CHAPTER 2

Basic Electromagnetics

Study of magnetic fields provides an explanation of how magnetic actuators and sensors work. Hence this chapter presents the basic principles of *electromagnetics*, a subject that includes magnetic fields.

In reviewing electromagnetic theory, this chapter also introduces various parameters and their symbols. The symbols and notations used in this chapter will be used throughout the book, and most are also listed in Appendix A along with their units.

2.1 VECTORS

Magnetic fields are vectors, and thus it is useful to review mathematical operations involving vectors. A *vector* is defined here as a parameter having both magnitude and direction. Thus it differs from a *scalar*, which has only magnitude (and no direction). In this book, vectors are indicated by **bold** type, and scalars are indicated by italic non-bold type.

To define *direction*, rectangular coordinates are often used. Also called *Cartesian coordinates*, the position and direction are specified in terms of x, y, and z. This book denotes the three rectangular direction unit vectors as $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$; they all have magnitude equal to one.

Common to several vector operations is the "del" operator (also termed "nabla"). It is denoted by an upside down (inverted) delta symbol, and in rectangular coordinates is given by:

$$\nabla = \frac{\partial}{\partial x}\mathbf{u}_x + \frac{\partial}{\partial y}\mathbf{u}_y + \frac{\partial}{\partial z}\mathbf{u}_z \tag{2.1}$$

2.1.1 Gradient

A basic vector operation is *gradient*, also called "grad" for short. It involves the del operator operating on a scalar quantity, for example, temperature T. In rectangular coordinates the gradient of T is expressed as:

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{u}_x + \frac{\partial T}{\partial y}\mathbf{u}_y + \frac{\partial T}{\partial z}\mathbf{u}_z \tag{2.2}$$

Magnetic Actuators and Sensors, Second Edition. John R. Brauer.

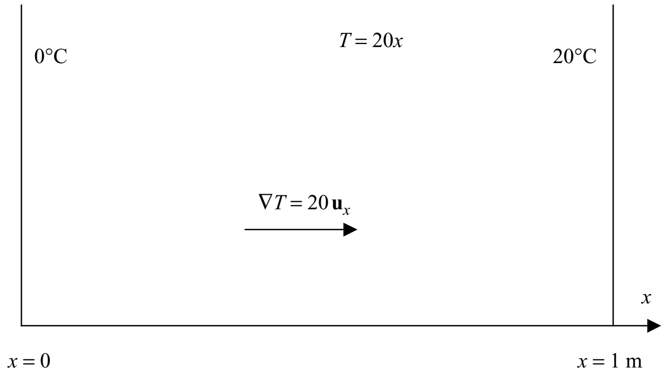

FIGURE 2.1 Temperature distribution and gradient versus position x.

An example of temperature gradient is shown in Figure 2.1. A block of ice is placed to the left of $x = 0$, for position x values less than zero. At $x = 1$ m, a wall of room temperature 20°C is located. Assuming the temperature varies linearly from $x = 0$ to $x = 1$ m, then:

$$T = 20x \tag{2.3}$$

To find the temperature gradient, substitute (2.3) into (2.2), obtaining:

$$\nabla T = 20\,\mathbf{u}_x\,^\circ\mathrm{C/m} \tag{2.4}$$

The direction of the gradient is the direction of maximum rate of change of the scalar (here temperature). The magnitude of the gradient equals the maximum rate of change per unit length. Since this book uses the SI (Système International) or metric system of units, all gradients here are per meter.

Two other vector operations involve multiplication with the del operator. Another word for multiplication is product, and there are two types of vector products.

Example 2.1 Gradient Calculations Find the gradient of the following temperature distribution at locations $(x,y,z) = (1,2,3)$ and $(4,-2,5)$:

$$T = 5x + 8y^2 + 3z \tag{E2.1.1}$$

Solution You must be careful in taking the partial derivatives in the gradient equation (2.2), and you must first find the expression for the gradient before evaluating it at any location. Thus the first step is to find the gradient expression:

$$\nabla T = \frac{\partial(5x + 8y^2 + 3z)}{\partial x}\mathbf{u}_x + \frac{\partial(5x + 8y^2 + 3z)}{\partial y}\mathbf{u}_y + \frac{\partial(5x + 8y^2 + 3z)}{\partial z}\mathbf{u}_z \tag{E2.1.2}$$

The partial derivative of y with respect to x is zero, and so are all other partial derivatives of non-alike variables, and thus we obtain:

$$\nabla T = \frac{\partial(5x)}{\partial x}\mathbf{u}_x + \frac{\partial(8y^2)}{\partial y}\mathbf{u}_y + \frac{\partial(3z)}{\partial z}\mathbf{u}_z \tag{E2.1.3}$$

Carrying out the derivatives gives:

$$\nabla T = 5\mathbf{u}_x + 16y\mathbf{u}_y + 3\mathbf{u}_z \tag{E2.1.4}$$

Finally, the gradient can be evaluated at the two specified locations:

$$\nabla T(1, 2, 3) = 5\mathbf{u}_x + 16(2)\mathbf{u}_y + 3\mathbf{u}_z = 5\mathbf{u}_x + 32\mathbf{u}_y + 3\mathbf{u}_z \tag{E2.1.5}$$

$$\nabla T(4, -2, 5) = 5\mathbf{u}_x + 16(-2)\mathbf{u}_y + 3\mathbf{u}_z = 5\mathbf{u}_x - 32\mathbf{u}_y + 3\mathbf{u}_z \tag{E2.1.6}$$

Recall that the gradient must always be a vector. Its magnitude is always the square root of the sum of the squares of its x, y, and z components.

2.1.2 Divergence

The *scalar product* or *dot product* obtains a scalar and is denoted by a "dot" symbol. Applying it to the del operator and a typical vector, here called **J,** obtains "del dot **J,**" called the *divergence* of the vector:

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \tag{2.5}$$

The divergence of a vector is its net outflow per unit volume, which is a scalar. In some cases, the divergence is zero, that is, the vector is *divergenceless*. For example, if **J** is current density (to be defined later), then Kirchhoff's law which shows that total current at a point is zero ($\sum I = 0$) can be expressed as a divergenceless **J**:

$$\nabla \cdot \mathbf{J} = 0 \tag{2.6}$$

Figure 2.2 shows typical fields with and without divergence.

2.1.3 Curl

The other type of vector product obtains a vector and is called the *vector product* or *cross product*. It is expressed using a cross or x sign. If it is the product of the del

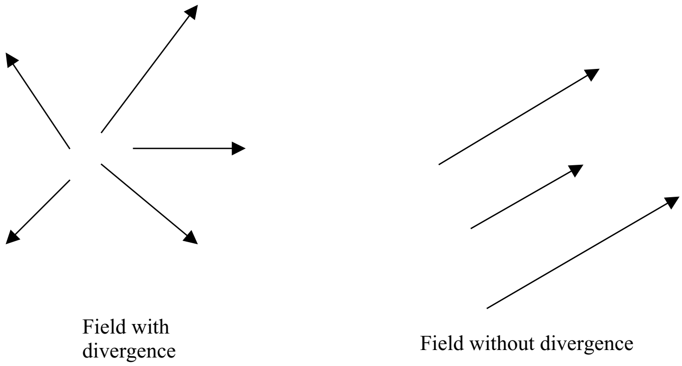

FIGURE 2.2 Field with and without divergence.

operator and a typical vector, here called **A**, one obtains a vector "del cross **A**," called the *curl* of a vector. It can be expressed as a 3 by 3 determinant:

$$\nabla \times \mathbf{A} = \begin{pmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \\ \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \end{pmatrix} \tag{2.7}$$

A 3 by 3 determinant is evaluated by the "basket-weave" method. The reader is recommended to write out such a 3 by 3 determinant and connect its elements with diagonal lines. Row 1 column 1 (the (1,1) or top left entry) is multiplied by row 2 column 2 and then by row 3 column 3, resulting in one of six terms of the cross product. The next term is found by multiplying the (1,2) entry by the (2,3) entry and the (3,1) entry. The next term multiplies the (1,3), (2,1), and (3,2) entries. The next three terms must be subtracted, and consist of (3,1) times (2,2) times (1,3), then (3,2) times (2,3) times (1,1), and finally (3,3) times (2,1) times (1,2). Thus (2.7) can be rewritten as:

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{u}_z \tag{2.8}$$

Besides the rectangular (x,y,z) coordinates analyzed above, engineers often use cylindrical coordinates or spherical coordinates. In cylindrical and spherical coordinates there are differences in the gradient, divergence, and curl equations [1].

In general, curl is analogous to a wheel that rotates, and thus curl is sometimes called *rot*. Figure 2.3 shows a water wheel and its curl. The wheel has paddles and is rotated by a stream of water. The stream may either be a river (mill stream) or may be a diversion channel inside a dam. Note that the wheel has curl (rotation) about its z axis because its velocities vary with position. Plots of the x and y components of velocity **v** are shown as functions of y and x, respectively. The partial derivatives of (2.8) produce a z-component of curl. The partial of velocity component v_y with

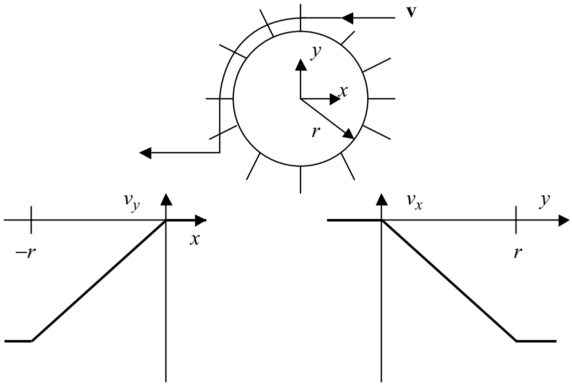

FIGURE 2.3 A rotating wheel is analogous to curl.

respect to x gives a positive contribution, and the negative partial of v_x with respect to y also gives a positive contribution, resulting in a curl of **v** with a large positive z component. Thus the curl is directed along the axis of rotation.

Figure 2.3 also indicates a relation between curl and the integral all around the circular path on the surface of the wheel. The velocity follows the outer circular path. The integral is called the *circulation*. In the next section, Stokes' law will mathematically define the relation between circulation and curl.

Example 2.2 Divergence and Curl of a Vector Find the divergence and curl at location (3,2,−1) of the vector:

$$\mathbf{A} = [8x^4 + 6(y^2 - 2)]\mathbf{u_x} + [9x + 10y + 11z]\mathbf{u_y} + [4x]\,\mathbf{u_z} \tag{E2.2.1}$$

Solution You must first find the expressions for divergence and curl, and then evaluate them at the desired location.

In finding the divergence using (2.5), only partial derivatives of *like variables* (such as x with respect to x) are involved. Thus we obtain:

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{\partial[8x^4 + 6(y^2 - 2)]}{\partial x} + \frac{\partial[9x + 10y + 11z]}{\partial y} + \frac{\partial[4x]}{\partial z}\end{aligned} \tag{E2.2.2}$$

Again, a partial derivative with respect to x treats y and z as constants, the partial with respect to y treats x and z as constants, and the partial with respect to z treats x and y as constants. Thus we obtain for the divergence expression:

$$\nabla \cdot \mathbf{A} = \frac{\partial[8x^4]}{\partial x} + \frac{\partial[10y]}{\partial y} + \frac{\partial[0]}{\partial z} = 32x^3 + 10 \tag{E2.2.3}$$

which evaluated at (3,2,−1) gives 874. Recall that the divergence is always a scalar.

In finding the curl using (2.8), only partial derivatives of *unlike variables* (such as y with respect to x) are involved. Thus (2.8) obtains:

$$\nabla \times \mathbf{A} = \left(\frac{\partial[4x]}{\partial y} - \frac{\partial[9x + 10y + 11z]}{\partial z} \right) \mathbf{u}_x + \left(\frac{\partial[8x^4 + 6y^2 - 12]}{\partial z} - \frac{\partial[4x]}{\partial x} \right) \mathbf{u}_y + \left(\frac{\partial[9x + 10y + 11z]}{\partial x} - \frac{\partial[8x^4 + 6y^2 - 12]}{\partial y} \right) \mathbf{u}_z \tag{E2.2.4}$$

Again, since partials of unlike variables are zero, a simplified equation is obtained:

$$\nabla \times \mathbf{A} = \left(\frac{\partial[0]}{\partial y} - \frac{\partial[11z]}{\partial z} \right) \mathbf{u}_x + \left(\frac{\partial[0]}{\partial z} - \frac{\partial[4x]}{\partial x} \right) \mathbf{u}_y + \left(\frac{\partial[9x]}{\partial x} - \frac{\partial[6y^2]}{\partial y} \right) \mathbf{u}_z \tag{E2.2.5}$$

which yields the curl expression:

$$\nabla \times \mathbf{A} = (-11)\mathbf{u}_x + (-4)\mathbf{u}_y + (9 - 12y)\mathbf{u}_z \tag{E2.2.6}$$

Substituting the point (3,2,−1) yields the final answer:

$$\nabla \times \mathbf{A} = (-11)\mathbf{u}_x + (-4)\mathbf{u}_y + (-15)\mathbf{u}_z \tag{E2.2.7}$$

2.2 AMPERE'S LAW

With the background in vector operations presented above, the fundamental source of magnetic fields can now be presented. The origin of magnetic fields is expressed by Ampere's law, named after André-Marie Ampere of France.

Ampere's law at any point in space states that the curl of static magnetic field intensity **H** equals current density **J**:

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2.9}$$

where **H** is magnetic field intensity in A/m, and current density **J** is in A/m^2. These units are SI, and will be used throughout this book as listed in Appendix A.

In air, related to **H** is magnetic flux density **B** by:

$$\mathbf{B} = \mu_o \mathbf{H} \tag{2.10}$$

where **B** is magnetic flux density in teslas and μ_o is the permeability of free space (vacuum) or air. The unit, tesla, has the symbol T and is named after the renowned American inventor Nikola Tesla. One tesla (1 T) equals one weber per square meter (1 Wb/m^2). The value $\mu_o = 12.57\times10^{-7}$ H/m. Thus in air, vector **B** equals vector **H** multiplied by a very small number. Often in this book and other books, **B** is simply called the magnetic field. A field in this context is any quantity that can vary over space, and **B** is a vector field.

Because the curl of **H** equals **J**, **H** "circles" around an axis consisting of current much as the wheel circles around its axis in Figure 2.3. Another way to express Ampere's law of (2.9) is to integrate it over a surface **S** to obtain:

$$\int (\nabla \times \mathbf{H}) \cdot \mathbf{dS} = \int \mathbf{J} \cdot \mathbf{dS} \tag{2.11}$$

The units of both sides of this equation are amperes. The surface **S** is a vector with direction normal (perpendicular) and magnitude equal to the surface area.

There is a purely mathematical vector identity that can be used to replace the surface integral of the curl in the left side of (2.11). As mentioned at the end of the preceding section, Stokes' law replaces the surface integral by a *closed* path (or line) integral, giving the most common expression for Ampere's law in integral form:

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int \mathbf{J} \cdot \mathbf{dS} = NI \tag{2.12}$$

where **l** is the vector path length in meters, and the path being closed is indicated by the circle on the integral sign. Note that the total current, the surface integral of current density, can also be written as the product of current I times the number of conductors N carrying that current. To create large **H** and **B** with reasonably small current I values, often a coil winding with many conductors (or turns) N is used in magnetic devices.

While **B** is magnetic flux density, its integral over any surface **S** is called magnetic flux. Flux is the Latin word for the English word flow, and magnetic flux flows around much like other fluids such as water. Since the SI units of **B** are teslas or webers per square meter, magnetic flux has units of webers. Using ϕ for flux, the surface integral can be written as:

$$\phi = \int \mathbf{B} \cdot \mathbf{dS} \tag{2.13}$$

An important property of magnetic flux is that if **S** is a *closed* surface, such as a spherical surface or any surface that completely encloses a volume, then the total magnetic flux through the closed surface is zero. A closed surface integral is indicated by a circle on the integral sign, and thus magnetic flux through a closed surface obeys:

$$\phi_c = \oint \mathbf{B} \cdot \mathbf{dS} = 0 \tag{2.14}$$

Since divergence has been previously defined as the net output flux per unit volume, the zero flux of (2.14) applied to a tiny volume and its closed surface means that:

$$\nabla \cdot \mathbf{B} = 0 \tag{2.15}$$

Thus magnetic flux density is always divergenceless. Since the total magnetic flux through any closed surface is zero, magnetic flux flows in a manner similar to that for an incompressible fluid. In air, since $\mathbf{B} = \mu_o \mathbf{H}$, both $\mathbf{B}$ and $\mathbf{H}$ must both "circle around" the "axis" of a current-carrying wire.

Example 2.3 Ampere's Law at a Point and Along a Closed Path Apply Ampere's law to two situations:

(a) Given the magnetic field intensity expression (in A/m):

$$\mathbf{H} = [8x^4 + 6(y^2 - 2)]\,\mathbf{u_x} + [9x + 10y + 11z]\,\mathbf{u_y} + [4x]\mathbf{u_z} \tag{E2.3.1}$$

Find the expression for current density $\mathbf{J}$ at location (2,4,6).

(b) Given a region of four conductors, each carrying current $I = 5$ A outward. As shown in Figure E2.3.1, two closed paths are defined, $\mathbf{l}_1$ and $\mathbf{l}_2$. Find the integral of $\mathbf{H}$ along each of the closed paths.

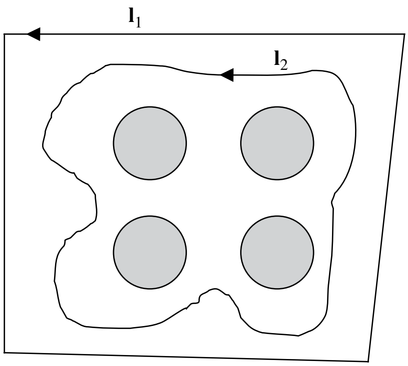

FIGURE E2.3.1 Two closed paths adjacent to a current-carrying region.

Solution

(a) The current density $\mathbf{J}$ at any point is the curl of $\mathbf{H}$ at that point. The $\mathbf{H}$ expression (E2.3.1) is recognized as identical to (E2.2.1) for $\mathbf{A}$, for which the curl has already been found. Thus:

$$\mathbf{J} = \nabla \times \mathbf{H} = (-11)\mathbf{u}_x + (-4)\mathbf{u}_y + (9 - 12y)\mathbf{u}_z \tag{E2.3.2}$$

which evaluated at (2,4,6) gives:

$$\mathbf{J} = (-11)\mathbf{u}_x + (-4)\mathbf{u}_y + (-39)\mathbf{u}_z \tag{E2.3.3}$$

in A/m^2.

(b) Both the number of conductors and their individual currents are known. Their product $NI = 20$ A, since number N is dimensionless. Since both closed paths enclose all conductors, the integral of **H** along each of the closed paths equals 20 A.

2.3 MAGNETIC MATERIALS

The main magnetic property that can vary among materials is permeability. The permeability of free space (vacuum), μ_o, is also applicable in air. Also, many other materials, including copper and aluminum, have free space permeability μ_o.

The general symbol for permeability is μ, and materials with permeability other than μ_o are often expressed in terms of relative permeability μ_r defined using:

$$\mu = \mu_r \, \mu_o \tag{2.16}$$

where we recall that $\mu_o = 12.57\text{E}{-7}$ H/m, where throughout this book yEz means $y \times 10^z$, and thus $\mu_o = 12.57 \times 10^{-7}$ H/m. Several materials have $\mu_r \gg 1$ and thus are much more permeable than air. Permeability is another word common to both magnetics and fluid flow. Highly fluid-permeable earth, such as sand, conducts water flow much better than does low fluid-permeable rock. Thus magnetic flux, like fluids, prefers to flow in materials of high permeability.

Magnetic materials with high relative permeability μ_r are said to be magnetically *soft*. The most common soft magnetic materials are *ferromagnetics,* including iron (Fe or ferrous material), cobalt (Co), and nickel (Ni). These elements are neighbors on the periodic table, with atomic numbers 26, 27, and 28, respectively. The Curie temperatures of these elements are 770, 1131, and 358°C, respectively, and the high permeability exists as long as the elements are kept below the Curie temperature. Besides these three pure elements, alloys containing these elements are usually also ferromagnetic with high permeability. The most common ferromagnetic alloys are steels, which contain both iron and carbon. The relative permeability of typical steel is often on the order of 2000. Steel is often used as the main inner or *core* material of magnetic devices.

The reason for the high permeability of certain materials is that they contain many *magnetic domains.* Each domain has **B** in a particular direction created by its atomic electron motion (to be discussed further in Chapter 10). As **H** is applied and increased, more domains rotate and/or expand in the direction of **H**, causing magnetization **M** that increases **B** in accordance with:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2.17}$$

where sometimes **M** is proportional to the applied **H**:

$$\mathbf{M} = \chi \mathbf{H} \tag{2.18}$$

where the dimensionless proportionality constant χ is called magnetic susceptibility. Thus in magnetic materials:

$$\mathbf{B} = \mu \mathbf{H} = \mu_o(1 + \chi)\mathbf{H} \tag{2.19}$$

Each magnetic domain is of size as large as 0.1 mm = 100 μm [2]. Since permeability μ is a macroscopic or average number over all domains, it usually applies only for materials with at least one dimension exceeding a few micrometers. The macroscopic concept of permeability therefore does not usually apply for material samples of nanometer dimensions in all three directions. Often nanotechnology relies instead on microscopic effects such as those of quantum mechanics, which is not covered in this book except briefly in Section 11.8.

As an example of the effect of permeability on magnetic field **B**, Figure 2.4 shows the application of Ampere's law to a single circular wire carrying current *I*. Ampere's law and material permeability are here used to find **B** at any location of radius $r > r_w$, the wire radius. First, Ampere's law over the closed path of radius r completely enclosing the current I gives:

$$\oint \mathbf{H} \cdot \mathbf{dl} = 2\pi r H = I \tag{2.20}$$

Note that the closed path integral can be replaced by the path length $2\pi r$ times a scalar constant *H,* the magnitude of the magnetic field intensity, because symmetry requires the field to be independent of angular position, that is to be a constant magnitude at any given radius. The direction of **H** must circle around the current as shown in Figure 2.4. Such a peripheral direction is usually called a circumferential polar direction $\mathbf{u}_{\varnothing}$. The direction follows the *right-hand rule:* For your right-hand

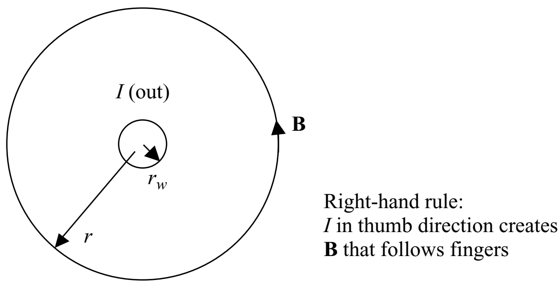

FIGURE 2.4 Magnetic field of a wire found using Ampere's law.

thumb pointing in the direction of the current, the direction of **H** circles around in the direction of the right-hand fingers. Then solving (2.20) gives the magnitude:

$$H = I/(2\pi r) \tag{2.21}$$

Finally, the magnetic flux density **B** is found by multiplying **H** by permeability μ of the material surrounding the wire. If the material is air, then the flux density magnitude is:

$$B = \mu_o H = \mu_o I/(2\pi r) \tag{2.22}$$

Note that the magnetic field is inversely proportional to radius from the current, assuming that the radius is no smaller than the wire radius.

If the material surrounding the wire is steel of relative permeability 2000, then the flux density magnitude is increased to:

$$B = \mu_r \mu_o H = 2000\mu_o I/(2\pi r) \tag{2.23}$$

Note that the direction of **B** is the same as the direction of **H**, which is true for all materials except *anisotropic* (directionally dependent) magnetic materials. Air and many ferromagnetic materials are magnetically *isotropic* with a scalar permeability. Anisotropic magnetic materials require a tensor permeability which is not discussed in this book but has been investigated elsewhere [3,4]. Tensor material properties are used in Chapter 10.

An important property of magnetic permeability to be discussed next is the *nonlinear B–H curve.* Such nonlinearity is exhibited by all ferromagnetic materials. Their high relative permeability, such as the 2000 assumed in (2.23), is only applicable for low values of flux density. For high values of flux density, *saturation* is said to occur. Flux no longer flows as easily, similar to a towel being saturated with water and no longer picking up as much liquid. The ratio *B/H* gradually decreases and thus the permeability is no longer a constant. Figure 2.5 shows a typical nonlinear *B–H* curve for steel. It is customarily plotted with *H* along the horizontal axis. Note that in the neighborhood of 1.5 T or so, the curve has a “knee” that transitions to a much flatter slope. The *incremental permeability,* defined as the slope, gradually decreases to the permeability of free space. Usually the slope equals μ_o at a *saturation flux density* somewhat above 2 T.

To obtain *B–H* curves, steel suppliers and other sources can be consulted. While their curves are often in the SI units of T and A/m, sometimes they are instead in CGS units. For flux density *B* values given in gauss (G), multiply by 1.E−4 to obtain teslas. For field intensity *H* values in oersteds (Oe), multiply by 79.577 to obtain amperes per meter. Steel suppliers can provide better *B–H* curves (with higher permeabilities) with more expensive heat treatment or by adding alloy materials such as Co.

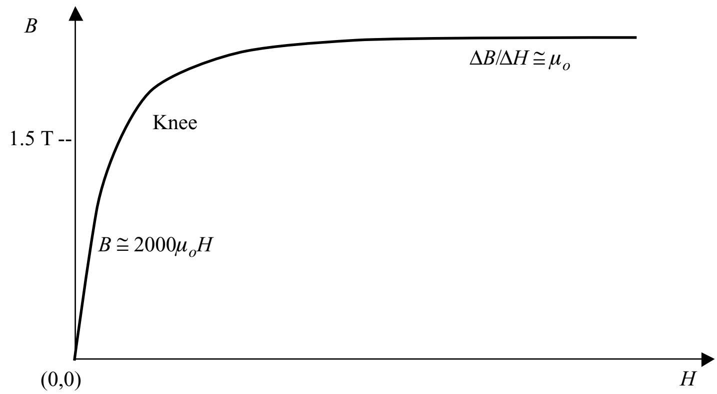

FIGURE 2.5 Typical nonlinear magnetization curve of steel.

A useful approximate expression for steel *B–H* curves uses three constants and was published in 1975 [5]:

$$H = (k_1 e^{k_2 B^2} + k_3)B \tag{2.24}$$

where values of the three constants are listed in Table 2.1 for typical types of steel. Cast steel is cast or forged into its desired shape. Cold rolled steel has been formed into a thin sheet in a steel rolling mill at typical room temperature. Annealed steel has been heat treated, where annealing involves high temperature heating often in low oxygen atmospheres, followed by slow cooling. The annealing temperature, length of time, atmosphere, cooling process, and other details all can significantly affect the *B–H* curve. The *B–H* relation also may actually be a set of multiple curves depending on history of applied *H*; such *hard magnetic* materials will be discussed in Chapter 5. Mechanical hardness also tends to follow the magnetic hardness (or softness) of materials. However, hard magnetic materials often are brittle, not nearly as mechanically strong as steel.

An improvement on the nonlinear *B–H* relation of (2.24) has been proposed recently by Mark A. Juds, who notes that (2.24) gives permeability:

$$\mu = B/H = 1/(k_1 e^{k_2 B^2} + k_3) \tag{2.25}$$

TABLE 2.1 Constants for (2.24) uncorrected *B–H* Curves of Three Typical Types of Steel

	k_1	k_2	k_3
Cast	49.4	1.46	520.6
Cold-rolled	3.8	2.17	396.3
Annealed	2.6	2.72	154.4

But since the slope (incremental μ) should approach μ_o at high B and high H, Juds adds one term to give [6]:

$$\mu = B/H = [1/(k_1 e^{k_2 B^2} + k_3)] + \mu_o \tag{2.26}$$

This new equation improves the *B–H* curve fit at high *B* and significantly enhances the numerical convergence of nonlinear iterative solutions. Appendix B lists the three constants k_1, k_2, and k_3 determined by Juds for a large number of different ferromagnetic materials for the *B–H* relation (2.26) with the corrected final permeability.

Another important property of magnetic materials is their electrical conductivity. Air has zero electrical conductivity (except in special cases such as lightning), but ferromagnetic materials usually have high electrical conductivities. To see why electrical conductivity is important, Faraday's law must be presented.

Example 2.4 Magnetic Flux Density in Various Materials Surrounding a Wire

(a) A copper wire of radius 1 mm is placed at location (1,0,0) m, carrying current of 100 kA in the $+z$ direction. Find the vector **B** at location (3,0,0) for the wire embedded in the following materials that extend infinitely far in all three directions: (1) air, (2) steel with constant (assuming no saturation) relative permeability = 2500, (3) steel with a *B–H* curve with the following (*B,H*) values: (0,0), (1.5,1000), (1.8,7958), . . .

(b) Repeat the above if the current direction is reversed to the $-z$ direction.

Solution

(a) The radius from (1,0,0) to (3,0,0) is 2 m. From the right-hand rule, the direction of **B** at point (3,0,0) is in the $+y$ direction. From (2.23), the magnitude is:

$$B = \mu_r \mu_o I/(2\pi r) = \mu_r(12.57\text{E}-7)(100\text{E}3)/(4\pi) = \mu_r(1.\text{E}-2) \tag{E2.4.1}$$

(1) Thus for air, $\mathbf{B} = 1.\text{E}-2\ \text{T}\ \mathbf{u}_y$

(2) For steel with relative permeability = 2500, $\mathbf{B} = 25\ \text{T}\ \mathbf{u}_y$

(3) For steel with the nonlinear *B–H* curve,

$$H = I/(2\pi r) = 1.\text{E}5/(4\pi) = 7958\ \text{A/m} \tag{E2.4.2}$$

From the (*B,H*) curve, the corresponding flux density $\mathbf{B} = 1.8\ \text{T}\ \mathbf{u}_y$

(b) When the current direction is reversed in the above three cases, the right-hand rule shows that the direction of **B** changes to $-\mathbf{u}_y$

2.4 FARADAY'S LAW

Following Ampere's law in importance for magnetic devices is Faraday's law. It was discovered by Michael Faraday of England. While Ampere's law deals primarily with current, Faraday's law deals primarily with voltage.

Faraday's law at any point in space equates the negative partial time derivative of **B** with the curl of the electric field intensity **E**:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.27}$$

where **E** is in volts per meter. The line or path integral of **E** is the negative of voltage V_1:

$$V_1 = -\int \mathbf{E} \cdot \mathbf{dl} \tag{2.28}$$

Taking the integral of both sides of (2.27) over any unchanging surface **S** (a surface changed by motion will be discussed later in this section) gives:

$$\int (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot \mathbf{dS} \tag{2.29}$$

As for Ampere's law in integral form, Stokes' law can be used to replace the surface integral of curl by a closed line integral. Thus (2.24) becomes:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \mathbf{B} \cdot \mathbf{dS} \tag{2.30}$$

which is Faraday's law in integral form. The electric field (intensity) is *induced* by a time-varying magnetic field (flux density).

As previously mentioned, often magnetic devices have N conductors, usually arranged in coils of wire with N turns. The voltage induced in such a coil, using the usual sign convention, is $-N$ times that of one turn, and thus from (2.28) the coil voltage V is:

$$V = N \oint \mathbf{E} \cdot \mathbf{dl} \tag{2.31}$$

where the line integral is over the closed path of one turn. Finally, multiplying both sides of (2.30) by N and using (2.31) gives the expression for induced coil voltage:

$$V = -N \frac{\partial}{\partial t} \int \mathbf{B} \cdot \mathbf{dS} \tag{2.32}$$

Recall from (2.13) that the surface integral of flux density is flux, giving:

$$V = -N\frac{\partial\varphi}{\partial t} \tag{2.33}$$

This form of Faraday's law states that voltage is ($-N$) times the time rate of change of magnetic flux. Note that the flux in a stationary coil must change with time in order for voltage to be induced.

Assuming that the number of turns does not change with time, another expression of Faraday's law is:

$$V = -\frac{\partial\lambda}{\partial t} \tag{2.34}$$

where λ is called *flux linkage* and is defined by:

$$\lambda = N\phi \tag{2.35}$$

Since N is dimensionless, flux linkage has units of webers.

If a conductor moves with a velocity $\mathbf{v}$ through a magnetic flux density $\mathbf{B}$, Faraday's law (for a changing surface) shows that a *motional electric field* is induced given by:

$$\mathbf{E}_{\text{motion}} = \mathbf{v} \times \mathbf{B} \tag{2.36}$$

The velocity $\mathbf{v}$ is assumed in this book to be much less than the speed of light, so that relativistic effects can be ignored. The motionally induced voltage is found by the line integral of (2.28). Note that a motional voltage can be produced even by a constant (DC) magnetic flux density.

Since a voltage is induced by a time-varying magnetic field, current I may also flow according to Ohm's law of electric circuits:

$$I = V/R \tag{2.37}$$

where R is resistance in ohms, named after Georg Ohm of Germany. Another form of Ohm's law is for fields:

$$\mathbf{J} = \sigma\mathbf{E} \tag{2.38}$$

where σ is electrical conductivity in the reciprocal of ohm-meters, also called siemens per meter (S/m). The reciprocal of conductivity is resistivity ρ in ohm-meters.

If $\mathbf{E}$ is induced by magnetic fields, then the current density $\mathbf{J}$ is also said to be induced, and in the same direction for isotropic materials. Induced current densities and induced currents occur in many magnetic devices with time-varying magnetic fields. These induced currents can be either desirable or undesirable.

A prime example of desirable induced current is the current in the secondary (or output) coil of a transformer. Transformers produce secondary voltage and current from Faraday's law as long as the primary (input) coil voltage and current are time varying. Usually the primary voltage and current are AC sinusoids, producing from Ampere's law a time-varying magnetic field and flux. The flux *links* or passes through both the primary and secondary windings. Faraday's law states that this flux induces a secondary voltage and current which depend in magnitude on the number of secondary turns. Thus transformers produce useful induced currents. The induced currents produce their own flux (from Ampere's law) which can be shown to oppose (by on opposite direction to) the original primary magnetic flux according to Lenz's law, a corollary of Faraday's law.

Induced currents may also be undesirable. For example, the transformer described above with its desirable secondary induced current may also have undesirable induced currents in other parts. The transformer is customarily surrounded by a steel case to confine its magnetic fields, and steel has high electrical conductivity. Typical steel conductivities range from about 5.E5 S/m to 1.E7 S/m. The lower values are for steels with high silicon content. Because of the high steel conductivity, the transformer housing often contains induced *eddy* currents, the name coming from their flow patterns which somewhat resemble the eddy patterns of turbulent rivers. The housing eddy currents are of no use and consume power (the product of voltage and current). They also produce heating. Thus eddy currents are usually undesirable.

A common way to reduce eddy currents in steel (or other conductive materials) is to *laminate*. Figure 2.6 shows typical steel *laminations,* or thin sheets. Due to the thin air spaces, surface oxidation, and/or surface treatments, electric current cannot flow from one lamination to another. Instead, any eddy currents are confined to flow within each lamination. Thus steel eddy current loss, often given in watts per cubic meter, is usually greatly reduced by using laminations.

Alternatives to steel laminations are *ferrites* and *composites.* Both have much lower conductivities than does steel, on the order of 1 S/m. However, their *B–H* curves are poorer than most steels. Ferrites saturate at only about 0.4 T. Composites, made of iron powder and insulating binders, have relative permeability of only a few hundred [7].

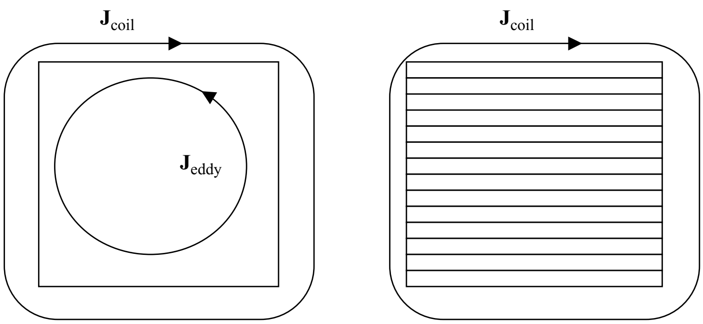

FIGURE 2.6 Comparison of eddy current flow patterns in conductive steel without (left) and with (right) laminations. Due to Lenz' law, the eddy currents tend to flow in the direction opposing the applied coil current.

Eddy currents and their effects will be considered in detail in Chapters 6–16.

Example 2.5 Induced Voltage and Current

(a) A coil called the primary coil establishes the magnetic flux density $\mathbf{B} = 1.1\sin(2\pi ft)\ \mathbf{u}_z$ T. The frequency $f = 60$ Hz. A secondary coil is of thin uniform copper (conductivity 5.8E6 S/m) wire and has resistance $R = 2\ \Omega$. Assume $I = V/R$ and that the magnetic field is not changed by the secondary current. The secondary coil is square in shape, connecting the points $(x,y,z) =$ (0,0,0), (0.5 m,0,0), (0.5 m,0.6 m,0), (0,0.6 m,0) and back to the origin, with a total of 50 turns.

Find the voltage and the current induced in the secondary, including their polarities (directions). Also find both **E** and **J** in the wire.

(b) Repeat for frequency $f = 50$ Hz.

Solution

(a) The coil has turns $N = 50$ and area $A = (0.5\text{ m})(0.6\text{ m}) = 0.3\text{ m}^2$. Faraday's law gives:

$$\begin{aligned} V &= -NA\frac{\partial B}{\partial t} = -50(0.3)\frac{\partial}{\partial t}(1.1\sin 377t) = -15(1.1)(377)\cos 377t \\ &= -6221\cos(2\pi 60t) \end{aligned} \tag{E2.5.1}$$

This negative voltage has polarity following the right-hand rule, which is counterclockwise when viewed from the $+z$ axis.

The secondary current is the voltage divided by $2\ \Omega$, or $-3111\cos(2\pi 60t)$ A. The electric field is the voltage divided by the wire length; the wire length is (50)(1 m + 1.2 m) = 110 m. Thus the electric field is $-56.55\cos(2\pi 60t)$ V/m, directed around the loop in the same direction as the current. To find the current density, multiply by the conductivity 5.8E7 S/m, giving 3.28E9 A/m^2 in the same direction as the current.

(b) The lower frequency changes both frequency and amplitude of the secondary voltage and current. The voltage becomes $-5184\cos(2\pi 50\text{t})$ V, and the current is $-2592\cos(\pi 50t)$ A. The electric field is $-47.13\cos(2\pi 50t)$ V/m and the current density is 2.73E9 A/m^2. Directions and polarities do not change.

2.5 POTENTIALS

Potentials are very useful in electrical engineering. The most common example is voltage, which is a scalar, not a vector.

A DC or static electric field is the negative gradient of electrostatic scalar potential, here denoted as ϕ_v:

$$\mathbf{E} = -\nabla\phi_v \tag{2.39}$$

where ϕ_v has units of volts. For time-varying fields, the induced electric field of Faraday's law must also be added.

Similar to magnetic flux density, there is an electric flux density $\mathbf{D}$ (in units of coulombs per square meter) defined by:

$$\mathbf{D} = \varepsilon\,\mathbf{E} \tag{2.40}$$

where ε is permittivity in farads per meter, a material property. Air and vacuum have $\varepsilon = \varepsilon_o = 8.854\text{E}{-12}$ F/m.

Materials with permittivity other than ε_o are often expressed in terms of relative permittivity, ε_r defined using:

$$\varepsilon = \varepsilon_r\,\varepsilon_o \tag{2.41}$$

Another common term for relative permittivity is dielectric constant.

There is an important relation between $\mathbf{D}$ and electric charge density:

$$\nabla\cdot\mathbf{D} = \rho_v \tag{2.42}$$

where ρ_v is electric charge density per unit volume in units of coulombs per cubic meter.

The relation between electric scalar potential, electric field, and charge is illustrated in the capacitor shown in Figure 2.7. The capacitor consists of two metal plates spaced 1 m apart in air. Because they are made of high conductivity copper or aluminum, there is essentially zero voltage drop along either plate. The lower plate at $y = 0$ is

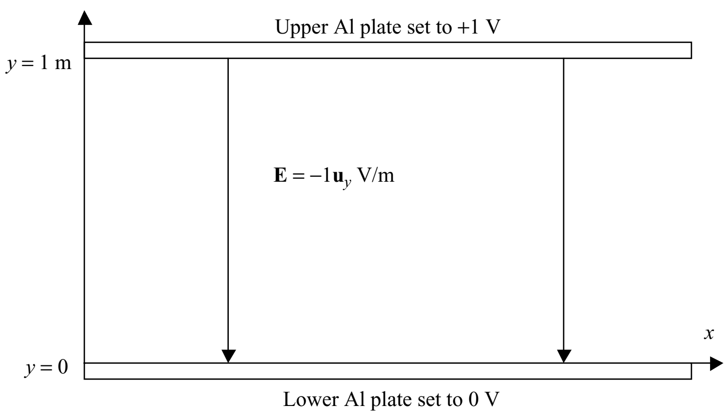

FIGURE 2.7 Two plates of a capacitor spaced 1 m apart and with 1 V DC applied.

grounded, that is, set to 0 V. The upper plate at $y = 1$ is attached to a 1 V DC battery. Thus the expression for the electric scalar potential in volts in Figure 2.6 is $\phi_v = y$ volts. Substituting into (2.39) gives:

$$\mathbf{E} = -\nabla y = -\frac{\partial y}{\partial x}\mathbf{u}_x - \frac{\partial y}{\partial y}\mathbf{u}_y - \frac{\partial y}{\partial z}\mathbf{u}_z = -1\mathbf{u}_y \text{ V/m} \tag{2.43}$$

Note that the electric field in Figure 2.7 points from high scalar potential in volts to low scalar potential. Since **D** is **E** times the permittivity of air, and from (2.42) is *not* divergenceless, both **D** and **E** terminate on the plates, which contain electric charge. Unlike magnetic flux lines, which must always "circle" due to their divergencelessness, the electric flux lines of Figure 2.7 can terminate or stop.

For magnetic fields, the potential required is a vector, not a scalar. Magnetic vector potential **A** is defined using:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.44}$$

Substituting into Faraday's law at a point (2.27) gives:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\left(\nabla \times \frac{\partial \mathbf{A}}{\partial t}\right) \tag{2.45}$$

where the above **E** is the induced electric field intensity:

$$\mathbf{E}_{\text{ind}} = -\frac{\partial \mathbf{A}}{\partial t} \tag{2.46}$$

Adding this induced electric field to the electric field due to electric scalar potential of (2.39) gives the total electric field expression:

$$\mathbf{E} = -\nabla\phi_v - \frac{\partial \mathbf{A}}{\partial t} \tag{2.47}$$

Example 2.6 Fields from Potentials Given the following potentials in a region, find its fields **B** and **E**:

$$\phi_v = 2x, \qquad \mathbf{A} = 0.2y\ \sin(2\pi 60t)\,\mathbf{u}_z \tag{E2.6.1}$$

Solution

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} = (\nabla \times 0.2y\mathbf{u}_z)\sin(2\pi 60t) = \frac{\partial(0.2y)}{\partial y}\mathbf{u}_x(\sin 2\pi 60t)\\ &= 0.2\sin(2\pi 60t)\mathbf{u}_x\end{aligned} \tag{E2.6.2}$$

$$\mathbf{E} = -\nabla\phi_v - \frac{\partial \mathbf{A}}{\partial t} = -2\mathbf{u}_x - (0.2)(2\pi 60)\cos(2\pi 60t)\mathbf{u}_z\,y \tag{E2.6.3}$$

2.6 MAXWELL'S EQUATIONS

James Clerk Maxwell of Scotland summarized all of electromagnetics in four equations. Expressed in point form, they are:

$$\nabla \cdot \mathbf{B} = 0 \tag{2.48}$$

$$\nabla \cdot \mathbf{D} = \rho_v \tag{2.49}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.50}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.51}$$

These four equations need not be in any particular order. As listed above, the first three have been previously given in this chapter: (2.48) expresses the continuity of magnetic flux, (2.49) expresses that electric flux terminates on charge, and (2.50) expresses Faraday's law. Note that the final equation, (2.51), is Ampere's law that has been enhanced by an additional term on its right-hand side. The additional term is called *displacement current density*:

$$\mathbf{J}_{\text{disp}} = \frac{\partial \mathbf{D}}{\partial t} \tag{2.52}$$

Displacement current density and its surface integral, displacement current, only exist for time-varying fields. Examples of displacement current include current through capacitors and coupled high frequency electromagnetic fields. Coupled electromagnetic fields [1, 8] are often of frequencies from 1 MHz to over 1 GHz, and are further discussed in Chapter 13.

Maxwell's equations can also be written in integral form by integrating the preceding four differential Maxwell equations, obtaining respectively:

$$\oint \mathbf{B} \cdot \mathbf{dS} = 0 \tag{2.53}$$

$$\oint \mathbf{D} \cdot \mathbf{dS} = \rho_v dV \tag{2.54}$$

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot \mathbf{dS} \tag{2.55}$$

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{dS} \tag{2.56}$$

This entire chapter is summarized by the above Maxwell's equations. Now that they are known and understood, their application to real world problems can begin in the next chapter.

Example 2.7 Displacement Current in a Capacitor The parallel-plate capacitor of Figure E2.7.1 has the following electric field **E** between its plates:

$$\mathbf{E} = 2500\ \sin(2\pi 60t)\,\mathbf{u}_y\ \mathrm{V/m}$$

where y is directed between the plates, which are separated by 50 mm of air. Each plate is of area 40 mm by 30 mm. Find the displacement current density and the total current in the capacitor.

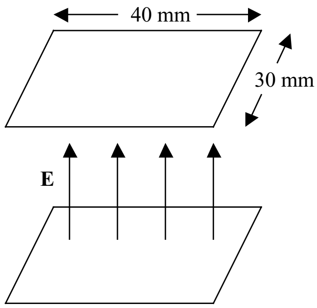

FIGURE E2.7.1 A capacitor made of two 30 by 40 mm plates spaced by 10 mm and with a known AC electric field intensity.

Solution

$$\begin{aligned}\mathbf{J}_{\mathrm{disp}} &= \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = (8.854\mathrm{E}{-}12)(2\pi 60)(2500)\cos(2\pi 60t)\mathbf{u}_y \\ &= 8.34\mathrm{E}{-}6\cos(2\pi 60t)\mathbf{u}_y\ \ \mathrm{A/m^2}\end{aligned} \tag{E2.7.1}$$

$$\begin{aligned}I_C &= \int \mathbf{J}_{\mathrm{disp}} \cdot \mathbf{dS} = (8.34\mathrm{E}{-}6\cos 377t)(0.04)(0.03) \\ &= 1.\mathrm{E}{-}8\cos 377t\ \mathrm{A}\end{aligned} \tag{E2.7.2}$$

PROBLEMS

2.1 A temperature distribution in Cartesian coordinates follows the equation:

$$T(x, y, z) = 10x + 20y^3 + 30z$$

Find the expression for the temperature gradient.

2.2 A mosquito flies in the direction of maximum rate of change of temperature, thereby seeking warm-blooded animals. If the temperature distribution obeys:

$$T(x, y, z) = 40x + 10y^2 + 30z$$

and the mosquito is sitting at location (1,2,3) m, find the direction it will fly initially.

2.3 Find the divergence and curl at location (1,2,−3) of the vector:

$$\mathbf{A} = [8x^4 + 6(y^2 - 2)]\,\mathbf{u}_x + [9x + 10y + 11z]\,\mathbf{u}_y + [4x]\,\mathbf{u}_z$$

2.4 Find the divergence and curl at location (2,−3,4) of the vector:

$$\mathbf{A} = [5x^4 + 6(y^3 - 2)]\,\mathbf{u}_x + [9x + 11y + 12z]\,\mathbf{u}_y + [4y]\mathbf{u}_z$$

2.5 Given the magnetic field intensity distribution (in A/m):

$$\mathbf{H} = [5x^4 + 6(y^3 - 2)]\,\mathbf{u}_x + [9x + 11y + 12z]\,\mathbf{u}_y + [4x]\,\mathbf{u}_z$$

Find the expression for current density **J** at location (2,4,3).

2.6 Given a region of 20 conductors, each carrying current $I = 10$ A. As shown in Figure E2.3.1, two closed paths are defined, $\mathbf{l}_1$ and $\mathbf{l}_2$. Find the integral of **H** along each of the closed paths.

2.7 A copper wire of radius 2 mm is placed at location (0,3,0) m, carrying current of 50 kA in the $+z$ direction. Find the vector **B** at location (4,0,0) for the wire embedded in the following materials that extend infinitely far in all the three directions: (1) air, (2) steel with constant relative permeability = 2500, (3) steel with a B–H curve with the following (B,H) values: (0,0), (1.5,1000), (1.55,1592), and (1.8,7958).

2.8 Same as the preceding problem, except the current direction is reversed to the $-z$ direction.

2.9 A coil called the primary coil establishes the magnetic flux density $\mathbf{B} = 1.0\sin(2\pi ft)\ \mathbf{u}_z$ T. The frequency $f = 60$ Hz. A secondary coil is of thin uniform aluminum (conductivity 3.54E7 S/m) wire and has resistance $R = 4\ \Omega$. Assume $I = V/R$ and that the magnetic field is not changed by the secondary current. The secondary coil is square in shape, connecting the points $(x,y,z) =$ (0,0,0), (0.5 m,0,0), (0.5 m,0.6 m,0), (0,0.6 m,0) and back to the origin, with a total of 40 turns.

Find the voltage and the current induced in the secondary, including their polarities (directions). Also find both **E** and **J** in the wire.

2.10 Same as preceding problem, except the frequency is changed to 400 Hz.

2.11 Given the following potentials in a region, find its fields **B** and **E**:

$$\phi_v = 4y, \quad \mathbf{A} = 0.2x \ \sin(2\pi 60t)\, \mathbf{u}_z$$

2.12 Given the following potentials in a region, find its fields **B** and **E**:

$$\phi_v = 4xy, \quad \mathbf{A} = 0.3x \ \sin(2\pi 50t)\, \mathbf{u}_z$$

2.13 The parallel-plate capacitor of Figure 2.7 has the following electric field **E** between its plates:

$$\mathbf{E} = 4500 \ \sin(2\pi 50t)\, \mathbf{u}_y \ \mathrm{V/m}$$

where y is directed between the plates, which are separated by 5 mm of air. Each plate is of area 70 mm by 50 mm. Find the displacement current density and the total current in the capacitor.

2.14 The parallel-plate capacitor of Figure 2.7 has the following electric field **E** between its plates:

$$\mathbf{E} = 3500 \ \sin(2\pi 400t)\, \mathbf{u}_y \ \mathrm{V/m}$$

where y is directed between the plates, which are separated by 8 mm of air. Each plate is of area 40 mm by 90 mm. Find the displacement current density and the total current in the capacitor.

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