# CHAPTER 4

# Finite-Element Method

Because of the limitations of the reluctance method of the preceding chapter, engineers have long sought more accurate ways to calculate magnetic fields. This chapter is devoted to the finite-element method, the most general and accurate method yet found [1–3].

Even before digital computers existed, a few scientists and engineers employed *relaxation techniques* [4] to iteratively solve governing differential equations of phenomena including heat and magnetics. For magnetic fields, for example, Ampere's law at a point can be solved using a method commonly called the *finite-difference method.* Whether solved on a computer or not, the finite-difference method involves replacing differentials such as $\partial x$ with differences such as $\Delta x$. Usually the geometric spacing, such as $\Delta x$ and $\Delta y$ in two dimensions, must be uniform everywhere, although some finite-difference software allows a nonuniform "grid" of spacing. The process of subdividing the problem region into such a gridwork is called *discretization*, and a major problem with the finite-difference method is that a uniform discretization is not appropriate for many engineering devices. Most devices have regions with intricate geometry requiring fine discretization, along with regions without intricate geometry not requiring such detailed modeling.

## 4.1 ENERGY CONSERVATION AND FUNCTIONAL MINIMIZATION

The *finite-element method* can be derived using the basic principle of conservation of energy. Assuming for now that the region analyzed has no power loss, then the energy input must equal the energy stored:

$$W_{\text{in}} = W_{\text{stored}} \tag{4.1}$$

The finite-element method has been applied to structural analysis using mechanical energies, to thermal analysis using thermal energies, and to fluid dynamics using fluidic energies [1].

*Magnetic Actuators and Sensors*, Second Edition. John R. Brauer.

The finite-element equation (4.1) is here applied to magnetic field problems. Energy is stored in magnetic fields and is input using a current density **J**. Thus expressions for the energies over the problem volume $v$ can be substituted to obtain:

$$\frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}dv = \int \frac{B^2}{2\mu}\,dv \tag{4.2}$$

where constant permeability $\mu$ (linear $B$–$H$) is assumed for now, and integrations are taken on both sides over the problem volume. Recall from Chapter 2 that **A** is magnetic vector potential, here unknown. Recall also that magnetic field (flux density) **B** is the curl of **A**:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.3}$$

The next step in finite-element analysis is to define an *energy functional* as the difference between stored energy and input energy [1]:

$$F = W_{\mathrm{stored}} - W_{\mathrm{input}} \tag{4.4}$$

For linear magnetic fields, substituting (4.2) gives the functional:

$$F = \int \left[\frac{B^2}{2\mu} - \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\right] dv \tag{4.5}$$

The Law of Energy Conservation requires the functional to be zero. However, rather than setting (4.5) directly to zero, the finite-element method minimizes the functional. That is, the correct solution for unknown fields **B** and **A** is found by setting the partial derivative of the functional to zero:

$$\frac{\partial F}{\partial A} = 0 \tag{4.6}$$

As shown in most calculus texts, the derivative is a slope. At a minimum of a function, the slope is zero, as shown for example in Figure 4.1.

Substituting the functional of (4.5) into (4.6) obtains:

$$\frac{\partial}{\partial A}\int \frac{B^2}{2\mu} dv = \int J dv \tag{4.7}$$

The above equation is the basis for finite-element analysis of linear magnetostatic (DC magnetic) fields.

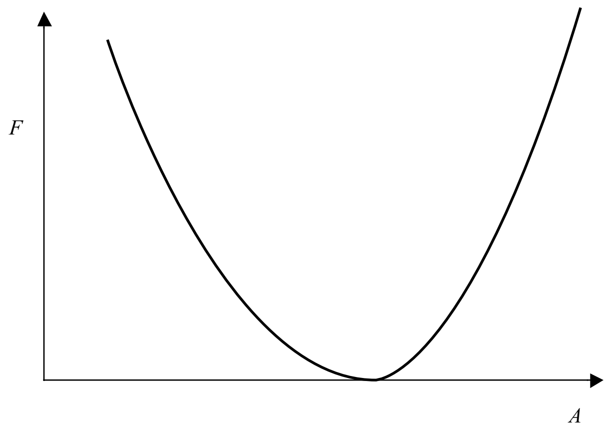


**FIGURE 4.1** Functional $F$ with a minimum having zero slope (derivative).

## 4.2 TRIANGULAR ELEMENTS FOR MAGNETOSTATICS

Besides the energy functional, the second main requirement of finite-element analysis is to discretize (break up) the volume analyzed into small pieces called finite elements. The most basic finite element is the triangle.

Figure 4.2 shows the simplest triangular finite element for magnetostatic fields. It lies in the $xy$ plane and its magnetic flux density **B** is assumed to lie in the same plane. As in the reluctance examples of the preceding chapter, the planar **B** is produced by a current coming out of the plane. Thus the current density **J** is in the $z$ direction normal to the plane of the page. Since **J** lies only in the $z$ direction (plus or minus), the unknown magnetic vector potential **A** lies only in the $z$ direction. In many magnetic devices, especially those made of steel laminations, **B** lies in the plane and the devices have geometry independent of the direction out of the plane as shown in Figure 4.2.

Note that the triangle of Figure 4.2 has three vertices called *nodes* labeled as $L$, $M$, $N$. The unknown $A_z = A$ must be found at the three nodes, and thus the triangle of Figure 4.2 is called a *nodal* finite element.

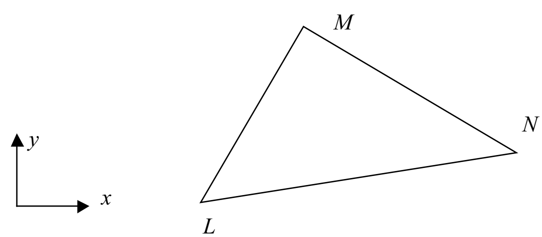


**FIGURE 4.2** Triangular finite element lying in the $xy$ plane.

The triangular finite element of Figure 4.2 is assumed to be a *first-order finite element* by assuming the following *shape function*:

$$A(x, y) = \sum_{k=L,M,N} [A_k(a_k + b_k x + c_k y)] \tag{4.8}$$

where the first-order shape function is a first-order polynomial over the plane of interest. From Figure 4.2, the unknown $A = A_K$ at $x = x_K$ and $y = y_K$, with two more such equalities at each of the other two nodes. Thus one obtains the matrix relation for the constants of (4.8):

$$\begin{pmatrix} a_L & a_M & a_N \\ b_L & b_M & b_N \\ c_L & c_M & c_N \end{pmatrix} = \begin{pmatrix} 1 & x_L & y_L \\ 1 & x_M & y_M \\ 1 & x_N & y_N \end{pmatrix}^{-1} \tag{4.9}$$

Hence for any triangle, the polynomial constants of (4.8) are known.

Substituting the shape function of (4.8) into the functional minimization Equation 4.7 yields:

$$\int \frac{\partial}{\partial A_k} \left( \frac{B^2}{2\mu} - \frac{1}{2} JA \right) dv = 0 \tag{4.10}$$

where evaluating the curl of **A** for the triangle gives:

$$B^2 = \left( \frac{\partial A}{\partial x} \right)^2 + \left( \frac{\partial A}{\partial y} \right)^2 \tag{4.11}$$

## 4.3 MATRIX EQUATION

The final step of finding the unknown vector potentials at the three nodes of the triangle of Figure 4.2 is to integrate over its volume, which is actually its area times the depth into the page. If the triangle area is denoted as $S$, one can show [1] that a 3 by 3 matrix equation is obtained:

$$(S/\mu) \begin{pmatrix} (b_L b_L + c_L c_L) & (b_L b_M + c_L c_M) & (b_L b_N + c_L c_N) \\ (b_M b_L + c_M c_L) & (b_M b_M + c_M c_M) & (b_M b_N + c_M c_N) \\ (b_N b_L + c_N c_L) & (b_N b_M + c_N c_{ML}) & (b_N b_N + c_N c_N) \end{pmatrix} \begin{bmatrix} A_L \\ A_M \\ A_N \end{bmatrix}$$
$$= (S/3) \begin{bmatrix} J \\ J \\ J \end{bmatrix} \tag{4.12}$$

which is commonly denoted as:

$$[K]\{A\} = \{J\} \tag{4.13}$$

The matrix $K$ is commonly called the *stiffness matrix* from its origin in structural finite-element analysis. The column vector $J$ is usually known, and then (4.13) enables the unknown column vector $A$ to be found.

When several triangles are used to discretize the region analyzed, then the 3 by 3 matrix equation of (4.12) and (4.13) becomes an $n$ by $n$ matrix equation, where $n$ is the total number of nodes. Usually many triangles are required, each of which may have a different constant permeability and applied current density (zero or nonzero). With modern software and hardware, matrices with thousands of rows and columns (and millions of matrix entries) can be solved within a few minutes. In general, both storage requirements and computer solution times are proportional to the number of unknowns $n$ to a power between 2 and 3.

After all nodal vector potentials are found using (4.13), the magnetic flux density in each triangle can be found using (4.11). Substituting the first-order shape function into (4.11), it is seen that the **B** in first-order triangles is a constant value. Since each first-order triangle has constant flux density, more and smaller triangular finite elements are needed wherever the magnetic field varies the most. Since the field is expected to vary the most near fine steel geometries such as corners, which must naturally be modeled with many triangles, the finite-element method is practical and can be very accurate.

To increase accuracy over each finite element, shape functions with higher polynomial order (called $p$) can be used. For example, a second-order or *quadratic* shape function for magnetic vector potential over a triangle is:

$$A(x, y) = \sum_{k=L,M,N,O,P,Q} [A_k(a_k + b_k x + c_k y + d_k x^2 + e_k y^2 + f_k xy)] \tag{4.14}$$

In most cases the three extra variables create three extra nodes placed at the middle of each triangle edge and thus called *midnode* variables.

Extensions of the above equations enable better analysis of real-world magnetic problems. For saturable nonlinear materials, Newton's iterative method is used [1]. For conducting materials with eddy current power losses, an additional energy term can be added to the energy functional. For true 3D problems, often the triangular finite element is replaced by a tetrahedron. With only corner nodes, the tetrahedron has 4 nodes, whereas with midedge nodes, it has 10 nodes. However, to allow discontinuity in the normal component of 3D magnetic vector potential, for highest accuracy the variables preferred in tetrahedrons are the tangential components of **A** along its edges. Such finite elements are called *edge elements* or tangential vector elements [5]. In some finite-element matrix derivations, as an alternative to the energy methods described here, the differential equation is directly used in Galerkin's method [3].

**Example 4.1 Matrix Equation for Two Finite Elements** Two first-order triangular finite elements are used to model a region made of copper wire and containing current density of 40,000 A/m$^2$ in the direction normal to the plane of the elements. The first element connects node numbers 1, 2, and 3. It has coefficient values:

$$b_1 = 1,\ b_2 = 2,\quad b_3 = 3,\ c_1 = 4,\ c_2 = 5,\ c_3 = 6,\quad \text{and area}\quad S = 1.\text{E–}4\,\text{m}^2$$

The second element connects node numbers 3, 4, and 5. It has coefficient values (for those nodes in that order):

$$b_1 = 1,\ b_2 = 2,\quad b_3 = 3,\ c_1 = 4,\ c_2 = 5,\ c_3 = 6,\ \text{and area}\ S = 2.\text{E–}4\,\text{m}^2$$

The matrix equation for all unknown vector potentials is to be written in form ready to be solved.

***Solution*** Set up (4.12) for each element, and then combine the matrices into a single 4 by 4 matrix equation. For the first element, (4.12) is:

$$(1.\text{E–}4/12.57\text{E–}7)\begin{pmatrix}(1+16) & (2+20) & (3+24)\\ (2+20) & (4+25) & (6+30)\\ (3+24) & (6+30) & (9+36)\end{pmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}$$
$$= (1.\text{E–}4/3)\begin{bmatrix}40.\text{E}3\\ 40.\text{E}3\\ 40.\text{E}3\end{bmatrix} \tag{E4.1.1}$$

$$\begin{pmatrix}(1352) & (1750) & (2148)\\ (1750) & (2307) & (2864)\\ (2148) & (2864) & (3580)\end{pmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix} = \begin{bmatrix}1.333\\ 1.333\\ 1.333\end{bmatrix} \tag{E4.1.2}$$

where the rows and columns correspond to node numbers 1, 2, and 3, respectively. For the second element, the rows and columns correspond to node numbers 3, 4, and 5, respectively, with the matrix equation:

$$(2.\text{E–}4/12.57\text{E–}7)\begin{pmatrix}(1+16) & (2+20) & (3+24)\\ (2+20) & (4+25) & (6+30)\\ (3+24) & (6+30) & (9+36)\end{pmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}$$
$$= (2.\text{E–}4/3)\begin{bmatrix}40.\text{E}3\\ 40.\text{E}3\\ 40.\text{E}3\end{bmatrix} \tag{E4.1.3}$$

$$\begin{pmatrix} (2704) & (3500) & (4296) \\ (3500) & (4614) & (5728) \\ (4296) & (5728) & (7160) \end{pmatrix} = \begin{bmatrix} 2.667 \\ 2.667 \\ 2.667 \end{bmatrix} \tag{E4.1.4}$$

Combining (E4.1.2) for rows and columns 1, 2, 3 with (E4.1.4) for rows and columns 3, 4, 5 gives the final 5 by 5 matrix equation:

$$\begin{pmatrix} 1352 & 1750 & 2148 & 0 & 0 \\ 1750 & 2307 & 2864 & 0 & 0 \\ 2148 & 2864 & (3580+2704) & 3500 & 4296 \\ 0 & 0 & 3500 & 4614 & 5728 \\ 0 & 0 & 4296 & 5728 & 7160 \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix} = \begin{bmatrix} 1.333 \\ 1.333 \\ 4 \\ 2.667 \\ 2.667 \end{bmatrix} \tag{E4.1.5}$$

Note that zero entries exist for nodes that are not directly connected by a finite element. As more elements are added, more zeros occur, and the matrix becomes dominated by zeros. A zero-dominated matrix is said to be *sparse*. Modern matrix solution software takes advantage of the *sparsity* to reduce storage requirements and solution time [6].

## 4.4 FINITE-ELEMENT MODELS

For the magnetic fields to be found in any magnetic device, a *finite-element model* of that device must be created. Making a finite-element model consists of the following four steps.

(1) **Geometry, subdivided into finite elements.** All 2D or 3D geometry must be specified, and the user and/or the software must subdivide it into finite elements. Geometry specification is difficult for many real-world devices of complicated shape. For example, Figure 4.3 shows a magnetic actuator with 2D geometry. If the geometry already exists in drawing software, then some finite-element software will accept it. However, either the finite-element software or the user must subdivide all geometric regions, including air for magnetic fields, into finite elements, such as those shown in Figure 4.4.

(2) **Materials.** For electromagnetic fields, the three material properties of Chapter 2 may be required: permeability, conductivity, and permittivity. Recall that for nonlinear magnetic materials, the *B–H* curve is required. For magnetostatics, (4.12) shows that conductivity and permittivity are not required.

(3) **Excitations.** For magnetic fields, the excitations or sources input are current densities and/or currents according to (4.12). As will be seen in later chapters, voltages and/or permanent magnets can be used instead if desired.

(4) **Boundary conditions.** On the outer boundary of the region analyzed, boundary conditions are required. For example, since Figure 4.4 is a one-half model

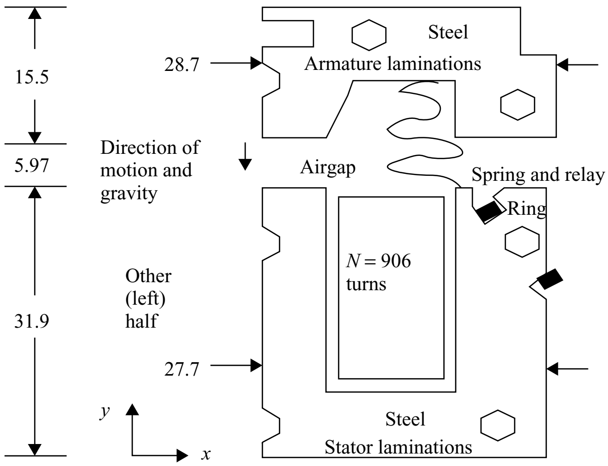


**FIGURE 4.3** Magnetic actuator manufactured by Eaton Corporation [7], showing the geometry and materials of the right half. Dimensions are in millimeters. The steel laminations are stacked to a depth of 28.5 mm into the page.

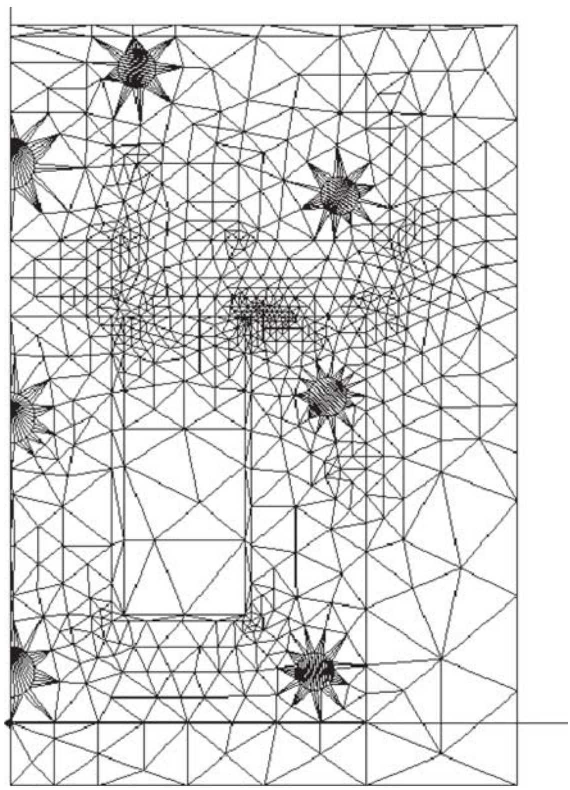

**FIGURE 4.4** Computer display of triangular finite elements used to model Figure 4.3.

of the right side of an actuator, the left (symmetry) boundary must be specified. On the other three sides of the outer boundary, one must specify whether magnetic flux will enter or not. If flux does not enter at the boundary of a 2D model, then the next chapter shows that boundary has constant $A_z$, usually set to zero. Modeling only one-half reduces the computer time substantially. In some magnetic devices with multiple poles, periodic boundary conditions enable even smaller fractions to be modeled. In many cases, the magnetic field extends beyond the region modeled into surrounding air space, and special *open boundaries* are appropriate.

In some finite-element software, the four steps listed above appear in a task menu. The four steps are often called *preprocessing*, because they must be accomplished in order for the matrix equation to be set up and solved for the magnetic vector potential distribution throughout the device analyzed. Most magnetic finite-element software available today, both paid and free, has preprocessing steps similar to the above four steps. At this writing, free magnetic finite-element software is available from FEMM and Quickfield™.

Detailed examples of software use are described at many websites, such as at www.ansys.com, where examples explain the use of the magnetic finite-element software Maxwell®. This book uses Maxwell for many examples in several chapters. The version of Maxwell used for solutions in this book is version 9 (formerly incorporating student version SV), but all Maxwell files in this book can presently be converted to current Maxwell version 16 software files. The website http://booksupport.wiley.com contains example input files for both versions 9 and 16 of Maxwell. The input procedures in this book are given for Maxwell version 9 except when stated that another version of Maxwell is used.

**Example 4.2 Finite-Element Analysis of the "C" Steel Path with Airgap of Example 3.1** The problem shown in Figure E3.1.1 and solved by the reluctance method in Example 3.1 is to be solved here using Maxwell.

***Solution*** Follow the above four steps to make a finite-element model of the problem in Maxwell SV.

(1) Enter the geometry. Maxwell requires a *background* or *region* which must be at least as big as the device modeled. The overall background chosen is the same as in Figure E3.1.1, 0.8 m horizontally and 0.6 m vertically. The geometry is entered as straight lines that are connected to form the various parts. Make sure that the *xy* plane is selected on the upper left tab on the main menu just below the tab selecting the *magnetostatic* problem type.

(2) Enter the materials. Since the steel is to have a relative permeability of 2000, you must add a special linear steel material.

(3) Enter the *sources* (version SV) or *excitations* (version 16) in the menu. The coil region in the center of the "C" is set to 10,000 ampere-turns. Its return on the left side of the "C" must then have −10,000 ampere-turns.

(4) Enter the boundary conditions. Here the flux is assumed confined to the 0.8 m by 0.6 m region analyzed, and thus the vector potential is set to zero on the entire outer boundary.

You should now note on the Maxwell SV main menu that the above steps all have check marks next to them, indicating that you can now click on its “Solve” command. (In version 16, you can click on “Validate” to ensure that all the steps have been completed, and then click on “Analyze” to initiate the solution.) Maxwell then automatically places triangular finite elements in the geometry you have entered. It adds elements in passes as shown in Table E4.2.1. For each pass it computes both the total stored energy and the estimated energy error in percent. Its default energy error is 1%, but in Table E4.2.1 it has been reduced to 0.1%, requiring 11 passes. The corresponding finite-element model or *mesh* is shown in Figure E4.2.1.

**TABLE E4.2.1 Convergence of “C” core example finite-element analysis in Maxwell**

| Pass | Triangles | Total Energy (J) | Energy Error (%) |
|---|---|---|---|
| 1 | 46 | 1.331176E+002 | 8.8755 |
| 2 | 124 | 1.38127 E+002 | 2.6007 |
| 3 | 160 | 1.38824 E+002 | 1.2903 |
| 4 | 206 | 1.39575 E+002 | 0.8061 |
| 5 | 262 | 1.40271 E+002 | 0.7050 |
| 6 | 340 | 1.40619 E+002 | 0.4578 |
| 7 | 440 | 1.41122 E+002 | 0.3302 |
| 8 | 572 | 1.41274 E+002 | 0.2237 |
| 9 | 737 | 1.41454 E+002 | 0.1833 |
| 10 | 965 | 1.41588 E+002 | 1.1220 |
| 11 | 1241 | 1.41651 E+002 | 0.0851 |

The energy listed in Table E4.2.1 is for a depth of 1 m. To compare its energy of 141.65 J with that of the reluctance solution of Example 3.1, use the right hand side of (4.2):

$$W = \int \frac{B^2}{2\mu} dv \tag{E4.2.1}$$

Since the reluctance method of Example 3.1 obtained $B = 0.125$ T, and most of the energy is in the airgap volume of size 0.1 m by 0.1 m by a depth of 1 m (to agree with Maxwell), (E4.2.1) becomes:

$$W = (0.125)^2/(12.57\text{E–}7)(0.1)(0.1)(1) = 124.3\,\text{J} \tag{E4.2.2}$$

This energy of the reluctance method is about 12% lower than the 141.65 J computed by Maxwell. Since it accurately accounts for fringing flux, the finite-element solution is thus considerably more accurate than the reluctance method solution.

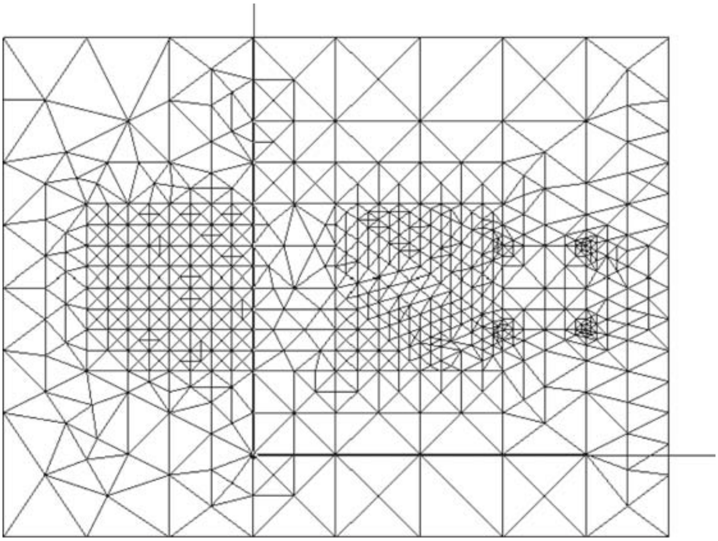

**FIGURE E4.2.1** Finite-element model of steel path with airgap of Examples 3.1 and 4.2 produced by 11 adaptive passes.

## PROBLEMS

**4.1** Repeat Example 4.1 with the second finite element (of area 2.E−4 $m^2$) changed to steel with relative permeability 1000 and containing no current. Assuming the first element remains the same, find the final matrix equation.

**4.2** Substitute the first-order triangular shape function of (4.8) into $\mathbf{B} = \nabla \times \mathbf{A}$ to show that **B** is constant in each such finite element.

**4.3** Use Maxwell to solve Example 4.2 with the steel having the nonlinear *B–H* curve called "steel_1010" in the Maxwell material list.

**4.4** Use Maxwell to solve Example 4.2 with the steel having the nonlinear *B–H* curve called "steel_1008" in the Maxwell material list.

## REFERENCES

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