MATHEMATICS FOR STATISTICIANS HOMEWORK 1

[35/35] Please upload a typewritten or (legible!) scanned copy of your homework on Brightspace. Collaboration is encouraged, but please ensure you state at the start of your homework with whom you've collaborated. For full marks you must provide a justification of your answers, i.e. only a (correct) answer without any work shown will not be considered sufficient.

1) [14/14]
   (a) [4/4] Rewrite the following expression into a single logarithm with a coefficient of 1:
$$2\log_3(x+y) + 6\log_3(x) - \frac{1}{3}.$$
SOLUTION: We note that $2\log_3(x+y) = \log_3((x+y)^2)$ [1 pt] and similarly $6\log_3(x) = \log_3(x^6)$ [1 pt].
Furthermore, note that $1/3 = \log_3(3^{1/3})$ [1 pt].
Hence, using the properties of the logarithm, we find
$$2\log_3(x+y)+6\log_3(x)-\frac{1}{3} = \log_3((x+y)^2)+\log_3(x^6)-\log_3(3^{1/3}) = \log_3(\frac{x^6(x+y)^2}{3^{1/3}}).[1pt]$$
   (b) A quantity of a radioactive substance is modelled by the equation $Q(t) = Q_0e^{kt}$, where t is time and Q_0 is the initial quantity. The constant k depends on the substance in question. A given radioactive substance begins with 2kg, and after 1750 years, there is 1.5kg remaining.
      (i) [4/4] Determine the exponential decay equation for this substance. i.e. What are Q_0 and k? Leave your answer in exact form.
SOLUTION: Clearly $Q_0 = 2$ [1 pt] and when $t = 1750$, we're told $Q = 1.5$. Thus, $1.5 = 2e^{1750k}$, or $0.75 = e^{1750k}$ [1 pt].
Take the logarithm of both sides to get $\log(0.75) = 1750k$ [1 pt], or $k = \log(0.75)/1750 = 1.644 \times 10^{-4}$ [1 pt].
      (ii) [3/3] How long will it take for half of the substance to decay? Leave your answer in exact form.
SOLUTION: i.e. how long will it take for $Q = 1$? [1 pt] In this case, we have $0.5 = e^{kt}$, or $\log(0.5) = kt$ [1 pt].
Thus $t = 1750\log(0.5)/\log(0.75) = 4216.486$ [1 pt].
      (iii) [3/3] How long will it take until there is only 250g remaining? Leave your answer in exact form.
SOLUTION: Now we're told $Q = 0.25$ [1 pt], or $0.125 = e^{kt}$ [1 pt]. Thus, proceeding as in part ii, we have $t = 1750\log(0.125)/\log(0.75) = 12649.459$ [1 pt].

2) [15/15] Determine the following limits. You do *not* have to use the epsilon-delta definition.
   (a) [5/5] $\lim\limits_{x\to\infty} \frac{3x^4+6x^2+3x+12}{-2x^4-6x^3+\pi x-27} + 2^{-x} + 3$
SOLUTION: [2 pts for the limit of the rational function] [2 pts for the other two limits]
$$\lim_{x\to\infty} \frac{3x^4+6x^2+3x+12}{-2x^4-6x^3+\pi x-27} + 2^{-x} + 3$$

$$= \lim_{x\to\infty} \frac{3x^4+6x^2+3x+12}{-2x^4-6x^3+\pi x-27} + \lim_{x\to\infty} 2^{-x} + \lim_{x\to\infty} 3$$
$$= \left(\lim_{x\to\infty} \frac{3+\frac{6}{x^2}+\frac{3}{x^3}+\frac{12}{x^4}}{-2-\frac{6}{x}+\frac{\pi}{x^3}-\frac{27}{x^4}}\right) + 0 + 3$$
$$= \frac{\lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty} \frac{6}{x^2} + \lim\limits_{x\to\infty} \frac{3}{x^3} + \lim\limits_{x\to\infty} \frac{12}{x^4}}{-\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{6}{x} + \lim\limits_{x\to\infty} \frac{\pi}{x^3} - \lim\limits_{x\to\infty} \frac{27}{x^4}} + 3$$
$$= \frac{3+0+0+0}{-2-0+0-0} + 3 = -\frac{3}{2} + 3 = \frac{3}{2}$$

[1 pt for final answer]

(b) [5/5] $\lim\limits_{x\to 3} \frac{1}{(x-3)^3}$

SOLUTION: $\lim\limits_{x\to 3} \frac{1}{(x-3)^3}$: For $x \to 3$ we have that $(x-3) \to 0$ [1 pt], and thus also $(x-3)^3 \to 0$. Hence the function f blows up if $x \to 3$ [1 pt].
Note that for all $x < 3$ we have that $f(x) < 0$, hence $\lim\limits_{x\uparrow 3} \frac{1}{(x-3)^3} = -\infty$. [1 pt]

Also note that for all $x > 3$ we have that $f(x) > 0$, hence $\lim\limits_{x\downarrow 3} \frac{1}{(x-3)^3} = +\infty$. [1 pt]

Therefore the limit $\lim\limits_{x\to 3} \frac{1}{(x-3)^3}$ does not exist [1 pt].

(c) [5/5] $\lim\limits_{x\to 2} \frac{(x^2-x-2)(x^2-5x+6)}{x^2-4x+4}$

SOLUTION: We see that [1 pt]
$$\frac{(x^2-x-2)(x^2-5x+6)}{x^2-4x+4} = \frac{(x+1)(x-2)(x-3)(x-2)}{(x-2)^2}[1pt].$$
Hence we can write, for $x \neq 2$ [1 pt]:
$$\frac{(x^2-x-2)(x^2-5x+6)}{x^2-4x+4} = (x+1)(x-3)[1pt],$$
and thus
$$\lim_{x\to 2} \frac{(x^2-x-2)(x^2-5x+6)}{x^2-4x+4} = \lim_{x\to 2}(x+1)(x-3)[1pt] = (2+1)(2-3) = 3\cdot -1 = -3.$$
[1 pt]

3) [6/6] Using the definition of the derivative as a limit, compute the derivative of

(a) [3/3] $f(x) = \frac{1}{x+1}$

SOLUTION: [1 pt per bullet]

- The limit is
$$\lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}.$$
- Rewrite the numerator as $\frac{x+1-(x+h+1)}{(x+1)(x+h+1)} = \frac{-h}{(x+1)(x+h+1)}$.
- Cancel the h's out, and let h to go 0 in the denominator to get the answer.

(a) [3/3] $h(t) = 2t^2 + 2t$ at the point $t = 7$.

SOLUTION: [1 pt per bullet]

- The limit is
$$\lim_{h\to 0} \frac{2(7+h)^2 + 2(7+h) - 2(7)^2 - 2(7)}{h}.$$
- Expand the numerator to get $2h^2 + 28h + 2h = 2h^2 + 30h$.
- Cancel the h's out, and let h to go 0 in the denominator to get 30.