MfS Lecture 6

Last time: Definite/indefinite integrals

Today: The substitution rule, multivariate functions

Recall the chain rule: $\frac{d}{dx}(F(g(x)) = F'(g(x))\,g'(x)$. Thus, $F(g(x))$ is an antiderivative of $F'(g(x))g'(x)$ i.e.

$$\int F'(g(x))\cdot g'(x)\,dx = F(g(x)) + C.$$

Suppose you want to integrate a fcn $h(x)$, and you recognize that $h(x) = F'(g(x))\cdot g'(x)$.

1. substitute $u = g(x)$, so $\frac{du}{dx} = g'(x)$ or $du = g'(x)dx$. Then
$$\int h(x)\,dx = \int F'(g(x))g'(x)\,dx = \int F'(u)\,du$$
2. integrate $F'(u)$ wrt u to get
$$\int F'(u)\,du = F(u) + C$$
for some constant C.
3. substitute back $u = g(x)$ to get
$$F(u) + C = F(g(x)) + C$$

The tricky part is choosing u. In general, choose u so that's its derivative appears up to a constant elsewhere in the integrand.

Example: Integrate $\int x^3 e^{x^4+\pi}\,dx$.

A: Note that the derivative of $x^4+\pi$ is almost x^3 – it only differs by a constant.

1. Let $u = x^4+\pi$, so $\frac{du}{dx} = 4x^3$, or $\frac{1}{4}du = x^3 dx$. Then
$$\int x^3 e^{x^4+\pi}\,dx = \int e^{x^4+\pi}(x^3 dx) = \int e^u\left(\tfrac{1}{4}du\right) = \tfrac{1}{4}\int e^u\,du$$
2. Thus,
$$\tfrac{1}{4}\int e^u\,du = \tfrac{1}{4}e^u + C.$$
3. Finally,
$$\tfrac{1}{4}e^u + C = \tfrac{1}{4}e^{x^4+\pi} + C.$$

Exercise: Try $\int x e^{x^2}\,dx$.

A: Note if $u = x^2$, $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$. Thus

$$\int x e^{x^2}\,dx = \tfrac{1}{2}\int e^u\,du$$
$$= \tfrac{1}{2}e^u + C$$
$$= \tfrac{1}{2}e^{x^2} + C.$$

Substitution rule for definite integrals

If we want to integrate $f(g(x))\cdot g'(x)$ over an an interval $[a,b]$, then we can either

a) integrate $f(g(x))\cdot g'(x)$ as before

b) or change the endpts as

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

That is, after substituting $u=g(x)$, we don't have to switch back to x at the end - we can just change our boundaries to $g(a)$ and $g(b)$!

Example: Calculate $\int_0^2 \sqrt{5x+1}\,dx$

A: Let $u=5x+1$, so $dx=\frac{1}{5}du$. We update our bounds of integration as $u(0)=1$ and $u(2)=11$. Then

$$\int_0^2 \sqrt{5x+1}\,dx = \frac{1}{5}\int_1^{11}\sqrt{u}\,du$$
$$= \frac{1}{5}\int_1^{11} u^{\frac{1}{2}}\,du = \frac{2}{15}\left[u^{3/2}\right]_1^{11}$$
$$= \frac{2}{15}\left[11^{3/2}-1\right].$$

Example: $\int_0^2 (x-1)^{25}\,dx$.

Let $u=x-1$, so $dx=du$. $u(0)=-1$, $u(2)=1$. Then

$$\int_0^2 (x-1)^{25}\,dx = \int_{-1}^{1} u^{25}\,du$$
$$= \frac{1}{26}\left[u^{26}\right]_{-1}^{1} = 0.$$

Ex: $\int_{e^2}^{e^6} \frac{(\ln(t))^4}{t}\,dt$.

A: $u=\ln(t)$, $u(e^2)=2$, $u(e^6)=6$, $\frac{dt}{t}=du$

$$\int_{e^2}^{e^6}\frac{(\ln(t))^4}{t}\,dt = \int_2^6 u^4\,du = \frac{1}{5}\left[u^5\right]_2^6 = \frac{1}{5}(6^5-6^2).$$

Ex: $\int_{\frac{1}{50}}^{2} \frac{e^{\frac{2}{w}}}{w^2}\,dw$.

A: $u=\frac{2}{w}$, $\frac{du}{dw}=-\frac{2}{w^2}$, or $-\frac{1}{2}du=\frac{dw}{w^2}$. $u(\frac{1}{50})=100$, $u(2)=1$.

$$\int_{\frac{1}{50}}^{2}\frac{e^{2/w}}{w^2}\,dw = -\frac{1}{2}\int_{100}^{1} e^u\,du$$
$$= \frac{1}{2}\int_1^{100} e^u\,du$$
$$= \frac{1}{2}\left(e^{100}-e\right).$$

Functions of multiple variables

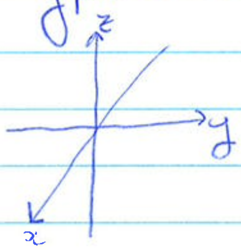
To model reality, we often have to work with multiple variables, i.e. the temperature in a specific location can depend on time, season, latitude, longitude, etc.

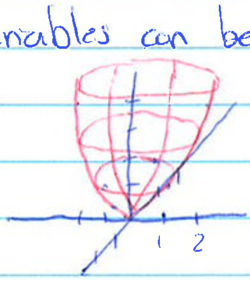
Definition: A multivariate function is a function $f(x_1,\dots,x_n)$ met meer then one variable in its definition

Examples: $f(x,y)=x+y$ $\quad f(t,k)=te^k$

$f(x,y,z)=x+z-y$ $\quad f(x,y)=2x^2$.

The graphs of such fcns of 2 variables can be drawn in 3 dimensions.





$z=x^2+y^2$.

Definition: The domain $D(f)$ of a function $f(x,y)$ is the set of all pairs (x,y) such that $f(x,y)$ exists

Example: Let $f(x,y)=\sqrt{x^2-y^2}$. Is $(0,-1)$ in $D(f)$? What is $D(f)$?

$f(0,-1)=\sqrt{-1}$ doesn't exist (in $\mathbb{R}$), so $(0,-1)\notin D(f)$.

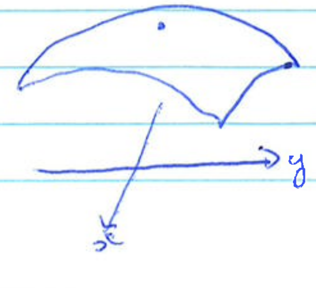
We require $x^2-y^2\geq 0$, or $|x|\geq|y|$. So,

$D(f)=\{(x,y)\in\mathbb{R}^2 : |x|\geq|y|\}$.

Partial derivatives

Recall: With 1 variable, the derivative $f'(x)$ of $f(x)$ is the instantaneous rate of change of f at x. If we move x by a small amount, $f'(x)$ is how fast the fcn grows.

Let $f(x,y)$ be a fcn of 2 variables. Now we can move in the x or y direction.



- $f(x+h,y)$ is a small shift in the direction of the x-axis
- $f(x,y+h)$ " " y-axis.

As we have 2 directions, we have 2 derivatives that we'll call partial derivatives.

Definition: The partial derivatives (with respect to x or y) of $f(x,y)$ are

$$\frac{\partial f}{\partial x} = \lim_{x \to 0} \frac{f(x+h,y) - f(x,y)}{h} = f_x(x,y)$$

$$\frac{\partial f}{\partial y} = \lim_{y \to 0} \frac{f(x,y+h) - f(x,y)}{h} = f_y(x,y).$$

Upshot/remarking: In practice, we compute the partial wrt x by treating y as a constant and taking the standard derivative wrt x.

Example: Compute f_x and f_y for:

a) $f(x,y) = x^2 + y^3 + 5xy^4$ $\quad f_x = 2x + 5y^4 \quad f_y = 3y^2 + 20xy^3$

b) $f(x,y) = x\ln(y)$ $\quad f_x = \ln(y) \quad f_y = \frac{x}{y}$

Higher order derivatives.

Just like w/ 1 variable, we can compute 2^{nd} order partial derivatives.

Ex: $f(x,y) = x^2 + y^3 + 5xy^4$.

We just saw $f_x(x,y) = 2x + 5y^4$ and $f_y(3y^2 + 20xy^3)$. Then

$$\frac{\partial f_x}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \cancel{\partial} f_{xx}(x,y) = 2 \qquad \frac{\partial f_x}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 20y^3$$

and

$$\frac{\partial f_y}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = 20y^3 \qquad \frac{\partial f_y}{\partial y} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = 6y^2 + 60xy^2.$$

So every fcn of 2 variables has possibly 4 second order derivatives.

We saw $f_{xy} = f_{yx}$. Is this always true?

Lemma: If f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.

Remark: Continuity is necessary, but for all fcns we see this will be the case.

Example: Compute f_{xy} and f_{yx} for $f(x,y) = x\ln(y) + e^{xy}$.

A: $f_x = \ln(y) + ye^{xy}$ $\qquad f_y = \frac{x}{y} + xe^{xy}$

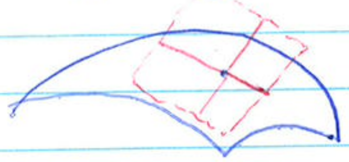
$f_{xy} = \frac{1}{y} + e^{xy} + xye^{xy}$ $\qquad f_{yx} = \frac{1}{y} + e^{xy} + xye^{xy}$.

Tangent planes

While in 1 variable we had the tangent line, now we have the tangent plane:

Def: Let $f(x,y)$ be a fcn and $P(a,b,f(a,b))$ a pt on the surface. The tangent plane of $f(x,y)$ at P is the surface that touches the graph of f at P and includes the tangent lines in the x- and y- directions.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



Ex: Compute the tangent plane of $f(x,y) = xy$ at $P = (1,-1,-1)$.

A: $f_x = y \quad f_y = x$.

$f_x(1,-1) = -1 \quad f_y(1,-1) = 1$.

so $z = f(1,-1) + f_x(x-1) + f_y(1,-1)(y+1) = -1 - 1(x-1) + 1(y+1)$

$= -1 - (x-1) + (y+1)$.

This generalises to more than 2 variables.

For next lecture, we need:

Definition: Let $f(x_1,\dots,x_n)$ be a fcn of n variables. The gradient of f, $\nabla f(x)$, is the vector

$$\nabla f(x)(x_1,\dots,x_n) = \langle f_{x_1}(x_1,\dots,x_n), \dots, f_{x_n}(x_1,\dots,x_n)\rangle$$

Remark: For now, a vector of length n is a list with n functions / numbers.

Definition: Let $(x_1,\dots,x_n) = \vec{x}$ and $\vec{y} = (y_1,\dots,y_n)$ be 2 vectors. The dot product of $\vec{x}$ and $\vec{y}$ is $\vec{x}\cdot\vec{y} = x_1y_1 + \dots + x_ny_n = \sum_{k=1}^{n} x_k y_k$.

Ex: $\vec{x} = (-1,3)$, $\vec{y} = (2,4)$ $\quad \vec{x}\cdot\vec{y} = (-1)(2) + (3)(4) = 10$.

Fix a pt $(a_1,\dots,a_n)$ in $D(f)$. Let $\Delta x_i = x_i - a_i$, set $\vec{h} = (\Delta x_1,\dots,\Delta x_n)$, $\vec{x} = (x_1,\dots,x_n)$. Then the 1st order app. is

$$f(\vec{x}+\vec{h}) \approx f(\vec{x}) + \nabla f(\vec{x})\cdot\vec{h}.$$