Exercises Lecture 3 - Part I:

1. A couple has two children.

   (a) What is the probability that both are girls given that the oldest one is a girl?

   (b) What is the probability that both are girls given that at least one of them is a girl?

   <u>**Solution:**</u>

   (a) Let A_1 the event of oldest child being a girl, A_2 the event of youngest child being a girl and B the event of 2 girls. We know that $P(A_1) = P(A_2) = \frac{1}{2}$ and $P(B) = P(A_1 \cap A_2) = \frac{1}{4}$ (there are 4 elements in the sample space for the sex of the 2 children each with probability $\frac{1}{4}$.
   We want to compute $P(A_2 \mid A_1)$ which based on the definition of the conditional probability is given by:

   $$P(A_2 \mid A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{1/4}{1/2} = 1/2.$$

   So the probability that they are both girls increases from 1/4 to 1/2 if we know that already the first one is a girl.

   (b) Let C the event of at least one girl. We want to compute

   $$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{P(C \mid B)P(B)}{P(C)}.$$

   We know that $P(B) = 1/4$ and $P(C \mid B) = 1$. We then need to compute $P(C)$.
   We can work with the compliment. We know that $P(\text{none is a girl}) = P(C^C) = \frac{1}{4}$.
   From this it follows that $1 - P(C) = \frac{1}{4}$ and thus $P(C) = 3/4$.
   $P(B \mid C) = \frac{1 \times 1/4}{3/4} = 1/3$.

2. Suppose that two dice were rolled and we note down the sum T of the two numbers.

   (a) If T is odd, what is the probability that T was less than 8?

   (b) Let $A = \{\text{outcomes of the 2 dice match}\}$ and $B = \{\text{sum of outcomes at least 8}\}$. Compute $P(A \mid B)$ and $P(B \mid A)$. *Hint:* Write down the sample space of this experiment and note that all elements are equally likely.

   <u>**Solution:**</u>

(a) If we let A be the event that $T < 8$ and let B be the event that T is odd, then $A \cap B$ is the event that T is 3, 5 or 7. From the probabilities for the outcomes of 2 dice discussed in Lectures 1 and 2 we can evaluate $P(A \cap B)$ and $P(B)$ as follows:

$$P(A \cap B) = \frac{2}{36} + \frac{4}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3},$$
$$P(B) = \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{18}{36} = \frac{1}{2}.$$

Then from the definition of the conditional probability we have:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{3}.$$

You may also try it in R, using the following code:

```
> x <- expand.grid(1:6, 1:6)
> x <- cbind(x, x[,1]+x[,2])
> x[x[,3]%%2 != 0, 3]

 [1]  3  5  7  3  5  7  5  7  9  5  7  9  7  9 11  7  9 11

>
```

(b) Let $A = \{\text{outcomes match}\}$ and $B = \{\text{sum of outcomes at least 8}\}$. We have $P(A) = \frac{6}{36}$, $P(B) = \frac{15}{36}$ and $P(A \cap B) = 3/36$. Finally, $P(A \mid B) = \frac{3/36}{15/36} = 1/5$ and $P(B \mid A) = \frac{3/36}{6/36}$.

3. We draw 2 cards from a standard playing deck. What is the probability that both are aces?

Solution:

Let A the event that the first card is an ace and B the event that the second card is an ace. $P(\text{both Aces}) = P(A \cap B) = P(A)P(B \mid A) = \frac{4}{52} \cdot \frac{3}{51} \approx 0.00452.$