$h(x)$: hazard rate is the instantaneous rate of experiencing the event at time x given that an individual is alive at time x

$$h(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x+\Delta x \mid X \geq t)}{\Delta x}$$

If Δx is very small we have:

$$P(x \leq X < x + \Delta x \mid X \geq x) \approx h(x) \Delta x .$$

The definition of the hazards implies that

$$h(x) = \frac{\lim_{\Delta x \to 0} P(x \leq X < x + \Delta x)}{P(X \geq x)} = \frac{f(x)}{S(x)}$$

(remember : $P(x \leq X < x+\Delta x) = f(x)\Delta x$)

$$f(x) = \frac{d F(x)}{dx} = -\frac{d S(x)}{dx} = -S'(x)$$

We can write then:

$$(*) \quad h(x) = \frac{f(x)}{S(x)} = -\frac{S'(x)}{S(x)} = -\frac{d \log\{S(x)\}}{dx}$$

From $(*)$ by integrating both side we get

$$H(x) = \int_0^x h(u)\,du = \int_0^x \frac{-S'(u)}{S(u)}\,du = -\log(S(u))\Big|_0^x$$

$$= -\left(\log(S(x)) - \log(\underbrace{S(0)}_{=1})\right)$$

$$\overset{!}{=} -\log\{S(x)\}$$

$$H(x) = \int_0^x h(u)\,du = -\log\{S(x)\}$$

cumulative hazard function

Hence: $S(x) = \exp(-H(x))$

$$h(x) = \frac{f(x)}{S(x)} \Rightarrow f(x) = h(x)\,S(x)$$
$$\stackrel{!}{=} h(x)\exp(-H(x))$$
$$\stackrel{!}{=} h(x)\exp\left(-\int_0^x h(u)\,du\right)$$

Exponential distribution:

$$f(x) = \lambda e^{-\lambda x} \quad x > 0 \quad ; \quad F(x) = \int_0^x \lambda e^{-\lambda u}\,du$$
$$h(x) = \frac{f(x)}{S(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda \qquad \stackrel{!}{=} 1 - e^{-\lambda x}$$

$$h(x) = \lambda \quad ; \quad S(x) = 1 - F(x) = e^{-\lambda x}$$

Mean survival time:

$$\mu = \int_0^\infty x f(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx = -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$
$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda} = EX = \int_0^\infty S(x)\,dx$$

It is easier to compute the mean survival time by:

$$EX = \int_0^\infty S(x)\,dx$$

For the exponential distribution:

$$EX = \int_0^\infty e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = -\frac{1}{\lambda}\left(e^{-\lambda x}\Big|_0^\infty\right)$$

$$= -\frac{1}{\lambda}$$

Median survival time for the exponential

$$S(x_{0.5}) = 0.5$$

$$me(x) = \int_0^x \lambda e^{-\lambda t}\,dt = 0.5$$

$$-e^{-\lambda t}\Big|_0^x = 0.5$$

$$1 - e^{-\lambda x} = 0.5 \Rightarrow e^{-\lambda x} = \frac{1}{2}$$

$$-\lambda x = \log\left(\frac{1}{2}\right) \Rightarrow \boxed{x = \log 2/\lambda}$$

Mean residual life function:

$$mrl(x) = E(X-x \mid X>x) = \frac{\int_x^\infty (t-x) f(t)\,dt}{P(X>x)}$$

$$= \frac{\int_x^\infty (t-x) f(t)\,dt}{S(x)}$$

Integrate by parts: (remember $f(t)dt = -dS(t)$)

$$E(X-x \mid X>x)\cdot S(x) = \int_x^\infty (t-x) \underline{f(t)dt}$$

$$= \underbrace{-(t-x)S(t)\Big|_x^\infty}_{=0} + \int_x^\infty S(t)\,dt$$

$$= \underbrace{\underbrace{S(\infty)}_{=0}\cdot\infty}_{=0} + \int_x^\infty S(t)\,dt$$

$$\Rightarrow E(X-x \mid X>x) = \frac{\int_x^\infty S(t)\,dt}{S(t)}$$

Mrl for exponential distribution:

$$E(X-x \mid X>x) = \frac{\int_x^\infty e^{-\lambda t}\,dt}{e^{-\lambda t}} = \frac{-\frac{1}{\lambda} e^{-\lambda t}\Big|_x^\infty}{e^{-\lambda t}} = \frac{1}{\lambda}$$

$$Var(X) = EX^2 - (EX)^2$$

find the relation between variance and survival

$$EX = \int_0^\infty S(t)\,dt$$

$$EX^2 = \int_0^\infty t^2 f(t)\,dt = \underbrace{-t^2 S(t)\Big|_0^\infty}_{=0} + 2\int_0^\infty t\,S(t)\,dt$$

$$= 2\int_0^\infty t\,S(t)\,dt$$

$$Var\,X = 2\int_0^\infty t\,S(t)\,dt - \left[\int_0^\infty S(t)\,dt\right]^2$$

$$EX = \int_0^\infty t\underbrace{f(t)\,dt}_{-S(t)} = \underbrace{-t\,S(t)\Big|_0^\infty}_{=0} + \int_0^\infty S(t)\,dt$$

$$= \int_0^\infty S(t)\,dt$$

Remember that $S(\infty) = 0$, $S(0) = 1$

$$\lim_{t\to\infty} t\,S(t) = 0$$

Memoryless property for the exponential distribution

$$P(X > s+t \mid X > s) = P(X > t) \quad \forall s, t \geq 0$$

$$P(X > s+t \mid X > s) = \frac{P(X > t+s,\ X > s)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

Solution excercise car on the slides:

average $= 10 \Rightarrow \lambda = \frac{1}{10}$ $\quad X \sim \exp(\lambda)$

$$P(\text{remaining lifetime} > 5) = 1 - F(5)$$
$$\overset{!}{=} e^{-5\lambda}$$
$$\overset{!}{=} e^{-1/2} \approx .604$$

if the lifetime distribution is not exponential then the relevant probability is:

$$P(X > t+5 \mid X > t) = \frac{P(X > 5+t,\ X > t)}{P(X > 5)}$$

$$= \frac{P(X > 5+t)}{P(X > 5)}$$

$$= \frac{1 - F(t+5)}{1 - F(t)} = \frac{S(t+5)}{S(t)}$$

$t =$ n° of Km that the battery had been

in use prior to the start of the trip.
If the distribution is not exponential ~~other~~ additional information is needed (namely t), before the probability can be calculated.