1. basic and free variables: $\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2t \\ x_2 = 3t \\ x_3 = t \end{cases}$, x_1, x_2 are basic variables, x_3 is free variable.

2. invertible condition:

① A invertible ② $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ ③ $A \xrightarrow{\text{row operation}} I_n$

④ $A\vec{x} = \vec{b} \Rightarrow \vec{x}$ is unique ⑤ $AC = I_n$

3. Homogeneous System: $A\vec{x} = \vec{b}$, $\vec{b} = \vec{0}$

4. Change of basis:

Grant basis: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, Jennifer basis: $\begin{pmatrix}2\\1\end{pmatrix} \begin{pmatrix}-1\\1\end{pmatrix}$

$\begin{pmatrix}5/3\\1/3\end{pmatrix}$ in Jennifer basis means $\begin{pmatrix}2\\1\end{pmatrix} \cdot \frac{5}{3} + \begin{pmatrix}-1\\1\end{pmatrix} \cdot \frac{1}{3} = \begin{pmatrix}3\\2\end{pmatrix}$ in Grant basis.

A: Jennifer basis, $A\vec{x}$: translate $\vec{x}$ into Grant basis

$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\x\end{pmatrix}$, $A_G = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ in Grant basis

$\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix} A_J = A_G \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}$ make transfer to Jennifer basis

translate A_J into Grant basis

$A_J = \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}^{-1} A_G \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}$

5. $A_{n\times n}$ has n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, $|A| = \lambda_1\lambda_2\cdots\lambda_n$, $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

6. diagonal condition: $A_{n\times n}$

① n distinct eigenvalues ② eigenvalues of m multiplicity yeilds m eigenvectors ③ $P^{-1}AP = D$, P is invertible

7. normal equation: project $\vec{x}$ on subspace U, $A = (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)$

suppose $\vec{p} = proj_U \vec{x}$ $\begin{cases} A^T(\vec{x} - \vec{p}) = \vec{0} \\ \vec{p} = A\vec{\alpha} \end{cases} \Rightarrow A^T(\vec{x} - A\vec{\alpha}) = \vec{0} \Rightarrow A^TA\vec{\alpha} = A^T\vec{x} \Rightarrow \vec{\alpha} = ?, \vec{p} = ?$

8. spectral theorem:

① n eigenvectors orthonormal set ② A: orthogonally diagonalizable

③ A: symmetric

$\{\lambda_1, \cdots, \lambda_n\}$ is spectrum

9. spectral decomposition:

$A = PDP^T$ P orthogonal matrix

10. positive-definite: A symmetric, $\lambda_i > 0$

positive semi-definite: A symmetric, $\lambda_i \geq 0$

11. SVD: $\overset{A_{m\times n}}{A^T}\overset{n\times n}{A}$, $\Sigma_A = \begin{pmatrix} \sqrt{\lambda_1} & 0 & & 0 & 0 \\ 0 & \sqrt{\lambda_2} & & & \\ & & \ddots & \sqrt{\lambda_r} & \\ 0 & & & 0 & 0 \\ 0 & & & & \end{pmatrix}$, $A = \overset{m\times m}{P}\overset{m\times n}{\Sigma_A}\overset{n\times n}{Q^T}$ $\quad \overset{\text{eigenvector of } A^TA,\ \text{orthogonal matrix}}{Q = (q_1, q_2, \cdots, q_n)}$

$r(A^TA) = r(A)$ $\quad P = \left(\frac{1}{\|Aq_1\|}Aq_1, \cdots, \frac{1}{\|Aq_r\|}Aq_r, p_{r+1}, \cdots, p_m\right)$

$\overset{m\times n}{A}\overset{n\times n}{Q} = \overset{m\times m}{P}\overset{m\times n}{\Sigma_A}$

12. rank geometric property:

$\text{rank}(UA) = \text{rank}(AV) = \text{rank}(A)$, U and V are invertible, $\text{rank}(A^T) = \text{rank}(A)$

13. Euclidean norm: $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, $\|\alpha\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$

14. Compute orthogonal projection: normal equation

15. square root matrix: $\Sigma = PDP^T$, $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$

$\Sigma^{\frac{1}{2}} = P\tilde{D}P^T$, $\tilde{D} = \text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$

$(\Sigma^{\frac{1}{2}})^2 = P\tilde{D}P^TP\tilde{D}P^T = P\tilde{D}^2P^T = PDP^T = \Sigma$