MFS Practice Exam

1) Suppose $p(t)$ is the position of the particle at t.

$p(t) = 0 + \int_0^t v(x)\,dx = \int_0^t x^2-2x-8\,dx$

$= \left(\frac{1}{3}x^3 - x^2 - 8x\right)\Big|_0^t$

$= \frac{1}{3}t^3 - t^2 - 8t$

When $p(t)=0$:

$\frac{1}{3}t^3 - t^2 - 8t = 0 \Rightarrow t\left(\frac{1}{3}t^2 - t - 8\right) = 0 \Rightarrow t = 0$ or $\frac{3-\sqrt{105}}{2}$ or $\frac{3+\sqrt{105}}{2}$

$\because t \geq 0 \quad \therefore t = 0$ or $\frac{3+\sqrt{105}}{2}$

$\therefore$ when $t = \frac{3+\sqrt{105}}{2}$, the particle has position 0.

2) $D(g) = \mathbb{R}$

$\because$ In each interval of $(-\infty, 2]$, $(2,4)$ and $[4,+\infty)$, $g(t)$ are basic functions which are continuous on their domains.

$\therefore$ We only need to focus on the point where $t=2$ and $t=4$

①: $\lim_{t\to 2^-} g(t) = \lim_{t\to 2^-} (2t - t^2) = 2\times 2 - 2^2 = 0$

$\lim_{t\to 2^+} g(t) = \lim_{t\to 2^+} (t-3) = 2-3 = -1$

$\lim_{t\to 2^-} g(t) \neq \lim_{t\to 2^+} g(t)$

$\therefore \lim_{t\to 2} g(t)$ DNE

$\therefore g(t)$ is not continuous at $t=2$

②: $\lim_{t\to 4^-} g(t) = \lim_{t\to 4^-} (t-3) = 1$

$\lim_{t\to 4^+} g(t) = \lim_{t\to 4^+} e^{t-4} = 1$

$\therefore \lim_{t\to 4} g(t) = \lim_{t\to 4^-} g(t) = \lim_{t\to 4^+} g(t) = 1$

$\because g(4) = e^{4-4} = 1 = \lim_{t\to 4} g(t)$

$\therefore g(t)$ is continuous at $t=4$

Based on ① and ②, $g(t)$ is not continuous on its domain.

3) (a) $\vec{u} = (1,2)$, $f(x,y) = e^{x^2} \ln y$, $P = (1,2)$

$\because f_x(x,y) = \ln(y)\cdot e^{x^2}(2x) = 2x\ln(y)e^{x^2}$

$f_y(x,y) = e^{x^2}\frac{1}{y} = \frac{e^{x^2}}{y}$

$\therefore \nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (2x\ln(y)e^{x^2}, \frac{e^{x^2}}{y})$

$\therefore \nabla f(1,2) = (2e\ln 2, \frac{1}{2}e)$

$\because \|\vec{u}\| = \sqrt{1^2+2^2} = \sqrt{5} \neq 1$

$\therefore \vec{u}$ is not a unit vector and needs to be normalized.

$\vec{u}_n = \frac{\vec{u}}{\|\vec{u}\|} = (\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$

$\therefore D_{\vec{u}}f(1,2) = \nabla f(1,2)\cdot\vec{u}_n = (2e\ln 2, \frac{e}{2})\cdot(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}) = \frac{e\sqrt{5}[2\ln(2)+1]}{5}$

(b) $f(x,y) = x^3+x^2y-2y^3+6y$

$\because f_x(x,y) = 3x^2+2xy$, $f_y(x,y) = x^2-6y^2+6$

$$\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases} \Rightarrow \begin{cases} 3x^2+2xy=0 \\ x^2-6y^2+6=0 \end{cases} \Rightarrow (0,1) \text{ or } (0,-1) \text{ or } (\tfrac{2\sqrt{3}}{5}, -\tfrac{3\sqrt{3}}{5}) \text{ or } (-\tfrac{2\sqrt{3}}{5}, \tfrac{3\sqrt{3}}{5})$$

$\therefore$ The critical points are $(0,1), (0,-1), (\frac{2\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}), (-\frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{5})$

$\because f_{xx}(x,y) = 6x+2y$, $f_{xy}(x,y) = 2x$

$f_{yy}(x,y) = -12y$

$\therefore D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2$

$= (6x+2y)(-12y) - (2x)^2$

$= -4x^2 - 72xy - 24y^2$

① At point $(0,1)$,

$D(0,1) = -24 < 0 \Rightarrow (0,1)$ is a saddle point.

② At point $(0,-1)$,

$D(0,-1) = -24 < 0 \Rightarrow (0,1)$ is a saddle point

③ At point $(\frac{2\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5})$,

$D(\frac{2\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}) = 24 > 0$

$2\sqrt{3}$

$-\frac{}{5}\times\frac{}{5}$

$f_{xx}\left(\frac{2\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}\right) = \frac{6\sqrt{3}}{5} > 0$

$\therefore f\left(\frac{2\sqrt{3}}{5}, -\frac{3\sqrt{3}}{5}\right)$ is a local minimum.

④ At point $\left(-\frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{5}\right)$,

$D\left(-\frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{5}\right) = 24 > 0$

$f_{xx}\left(-\frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{5}\right) = -\frac{6\sqrt{3}}{5} < 0$

$\therefore f\left(-\frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{5}\right)$ is local maximum

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$-4\times\frac{12}{25} + 72\times\frac{18}{25} - 24\times\frac{27}{25}$

$\frac{600}{25}$