Exercises Lecture 8: Limiting Theorems

1. Let $X_1, \dots, X_{25}$ be a random sample from a $N(\mu = 37, \sigma = 45)$ distribution, and let $\bar{X}$ be the sample mean of these $n = 25$ observations. Reply to the following questions.

   (a) How is $\bar{X}$ distributed?

   (b) Find $P(\bar{X} > 43.1)$ using the tables of the standard normal distrbution. [You may check your answer with R.]

   Solution:

   (a) $\bar{X} \sim N(\mu = 37, \sigma = 45/\sqrt{25})$

   (b) The number from the Table of the Standard Normal should be close to the value printed below.

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   > pnorm(43.1, mean = 37, sd = 9, lower.tail = FALSE)
   [1] 0.2489563
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2. Suppose a certain manufacturer produces steel shafts for which the diameter has mean 0.3 inches and standard deviation 0.05 inches.

   (a) Determine the mean and standard deviation of the average diameter of 30 shafts.

   (b) Use the Central Limit Theorem to determine the probability that the average diameter of 30 shafts is greater than 0.31. For the computation use the tables of the standard normal distrbution. [You may check your answer with R.]

   Solution:

   (a) The mean of the average diameter of 30 shafts is 0.3 and the standard deviation is $\sigma/\sqrt{n} = 0.0091287$

   (b) Let $X_1, \dots, X_{30}$ be the diameter of the 30 shafts. We want to find $P(\bar{X} > 0.31)$, where $\bar{X}$ is approximately normally distributed with mean 0.3 and standard deviation $= 0.0091287$ by the Central limit theorem. Using R:

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   > pnorm(0.31, 0.3, 0.0091287, lower.tail=FALSE)
   [1] 0.1366606
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   Thus, $P(\bar{X} > 0.31) = 0.137$.

3. Suppose that the number of insurance claims, X, filled in a year is Poisson distributed with $E(X) = 10000$. Use the normal approximation to the Poisson to approximate $P(X > 10200)$. For the computation use the tables of the standard normal distrbution. [You may check your answer with R.]

Solution:
Limiting distribution is $X \sim N(mean = 10000, sd = 100)$, since we know for the Poisson $E(X) = Var(X) = 10000$. Then $P(X > 10200)$ is:

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[1] 0.02275013
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4. Consider 12 independent random variables $U_1, \ldots, U_{12}$ following a uniform distribution on $[0, 1]$. Let $X_i = (U_i - 0.5)$.
(1) Find the mean of X_i and the variance of X_i.
(2) Find the mean of $\sum_{i=1}^{12} X_i$ and the variance of $\sum_{i=1}^{12} X_i$.

Solution:
(1) $E(X_i) = E(U_i - 0.5) = 0.5 - 0.5 = 0$, and $Var(X_i) = Var(U_i) = 1/12$.
(2) Consider the sum of these 12 X_i. $E(\sum_{i=1}^{12} X_i) = \sum_{i=1}^{12} E(X_i) = 12 \times 0 = 0$.
$Var(\sum_{i=1}^{12} X_i) = \sum_{i=1}^{12} Var(X_i) = 12 \times 1/12 = 1$.

5. Suppose that a company ships packages that are variable in weight, with an average weight of 15kg and a standard deviation of 10. Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variables. Find the probability that 100 packages will have a total weight that does not exceed 1700 kg.

Solution:
Let X_i denote the ith package weight and $S = X_1 + X_2 + \ldots + X_{100}$. We want $P(S \leq 1700)$. We have that $E(S) = 100 \times E(X_i) = 100 \times 15 = 1500$. ALso $\mathrm{Var}(S) = \sum_{i=1}^{100} \mathrm{Var}(X_i) = 100 \times 10^2 = 10000$.

From the CLT we know that $S \sim N(100 * 15, 100 * 10^2)$. Then

$$P(S \leq 1700) = P(\frac{S - 1500}{\sqrt{10000}} \leq \frac{1700 - 1500}{\sqrt{10000}}) \approx P(Z \leq 2) = 0.9772.$$

6. Each minute a machine produces a length of rope with mean of 4 feet and standard deviation of 5 inches. Assuming that the amounts produced in different minutes are independent and

identically distributed, approximate the probability that the machine will produce at least 250 feet in one hour.

Solution:

Let Y the length of the rope in a minute. We know $E(Y) = 4$ feet and $Var(Y) = 5$ inches or $5/12$ feet. The length of rope produced in an hour $X = 60 \times Y$ has mean $60 \times 4=$ 240 feet and standard deviation is $\sqrt{60} \times 5/12$ feet. The probability that $X \geq 250$ is $1 - \Phi([250 - 240]/\sqrt{60} \times 5/12) = 1 - \Phi(3.098387) = 0.0009728858$.

7. Suppose that 75 percent of the people in a certain metropolitan area live in the city and 25 percent of the people live in the suburbs. If 1200 people attending a certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270?

   Solution:

   The total number of people from the subperbs attending the concert can be regarded as the sum of 1200 independent random variables, each with a Bernoulli distribution with $p = 1/4$. X will be approximately a normal with mean 1200/4=300 and variance $1200(1/4)(3/4) =$ 225. If we let $Z = (X - 300)/15$, then Z is approximately standard normal. Thus $P(X < 270) = P(Z < -2) \approx 1 - \Phi(2) = 0.0227$.

8. Suppose that the distribution of the number of defects on any given bolt of cloth is the Poisson distribution with mean 5, and the number of defects on each bolt is counted for a random sample of 125 bolts. Determine the probability that the average number of defects per bolt in the sample will be less than 5.5.

   Solution:

   Since the mean and the variance of a Poisson distribution are equal, the number of defects on any bolt has mean 5 and variance 5. Thus $\bar{X}_{125}$ on the 125 bolts will be approximately normal with mean 5 and variance $5/125 = 1/25$. If we let $Z = (\bar{X}_{125} - 5)/(1/5)$ then Z is approximately a standard normal: $P(\bar{X}_{125} < 5.5) = P(Z < 2.5) \approx \Phi(2.5) = 0.9938$.

9. Suppose that the number of minutes required to serve a customer at the checkout counter of a supermarket has an exponential distribution for which the mean is 3. Using the central limit theorem, approximate the probability that the total time required to serve a random sample of 16 customers will exceed one hour.

   Solution:

Let $X_1, \ldots, X_{16}$ the times to serve 16 customers. The parameter of the exponential distribution is $1/3$. The mean and variance of each Xi is 3 and 9 respectively. Let $Y = \sum_{i=1}^{16} X_i$ the total time. Then according to CLT Y is approximately normal with mean $16 \times 3 = 48$ and variance $16 \times 9 = 144$. $P(Y > 60) = 1 - \Phi(\frac{60-48}{144^{1/2}}) = 1 - \Phi(1) = 0.1587$. The actual distribution of Y is Gamma with parameters 16 and $1/3$. Then this probability is 0.1565.

10. Suppose that we model the ocurrence of defects on a fabric manufacturing line as a Poisson process with rate 0.01 per square foot. Use the central limit theorem to approximate the probability that one would find at least 15 defects in 2000 square feet of fabric.

    Solution:

    The number of defects X in 2000 square feet has the Poisson distribution with mean $2000 \times 0.01 = 20$. Accoring to the CLT, we will use a normal with mean 20 and variance 20. $P(X \geq 15) = 1 - \Phi(\frac{15-20}{20^{1/2}}) = 1 - \Phi(-1.1180) = 0.8682$. The actual Poisson probability is 0.8951.