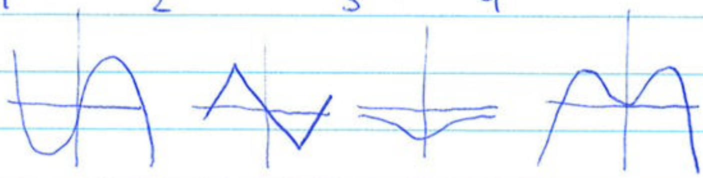
MfS Lecture 3

Last time: Limits, continuity, derivatives

Today: The rules of differentiation, Taylor approximations

Exercise: Match each graph with its derivative:



Recall: The sum rule: if $f(x)=g(x)+h(x)$ and g, h are diff., then so is f, and
$f' = g' + h'$.

Example: Let $h(x) = 3x^2 + 3^x$. Find h':
Answer: $h'(x) = 6x + 3^x \ln(3)$.

Today we'll see 3 more rules of differentiation.

Theorem (The Product Rule): If f, g are diff. functions, so too is the product, and

$$(f \cdot g)' = f g' + f' g.$$

Remark: Our general strategy is thus:
1. identify two functions f, g, s.t. $h = f \cdot g$.
2. compute f' and g'
3. compute $h' = f g' + f' g$

Example: $h(x) = x^2 e^x$. Find h'.
Answer: $h'(x) = 2x e^x + x^2 e^x$.

Example: $h(x) = \sqrt{x} \ln(x)$. Find h'.
Answer: $h'(x) = \frac{1}{2} x^{-\frac{1}{2}} \ln(x) + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \left(\frac{1}{2} \ln(x) + 1\right)$.

Theorem (The Quotient Rule): If f and g are diff functions, and $g(x) \neq 0$, then $\left(\frac{f}{g}\right)' = \frac{1}{g^2}(f'g - fg')$.

Remark: The numerator looks very similar to the product rule, albeit with a negative sign. Given a fcn h we suspect is a quotient of fcns, our strategy is thus

1. identify 2 fcns f, g s.t. $h = \frac{f}{g}$.
2. compute f' and g'
3. compute $\left(\frac{f}{g}\right)' = \frac{1}{g^2}(f'g - fg')$.

Example: Compute $h'(x)$ if a) $h(x) = \frac{x^2+1}{2x-3}$

b) $h(x) = \frac{10^x}{5x^{10}}$.

Answer: a) $h'(x) = \frac{2x(2x-3) - 2(x^2+1)}{(2x-3)^2} = \frac{2x^2 - 6x - 2}{(2x-3)^2}$

b) $h'(x) = \frac{\ln(10)10^x(5x^{10}) - 10^x 50x^9}{(5x^{10})^2} = \frac{\ln(10)x10^x - 10^x 10}{5x^{11}}$

Exercise: Find and interpret the derivative of each function:

1) $f(t) = {}^{1/t}/2^t$, $t \geq 1$, is the position of an object.

2) $g(x) = xe^x + \frac{x}{e^x}$ is the concentration of a chemical in the ocean after a spill, where x is the distance from the centre of the spill.

3) $C(x) = 2^x \ln(x) + 20$ is the cost of producing x laptops for a company, w/ $x > 1$.

Answer: 1) $\frac{1}{(\ln(2)2^t)^2}\left[-\frac{1}{t^2}2^t - \frac{\ln(2)2^t}{t}\right]$ is the velocity of the object.

2) $g'(x) = e^x + xe^x + \frac{e^x - xe^x}{e^{2x}}$ is the change in concentration of the spill wrt distance from the centre of the spill. ←

3) $C'(x) = 2^x \ln(2) \ln(x) + \frac{2^x}{x}$ is the marginal cost

We now turn to the last - and perhaps most common - rule of differentiation.

Theorem (The Chain Rule): If f, g are diff. fcns, so is $(f \circ g)$ and $(f(g(x)))' = f'(g(x))g'(x)$.

Remark: If you suspect a fcn is a composition of fcns, our strategy for computing its derivative is

1) identify 2 fcns f, g s.t. $h(x) = f(g(x))$.
2) compute $f'(x)$, $g'(x)$, and $f'(g(x))$.
3) compute $f'(g(x))g'(x) = h'(x)$.

Example: Let $h(x) = e^{2x^3+5x}$. What is $h'(x)$?
Answer: $(6x^2+5)e^{2x^3+5x}$.

Example: Let $h(x) = (2x-3)^9$. What is $h'(x)$?
Answer: $9(2x-3)^8 \cdot 2$

More complicated functions often require that we combine rules:

Example: Compute $h'(x)$, where $h(x) = x^2 e^{3x}$.
Answer: $h'(x) = 2xe^{3x} + 3x^2 e^{3x}$.

Exercise: The depth of water at a dock changes over time. It's depth is given by

$$D(t) = 5\sin\left(\frac{\pi}{6}t \cancel{+} - \frac{7\pi}{6}\right) + 8,$$

where t is hours after midnight. Find the rate at which the depth is changing at 6.00.
Hint: $\sin(t)$ has derivative $\cos(t)$ - you don't have to know this

$$D'(t) = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t \cancel{+} - \frac{7\pi}{6}\right).$$

$$D'(6) = \frac{5\sqrt{3}}{12}\pi \approx 2.3 \frac{m}{hr}.$$

Tangent lines

Let $f(x)$ be a function. Recall that the derivative of f is the slope of the tangent line of the fcn at a pt.
i.e. if $f(x) = x^2$, the slope of the tangent line to $f(x)$ at any pt is given by $f'(x) = 2x$.

Let $T_1(x)$ be the tangent line at a pt p of a fcn $f(x)$ so $T_1(x) = mx + b$.
- by definition, we have $m = f'(p)$.
- how can we find b?

Idea: We know the pt $(p, f(p))$ lies on the tangent line, so

$$f(p) = f'(p) \cdot p + b,$$

so

$$b = f(p) - f'(p) \cdot p.$$

Thus

$$T_1(x) = f'(p)x + (f(p) - f'(p) \cdot p) = f(p) + f'(p)(x-p).$$

Example: Find the tangent line to $f(x) = \sqrt{x+5}$ at $x=3$.
Answer: $f'(x) = \frac{1}{2}(x+5)^{-\frac{1}{2}}$. When $x=3$, $f'(3) = \frac{1}{2\sqrt{8}}$.
Also, $f(3) = \sqrt{8}$, so

$$T_1(x) = f(3) + f'(3)(x-3) = \sqrt{8} + \frac{1}{2\sqrt{8}}(x-3).$$

(show!)

Remark: We often refer to the tangent line as the 1st order approximation to $f(x)$ at p, or the 1st Taylor polynomial to f at p.

Higher order derivatives

Let $f(x)$ be a fcn, $f'(x)$ its derivative. Note that $f'(x)$ is also a fcn, so we can take its derivative to get the second order approximation of f, denoted $f''(x)$ or $\frac{d^2 f}{dx^2}$.

Example: $f(x) = 2x^3 + 5$. Find f''.
Answer: $f'(x) = 6x^2$

$$f''(x) = 12x.$$

Remark: Next lecture we'll interpret the second derivative, but for now we only need to know how to compute it.

We can keep going - we can find $f'''(x)$, $f''''(x)$, etc.
Normally, we write $f^{(n)}(x)$ to denote the 3rd or 4th derivatives or higher, as

$$f''''''''''(x)$$

is hard to look at.

Second order approximations

Definition: The second order approximation to a fcn $f(x)$ at a pt $x=p$ is

$$T_2(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2}(x-p)^2$$
$$= T_1(x) + \frac{f''(p)}{2}(x-p).$$

Example: Let $f(x) = x3^x$, $x>0$. Find $T_2(x)$ at $x=2$.

Answer $f(2) = 2\cdot 3^2 = 18$

$f'(x) = 3^x + x\ln(3)3^x$ $\quad f'(2) = 9 + 18\ln(3)$

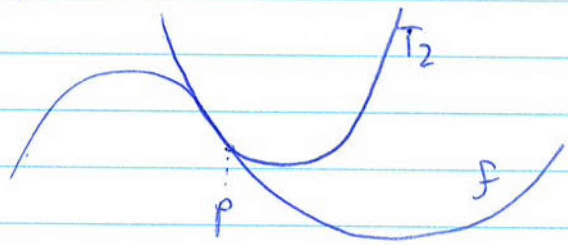
$f''(x) = \ln(3)3^x + \ln(3)3^x + x(\ln(3))^2 3^x$

$f''(2) = 18\ln(3) + 18(\ln(3))^2.$

Thus

$$T_2(x) = 18 + (9+18\ln(3))(x-2) + (9\ln(3) + 9(\ln(3))^2)(x-2)^2.$$

Remark: This is a parabola that approximates the function at p if x is close to p.



Because it can account for more "movement", i.e. the 2^{nd} derivative, it can often be more accurate than T_1.

We can keep adding higher order derivatives to get a (hopefully) better approximation.

i.e. ~~the 3rd order~~ Definition: Let k be a positive integer. We define $k!$ (k factorial) as $k! = k(k-1)(k-2)\cdots 2\cdot 1$.

If $k=0$, we set $0! = 1$.

Examples: $1! = 1$ $\quad 2! = 2\cdot 1 = 2$ $\quad 3! = 3\cdot 2\cdot 1 = 6$ $\quad 4! = 4\cdot 3\cdot 2\cdot 1 = 24$, etc.

Definition: The 3^{rd} order app. T_3 to f at p / the 3^{rd} Taylor polynomial to f at p is

$$T_3(x) = T_2(x) + \frac{f'''(p)}{3!}(x-p)^3.$$

What do you think T_4 is? $= T_3 + \frac{f^{(4)}(p)}{4!}(x-p)^4.$

In general: $T_n(x) = f(p) + f'(p)(x-p) + \ldots + \frac{f^{(n)}(p)}{n!}(x-p)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(x-p)^k.$$

Ex: Find the fourth Taylor polynomial to $f(x)=e^x$ at $x=0$.

Answer: $f^{(k)}=e^x \quad \forall\ k$, so

$$f^{(k)}(0)=e^0=1.$$

Thus

$$T_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$