1.3.5 a. $\begin{cases} x_1+2x_2 - \frac{1}{3}x_5=0 \\ x_3 + \frac{2}{3}x_5=0 \\ x_4+x_5=0 \end{cases} \Rightarrow \begin{cases} x_1=-2s+\frac{1}{3}t \\ x_2=s \\ x_3=-\frac{2}{3}t \\ x_4=-t \\ x_5=t \end{cases} \Rightarrow \vec{x}=s\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}\frac{1}{3}\\0\\-\frac{2}{3}\\-1\\1\end{pmatrix}=s\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\-2\\-3\\3\end{pmatrix}$ ✓

b. $\begin{cases} x_1+2x_2+2x_4+3x_5=0 \\ x_3+x_4+2x_5=0 \end{cases} \Rightarrow \begin{cases} x_1=-2r-2s-3t \\ x_2=r \\ x_3=-s-2t \\ x_4=s \\ x_5=t \end{cases} \Rightarrow \vec{x}=r\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+s\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-3\\0\\-2\\0\\1\end{pmatrix}$ ✓

c. $\begin{cases} x_1+2x_4=0 \\ x_2-x_4=0 \\ x_3-x_4-x_5=0 \end{cases} \Rightarrow \begin{cases} x_1=-2s \\ x_2=\cancel{t}\,s \\ x_3=s+t \\ x_4=s \\ x_5=t \end{cases} \Rightarrow \vec{x}=s\begin{pmatrix}-2\\\cancel{0}\,1\\1\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\\cancel{1}\,0\\1\\0\\1\end{pmatrix}$

d. $\begin{cases} x_1+x_4=0 \\ x_2-2x_3-3x_4=0 \\ x_5=0 \end{cases} \Rightarrow \begin{cases} x_1=-\cancel{s}\,t \\ x_2=2s+3t \\ x_3=s \\ x_4=t \\ x_5=0 \end{cases} \Rightarrow \vec{x}=s\begin{pmatrix}\cancel{-1}\,0\\2\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}\cancel{0}\,-1\\3\\0\\1\\0\end{pmatrix}$

5.1.1 a. $U=\{(1,s,t)\mid s,t\in\mathbb{R}\}$

$(0,0,0)\notin U \Rightarrow U$ not subspace ✓

b. $U=\{(0,s,t)\mid s,t\in\mathbb{R}\}$

① $(0,0,0)\in U$

② $(0,s_1,t_1)+(0,s_2,t_2)=(0,s_1+s_2,t_1+t_2)\in U$

③ $a(0,s,t)=(0,as,at)\in U$

$\Rightarrow U$ is subspace ✓

c. $U=\{(r,s,t)\mid -r+3s+2t=0,\ r,s,t\in\mathbb{R}\}$

① $-0+3\cdot0+2\cdot0=0\Rightarrow(0,0,0)\in U$

② $(r_1,s_1,t_1)+(r_2,s_2,t_2)=(r_1+r_2,s_1+s_2,t_1+t_2)$, $-(r_1+r_2)+3(s_1+s_2)+2(t_1+t_2)=(-r_1+3s_1+2t_1)$

$+(-r_2+3s_2+2t_2)=0+0=0\Rightarrow(r_1,s_1,t_1)+(r_2,s_2,t_2)\in U$

③ $a(r,s,t)=(ar,as,at)$, $-ar+3as+2at=a(-r+3s+2t)=a\cdot0=0\Rightarrow$

$a(r,s,t)\in U$

①+②+③ $\Rightarrow U$ is subspace ✓

d. $U=\{(r,3s,r-2)\mid r,s\in\mathbb{R}'\}$

① $(0,0,0)\notin U \Rightarrow U$ is not subspace ✓

e. $U=\{(r,0,s)\mid r^2+s^2=0,\ r,s\in\mathbb{R}\}$ $r^2+s^2=0 \Rightarrow r=s=0 \Rightarrow U=\{(0,0,0)\}$ is a subspace

① $0^2+0^2=0 \Rightarrow (0,0,0)\in U$

② $(r_1,0,s_1)+(r_2,0,s_2)=(r_1+r_2,0,s_1+s_2)$, $(r_1+r_2)^2+(s_1+s_2)^2=r_1^2+r_2^2+s_1^2+s_2^2+2r_1r_2+2s_1s_2$

$=2r_1r_2+2s_1s_2\neq 0 \Rightarrow (r_1,0,s_1)+(r_2,0,s_2)\notin U \Rightarrow U$ is not subspace ✗

f. $U=\{(2r,-s^2,t)\mid r,s,t\in\mathbb{R}\}$

???

① $r=s=t=0 \Rightarrow (0,0,0)\in U$

② $(2r_1,-s_1^2,t_1)+(2r_2,-s_2^2,t_2)=(2(r_1+r_2),-(\sqrt{s_1^2+s_2^2})^2,t_1+t_2)\in U$

③ $a(2r,-s^2,t)=(2(ar),-(\sqrt{as^2})^2,at)\in U$ $a<0,\ -as^2\geq 0$ can't be written as $-s^2$

①+②+③ $\Rightarrow U$ is subspace $a(2r,-s^2,t)\notin U$

5.1.2. a. $k_1\vec{y}+k_2\vec{z}=\vec{x} \Rightarrow \begin{cases} k_1 = 2 \\ k_2 = -1 \\ 0 = 0 \\ k_1+k_2 = 1 \end{cases} \Rightarrow \begin{cases} k_1=2 \\ k_2=-1 \end{cases} \Rightarrow \vec{x}=2\vec{y}-\vec{z}$ ✓

b. $k_1\vec{y}+k_2\vec{z}=\vec{x} \Rightarrow k_1\begin{pmatrix}2\\-1\\0\\2\end{pmatrix}+k_2\begin{pmatrix}1\\-1\\-3\\1\end{pmatrix}=\begin{pmatrix}1\\2\\15\\11\end{pmatrix} \Rightarrow \begin{cases} 2k_1+k_2=1 \\ -k_1-k_2=2 \\ -3k_2=15 \\ 2k_1+k_2=11 \end{cases} \Rightarrow$ no solution $\Rightarrow \vec{x}\notin \operatorname{span}\{\vec{y},\vec{z}\}$ ✓

c. $k_1\vec{y}+k_2\vec{z}=\vec{x} \Rightarrow k_1\begin{pmatrix}2\\1\\-3\\5\end{pmatrix}+k_2\begin{pmatrix}-1\\0\\2\\-3\end{pmatrix}=\begin{pmatrix}8\\3\\-13\\20\end{pmatrix} \Rightarrow \begin{cases} 2k_1-k_2=8 \\ k_1=3 \\ -3k_1+2k_2=-13 \\ 5k_1-3k_2=20 \end{cases} \Rightarrow$ no solution $\Rightarrow \vec{x}\notin \operatorname{span}\{\vec{y},\vec{z}\}$

d. $k_1\vec{y}+k_2\vec{z}=\vec{x} \Rightarrow k_1\begin{pmatrix}2\\-1\\0\\5\end{pmatrix}+k_2\begin{pmatrix}-1\\2\\2\\-3\end{pmatrix}=\begin{pmatrix}2\\5\\8\\3\end{pmatrix} \Rightarrow \begin{cases} 2k_1-k_2=2 \\ -k_1+2k_2=5 \\ 2k_2=8 \\ 5k_1-3k_2=3 \end{cases} \Rightarrow \begin{cases} k_1=3 \\ k_2=4 \end{cases} \Rightarrow \vec{x}=3\vec{y}+4\vec{z}$ ✓

5.1.3. a. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}+k_4\vec{x_4}=\vec{0} \Rightarrow \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \mathbb{R}=\operatorname{span}\{\vec{x_1},\dots,\vec{x_4}\}$ ✓

$a\vec{x_1}+b\vec{x_2}+c\vec{x_3}+d\vec{x_4}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4$

b. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}+k_4\vec{x_4}=\vec{0} \Rightarrow \begin{pmatrix}1&-2&0&1\\3&1&2&-4\\-5&0&1&5\\0&0&-1&0\end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}t\\t\\0\\t\end{pmatrix}=t\begin{pmatrix}1\\1\\0\\1\end{pmatrix} \Rightarrow \mathbb{R}\neq\operatorname{span}\{\vec{x_1},\dots,\vec{x_4}\}$ ✓

5.2.1 a. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}=\vec{0} \Rightarrow \begin{pmatrix}1&3&3\\-1&2&5\\0&-1&-2\end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$ independent ✓

b. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$ independent

c. $\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=t\begin{pmatrix}-2\\1\\1\end{pmatrix} \Rightarrow -2\vec{x_1}+\vec{x_2}+\vec{x_3}=\vec{0}$ dependent

d. $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=t\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix} \Rightarrow -\vec{x_1}+\vec{x_2}-\vec{x_3}+\vec{x_4}=\vec{0}$ dependent

5.2.3. a. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}=\vec{0} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 9 \\ 2 & 0 & -6 \\ 0 & 3 & 6 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=t\begin{pmatrix}3\\-2\\1\end{pmatrix} \Rightarrow 3\vec{x_1}-2\vec{x_2}+\vec{x_3}=\vec{0}$

dimension $=2$, basis $=\{\vec{x_1},\vec{x_2}\}$

b. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}=\vec{0} \Rightarrow \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 7 \\ 0 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=t\begin{pmatrix}-3\\-4\\1\end{pmatrix} \Rightarrow -3\vec{x_1}-4\vec{x_2}+\vec{x_3}=\vec{0}$

dimension $=2$, basis $=\{\vec{x_1},\vec{x_2}\}$

5.2.6 a. $k_1\vec{x_1}+k_2\vec{x_2}=\vec{0} \Rightarrow \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \vec{x_1},\vec{x_2}$ are independent $\Rightarrow$ be a basis

b. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}=\vec{0} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \vec{x_1},\vec{x_2},\vec{x_3}$ are a basis

c. $k_1\vec{x_1}+k_2\vec{x_2}+k_3\vec{x_3}=\vec{0} \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}\vec{k}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \vec{k}=\begin{pmatrix}t\\t\\0\end{pmatrix}=t\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow \vec{x_1}=-\vec{x_2} \Rightarrow \vec{x_1},\vec{x_2},\vec{x_3}$ are not a basis.

5.4.1 a. $\begin{pmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ rank(col A) $=2$, basis col A $=$?? $\left\{\begin{pmatrix}2\\2\\4\\0\end{pmatrix},\begin{pmatrix}-4\\-1\\-5\\-1\end{pmatrix}\right\}$

b. $\begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -0.5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ rank(col A) $=2$, basis of col A $=\left\{\begin{pmatrix}2\\-2\\4\\-6\end{pmatrix},\begin{pmatrix}1\\1\\3\\0\end{pmatrix}\right\}$

c. $\begin{pmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1.75 & 0.75 \\ 0 & 0 & 1 & -0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank(col A) $=2$, basis of col A $=\left\{\begin{pmatrix}1\\2\\0\\-1\end{pmatrix},\begin{pmatrix}5\\-2\\-12\\7\end{pmatrix}\right\}$

d. $\begin{pmatrix} 1 & 2 & -1 & 3 \\ -3 & -6 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, rank(col A) $=2$, basis of col A $=\left\{\begin{pmatrix}1\\-3\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix}\right\}$

5.4.2 a. $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, basis $=\left\{\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}$

$$\begin{pmatrix} \cdots \\ 0 & 5 & 5 \\ 3 & 1 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} \cdots \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \ldots \left\{ \begin{pmatrix} \vdots \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} \vdots \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} \vdots \\ 5 \\ 7 \end{pmatrix} \right\}$$

b. $$\begin{pmatrix} 1 & 3 & 1 & 5 \\ -1 & 1 & 1 & 1 \\ 2 & 4 & 0 & 6 \\ 5 & 2 & 0 & 7 \\ 1 & 7 & 0 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{basis} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$ ✓

5.4.3 a. $\begin{cases} \text{rank}(A_{3\times 4}) \le 3 \\ \text{rank}(A_{3\times 4}) \le 4 \end{cases} \Rightarrow \text{rank}(A_{3\times4}) \le 3 \Rightarrow \dim\{c_1, c_2, c_3, c_4\} \le 3 \Rightarrow$ not ind ✓

$\dim(\{r_1, r_2, r_3\}) \le 3 \Rightarrow$ when $\dim = 3$, r_1, r_2, r_3 are ind.

b. $\dim\{c_1, c_2, c_3\} = \text{rank}(A) = 2 \Rightarrow$ not ind ✓

$\dim\{r_1, r_2, r_3, r_4\} = \text{rank}(A) = 2 \Rightarrow$ not ind

c. $\text{rank}(A) \le n$, $\text{rank}(A) = m \Rightarrow m \le n$ ✓

d. $A = A_{m\times n}$, $m \ne n$, $\text{rank}(A) = r$, $r \le m$, $r \le n$

Suppose $m > n$:

if $r < n$, $\dim\{c_1, c_2, \cdots, c_n\} = r < n$, $\{c_i\}$ is not ind,

$\dim\{r_1, r_2, \cdots, r_m\} = r < m$, $\{r_j\}$ is not ind.

if $r = n$, $\dim\{c_1, c_2, \cdots, c_n\} = r = n$, $\{c_i\}$ is ind

$\dim(\{r_1, r_2, \cdots, r_m\}) = r = n < m$, $\{r_j\}$ is not ind

Thus, $\{c_i\}$ and $\{r_j\}$ are both ind is impossible. ✓

e. $\text{rank}(A_{3\times 6}) \le 3 \Rightarrow \dim(\text{null } A_{3\times6}) = 6 - \text{rank}(A_{3\times6}) \ge 3$ ✓

$\therefore \dim(\text{null } A_{3\times 6}) \ne 2$

f. $\dim(\text{null } A) = 1$, $\text{rank}(A) = 4 - \dim(\text{null } A) = 3$

$\therefore \dim(\text{im } A) = \dim(\text{col } A) = \text{rank}(A) = 3 \ne 2$ ✓