Mathematics For Statisticians Homework 2

1) (a) Use implicit differentiation to equation $x^2+y^2=1$:

$2x+2y\cdot y'=0$

Simplify above equation to get y':

$y'=-\frac{x}{y}$

(b) Use implicit differentiation to equation $x^2+y^2=9$:

$2x+2y\,y'=0$

So $y'=-\frac{x}{y}$

At point $(-2,\sqrt{5})$, we can get $y'=-\frac{x}{y}=-\frac{-2}{\sqrt{5}}=\frac{2}{\sqrt{5}}$, which is also the slope of tangent line at point $(-2,\sqrt{5})$

Use the point-slope equation to get the result:

$y-\sqrt{5}=\frac{2}{\sqrt{5}}[x-(-2)] \Rightarrow y=\frac{2\sqrt{5}}{5}x+\frac{9\sqrt{5}}{5}$

(c) Use Chain Rule to calculate $\frac{dV}{dt}$:

$$\frac{dV}{dt}=\frac{d(\frac{4}{3}\pi r^3)}{dt}=\frac{4}{3}\pi\cdot 3r^2\cdot\frac{dr}{dt}=4\pi r^2\cdot\frac{dr}{dt}$$

According to the condition $\frac{dV}{dt}=5$,

we can get $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}=5$

Simplify the equation: $\frac{dr}{dt}=\frac{5}{4\pi r^2}$

When $r=5$, $\frac{dr}{dt}=\frac{5}{4\pi r^2}=\frac{5}{4\pi\times 5^2}=\frac{1}{20\pi}$

(d) Suppose $S(t)$ is the distance between rocket and me

When $0\le t<3$, I move and rocket doesn't move, so $S(t)=4t$

When $t\ge 3$, both rocket and I move, the distance between rocket and me is the hypotenuse of right triangle. So we use Pythagorean theorem,

$$S(t)=\sqrt{(4t)^2+[20(t-3)]^2}=\sqrt{416t^2-2400t+3600}$$

Thus, $S(t)=\begin{cases} 4t & , 0\le t<3 \\ \sqrt{416t^2-2400t+3600}, & t\ge 3 \end{cases}$

so the rate of change of the distance is $S'(t)$:

$$S'(t)=\begin{cases} 4 & , 0\le t<3 \\ \dfrac{1}{2\sqrt{416t^2-2400t+3600}}\cdot(416\times 2t-2400), & t\ge 3 \end{cases}$$

$$=\begin{cases} 4 & , 0\le t<3 \\ \dfrac{416t-1200}{\sqrt{416t^2-2400t+3600}}, & t\ge 3 \end{cases}$$

therefore, $S'(1)=4$, $S'(4)=\dfrac{416\times 4-1200}{\sqrt{416\times 4^2-2400\times 4+3600}}=\dfrac{116\sqrt{41}}{41}\approx 18.12$

2)(a) Take the natural logarithm of both sides of the equation:

$\ln y=\ln x^x \Rightarrow \ln y=x\ln x$

Differentiate both side of the equation:

$\frac{1}{y}\cdot y'=1\cdot\ln x+x\cdot\frac{1}{x} \Rightarrow \frac{1}{y}\cdot y'=\ln x+1$

Multiply both sides of the equation by y:

$y\cdot\frac{1}{y}\cdot y'=y\cdot(\ln x+1) \Rightarrow y'=y\ln x+y$

Replace y by $y=x^x$:

$y'=x^x\ln x+x^x$

(b) Take the natural logarithm of both sides of the equation:

$\ln y=\ln x^{(\ln x)} \Rightarrow \ln y=(\ln x)^2$

Differentiate both side of the equation:

$\frac{1}{y}\cdot y'=2\ln(x)\cdot\frac{1}{x} \quad \Rightarrow \frac{1}{y}\cdot y'=\frac{2\ln x}{x}$

Multiply both sides of the equation by y:

$y\cdot\frac{1}{y}\cdot y'=\frac{2y\ln x}{x} \Rightarrow y'=\frac{2y\ln x}{x}$

Replace y by $y=x^{\ln(x)}$

$y'=\frac{2x^{\ln(x)}\ln x}{x}=2x^{\ln(x)-1}\ln x$

3) (a) According to the definition of concave up and concave down:

if f' is increasing over interval I, f is concave up over I

if f' is decreasing over interval I, f is concave down over I

Over interval $(-2,0),(2,4)$, y' is increasing, y is concave up

Over interval $(-4,-2),(0,2)$, y' is decreasing, y is concave down

(b) Over interval $(-4,-1),(1,3)$, y' is increasing, y is concave up

Over interval $(-6,-4),(-1,1)$, y' is decreasing, y is concave down

(c) $f(x)=3xe^{1-\frac{1}{4}x^2}$

(i) $f'(x)=3\cdot e^{1-\frac{1}{4}x^2}+3x\cdot e^{1-\frac{1}{4}x^2}\cdot(-\frac{1}{4}2x)$ Use product and chain rule

$=(-\frac{3}{2}x^2+3)e^{1-\frac{1}{4}x^2}$

According to the definition of critical point, we need to find points that let $f'(x)=0$ or $f'(x)$ undefined.

$f'(x)=(-\frac{3}{2}x^2+3)e^{1-\frac{1}{4}x^2}=0 \Rightarrow x=\pm\sqrt{2}$

The domain of $f(x)$ and $f'(x)$ is $\mathbb{R} \Rightarrow f'(x)$ is defined on everywhere

Thus, $x=\sqrt{2}$ and $x=-\sqrt{2}$ are critical points.

According to the expression of $f'(x)$, $-\frac{3}{2}x^2+3>0$ over $(-\sqrt{2},\sqrt{2})$, $-\frac{3}{2}x^2+3<0$ over $(-\infty,-\sqrt{2})$ and $(\sqrt{2},+\infty)$, $e^{1-\frac{1}{4}x^2}>0$ over $\mathbb{R}$,

so $f'(x)<0$ over $(-\infty,-\sqrt{2})$, $f'(x)>0$ over $(-\sqrt{2},\sqrt{2})$ and $f'(x)<0$ over $(\sqrt{2},+\infty)$,

$x=-\sqrt{2}$ is local minima and $x=\sqrt{2}$ is local maxima.

(ii) According to the solution of question (i), $f(x)$ is increasing over $(-\sqrt{2},\sqrt{2})$, $f(x)$ is decreasing over $(-\infty,-\sqrt{2})$ and $(\sqrt{2},+\infty)$

(iii) The first derivative is $f'(x)=(-\frac{3}{2}x^2+3)e^{1-\frac{1}{4}x^2}$, so the second derivatives is $f''(x)=-\frac{3}{2}\cdot 2x\cdot e^{1-\frac{1}{4}x^2}+(-\frac{3}{2}x^2+3)\cdot e^{1-\frac{1}{4}x^2}\cdot -\frac{1}{4}\cdot 2x$

$=(\frac{3}{4}x^3-\frac{9}{2}x)e^{1-\frac{1}{4}x^2}$.

At inflection point, $f''(x)=0$ or $f'(x)$ is undefined.

Since $f''(x)$ is defined for all real number, we only need to find where $f''(x)=0$. Solve the equation $(\frac{3}{4}x^3-\frac{9}{2}x)e^{1-\frac{1}{4}x^2}=0$, we get the root $x=0$, $x=-\sqrt{6}$ and $x=\sqrt{6}$.

Interval	Test Point	Sign of $f''(x)$	Conclusion
$(-\infty,-\sqrt{6})$	-3	$-$	$f(x)$ is concave down
$(-\sqrt{6},0)$	-1	$+$	$f(x)$ is concave up
$(0,\sqrt{6})$	1	$-$	$f(x)$ is concave down
$(\sqrt{6},+\infty)$	3	$+$	$f(x)$ is concave up

Thus, $(-\sqrt{6}, f(-\sqrt{6}))=(-\sqrt{6},-3\sqrt{6}e^{-\frac{1}{2}})$, $(0,f(0))=(0,0)$ and $(\sqrt{6},f(\sqrt{6}))=(\sqrt{6},3\sqrt{6}e^{-\frac{1}{2}})$ are inflection points.

(iv) According to the solution of (iii), the inflection points are $(-\sqrt{6},-3\sqrt{6}e^{-\frac{1}{2}})$, $(0,0)$ and $(\sqrt{6},3\sqrt{6}e^{-\frac{1}{2}})$

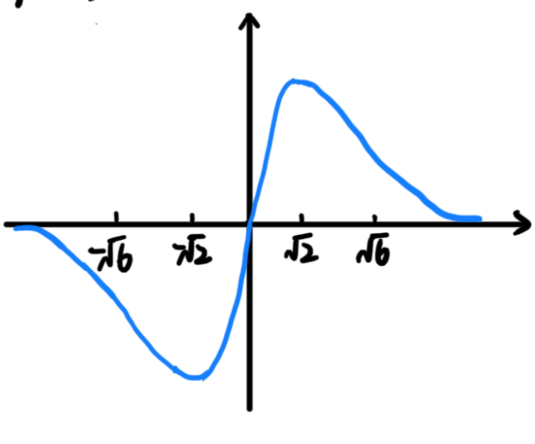
(v) Calculate x and y intercepts:

$x=0 \Rightarrow f(0)=0$; $f(x)=0 \Rightarrow x=0$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 3xe^{1-\frac{1}{4}x^2} = \lim_{x\to+\infty}\frac{3x}{e^{\frac{1}{4}x^2-1}} = \lim_{x\to+\infty}\frac{3}{e^{\frac{1}{4}x^2-1}\cdot\frac{1}{2}x} \text{ (L'Hôpital's rule)}$$
$$= 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3xe^{1-\frac{1}{4}x^2} = \lim_{x\to-\infty}\frac{3x}{e^{\frac{1}{4}x^2-1}} = \lim_{x\to-\infty}\frac{3}{e^{\frac{1}{4}x^2-1}\cdot\frac{1}{2}x} \text{ (L'Hôpital's rule)}$$
$$= 0$$

$f(-\sqrt{2})=3\cdot(-\sqrt{2})e^{1-\frac{1}{4}(-\sqrt{2})^2}=-3\sqrt{2}e^{\frac{1}{2}}$, $f(\sqrt{2})=3\sqrt{2}e^{1-\frac{1}{4}(\sqrt{2})^2}=3\sqrt{2}e^{\frac{1}{2}}$



(4) Suppose the length and width of printed area are x and y.

$x>0$, $y>0$

$\max\limits_{x,y} \; xy$

s.t. $(x+2\times2)(y+2\times3)=500$

We can get $y=\frac{500}{x+4}-6$

Replace y by $\frac{500}{x+4}-6$:

$\max\limits_{x} \; x(\frac{500}{x+4}-6)$

Suppose $g(x)=x(\frac{500}{x+4}-6)$,

Solve $g'(x)=0$, $1\times(\frac{500}{x+4}-6)+x\cdot500[-\frac{1}{(x+4)^2}]=0 \Rightarrow x=-4\pm\frac{10\sqrt{10}}{\sqrt{3}}$

$\because x>0 \;\therefore x=-4+\frac{10\sqrt{10}}{\sqrt{3}}\approx14.26 \;\therefore y=\frac{500}{x+4}-6=\frac{50\sqrt{3}}{\sqrt{10}}-6\approx21.39$

Thus, the optimal dimensions are approximately $14.26+2\times2=18.26$ cm and $21.39+2\times3=27.39$ cm.