MfS Formula Sheet

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Basic functions

Polynomials:

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Exponential properties:

$c^{a+b} = c^a e^b$

$(c^a)^b = c^{ab}$

$c^{-a} = \frac{1}{c^a}$

$c^0 = 1$

$c^x > 0$ for all x

Logarithms:

$\log_b(b^x) = x$

$b^{\log_b(x)} = x$

$\log_b(1) = 0$

$\log_b(b) = 1$

$\log_b(x^r) = r \log_b(x)$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$

Limits and continuity

Limits:

$\lim_{x \to a} f(x)$ exists and equals L if and only if the left and right-sided limits exist and equal L.

Continuity:

A function is continuous at a point p if $f(p)$ exists and equals $\lim_{x \to p} f(x)$.

Continuous functions include: polynomials

Derivatives

Definition:

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

Standard derivatives:

$f(x) = x^n$, then $f'(x) = nx^{n-1}$.

$f(x) = b^x$, then $f'(x) = \ln(b) b^x$.

$f(x) = \log_b(x)$ then $f'(x) = \frac{1}{\ln(b)x}$.

Rules of derivation:

Sum rule: $(f(x) + g(x))' = f'(x) + g'(x)$.

Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Quotient rule: $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

Chain rule: $(f(g(x))' = f'(g(x))g'(x)$

Approximations

Tangent lines:

At a point p: $T_1(x) = f(p) + f'(p)(x - p)$.

Taylor polynomial at a point p:

$\sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(x-p)^k$.

Optimisation

Critical point definition:

$f'(x) = 0$ *or* $f'(x)$ is not defined.

Fermat's theorem:

If p is a local extremum, then p is a critical point.

First derivative test: If $f'(x)$ switches signs

from + to - at p, p is a loc. max.

from - to + at p, p is a loc. min.

If $f'(x)$ doesn't change sign at p, p is neither.

Second derivative test:

$f'(x) = 0$ and $f''(x) > 0$: local min.

$f'(x) = 0$ and $f''(x) < 0$: local max.

otherwise inconclusive

Concavity definition:

Concave up: $f''(x) > 0$.

Concave down: $f''(x) < 0$

Inflection point: $f''(x)$ changes sign at p.

Integration

F is an antideritavive of f if :

$F'(x) = f(x)$.

Standard integrals:

$\int x^{-1} dx = \ln(|x|) + C$.

$\int x^n = \frac{x^{n+1}}{n+1} + C$.

$\int b^x = \frac{b^x}{\ln(b)} + C$.

Indefinite integrals: If F is any antiderivative of f, then

$\int_a^b f(x)dx = [F(b)]_{x=b}^{x=a} = F(b) - F(a)$.

Properties:

$\int_a^b f(x)dx = -\int_b^a f(x)dx$.

$\int_a^a f(x)dx = 0$.

Substitution rule:

$\int F'(g(x))g'(x)dx = \int F'(u)du$ where $u = g(x)$.

$\int_a^b F'(g(x))g'(x)dx = \int_{u(b)}^{u(a)} F'(u)du$.

Multivariate calculus

$\frac{\partial f}{\partial x} = f_x(x, y)$ represents

the change in f if only x moves and y is kep constant.

The functions f_{xy} and f_{yx} are equivalent (for functions we look at).

Equation of a tangent plane:

$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$

Gradient of f:

$\nabla f = (f_x(x, y), f_y(x, y))$.

Dot produce of two vectors:

If $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, then

$\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2$.

The slope of a surface given by $z = f(x, y)$ in the direction of a vector $\bar{u}$ is called the directional derivative of f, written $D_u f$

$D_{\bar{u}} f = \nabla f \cdot \bar{u}$.

The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{v})$ is $|\nabla f(\mathbf{v})|$and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{v})$.

Discriminant of $f(x, y)$:

$$D(x, y) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx}.$$

If $f_{xx} > 0$ or $f_{yy} > 0$ and $D(a, b) > 0$, then $f(a, b)$ is a local minimum.

If $f_{xx} < 0$or $f_{yy} < 0$ and $D(a, b) > 0$, then $f(a, b)$ is a local maximum.

If $D(a, b) < 0$, then $f(a, b)$ is a saddle point.

If $D(a, b) = 0$, no information is given.