MfS Lecture 1.1

Today: Intro, sets and functions, review of elementary functions

Sets

Definition: A set is a collection of numbers, either finite or infinite.

Example: integers: $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$
natural numbers: $1, 2, 3, 4, \dots$
rational numbers: $\frac{a}{b}$, where a, b are integers
real numbers: any non-complex number, i.e. π, e^2, $\frac{1}{2}$, 0, etc.

Remark: The real numbers come up all the time! We give them a special symbol, $\mathbb{R}$.

Definition: A special kind of set are intervals, which are all the real numbers between 2 numbers, i.e. all the numbers between a and b.

For example, an interval is all the numbers between 0 and 1.
When it comes to intervals, we have to choose to include the endpts or not. For this we use the following notation:
- an open bracket '(' or ')' indicates an endpt is NOT included
- a closed/square bracket '[' or ']' indicates an endpt is included.

Example:
- $(0,1)$ is all numbers between 0 and 1, NOT including 0 or 1.
- $[0,1]$ " " , including 0 and 1.
- $(0,1]$ " " , NOT including 0 but including 1.
- $[0,1)$ " " " 1 " 0.

Exercise: Write, in interval notation, all the numbers between $-\pi$ and $+10$, not including $-\pi$ but including 10.
Answer: $(-\pi, 10]$

Remark: We use $\pm\infty$ to denote positive ∞ or negative infinity, so i.e.
- $(-\infty, 0]$ is all the numbers less than or equal to 0
- $(5, \infty)$ " larger than 5.

If $\pm\infty$ is an endpt, you must use an <u>open</u> bracket.

We also use set notation a lot, where all the elements of a set can be written between curly brackets, i.e. $\{1,2,3\}$ is the set containing 1, 2, and 3.

Example: We can write the interval $[0,1]$ as

$$\{x \in \mathbb{R} : 0 \leq x \leq 1\}$$ ← a sentence!

- $\in$ means "in", so $x \in \mathbb{R}$ is "the set of ~~numbers~~ elements contained in the real numbers
- $:$ means "such that"
- $0 \leq x \leq 1$ defines the elements we want.

Thus, $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ reads,

"the set of elements in the real numbers such that 0 is less than or equal to x, which is less than or equal to 1".

Exercise: Use set notation to write the intervals

1. $[-20, \pi)$ Answer: $\{x \in \mathbb{R} : -20 \leq x < \pi\}$
2. $(-\infty, 4)$ $\{x \in \mathbb{R} : x < 4\}$.

Remark: If it's understood $x \in \mathbb{R}$, we often leave this part out, i.e.

$\{x < 4\}$.

Functions:

Definition: A function is a map between 2 sets that assigns a unique output to each input. The domain of a fcn is the set of all values that can be inputs for the fcn. The range is the set of all output values.

Notation: We often call functions names like f or g. We can represent them by writing how they act on an input value, i.e.

$f : x \mapsto x^2$ sends x to x^2, or $f(x) = x^2$.

Domain $= D(f)$, range $= R(f)$.

Examples:
- $f(x) = x^2$ has domain $D(f) = \mathbb{R}$, $R(f) = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$.
- $f(x) = \sqrt{x}$: $D(f) = [0, \infty) = R(f)$.
- $g(t) = \frac{1}{t}$ $D(g) = \{x \neq 0\}$. $R(g) = \{x \neq 0\}$.
- $h(t) = \sqrt{t-1}$ $D(h) = \{t \geq 1\}$, $R(h) = [0, \infty)$.

- $f(x) = \begin{cases} 0 & x \text{ is odd} \\ 1 & x \text{ is even} \end{cases}$ with $D(f)$ = integers and $R(f) = \{0, 1\}$.

Remark: We can always change a function name or a variable name. $g(t) = t^2$ is the same map as $f(x) = x^2$. We normally use f, g, and h for fns and x, y, or z for variables, but don't get thrown off if you see other names.

Definition: Let $f(x)$ and $g(x)$ be two functions such that the range of g is contained in the domain of f. The composition of $f(x)$ with $g(x)$ is the function

$$(f \circ g)(x) = f(g(x)).$$

Remark: That is, just replace all instances of x in $f(x)$ with $g(x)$.

Example: If $f(x) = x^2 + 1$ and $g(x) = x - 2$, find

(i) $(f \circ g)(x)$

(ii) $(f \circ f)(x)$.

Answer: (i) $= (g(x))^2 + 1 = (x-2)^2 + 1 = x^2 - 4x + 5$.

(ii) $= (f(x))^2 + 1 = (x^2+1)^2 + 1 = x^4 + 2x^2 + 2$.

Exercise: Find both $(f \circ g)(x)$ and $(g \circ f)(x)$ if

(i) $f(x) = \frac{1}{3}x$, $g(x) = 3x$

(ii) $f(x) = x^2$, $g(x) = \sqrt{x}$.

Answer: (i) $f(g(x)) = x$.

(ii) $f(g(x)) = (\sqrt{x})^2 = x$, $g(f(x)) = \sqrt{x^2} = |x|$, the absolute value of x. i.e. if $x = -3$, $x^2 = 9$, so $\sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$.

Definition: If f and g are two functions such that $(f(g(x)) = x$ AND $g(f(x)) = x$, the f and g are inverse functions, and we can write $g = f^{-1}$.

Basic Functions Review

Polynomials: A polynomial of deg. n is a function of the form

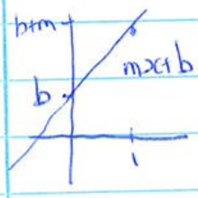
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

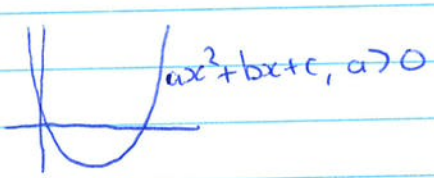
where the a_i's are in $\mathbb{R}$ and $a_n \neq 0$. ~~The~~

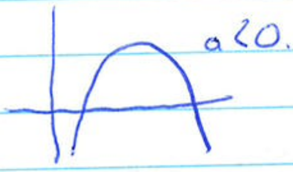
Examples: $f(x) = x^3 + x^2 + x + 1$ has degree 3.

Remark: A polynomial of degree 1 is a line: $f(x) = mx + b$. m is the slope and b is the y-intercept.

A polynomial of degree 2 is a quadratic, $f(x) = ax^2 + bx + c$. It's graph is a parabola that goes up if $a > 0$ and down if $a < 0$.







Definition: The roots/zeroes of a function $f(x)$ are the x-values for which $f(x) = 0$.

Remark: To find the roots of a quadratic polynomial $ax^2 + bx + c$, you can use the quadratic formula, which says the roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

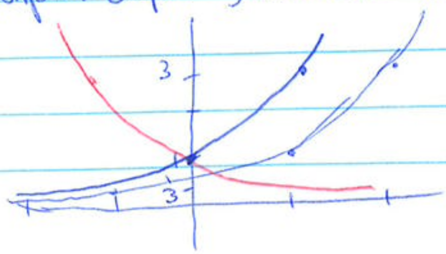
Exercise: Find the roots of (i) $x^2 + x - 6$ and (ii) $-x^2 - 1$.

Answer: (i) $2, -3$, (ii) none.

Exponential functions: Let $b > 0$, $b \neq 1$ be a real number. An exponential fcn is a fcn of the form

$$f(x) = b^x.$$

Example: Graph $f(x) = 3^x$ and $g(x) = (\frac{1}{3})^x$.



In general:

- $b^0 = 1$ for all $b > 0$
- $b^x \neq 0$ for all $b > 0$
- $b^x > 0$ for all $b > 0$.
- $R(b^x) = (0, \infty)$
- $D(b^x) = \mathbb{R}$
- $f(x) \to \infty$ as $x \to \infty$ if $b > 1$
and $f(x) \to 0$ as $x \to -\infty$
- $f(x) \to \infty$ $x \to -\infty$, $0 < b < 1$
and $f(x) \to 0$ as $x \to \infty$.
- $b^1 = b$
- $b^{x_1} b^{x_2} = b^{x_1 + x_2}$
- $(b^{x_1})^{x_2} = b^{x_1 x_2}$
- $b^{-x} = \frac{1}{b^x}$

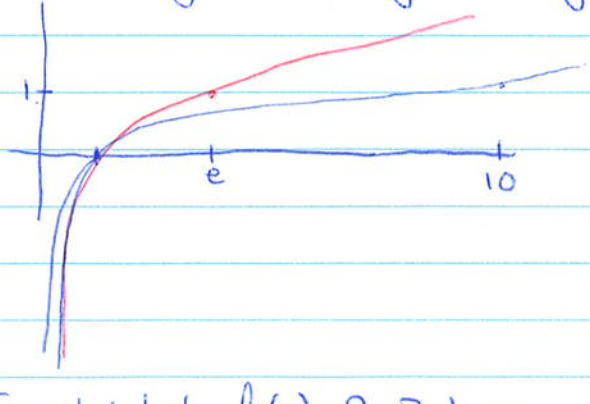
We are most about $b = e = 2.718\ldots$ In this case, $f(x) = e^x$ is called the natural exponential function, or just <u>the</u> exponential function.

Remark: We'll see why it's so important once we get to derivatives!

Example: Simplify $(e^x)^2 e^{-3x}$
Answer: $e^{2x} e^{-3x} = e^{-x}$. $(= \frac{1}{e^x})$.

Logarithms: The inverse function of b^x is the logarithm w/ base b, written $\log_b(x)$.
If $b = e$, we write $\log_e(x) = \ln(x)$ and call it the natural logarithm.

If $b > 1$, $\log_b(x)$ is always increasing, and only defined if $x > 0$.



As b^x and $\log_b(x)$ are inverses, we have
$$\log_b(b^x) = x$$
$$b^{\log_b(x)} = x.$$
That is, $y = \log_b(x)$ is equal to $x = b^y$.

Example: Let $\ln(x) = 2$. Find x.
Answer: $e^{\ln(x)} = x = e^2$.

In general

- $\log_b(1) = 0$
- $\log_b(b) = 1$,
- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x^r) = r \log_b(x)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Remark: We'll never use it, but another property is

$\log_b(x) = \cancel{\ln(x)}\ \log_a(x) / \log_a(b)$.

Example: Solve $7 + 15e^{1-3x} = 10$

$$e^{1-3x} = \frac{1}{5}$$

$$1 - 3x = \ln\left(\frac{1}{5}\right) = -\ln(5)$$

$$1 + \ln(5) = 3x$$

$$x = \frac{1+\ln(5)}{3}.$$

Sum notation:

We'll often add a lot of things at once.

Ex: $1+2+3+4+5+6+7+8+9+10 = \sum_{k=1}^{10} k$

$\sum_{k=1}^{10} k$: $\sum$ (the) sum from $k=1$ to 10 of k.

Exercise: What is $\sum_{k=5}^{8} k$? $\quad 5+6+7+8$

What is $\sum_{k=0}^{4} k^2$? $\quad 0^2+1^2+2^2+3^2+4^2$

Remark: You'll later see this with variables. If you have 5 variables x_1, x_2, x_3, x_4, x_5, we can add them via $\sum_{k=1}^{5} x_k$.

What is $\frac{1}{5}\sum_{k=1}^{5} x_k$?