Mathematics For Statisticians Homework3

1)(a) $\int \frac{1}{x+x\sqrt{x}}dx$

Choose $u=\sqrt{x}$, so $du=\frac{1}{2\sqrt{x}}dx \Rightarrow dx=2\sqrt{x}\,du$

$$\int \frac{1}{x+x\sqrt{x}}dx=\int \frac{2\sqrt{x}}{x+x\sqrt{x}}du=\int \frac{2}{\sqrt{x}+x}du=\int \frac{2}{u+u^2}du=\int \frac{2}{u(u+1)}du$$

$$=\int \frac{2}{u}-\frac{2}{u+1}du=2\ln u-2\ln(u+1)+C$$

Replace $u=\sqrt{x}$, thus $\int \frac{1}{x+x\sqrt{x}}dx=2\ln\sqrt{x}-2\ln(\sqrt{x}+1)+C$

(b) $\int \frac{e^{5x}}{1+e^{5x}}dx$

Choose $u=e^{5x}$, so $du=5e^{5x}dx \Rightarrow dx=\frac{1}{5e^{5x}}du$

$$\int \frac{e^{5x}}{1+e^{5x}}dx=\int \frac{e^{5x}}{1+e^{5x}}\frac{1}{5e^{5x}}du=\frac{1}{5}\int\frac{1}{1+u}du=\frac{1}{5}\ln(1+u)+C=\frac{1}{5}\ln(1+e^{5x})+C$$

2)(a) $\int \frac{1}{x^2}dx=-\frac{1}{x}+C$

Suppose $x=a$ bisects the area, so $\int_1^a \frac{1}{x^2}dx=\int_a^4\frac{1}{x^2}dx$, $1<a<4$

$$\int_1^a\frac{1}{x^2}dx=(-\frac{1}{x}+C)\Big|_1^a=-\frac{1}{a}-(-1)=1-\frac{1}{a}$$

$$\int_a^4\frac{1}{x^2}dx=(-\frac{1}{x}+C)\Big|_a^4=-\frac{1}{4}-(-\frac{1}{a})=\frac{1}{a}-\frac{1}{4}$$

Thus, $1-\frac{1}{a}=\frac{1}{a}-\frac{1}{4}\Rightarrow a=\frac{8}{5}$

(b) $y=\frac{1}{x^2}$, $1\le x\le 4 \Rightarrow x=\frac{1}{\sqrt{y}}$, $\frac{1}{16}\le y\le 1$

The area under $y=\frac{1}{x^2}$ between $1\le x\le 4$ can be expressed as the area between $x=f(y)=\begin{cases}4, & 0\le y<\frac{1}{16}\\ \frac{1}{\sqrt{y}}, & \frac{1}{16}\le y\le 1\end{cases}$ and $x=1$ which range in $0\le y\le 1$.

Suppose $y=b$ bisects the area, so $\int_0^b f(y)-1\,dy=\int_b^1 f(y)-1\,dy$, $0<b<1$

① If $0<b<\frac{1}{16}$, $\int_0^b(4-1)dy=\int_b^{\frac{1}{16}}(4-1)dy+\int_{\frac{1}{16}}^1(\frac{1}{\sqrt{y}}-1)dy$

$$(3y)\Big|_0^b=(3y)\Big|_b^{\frac{1}{16}}+(2y^{\frac{1}{2}}-y)\Big|_{\frac{1}{16}}^1\Rightarrow 3b=\frac{3}{16}-3b+(2-1)-(\frac{2}{4}-\frac{1}{16})\Rightarrow b=\frac{1}{8}$$

Because $b=\frac{1}{8}\notin(0,\frac{1}{16})$, this situation is impossibe.

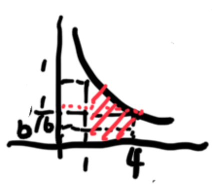
② If $\frac{1}{16}\le b<1$, $\int_0^{\frac{1}{16}}(4-1)dy+\int_{\frac{1}{16}}^b(\frac{1}{\sqrt{y}}-1)dy=\int_b^1(\frac{1}{\sqrt{y}}-1)dy$

$$(3y)\Big|_0^{\frac{1}{16}}+(2y^{\frac{1}{2}}-y)\Big|_{\frac{1}{16}}^b=(2y^{\frac{1}{2}}-y)\Big|_b^1$$

$$\Rightarrow\frac{3}{16}+(2b^{\frac{1}{2}}-b)-(\frac{2}{4}-\frac{1}{16})=(2-1)-(2b^{\frac{1}{2}}-b)\Rightarrow 4\sqrt{b}-2b=\frac{5}{4}\Rightarrow b=\frac{11-4\sqrt{6}}{8}$$

$\Rightarrow b=\frac{11-4\sqrt{6}}{8}$ or $\frac{11+4\sqrt{6}}{8}$

$\because \frac{1}{16}\le b<1 \therefore b=\frac{11-4\sqrt{6}}{8}$



3) $W=13.12+0.6215T-11.37v^{0.16}+0.3965Tv^{0.16}$

$\frac{\partial W}{\partial T}=0.6215+0.3965v^{0.16}$

$\frac{\partial W}{\partial T}\Big|_{T=-15,v=25}=0.6215+0.3965\cdot 25^{0.16}\approx 1.285$

If the actual temperature T decreased by 1°C, the expective temperature will drop by 1.285°C.

$\frac{\partial W}{\partial v}=-11.37\times 0.16v^{0.16-1}+0.3965T\times 0.16v^{0.16-1}$

$=-1.8192v^{-0.84}+0.06344Tv^{-0.84}$

$\frac{\partial W}{\partial v}\Big|_{T=-15,v=25}=-1.8192\times(25)^{-0.84}+0.06344\times(-15)\times(25)^{-0.84}\approx -0.185$

If the wind speed drops by 1km/h, the expective temperature will increase by 0.185°C.

$W(-16,25)=13.12+0.6215\times(-16)-11.37\times 25^{0.16}+0.3965\times(-16)\times 25^{0.16}$

≈ -26.471

$W(-15,24)=13.12+0.6215\times(-15)-11.37\times 24^{0.16}+0.3965\times(-15)\times 24^{0.16}$

≈ -24.998

$W(-15,25)=13.12+0.6215\times(-15)-11.37\times 25^{0.16}+0.3965\times(-15)\times 25^{0.16}$

≈ -25.186

$W(-15,25)-W(-16,25)\approx 1.285=\frac{\partial W}{\partial T}\Big|_{T=-15,v=25}$

$W(-15,25)-W(-15,24)\approx -0.188\approx -\frac{\partial W}{\partial v}\Big|_{T=-15,v=25}$

4) (a) ① When $t\ge 0$, $f(t)=\frac{1}{4}te^{-\frac{t}{2}}\ge 0$; when $t<0$, $f(t)=0$. Thus, $f(t)\ge 0$.

② $\int_{-\infty}^{+\infty}f(t)\,dt=\int_0^{+\infty}\frac{1}{4}te^{-\frac{t}{2}}dt$

Suppose $u=-\frac{t}{2}$, so $du=-\frac{1}{2}dt\Rightarrow dt=-2du$, $u=-\frac{t}{2}\Rightarrow t=-2u$

$\int\frac{1}{4}te^{-\frac{t}{2}}dt=\int\frac{1}{4}(-2u)e^{u}(-2)du=\int ue^{u}du=(u-1)e^{u}+C$

Replace by $u=-\frac{t}{2}$, $\int\frac{1}{4}te^{-\frac{t}{2}}dt=(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C$

Thus, $\int_0^{+\infty}\frac{1}{4}te^{-\frac{t}{2}}dt=\left[(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C\right]\Big|_0^{+\infty}=\lim_{t\to+\infty}-\frac{\frac{t}{2}+1}{e^{\frac{t}{2}}}-(-1)$

$=\lim_{t\to+\infty}-\frac{\frac{1}{2}}{e^{\frac{t}{2}}\cdot\frac{1}{2}}+1$ (L'Hôpital's rule) $=1$

Based on ① and ②, f(t) satisfies the properties of probability density function, so f(t) is a probability density function

For the following answers, suppose $F(t)=\int_{-\infty}^{t}f(x)dx$, F(t) is the cumulative density function of random variable t.

(b) $P(t\le 0.5)=F(0.5)=\int_{-\infty}^{0.5}f(t)dt=\int_0^{0.5}\frac{1}{4}te^{-\frac{t}{2}}dt=\left[(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C\right]\Big|_0^{0.5}$

$=(-\frac{0.5}{2}-1)e^{-\frac{0.5}{2}}-(-1)\approx 0.026$

(c) $P(2\le t\le 4)=F(4)-F(2)=\int_2^4 f(t)dt=\left[(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C\right]\Big|_2^4\approx 0.330$

(d) $P(t>5)=1-F(5)=1-\int_0^5 f(t)dt=1-\left[(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C\right]\Big|_0^5\approx 0.287$

(e) $E(t)=\int_{-\infty}^{+\infty}t f(t)dt=\int_0^{+\infty}\frac{1}{4}t^2e^{-\frac{t}{2}}dt$

Use Integration by Part: $\int_a^b u(x)v'(x)dx=u(x)v(x)\Big|_a^b-\int_a^b v(x)u'(x)dx$

Suppose $u(t)=t^2$, $v(t)=e^{-\frac{t}{2}}$, so $u'(t)=2t$, $v'(t)=-\frac{1}{2}e^{-\frac{t}{2}}$

$\int_0^{+\infty}\frac{1}{4}t^2e^{-\frac{t}{2}}dt=\int_0^{+\infty}\frac{1}{4}u(t)(-2)\times v'(t)dt=-\frac{1}{2}\int_0^{+\infty}u(t)v'(t)dt$

$=\left[-\frac{1}{2}u(t)v(t)\right]\Big|_0^{+\infty}-(-\frac{1}{2})\int_0^{+\infty}u'(t)v(t)dt$

$=-\frac{1}{2}t^2e^{-\frac{t}{2}}\Big|_0^{+\infty}+\frac{1}{2}\int_0^{+\infty}2te^{-\frac{t}{2}}dt$

L'Hôpital's rule $\to$ $=-\frac{1}{2}\lim_{t\to+\infty}\frac{t^2}{e^{\frac{t}{2}}}+\frac{1}{2}\int_0^{+\infty}2te^{-\frac{t}{2}}dt$

$=\int_0^{+\infty}te^{-\frac{t}{2}}dt$

Suppose $w=-\frac{t}{2}$, so $dw=-\frac{1}{2}dt$, $dt=-2dw$, $t=-2w$

$\int te^{-\frac{t}{2}}dt=\int -2we^{w}(-2)dw=4\int we^{w}dw=4(w-1)e^{w}+C$

Replace by $w=-\frac{t}{2}$, $\int te^{-\frac{t}{2}}dt=4(-\frac{t}{2}-1)e^{-\frac{t}{2}}+C$

Thus, $\int_0^{+\infty}te^{-\frac{t}{2}}dt=\left[4(-\frac{t}{2}-1)e^{-\frac{t}{2}}\right]\Big|_0^{+\infty}=0-4\times(-1)\times 1=4$

L'Hôpital's rule

So, $E(t)=4$