CHAPTER 3

Reluctance Method

The reluctance method is a way of using Ampere's law of the preceding chapter to solve for magnetic fluxes and magnetic fields. Simplifying assumptions must be made, and thus the reluctance method cannot always obtain accurate results. For very simple problems, its results are often reasonably accurate, and can be quickly obtained "on the back of an envelope." Thus it often serves as a first step in the process of designing magnetic actuators and sensors.

3.1 SIMPLIFYING AMPERE'S LAW

The reluctance method begins with Ampere's law in integral form from the preceding chapter:

$$\oint \mathbf{H} \cdot \mathbf{dl} = NI \tag{3.1}$$

where ampere-turns NI are assumed given, and magnetic field intensity $\mathbf{H}$ and magnetic flux density $\mathbf{B}$ are to be found. The closed line integral is replaced by a summation:

$$\sum_k H_k l_k = NI \tag{3.2}$$

where the closed path consists of line segments of subscript k, each of which is in the direction of the field intensity. Thus the direction of the field and flux is assumed known. Because flux density $\mathbf{B}$ prefers to flow through high permeability materials, the line segments (straight or curved) are assumed to follow a path through high permeability materials such as steel. If the steel has a gap made of air, the closed path follows the shortest part of the airgap.

Magnetic Actuators and Sensors, Second Edition. John R. Brauer.
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To account for the permeability of the closed path of (3.2), recall that **B** is permeability μ times **H**, giving:

$$\sum_k (B_k/\mu_k)l_k = NI \tag{3.3}$$

where μ_k is the permeability of path segment k. Next, recall from the preceding chapter that flux is the surface integral of flux density:

$$\phi = \int \mathbf{B} \cdot \mathbf{dS} \tag{3.4}$$

Assuming that each path segment has cross-sectional surface area S_k normal to the segment direction carrying B_k, each segment carries the flux:

$$\varphi_k = B_k S_k \tag{3.5}$$

Substituting into (3.3) gives:

$$\sum_k [(\varphi_k/(\mu_k S_k)]l_k = NI \tag{3.6}$$

Recall from the preceding chapter that flux is continuous (because the divergence of flux density is zero). Thus the flux through all segments of (3.6) is the same value, giving:

$$\varphi \sum_k [(l_k/(\mu_k S_k)] = NI \tag{3.7}$$

The term being summed is called *reluctance*, symbolized by the script letter $\mathcal{R}$. Thus (3.7) becomes:

$$\varphi \sum_k \mathcal{R}_k = NI \tag{3.8}$$

Units of reluctance must be amperes per weber; see Appendix A for alternative expression of its units.

From the preceding two equations, *reluctance* is defined as:

$$\mathcal{R} = l/(\mu S) \tag{3.9}$$

If all path reluctances are known, then (3.8) can be used to find the unknown flux:

$$\varphi = (NI)/\left(\sum_k \mathcal{R}_k\right) \tag{3.10}$$

TABLE 3.1 Basic Analogous Parameters of Electric Circuits and Magnetic Circuits

Parameter	Electric Circuit	Magnetic Circuit
Flow	Current I in amperes (A)	Flux ϕ in webers (Wb)
Potential	EMF in volts (V)	MMF in ampere-turns
Potential/flow	Resistance R in ohms (Ω)	Reluctance $\mathcal{R}$ in A/Wb
Flow/potential	Conductance G in siemens (S)	Permeance $\mathcal{P}$ in Wb/A
Flow density	Current density $\boldsymbol{J}$ in A/m^2	Flux density $\boldsymbol{B}$ in teslas (T)

With flux known, individual flux densities can be found using (3.5) to give:

$$B_k = \varphi_k / S_k \tag{3.11}$$

Thus reluctances can be used in the reluctance method to solve for flux and flux density everywhere along the closed flux path. The simplest equation for the reluctance method is a simplification of (3.8):

$$\varphi \mathcal{R} = NI \tag{3.12}$$

which is analogous to the familiar Ohm's law of electric circuits:

$$IR = V \tag{3.13}$$

Thus the reluctance method is also called *the magnetic circuit method.* In magnetic circuits, reluctance is analogous to resistance, and flux is analogous to current. The excitation of magnetic circuits is ampere-turns, analogous to voltage. Since voltage is sometimes called electromotive force (EMF), NI is sometimes called *magnetomotive force* (MMF). Also, just as any voltage drop in an electric circuit can be called an *EMF drop*, the corresponding drop in a magnetic circuit is an *MMF drop*. Table 3.1 summarizes the basic analogies between electric and magnetic circuits. Figure 3.1 shows the magnetic circuit parameters.

The reluctance method or magnetic circuit method is also sometimes called the *permeance method.* Permeance is defined as the reciprocal of reluctance:

$$\mathcal{P} = 1/\mathcal{R} \tag{3.14}$$

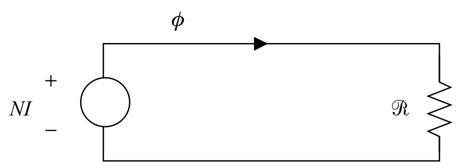
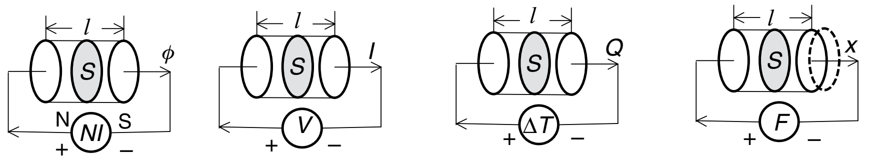


FIGURE 3.1 Basic magnetic circuit parameters.



Magnetic circuit	Electric circuit	Heat conduction	Mechanical structure
Reluctance $\mathcal{R} = l/(\mu S)$	Resistance $R = l/(\sigma S)$	Resistance $R_T = l/(kS)$	Stiffness $K = (ES)/l$
Permeance $\mathcal{P} = (\mu S/l)$	Conductance $G = (\sigma S)/l$	Conductance $G_T = (kS)/l$	Compliance $C = l/(ES)$
$NI = \phi\mathcal{R}$	$V = IR$	$\Delta T = QR_T$	$F = Kx$
Flux density $B = \phi/S$	Current dens. $J = I/S$	Heat flux dens. $q = Q/S$	Stress $\sigma = F/S$
Field intensity $H = NI/l$	Field intensity $E = I/V$	Temperature gradient $= \Delta T/l$	Strain $\varepsilon = x/l$
$\mu = B/H$	$\sigma = J/E$	$k = q/(\Delta T/l)$	Modulus $= \sigma/\varepsilon$
Energy $W = \phi NI/2$	Power $P = VI$	Power $P = Q$	Energy $W = Fx/2$
Energy/vol. $w = BH/2$	Power/vol. $p = EJ$	Power/vol. $p = Q/(Sl)$	Energy/vol. $w = \sigma\varepsilon/2$

FIGURE 3.2 Analogous static (DC) parameters for magnetic circuits, electric circuits (Section 2.4), heat conduction (Section 12.4.1), and mechanical structures (Section 14.3). For more information on units, see Appendix A.

and thus permeance is proportional to permeability. Figure 3.2 shows reluctance and permeance (and other parameters) in magnetic circuits and how they compare with analogous parameters in electric circuits, thermal circuits, and mechanical structures. Thermal and mechanical problems will be discussed further in Part IV of this book.

The reluctance method consists of the following steps.

(1) Find the closed flux path (or paths) that "circles" the given ampere-turns NI. This path is usually through high permeability materials such as steel, but may also contain air segments, which should be as short as possible. The path direction must follow the right-hand rule, where your thumb is in the direction of NI.

(2) Find the lengths and cross-sectional areas of all path segments, assuming average values of lengths and the inverses of areas. Also note the permeability of each path segment. For nonlinear materials, assume initially that their permeability is the constant value below their "knee."

(3) Find the reluctance of each path segment using (3.9).

(4) Combine the reluctances and use (3.10) or (3.8) to find the flux. Reluctances in series add directly. Reluctances in parallel combine just like resistances in parallel; the combined value is the reciprocal of the sum of reciprocals.

(5) Find the flux density of each path segment using (3.11).

(6) If the flux density in any nonlinear material is beyond the knee, then make a new (lower) assumption of its permeability and repeat steps (3), (4), and (5) until the calculated flux densities match the assumed nonlinear permeabilities. For details and various examples consult books such as that by Roters [1].

3.2 APPLICATIONS

Example 3.1 Reluctance Method for “C” Steel Path with Airgap The first example of the reluctance method is shown in Figure E3.1.1. A “C”-shaped piece of steel of uniform thickness 0.1 m lies in the plane of the paper. The steel is of depth 0.1 m into the page. The opening of the “C” is an airgap of length 0.1 m between the steel poles. (A magnetic *pole* is a surface where magnetic flux leaves to form a North pole or enters to form a South pole.) The object is to find magnetic flux density **B** throughout the steel and airgap produced by the 10,000 ampere-turns directed out of the page in a coil inside the “C” and returning to its left. The steel is assumed to have a relative permeability of 2000.

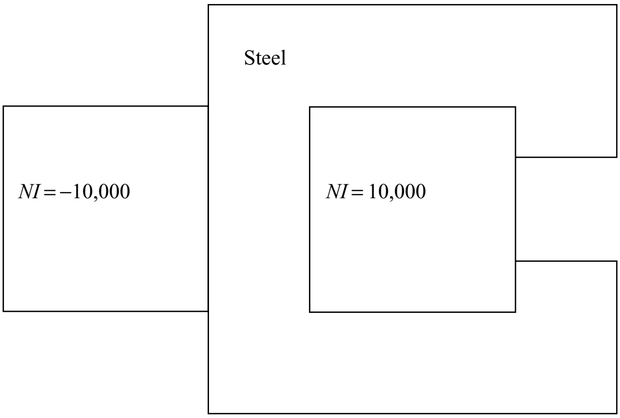


FIGURE E3.1.1 Steel magnetic structure with a single airgap. Each side of the steel “C” core is 0.4 m long and the steel thickness and depth are 0.1 m.

Solution Follow the six steps at the end of Section 3.1.

(1) From the right-hand rule, the flux flows counterclockwise (CCW) around the steel “C” as shown in Figure E3.1.2. The flux path shown is in the middle of the steel and airgap, and thus is of average length. The longest flux path (not shown) follows the outer surface of the steel, while the shortest flux path follows the interface between the winding and the steel.

(2) The steel segment length along the flux path is:

$$\ell_{\mathrm{Fe}} = (0.4 + 0.4 + 0.4 + 0.15 + 0.15)\,\mathrm{m} = 1.5\,\mathrm{m}$$

The steel segment cross-sectional area is (0.1 m)(0.1 m) = 0.01 m^2. (At the four corners the area is bigger, but the corners are only a small portion of the steel.) The steel relative permeability is given as 2000.

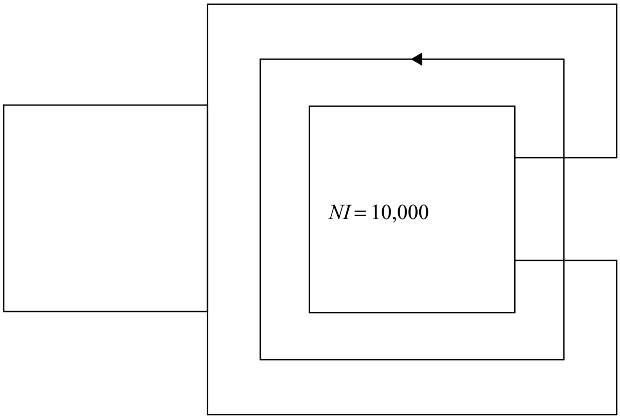


FIGURE E3.1.2 Flux path of Figure E3.1.1 following the right-hand rule.

The air segment length, called the airgap, is 0.1 m. The airgap cross-sectional area is (0.1 m)(0.1 m) = 0.01 m^2. Actually, as will be discussed in the next section, because air extends beyond the steel poles, the flux can expand or "fringe" to a larger area. For now, however, fringing is ignored and hence 0.01 m^2 is assumed. The relative permeability of air (or vacuum) is 1.

(3) The reluctances are

$$\mathcal{R}_{\mathrm{Fe}} = 1.5/[(20{,}000)(12.57\mathrm{E}\text{–}7)(0.01)] = 5966 \tag{E3.1.1}$$

$$\mathcal{R}_{\mathrm{air}} = 0.1/[(12.57\mathrm{E}\text{–}7)(0.01)] = 7.955\mathrm{E}6 \tag{E3.1.2}$$

(4) The flux obeys:

$$\phi(59{,}666 + 7.955\mathrm{E}6) = 10{,}000 \tag{E3.1.3}$$

$$\varphi = 12.477\mathrm{E}\text{–}4\ \mathrm{Wb} \tag{E3.1.4}$$

(5) Since the cross-sectional area is approximately the same everywhere, everywhere the flux density is the same:

$$B = \frac{\varphi}{S} = \frac{12.477\mathrm{E}\text{–}4}{1.\mathrm{E}\text{–}2} = 0.125\ \mathrm{T} \tag{E3.1.5}$$

(6) Since the above flux density is less than 1.5 T, which is the approximate knee of steel *B–H* curves, the above calculated values are valid. However, one must remember that all reluctance method calculations are approximate.

Example 3.2 Reluctance Method for Sensor with Variable Airgap The second example of the reluctance method is shown in Figure E3.2.1. It shows a simplified magnetic sensor with stationary stator and movable armature. Both stator and armature are made of steel, with the stator also having a coil made of copper or aluminum. The armature is shown in two positions. In the first position the armature and stator teeth are aligned. In the second position the armature teeth are moved halfway between the stator teeth, so they are as misaligned as possible.

The stator coil in Figure E3.2.1 has 1000 ampere-turns. The stator and armature teeth are shown to have width equal to 1 cm. The airgap between the stator and rotor teeth is 1 mm when they are aligned. In the misaligned position, the airgap is assumed to increase to 1 cm. The depth into the page of the sensor of Figure E3.2.1 is given as 10 cm.

The steel permeability is given as infinitely high, and thus only airgap reluctances are needed. Hence dimensions of the steel are not given.

The airgap flux densities are to be found for both armature positions in Figure E3.2.1.

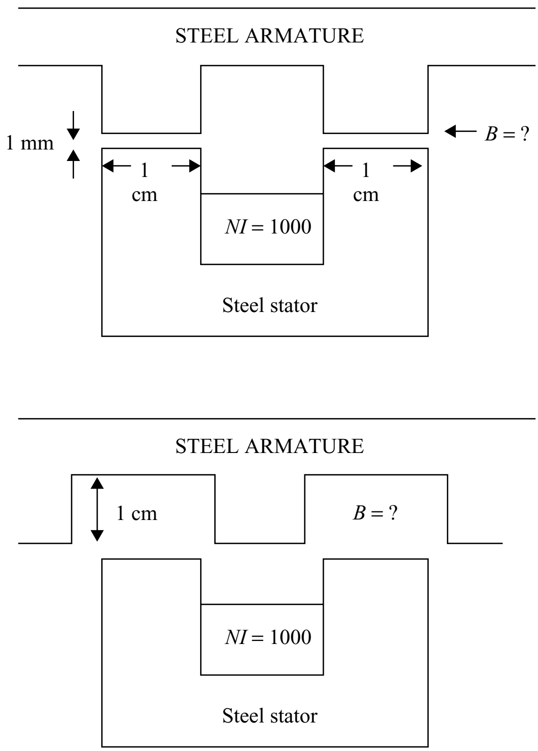


FIGURE E3.2.1 Portion of simple magnetic sensor.

Solution Follow the six steps at the end of the preceding Section 3.1.

(1) From the right-hand rule, the flux flows CCW around the steel stator "U" shape. It must cross the airgap twice, once entering the armature on its right and then leaving the armature on its left.

(2) The air segment length is total air length over the entire closed path. Thus it equals the right airgap plus the left airgap. Hence in the aligned position the air length is 2 mm, while in the misaligned position the air length is 2 cm = 20 mm.

(3) The reluctances are airgap reluctances only. For the two positions, aligned and misaligned, they are proportional to total air lengths:

$$\mathcal{R}_{\text{align}} = 0.002/[(12.57\text{E}-7)(0.01)(0.10)] = 1.591\text{E}6 \tag{E3.2.1}$$

$$\mathcal{R}_{\text{misalign}} = 0.02/[(12.57\text{E}-7)(0.01)(0.10)] = 15.91\text{E}6 \tag{E3.2.2}$$

(4) The flux obeys:

$$\phi_{\text{align}} = (1000/\mathcal{R}_{\text{align}}) = 628.5\text{E}-6\ \text{Wb} \tag{E3.2.3}$$

$$\phi_{\text{misalign}} = (1000/\mathcal{R}_{\text{misalign}}) = 62.85\text{E}-6\ \text{Wb} \tag{E3.2.4}$$

(5) Assuming the cross-sectional area for the airgap flux is always 1 cm by 10 cm, the airgap flux density is the same on both sides. For the two positions, B varies as:

$$B_{\text{align}} = \phi_{\text{align}}/[(0.01)(0.10)] = 1000\phi_{\text{align}} = 0.6285\,\text{T} \tag{E3.2.5}$$

$$B_{\text{misalign}} = \phi_{\text{misalign}}/[(0.01)(0.10)] = 1000\phi_{\text{misalign}} = 0.06285\,\text{T} \tag{E3.2.6}$$

(6) Since the above flux densities are less than 1.5 T, which is the approximate knee of steel *B–H* curves, the assumption of infinite steel permeability might be approximately true. However, one must remember that all reluctance method calculations are approximate, especially if the steel reluctance is ignored.

3.3 FRINGING FLUX

As mentioned in solving Example 3.1, *fringing* is the expansion of flux in air. Magnetic flux paths in steel can expand outward when entering air, much like water flow paths through a hose can expand outward in air at the end of the hose or nozzle. Thus the area seen by the flux or flow increases, and fringing can cause the airgap reluctance to decrease significantly. Since, as in Examples 3.1 and 3.2, airgap reluctance is often the largest reluctance determining the flux, ignoring fringing can reduce the accuracy of the reluctance method.

In general, fringing flux can be ignored when the airgap path length is small compared with the airgap width. One can define a *fringing factor* as:

$$K_{\text{fringe}} = \mathcal{R}_{\text{nofringe}}/\mathcal{R}_{\text{fringe}} \tag{3.15}$$

where the factor is usually greater than 1, because the reluctance with fringing is less than that without fringing. For example, if a steel leg is cylindrical with diameter D and the airgap length is g, then K_{fringe} is a function of the ratio g/D that increases with the ratio. For g/D less than approximately 0.04, K_{fringe} is close to 1 and fringing can usually be ignored. For larger g/D ratios, however, fringing can cause greater than 5% changes in magnetic parameters such as reluctance and force.

Reluctances or permeances due to fringing have been derived for several common geometries. Examples include quarter and half cylinders and shells [1–3]. Example 5.2 will display fringing flux. In many geometries, parallel flux paths mean that reluctances or permeances in magnetic circuits must be placed in parallel as well as in series [4], analogous to parallel and series electric circuits.

3.4 COMPLEX RELUCTANCE

The reluctance derived above is a real number and is based upon Ampere's law. To include AC eddy losses produced by Faraday's law currents in conducting materials such as steel, the reluctance becomes a complex number. Complex numbers are customarily used in AC electrical engineering to account for phase shifts between AC sinusoids. Complex permeability, which produces complex permeance and complex reluctance, is often specified for semiconducting ferrites [5, 6].

In general, the reluctance $\mathcal{R}$ is the sum of ferromagnetic (Fe) and airgap reluctances, thus:

$$\phi = NI/(\mathcal{R}_{\text{Fe}} + \mathcal{R}_{\text{gap}}) \tag{3.16}$$

The airgap reluctance is real (lossless), but the steel reluctance may be complex in order to include eddy current losses and/or other power losses [7]. Hence $\mathcal{R}_{\text{Fe}}$ has both a real part and a positive imaginary part:

$$\mathcal{R}_{\text{Fe}} = \mathcal{R}_{\text{RE}} + \mathrm{j}\mathcal{R}_{\text{IM}} = |\mathcal{R}_{\text{Fe}}|\angle\theta \tag{3.17}$$

where θ is the phase angle. Chapter 8 will discuss and apply complex reluctance to AC devices.

3.5 LIMITATIONS

The reluctance method described above has several limitations. The limitations and ways to address them are the following.

(1) Fringing flux in air is either ignored or approximated with fringing factors, or derived assuming the shape of the flux path.

(2) The path area of each part is often assumed to be average path area. A more accurate method is to use the reciprocal of the average reciprocal area, since reluctance is proportional to the reciprocal of area.

(3) Since reluctance is inversely proportional to permeability, predicting nonlinear *B–H* effects in steel is difficult. However, *B–H* curves can be used in the reluctance method [1, 3].

(4) As discussed in the preceding section, losses in steel can be represented by the use of complex reluctance. However, obtaining values of complex reluctance is often very difficult.

(5) Most real world devices have complicated geometries with multiple flux paths that are difficult to analyze. However, magnetic circuits can be constructed with series and parallel reluctances to more accurately model such devices.

(6) For many magnetic devices, however, the necessity of assuming flux paths makes the reluctance method inaccurate. Thus the reluctance method cannot be accurately applied to many devices.

PROBLEMS

3.1 Redo Example 3.1 with steel relative permeability equal to 100.

3.2 Change both the tooth width and the airgap in Figure E3.2.1 of Example 3.2. The tooth width on both sides is reduced to 8 mm. The airgap is 2 mm in the aligned position and 11 mm in the misaligned position.

3.3 Apply Ampere's law directly to Example 3.2 to obtain flux density in both airgaps for both positions. Ampere's law can be applied directly because *H* and *B* are the same on both left and right sides due to their exact symmetry, and the steel is assumed to have zero reluctance.

3.4 Apply Ampere's law directly to Problem 3.2.

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