① a. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$xI - A$

$Ax = \lambda x$

$(\lambda I - A)\vec{x} = \vec{0}$

b. $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3.3.1 a. $|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-2 \end{vmatrix} = (\lambda-4)(\lambda+1) = 0 \Rightarrow \lambda_1 = 4, \lambda_2 = -1$

solve $(\lambda_1 I - A)\vec{x} = \vec{0}$, $\left(\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{cc|c} 1 & -2/3 & 0 \\ -3 & 2 & 0 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$

$x_1 - \frac{2}{3}x_2 = 0 \Rightarrow \vec{x} = t\begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

solve $(\lambda_2 I - A)\vec{x} = \vec{0}$, $\left(\begin{array}{cc|c} -2 & -2 & 0 \\ -3 & -3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$

$x_1 + x_2 = 0 \Rightarrow \vec{x} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b. $|\lambda I - A| = \begin{vmatrix} \lambda-2 & 4 \\ 1 & \lambda+1 \end{vmatrix} = (\lambda+2)(\lambda-3) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 3$

solve $(\lambda_1 I - A)\vec{x} = \vec{0}$, $\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow$

$x_1 - x_2 = 0 \Rightarrow \vec{x} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

solve $(\lambda_2 I - A)\vec{x} = \vec{0}$, $\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 1 & 4 & 0 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow x_1 + 4x_2 = 0 \Rightarrow \vec{x} = t\begin{pmatrix} -4 \\ 1 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

C. $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 2$

$\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \vec{x} = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \begin{cases} x_1 - x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \vec{x} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\begin{cases} x_1 - 0.8x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \vec{x} = t\begin{pmatrix} 0.8 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x_3} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$

d. $\lambda_1 = \lambda_2 = \lambda_3 = 2$

$x_1 - x_2 + 3x_3 = 0 \Rightarrow \vec{x} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{x_2} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

e. no eigenvalue

f. $\lambda_1 = 2, \lambda_2 = \lambda_3 = -1$

$\begin{cases} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \Rightarrow \vec{x} = t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad \begin{cases} x_1 + 0.5x_3 = 0 \\ x_2 - 0.5x_3 = 0 \end{cases} \Rightarrow \vec{x} = t\begin{pmatrix} -0.5 \\ 0.5 \\ 1 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

3.3.3 $A\vec{v}=\lambda\vec{v} \Rightarrow (\lambda I-A)\vec{v}=\vec{0}$, when $\lambda=0$, $|\lambda I-A|=-|A|=0 \Rightarrow |A|=0 \Rightarrow A$ not inv

3.3.19 a. $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\vec{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $A\vec{v}=\begin{pmatrix} a_{11}v_1+a_{12}v_2+a_{13}v_3 \\ a_{21}v_1+a_{22}v_2+a_{23}v_3 \\ a_{31}v_1+a_{32}v_2+a_{33}v_3 \end{pmatrix}=\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$

When $\vec{v}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A\vec{v}=\begin{pmatrix} a_{11}+a_{12}+a_{13} \\ a_{21}+a_{22}+a_{23} \\ a_{31}+a_{32}+a_{33} \end{pmatrix}=\begin{pmatrix} s \\ s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=s\vec{v}$, s is an eigenvalue.

b. $A^T=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$, $A^T v=\begin{pmatrix} a_{11}v_1+a_{21}v_2+a_{31}v_3 \\ a_{12}v_1+a_{22}v_2+a_{32}v_3 \\ a_{13}v_1+a_{23}v_2+a_{33}v_3 \end{pmatrix}=\begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$

when $\vec{v}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A^T v=\begin{pmatrix} a_{11}+a_{21}+a_{31} \\ a_{12}+a_{22}+a_{32} \\ a_{13}+a_{23}+a_{33} \end{pmatrix}=\begin{pmatrix} s \\ s \\ s \end{pmatrix}=s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=s\vec{v}$, s is an eigenvalue of A^T, also for A