

# Probability Practice Exam 2021

$$1.(a) P(X=1) = \frac{2}{36} + \frac{1}{36} + \frac{4}{36} = \frac{7}{36}$$

$$P(X=2) = \frac{2}{36} + \frac{10}{36} + \frac{5}{36} = \frac{17}{36}$$

$$P(X=3) = \frac{2}{36} + \frac{2}{36} + \frac{6}{36} = \frac{10}{36}$$

$$(b) P(Y=2) = \frac{1}{36} + \frac{10}{36} + \frac{3}{36} = \frac{14}{36}$$

$$P(X=1|Y=2) = \frac{\frac{1}{36}}{\frac{14}{36}} = \frac{1}{14}$$

$$P(X=2|Y=2) = \frac{\frac{10}{36}}{\frac{14}{36}} = \frac{10}{14}$$

$$P(X=3|Y=2) = \frac{\frac{3}{36}}{\frac{14}{36}} = \frac{3}{14}$$

$$(c) E(X|Y=2) = 1 \times \frac{1}{14} + 2 \times \frac{10}{14} + 3 \times \frac{3}{14} = \frac{15}{7}$$

$$E[E(X|Y)] = E(X) = 1 \times \frac{7}{36} + 2 \times \frac{17}{36} + 3 \times \frac{10}{36} = \frac{77}{36}$$

$$(d) P(Y>1|X \leq 1) = \frac{P(Y>1, X \leq 1)}{P(X \leq 1)} = \frac{\frac{1}{36} + \frac{4}{36}}{\frac{7}{36}} = \frac{5}{7}$$

$$(e) \text{Var}(X) = E(X^2) - (E(X))^2 = 1 \times \frac{2}{36} + 4 \times \frac{17}{36} + 9 \times \frac{10}{36} - \left(\frac{77}{36}\right)^2 = \frac{659}{1296}$$

$$\text{Var}(2X+5) = 4\text{Var}(X) = \frac{659}{324}$$

$$(f) E(XY) = 1 \times \frac{2}{36} + 2 \times \frac{3}{36} + 3 \times \frac{7}{36} + 4 \times \frac{10}{36} + 6 \times \frac{8}{36} + 9 \times \frac{6}{36} = \frac{19}{4}$$

$$E(X) = \frac{77}{36}$$

$$E(Y) = 1 \times \frac{7}{36} + 2 \times \frac{14}{36} + 3 \times \frac{15}{36} = \frac{20}{9}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{19}{4} - \frac{77}{36} \times \frac{20}{9} = -\frac{1}{324}$$

2.  $X_i$  number of infected vol in Group  $i$

$$(a) P(X_1=0) = (1-0.7)^3 = 0.027$$

$$(b) P(X_3=9 | X_1+X_2=5) = P(X_3=9) = C_{10}^9 0.7^9 0.3 = 0.121$$

3.  $X$ : who file,  $Y$ : mistake

$$P(X=A)=0.6, P(X=B)=0.3, P(X=C)=0.1$$

$$P(Y=T|X=A)=0.003, P(Y=T|X=B)=0.007, P(Y=T|X=C)=0.001$$

$$P(X=A|Y=T) = \frac{P(X=A, Y=T)}{P(Y=T)}$$

$$= \frac{P(Y=T|X=A) \times P(X=A)}{P(Y=T)}$$

$$\begin{aligned}
 &= \frac{P(Y=T|X=A) \times P(X=A) + P(Y=T|X=B) \times P(X=B) + P(Y=T|X=C) \times P(X=C)}{0.003 \times 0.6} \\
 &= \frac{0.003 \times 0.6 + 0.007 \times 0.3 + 0.001 \times 0.1}{0.003 \times 0.6 + 0.007 \times 0.3 + 0.001 \times 0.1} \\
 &= 0.45
 \end{aligned}$$

4. 11选3, 7选4, 3排3, 31排3

$$\begin{aligned}
 C_{11}^3 \times C_7^4 \times A_3^3 \times A_{31}^3 &= \frac{11!}{(11-3)! \times 3!} \times \frac{7!}{(7-4)! \times 4!} \times 3! \times \frac{31!}{(31-3)!} \\
 A_{11}^3 \times A_7^4 \times A_3^3 \times 31^3 &= 934510500
 \end{aligned}$$

$$5. (a) P = \frac{4}{52} = \frac{1}{13}$$

$$(b) P(X=k) = C_n^k \left(\frac{1}{13}\right)^k \left(\frac{12}{13}\right)^{n-k}, k=0,1,2,\dots$$

$$(c) P = \sum_{k=65}^{90} P(X=k) = \sum_{k=65}^{90} C_{1000}^k \left(\frac{1}{13}\right)^k \left(\frac{12}{13}\right)^{1000-k}$$

$$(d) Y_i \sim \text{Bern}\left(\frac{1}{13}\right), E(Y_i) = \frac{1}{13}, \text{Var}(Y_i) = \frac{12}{13^2}$$

$$Z = \sum_{i=1}^{1000} Y_i \sim N\left(\frac{1000}{13}, \frac{12000}{13^2}\right)$$

$$P(65 \leq Z \leq 90) = \Phi\left(\frac{90 - \frac{1000}{13}}{\sqrt{\frac{12000}{13^2}}}\right) - \Phi\left(\frac{65 - \frac{1000}{13}}{\sqrt{\frac{12000}{13^2}}}\right)$$

$$= \Phi\left(\frac{1.55}{0.43}\right) - \Phi\left(\frac{-1.41}{0.39}\right)$$

$$= \Phi(1.55) - 1 + \Phi(1.41)$$

$$= 0.31512 + 0.86016$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$n\bar{X} \sim N(n\mu, n\sigma^2)$$

$$(e) Y_i \sim \text{Bern}(0.1), E(Y_i) = 0.1, \text{Var}(Y_i) = 0.09$$

$$Z = \sum_{i=1}^{1000} Y_i \sim N(100, 90)$$

$$P(X \leq c) = P\left(\frac{X-100}{\sqrt{90}} \leq \frac{c-100}{\sqrt{90}}\right) = \Phi\left(\frac{c-100}{\sqrt{90}}\right) \approx 0.1$$

$$\frac{c-100}{\sqrt{90}} = -1.29 \Rightarrow c = 87.76 \approx 88$$

$$\frac{\sum_{i=1}^7 7 \times 6 \times 5 \times 3}{11 \times 10 \times 9 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0 \times 29}$$

$$E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$$

$$\begin{aligned}
&= [1 \times P(X=1|Y=1) + 2 \times P(X=2|Y=1) + 3 \times P(X=3|Y=1)] \times P(Y=1) + \dots + \\
&= 1 \times P(X=1, Y=1) + 2 \times P(X=2, Y=1) + 3 \times P(X=3, Y=1) + \dots + \\
&\quad 1 \times P(X=1, Y=3) + 2 \times P(X=2, Y=3) + 3 \times P(X=3, Y=3) \\
&= E(X)
\end{aligned}$$