Exercises Lecture 3 - Part II:

1. Urn A contains three red balls and two white balls, urn B has two red balls and five white balls. A fair coin is tossed; if it lands heads up, a ball is drawn from urn A, and otherwise a ball is drawn from urn B.

   a. What is the probability that a red ball is drawn?

   b. If a red ball is drawn, what is the probability that the coin landed heads up.

   Solution:

   Let H be the event that we get heads and T the event that we get tails. Let also A be the event that we draw from urn A and B the event that we draw from urn B. We know that $P(H) = P(A) = 1/2$ and $P(T) = P(B) = 1/2$. Let R be the event that we draw a red ball, and W the event that we draw a white ball. Based on the information provided by the exercise, we have the following probabilities: $P(R \mid A) = 3/5$, $P(W \mid A) = 2/5$, $P(R \mid B) = 2/7$, $P(W \mid B) = 5/7$.

   a. From the law of total probability we have: $P(R) = P(R \mid A) * P(A) + P(R \mid B) * P(B) = 3/5 \times 1/2 + 2/7 \times 1/2 = 0.44$

   b. From Bayes theorem: $P(A \mid R) = P(R \mid A) * P(A)/P(R) = 3/5 \times 1/2/0.44 = 0.68$.

2. Suppose a statistics class contains 70% male and 30% female students. It is known that in a test, 5% of males and 10% of females got an "A" grade. If one student from this class is randomly selected and observed to have an "A" grade, what is the probability that this is a male student?

   Solution:

   Let A_1 denote that the selected student is a male, and A_2 denote that the selected student is a female. Here the sample space $S = A_1 \cup A_2$. Let D denote that the selected student has an "A" grade. We are given $P(A_1) = 0.7$, $P(A_2) = 0.3$, $P(D \mid A_1) = 0.05$ and $P(D \mid A_2) = 0.10$. Then by the law of total probability we have: $P(D) = P(A_1)P(D \mid A_1) + P(A_2)P(D \mid A_2) = 0.035 + 0.030 = 0.065$. Now by the Bayes' rule:

$$\begin{aligned} P(A_1 \mid D) &= \frac{P(A_1)P(D \mid A_1)}{P(A_1)P(D \mid A_1) + P(A_2)P(D \mid A_2)} \\ &= \frac{0.7 \cdot 0.05}{0.065} = 0.538. \end{aligned}$$

This shows that even though the probability of a male student getting "A" grade is smaller than that for a female student, because of the larger number of male students in the class, a male student with an "A" grade has a greater probability of being selected than a female student with an "A" grade.

3. A box has three coins. One has two heads, one has two tails and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped and comes up heads.

   (a) What is the probability that the coin chosen is the two-headed coin?

   (b) What is the probability that if it thrown another time it will come up heads?

   Solution:

   Let H denote the coin with two heads, T denotes the coin with two tails and F denotes the fair coin. The coin is chosen at random and this means that $P(H) = P(T) = P(F) = 1/3$. Let R be the result of flipping the randomly selected coin, i.e., head or tails. Then the conditional probabilities of getting heads for the three coins are: $P(R \mid H) = 1$, $P(R \mid T) = 0$ and $P(R \mid F) = 0.5$.

   (a) We need the conditional probability: $P(H \mid R)$ which we can compute using the Bayes' theorem.

$$\begin{aligned} P(H \mid R) &= \frac{P(R \mid H)P(H)}{P(R)} \\ &= \frac{P(R \mid H)P(H)}{P(R \mid H)P(H) + P(R \mid T)P(T) + P(R \mid F)P(F)} \\ &= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 0 \cdot 1/3 + 1/2 \cdot 1/3} = 2/3. \end{aligned}$$

   (b) Let R_1 the result from flipping the randomly chosen coin the first time, heads or tails. Let R_2 the result from flipping the randomly chosen coin the second time, heads or tails. We want to compute: $P(R_2 \mid R_1)$. We will use again the Bayes' rule.

$$P(R_2 \mid R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)},$$

   where $P(R_1 \cap R_2) = P(R_1 \cap R_2 \mid F)P(F) + P(R_1 \cap R_2 \mid H)P(H) + P(R_1 \cap R_2 \mid T)P(T) = 1/2 \cdot 1/2 \cdot 1/3 + 1 \cdot 1/3 + 0 \cdot 1/3 = 1/12 + 1/3 = 5/12$ and $P(R_1) = 1 \cdot 1/3 + 0 \cdot 1/3 + 1/2 \cdot 1/3 = 3/6$.

   Then $P(R_2 \mid R_1) = (5/12)/(3/6) = 5/6$.

4. Toss ten coins. What is the probability of observing at least one Head?

Solution:

Let A the event that the ith coin shows H, $i = 1, 2, \ldots, 10$. Assuming that we toss the coins in such a way that they do not interfere with each other, this is one of the situations where all of the A_i may be considered mutually independent due to the nature of the tossing. Of course, the only way that there will not be at least one Head showing is if all tosses are Tails. Therefore,

$$\begin{aligned}
P(\text{at least one H}) &= 1 - P(\text{all T}) \\
&= 1 - P(A_1^C \cap A_2^C \cap \ldots A_{10}^C) \\
&= 1 - P(A_1^C) \cdot P(A_2^C) \cdot \ldots P(A_{10}^C) \\
&= 1 - (1/2)^{10}.
\end{aligned}$$