Examination Linear Algebra (4433LALG3)

Statistics and Data Science

Wednesday, 22 November 2023, 9:00–12:00

Name : ....

Student number : ....

- This is a closed-book exam.
- The exam consists of 15 questions.
- Your answers to questions 1–10 should be written down on the exam itself. You do not need to justify your answer unless explicitly instructed to do so.
- For the multiple choice questions, only one answer is correct.
- Answers to questions 11–15 must be carefully written down in the separate paper provided for this purpose, including argumentation and computations.
- A simple scientific calculator (not graphic/programmable) is allowed. Other electronic devices are not allowed.
- For each question, the maximum score is indicated on the margin.
- The exam total is 90 points. Your grade is computed as $1 + \frac{\#\text{ points}}{10}$.

Please do not open nor flip this booklet until instructed

Part I: Multiple-choice questions

Please mark your choice directly on this paper (only one option is correct).

(3) **1.** Consider the transformation $T(x,y) = \begin{bmatrix} x+y \\ -2xy \end{bmatrix}$. Which is the matrix A associated to the transformation T via the correspondence $T(\mathbf{x}) = A\mathbf{x}$?

(a) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$

(f) A does not exist because T is not linear

(3) **2.** Consider the vector space of 4×4 matrices $V = \mathrm{Mat}(4,4)$. Which of the following is a subspace of V?

$$U_1 = \{A \in V \mid \mathrm{rank}(A) = 4\}, \quad U_2 = \{A \in V \mid A \text{ is invertible}\}, \quad U_3 = \{A \in V \mid \mathrm{tr}(A) = 0\}.$$

(a) Only U_1

(b) Only U_2

(c) Only U_3

(d) All except U_1

(e) All except U_2

(f) All except U_3

(3) **3.** A linear transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ is obtained by projecting orthogonally onto the plane $U = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$. What can be said about the matrix A?

(a) $\det A = 0$, because $\mathrm{rank}\, A = 0$

(b) $\det A = 0$, because $\mathrm{rank}\, A = 2$

(c) $\det A = 1$, because $A^2 = A$

(d) $\det A = 1$, because $\mathrm{rank}\, A = 1$

(e) $\det A$ cannot be determined from the information provided

(f) $\det A$ does not exist because A is a 2×3 matrix

(3) **4.** Below are given three matrices:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Which of these matrices has an inverse?

(a) Only matrix A

(b) Only matrix B

(c) Only matrix C

(d) All except matrix A

(e) All except matrix B

(f) All except matrix C

(3) **5.** Consider the transformation T on $\mathbb{R}^2$ which first rotates the plane 45° counter-clockwise around the origin, and then reflects the plane in the line $y = x$. Which is the matrix A associated to the transformation T via the correspondence $T(\mathbf{x}) = A\mathbf{x}$?

(a) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(b) $A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(c) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(d) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(e) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(f) A does not exist because T is not linear

(3) **6.** In the videos by 3Blue1Brown, it is explained how vectors can be represented with respect to a different basis. In Chapter 13, we encounter a situation in which Grant represents vectors using the standard basis $\mathbf{e}_1, \mathbf{e}_2$, and Jennifer uses a different basis $\mathbf{b}_1, \mathbf{b}_2$.

Consider a linear transformation which is represented in Grant's language by the matrix A_{G}.

Given: $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$, which is the correct expression for the matrix A_{J},
representing the same transformation but in Jennifer's language?

(a) $A_{\mathrm{J}} = A_{\mathrm{G}}$

(b) $A_{\mathrm{J}} = A_{\mathrm{G}}B$

(c) $A_{\mathrm{J}} = A_{\mathrm{G}}B^{-1}$

(d) $A_{\mathrm{J}} = BA_{\mathrm{G}}B$

(e) $A_{\mathrm{J}} = BA_{\mathrm{G}}B^{-1}$

(f) $A_{\mathrm{J}} = B^{-1}A_{\mathrm{G}}B$

Part II: Short-answer questions

Please fill in your answer in the space provided. Do not include intermediate steps.

(8) **7.** Consider $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 3 & 5 \\ 1 & 1 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

For each of the following expressions, provide the **size** of the matrix/vector defined by the expression (you do not need to perform the computations).

If the expression provided is not well defined (e.g. it is not possible to perform one of the operations), explain which operation is not possible.

a. $A^\top BA$ ....

b. $BA + CB$ ....

c. $(A^\top + C)\,\mathbf{x}$ ....

d. $(\mathbf{x}^\top C\mathbf{z})^\top$ ....

(6) **8.** A study aims to examine the relationship between the length of yellow-spotted salamanders and the amount of tetrodotoxin excreted when threatened. Twelve salamanders are captured and studied. The variables body length, tail length, and total length (in cm), together with tetrodotoxin (in μg) are shown below (first 5 rows only).

```
> head(data, n=5L)
   body  tail  total   ttx
1   9.3   6.6   15.9  8.12
2   7.5   4.8   12.3  6.91
3  11.1   8.7   19.8  9.83
4   7.8   5.4   13.2  7.59
5  10.2   7.3   17.5  7.19
```

a. Based on the information available, what value do you **expect** for the rank of the 12×4 matrix containing the measurements? Justify your reasoning (you are not asked for a numerical computation).

....

b. Suppose we want to fit a simple linear regression model (including an intercept), for `ttx` as response variable and `total` as explanatory variable. Write down a formula for the least-squares estimates $\hat{\boldsymbol{\beta}}$ in terms of the 12×2 matrix X, whose columns are $\mathbf{1}_{12}$ and $\mathbf{x}_{\text{total}}$.

....

c. Write down a formula in terms of X for the 12×12 matrix P which represents orthogonal projection onto the subspace $U = \text{span}\,\{\mathbf{1}_{12}, \mathbf{x}_{\text{total}}\}$.

....

(8) **9.** Consider a singular value decomposition of some matrix A, given by $A = U\Sigma_A V^\top$, where

$$\Sigma_A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on the above information, give:

a. The size of A and of the Gram matrix $A^\top A$.

....

b. The rank of A.

....

c. The eigenvalues of $A^\top A$.

....

d. The assumptions made on the matrices U and V so that $A = U\Sigma_A V^\top$ is indeed a singular value decomposition of A. In other words, what does the statement of the singular value decomposition theorem say about U and V?

....

(6) **10.** Consider 3×3 matrices A, B, X, and the following R–output:

```
> A %*% B
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1

> D <- A %*% X %*% B
> D
     [,1] [,2] [,3]
[1,]   -2    0    0
[2,]    0    1    0
[3,]    0    0    5

> det(B)
[1] 0.5
```

Based on the above information, provide the following:

a. $\det A$ ....

b. $\operatorname{rank} A$ ....

c. $\det X$ ....

Part III: Long-answer questions

Please write down your solutions carefully in the separate paper provided. You must include argumentation and computations. Your solution must be organized and easy to follow.

(4) **11.** Suppose we are given $n \times n$ matrices A, B, C, X satisfying the equation

$$AX^\top B + C = I_n.$$

Moreover, it is known that A, B are invertible. Solve the equation for X in terms of A, B, C.

(10) **12.** Consider the following 4×6 matrix and 4–dimensional vector:

$$A = \begin{bmatrix} -1 & 2 & 1 & 2 & 1 & -1 \\ 1 & -2 & 2 & 7 & 2 & 4 \\ -2 & 4 & 3 & 7 & 1 & 0 \\ 3 & -6 & 1 & 6 & 4 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 1 \end{bmatrix}.$$

For the next questions, you may use the following R–output.

```
> A <- matrix(c(
    -1, 2, 1, 2, 1,-1,
     1,-2, 2, 7, 2, 4,
    -2, 4, 3, 7, 1, 0,
     3,-6, 1, 6 ,4, 1), byrow=TRUE, ncol=6)
> b <- c(0,3,3,1)
> gaussianElimination(A, b)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7]
[1,]    1   -2    0    1    0    0    1
[2,]    0    0    1    3    0    0    2
[3,]    0    0    0    0    1    0   -1
[4,]    0    0    0    0    0    1    0
```

a. Find <u>**one**</u> solution to the linear system $A\mathbf{x} = \mathbf{b}$.

b. Compute the rank and nullity of A.

c. Find a basis for the column space of A.

d. Find a basis for the null space of A.

(10) **13.** A matrix M is given by $M = \begin{bmatrix} 9 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 3 & 9 \end{bmatrix}$. Consider the vector $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

a. Show that $\mathbf{u}_1$ is an eigenvector of M and determine the corresponding eigenvalue.

b. The numbers $\lambda_2 = 9$ and $\lambda_3 = 5$ are also eigenvalues of M. Determine an eigenvector for each of these eigenvalues. You may use the following R output, but your answer must be **exact** (no decimal approximations).

```
> M <- matrix(c(9,1,0, 1,6,3, 0,1,9), ncol=3)

> gaussianElimination(M-9*diag(3))
     [,1] [,2] [,3]
[1,]    1    0    1
[2,]    0    1    0
[3,]    0    0    0

> gaussianElimination(M-5*diag(3))
     [,1] [,2]       [,3]
[1,]    1    0 -0.3333333
[2,]    0    1  1.3333333
[3,]    0    0  0.0000000
```

c. Find, if possible, a matrix P such that $P^{-1}MP$ is diagonal.

d. Answer TRUE or FALSE: There exists an orthogonal matrix Q such that $Q^\top MQ$ is diagonal. Justify your answer (you are <u>not</u> asked to do computations).

(12) **14.** Consider the following four-dimensional vectors:

$$\mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -2 \\ 3 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}.$$

a. Calculate the length of the vector **a**.

b. Are the vectors **a** and **b** orthogonal? Justify.

c. Find the projection of the vector **b** onto (the span of) the vector **a**.

d. Using the output provided, find a basis for the subspace $U = \text{span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

```
> a <- c(-1,1,2,0)
> b <- c(0,1,1,1)
> c <- c(-2,3,5,1)
> d <- c(2,-2,-4,0)
> X <- matrix(c(a,b,c,d), ncol=4)
> gaussianElimination(X)
     [,1] [,2] [,3] [,4]
[1,]    1    0    2   -2
[2,]    0    1    1    0
[3,]    0    0    0    0
[4,]    0    0    0    0
```

e. Use your answer in (d) to compute the orthogonal projection of **y** onto U (no extra R output is provided: you are asked to do this by hand).
*If you did **not** succeed in part (d), you may use the following as a basis for* U:

$$\{\begin{bmatrix} -1 & 0 & 1 & -1 \end{bmatrix}^\top, \begin{bmatrix} 1 & -2 & -3 & -1 \end{bmatrix}^\top\}.$$

(8) **15.** A random vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ can be transformed into a *standard normal vector*, $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, I_n)$, by setting

$$\mathbf{Z} = \Sigma^{-\frac{1}{2}}(\mathbf{X} - \boldsymbol{\mu}).$$

In the above equation, $\Sigma^{-\frac{1}{2}}$ denotes the (symmetric) square root of the precision matrix $Q = \Sigma^{-1}$.

Consider the positive-definite symmetric matrix $\Sigma = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

The eigenvalues and eigenvectors of Σ, and its inverse, are given below.

```
> Sigma <- matrix(c(3,2,0, 2,3,0, 0,0,4), 3,3)

> eigen(Sigma)$values
[1] 5  4  1

> eigen(Sigma)$vectors
          [,1] [,2]       [,3]
[1,] 0.7071068    0  0.7071068
[2,] 0.7071068    0 -0.7071068
[3,] 0.0000000    1  0.0000000

> Q <- Inverse(Sigma); Q
     [,1] [,2] [,3]
[1,]  0.6 -0.4 0.00
[2,] -0.4  0.6 0.00
[3,]  0.0  0.0 0.25
```

a. What does it mean when we say that a matrix is *positive-definite*?

b. Verify, by multiplying the matrices, that the above computation for $Q = \Sigma^{-1}$ is correct.

c. We can use the spectral decomposition $\Sigma = PDP^\top$ to write down a formula for $\Sigma^{-\frac{1}{2}}$. Indeed,

$$\Sigma^{-\frac{1}{2}} = P\widetilde{D}P^\top.$$

Write down explicitly what is the matrix $\widetilde{D}$ in the above equation.

d. Expand $\mathbf{x} = \begin{bmatrix} 2 & -1 & 8 \end{bmatrix}^\top$ in terms of the orthonormal basis given by the columns of P.