2)}{P(Y=2)} = 0.05/0.32,$
$P(X = 2 \mid Y = 2) = 0.20/0.32,$
$P(X = 3 \mid Y = 2) = 0.05/0.32,$
$P(X = 4 \mid Y = 2) = 0.02/0.32.$
$E(X \mid Y = 2) = 1 \cdot P(X = 1 \mid Y = 2) + 2 \cdot P(X = 2 \mid Y = 2) + 3 \cdot P(X = 3 \mid Y = 2) + 4 \cdot P(X = 4 \mid Y = 2) = 1 * 0.05/0.32 + 2 * 0.20/0.32 + 3 * 0.05/0.32 + 4 * 0.02/0.32 = 2.125.$

(c) We want to compute $E(X)$. So we need to derive first the frequency function $P(X = x)$:
$P(X = 1) = \sum_y P(X = 1, Y = y) = 0.10 + 0.05 + 0.02 + 0.02 = 0.19,$
$P(X = 2) = 0.32,$
$P(X = 3) = 0.31,$
$P(X = 4) = 0.18.$
Thus, $E(X) = 1 * 0.19 + 2 * 0.32 + 3 * 0.31 + 4 * 0.18 = 2.48.$

(d) We have computed the conditional frequency function $P(X = x \mid Y = 2)$ above. $E(X^2 \mid Y = 2) = 1^2 * 0.05/0.32 + 2^2 * 0.20/0.32 + 3^2 * 0.05/0.32 + 4^2 * 0.02/0.32 = 5.0625.$

(e) We want to compute: $Var(X \mid Y = 2)$. We have already computed $E(X^2 \mid Y = 2)$ and $E(X \mid Y = 2)$. We know that $Var(X \mid Y = 2) = E(X^2 \mid Y = 2) - [E(X \mid Y = 2)]^2. = 5.0625 - 2.125^2 = 0.546875.$

2. Each student in a certain high school was classified according to her year in school (freshman, sophomore, junior, or senior) and according to the number of times that she had visited a certain museum (never, once, or more than once). The proportions of students in the various classifications are given in the following table:

	Never	Once	More than once
Freshmen	0.08	0.10	0.04
Sophomores	0.04	0.10	0.04
Juniors	0.04	0.20	0.09
Seniors	0.02	0.15	0.10

(a) If a student selected at random from the high school is a junior, what is the probability that she has never visited the museum?

(b) If a student selected at random from the high school has visited the museum three times, what is the probability that she is a senior?

Solution:

(a) We have $P(Junior) = 0.04 + 0.2 + 0.09 = 0.33$. Thus, $P(Never \mid Junior) = \frac{P(Never, Junior)}{P(Junior)} = \frac{0.04}{0.33} = 4/33$.

(b) The selected student belongs to the students that have visited a museum more than once. We have $P(\text{More than once}) = 0.04 + 0.04 + 0.09 + 0.10 = 0.27$. Then $P(\text{senior} \mid \text{More than once}) = \frac{P(\text{senior and More than once})}{P(\text{More than once})} = \frac{0.10}{0.27} = 10/27$.