MFS Lecture 2

Last time: Sets, functions review

Today: Limits and continuity, derivatives

Goal: define derivative of a fn, but for that we need limits.

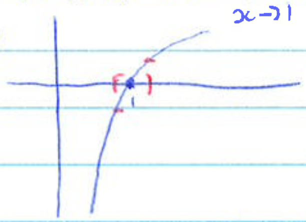
Idea: We say a number L is the limit of a function $f(x)$ at as x approaches a, if $f(x)$ gets arbitrarily close to L as x approaches a. Notation: $\lim_{x\to a} f(x) = L$.

Remark: There's a more formal definition of limits - we do not need this!

Limits are best understood by examples:

Example 1: What is $\lim_{x\to 1} \ln(x)$?

Answer:

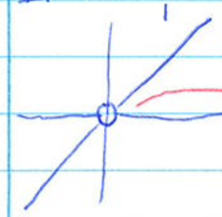


Visually, as x gets very close to 1, $\ln(x)$ gets very close to $\ln(1) = 0$, so we suspect $\lim_{x\to 1} \ln(x) = \ln(1) = 0$.

Example 2: What is $\lim_{x\to 0} \frac{x^2}{x}$?

Answer: Note $\frac{x^2}{x}$ is not defined if $x = 0$! So we can't just plug in 0.

If $x \neq 0$, $\frac{x^2}{x} = x$. The graph of $\frac{x^2}{x}$ is:



- looks like x except at $x = 0$!

open circle to indicate the function isn't defined.

As $x \to 0$, $\frac{x^2}{x} = x$ approaches 0, but is never actually 0.

So $\lim_{x\to 0} \frac{x^2}{x} = 0$.

Example 3: What is $\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$?

Answer: Note if we plug in $x = 2$ we get $\frac{0}{0}$, which is bad. But we can factor the top as $(x-2)(x-1)$, so if $x \neq 2$, we have

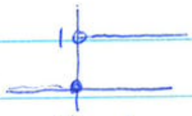
$$\frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}} = x-1,$$

which approaches 1 as $x \to 2$.

Of course, limits don't always exist as the next example shows.

Example 4: Find $\lim_{t\to 0} H(t)$, where $H(t) = \begin{cases} 0 & , \text{if } x \leq 0 \\ 1 & , \text{if } x > 0. \end{cases}$

$\sqrt{x} = x^{\frac{1}{2}}$ $\log(ab) = \log(a) + \log(b)$

Answer:  If $\overset{t}{\cancel{x}} < 0$, $H(t) = 0$, but if $x > 0$, $H(t) = 1$.
So we can't choose one value L for the limit - in this case, we say the limit does not exist.

This leads to another notion - that of one-sided limits.

Idea/notation: Suppose a fcn $f(x)$ is defined on an interval to the right (resp. left) of a. We say L is the right-hand limit (resp. left hand limit) of $f(x)$ as x approaches a from the right (resp. left) and we write

$$\lim_{x\to a^+} f(x) = L \qquad (\text{resp. } \lim_{x\to a^-} f(x) = L)$$

if we can make $f(x)$ arbitrarily close to L by choosing x arbitrarily close to the right (resp. left) of a.

Remark: Hence, for $H(t)$ from Example 4, $\lim_{x\to 0^+} H(t) = 1$ and $\lim_{x\to 0^-} H(t) = 0$.

We'll now see a nice ~~result of~~ theorem on left/right-sided limits and limits. A theorem is an important result in mathematics that has been proven.

Theorem: Let $f(x)$ be a fcn and suppose $f(x)$ is defined at all pts in an open interval containing a pt a, but not necessarily at a. Then

$$\lim_{x\to a} f(x)$$

exists and equals a value L if and only if (iff) the one-sided limits

$$\lim_{x\to a^+} f(x) \text{ and } \lim_{x\to a^-} f(x)$$

both exist and equal L.

Remark: iff essentially means two statements are equal. i.e. if you have human DNA, you're a human, and if you ~~h~~ are human, you have human DNA.

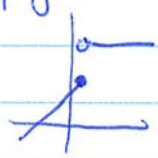
Limits at infinity:
Definition: If f is a fcn, f is said to have a limit at infinity (negative) equal (and neg.) to L if $f(x)$ gets arbitrarily close to L as x gets arbitrarily large. We write $\lim_{x\to\infty} f(x) = L$.

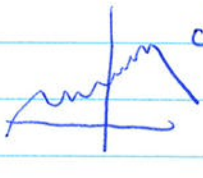
Example 5: What is $\lim_{x\to\infty} \frac{1}{x}$? $\lim_{x\to-\infty} \frac{1}{x}$?
Answer: Both are 0 - the denominators get arbitrarily large.

Example 6: Find $\lim_{x\to\infty} x^2+6x+12 / 3x^2+8x-10$.
Answer: divide by x^2 on the top and bottom:
$1+\frac{6}{x}+\frac{12}{x^2} / 3+\frac{8}{x}-\frac{10}{x^2}$.
As $x\to\infty$, $\frac{6}{x}$, $\frac{8}{x}$, $\frac{12}{x^2}$, and $-\frac{10}{x^2}$ all go to 0, and we're left with $\frac{1}{3}$.

In general, consider $\frac{f(x)}{g(x)}$ where f and g are only comprised of powers of x.
- if the numerator has a larger power of x, the fcn goes to ∞ as $x\to\infty$, i.e. $\lim_{x\to\infty} x^{6.1}/x^6 = \infty$.
- if the denominator has a larger power of x, the fcn goes to 0 as $x\to\infty$, i.e. $\lim_{x\to\infty} x^{10}/x^{11.5} = 0$.
- if the largest powers of x are equal in f and g, we take the corresponding coefficients, i.e. $\lim_{x\to\infty} (2x^{e^2}+x^e)/(3x^{e^2}-10e^2x) = \frac{2}{3}$.

Continuity
Idea: A fcn is cts if you can draw its graph without your pen leaving the page:


not cts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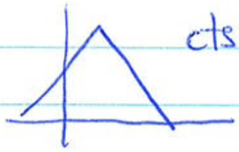

cts.

Formally:
Def: Suppose a fcn f is defined on an open interval containing a pt a. We say f is cts at a if $\lim_{x\to a} f(x)$ exists and equals $f(a)$. We say f is cts if its cts on its domain.

Remark: So if a fcn is continuous, you can find a limit at a pt a by just plugging a into f. You can use without proof the fact the following fcns are cts:
- polynomials
- exponentials
- logarithms
- power functions (including square roots)
- quotients of these fcns

where defined!

Slopes and lines

Let $f(x)=mx+b$ be a line. Recall that m is the slope, or how fast y is changing with respect to (wrt) x.

Given any 2 pts (x_1, y_1) and (x_2, y_2) on the line, we can find m via
$m = \frac{\Delta y}{\Delta x} = {y_2-y_1}/{x_2-x_1}$.

i.e. if a line has two pts $(0,1)$ and $(1,3)$, then $m = \frac{1-3}{0-1} = 2$.

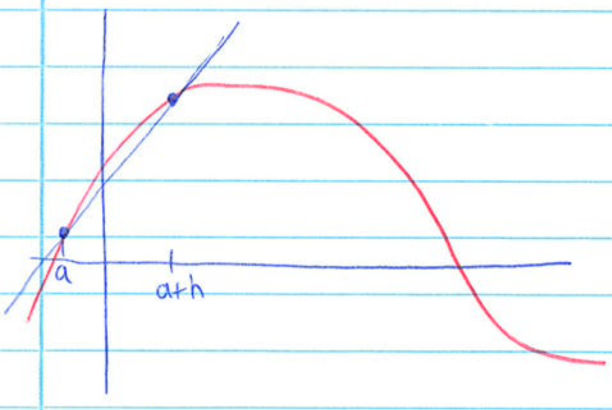
Whenever

- $m>0$, $f(x)=mx+b$ is increasing in x
- $m<0$, $f(x)=mx+b$ is decreasing in x.

So m tells us if f is increasing/decreasing and how fast.

Tangent lines and derivatives

Let $f(x)$ be a fcn. How is f changing at a pt a? Is it increasing/decreasing, and how fast or slow?



Let's pick another pt $a+h$ close to a and draw a line through $(a, f(a))$ and $(a+h, f(a+h))$. The slope of this line gives us an idea how the function is changing at a, i.e.
$m = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$.

As $h \to 0$, this will (hopefully) become more + more accurate. That is, let's take the limit ~~of~~ as $h \to 0$ and find the slope of the resulting line.

Definition: Let f be a fcn defined on an open interval containing a pt a. If $\lim_{h\to 0} {f(a+h)-f(a)}/{h}$ exists,
we say f is differentiable at a and write $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.
If f is diff. at all pts in its domain, we say f is differentiable, and we call the function $f' : x \to f'(x)$ the derivative of f.

10

Remark: The derivative of f at a is the instantaneous rate of change of f at a.

- another common notation for the derivative is $\frac{df}{dx}$.

Example: What is the derivative of $f(x) = 2x$ at $x=0$?
What about $f(x) = mx+b$?

Answer: It's 2 - the slope of the line itself.
It's m.

Example: Find f' if $f(x)=x^2$.

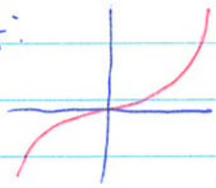
Answer: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} (2x+h) = 2x.$

Later, we'll need:

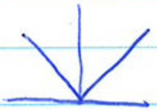
Definition: Let f be differentiable at a pt a. The tangent line of f at a is the line going through ~~x=a~~ $(a, f(a))$ w/ slope $f'(a)$.

Ex: Without calculating f', draw the tangent line of $f(x)=x^3$ at $x=0$.

A:



Ex: Let $f(x)=|x|$, whose graph is below. What is the derivative at $x=0$?



A: To the right we have the line x and to the left the line $-x$, so the derivative isn't defined.

Remark: In general, the derivative requires a function doesn't have any sharp corners. It should be "smooth".

In general we won't calculate derivatives using the definition. Instead, we'll use:

- if $f(x)=ax^n$ where a, n are constants, then $f(x) = nax^{n-1}$.
- if $f(x)=b^x$ with $b>0$, $b\neq 1$, then $f'(x) = \ln(b)\, b^x$.
  Hence if $b=e$, $f'(x) = \ln(e)\, e^x = e^x = f(x)$.
- if $f(x) = \log_b(x)$, $b>1$, then $f'(x) = \frac{1}{x\ln(b)}$.
  Hence if $b=e$, $f'(x) = \frac{1}{x\ln(e)} = \frac{1}{x}$.

MUST KNOW!

Example: Find ~~$f'(x)$~~: the derivatives:

1) $f(x) = -x^{5.2}$ $\qquad f' = -5.2x^{4.2}$

2) $g(t) = t^{-1} = \frac{1}{t}$. $\qquad g' = -t^{-2} = -\frac{1}{t^2}$

3) $h(y) = y\sqrt{y}$. $\qquad h' = \frac{3}{2}\sqrt{y}$.

Theorem (The Sum Rule): If $f(x) = g(x) + h(x)$ and g and h are diff., then
$f'(x) = g'(x) + h'(x)$.

Ex: If $f(x) = x + 5x^2 + \ln(x)$, $f'(x) = 1 + 10x + \frac{1}{x}$.

Application: Velocity and Acceleration

Let $p(t)$ be the distance travelled ~~on a bike~~ wrt time, i.e. distance on your bike, etc.
The rate of change of position wrt time is velocity, i.e. $p'(t) = v(t)$.

Ex: Suppose you throw a ball up and its height is modelled by
$p(t) = -4.9t^2 + 10t + 1$.
What is the velocity at $t=1$ and $t=2$?
A: $v(t) = -9.8t + 10$ so $v(1) = 0.2$ and $v(2) = -9.6$.
Hence at $t=1$ the ball is still going up but at $t=2$ it's falling.

The change in velocity wrt time is acceleration, i.e. $a(t) = v'(t)$ $(= p''(t))$.
In the above example, $v'(t) = -9.8 = a(t)$.

Ex: Newton's 2nd law says the force F applied to an object of mass m and acceleration a is $F = ma$. If you push a cart that weighs 20kg s.t. the position of the cart is $p(t) = t^3 + 20t$, what force do you apply to the cart at $t=1$?

A: $v(t) = 3t^2 + 20$

$a(t) = v'(t) = 6t$

$a(1) = 6$ $\qquad F = ma = 6 \cdot 20 = 120$.