8.2.1 a. $\vec{c_1} = \begin{pmatrix}1\\1\end{pmatrix}^T$, $\frac{1}{\|\vec{c_1}\|}\vec{c_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^T$ $\quad \vec{c_1}\cdot\vec{c_2} = 1\times(-1)+1\times1=0$

$\vec{c_2} = \begin{pmatrix}-1\\1\end{pmatrix}^T$, $\frac{1}{\|\vec{c_2}\|}\vec{c_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}^T$

$$\begin{pmatrix}\frac{1}{\|\vec{c_1}\|}\vec{c_1}\\ \frac{1}{\|\vec{c_2}\|}\vec{c_2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

b. $\vec{c_1}\cdot\vec{c_2} = 3\times4+(-4)\times3=0$

$\frac{1}{\|\vec{c_1}\|}\vec{c_1} = \frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix}^T$, $\frac{1}{\|\vec{c_2}\|}\vec{c_2} = \frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}^T$, $\begin{pmatrix}\frac{1}{\|\vec{c_1}\|}\vec{c_1}\\ \frac{1}{\|\vec{c_2}\|}\vec{c_2}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$

f. $\vec{c_1}\cdot\vec{c_2} = 2\times1+1\times(-1)+(-1)\times1=0$, $\vec{c_2}\cdot\vec{c_3} = 1\times0+(-1)\times1+1\times1=0$, $\vec{c_1}\cdot\vec{c_3} = 2\times0+1\times1+(-1)\times1=0$

$\frac{1}{\|\vec{c_1}\|}\vec{c_1} = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}^T$, $\frac{1}{\|\vec{c_2}\|}\vec{c_2} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}^T$, $\frac{1}{\|\vec{c_3}\|}\vec{c_3} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}^T$

$$\begin{pmatrix}\frac{1}{\|\vec{c_1}\|}\vec{c_1}\\ \frac{1}{\|\vec{c_2}\|}\vec{c_2}\\ \frac{1}{\|\vec{c_3}\|}\vec{c_3}\end{pmatrix} = \begin{pmatrix}2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$$

g. $\vec{c_1}\cdot\vec{c_2} = -1\times2+2\times(-1)+2\times2=0$, $\vec{c_1}\cdot\vec{c_3} = -1\times2+2\times2+2\times(-1)=0$, $\vec{c_2}\cdot\vec{c_3} = 2\times2+(-1)\times2+2\times(-1)=0$

$\frac{1}{\|\vec{c_1}\|}\vec{c_1} = \frac{1}{3}\begin{pmatrix}-1\\2\\2\end{pmatrix}^T$, $\frac{1}{\|\vec{c_2}\|}\vec{c_2} = \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\frac{1}{\|\vec{c_3}\|}\vec{c_3} = \frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$, $\begin{pmatrix}\frac{1}{\|\vec{c_1}\|}\vec{c_1}\\ \frac{1}{\|\vec{c_2}\|}\vec{c_2}\\ \frac{1}{\|\vec{c_3}\|}\vec{c_3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}$

8.2.4 $\vec{x} = (x_1, x_2, x_3)$, $\begin{cases}\vec{r_1}\cdot\vec{x}=0\\ \vec{r_2}\cdot\vec{x}=0\end{cases} \Rightarrow \begin{cases}\frac{1}{3}x_1+\frac{2}{3}x_2+\frac{2}{3}x_3=0\\ \frac{2}{3}x_1+\frac{1}{3}x_2-\frac{2}{3}x_3=0\end{cases} \Rightarrow \begin{cases}x_1=2t\\ x_2=-2t\\ x_3=t\end{cases}$

$\because \|\vec{x}\|=1$ $\therefore (2t)^2+(-2t)^2+t^2=1$ $\therefore t=\frac{1}{3}$ or $-\frac{1}{3}$ $\therefore \vec{x} = \begin{pmatrix}2/3\\-2/3\\1/3\end{pmatrix}$ or $\begin{pmatrix}-2/3\\2/3\\-1/3\end{pmatrix}$

8.2.5 a. $C_A(x) = \begin{vmatrix}x & -1\\ -1 & x\end{vmatrix} = x^2-1=0 \Rightarrow x=1$ or $-1 \Rightarrow \lambda_1=1, \lambda_2=-1$

$(\lambda_1 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix}t\\t\end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix}1\\1\end{pmatrix}$

$(\lambda_2 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix}-t\\t\end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\vec{x_1}\cdot\vec{x_2} = 1\times(-1)+1\times1=0$, $\frac{1}{\|\vec{x_1}\|}\vec{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\frac{1}{\|\vec{x_2}\|}\vec{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$, $P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

b. $C_A(x) = \begin{vmatrix} x-1 & 1 \\ 1 & x-1 \end{vmatrix} = (x-2)x = 0 \Rightarrow x = 2, 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 0$

$(\lambda_1 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} -t \\ t \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$(\lambda_2 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} t \\ t \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x_1} \cdot \vec{x_2} = -1\times 1 + 1\times 1 = 0$, $\frac{1}{\|\vec{x_1}\|}\vec{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\frac{1}{\|\vec{x_2}\|}\vec{x_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $P = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ ✓

d. $C_A(x) = \begin{vmatrix} x-3 & 0 & -7 \\ 0 & x-5 & 0 \\ -7 & 0 & x-3 \end{vmatrix} = (x-10)(x-5)(x+4) = 0 \Rightarrow x = 10, 5, -4 \Rightarrow \lambda_1 = 10, \lambda_2 = 5, \lambda_1 = -4$

$(\lambda_1 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$(\lambda_2 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 & -7 \\ 0 & 0 & 0 \\ -7 & 0 & 2 \end{pmatrix}$

$(\lambda_3 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} -r \\ 0 \\ r \end{pmatrix} \Rightarrow \vec{x_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$2x_1 - 7x_3 = 0$
$-7x_1 + 2x_3 = 0$

$\vec{x_1} \cdot \vec{x_2} = 1\times 0 + 0\times 1 + 1\times 0 = 0$, $\vec{x_1} \cdot \vec{x_3} = 1\times(-1) + 0\times 0 + 1\times 1 = 0$, $\vec{x_2} \cdot \vec{x_3} = 0\times(-1) + 1\times 0 + 0\times 1 = 0$

$\frac{1}{\|\vec{x_1}\|}\vec{x_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\frac{1}{\|\vec{x_2}\|}\vec{x_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\frac{1}{\|\vec{x_3}\|}\vec{x_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $P = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$ ✓

g. $C_A(x) = \begin{vmatrix} x-5 & -3 & 0 & 0 \\ -3 & x-5 & 0 & 0 \\ 0 & 0 & x-7 & -1 \\ 0 & 0 & -1 & x-7 \end{vmatrix} = (x-8)^2(x-6)(x-2) = 0 \Rightarrow x = 8, 6, 2 \Rightarrow \lambda_1 = \lambda_2 = 8, \lambda_3 = 6, \lambda_4 = 2$

$(\lambda_1 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} s \\ s \\ t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{x_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$(\lambda_3 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \\ -r \\ r \end{pmatrix} \Rightarrow \vec{x_3} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$(\lambda_4 I - A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} -q \\ q \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x_4} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{x_1} \cdot \vec{x_2} = 0$, $\vec{x_1} \cdot \vec{x_3} = 0$, $\vec{x_1} \cdot \vec{x_4} = 0$, $\vec{x_2} \cdot \vec{x_3} = 0$, $\vec{x_2} \cdot \vec{x_4} = 0$, $\vec{x_3} \cdot \vec{x_4} = 0$

$\frac{1}{\|\vec{x_1}\|}\vec{x_1} = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ 1 \end{vmatrix}$, $\frac{1}{\|\vec{x_2}\|}\vec{x_2} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 \\ 0 \end{vmatrix}$, $\frac{1}{\|\vec{x_3}\|}\vec{x_3} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 \\ 0 \end{vmatrix}$, $\frac{1}{\|\vec{x_4}\|}\vec{x_4} = \frac{1}{\sqrt{2}}\begin{vmatrix} -1 \\ 1 \end{vmatrix}$, $P = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$

$$\begin{pmatrix}0\\0\end{pmatrix} \quad \begin{pmatrix}1\\1\end{pmatrix} \quad \begin{pmatrix}-1\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\end{pmatrix} \quad \begin{pmatrix}0&1&-1&0\\0&1&1&0\end{pmatrix}$$

h. $$C_A(x)=\begin{vmatrix}x-3&-5&1&-1\\-5&x-3&-1&1\\1&-1&x-3&-5\\-1&1&-5&x-3\end{vmatrix}=(x-8)^2x(x+4)=0 \Rightarrow x=8,0,-4 \Rightarrow \lambda_1=\lambda_2=8,\ \lambda_3=0,\ \lambda_4=-4$$

$$(\lambda_1 I-A)\vec{x}=\vec{0} \Rightarrow \vec{x}=\begin{pmatrix}s\\s\\t\\t\end{pmatrix}=s\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\1\\1\end{pmatrix} \Rightarrow \vec{x_1}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \vec{x_2}=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$(\lambda_3 I-A)\vec{x}=\vec{0} \Rightarrow \vec{x}=\begin{pmatrix}r\\-r\\-r\\r\end{pmatrix} \Rightarrow \vec{x_3}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$(\lambda_4 I-A)\vec{x}=\vec{0} \Rightarrow \vec{x}=\begin{pmatrix}-q\\q\\-q\\q\end{pmatrix} \Rightarrow \vec{x_4}=\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$$

$$\vec{x_1}\cdot\vec{x_2}=0,\ \vec{x_1}\cdot\vec{x_3}=0,\ \vec{x_1}\cdot\vec{x_4}=0,\ \vec{x_2}\cdot\vec{x_3}=0,\ \vec{x_2}\cdot\vec{x_4}=0,\ \vec{x_3}\cdot\vec{x_4}=0$$

$$\frac{1}{\|\vec{x_1}\|}\vec{x_1}=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \frac{1}{\|\vec{x_2}\|}\vec{x_2}=\frac{1}{\sqrt2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\ \frac{1}{\|\vec{x_3}\|}\vec{x_3}=\frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix},\ \frac{1}{\|\vec{x_4}\|}\vec{x_4}=\frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix},\ P=\begin{pmatrix}1/\sqrt2&0&1/2&-1/2\\1/\sqrt2&0&-1/2&1/2\\0&1/\sqrt2&-1/2&-1/2\\0&1/\sqrt2&1/2&1/2\end{pmatrix}$$