Limit and Continuity
(ASSIGNMENT)

1. Explain in your own words the meaning of each of the following:

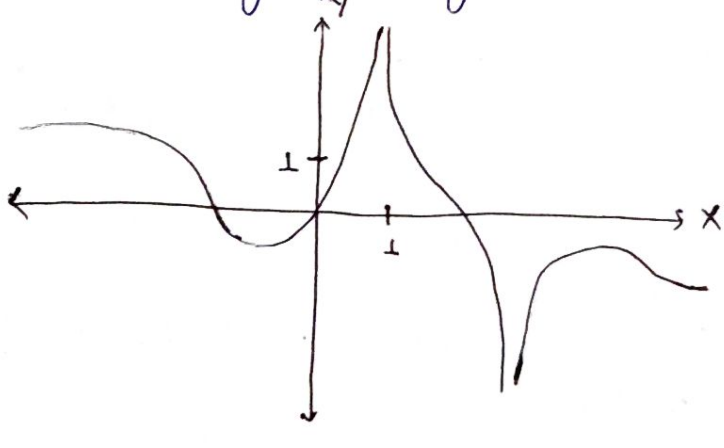
a) $\lim_{x \to \infty} f(x) = 5$

Ans: The limit of the function f, as x approaches infinity is 5. That means, as the x approaches infinity, the y approaches 5 but not equal to 5. Alternatively, $y=5$ will be a horizontal asymptote of $f(x)$. It will get close to $y=5$ but will not be equal.

b) $\lim_{x \to \infty} f(x) = 3$

Ans: The limit of the function f as x approaches infinity is 3. That is, as the x tends to be infinity, the value of $f(x) = y$ will be as close to 3 as it demands. but not equal. Alternatively, $y=3$ will be asymptote of $f(x)$.

2. For the function f whose graph is given, state the following:



a) $\lim_{x \to \infty} f(x)$

$\Rightarrow$ According to the graph, when the function 'f' moves to right, the curve appears to get closer and closer to $y = -2$

i.e. $$\lim_{x \to \infty} f(x) = -2$$

$y=-2$ is the limit of the function f when the function moves to right and approaches infinity.

b) $\lim_{x \to -\infty} f(x)$

⇒ According to the graph, as the function 'f' moves left, it gets closer and closer to $y = 2$ but not equal.

i.e.

$$\lim_{x \to -\infty} f(x) = 2$$

∴ $y = 2$ is the limit of the function 'f' when the function moves right and approaches infinity at left($-\infty$).

c) $\lim_{x \to 1} f(x)$

⇒ According to the graph, as x tends to 1, $f(x)$ goes up towards infinity but not exactly at infinity

i.e.

$$\lim_{x \to 1} f(x) = \infty$$

∴ Infinity (∞) is the limit of function f when x tends to 1.

d) $\lim_{x \to 3} f(x)$

⇒ From the graph, we can observe that as the x tends to 3, $f(x)$ goes down towards infinity but not exactly at negative infinity

i.e.

$$\lim_{x \to 3} f(x) = -\infty$$

∴ $-\infty$ (negative infinity) is the limit of the function f when x tends to 3.

e) the equation of asymptotes

⇒ From the graph, we can see that,

the horizontal asymptotes are

$y = -2$ and $y = 2$.

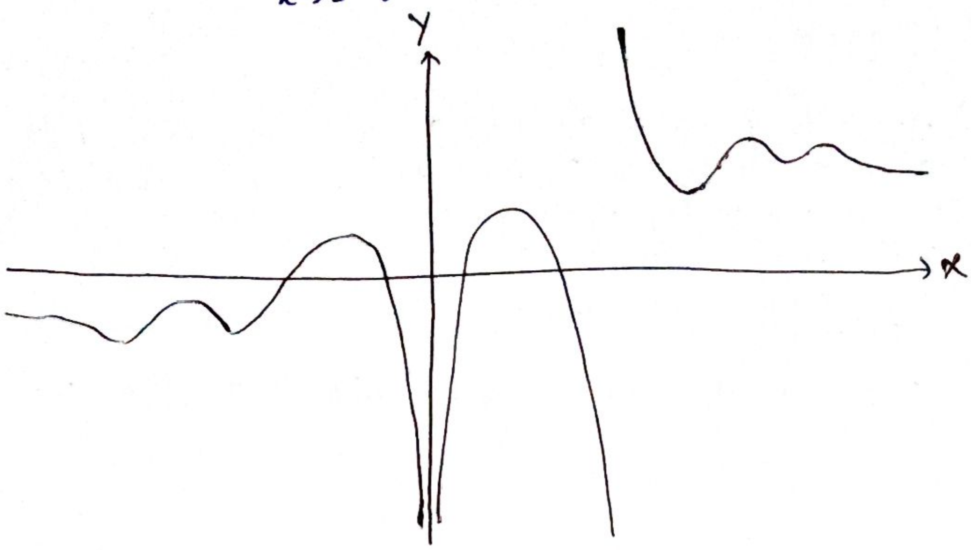
and, the vertical asymptotes are

$x = 1$ and $x = 3$.

3. From the function of whose graph is given, state the following:

a) $\lim_{x \to \infty} g(x)$ b) $\lim_{x \to -\infty} g(x)$ c) $\lim_{x \to 0} g(x)$

d) $\lim_{x \to 2^-} g(x)$ e) $\lim_{x \to 2^+} g(x)$ f) the equation of the asymptotes



a) $\lim_{x \to \infty} g(x)$

$\Rightarrow$ From the graph, we can see that, as the curve moves to right, the curve is becoming a straight line as it's approaching $y=2$. Also, if we suppose, the graph is representation of amplitude and frequency, then it is decreasing as curve moves right and approaches $y=2$.

i.e.

$$\lim_{x \to \infty} g(x) = 2.$$

$\therefore y=2$ is the limit of g, as $x \to \infty$

b) $\lim_{x \to -\infty} g(x)$

$\Rightarrow$ From the graph, we can see that, as the curve moves left, the curve is becoming a straight line as it is approaching $y=-1$. Also the curve is decaying.

i.e.

$$\lim_{x \to -\infty} g(x) = -1.$$

$\therefore y=-1$ is the limit of g, as $x \to -\infty$

c) $\lim_{x \to 0} g(x)$

$\Rightarrow$ From the graph, we observe that as x tends to zero, the curve from both right and left side ~~gets move~~ goes down ~~up~~ towards infinity but not exactly at infinity.

i.e.

$$\lim_{x \to 0} g(x) = -\infty$$

$\therefore -\infty$ is the limit of g as $x \to 0$.

d) $\lim_{x \to 2^-} g(x)$

$\Rightarrow$ According to the graph, as we move from left to right, and x tends to 2, the curve drops down towards infinity but not exactly at infinity.

i.e.

$$\lim_{x \to 2^-} g(x) = -\infty$$

$\therefore -\infty$ is the left hand limit of $g(x)$ as $x \to 2$

e) $\lim_{x \to 2^+} g(x)$

$\Rightarrow$ According to the graph, as curve moves from right to left and x tends to 2, the curve rises up towards infinity but never ~~reaches~~ exactly at infinity.

i.e.

$$\lim_{x \to 2^+} g(x) = \infty$$

f) the equation of the asymptotes

$\Rightarrow$ The vertical asymptotes are:

$x = 0$ [from graph and c]

$x = 2$ [from graph and d & e]

$\Rightarrow$ The horizontal asymptotes are:

$y = 2$ [from graph and a]

$y = -1$ [from graph and b]

4. Sketch the graph of an example of a function f that satisfies all of the given conditions.

i) $\lim_{x\to 0} f(x) = -\infty$ ii) $\lim_{x\to -\infty} f(x) = 5$ iii) $\lim_{x\to \infty} f(x) = -5$

<u>Solution</u>:

The given conditions for $f(x)$ are:

i) $\lim_{x\to 0} f(x) = -\infty$

i.e. when $x \to 0$, $f(x)$ drops to negative infinity. In other words, $x=0$ is the vertical asymptote of $f(x)$.

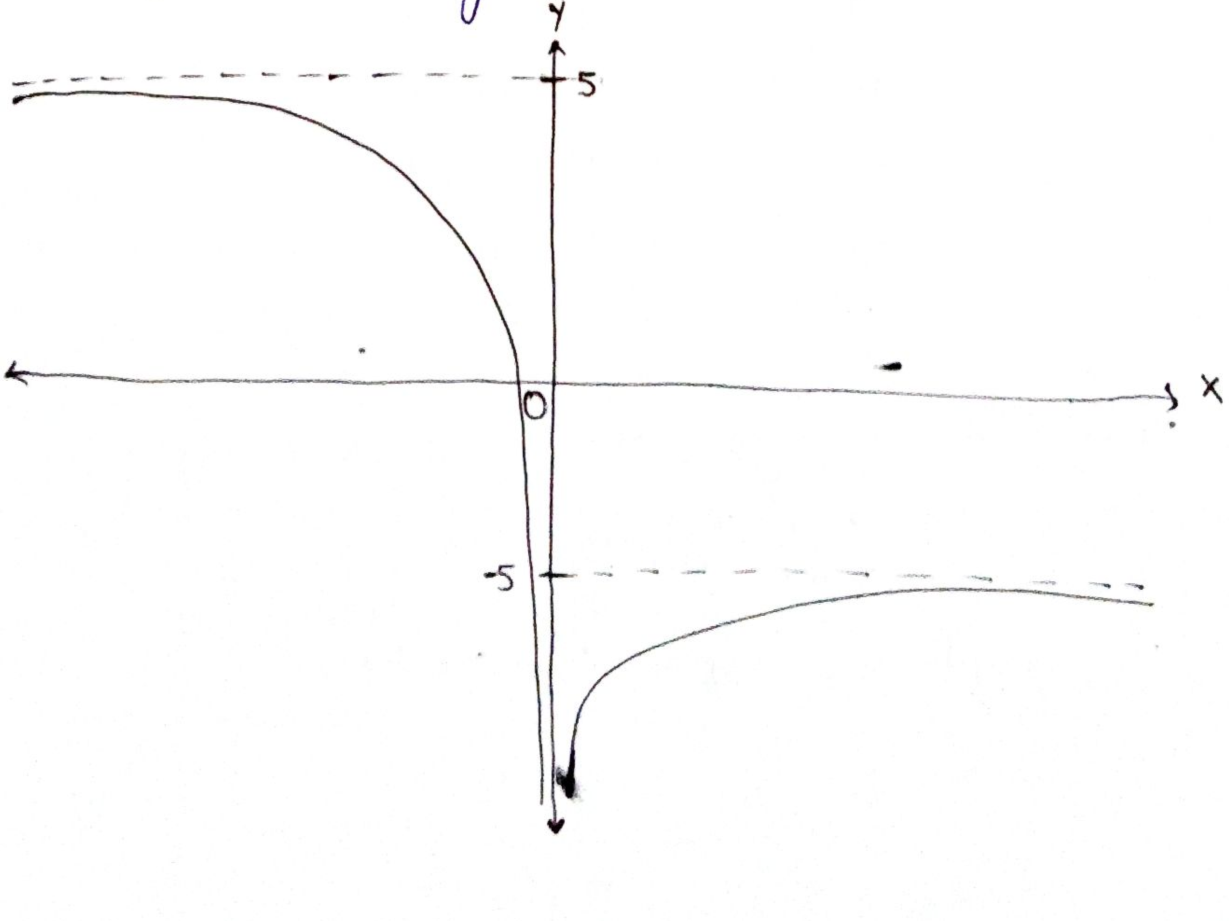
ii) $\lim_{x\to -\infty} f(x) = 5$

i.e. when $x \to -\infty$, $f(x)$ approaches as close as $y=5$ but not equal. In other words, $y=5$ is the horizontal asymptote of $f(x)$.

iii). $\lim_{x\to \infty} f(x) = -5$

i.e. when $x\to\infty$, $f(x)$ approaches as close as $y=-5$, but not equal. In other words, $y=-5$ is the horizontal asymptote of $f(x)$.

Since, there are many possible graphs which might satisfy the above conditions, the graph of an example of function f that satisfies all the given conditions is as below:



5. Guess the value of limit $\lim_{x \to \infty} \frac{x^2}{2^x}$ by evaluating the function $f(x) = \frac{x^2}{2^x}$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$ and 100.

Then use the graph to support your guess.

Solution:

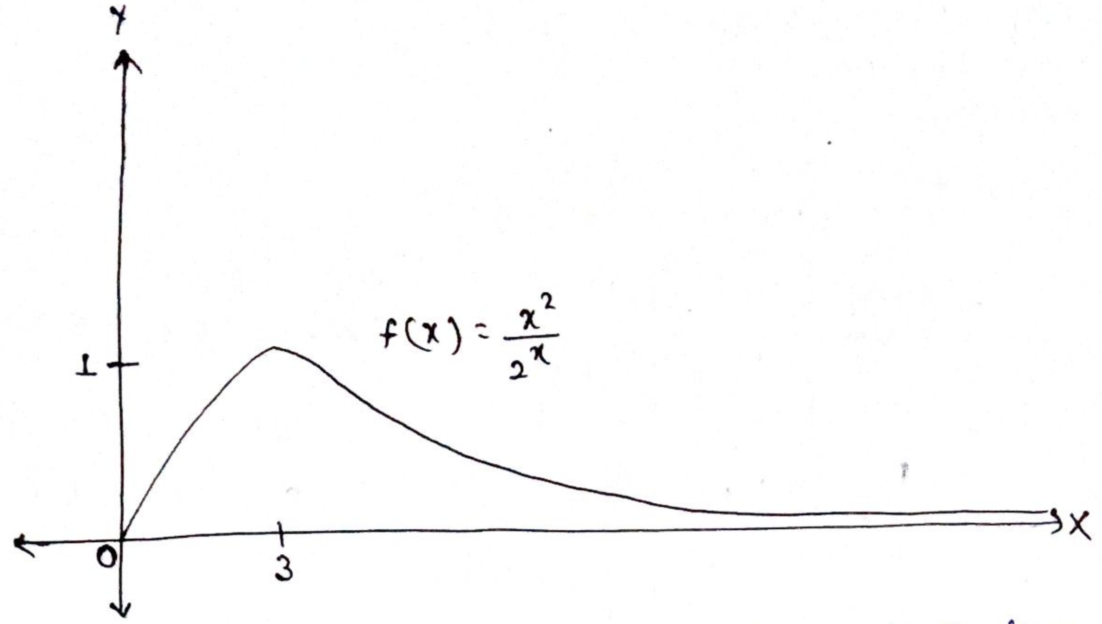
For the given set of values of x, corresponding values of $f(x)$ are as follows:

x	$f(x) = \frac{x^2}{2^x}$
0	0
1	$1/2 = 0.5$
2	1
3	1.125
4	1
5	0.78125
6	0.5625
7	0.382812
8	0.25
9	0.158203
10	0.097656
20	0.000381
50	2.2×10^{-12}
100	0

Thus, from the above table, we can conclusively guess that

$$\lim_{x \to \infty} \frac{x^2}{2^x} = 0$$

The graph also supports this conclusive guess.



6. Evaluate the limit and justify each step by indicating the properties of limit.

a) $\lim\limits_{x \to \infty} \dfrac{3x^2 - x + 14}{2x^2 + 5x - 8}$

<u>Solution:</u>

$$\lim_{x \to \infty} \frac{3x^2 - x + 14}{2x^2 + 5x - 8}$$

$$= \frac{\lim\limits_{x \to \infty} (3x^2 - x + 14)}{\lim\limits_{x \to \infty} (2x^2 + 5x - 8)}$$ [Quotient law of limit]

$$= \frac{\lim\limits_{x \to \infty} \left(3 - \frac{1}{x} + \frac{14}{x^2}\right)}{\lim\limits_{x \to \infty} \left(2 + \frac{5}{x} - \frac{8}{x^2}\right)}$$ [Dividing denominator and numerator by the greatest power : x^2]

$$= \frac{\lim\limits_{x \to \infty} 3 - \lim\limits_{x \to \infty} \frac{1}{x} + \lim\limits_{x \to \infty} \frac{14}{x^2}}{\lim\limits_{x \to \infty} 2 + \lim\limits_{x \to \infty} \frac{5}{x} - \lim\limits_{x \to \infty} \frac{14}{x^2}}$$ [Sum law of limit]

$$= \frac{3 - 0 + 0}{2 + 0 - 0}$$ [$\lim\limits_{x \to \infty} \frac{a}{x^n} = 0, n \in \mathbb{R}$]

$$= \frac{3}{2}$$

$$\therefore \lim_{x \to \infty} \frac{3x^2 - x + 14}{2x^2 + 5x - 8} = \frac{3}{2}$$

b) $\lim_{x \to \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

Solution:

$$\lim_{x \to \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$$

$$= \lim_{x \to \infty} \sqrt{\frac{12 - \frac{5}{x^2} + \frac{2}{x^3}}{\frac{1}{x^3} + \frac{4}{x} + 3}}$$ [Dividing numerator and denominator by x^3]

$$= \sqrt{\lim_{x \to \infty} \left(\frac{12 - \frac{5}{x^2} + \frac{2}{x^3}}{\frac{1}{x^3} + \frac{4}{x} + 3} \right)}$$ [Square root is continuous]

$$= \sqrt{\frac{\lim_{x \to \infty} \left(12 - \frac{5}{x^2} + \frac{2}{x^3}\right)}{\lim_{x \to \infty} \left(\frac{1}{x^3} + \frac{4}{x} + 3\right)}}$$ [Quotient law of limit]

$$= \sqrt{\frac{\lim_{x \to \infty} 12 - \lim_{x \to \infty} \frac{5}{x^2} + \lim_{x \to \infty} \frac{2}{x^3}}{\lim_{x \to \infty} \frac{1}{x^3} + \lim_{x \to \infty} \frac{4}{x} + \lim_{x \to \infty} 3}}$$ [Sum law of limit]

$$= \sqrt{\frac{12 - 0 + 0}{0 + 0 + 3}}$$ $\left[\lim_{x \to \infty} \frac{a}{x^n} = 0, \ \forall n \in \mathbb{R}\right]$

$$= \sqrt{\frac{12}{3}}$$

$$= \sqrt{4}$$

$$= 2$$

$$\therefore \lim_{x \to \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}} = 2$$

7. Find the limit or show that it does not exist:

a) $\lim_{x\to\infty} \frac{3x-2}{2x+1}$

Solution:

$$\lim_{x\to\infty} \frac{3x-2}{2x+1}$$

$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}}{2+\frac{1}{x}} \quad \left[\text{Dividing numerator and denominator by } x\right]$$

$$= \frac{\lim_{x\to\infty}\left(3-\frac{2}{x}\right)}{\lim_{x\to\infty}\left(2+\frac{1}{x}\right)} \quad \left[\text{Quotient law of limit}\right]$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{2}{x}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x}} \quad \left[\text{Sum law of limit}\right]$$

$$= \frac{3-0}{2+0} \quad \left[\lim_{x\to\infty}\frac{a}{x^n} = 0, \forall n \in \mathbb{R}\right]$$

$$= \frac{3}{2}$$

$$\therefore \lim_{x\to\infty}\frac{3x-2}{2x+1} = \frac{3}{2}$$

b) $\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$

Solution:

$$\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$$

$$= \frac{\lim_{t\to\infty}\left(\frac{1}{\sqrt{t^3}}+1\right)}{\lim_{t\to\infty}\left(\frac{2}{t}-1\right)}$$

$$= \frac{0+1}{0-1}$$

$$= -1$$

$$\therefore \lim_{t\to\infty}\frac{\sqrt{t}+t^2}{2t-t^2} = -1$$

c) $\lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

Solution:

$$\lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt{9x^6 - x}}{x^3}}{\frac{x^3 + 1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}}$$

$$= \frac{\sqrt{\lim_{x \to \infty} \left(9 - \frac{1}{x^5}\right)}}{\lim_{x \to \infty} \left(1 + \frac{1}{x^3}\right)}$$

$$= \frac{\sqrt{9 - 0}}{1 + 0}$$

$$= 3$$

$$\therefore \lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} = 3$$

d) $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

Solution:

$$\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$$

$$= \lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right) \frac{\left(\sqrt{9x^2 + x} + 3x\right)}{\left(\sqrt{9x^2 + x} + 3x\right)}$$

$$= \lim_{x \to \infty} \frac{\cancel{9x^2} + x - \cancel{9x^2}}{\sqrt{9x^2 + x} + 3x}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{9x^2 + x} + 3x}$$

$$= \lim_{x \to \infty} \frac{\frac{x}{x}}{\frac{\sqrt{9x^2 + x}}{x} + \frac{3x}{x}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{9 + \frac{1}{x}} + 3}$$

$$= \frac{1}{\sqrt{9+0}+3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

$$\therefore \lim_{x \to \infty} \sqrt{9x^2+x} - 3x = \frac{1}{6}$$

e) $\lim_{x \to \infty} \arctan(e^x)$

Solution:

Since arc tan is a continuous function as it is an inverse trigonometric function, so

$$\lim_{x \to \infty} \arctan(e^x)$$

$$= \arctan\left(\lim_{x \to \infty} e^x\right) \quad \left[\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)\right]$$

Here, $\lim_{x \to \infty} e^x = \infty$ because, as x increases without bound, e^x increases without bound.

Thus,

$$\arctan\left(\lim_{x \to \infty} e^x\right)$$

$$= \arctan(\infty)$$

$$= \frac{\pi}{2} \quad \left[\because \tan\left(\frac{\pi}{2}\right) = \infty\right]$$

$$\therefore \lim_{x \to \infty} \arctan(e^x) = \frac{\pi}{2}$$

8. Find the horizontal and vertical asymptotes of each curve:

a) $y = \frac{2x+1}{x-2}$

Solution:

Here, the given function, $f(x) = y = \frac{2x+1}{x-2}$

For Horizontal asymptotes:

Case-I: If $x \to \infty$ and $x > 0$, then,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+1}{x-2} = \lim_{x\to\infty}\left(\frac{2+1/x}{1-2/x}\right) = \frac{2+0}{1-0} = 2$$

Case-II: If $x \to -\infty$ and $x < 0$, then

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x+1}{x-2} = \lim_{x\to-\infty}\left(\frac{2+1/x}{1-2/x}\right) = \frac{2-0}{1+0} = 2$$

$\therefore$ From Case-I and Case-II, $y = 2$ is the horizontal asymptote.

For vertical asymptotes:

Vertical asymptote might occur if the denominator is 0

i.e. $x - 2 = 0$ i.e. $x = 2$

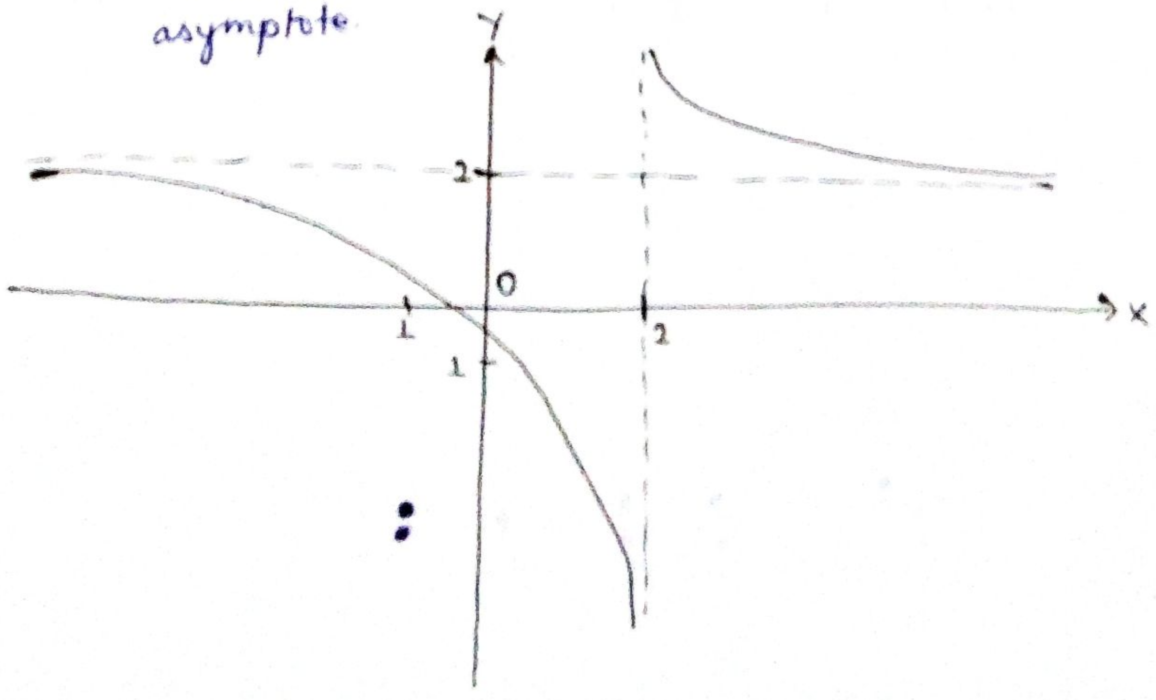
Case-I: If $x \to 2$ and $x > 2$, then

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{2x+1}{x-2} = \frac{2\times2+1}{2-2} = \frac{5}{0} = \infty$$

Case-II: If $x \to 2$ and $x < 2$, then

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{2x+1}{x-2} = \frac{2\times2+1}{2-2} = \frac{-5}{0} = -\infty$$

$\therefore$ From case-I and Case-II, $x = 2$ is the vertical asymptote



b) $y = \frac{x^2+1}{2x^2-3x-2}$

Solution:

Here, the given function is $f(x) = y = \frac{x^2+1}{2x^2-3x-2}$

For Horizontal Asymptote,

Case-I: If $x \to \infty$ and $x > 0$, then,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+1}{2x^2-3x-2} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}} = \frac{1+0}{2-0-0} = \frac{1}{2}$$

Case-II: If $x \to -\infty$ and $x < 0$, then,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{2x^2-3x-2} = \lim_{x\to-\infty} \frac{1+1/x^2}{2-3/x-2/x^2} = \frac{1+0}{2+0+0} = \frac{1}{2}$$

From case-I and case-II,

$y = 1/2$ is the horizontal asymptote.

For Vertical Asymptote,

Vertical asymptote might occur if the denominator is 0.

i.e. $2x^2-3x-2=0$

i.e. $x = \frac{-1}{2}$ or 2.

Case-I: If $x \to \frac{-1}{2}$ and $x > \frac{-1}{2}$, then,

$$\lim_{x\to\frac{-1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^+}\left(\frac{x^2+1}{2x^2-3x-2}\right) = \frac{(-1/2)^2+1}{2\times\left(\frac{-1}{2}\right)^2-3\times\left(\frac{-1}{2}\right)-2} = \infty$$

Case-II: If $x \to \frac{-1}{2}$ and $x < \frac{-1}{2}$, then,

$$\lim_{x\to\frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-}\left(\frac{x^2+1}{2x^2-3x-2}\right) = \frac{\left(\frac{-1}{2}\right)^2+1}{2\times\left(\frac{-1}{2}\right)^2-3\times\left(\frac{-1}{2}\right)-2} = -\infty$$

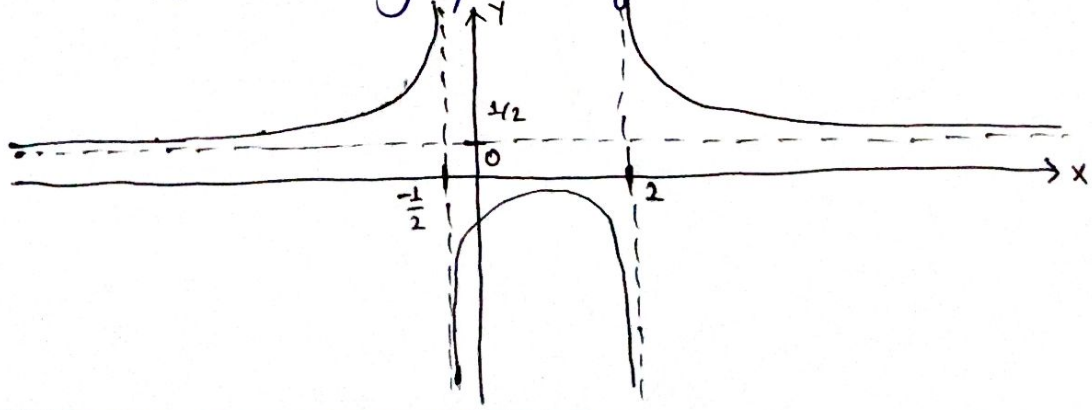
Case-III: If $x \to 2$ and $x > 2$, then

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{x^2+1}{2x^2-3x-2} = \cancel{\lim} \frac{2^2+1}{2\times2^2-3\times2-2} = \infty$$

Case-IV: Similarly as case-III, if $x \to 2$ and $x < 2$, then

$$\lim_{x\to2^-} f(x) = -\infty$$

From above cases, we conclude that $x = \frac{-1}{2}$ and $x = 2$ are the vertical asymptotes of the curve.



c) $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

Solution:

The given function is $f(x) = y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

For horizontal asymptote,

If $x \to \infty$ and $x > 0$, then,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{2x^2+x-1}{x^2+x-2}\right) = \lim_{x\to\infty}\left(\frac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}\right) = \frac{2+0-0}{1+0-0} = 2$$

If $x \to -\infty$ and $x < 0$, then,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{2x^2+x-1}{x^2+x-2}\right) = \lim_{x\to-\infty}\left(\frac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}\right) = \frac{2+0+0}{1+0} = 2$$

$\therefore y = 2$ is the horizontal asymptote.

For vertical asymptote,

vertical asymptote might occur if the denominator is 0

i.e. $x^2 + x - 2 = 0$

i.e. $x = -2$ or 1

If $x \to -2$ and $x > -2$, then,

$$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} \frac{2x^2+x-1}{x^2+x-2} = \frac{2(-2)^2+(-2)-1}{(-2)^2+(-2)-2} = \infty$$

Similarly, when $x \to -2$ and $x < -2$, then,

$$\lim_{x\to 2^-} f(x) = -\infty$$

If $x \to 1$ and $x > 1$, then,

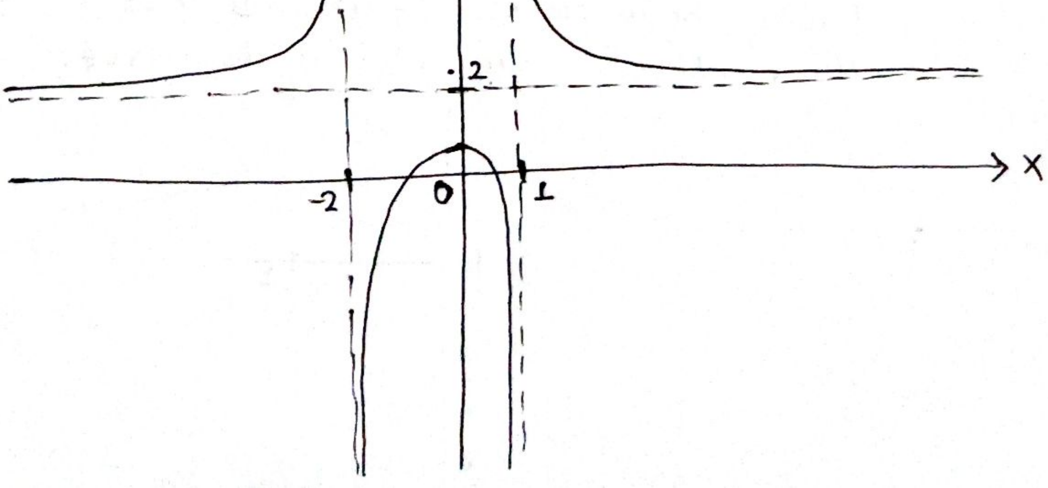
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2x^2+x-1}{x^2+x-2} = \frac{2+1-1}{1+1-2} = \infty$$

Similarly, when $x \to 1$ and $x < 1$, then,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{2x^2+x-1}{x^2+x-2} = -\infty$$

From above cases,

$x = 1$ and $x = -2$ are the vertical asymptotes of the curve.



d) $y = \frac{1+x^4}{x^2-x^4}$

Solution:

The given function is $f(x) = y = \frac{1+x^4}{x^2-x^4}$

For horizontal asymptote,

If $x \to \infty$ and $x > 0$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{1+x^4}{x^2-x^4}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1}\right) = \frac{0+1}{0-1} = -1$$

If $x \to -\infty$ and $x < 0$, then,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{1+x^4}{x^2-x^4}\right) = \lim_{x\to-\infty}\left(\frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1}\right) = \frac{-0+1}{0-1} = -1$$

$\therefore y = -1$ is the horizontal asymptote of the curve.

For vertical asymptote,

~~If~~ vertical asymptote might occur if the denominator is 0,

i.e. $x^2 - x^4 = 0$

i.e. $x = 0, 1$ or -1

If $x \to 0$ and $x > 0$, then,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{1+x^4}{x^2-x^4}\right) = \frac{1+0^4}{0-0} = \infty$$

similarly, if $x \to 0$ and $x < 0$, then,

$$\lim_{x\to 0^-} f(x) = -\infty$$

If $x \to 1$ and $x > 1$, then,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(\frac{1+x^4}{x^2-x^4}\right) = \frac{1+1}{1-1} = \infty$$

similarly, if $x \to 1$ and $x < 1$, then,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(\frac{1+x^4}{x^2-x^4}\right) = -\infty$$

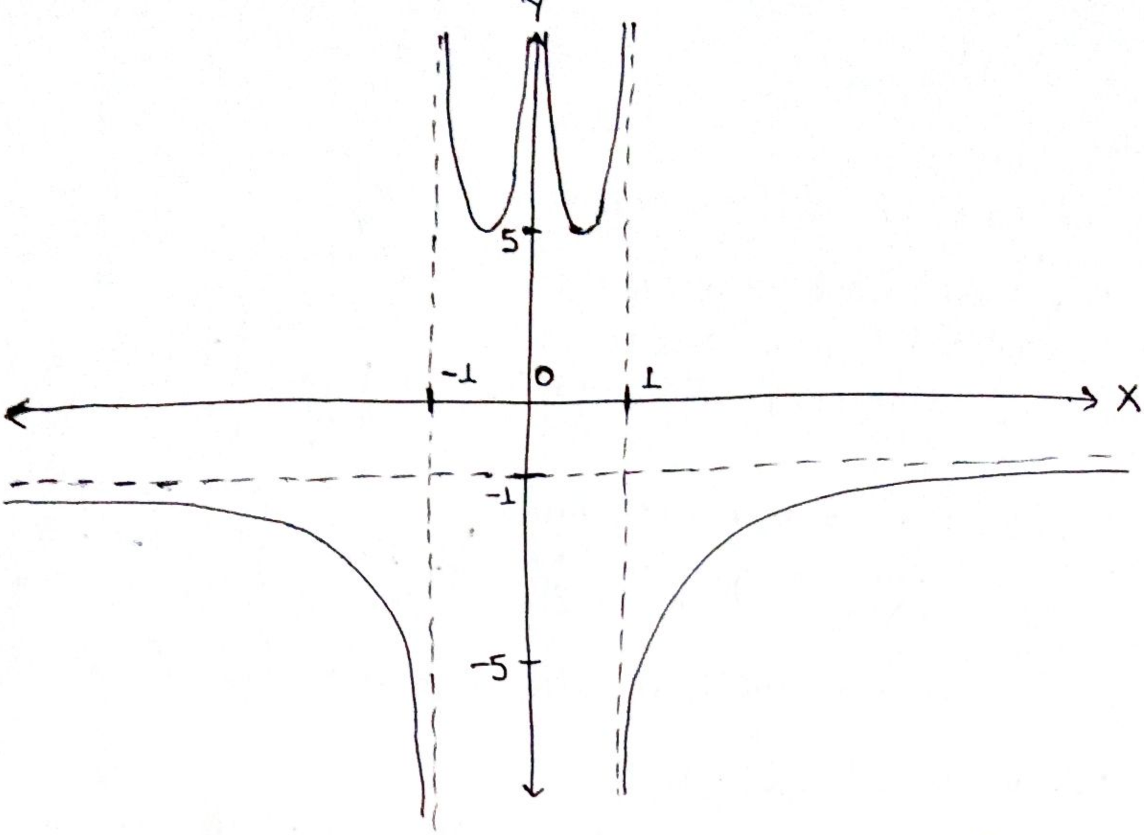
If $x \to -1$ and $x > -1$, then,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to -1^+}\left(\frac{1+x^4}{x^2-x^4}\right) = \frac{1+1}{1-1} = \infty$$

similarly, if $x \to -1$ and $x < -1$, then,

$$\lim_{x\to -1^-} f(x) = -\infty$$

$\therefore x = 0$, $x = 1$ and $x = -1$ are the vertical asymptotes of the curve.



9. Find the vertical and horizontal asymptote : $y = \frac{x^3 - x}{x^2 - 6x + 5}$

Solution:

The given function is $f(x) = y = \frac{x^3 - x}{x^2 - 6x + 5}$

For vertical asymptote,

vertical asymptote might occur if the denominator is 0

i.e. $x^2 - 6x + 5 = 0$

i.e. $x = 1$ ~~and~~ or 5

If $x \to 1$ and $x > 1$, then,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \frac{1^3 - 1}{1 - 6 + 5} = \frac{0}{0} \text{ (undefined)}$$

If $x \to 1$ and $x < 1$, then,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \frac{1^3 - 1}{1^2 - 6 + 5} = \frac{0}{0} \text{ (undefined)}$$

If $x \to 5$ and $x > 5$, then,

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \frac{5^3 - 5}{5^2 - 6 \times 5 + 5} = \infty$$

If $x \to 5$ and $x < 5$, then,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \frac{5^3 - 5}{5^2 - 6 \times 5 + 5} = -\infty$$

$\therefore x = 5$ is the only vertical asymptote of the curve.

For horizontal asymptote,

If $x \to \infty$ and $x > 0$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \lim_{x\to\infty}\left(\frac{x - \frac{1}{x}}{1 - \frac{6}{x} + \frac{5}{x^2}}\right) = \frac{\infty}{1} = \infty$$

If $x \to -\infty$ and $x < 0$, then,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{x^3 - x}{x^2 - 6x + 5}\right) = \lim_{x\to-\infty}\left(\frac{x - \frac{1}{x}}{1 - \frac{6}{x} + \frac{5}{x^2}}\right) = \infty$$

$\therefore$ The curve has no horizontal asymptotes.

10. Find the slant asymptotes of the following curves if it exists:

a) $f(x) = \dfrac{2x^2}{1-x}$

Solution:

Here, the given function is,

$$f(x) = \frac{2x^2}{1-x} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, degree of $P(x)$ is one more than $Q(x)$, it has slant asymptote.

Now,

$$1-x\overline{)\,2x^2\,}\left(-2x\right.$$
$$\frac{2x^2 - 2x}{2x}$$

$$\therefore f(x) = -2x + \frac{2x}{1-x}$$

Taking $x \to \pm\infty$, we have, $\frac{2x}{1-x} \to 0$. Hence y and $-2x$ are very close to each other as $x \to \pm\infty$.

Hence, $y = -2x$ is an oblique or slant asymptote of the curve.

b) $f(x) = \dfrac{x^3 - 3x^2}{x^2 - 1}$

Solution:

Here, the given function is.

$$f(x) = \frac{x^3 - 3x^2}{x^2 - 1} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since degree of $P(x)$ is one more than the degree of $Q(x)$, it has slant asymptote.

Now,

$$
\begin{array}{r l}
x^2-1 \,\big)\, x^3-3x^2 & \big(\, x-3 \\
x^3 \quad - x & \\
- \qquad + & \\
\hline
-3x^2+x & \\
-3x^2 \quad +3 & \\
+ \qquad - & \\
\hline
x-3 &
\end{array}
$$

$$\therefore f(x) = y = \frac{x^3-3x^2}{x^2-1} = (x-3) + \frac{x-3}{x^2+1}$$

Taking $x \to \pm\infty$, we have, $\frac{x-3}{x^2+1} \to 0$. Hence, y and $(x-3)$ are very close to each other as $x \to \pm\infty$.
Hence $y = x-3$ is an oblique or slant asymptote of the curve.

c) $f(x) = \frac{(2+x)(2-3x)}{(2x+3)^2}$

Solution:
Here, the given function is,

$$f(x) = \frac{(2+x)(2-3x)}{(2x+3)^2} = \frac{4-4x-3x^2}{4x^2+12x+9} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, the degree of $P(x)$ and $Q(x)$ are equal, $f(x)$ won't have any oblique or slant asymptote.

d) $f(x) = \frac{x^3-1}{x^2-x-2}$

Solution:
Here, the given function is,

$$f(x) = \frac{x^3-1}{x^2-x-2} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, degree of $P(x)$ is one more than degree of $Q(x)$, so it has slant asymptote.

Now,

$$
\begin{array}{r l}
x^2-x-2 \,\big)\, x^3-1 & \big(\, x+1 \\
x^3-x^2-2x & \\
- \quad + \quad + & \\
\hline
x^2+2x-1 & \\
x^2-x-2 & \\
- \quad + \quad + & \\
\hline
3x+1 &
\end{array}
$$

$$\therefore f(x) = \frac{x^3-1}{x^2-x-2} = (x+1) + \frac{3x+1}{x^2-x-2}$$

Taking $x \to \pm\infty$, we have $\frac{3x+1}{x^2-x-2} \to 0$. Hence y and $(x+1)$ are very close to each other as $x \to \pm\infty$. ~~Here~~ Hence $y = x+1$ is the slant or oblique asymptote of the curve.

e) $f(x) = \frac{x^2-2x}{x^3+1}$

Solution:

Here, the given function is,

$$f(x) = \frac{x^2-2x}{x^3+1} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, the degree of $Q(x)$ is higher than $P(x)$, $f(x)$ doesn't have any slant or oblique asymptote.

f) $f(x) = \frac{1-x^3}{x}$

Solution:

Here, the given function is,

$$f(x) = \frac{1-x^3}{x} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, the degree of $P(x)$ is larger than $Q(x)$, it has slant or oblique asymptote.

Now,

$$x \overline{)\, -x^3+1 \,}(x^2$$
$$-x^3$$
$$+$$
$$\overline{\quad 1 \quad}$$

$$\therefore f(x) = \frac{-x^3+1}{x} = -x^2 + \frac{1}{x}$$

Taking $x \to \pm\infty$, we have $\frac{1}{x} \to 0$. Hence y and $-x^2$ are very close to each other as $x \to \pm\infty$.

Hence, $y = -x^2$ is the slant or oblique asymptote of the curve.

g) $f(x) = \frac{x^3-1}{2(x^2-1)}$

Solution:

Here, the given function is,

$$f(x) = \frac{x^3-1}{2(x^2-1)} = \frac{x^3-1}{2(x^2-1)} = \frac{x^3-1}{2x^2-2} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, the degree of $P(x)$ is larger than $Q(x)$, it has a slant or oblique asymptote.

Now,

$$2x^2-2 \overline{)\, x^3-1 \,}(\frac{x}{2}$$
$$x^3 - x$$
$$- \quad +$$
$$\overline{\quad x-1 \quad}$$

$$\therefore f(x) = \frac{x^3-1}{2x^2-2} = \frac{x}{2} + \frac{x-1}{2x^2-2}$$

Taking $x \to \pm\infty$, we have $\frac{x-1}{2x^2-2} \to 0$. Hence y and $\frac{x}{2}$ are close to each other, as $x \to \pm\infty$.

Hence, $y = \frac{x}{2}$ is the slant or oblique asymptote of the curve.

h) $f(x) = \dfrac{x^4 - 2x^3 + 1}{x^2}$

Solution:

Here, the given function is,

$$f(x) = \frac{x^4 - 2x^3 + 1}{x^2} = \frac{P(x)}{Q(x)} \text{ (let)}$$

Since, the degree of $P(x)$ is more than $Q(x)$, so it has slant or oblique asymptote.

Now,

$$
\begin{array}{r|l}
x^2 \,\big)\; x^4 - 2x^3 + 1 & x^2 - 2x \\
-x^4 & \\
\hline
-2x^3 + 1 & \\
-2x^3 & \\
+ & \\
\hline
1 &
\end{array}
$$

$$\therefore f(x) = \frac{x^4 - 2x^3 + 1}{x^2} = (x^2 - 2x) + \frac{1}{x^2}$$

Taking $x \to \pm\infty$, we have $\frac{1}{x^2} \to 0$. Hence, $f(x)$ and $(x^2 - 2x)$ are close to each other, as $x \to \pm\infty$.

Hence, $y = x^2 - 2x$ is the slant or oblique asymptote of the curve.

11. Evaluate the following limits:

a) $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x$

Solution:

$$\lim_{x\to\infty} \sqrt{x^2+1} - x$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)$$

$$= \lim_{x\to\infty} \frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x}$$

$$= \frac{1}{\sqrt{\infty + 1} + \infty}$$

$$= \frac{1}{\infty}$$

$$= 0$$

$$\therefore \lim_{x\to\infty} \sqrt{x^2+1} - x = 0$$

b) $\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$

Solution:

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$$

Let $t = \frac{1}{x-2}$, such that $t \to \infty$ when $x \to 2^+$

Thus,

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \arctan\left(\lim_{t \to \infty}(t)\right)$$

$$= \arctan(\infty)$$

$$= \frac{\pi}{2}$$

c) $\lim_{x \to 0^-} e^{4/x}$

Solution:

Let $t = \frac{1}{x}$, such that when $x \to 0^-$, $t \to -\infty$,

$$\lim_{x \to 0^-} e^{4/x} = \lim_{t \to -\infty} e^{t}$$

$$= e^{-\infty}$$

$$= \frac{1}{e^{\infty}}$$

$$\therefore \lim_{x \to 0^-} e^{4/x} = 0$$

d) $\lim_{x \to \infty} \sin x$

Solution:

Since $\sin x \leq |1|$, the value of $\sin x$ oscillates between -1 and 1.
Thus when x increases infinitely, the value of $\sin x$ will also change infinitely and won't approach any definite value.

Thus, $\lim_{x \to \infty} \sin x$ does not exist.

e) $\lim\limits_{x \to \infty} x^3$ and $\lim\limits_{x \to -\infty} x^3$

Solution:

Case - I: When x increases, x^3 also increases.

For example,

$$10^3 = 1000, \quad 100^3 = 1000000 \text{ and so on.}$$

Thus, x^3 will keep enlarging as we take x large enough.

Hence, we can say that

$$\lim_{x \to \infty} x^3 = \infty$$

Case - II: When x decreases, x^3 also decreases. And if we take a negative x, then the x^3 will be much larger negative value.

For example,

$$(-10)^3 = -1000, \quad (-100)^2 = -1000000 \text{ and so on.}$$

Therefore, we can say,

$$\lim_{x \to -\infty} x^3 = -\infty$$

Hence, $\lim\limits_{x \to \infty} x^3 = \infty$ and $\lim\limits_{x \to -\infty} x^3 = -\infty$

f) $\lim\limits_{x \to \infty} (x^2 - x)$

Solution:

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x-1)$$

When x becomes large, their product will become larger too. ~~Hotto~~ Also, taking 1 out of an infinitely large value of x won't make any significant change to the final result.

Thus,

$$\lim_{x \to \infty} x(x-1) = \cancel{\lim_{x \to \infty}} \infty(\infty - 1) = \infty \times \infty = \infty$$

$$\therefore \lim_{x \to \infty} x(x-1) = \infty$$

g) $\lim_{x\to\infty} \frac{x^2+x}{3-x}$

Solution:

$$\lim_{x\to\infty} \frac{x^2+x}{3-x}$$

$$= \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1}$$ [Dividing numerator and denominator by x]

$$= \frac{\infty+1}{\frac{3}{\infty}-1}$$

$$= \frac{\infty}{-1}$$

$$= -\infty$$

$$\therefore \lim_{x\to\infty} \frac{x^2+x}{3-x} = -\infty$$

12. Use the continuity to evaluate $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$

Solution:

Here, the given limit is $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$.

Let $f(x) = \frac{\sin x}{2+\cos x} = \frac{p(x)}{g(x)}$

Here, $p(x) = \sin x$ is a trigonometric function and it is continuous in its domain.

Also, $g(x) = 2+\cos x$ is a continuous function as it is sum of two continuous functions: linear and trigonometric.

Thus,

$f(x) = \frac{\sin x}{2+\cos x}$ is continuous everywhere as $2+\cos x \neq 0$.

Then, by the definition of continuous function,

$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = f(\pi)$$

$$= \frac{\sin\pi}{2+\cos\pi}$$

$$= \frac{0}{2-1}$$

$$= 0$$

$$\therefore \lim_{x\to\pi} \frac{\sin x}{2+\cos x} = 0$$

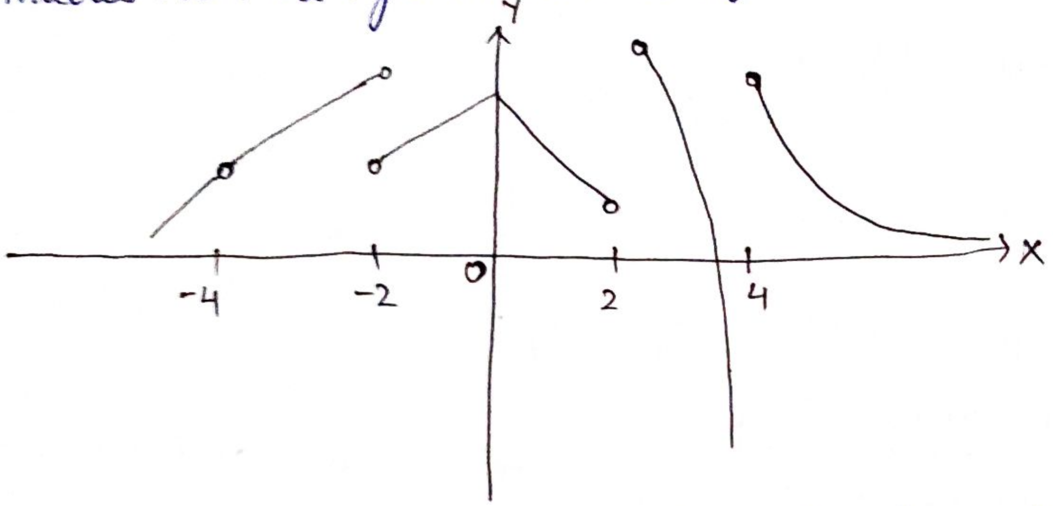
13. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

$\Rightarrow$ If a function f is continuous on $(-\infty, \infty)$, then the curve of the function will be without any breaks i.e. the function won't have any gaps or breaks in its graph.

Alternatively, we can say that the graph can be drawn for f in interval $(-\infty, \infty)$ without lifting the pen/pencil or rubbing the traced line at any point on the curve.

14. a) From the graph of f, state the number at which f is discontinuous and explain why.

b) For each of the numbers stated in part (a), determine whether f is continuous from the right or from the left.



a) $\Rightarrow$ f is undefined at $x=-4$, hence f is discontinuous.

$\rightarrow$ At $x=2$ and $x=-2$, limits from right and left are unequal hence f is also discontinuous at $x=-2$ and $x=2$.

$\rightarrow$ At $x=4$, the limit from left goes to infinity.

Hence,

$f(-4)$ is undefined and $\lim_{x \to a} f(x)$ for $a=-2, 2, 4$ doesn't exist.

Thus, $f(x)$ is discontinuous at $x = \pm 2, \pm 4$.

b) $\Rightarrow$ As we know, f is continuous from the left if $\lim_{x \to a^-} f(x) = f(a)$ and f is continuous from the right if $\lim_{x \to a^+} f(x) = f(a)$.

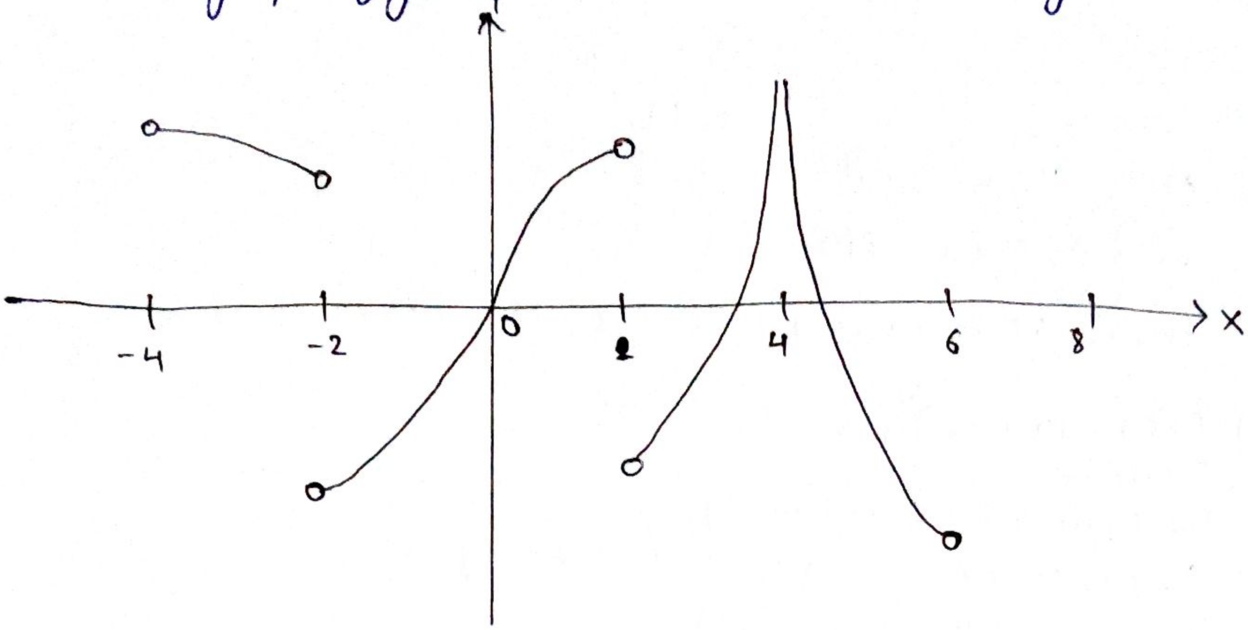
Thus,

For $x=-4$, neither right nor left limit exists ~~at~~ as $f(-4)$ is undefined.

At $x=-2$ and $x=2$, f is continuous from left.

At $x=4$, f is continuous from the right.

15. From the graph of g, state the intervals on which g is continuous.



Solution:

At $x=-4$, graph is continuous from right.
At $x=-2$, graph is neither continuous from left nor right
At $x=2$, graph is continuous from right, but not left.
At $x=4$, graph is neither continuous from left nor right.
At $x=6$, graph is neither continuous from light nor right.
At $x=8$, graph is not continuous from left.

$\therefore$ The interval of continuity is

$$x \in [-4,2) \cup (-2,2) \cup (2,4) \cup (4,6) \cup (6,8)$$

16. Use the definition of continuity and properties of the limits to show that the function is continuous at given number 'a'.

i) $f(x) = 3x^4 - 5x + 3\sqrt{x^2+4}$, $a=2$

Solution:

By definition of continuity,
$f(x)$ is continuous at $x=a$, if and only if

$$\lim_{x\to a} f(x) = f(a)$$

Now,

$$\lim_{x\to a} f(x) = \lim_{x\to 2} (3x^4 - 5x + 3\sqrt{x^2+4})$$

$$= \lim_{x\to 2} 3x^4 - \lim_{x\to 2} 5x + \lim_{x\to 2} 3\sqrt{x^2+4} \quad \text{(By sum law of limit)}$$

$$= 3.\lim_{x\to 2} x^4 - 5.\lim_{x\to 2} x + 3.\lim_{x\to 2}\sqrt{x^2+4} \quad \text{(By constant law of limit)}$$

$$= 3.\lim_{x\to 2} x^4 - 5.\lim_{x\to 2} x + 3.\sqrt{\lim_{x\to 2} x^4 + \lim_{x\to 2} 4}$$

($\because$ A ~~continu~~ function ~~is~~ of a continuous function is continuous and by sum law of limit

$= 3 \cdot 2^4 - 5 \cdot 2 + 3\sqrt{2^2+4} \quad \left(\because \lim_{x \to a} x^n = a^n\right)$

$= 38 + 6\sqrt{2}$

$= 46.49$

Also, $f(2) = 3 \times 2^4 - 5 \times 2 + 3\sqrt{2^2+4} = 38 + 6\sqrt{2} = 46.49$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Hence, it is proved that $f(x)$ is continuous at $x = 2$.

ii) $f(x) = (x + 2x^3)^4$, $a = -1$

Solution:

By definition of continuity,

$f(x)$ is continuous at $x = a$ if and only if

$$\lim_{x \to a} f(x) = f(a)$$

Now,

$$\lim_{x \to a} f(x) = \lim_{x \to -1} (x + 2x^3)^4$$

$= \left(\lim_{x \to -1} (x + 2x^3)\right)^4$ (Power law of limit)

$= \left(\lim_{x \to -1} x + \lim_{x \to -1} 2x^3\right)^4$ (Sum law of limit)

$= \left(\lim_{x \to -1} x + 2 \cdot \lim_{x \to -1} x^3\right)^4$ (Constant law of limit)

$= \left(-1 + 2(-1)^3\right)^4$ $\left(\lim_{x \to a} x = a \text{ \& } \lim_{x \to a} x^n = a^n\right)$

$= (-1-2)^4$

$= (-3)^4$

$= 81$

Also, $f(-1) = (-1 + 2 \times (-1)^3)^4 = (-1-2)^4 = 81 = \lim_{x \to 2} f(x)$

Hence, it is proved that $f(x)$ is continuous at $x = -1$

iii) $h(t) = \dfrac{2t - 3t^2}{1 + t^3}$, $a = 1$

Solution:

By definition of continuity,

$h(t)$ is continuous if and only if,

$$\lim_{t \to a} h(t) = h(a).$$

Now,

$$\lim_{t \to a} h(t) = \lim_{t \to 1} \frac{2t - 3t^2}{1 + t^3}$$

$$= \frac{\lim_{t\to1}(2t-3t^2)}{\lim_{t\to1}(1+t^3)}$$ (By quotient law of limit)

$$= \frac{2.\lim_{t\to1} t - 3\lim_{t\to1} t^2}{1+\lim_{t\to1} t^3}$$ (By sum law and constant law of limit)

$$= \frac{2\times1-3\times1^2}{1+1^3}$$

$$= \frac{2-3}{2}$$

$$= \frac{-1}{2}$$

Also,

$$h(t) = \frac{2\times1-3\times1^2}{1+1^3} = \frac{2-3}{2} = \frac{-1}{2} = \lim_{t\to1} h(t)$$

Hence, it is proved that $h(t)$ is continuous at $t=1$.

17. Explain using the theorem of continuous function why the functions are continuous ~~of every~~ in every number in its domain. State the domain.

a) $G(x) = \frac{x^2+1}{2x^2-x-1}$

Solution:

Here, the given function is $G(x) = \frac{x^2+1}{2x^2-x-1} = \frac{P(x)}{Q(x)}$ (let)

where, $P(x) = x^2+1$

$Q(x) = 2x^2-x-1$

$P(x)$ is a polynomial function of degree 2, and the domain of a polynomial function is $(-\infty,\infty)$.

Thus, $P(x)$ is continuous in its domain as polynomial functions are continuous in their domains.

$Q(x)$ is also continuous as it is a polynomial function and its domain is $(-\infty,\infty)$.

Thus, the function ~~Gx~~ $G(x)$ is also continuous, as it is a rational function.

The domain of $G(x)$ is $\{x : x \in \mathbb{R}$ but $Q(x) \neq 0\}$

i.e. $2x^2-x-1 \neq 0 \Rightarrow x \neq 1$ or $\frac{-1}{2}$ for $Q(x)$

Hence, $G(x)$ is continuous in its domain $\{x : x \in \mathbb{R} - \{1, -\frac{1}{2}\}\}$

b) $f(x) = \dfrac{\sqrt[3]{x-2}}{x^3-2}$

Solution:

Here, the given function is,

$$f(x) = \frac{\sqrt[3]{x-2}}{x^3-2} = \frac{P(x)}{Q(x)} \text{ (let)}$$

where, $P(x) = \sqrt[3]{x-2}$

$Q(x) = x^3 - 2$

$P(x)$ is continuous in its domain as it is a root function and every root function is continuous in its domain.

Domain of $P(x) = \sqrt[3]{x-2} = (-\infty, \infty)$

$Q(x)$ is continuous in its domain as it is a polynomial function.

Domain of $Q(x) = x^3 - 2 = (-\infty, \infty)$

Thus, $f(x)$ is also continuous, $f(x)$ being a rational function of two continuous functions $P(x)$ and $Q(x)$.

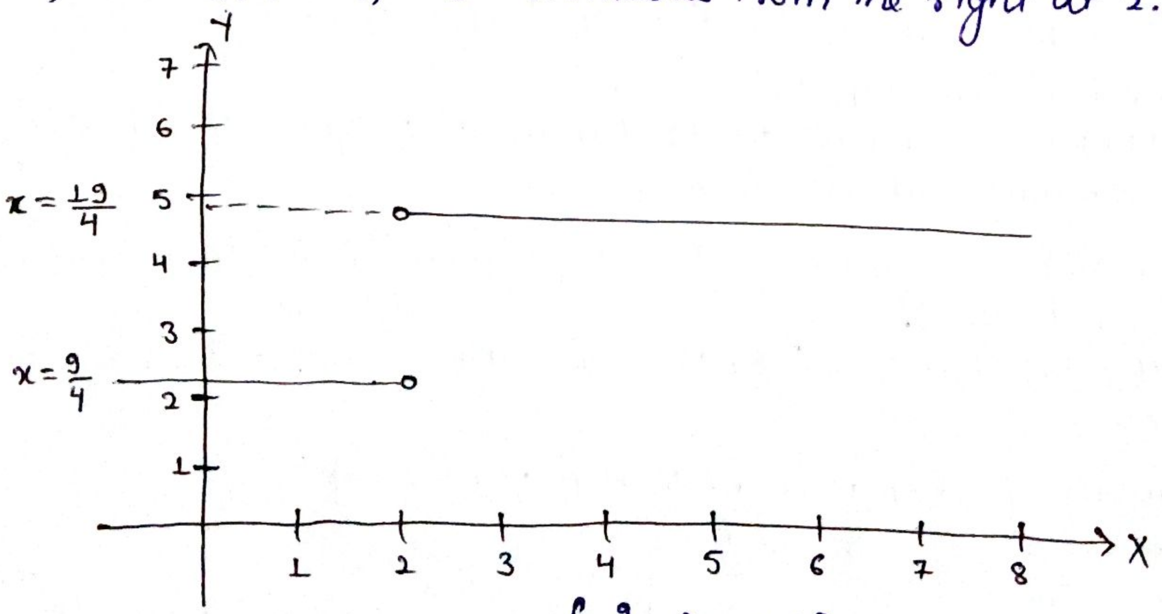
Domain of $f(x)$ is $\{x : x \in \mathbb{R} \text{ but } Q(x) \neq 0\}$

~~i.e.~~ Here, $x^3 - 2 \neq 0 \Rightarrow x \neq \sqrt[3]{2} \approx 1.26$ for $Q(x)$

Hence, $f(x)$ is continuous in its domain $\{x : x \in \mathbb{R} - \{1.26\}\}$.

18. Sketch graph of a function f that is continuous except for the stated discontinuity:

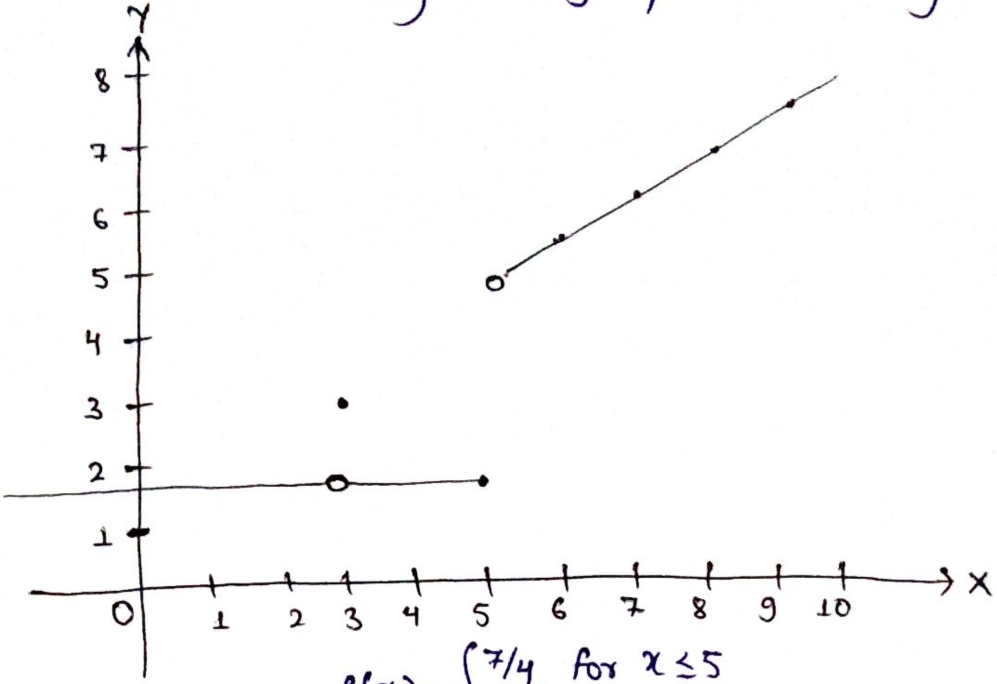
a) Discontinuous, but continuous from the right at 2.



$$f(x) = \begin{cases} \frac{9}{4} & \text{for } x < 2 \\ \frac{19}{4} & \text{for } x \geq 2 \end{cases}$$

~~a) Discor Remova~~

b) Removable discontinuity at 3, jump discontinuity at 5.



$$f(x) = \begin{cases} 7/4 & \text{for } x \leq 5 \\ x & \text{for } x > 5 \\ 3 & \text{for } x = 3 \end{cases}$$