MfS Lecture 4

Last time: Rules of diff, approximations

Today: Drawing functions and optimisation problems

Extreme points

Definition: Let $f(x)$ be a fcn defined on an interval I and $p \in I$ be a pt. Then p is a global maximum (resp. min.) of f on I if

$$f(p) \geq f(x) \ \forall\, x \in I \qquad (f(p) \leq f(x) \ \forall\, x \in I, \text{resp.}).$$

Example: $f(x) = x^2$ has a global minimum on $I = \mathbb{R}$ at $(0,0)$, as $f(0) < f(x)$ for all $x \in I$.

Example: $f(x) = \sin(x)$ has a global max at $x = \frac{\pi}{2}$

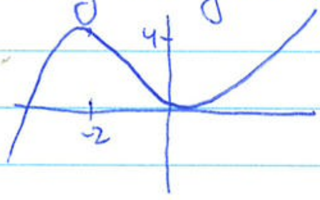
Key pt: $f(\frac{\pi}{2}) \geq f(x) \ \forall\, x$ - greater than or equal to.

Finding a global max/min can be hard, but we can often find local extrema:

Definition: Let $f(x)$ be a fcn defined on an interval I, and $p \in I$ a pt. We say p is a local max/(resp min) of f on I if $\exists$ an open interval $(a,b) \subseteq I$ s.t.

$$f(p) \geq f(x) \ \forall\, x \in (a,b) \quad (\text{resp., } f(p) \leq f(x) \ \forall\, x \in (a,b)).$$

Ex: If $f(x) = x^3 + 3x^2$, as shown, then $x = -2$ is a local max and $x = 0$ is a ~~global~~ local min. Are they also global extrema?



Let's look at 2 ~~egs~~ local minima: $f(x) = x^2$ and ~~f~~ $g(x) = |x|$ at $x=0$. What can we say about the derivatives at these pts?

either $f'(p) = 0$ or $g'(p)$ doesn't exist! This leads us to define these pts:

Definition: Let f be a diff fcn, and p in the domain of f a pt. We say

p is a critical pt if either
- $f'(p)=0$; or
- $f'(p)$ is not defined.

Theorem: (Fermat): If f has a local extremum at a pt p, and then p is a critical pt.

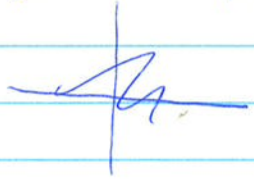
Remark: The opposite is NOT true! Think of $f(x)=x^3$. $f'(x)=3x^2$, so $x=0$ is a critical pt, but not an extremum.

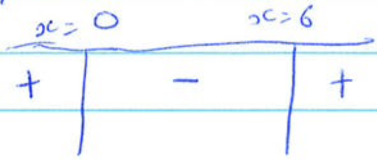
Hence to find critical pts, we:
1. Compute $f'(x)$
2. Set $f'(x)=0$ and solve for x.
3. Check if there are pts where f' is not defined. (but are in the domain of f)

Ex: Find the critical pts of: a) $f(x)=\frac{1}{3}x^3-3x^2+6$, b) $g(t)=\sqrt{t}$.
Answer: a) $f'(x)=x^2-6x$, so $x=0$ or $x=6$.
b) $g'(t)=\frac{1}{2\sqrt{t}}$ is not defined at $t=0$.

How do we tell if a critical pt is an extreme pt? Suppose p is a crit pt for f.
- if f' switches signs from +'ve to -'ve at p, p is a loc. max.
- if f' " -'ve to +'ve at p, loc. min.
- if f' doesn't change sign at p, p is neither.

Ex: $f'(x)=x^2-6x$, $x=0$, $x=6$. so $x=0$ is a max, 6 is a min.



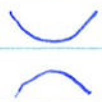


The 2nd derivative test only works if $f'(p)=0$. Then,
- if $f''(p)>0$, p is a loc. min
- if $f''(p)<0$ " loc. max
- if $f''(p)=0$ or DNE, use the 1st der test.

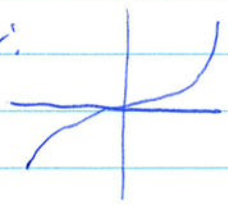
Concavity:

Definition: Let f be a twice diff. fcn. f is concave up at a pt if $f''(p)>0$ and concave down if $f''(p)<0$.

Being concave up means you look like an upwards facing parabola.
down downwards



Ex:



Where is $f(x)=x^3$ concave up/down from its graph. Confirm by computation.

$$f''(x)=6x \begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$$

So indeed, $f(x)$ is concave up if $x>0$, down if $x<0$.

Example/warning: Where is $\sqrt{x}$ concave up/down?
$f''(x) = -\frac{1}{4}x^{-3/2}$ is always negative if $x>0$.
Thus $\sqrt{x}$ is always concave down and always increasing.
Concavity is thus not ~~related~~ synonymous w/ increasing/decreasing.

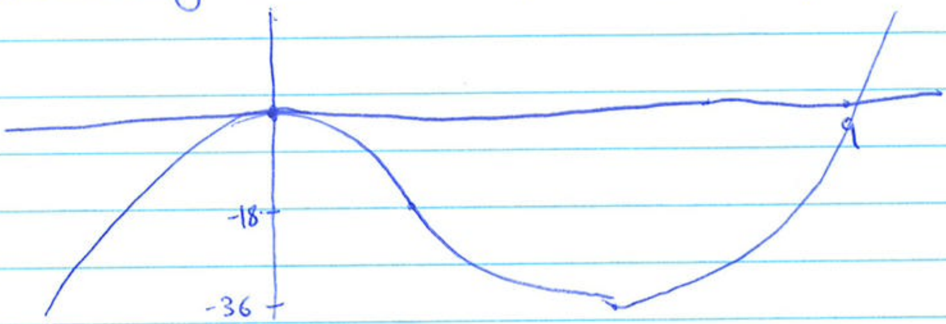
Def ~~Example~~: Let $f(x)$ be a fcn and f'' its second derivative. A pt p where f'' changes sign from positive to negative or vize-versa is called an inflection pt.

Remark: You must check if f'' changes sign at p — $f''(p)=0$ is NOT enough.

Ex: If $f(x)=\frac{1}{3}x^3-3x^2+6$, we found $f''(x)=2x-6$. Thus $f''(x)=0$ implies $x=3$, and f'' switches signs at $x=3$, so $x=3$ is an inflection pt.

Drawing functions.
Let $f(x)$ be a fcn. To draw it, follow these steps:
1. Find the local extrema of f and where f is increasing/decreasing.
2. Find where f is concave up/down and the inflection pts.
3. Find the x and y-intercepts.
4. Find $\lim_{x\to\infty} f$ and $\lim_{x\to-\infty} f$.

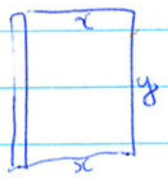
Ex: $f(x) = \frac{1}{3}x^3 - 3x^2$ has local max $(0,0)$ and min $(6,-36)$.
It's concave up if $x > 3$ and concave down otherwise.
$f(x) = 0$ only if $x = 0$ or $x = 9$, and $f(0) = (0,0)$.



Optimisation problems
Suppose you have some quantity you want to optimise, i.e.
- profit
- likelihood
- perimeter
- cost
- area
- volume

1. draw the problem, if possible
2. introduce variables of interest, i.e. length, width, height, etc
3. determine which variable you want to optimise
4. write an equation relating the variable you want to optimise in terms of the other variables to end up w/ an equation w/ 1 variable
5. identify the domain of your fcn
6. find the extrema

Example 1: A rectangular garden is to be constructed with one side a rock wall. You have 100m of wire. What dimensions give the largest garden?



$A = xy$

$2x + y = 100$

$y = 100 - 2x$

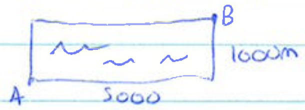
$A = x(100 - 2x) = 100x - 2x^2$

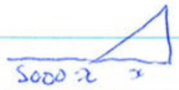
$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$

global max - why?

$y = 50$.

Ex: Power line from station to facility. 50 €/m beach, 130 €/m water.



A: 

$0 \le x \le 5000$. $C(x) = (5000-x)(50) + \sqrt{x^2+1000^2}(130)$.

$C'(x) = -50 + \dfrac{130x}{\sqrt{x^2+1000^2}}$, $x = 416.67$.

Ex: Operating rate of a factory over a 365 day period is

$f(t) = 100 + 800t/(t^2+90000)$, $0 \le t \le 365$.

On which day(s) is the rate maximised?

A: $f'(t) = \dfrac{800(t^2+90000) - 2t(800t)}{(t^2+90000)^2} = \dfrac{-800t^2+800(90000)}{(t^2+90000)^2}$

So $t = \sqrt{90000} = 300$ is a max.