Diagonalization

4433LALG3: Linear Algebra

Week 2, Lecture 6, Valente Ramírez

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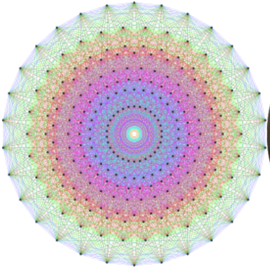

Overview

- Diagonal representation
- Conditions for diagonalization
- Supplementary material

References:

- Nicholson §3.3: *Diagonalization* and *Linear Dynamical Systems*
- Nicholson §5.5 (up to *Diagonalization Revisited*)
- 3Blue1Brown Ch.14

Important remark

Warning

In this lecture we will discuss exclusively maps $\mathbb{R}^n \to \mathbb{R}^n$, and thus square matrices.

Section 1

Diagonal representation

Motivation

We have seen that *the same* linear transformation can be represented by different matrices, depending on what basis we use as reference.

The obvious question is: what is the best representation we could have?

The answer: a diagonal matrix is as easy as it gets.

The dream is to find a change of coordinates P such that $P^{-1}AP$ is diagonal.

Example

The matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ is similar to the diagonal matrix $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.

The change-of-coordinates matrix is $P = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$.

Let's check that:

```
# Define matrices A and P
> A <- matrix(c(3,1,5,-1), nrow=2)
> P <- matrix(c(5,1,-1,1), nrow=2)

# Compute inverse of P
> Pinv <- Inverse(P)

# Compute Pinv A P (rounded to 6 decimal places)
> round(Pinv %*% A %*% P, 6)
     [,1] [,2]
[1,]    4    0
[2,]    0   -2
```

Example

The matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ is similar to the diagonal matrix $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.

We already know two useful facts:

- Eigenvalues are intrinsic/absolute properties of the transformation.
  - E.g. they do not depend on the choice of basis.
  - Thus all *similar* matrices have the same eigenvalues.
- The eigenvalues of a diagonal matrix are exactly the diagonal elements.

Conclusion:

The eigenvalues of A are 4 and -2.

Example

We can also justify algebraically that 4 and -2 are the eigenvalues.

The equation $P^{-1}AP = D$ can be expressed as $AP = PD$.

$$AP = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} A\begin{bmatrix} 5 \\ 1 \end{bmatrix} & A\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$PD = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4\begin{bmatrix} 5 \\ 1 \end{bmatrix} & -2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

Conclusion:

$$A\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \qquad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Relation to eigenvalues

It is always true that if A is similar to a diagonal matrix D, then the entries of D are precisely the eigenvalues of A.

This gives us a hint on *how* to find the change-of-coordinates matrix.

Theorem

Let A be an $n \times n$ matrix. Suppose there is a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of A.

Then $P = \begin{bmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_n \end{bmatrix}$ is invertible, and $P^{-1}AP$ is diagonal.

In that case, $P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i is the eigenvalue corresponding to the eigenvector $\mathbf{x}_i$.

Example

We can check that the theory works:

The first column of $P^{-1}AP$ is:

$$\begin{aligned} P^{-1}AP\mathbf{e}_1 &= P^{-1}A\mathbf{p}_1 \\ &= P^{-1}(4\mathbf{p}_1) \\ &= 4(P^{-1}\mathbf{p}_1) \\ &= 4\mathbf{e}_1 \end{aligned}$$

The second column of $P^{-1}AP$ is:

$$\begin{aligned} P^{-1}AP\mathbf{e}_2 &= P^{-1}A\mathbf{p}_2 \\ &= P^{-1}(-2\mathbf{p}_2) \\ &= -2(P^{-1}\mathbf{p}_2) \\ &= -2\mathbf{e}_2 \end{aligned}$$

We conclude that $P^{-1}AP = \begin{bmatrix} 4\mathbf{e}_1 & -2\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.

Remark

Warning

There are two equivalent statements:

$$P^{-1}AP = D \quad \text{and} \quad A = PDP^{-1}.$$

It is easy to confuse P and P^{-1}.

If you understand the steps on the previous slide you will not make this mistake.

Example

$$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

Which description is easier to visualize?

1. T_A is the transformation that maps: $\mathbf{p}_1 \mapsto 4\mathbf{p}_1$ and $\mathbf{p}_2 \mapsto -2\mathbf{p}_2$
2. T_A is the transformation that maps: $\mathbf{e}_1 \mapsto 3\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_2 \mapsto 5\mathbf{e}_1 - \mathbf{e}_2$

Application: diagonalization

Example (Nicholson Example 3.3.1)

We are interested in the evolution of the population of a species of birds. We count only females. A female remains juvenile (cannot reproduce) for one year, and then becomes adult (is able to produce offspring).

We assume:

1. The number of juveniles hatched in any year is twice the number of adult females alive the year before,
2. Half of the adults survive to the next year,
3. One quarter of juveniles survive into adulthood.

If there were 100 adult females and 40 juvenile females on a given year, compute the population of females 20 years later.

Application: analysis

The number of birds on a given year is a linear transformation of the number of birds on the year before.

We set: a_k number of adults on year k, j_k number of juveniles on year k.

We group them into a vector $\mathbf{v}_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$.

From the information given, we can set up a *linear discrete dynamical system*:

$$\mathbf{v}_{k+1} = A\mathbf{v}_k, \quad \text{where} \quad A = \begin{bmatrix} 1/2 & 1/4 \\ 2 & 0 \end{bmatrix}.$$

It follows that $\mathbf{v}_1 = A\mathbf{v}_0$, $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, $\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$, and so on.

We need to compute $\mathbf{v}_{20} = A^{20}\mathbf{v}_0$ (with $\mathbf{v}_0 = \begin{bmatrix} 100 & 40 \end{bmatrix}^\top$).

Direct computation of A^{20} is not practical.

Interlude: computing powers of a matrix

In many applications, we are interested in the **powers** of a matrix:

$$A, \;\; A^2, \;\; A^3, \; \dots, \; A^k, \; \dots$$

First, note that this only makes sense if A is square.

Computing A^k is usually slow and computationally expensive.

There is one exception: *diagonal matrices*. Suppose $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$.

Then:

$$D^2 = \begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & d_2^2 & 0 \\ 0 & 0 & d_3^2 \end{bmatrix}, \;\; D^3 = \begin{bmatrix} d_1^3 & 0 & 0 \\ 0 & d_2^3 & 0 \\ 0 & 0 & d_3^3 \end{bmatrix}, \;\; \dots, \;\; D^k = \begin{bmatrix} d_1^k & 0 & 0 \\ 0 & d_2^k & 0 \\ 0 & 0 & d_3^k \end{bmatrix}.$$

Application: analysis

Direct computation of A^{20} is not practical. Instead, we use:

Fact

If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, for any $k = 1, 2, 3, \ldots$

Obviously, the above fact is most useful when D is a diagonal matrix.

We now find D and P by finding the eigenvalues and eigenvectors of A.

```
> library(matlib)
> A <- matrix(c(1/2,2, 1/4,0), ncol=2)
> eigen(A)$values
[1]  1.0 -0.5

> gaussianElimination(A-diag(2))
     [,1] [,2]
[1,]    1 -0.5
[2,]    0  0.0

> gaussianElimination(A+0.5*diag(2))
     [,1] [,2]
[1,]    1 0.25
[2,]    0 0.00
```

Application: analysis

We conclude that if $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$.

Now, $A^{20} = PD^{20}P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 1/2^{20} \end{bmatrix} P^{-1} \approx P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$.

A short computation gives: $P^{-1} = \begin{bmatrix} 2/3 & 1/6 \\ -1/3 & 1/6 \end{bmatrix}$,

and $A^{20} \approx P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 2/3 & 1/6 \\ 4/3 & 1/3 \end{bmatrix}$.

Finally, $\mathbf{v}_{20} = A^{20}\mathbf{v}_0 = \begin{bmatrix} 2/3 & 1/6 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} 100 \\ 40 \end{bmatrix} = \begin{bmatrix} 73.3 \\ 146.6 \end{bmatrix}$.

After 20 years we'll have approximately 73 adults and 147 juvenile birds.

Section 2

Conditions for diagonalization

Diagonalizability

Our hope is to find n linearly independent eigenvectors of A.

However, this is not always possible: for example, a 90° rotation has no eigenvectors at all!

So not all square matrices are *diagonalizable*.

Definition

A square matrix A is called **diagonalizable** if there exists an invertible matrix P (of the same size) such that $P^{-1}AP$ is diagonal.

Why are some matrices not diagonalizable?

There are mainly two obstructions:

1. There are not enough eigenvalues
2. There are not enough eigenvectors

Not enough eigenvalues

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We look for the eigenvectors of A by setting $\det(A - \lambda I_2) = 0$.

$$\det(A - \lambda I_2) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

The equation $\lambda^2 + 1 = 0$ has no (real) solutions!!

A is not diagonalizable!

Not enough eigenvectors

Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

We look for the eigenvectors of A by setting $\det(A - \lambda I_2) = 0$.

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2.$$

The equation $(1-\lambda)^2 = 0$ has solutions $\lambda_1 = 1$ and $\lambda_2 = 1$. This is called an eigenvalue of (algebraic) **multiplicity two**.

We look for the eigenvectors by solving $(A - I_2)\mathbf{x} = \mathbf{0}$.

$$(A - I_2)\mathbf{x} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}. \qquad \text{If } \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ then } x_2 = 0.$$

The solutions are $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and all its scalar multiples.

But we cannot find **two** linearly independent eigenvectors!

A is not diagonalizable!

A fundamental result

Theorem

*Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be eigenvectors corresponding to **different** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of an $n \times n$ matrix A.*

Then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a linearly independent set.

Corollary

*If A is an $n \times n$ matrix having n **distinct** eigenvalues, then A is diagonalizable.*

Section 3

Supplementary material

Similar matrices

Recall that A and B are said to be **similar** if there exists an invertible matrix C such that $B = C^{-1}AC$.

Theorem

If A and B are similar $n \times n$ matrices, then A and B have the same:

- *rank*
- *determinant*
- *trace*
- *eigenvalues*

Trace and determinant

Corollary

Let A be an $n \times n$ matrix. Suppose A is diagonalizable[1]*. Then:*

- *The trace of A is equal to the* ***sum*** *of all eigenvalues of A.*
- *The determinant of A is equal to the* ***product*** *of all eigenvalues of A.*

[1] Note: The assumption that A is diagonalizable can be relaxed a bit. It is enough to assume A has n eigenvalues. Repeated eigenvalues are OK, as long as we take into account their *multiplicity*.

Trace and determinant

It is easy to justify the claims on the previous slide.

Suppose A is diagonalizable. Then A is similar to the diagonal matrix $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, where the λ_i are the eigenvalues of A.

For a diagonal matrix, we know that $\det D = \prod_i \lambda_i$, and $\operatorname{tr} D = \sum_i \lambda_i$.

By similarity, $\det A = \det D$, and $\operatorname{tr} A = \operatorname{tr} D$.

Therefore, $\det A = \prod_i \lambda_i$, and $\operatorname{tr} A = \sum_i \lambda_i$.