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Derivatives

Formulae :-

i) Power Rule :-

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

e.g. $\frac{dx^8}{dx} = 8x^7$

ii) Constant Rule :-

$$\frac{d(a)}{dx} = 0 \quad , \text{e.g. } \frac{d(8)}{dx} = 0$$

iii) Addition or Subtraction Rule :-

$$\frac{d(4x^3+3)}{dx}$$
$$= \frac{d(4x^3)}{dx} + \frac{d(3)}{dx}$$
$$= 12x^2 + 0$$
$$= 12x^2$$

$$\frac{d(ax^2+bx+c)}{dx}$$
$$= \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx} + \frac{d(c)}{dx}$$
$$= 2ax + b + 0$$
$$= 2ax + b$$

iv) General Power Rule :-

$$\frac{d(ax+b)^n}{dx} = \frac{d(ax+b)^n}{d(ax+b)} \times \frac{d(ax+b)}{dx}$$
$$= n(ax+b)^{n-1} \cdot \frac{d(ax+b)}{dx}$$

e.g.

$$\frac{d(2x+3)^{1/2}}{dx} = \frac{d(2x+3)^{1/2}}{d(2x+3)} \times \frac{d(2x+3)}{dx}$$
$$= \frac{1}{2}(2x+3)^{-1/2} \times \frac{d(2x)}{dx} + \frac{d(3)}{dx}$$
$$= \frac{1}{\cancel{2}\sqrt{2x+3}} \times \cancel{2} + 0$$
$$= \frac{1}{\sqrt{2x+3}}$$

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e.g.

$$\frac{d(\sqrt{ax^2+bx+c})}{dx} = \frac{d(ax^2+bx+c)^{1/2}}{dx}$$

$$= \frac{d(ax^2+bx+c)^{1/2}}{d(ax^2+bx+c)} \times \frac{d(ax^2+bx+c)}{dx}$$

$$= \frac{1}{2}(ax^2+bx+c)^{-1/2} \times \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx} + \frac{d(c)}{dx}$$

$$= \frac{1}{2\sqrt{ax^2+bx+c}} \times (2ax+b)$$

v) Product Rule :-

$$\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

eg. $\frac{d(4x^2+8)\cdot(3x^3+6x)}{dx}$

$$= (4x^2+8)\cdot\frac{d(3x^3+6x)}{dx} + (3x^3+6x)\frac{d(4x^2+8)}{dx}$$

$$= (4x^2+8)(9x^2+6) + (3x^3+6x)(8x)$$

$$= (4x^2+8)(9x^2+6) + (24x^3+48x^2)$$

vi) Quotient Rule :-

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

eg. $\frac{d\left(\frac{3x^2+2}{4x+7}\right)}{dx} = \frac{(4x+7)\frac{d(3x^2+2)}{dx} - (3x^2+2)\frac{d(4x+7)}{dx}}{(4x+7)^2}$

$$= \frac{(4x+7)(6x) - (3x^2+2)\cdot 4}{(4x+7)^2}$$

$$= \frac{24x^2 + 42x - \cancel{3x^2}\, 12x^2 + 8}{(4x+7)^2}$$

$$= \frac{12x^2 + 42x - 8}{(4x+7)^2}$$

e.g. $\dfrac{d\left(\sqrt{\frac{1+x}{1-x}}\right)}{dx}$

$$= \frac{d\left(\frac{1+x}{1-x}\right)^{1/2}}{d\left(\frac{1+x}{1-x}\right)} \times \frac{d\left(\frac{1+x}{1-x}\right)}{dx}$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \times \frac{(1-x)\frac{d(1+x)}{dx} - (1+x)\frac{d(1-x)}{dx}}{(1-x)^2}$$

$$= \frac{1}{2\sqrt{\left(\frac{1+x}{1-x}\right)}} \times \frac{(1-x)\cdot 1 - (1+x)(-1)}{(1-x)^2}$$

$$= \frac{1-x-(-1-x)}{2\sqrt{\left(\frac{1+x}{1-x}\right)}\cdot(1-x)^2}$$

$$= \frac{1-\cancel{x}+1+\cancel{x}}{2\sqrt{\left(\frac{1+x}{1-x}\right)}\cdot(1-x)^2} = \frac{\cancel{2}}{\cancel{2}\sqrt{\left(\frac{1+x}{1-x}\right)}\cdot(1-x)^2}$$

$$= \frac{1}{\sqrt{\left(\frac{1+x}{1-x}\right)}\cdot(1-x)^2} \quad \#.$$

vii) Trigonometric:

a) $\dfrac{d(\sin x)}{dx} = \cos x$

b) $\dfrac{d(\cos x)}{dx} = -\sin x$

c) $\dfrac{d(\tan x)}{dx} = \sec^2 x$

d) $\dfrac{d(\sec x)}{dx} = \sec x \cdot \tan x$

e) $\dfrac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cdot \cot x$

f) $\dfrac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

① e.g. $\frac{d(\sin^2 x + 8\cos^2 x + 2)}{dx}$

$= \frac{d(\sin^2 x)}{d\sin x} \times \frac{d\sin x}{dx} + 8\left(\frac{d\cos^2 x}{d\cos x} \times \frac{d\cos x}{dx}\right) + \frac{d(2)}{dx}$

$= 2\sin x.\cos x + 8 \cdot 2\cos x \cdot (-\sin x) + 0$

$= 2\sin x.\cos x - 16\sin x.\cos x$

$= -14\sin x.\cos x$ $[\because \sin 2x = 2\sin x.\cos x]$

$= -7\sin 2x$ #.

② e.g. $\frac{d(\tan^2 x + \sec x)}{dx}$

$= \frac{d(\tan^2 x)}{d\tan x} \times \frac{d(\tan x)}{dx} + \frac{d\sec x}{dx}$

$= 2\tan x \cdot \sec^2 x + \sec x.\tan x$

$= \tan x.\sec x(2\sec x + 1)$ #

③ e.g. $\frac{d\sqrt{\sin^3 x + 2\sin^2 x}}{dx}$

$= \frac{d}{dx}(\sin^3 x + 2\sin^2 x)^{\frac{1}{2}}$

$= \frac{d(\sin^3 x + 2\sin^2 x)^{\frac{1}{2}}}{d(\sin^3 x + 2\sin^2 x)} \times \frac{d(\sin^3 x + 2\sin^2 x)}{dx}$

$= \frac{1}{2}(\sin^3 x + 2\sin^2 x)^{-\frac{1}{2}} \times \frac{d\sin^3 x}{d\sin x} \times \frac{d\sin x}{dx} + 2\left[\frac{d\sin^2 x}{d\sin x} \times \frac{d\sin x}{dx}\right.$

$= \frac{1}{2\sqrt{\sin^3 x + 2\sin^2 x}} \times 3\sin^2 x.\cos x + 2 \cdot 2\sin x.\cos x$

$= \frac{\sin x.\cos x(3\sin x + 4)}{2\sqrt{\sin^3 x + 2\sin^2 x}}$

$= \frac{\sin x.\cos x.(3\sin x + 4)}{2\sqrt{\sin^3 x + 2\sin^2 x}}$ #.

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(4) e.g. $\dfrac{d\left(\dfrac{\sin 4x}{\tan 6x}\right)}{dx}$

$= \dfrac{\tan 6x \dfrac{d(\sin 4x)}{dx} - \sin 4x \dfrac{d(\tan 6x)}{dx}}{(\tan 6x)^2}$

$= \tan 6x \dfrac{d \sin 4x}{d 4x} \times \dfrac{d 4x}{dx} - \sin 4x \cdot \dfrac{d \tan 6x}{d 6x} \times \dfrac{d 6x}{dx} \Big/ \tan^2 6x$

$= \tan 6x \cdot \cos 4x \cdot 4 - \sin 4x \cdot \sec^2 6x \cdot 6 \Big/ \tan^2 6x$

$= \dfrac{4 \tan 6x \cdot \cos 4x - 6 \sin 4x \cdot \sec^2 6x.}{\tan^2 6x}$ //

viii) Logarithmic function:

$\dfrac{d(\log x)}{dx} = \dfrac{1}{x}$

(1) eg. $\dfrac{d(\log x^2)}{d(x^2)} \times \dfrac{d(x^2)}{dx}$

$= \dfrac{1}{x^{\cancel{2}}} \cdot 2\cancel{x}$

$= \dfrac{2}{x}$ #

(2) e.g. $\dfrac{d(x + 4\log 4x)}{dx}$

$= \dfrac{dx}{dx} + 4 \dfrac{d \log 4x}{d(4x)} \times \dfrac{d(4x)}{dx}$

$= 1 + 4 \times \dfrac{1}{\cancel{4}x} \cdot \cancel{4}$

$= 1 + \dfrac{4}{x}$ #

(3) eg. $\dfrac{d(\sin x + \log(\tan x))}{dx}$

$= \dfrac{d(\sin x)}{dx} + \dfrac{d \log(\tan x)}{d \tan x} \times \dfrac{d \tan x}{dx}$

$= \cos x + \dfrac{1}{\tan x} \times \sec^2 x$

$= \cos x + \dfrac{\sec^2 x}{\tan x}$ #

④ e.g. $\frac{d(\tan x + \log\sqrt{\sin x})}{dx}$

$= \frac{d(\tan x)}{dx} + \frac{d \log (\sin x)^{1/2}}{d(\sin x)^{1/2}} \times \frac{d(\sin x)^{1/2}}{d \sin x} \times \frac{d \sin x}{dx}$

$= \sec^2 x + \frac{1}{\sqrt{\sin x}} \times \frac{1}{2} \sin^{-1/2} x \cdot \cos x$

$= \sec^2 x + \frac{1}{\sqrt{\sin x}} \cdot \frac{\cos x}{2 \cdot \sqrt{\sin x}}$

$= \sec^2 x + \frac{\cot x}{2}$ #.

⑤ e.g. $\frac{d(\cot x + \log\sqrt{\cos x})}{dx}$

$= \frac{d(\cot x)}{dx} + \frac{d \log \sqrt{\cos x}}{d\sqrt{\cos x}} \times \frac{d\sqrt{\cos x}}{d \cos x} \times \frac{d \cos x}{dx}$

$= -\operatorname{cosec}^2 x + \frac{1}{\sqrt{\cos x}} \cdot \frac{1}{2} \cos^{-1/2} x \cdot -\sin x$

$= -\operatorname{cosec}^2 x + (-)\frac{\sin x}{2 \cos x}$

$= -\operatorname{cosec}^2 x - \frac{\tan x}{2}$ #.

ix) Exponential Function :-

$$\frac{d(e^x)}{dx} = e^x$$

e.g.

① $\frac{d(e^{4x})}{dx} = \frac{d(e^{4x})}{d(4x)} \times \frac{d(4x)}{dx} = e^{4x} \cdot 4 = 4e^{4x}$ #.

② $\frac{d(e^{\sin x + \tan x})}{dx} = \frac{d(e^{\sin x + \tan x})}{d(\sin x + \tan x)} \times \frac{d(\sin x + \tan x)}{dx}$

$= e^{\sin x + \tan x} \cdot (\cos x + \sec^2 x)$ #.

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③ $\frac{d(e^{1/x})}{dx} = \frac{d(e^{1/x})}{d(1/x)} \times \frac{dx^{-1}}{dx} = e^{1/x}(-1)x^{-2} = \frac{-e^{+1/x}}{x^2}$ #.

④ $\frac{d(e^{x^2+4x})}{dx} = \frac{d(e^{x^2+4x})}{d(x^2+4x)} \times \frac{d(x^2+4x)}{dx}$

$= e^{x^2+4x}.(2x+4)$

⑤ $\frac{d(e^{\sqrt{\cos x}})}{dx} = \frac{d(e^{\sqrt{\cos x}})}{d(\cos x)^{1/2}} \times \frac{d(\cos x)^{1/2}}{d\cos x} \times \frac{d\cos x}{dx}$

$= e^{\sqrt{\cos x}} . \frac{1}{2} \cos^{-1/2}. -\sin x$

$= -\frac{1}{2} e^{\sqrt{\cos x}} . \cos^{-1/2} x \ (+\sin x)$

$= e^{\sqrt{\cos x}} . \frac{1}{2} \cos^{-1/2} x(-\sin x)$ #.

x.) Implicit Differentiation :-

Find $\frac{dy}{dx}$ in, ⓘ eqⁿ of circle, $x^2+y^2=4$

ⓘⓘ eqⁿ of ellipse, $\frac{x^2}{9}+\frac{y^2}{16}=1$

ⓘⓘⓘ eqⁿ of Parabola, $y^2=16x$

ⓐ Eqⁿ of circle,

$x^2+y^2=4$

differentiating both sides w.r.to x

$\frac{d(x^2+y^2)}{dx} = \frac{d(4)}{dx}$

or, $\frac{d(x^2)}{dx} + \frac{dy^2}{dx} = 0$

or, $2x + \frac{dy^2}{dy} \times \frac{dy}{dx} = 0$

or, $2x + 2y \times \frac{dy}{dx} = 0$

or, $-2x = 2y\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$ #

(b) eq^n of ellipse,

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

differentiating both sides with respect to x.

$$\frac{d\left(\frac{x^2}{9} + \frac{y^2}{16}\right)}{dx} = \frac{d(1)}{dx}$$

or, $\frac{d(16x^2+9y^2)}{dx} = \frac{d(9\times 16)}{dx}$

or, $16\frac{dx^2}{dx} + 9\frac{dy^2}{dy} \times \frac{dy}{dx} = 0$

or, $16\cdot 2x + 9\cdot 2y\frac{dy}{dx} = 0$

or, $18y \cdot \frac{dy}{dx} = -32x$

$\therefore \frac{dy}{dx} = \frac{-16x}{9y}$ #.

(c) eq^n of Parabola,

$$y^2 = 16x$$

differentiating both sides w.r.t x.

w.r.t.y

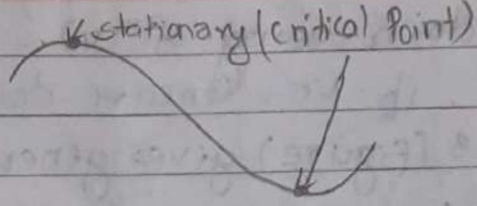
$\frac{d(y^2)}{dx} = \frac{d(16x)}{dx}$ $\cancel{2y = \frac{\partial x}{\partial y} \times \frac{\partial y}{\partial x}}$

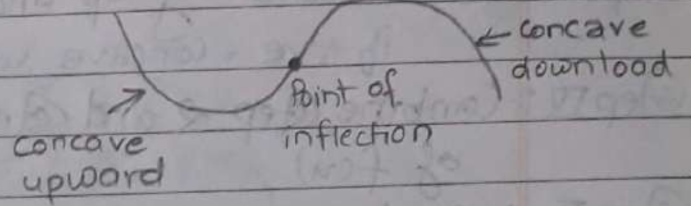
or, $\frac{dy^2}{dy} \times \frac{dy}{dx} = 16\frac{dx}{dx}$

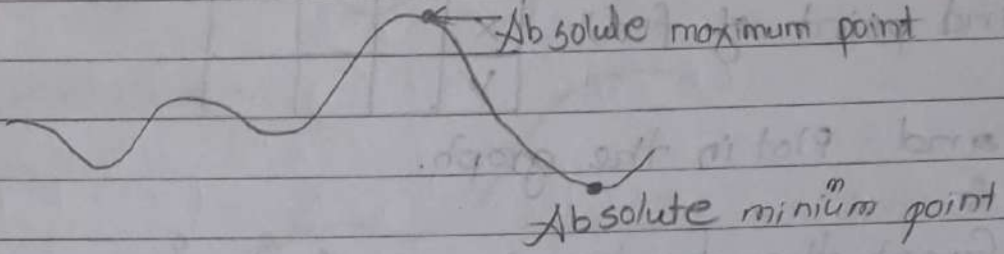
or, $\frac{dy}{dx} = \frac{16}{2y}$

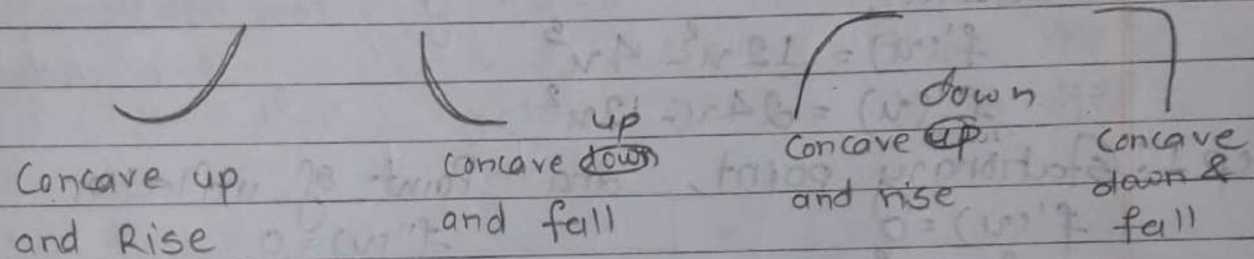
$\therefore \frac{dy}{dx} = \frac{8}{y}$ #

Maxima And Minima :-









Rise is denoted by $\nearrow$ and fall is denoted by $\searrow$

i> Stationary (critical point)

$f'(x) = 0$ and solve for x

$x = a, b, c$ (Stationary point)

ii> Point of inflection :

$f''(x) = 0$ and solve for x

check for maxima and minima

At $x = 0$, $f''(a) > 0$ (minima)

$f''(a) < 0$ (maxima)

$f''(a) = 0$ (neigther max nor min)

Graph Sketching :-

Step I :- Find stationary / critical point and point of inflection.

Step II :- Sketch Rise ($\rightarrow$) and fall ($\searrow$) pattern using $f'(x)$.

if +ve rise if -ve fall

Step III : Sketch concavity up and down using $f''(x)$.
if +ve = concave up , if -ve = concave down

Step IV : Combine step 2 and step 3 (figure) gives general shape of $f(x)$

Step V : find point of $f(x)$

x			
y			

and plot in the graph.

example 1 :-

Graph the function : $y = 4x^3 - x^4$

→ Solution,

$f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2$

Step I : Stationary point, and Point of inflection

$f'(x) = 0$ | $f''(x) = 0$

$12x^2 - 4x^3 = 0$ | $24x - 12x^2 = 0$

$4x^2(3 - x) = 0$ | $12x(2 - x) = 0$

Either $x = 3$, $x = 0$ | Either, $x = 2$, $x = 0$

Step II : Rise and fall pattern using $f'(x)$

$12x^2 - 4x^3$	+	+	−
	rise ↗ 0	rise ↗ 3	fall ↘

to check +ve, -ve Put value infront of 0 in $f'(x)$

Sup. (-1) $= 12(-1)^2 - 4(-1)^3$
$= 12 + 4 = 16$ ∴ +ve, so +

4 $= 12(4)^2 - 4(4)^3 = 192 - 256 = -64$ ∴ -ve

if +ve, Rise ↗

if -ve, fall ↘

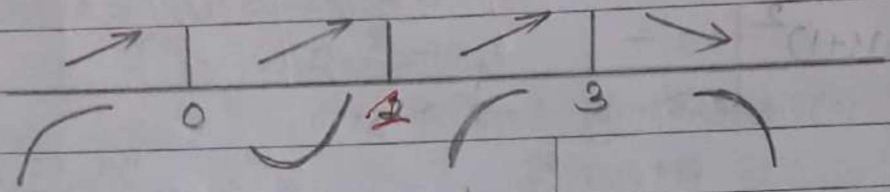
Step III : concavity using $f''(x)$; same as step II to check +ve, -ve

$24x - 12x^2$	−	+	−
	concave down 0	concave up 2	concave down

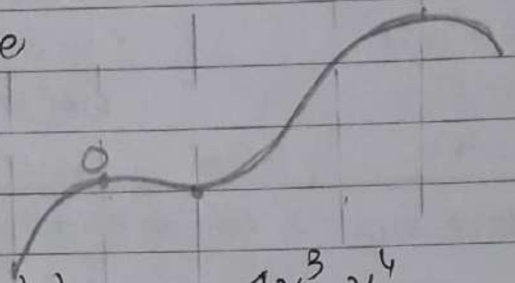
Step IV: Combine Step II and Step III.

rise/fall

concave up/down



general Shape



Step V: Find points of $f(x)$ and plot. $y = 4x^3 - x^4$

when $x = 1$

$y = 4 \cdot 1^3 - 1^4$

$= 3$

$x = 2$, $y = 16$ $x = 3$, $y = 27$ $x = -1$, $y = -5$ $x = 4$, $y = 0$

x	1	2	3	-1	4
y	3	16	27	-5	0

In graph

example: 2.

$$y = 1 - (x+1)^3$$

→ Solution,

$f'(x) = -3(x+1)^2$

$f''(x) = -6(x+1)$

$\frac{d(1)}{dx} - \frac{d(x+1)^3}{d(x+1)} \times \frac{d(x+1)}{dx}$

$= 0 - 3(x+1)^2 \times 1$

$= -3(x+1)^2$ //.

$-3 \frac{d(x+1)^2}{d(x+1)} \times \frac{d(x+1)}{dx}$

$= -3 \cdot 2(x+1)$

$= -6(x+1)$

Step I: Stationary point,

$f'(x) = 0$

$-3(x+1)^2 = 0$

$(x+1) = 0$

$\therefore x = -1$

Point of inflection,

$f''(x) = 0$

$-6(x+1) = 0$

$x + 1 = 0$

$\therefore x = -1$

Step 2: Rise and fall pattern :-

Put 0 in $f'(x)$

$-3(0+1)^2 = -3$

$-3(x+1)^2$	$-$		$-$
	$\searrow$	-1	$\searrow$

Put -2 in $f'(x)$

$= -3(-2+1)^2 = -3(-1)^2 = -3$

Step 3: Concavity

$-6(x+1)$	$+$		$-$
	Concave up	-1	Concave down

Put -2 in $f''(x)$

$= -6(-2+1) = (-6)(-1) = +7$

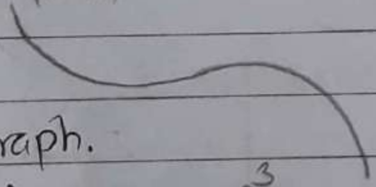
Put 0 in $f''(x)$

$= -6(0+1) = -6$

Step 4:- Combine Step 2 and 3

rise/fall	fall $\searrow$		fall $\searrow$
up/down		-1	

general form



Step 5: plot the in graph.

find point $f(x)$ $y = 1-(x+1)^3$

when $x = -1$

$y = 1-(-1+1)^3$

$y = 1$

when $x = 0$

$y = 1-(0+1)^3$

$y = 1-1 = 0$

when $x = 1$

$y = 1-(1+1)^3$

$= 1-8$

$= -7$

when $x = 2$

$y = 1-(2+1)^3$

$= -8$

X	-1	0	1	2	-2	-3
Y	1	0	-7	-8	2	9

In graph

example: 3

$y = x^4 + 2x^3$

Solution, $f'(x) = 4x^3 + 6x^2$

$f''(x) = 12x^2 + 12x$

Step 1: Stationary point

$f'(x) = 0$

$4x^3 + 6x^2 = 0$

or, $2x^2(2x+3) = 0$

either $x = 0$

$x = -\frac{3}{2} = -1.5$

Point of inflection,

$f''(x) = 0$

$12x^2 + 12x = 0$

$12x(x+1) = 0$

either, $x = 0$

$x = -1$

Step 2:

Rise and fall pattern from $f'(x)$

$4x^3+6x^2$	−		+		+
	$\searrow$	-1.5	$\nearrow$	0	$\nearrow$

$= 4(-2)^3 + 6(-2)^2$

$= 4(-8) + 6 \cdot 4$

$= -32 + 24 = -8$

Step 3:

Concavity using $f''(x)$

$12x^2+12x$	+		−		+
	up	-1	down	0	up

$12(-2)(-2+1)$

$= -24 \times (-1)$

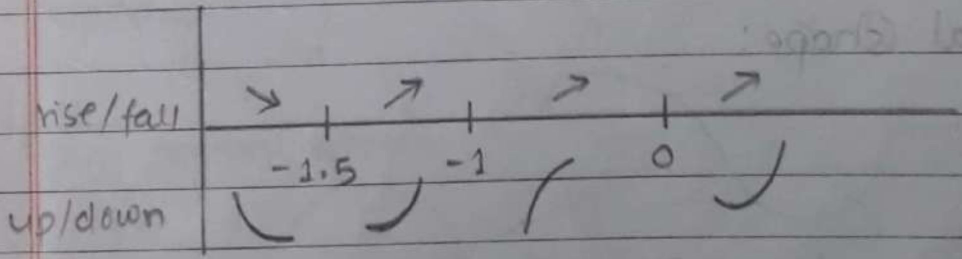
$= +24$

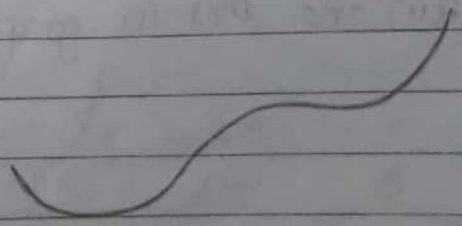
$12(-0.5)(-0.5+1)$

$= -6 \times 0.5$

$= -3$

Step 4: Combining Step 2 and 3





Step 5: Find the point $f(x)$ and plot the graph. $y = x^4 + 2x^3$

X	0	1	−1	−2	+2
Y	0	3	−1	0	32

In graph

HOW

Assignment # 1

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Graph the function:

ex. 1 (a) $y = -x^3 + 12x + 5$

→ Solution,

$f'(x) = -3x^2 + 12$, $f''(x) = -6x$

1. Stationary point

$f'(x) = 0$

$-3x^2 + 12 = 0$

$3(-x^2 + 4) = 0$

or, $-x^2 + 4 = 0$

or, $x^2 = 4$

$\therefore x = \pm 2$

Point of inflection

$f''(x) = 0$

$-6x = 0$

$\therefore x = 0$

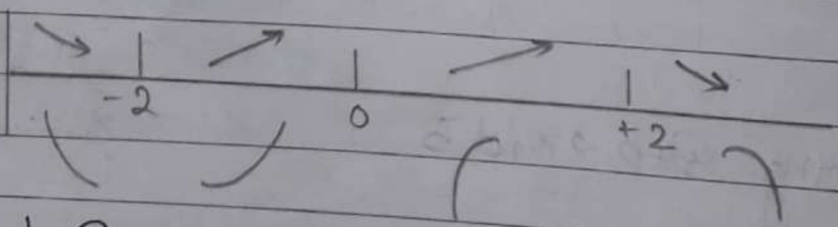
2. Rise and fall pattern:

$-3x^2+12$	−	−2	+	+2	−

3. Concavity

$-6x$	+ (up)	0	− (down)

4. Combine 2 and 3



general shape:

5. find point x and y in f(x) and plot in graph

x	-1	0	-2	2	-3
y	-6	5	-11	21	-4

in graph copy

ex. 5

(a) $y = x^3 - 3x + 3$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

1. Stationary point.

$f'(x) = 0$

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x = \sqrt{1}$

$\therefore x = \pm 1$

Point of inflection.

$f''(x) = 0$

$6x = 0$

$\therefore x = 0$

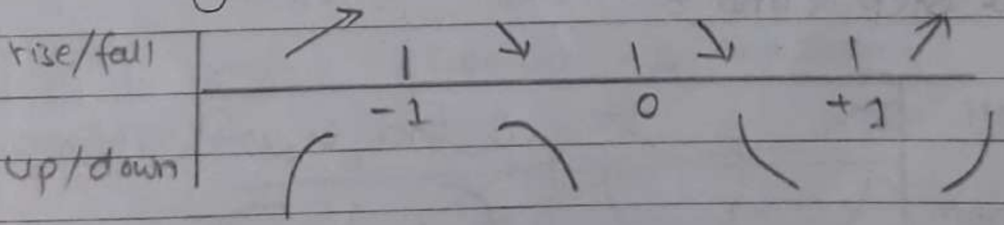
2. Rise and fall Pattern.

$3x^2-3$	+	-1	−	+1	+
	↗		↘		↗

3. Concavity

$6x$	−	0	+
	concave down		up

4. Combining 2 and 3



general shape

5. find point x and y and plot in graph.

x	0	1	-1	2	-2
y	+3	1	5	5	1

In graph

Q.No: 6 $y = x(6-2x^2)$

→ Solution,

$f(x) = 6x - 2x^3$

$f'(x) = 6 - 6x^2$

$f''(x) = -12x$

Step 1: Stationary point is given by $f'(x) = 0$

$6 - 6x^2 = 0$

or, $6(1-x^2) = 0$

or, $x^2 = 1$

$\therefore x = \pm 1$

Point of inflection is given by f''

$-12x = 0$

$\therefore x = 0$

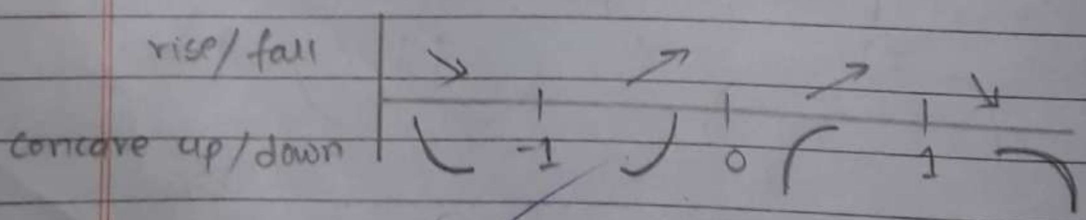
Step 2: Rise and fall pattern:

$6-6x^2$	$-$	-1	$+$	1	$-$
	↘		↗		↘

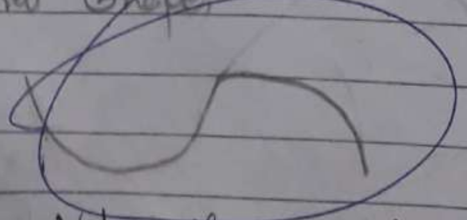
Step 3: Concavity

$-12x$	$+$	0	$-$
	Conc. up		Conc. down

Step 4: Combine Step 2 and 3.



general shape



Step 5: find the value of X and Y and plot in graph.

X	1	2	-1	-2	0
Y	4	-4	-4	+4	0

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Q.NO:7 $y = 1 - 9x - 6x^2 - x^3$

→ Solution,

$f'(x) = -9 - 12x - 3x^2$

$f''(x) = -12 - 6x$

Step 1: Stationary point from $f'(x)=0$ Critical point from $f''(x)=0$

$f'(x) = 0$
$-9 - 12x - 3x^2 = 0$
$3x^2 + 12x = -9$
$x^2 + 4x = -3$
$x^2 + 4x + 3 = 0$
$x^2 + x + 3x + 3 = 0$
or, $x(x+1) + 3(x+1) = 0$
Either $x = -1$
$x = -3$

$f''(x) = 0$
$-12 - 6x = 0$
$6(-2 - x) = 0$
$x = -2$

Step 2: Rise and fall pattern from $f'(x)$

$-9-12x-3x^2$	-ve	-3	+ve	-1	-ve
	↗	-3	↘	-1	↗

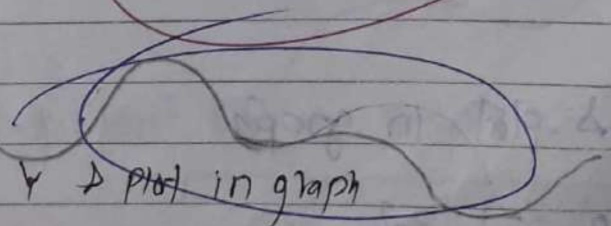
Step 3: Concavity from $f''(x)$

$-12-6x$	+ve		-ve
	Conc. up	-2	Conc. down

Step 4: combine Step 2 and 3.

rise/fall	↗	↘	↘	↗
up/down	-3	-2	-1	

general form



Step 5: find X and Y & plot in graph

X	0	-1	-2	-3	-4	1
Y	1	5	3	1	5	-15

8. $y = 1-(x+1)^3$

$f'(x) = -3(x+1)^2$

$f''(x) = -6(x+1)$

Step 1: Point of Stationary

$f'(x) = 0$

$-3(x+1)^2 = 0$

$(x+1) = 0$

$x = -1$

Point of inflection

$f''(x) = 0$

$-6(x+1) = 0$

or, $x+1 = 0$

$\therefore x = -1$

Step 2: Rise and fall pattern

$-3(x+1)^2$	$-$		$-$
	$\searrow$	-1	$\searrow$

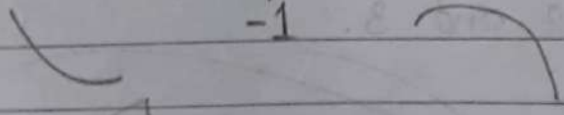
Step 3: Concavity

$-6(x+1)$	$+$		$-$
	up	-1	down

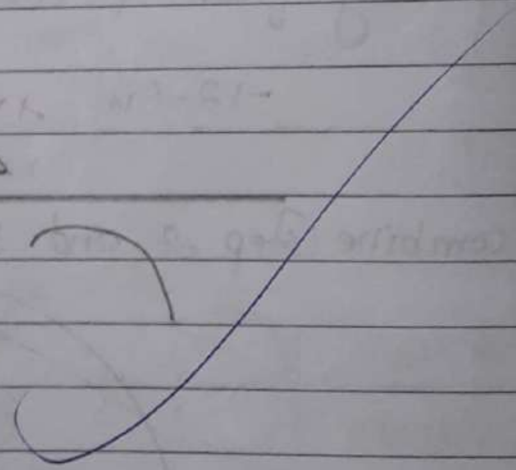
Step 4:

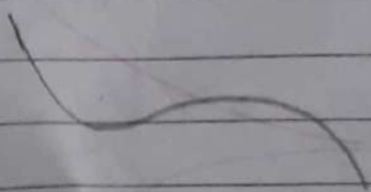
Combine Step 2 and 3

rise/fall	$\searrow$		$\searrow$
up/down		-1	



general shape





Step 5:

find x and y & plot in graph $y = 1-(x+1)^3$

x	1	0	-1	-2	-3
y	-7	0	1	2	9

9. $y = (x-2)^3 + 1$

→ Solution,

$f'(x) = 3(x-2)^2$

$f''(x) = 6(x-2)$

$\frac{d(x-2)^3}{d(x-2)} \times \frac{d(x-2)}{d(x-2)}$

$= 3(x-2)^2 \times 1$

$3 \times 2 \frac{d(x-2)^{2-1}}{d(x-2)}$

$6(x-2)$

step 1: Stationary Point

$f'(x) = 0$

$3(x-2)^2 = 0$

$(x-2) = 0$

$\therefore x = 2$

Point of inflection

$f''(x) = 0$

$6(x-2) = 0$

$x = 2$

Step 2: Rise and Fall pattern

$3(x-2)^2$	+	2	+
	↗		↗

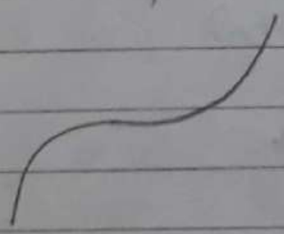
Step 3: Concavity

$6(x-2)$	−	2	+
	down		up

Step 4: Combining Step 2 and 3

rise/fall	↗	2	↗
up/down			

general form



Step 5: Find x and y and plot in graph

X	1	2	0	3	4
Y	0	1	-7	2	9

10. $y = -x^4 + 6x^2 - 4$

→ Solution,

$f'(x) = -4x^3 + 12x$

$f''(x) = -12x^2 + 12$

Step 1 Stationary point, $f'(x) = 0$

$-4x^3 + 12x = 0$

$4x(-x^2+3) = 0$

$-x^2 + 3 = 0$

$x = \sqrt{3} = 1.73$

Point of inflection,

$-12x^2 + 12 = 0$

$12(-x^2+1) = 0$

$-x^2 + 1 = 0$

$x = 1$

Step 2 :- Rise and fall pattern,

$-4x^3+12x$	+	1.73	−
	↗		↘

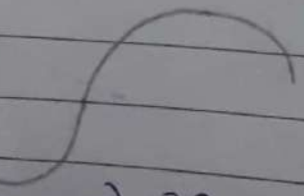
Step 3 :- Concavity

$-12x^2+12$	+	1	−
	up		down

Step 4 :- Combine 2 and 3

rise/fall	↗	1	↗	1.73	↘
up/down					

general form



Steps :- find x and y and plot in graph

x	1	−1	0	2
y	1	1	−4	4

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Graph the function.

11. $y = x^{5/3} - 5x^{2/3}$

→ Solution,

$f'(x) = \frac{5}{3} x^{5/3 - 1} - 5 \times \frac{2}{3} x^{2/3 - 1}$

$= \frac{5}{3} x^{2/3} - \frac{10}{3} x^{-1/3}$

$f''(x) = \frac{5}{3} \times \frac{2}{3} x^{2/3 - 1} - \frac{10}{3} \times (-1/3) x^{-1/3 - 1}$

$= \frac{10}{9} x^{-1/3} + \frac{10}{9} x^{-4/3}$

Stationary Point is given by $f'(x) = 0$

$5/3\, x^{2/3} - 10/3\, x^{-1/3} = 0$

$5/3\, x^{-1/3} (x - 2) = 0$

$x = 0$

$\therefore x = +2$

Point of inflection is given by $f''(x) = 0$

$\frac{10}{9} x^{-1/3} + \frac{10}{9} x^{-4/3} = 0$

$\frac{10}{9} x^{-4/3} (x + 1) = 0$

Either $x = 0$

$x = -1$

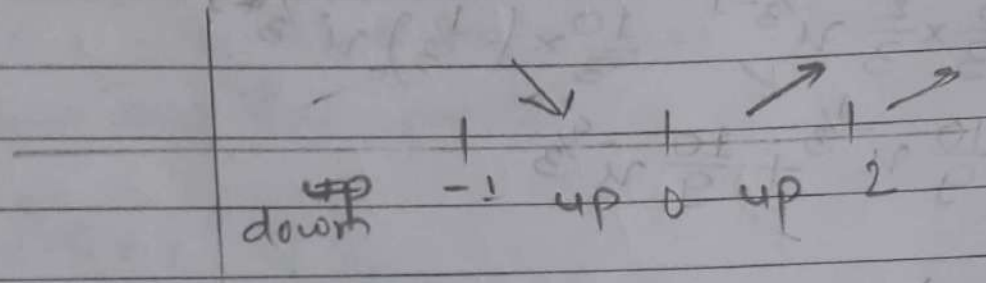
Rise and fall Pattern

$f'(x)$	+	−	+
	rise	0 fall	2 rise

Step 3: sketch concave up and concave down using f''(x)

f''(x)	-	+	+
	down -1	up 0	up

Step 4:- Combine Step 2 and 3.

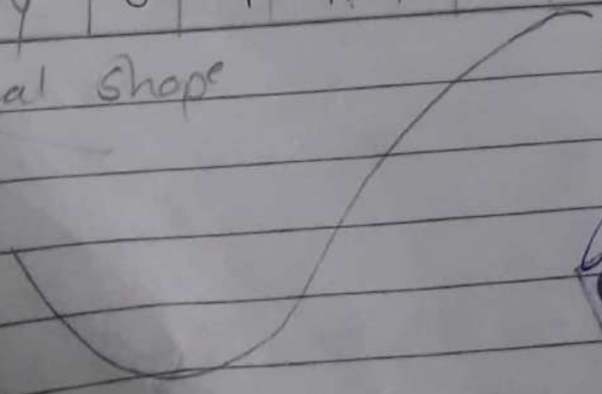


general form,

plot in graph,

x	0	1	2	3	4	5	6	7	8
y	0	-4	-4.7	-4.16	-2.5	0	3.3	7.3	12

general shape



Integration

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (c = constant of integration)

e.g. (a) $\int (x^{2/3} + x) dx$

$= \int x^{2/3} dx + \int x dx$

$= \frac{x^{2/3+1}}{\frac{2}{3}+1} + \frac{x^2}{2} + c$

$= \frac{x^{5/3}}{\frac{5}{3}} + \frac{x^2}{2} + c$ #.

2. $\int dx = x + c$

3. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$

e.g.

(a) $\int (4x+3)^5 dx = \frac{(4x+3)^6}{6 \times 4} + c$ #.

(b) $\int (4-6x)^{2/3} dx = \frac{(4-6x)^{2/3+1}}{(\frac{2}{3}+1) \times (-6)} + c = \frac{(4-6x)^{5/3}}{-\frac{30}{3}} + c$

$= -\frac{(4-6x)^{5/3}}{10} + c$ #.

4. $\int \frac{1}{x} dx = \log x + c$

e.g. (a) $\int (ax^2 + bx + x^{-1}) dx$ ($\log_e : \ln$)

$= a\frac{x^3}{3} + \frac{bx^2}{2} + \log x + c$

$\int \frac{1}{(1-x)} dx$

$- \log(1-x) + c$

$$\frac{(x-3)+6}{x-3}$$

ⓑ $\int\left(\frac{1-x}{1+x}\right)dx$

$=$

$= \int \frac{(1+x)-2x}{(1+x)}\,dx$

$= \int \frac{(1+x)}{(1+x)}\,dx - \int \frac{2x}{(1+x)}\,dx$

$= \int dx - 2\int\left(\frac{x+1)-1}{x+1}\right)dx$

$= x - 2\left\{\int \frac{(x+1)}{(x+1)}\,dx - \int \frac{1 \cdot 2\,dx}{(x+1)}\right.$

$= x - 2x - 2\log(x+1) + C$ # $= -x - 2\log(1+x) + C$ #.

ⓒ $\int \frac{(x+3)}{(x-3)}\,dx$

$= \int \frac{(x-3)+6}{(x-3)}\,dx$

$= \int\left(\frac{(x-3)}{(x-3)} + \frac{6}{(x-3)}\right)dx \quad = \int 1\,dx + 6\int\frac{1}{(x-3)}\,dx$

$= x + 6\log(x-3) + C$

ⓑ → OR

$\int \frac{1}{1+x} - \int \frac{x+1-1}{1+x}$

$\log(x+1) \quad - \frac{x+1}{x+1} + \frac{1}{1+x}$

5.) $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$

e.g. a.) $\int e^{5x}\,dx = \dfrac{e^{5x}}{5} + C$ #.

b.) $\int e^{-4/3 x}\,dx = \dfrac{e^{-4/3 x}}{-4/3} + C$ #.

c.) $\int (e^{-px} + e^{qx})\,dx = \dfrac{e^{-px}}{-p} + \dfrac{e^{qx}}{q} + C$ #.

6. Trigonometric

i.) $\int \cos ax\,dx = \dfrac{\sin ax}{a} + C$ #

ii.) $\int \sin ax\,dx = -\dfrac{\cos ax}{a} + C$ #.

iii.) $\int \sec^2 ax\,dx = \dfrac{\tan ax}{a} + C$ #.

iv.) $\int \sec ax \times \tan ax\,dx = \dfrac{\sec ax}{a} + C$ #

v.) $\int \operatorname{cosec}^2 ax\,dx = -\dfrac{\cot ax}{a} + C$ #.

vi.) $\int \operatorname{cosec} ax \cdot \cot ax\,dx = -\dfrac{\operatorname{cosec} ax}{a} + C$ #.

Trigonometric Substitution :-

i.) $a^2 - x^2$,

Put $x = a\sin\theta$

(2) x^2+a^2,

Put $x = a\tan\theta$

$= a^2(1+\tan^2\theta)$

$= a^2\sec^2\theta$

(3) x^2-a^2

put $x = a\sec\theta$

7. Definite Integral:-

$$\int_{x=a}^{x=b} x\,dx = \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2}$$

e.g.

(a) $$\int_{x=0}^{x=\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2}$$

$$= [-\cos \pi/2 + \cos 0^\circ]$$

$$= 1 \#.$$

(b) $$\int_{x=0}^{x=3} \sqrt{9-x^2}\,dx$$

$\sqrt{a^2-x^2}$

Put $x = a\sin\theta$ $\therefore a = 3$

$x = 3\sin\theta$

differentiating both side by θ.

2 limit change

1

$$\frac{dx}{d\theta} = 3\cos\theta$$

$dx = 3\cos\theta\, d\theta$

$x = 3\sin\theta$	when $x=3$
when $x=0$	$3 = 3\sin\theta$
$0 = 3\sin\theta$	$\sin\theta = 1$
$\theta = 0$	$\therefore \theta = \pi/2$

$$= \int_{\theta=0^\circ}^{\theta=\pi/2} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$\sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9}\,\cos^2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int_0^{\pi/2} 3\cos\theta \cdot 3\cos\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cdot\cos\theta$

$$= \int_0^{\pi/2} 9\cos^2\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int_0^{\pi/2} 9\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - 1 + \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{9}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

integrate

$$= \frac{9}{2}.\left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \sin\frac{2.\pi/2}{2} - 0\right]$$

$\sin \pi/2 = 0$

$$= \frac{9\pi}{4}$$

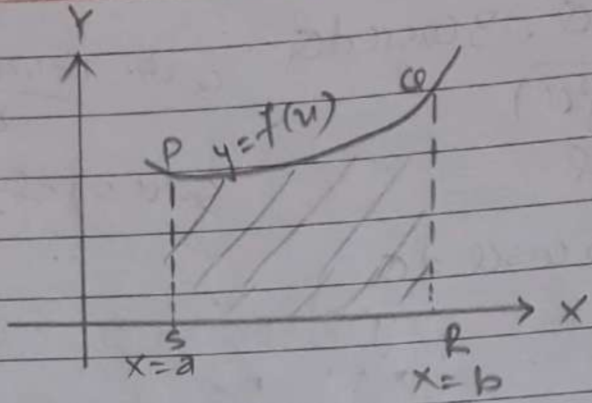
$$= \frac{9\pi}{4}$$

Answer.

HoW Assignment 2

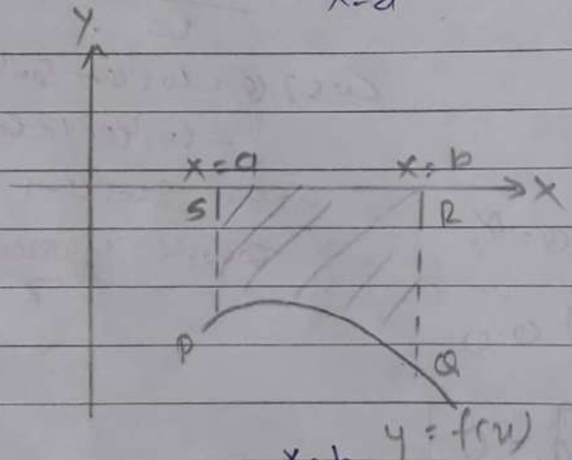
2073/08/30, Thu.

8. Area

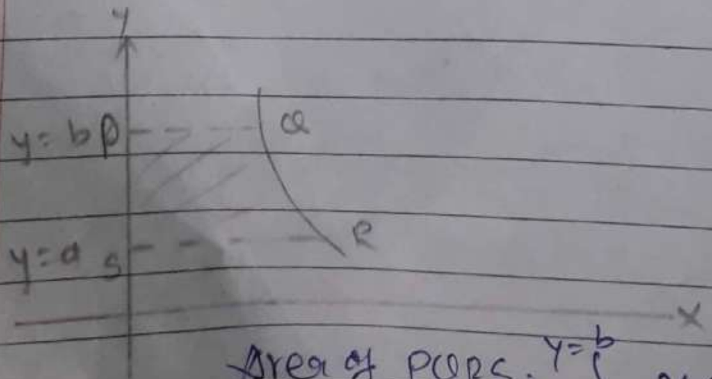


$$\text{Area } PQRS = \int_{x=a}^{x=b} y\,dx$$

$$= \int_{x=a}^{x=b} f(x)\,dx$$

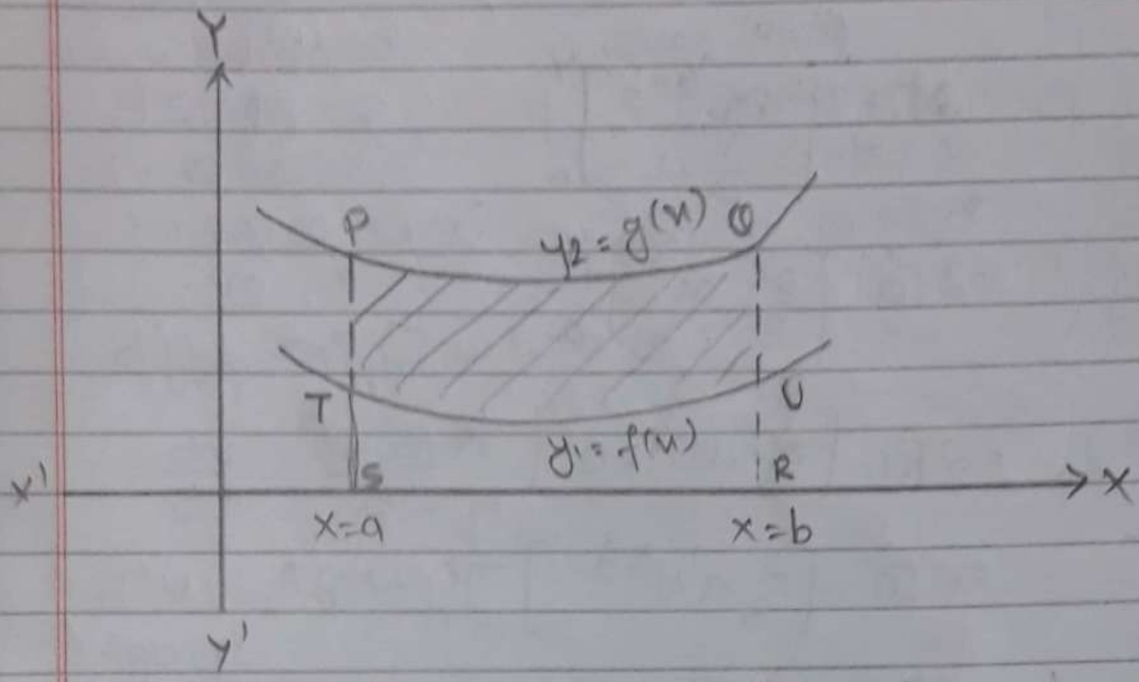


$$\text{Area of } PQRS = \int_{x=a}^{x=b} -y\,dx = \int_{x=a}^{x=b} -f(x)\,dx$$



$$\text{Area of } PQRS = \int_{y=a}^{y=b} x\,dy = \int_{y=a}^{y=b} f(y)\,dy$$

Area between two curve



Area between two curve (PQUT)

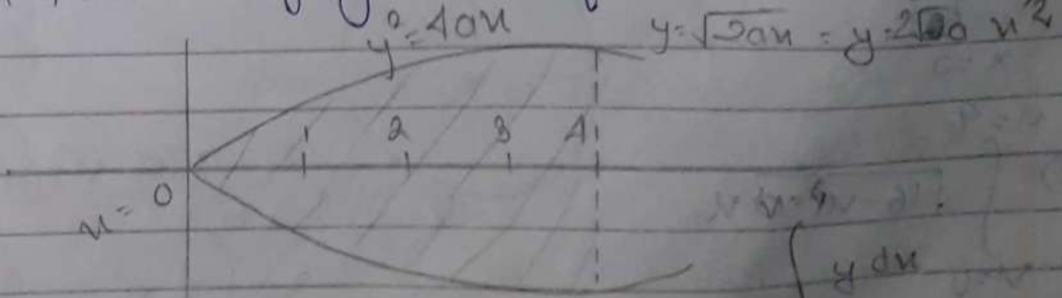
= Area PQRS − Area TURS

$$= \int_{x=a}^{x=b} y_2\,dx - \int_{x=a}^{x=b} y_1\,dx$$

$$= \int_{x=a}^{x=b} (y_2 - y_1)\,dx$$

How

Q. Find the area of $y^2 = 4ax$ from $x=0$ to $x=4$



$$\int_{x=0}^{x=4} y\,dx$$

$$= 2\sqrt{a}\int_{x=0}^{x=4} x^{1/2}\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^4$$

$$= 2\sqrt{a}\left[\frac{2}{3}\times(4)^{3/2} - 0\right]$$

$$= 2\sqrt{a}\left[\frac{2}{3}\times(2)^{2\cdot\frac{3}{2}}\right]$$

$$= 2\sqrt{a}\;\frac{2\times 8}{3}$$

$$= \frac{32\sqrt{a}}{3}$$

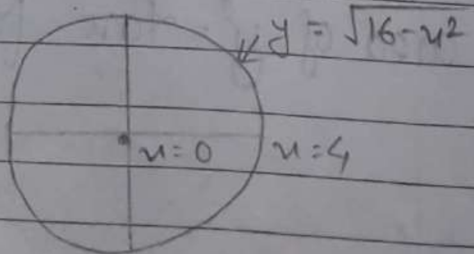
$\therefore$ Area of Parabola $= \dfrac{2\times 32\sqrt{a}}{3}$

$= \dfrac{64\sqrt{a}}{3}$ Sq. units

Area of circle:-

$x^2+y^2 = 16 \quad r=4$

$$\Rightarrow \int_{x=a}^{x=b} y\,dx$$



$$\Rightarrow \int_{x=0}^{x=4} \sqrt{16-x^2}\,dx$$

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Now,

Put $x = 4\sin\theta$ --- (1)

when $x = 0$

$4\sin\theta = 0$

$\therefore \sin\theta = 0$

$\therefore \sin\theta = \sin 0^\circ$

$\therefore \theta = 0^\circ$

when $x = 4$

$4\sin\theta = 4$

$\therefore \sin\theta = 1$

$\therefore \theta = 90^\circ$

diff. eqn (1) w.r.to θ

$$\frac{dx}{d\theta} = \frac{4\sin\theta}{d\theta}$$

$$dx = 4\cos\theta\, d\theta$$

Again,

$$\Rightarrow \int_{x=0}^{x=4} \sqrt{16 - x^2}\, dx$$

$$\Rightarrow \int_{0^\circ}^{\pi/2} \sqrt{16 - 16\sin^2\theta}\,.\,4\cos\theta\, d\theta$$

$$\sqrt{16(1-\sin^2\theta)} = 4\sqrt{\cos^2\theta} = 4\cos\theta$$

$$\Rightarrow \int_{0}^{\pi/2} 4\cos\theta\,.\,4\cos\theta\, d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$$

$$\therefore \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$\Rightarrow \int_{0}^{\pi/2} 16\cos^2\theta\, d\theta$$

$$\Rightarrow \int_{0}^{\pi/2} 16 \cdot \frac{\cos 2\theta + 1}{2}$$

$$= \frac{16}{2}\int_0^{\pi/2} \cos 2\theta + \int_0^{\pi/2} d\theta$$

$$= \frac{16}{2}\left\{\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} + \theta^{\pi/2}\right\}$$

$$= 8\left\{\left[\frac{\sin 2.\frac{\pi}{2}}{2} - 0\right] + \frac{\pi}{2}\right\}$$

$$= 8\left\{\left[\sin\frac{\pi}{2}\right] + \frac{\pi}{2}\right\}$$

$$= \frac{8\pi}{2} = 4\pi$$

$\therefore$ Area of circle $= 4\pi \times 4$

$= 16\pi$ #

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

or, $\frac{y^2}{9} = 1 - \frac{x^2}{16}$

or $\frac{y^2}{9} = \frac{16-x^2}{16}$

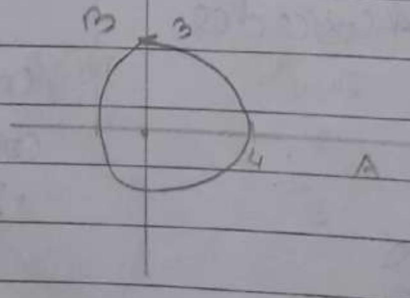
$y = \sqrt{\frac{9}{16}(16-x^2)}$

$y = \frac{3}{4}\sqrt{16-x^2}$

$a^2 = x^2$

put ~~mean~~ $a = 4$, $x = 4\sin\theta$

Q. Find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$



$9x^2 + 16y^2 =$

$\frac{y^2}{9} = 1 - \frac{x^2}{16}$

$\frac{y^2}{9} = \frac{16-x^2}{16}$

$y = \sqrt{\frac{9}{16}(16-x^2)}$

$\Rightarrow$ Solution :-

given Equation of ellipse

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\frac{y^2}{9} = 1 - \frac{x^2}{16}$

or, $\frac{y^2}{9} = \frac{16-x^2}{16}$

or, $y = \sqrt{\frac{9}{16}}\sqrt{16-x^2}$

$\therefore y = \frac{3}{4}\sqrt{16-x^2}$

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Now,

Put $x = a\sin\theta$ — when $x=0$, $0 = 4\sin\theta$, $\therefore \sin\theta = \sin 0^\circ$, $\therefore \theta = 0^\circ$ — when $x = 4$, $4 = 4\sin\theta$, $\therefore \sin\theta = \sin\pi/2$, $\theta = \pi/2$

diff. w. r. to $d\theta$

$$\frac{dx}{d\theta} = a\frac{\sin\theta}{d\theta}$$

so, $dx = a\cos\theta\, d\theta$

we have,

$$= \int_0^{\pi/2} \frac{3}{4}\sqrt{16 - 16\sin^2\theta}\, dx$$

$$= \frac{3}{4}\int_0^{\pi/2} \sqrt{16(1-\sin^2\theta)}\, dx$$

$$= \frac{3}{4}\int_0^{\pi/2} 4\cos\theta \cdot 4\cos\theta\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$$= \frac{3}{4}\int_0^{\pi/2} 16\cos^2\theta\, d\theta$$

$$= \frac{3\times16}{4}\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 12\int_0^{\pi/2} \frac{\cos 2\theta + 1}{2}\, d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= 12\int_0^{\pi/2} \frac{\cos 2\theta}{2}\, d\theta + \int_0^{\pi/2} d\theta$$

$$= 12.\left\{\left[\frac{\sin 2.\pi/2}{2} + 0\right] + \left[\frac{\pi/2 - 0}{2}\right]\right\}$$

$$= 12.\times 0 + \cancel{12}^{6}.\frac{\pi}{2}\times\frac{1}{2}$$

$$= 6\pi/2 = 3\pi$$

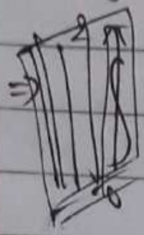
$\therefore$ Area of ellipse (A) $= 4\times 3\pi$

$= 12\pi$ Sq. unit

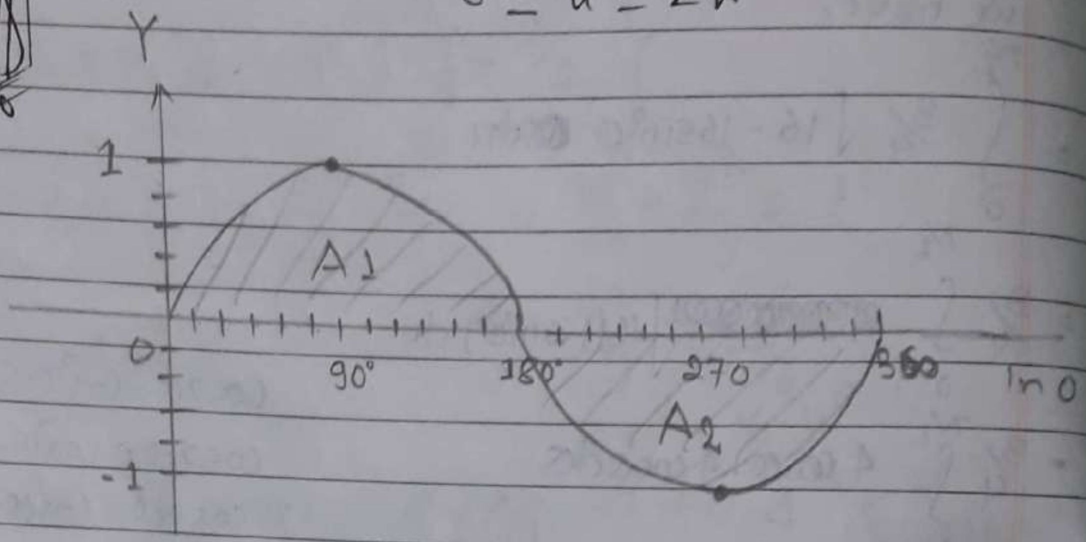
* Find the area of $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$

→ Solution,

given function $f(x) = \sin x$



$0 \le x \le 2\pi$



$$\because A_1 = \int_{x=0}^{x=\pi} \sin x \, dx$$

$$\because A_2 = -\int_{\pi}^{2\pi} \sin x \, dx$$

$$\because A = A_1 + A_2$$

Now,

$$A_1 = \int_0^{\pi} \sin x \, dx$$

$$= \Big[-\cos x \, dx\Big]_0^{\pi}$$

$= (-\cos \pi + \cos 0^\circ)$

$= -(-1) + 1$

$= 1 + 1$

$= 2$

$$A_2 = -\int_{\pi}^{2\pi} \sin x \, dx$$

$$= -\left[-\cos x\right]_{\pi}^{2\pi}$$

$$= -\left[-\cos 2\pi + \cos \pi\right]$$

$$= -[-1 + (-1))$$

$$= - \times - 2$$

$$= 2$$

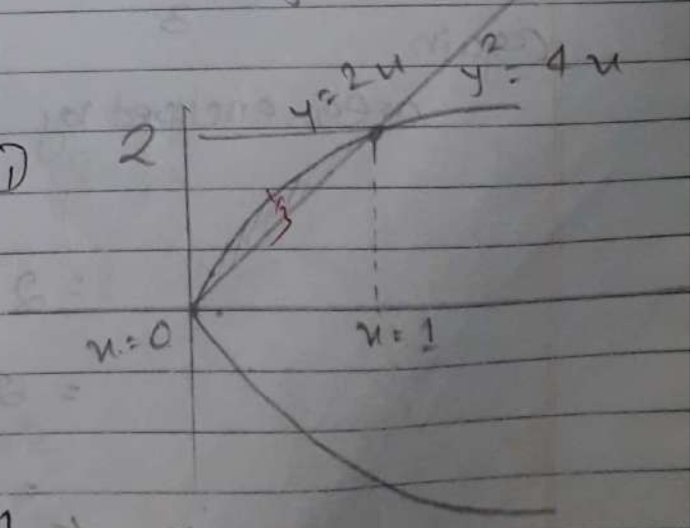
$\therefore$ Total area of $\sin x = 2 + 2$
$= 4$

* Find the area enclosed by the curve $y^2 = 4x$ and the line $y = 2x$

→ solution:

$y^2 = 4x$ ---(i)

$y = 2x$ ---(ii)



Solving eq^n (i) and (ii)

$(2x)^2 = 4x$

$4x^2 = 4x$

$4x^2 - 4x = 0$

$4x(x-1) = 0$

$x = 0$ and $x = 1$

$y = 0$ $\quad$ $y = 2$

Area enclosed by parabola $(A_1) = \int_{x=0}^{x=1} 2\sqrt{x}\,dx$

Area enclosed by line $(A_2) = \int_{x=0}^{x=1} 2x\,dx$

$\therefore$ Area enclosed $(A) = A_1 - A_2$.

here,

Area enclosed by parabola $(A_1) = \int_0^1 2\sqrt{x}\,dx$

$$= 2\int_0^1 x^{1/2}\,dx$$

$$= 2\int_0^1 \left[\frac{x^{3/2}}{3/2}\right]_0^1$$

$$= 2\times\frac{2}{3}\left[x^{3/2}\right]_0^1$$

$$= \frac{4}{3}\left[1^{3/2}\right]$$

$$= \frac{4}{3}$$

again,

area enclosed by line $(A_2) = \int_0^1 2x\,dx$

$$= 2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2.\left(\frac{1}{2}\right)$$

$$= 1$$

$\therefore$ Area enclosed $(A) = A_1 - A_2$

$$= \frac{4}{3} - 1$$

$$= 1/3 \text{ Ans.}$$

* Find the area of the region between the x-axis and the graph $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$

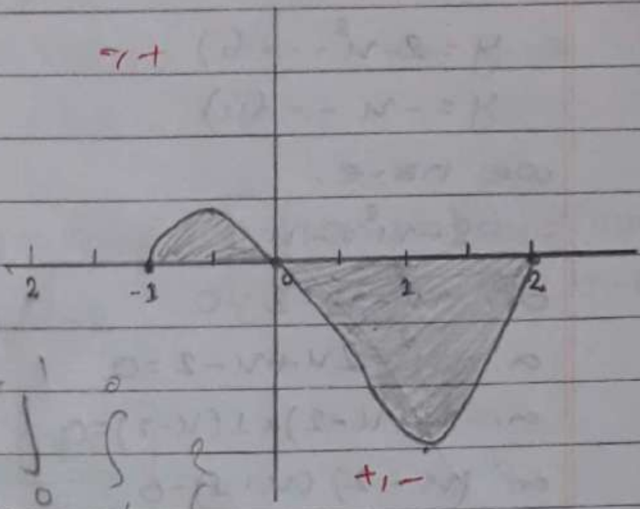
→ Solution,

When, $x = 0$, $y = 0$

$x = 1$, $y = 1 - \cancel{2} - 2 = -\cancel{3}$

$x = -1$, $y = -1 + 1 + 2 = 0$

$x = 2$, $y = 0$



Now,

$$\text{Area}(A_1) = -\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx$$

$$= -\left[\frac{x^4}{4}\right]_{-1}^{0} - \left[\frac{x^3}{3}\right]_{-1}^{0} - \left[\frac{2x^2}{\cancel{2}}\right]_{-1}^{0}$$

$$= \left(\frac{0}{4} - \frac{+1}{4}\right) - \left(\frac{0}{3} - \frac{-1}{3}\right) - (0^2 - 1) = \left(\frac{0}{4}\right) - \left(\frac{+1}{4}\right) - \left(\frac{0}{3}\right) + \left(\frac{-1}{3}\right) - 0 + (1)$$

$$= -\left(\frac{1}{4} - \frac{1}{3} + 1\right) \qquad = -\frac{1}{4} - \frac{1}{3} + 1$$

$$= -\left(\frac{1}{4} - \frac{4}{3}\right) \qquad = \frac{-3 - 4 + 12}{12} = \frac{5}{12}$$

$$= -\left(\frac{3 - 16}{12}\right) = +\frac{13}{12} \qquad \therefore \text{Area}(A_1) = -\frac{13}{12} \times -1 = \frac{13}{12} \quad = \frac{5}{12}$$

$$\text{Area}(A)_2 = -\int_{0}^{2} (x^3 - x^2 - 2x)\,dx$$

$$= -\left[\frac{x^4}{4}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2 - \left[2x\right]_0^2$$

$$= -\left(\frac{\cancel{16}}{4} - \frac{8}{3} - 4\right)$$

$$= -\left(-\frac{8}{3}\right) \qquad = \frac{32}{12}$$

$$= \frac{8}{3}$$

$$\therefore \text{total area} = \left(\frac{\cancel{13}\,5}{12} + \frac{8}{3}\right) = 3.\cancel{07}$$

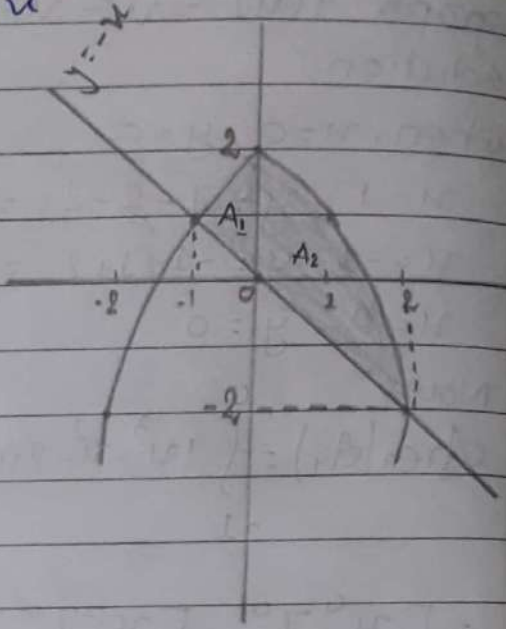
$$= 1.083 + 2.667 = 3.749$$

* Find the area of region enclosed by parabola $y = 2-x^2$ and the line $y = -x$

→ Solution:-

$y = 2-x^2$ --- (i)

$y = -x$ --- (ii)



we have,

$2-x^2 = -x$

or, $x^2-x-2=0$

or, $x^2-2x+x-2=0$

or, $x(x-2)+1(x-2)=0$

or $(x-2)(x+1)=0$

Either $x=2$, $x=-1$

$y=-2$, $y=1$

For parabola,

$y = 2-x^2$

when $x=0$, $y=2$

when $x=\pm1$, $y=1$

when $x=\pm2$, $y=-2$

Now,

$$\text{Area}(A) = \int_{-1}^{0}(2-x^2)\,dx - \int_{-1}^{0} -x\,dx$$

$$= \left[2x-\frac{x^3}{3}\right]_{-1}^{0} - \left[-\frac{x^2}{2}\right]_{-1}^{0}$$

$$= 0-\left\{2\times(-1)-\frac{(-1)^3}{3}\right\}+\left(\frac{0}{2}-\frac{(-1)^2}{2}\right)$$

$$= 2-\frac{1}{3}-\frac{1}{2}$$

$$= \frac{5}{3}-\frac{1}{2}$$

$$= \frac{10-3}{6}$$

$$= \frac{7}{6} \text{ sq. unit}$$

In y-axis,

$y = 2-x^2$

$x^2 = 2-y$

$\therefore x = \sqrt{2-y}$ — ①

$x = -y$ — ②

Again,

$$\text{Area}(A_2) = \int_{-2}^{2} \sqrt{2-y}\, dy - \int_{-2}^{0} -y\, dy$$

$$= \int_{-2}^{2} (2-y)^{1/2} dy + \int_{-2}^{0} y\, dy$$

$$= \left[\frac{(2-y)^{\frac{1}{2}+1}}{(\frac{1}{2}+1)(-1)}\right]_{-2}^{2} + \left[\frac{y^2}{2}\right]_{-2}^{0} \qquad \left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}\right]$$

$$= \left[\frac{(2-y)^{3/2}}{-3/2}\right]_{-2}^{2} + \left[\frac{0}{2} - \frac{(-2)^2}{2}\right]$$

$$= \frac{(2-2)^{3/2}}{-3/2} - \frac{(2+2)^{3/2}}{-3/2} + \left(-\frac{4}{2}\right)$$

$$= 0 - x - \frac{2}{3}(4)^{3/2} + (-2)$$

$$= \frac{2}{3} \times 2^{3} - 2$$

$$= \frac{16}{3} - 2 \qquad = \frac{16}{3} - 2 = \frac{10}{3}$$

$$= \frac{10}{3} \text{ sq. unit.}$$

Hence the area enclosed by parabola $y = 2 - x^2$ and line $y = -x$ is $A_1 + A_2$

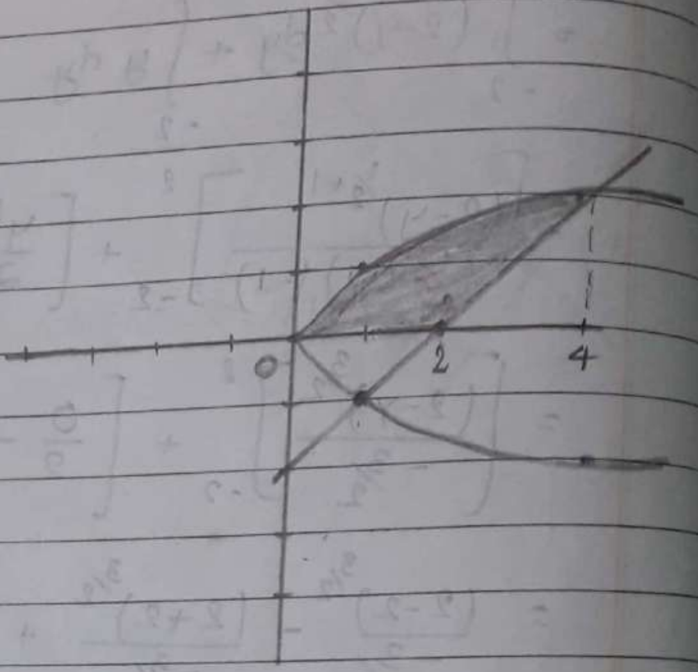
$$A = \left(\frac{7}{6} + \frac{10}{3}\right) \text{sq. unit}$$

$$= \left(\frac{27}{6}\right) \text{sq. units.}$$

$$= 4.5 \text{ sq. units.}$$

* Find the area of the region in the first quadrant that is bounded above $y=\sqrt{x}$ and below by the x-axis and the line $y = x-2$.

→ Solution.



$y=\sqrt{x}$. Squaring

$y^2 = x$ - - - (i)

$y = x-2$ - - - (ii)

Solving

$y = y^2 - 2$

or, $y^2 - y - 2 = 0$

or, $y^2 - 2y + y - 2 = 0$

or $y(y-2)+1(y-2)=0$

$\therefore y = +2, -1$

$x = 4, 1$

$\therefore x = 4, 1$

$y = 2, -1$

For $y^2 = x$

when $x=0, y=0$

when $x=1, y=1$

when $x=4, y=2$

Now,

$$\text{Area}(A) = \int_0^4 (\sqrt{x})\,dx - \int_2^4 (x-2)\,dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^4 - \left\{\left[\frac{x^2}{2}\right]_2^4 - \left[2x\right]_2^4\right.$$

$$= \frac{2}{3}\times 2^{2\cdot 3/2} - \left\{\left(\frac{\overset{8}{16}}{2} - \frac{4}{2}\right) - (2\times4 - 2\times2)\right\}$$

$$= \frac{16}{3} - (8 - \cancel{2} - 8 + 4) \qquad -2$$

~~$= \frac{16-2}{3}$ 6 Sq. unit~~

$= \frac{10}{3} = 3.33$ Sq. unit

$\therefore$ Area enclosed $= 3.33$ Sq. unit

~~* Find the area between $y=x$ and $y=x^3$ from $x=-1$ to $x=1$~~

~~→ Solution,~~

* Find the area of region enclosed by line $x=0$, $y=3$ and curve $x=2y^2$.

→ Solution,

$$x = 2y^2 \quad ----(i)$$

when $x=2y^2$

$x=0$	$y=0$
$x=2$	$y=1$
$x=8$	$y=2$
~~$x=27$~~	$y=3$

when $y=3$

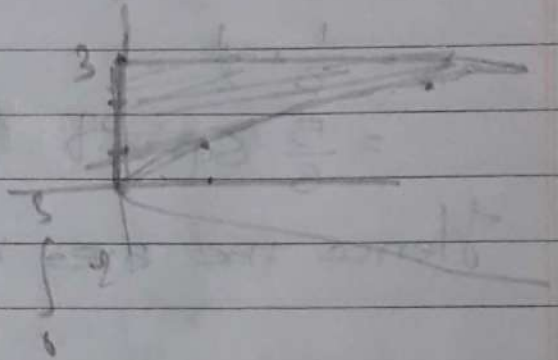
$$x = 3^2 \times 2 = 18$$

when $y=0$

$$x=0$$

Now,

$$\text{Area }(A) = \int_0^3 2y^2 \cdot dy$$



$$= 2 \cdot \left[\frac{y^3}{3}\right]_0^3$$

$$= 2 \cdot \left[\frac{3^3}{3} - \frac{0^3}{3}\right]$$

$$= 2 \times 9$$

$$= 18 \text{ sq. unit } \#.$$

Hence area enclosed is 18 sq. unit

* Find the area bounded on right by line $x+y=2$ and left $y=x^2$ below by the x-axis.

→ Solution,

$$x+y=2 \quad ---(i)$$

$$y=x^2 \quad ---(ii)$$

Solving eq^n (i) & (ii), we get

$$x+x^2=2$$

$$x^2+x-2=0$$

$$x^2+2x-x-2=0$$

or, $x(x+2)-1(x+2)=0$

or $(x+2)(x-1)=0$

$y = x^2$

$x=1$	$y=1$
$x=0$	$y=0$
$x=2$	$y=4$
$x=3$	$y=9$

$\therefore x = +1, -2$

$y = 1, 4$

Now,

$$\text{Area (A)} = \int_0^1 x^2 dx + \int_1^2 (2-x)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x\right]_1^2 - \left[\frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left(2\times2 - 2\right) - \left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= \frac{1}{3} + 2 - \frac{3}{2}$$

$$= \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6} \text{ Sq. unit}$$

Hence the area bounded is $5/6$ sq. unit

* Find the area of region in first quadrant bounded by the line $y = x$, that line $x = 2$ and curve $y = \frac{1}{x^2}$, and x-axis.

→ Solution,

$y = x$ and $y = \frac{1}{x^2}$

$y = x$

$\frac{1}{x^2} = x$

$\therefore x^3 = 1$

$\therefore x = 1$

$y = 1$

for curve,

$y = \frac{1}{x^2}$

when $x = 1$ $y = 1$

$x = 2$ $y = 0.25$

$$\text{Area (A)} = \int_0^1 x\,dx + \int_1^2 x^{-2} dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{1}{2} - \frac{0}{2}\right] + \left[\frac{-1}{2} + \frac{1}{1}\right]$$

$$= \cancel{\frac{1}{2}} + \cancel{\left(-\frac{1}{2}\right)} + 1 = 1 \text{ Sq. unit}$$

Hence the are in 1^{st} quadrant bounded by line $y=x$, line $x=2$, curve $y=\frac{1}{x^2}$ is 1 sq. unit.

* Find the area of triangular region in the first quadrant bounded by the Y-axis and the curves $y=\sin x$, $y=\cos x$.

→ Solution,

$y=\sin x$ ---(i) $\quad y=\cos x$ ---(ii)

$\sin x=\cos x$

or, $\tan x=1$

or, $\tan x=\tan \frac{\pi}{4}$

$\therefore x=\frac{\pi}{4}$

$$\text{Required Area} = \int_0^{\pi/4} \cos x\,dx - \int_0^{\pi/4} \sin x\,dx$$

$$= \left[\sin x+\cos x\right]_0^{\pi/4}$$

$$= \sin\frac{\pi}{4}+\cos\frac{\pi}{4}-(\sin 0^\circ+\cos 0^\circ)$$

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-(0+1)$$

$$= \frac{2}{\sqrt{2}}-1$$

$$= \frac{\sqrt{2}\times\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}-1$$

$$= \sqrt{2}-1 \text{ sq. unit}$$

Hence the area of triangular region is $\sqrt{2}-1$ sq. unit

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* Find the area betⁿ $y=x$ and $y=x^3$ from $x=-1$ to $x=1$

→ Solution,

when $x=1$, $x=-1$
$y=1$ $y=-1$

for, $y=x^3$
when $x=0$ $y=0$
$x=1$ $y=1$
$x=2$ $y=8$

Now,

$$\text{Area }(A) = \int_0^1 x\,dx - \int_0^1 x^3\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{2}-0\right) - \left(\frac{1}{4}-0\right)$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{2-1}{4}$$

$$= \frac{1}{4}$$

$$\cancel{= \int_0^1 y^{1/3}\,dy - \int_0^1 y\,dy}$$

$$\cancel{= \frac{3}{4}\left[y^{4/3}\right]_0^1 - 2\left(y\right)_0^1}$$

$$\cancel{= \frac{3}{4}\times 1 - 2}$$

$$\cancel{= \frac{3-8}{4}}$$

$$\cancel{= -\frac{5}{4}}$$

∴ Total area betⁿ $y=x$ and $y=x^3$ from $x=-1$ to $x=1$ is $= \frac{1}{4}\times 2$

$$= \frac{1}{2}\text{ sq. units}$$

* Find the area between two parabola $y^2=4ax$ and $x^2=4ay$.

→ Solution,

$y^2 = 4ax$ --(i)
$x^2 = 4ay$ --(ii)

or, $y = \frac{x^2}{4a}$

Now,

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

or, $\frac{x^4}{16a^2} = 4ax$ or, $x^4 = 64a^3x$

or, $\cancel{x^3 = 4\times 16a^3}$ or, $x^4 - 64a^3x = 0$

or, $\cancel{x^3 = 64a^3}$ or $x(x^3 - 64a^3) = 0$

Either $x = 0$

$= x^3 - 64a^3$

or, $x^3 = 64a^3$

put Cube root on both side

$x = 4a$

Now,

$$\text{Area (A)} = \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx$$

$$= \int_0^{4a} 2\sqrt{a}\, x^{1/2}\, dx - \int_0^{4a} \frac{1}{4a} x^2\, dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\times\frac{2}{3}\left[4a\right]^{3/2} - \frac{1}{4a\times\cancel{3}\times 3}\left[4a\right]^3$$

$$= \frac{4\sqrt{a}}{3}\, 2^{2\cdot\frac{3}{2}}\, a^{3/2} - \frac{1}{3\times\cancel{4a}}\, 64a^{\cancel{3}}$$

$$= \frac{4\sqrt{a}}{3}\times 8a^{3/2} - \frac{16a^2}{3}$$

$$= \frac{32a^{1/2+3/2}}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2 - 16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. unit}$$

$\therefore$ Req. Area is $\frac{16a^2}{3}$ sq. unit.

* Find the area bound by x-axis and parabola $y = 4 - x^2$.

→ Solution,

In x-axis $y = 0$

$0 = 4 - x^2$

$\therefore x = \sqrt{4}$

$\therefore x = \pm 2$

when $x = 0$

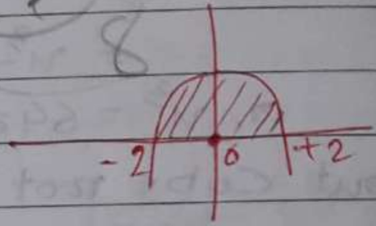
$y = 4$

$y = 4 - x^2$

when $x = 0$ $y = 4$

$x = 1$ $y = 3$

$x = 2$ $y = 0$



Now,

$$\text{Area } (A) = \int_0^2 (4 - x^2)\,dx$$

$$= \int_0^2 4\,dx - \int_0^2 x^2\,dx$$

$$= [4x]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= [4 \times 2 - 0] - \left[\frac{2^3}{3} - \frac{0}{3}\right]$$

$$= 8 - \frac{8}{3}$$

$$= \frac{24 - 8}{3}$$

$$= \frac{16}{3}$$

$$\therefore \text{Total area } (A) = \frac{16}{3} \times 2$$

$$= \frac{32}{3} \text{ sq. unit}$$

Hence the area bound by x-axis and parabola $y = 4 - x^2$ is $\frac{32}{3}$ sq. unit.

i) Even and odd Function:-

A function f is even if the graph of f is symmetric with respect to y-axis. Algebraically, f is even if and only if $f(-x) = f(x)$ for all x in the domain of f.

A function f is odd if the graph of f is symmetric with respect to the origin. Algebraically, f is odd if and only if $f(-x) = -f(x)$ for all x in the domain of f.

ii) One to One and Onto function:-

The function is one to one (injective) if every element of the codomain is mapped to by at most one element of domain.

The function is onto (surjective) if every element of the codomain is mapped to by at least one element of the domain. i.e. the image of and codomain of function are equal.

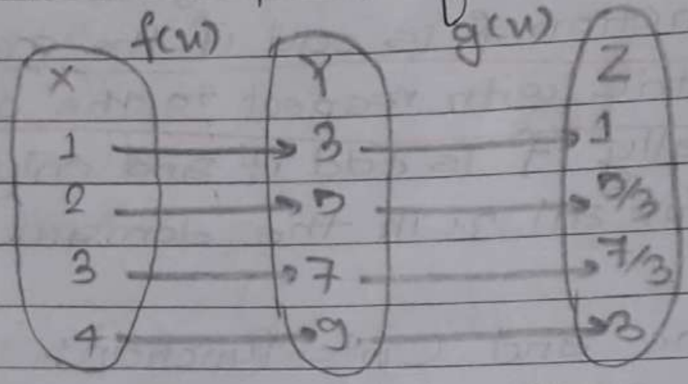
iii) Piecewise define function: (hybrid function)

A piecewise-defined function or hybrid function is a function which is defined by multiple sub-functions, each sub-function applying to a certain interval of the main function's domain (a sub domain).

iv) Composite function:-

A function whose values are found from two given functions by applying one function to an independent variable and then applying the second function to the result and whose domain consists of those values of the independent variable for which the result yielded by the first function lies in the domain of second function is called composite function.

e.g.

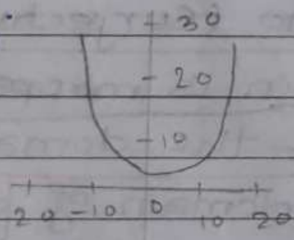


$g(f(x))$ where
$f(x) = y = 2x+1$
$g(x) = Z = \frac{x}{3}$

Composite function

i> Even function:

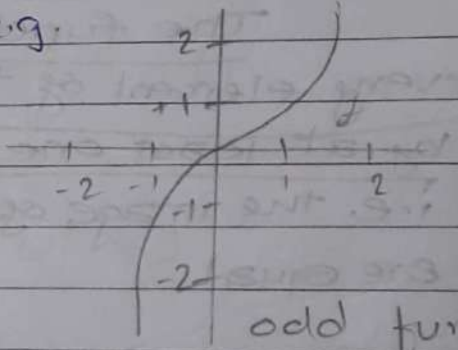
e.g.



even function

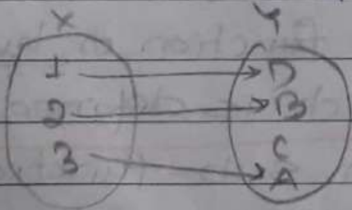
ii> odd function

e.g.



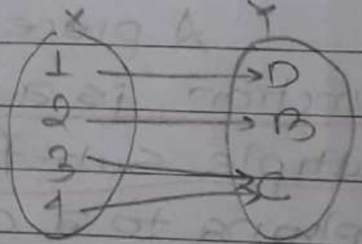
odd function

iii> one to one function



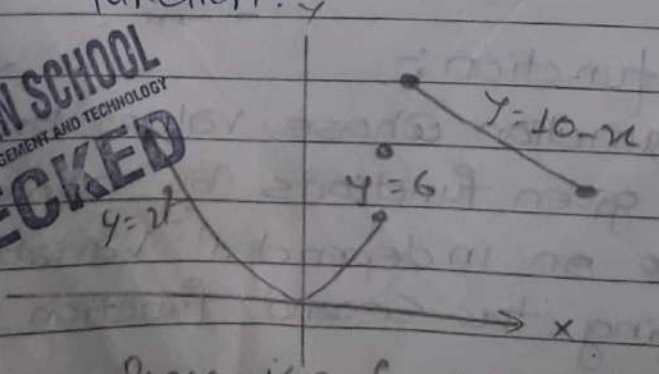
one to one function

iv> onto function



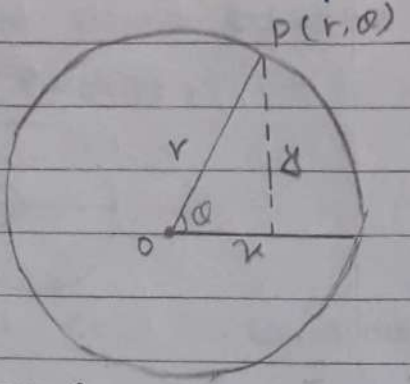
onto function

v> Piecewise function:



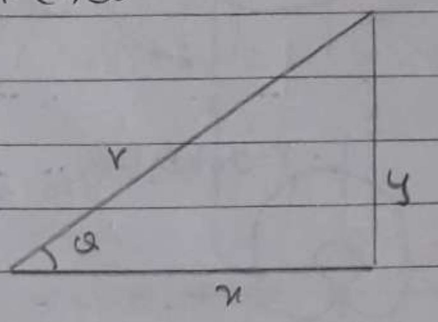
Piecewise function

Polar Equation



Polar co-ordinate Point

p(r,θ)



$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\tan\theta = \frac{y}{x}$

$$\boxed{\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

$\sin x = \frac{p}{h} = \frac{y}{r}$

$$\boxed{\therefore y = r\sin\theta}$$

$\cos x = \frac{b}{h} = \frac{x}{r}$

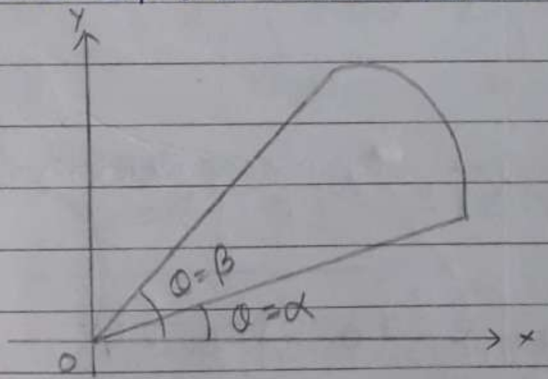
$$\boxed{\therefore x = r\cos\theta}$$

$x^2 + y^2 = r^2\sin^2\theta + r^2\cos^2\theta$

or, $x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta)$

$$\boxed{\text{or, } r^2 = x^2 + y^2}$$

Area of the Polar curve:



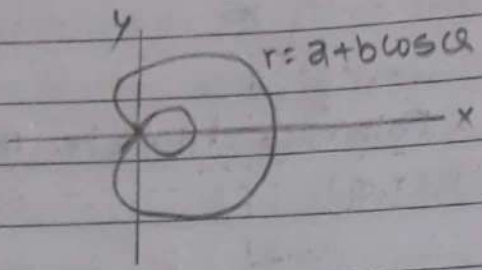
$$\therefore \text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2 \, d\theta$$

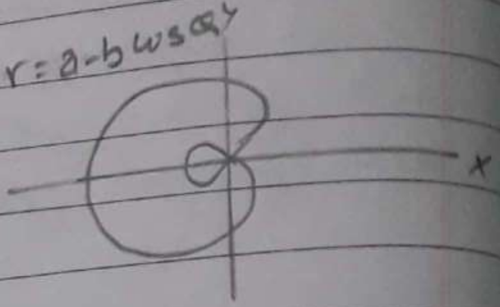
Polar Equation:

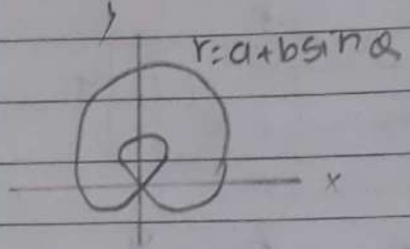
Limacons $(r = a \pm b\cos\theta,\ r = a \pm b\sin\theta)$

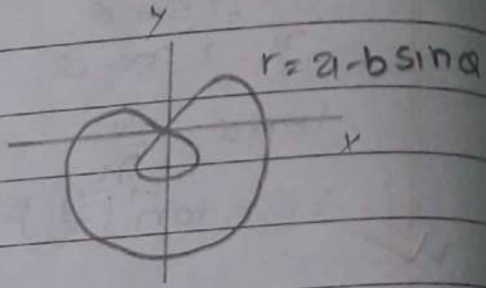
- → Inner loops : $\left(\frac{a}{b} < 1\right)$
- → Cardiode : $\left(\frac{a}{b} = 1\right)$
- → Dimple : $\left(1 < \frac{a}{b} < 2\right)$
- → Convex (oval) : $\left(\frac{a}{b} \geq 2\right)$

1) Inner loop $\left(\frac{a}{b}<1\right)$

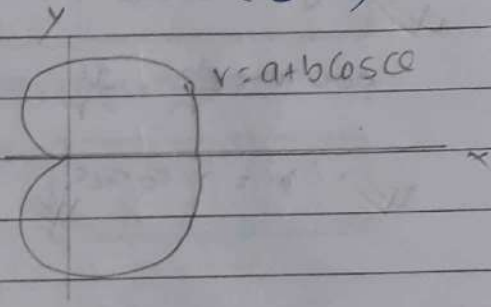


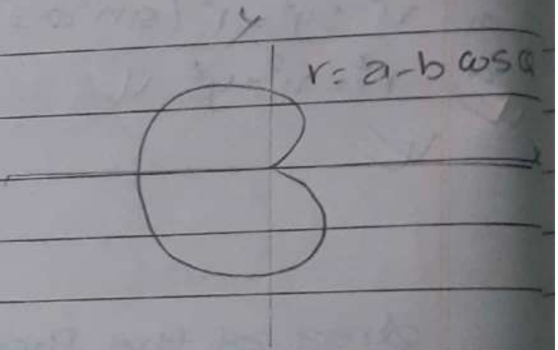




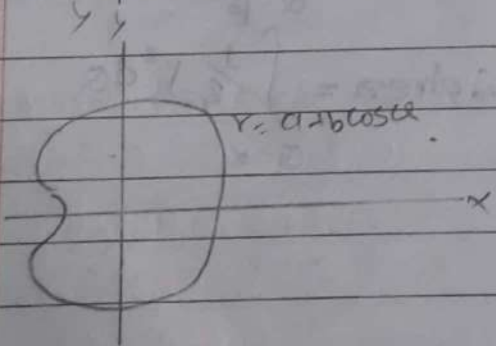


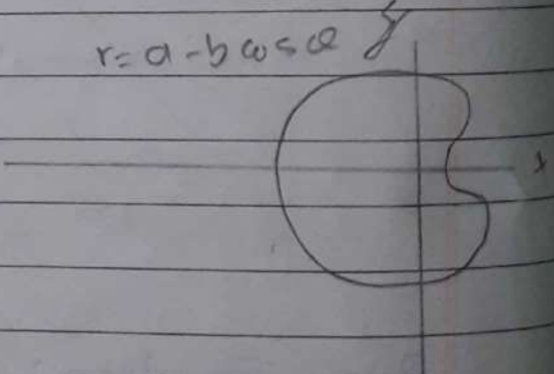
2) cardiode : $\left(\frac{a}{b}=1\right)$



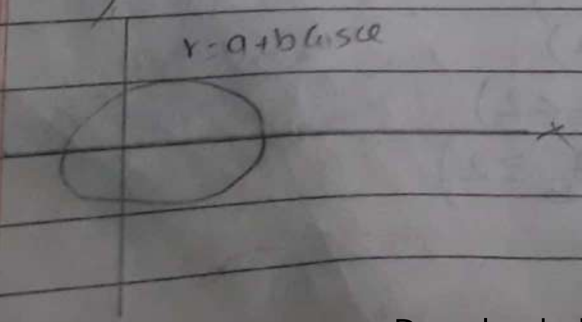


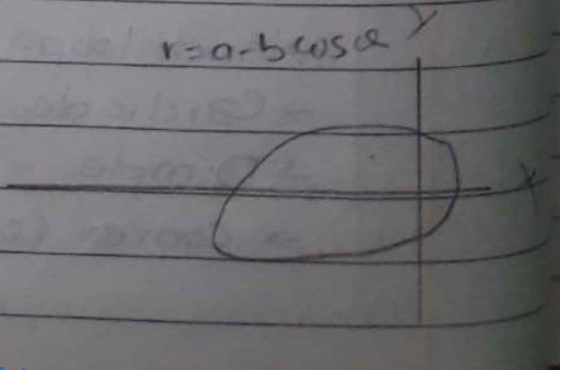
3. Dimple $\left(1<\frac{a}{b}<2\right)$





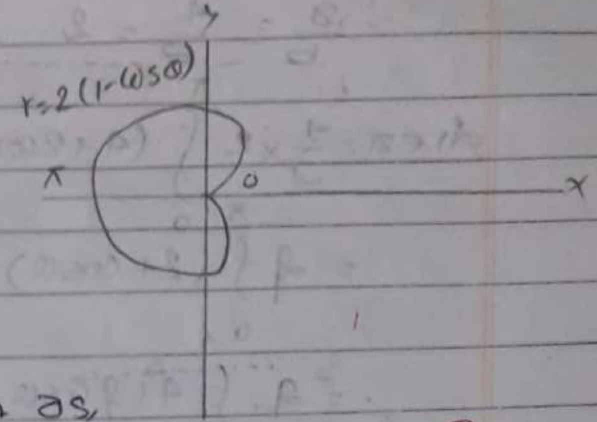
4. Convex (oval) $\left(\frac{a}{b}\geq 2\right)$





* Find the area of the cardioide: $r = 2(1-\cos\theta)$

→ Solution,



$r = 2(1-\cos\theta)$

$r = 2 - 2\cos\theta$

$a = 2 \quad b = 2$

$\therefore \frac{a}{b} = 1$, so it is cardiode.

Now,

Area of cardiode is given as,

$$\boxed{\text{Area}(A) = \int_{\theta=0}^{\theta=\pi} \frac{1}{2} r^2 \, d\theta}$$

Cardiodes $0 \to \pi$

Area is symmetrical divided so,

$$\text{Area}(A) = \frac{1}{2} \times 2 \int_{\theta=0}^{\theta=\pi} 2^2(1-\cos\theta)^2 \, d\theta$$

$$= 4\int_{\theta=0}^{\theta=\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 4\left[\int_{\theta=0}^{\pi} 1\, d\theta - 2\int_{\theta=0}^{\theta=\pi} \cos\theta + \int_{\theta=0}^{\theta=\pi} \frac{1+\cos 2\theta}{2}\, d\theta\right.$$

$$= 4\left\{[\theta]_0^{\pi} - 2[\sin\theta]_0^{\pi} + \frac{\cancel{1}}{2}\left\{[\theta]_0^{\pi} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}\right\}\right\}$$

$$= 4\left\{(\pi - 0) - 2(\sin\pi - \sin 0) + \frac{\cancel{1}}{2}(\pi - 0) + \left(\frac{\sin 2\pi}{2} - \frac{\sin 2 0}{2}\right)\right.$$

$$= 4\left\{(\pi - 0) - 2(0-0) + \frac{\cancel{1}}{2}\pi + \left(\frac{0}{2} - \frac{0}{2}\right)\right.$$

$$= 4\left\{\pi + \frac{\cancel{1}}{2}\pi\right\}$$

$$= \cancel{\frac{6\pi}{2}} = 6\pi$$

$\therefore$ Area of Cardiode is $\cancel{\frac{6\pi}{2}} = 6\pi$ sq. unit

$= \frac{a^2 3\pi}{2}$

* Find the area of $r = 4 + 2\cos\theta$

→ Solution,

$= \frac{\pi}{3} + \overset{2}{\cancel{4}}\frac{\sqrt{3}}{\cancel{2}} + 2\left\{\frac{\pi}{3} + \frac{0}{2} - \frac{-\frac{\sqrt{3}}{2}}{2}\right]$

$= \frac{\pi}{3} + 2\sqrt{3} + \frac{2\pi}{3} + \cancel{2}\left(+\frac{\sqrt{3}}{2}\right) \times \frac{1}{\cancel{2}}$

$= \frac{\pi}{3} + \frac{2\pi}{3} - 2\sqrt{3} + \frac{\sqrt{3}}{2}$

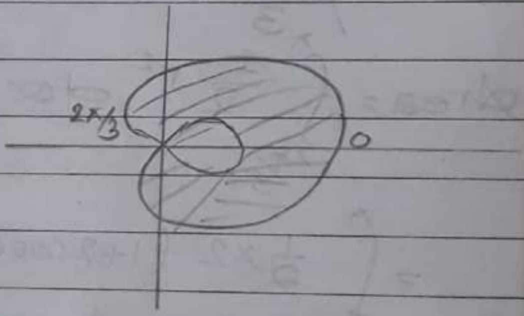
$= \frac{3\pi}{3} - \frac{4\sqrt{3}+\sqrt{3}}{2}$

$= \frac{\cancel{3}\pi}{\cancel{3}} - \frac{3\sqrt{3}}{2}$

$= \pi - \frac{3\sqrt{3}}{2}$ Answer. Hence the area inside the limacon is $\pi - \frac{3\sqrt{3}}{2}$ sq. units.

* find the area outside the loop of limacon.
$r = 2\cos\theta + 1$.

→ Solution,
we have,



$\text{Area} = \int_0^{2\pi/3} \frac{1}{\cancel{2}} \times \cancel{2}\, r^2 \, d\theta$

$= \int_0^{2\pi/3} (1 + 2\cos\theta)^2 \, d\theta$

$= \int_0^{2\pi/3} (1 + 4\cos\theta + 4\cos^2\theta) \, d\theta$

$= \int_0^{2\pi/3} 1 \, d\theta + 4\int_0^{2\pi/3} \cos\theta \, d\theta + 4\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= [\theta]_0^{2\pi/3} + 4[\sin\theta]_0^{2\pi/3} + \frac{4}{2}\left\{[\theta]_0^{2\pi/3} + \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi/3}\right\}$

$= \frac{2\pi}{3} - 0 + 4\left[\sin\frac{2\pi}{3} - \sin 0\right] + 2\left\{\frac{2\pi}{3} - 0 + \frac{\sin 2\cdot\frac{2\pi}{3}}{2} - \frac{\sin 2\cdot 0}{2}\right\}$

$= \frac{2\pi}{3} + \overset{2}{\cancel{4}}\frac{\sqrt{3}}{\cancel{2}} + 2\left\{\frac{2\pi}{3} + -\frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{0}{2}\right\}$

$= \frac{2\pi}{3} + \frac{2\sqrt{\cancel{3}}}{} + \frac{4\pi}{3} - \frac{2\sqrt{\cancel{3}}}{\cancel{4}2}$

$= \frac{2\pi + 4\pi}{3} + 2\sqrt{3} - \frac{\sqrt{3}}{2}$

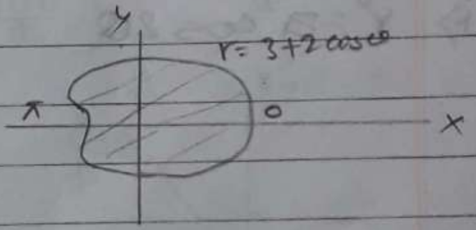
$= \frac{6\pi}{3} + \frac{4\sqrt{3} - \sqrt{3}}{2}$

$= 2\pi + \frac{3\sqrt{3}}{2}$ sq. unit.

∴ Hence Area outside the loop of limacon is $2\pi + \frac{3\sqrt{3}}{2}$ sq. unit

* Find the area of curve $r = 3+2\cos\theta$.

→ Solution,



Area of Curve is given by,

$= \int_0^{\pi} \frac{1}{\cancel{2}} \times \cancel{2} (r)^2 \, d\theta$

$= \int_0^{\pi} (3+2\cos\theta) \, d\theta$

$= \int_0^{\pi} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$

$= \int_0^{\pi} 9 \, d\theta + \int_0^{\pi} 12\cos\theta \, d\theta + \int_0^{\pi} 4\cos^2\theta \, d\theta$

$= 9\left[\theta\right]_0^{\pi} + 12\left[\sin\theta\right]_0^{\pi} + 4\int_0^{\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= 9(\pi - 0) + 12(\sin\pi - \sin 0^\circ) + \frac{4}{2}[\theta]_0^\pi + \left[\frac{\sin 2\theta}{2}\right]_0^\pi$

$= 9\pi + 0 + 2\left\{(\pi - 0) + \frac{\sin 2.\pi}{2} - \frac{\sin 0^\circ}{2}\right.$

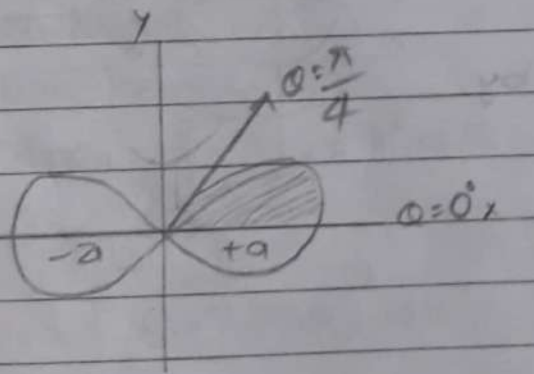
$= 9\pi + 2\pi + 0 + 0$

$= 11\pi$ (Sq. unit)

Hence the area of curve is 11π (square Unit)

Loop of leminiscate (Bernouli)

i) $r^2 = a^2\cos 2\theta$



when $\theta = 0$, the value of $r = max$

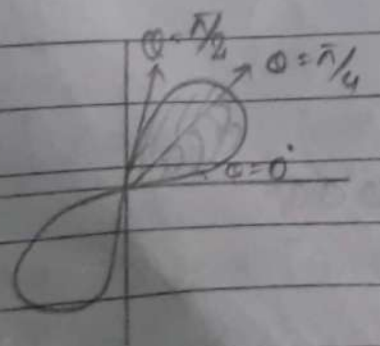
$r = \pm a$

when $\theta = \pi/4$ $r = 0$ min

$\theta_1 = 0^\circ$

$\theta_2 = \frac{\pi}{4}$

ii) $r^2 = a^2\sin 2\theta$



$\therefore \theta_1 = 0^\circ$

$\therefore \theta_2 = \frac{\pi}{2}$

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* Find the area of $r^2 = a^2 \cos 2\theta$.

→ solution,

$r^2 = a^2 \cos 2\theta$, it is bernauli, so

we have,

$$\text{Area } (A) = \int_{\theta=0^\circ}^{\theta=\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{\cancel{2}} \times \cancel{4} \int_{0^\circ}^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \int_0^{\pi/4} \frac{\sin 2\theta}{2} \, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= 2a^2 \left[\frac{\sin \cancel{2} \cdot \frac{\pi}{\cancel{4}2}}{2} - \frac{\sin 2 0^\circ}{2}\right]$$

$$= 2a^2 \left(\frac{1}{2} - 0\right)$$

$$= 2a^2 \times \frac{1}{\cancel{2}}$$

$$= a^2 \text{ Sq. unit}$$

∴ Area of $r^2 = a^2 \cos 2\theta$ is a^2 Sq. unit

* Find the area of $r^2 = a^2 \sin 2\theta$.

→ Solution,

we know, Area of bernouli (A) $= \int_{\theta=0^\circ}^{\theta=\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta$

$$= \int_{\theta=0^\circ}^{\theta=\pi/2} \frac{1}{2} \times 2 \, a^2 \sin 2\theta \, d\theta$$

$$= a^2 \left[- \frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

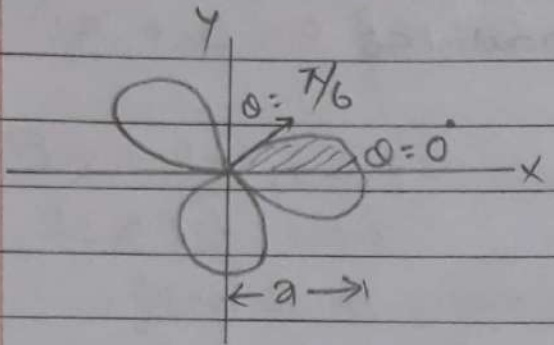
$$= a^2 \left[- \frac{\cos 2 \cdot \frac{\pi}{2}}{2} - \frac{\cos 2 0^\circ}{2}\right]$$

$$= a^2 \left[- -\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{a^2}{2} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} = a^2 \text{ Sq. unit}$$

* Three leaved Rose:

i) $r = a\cos 3\theta$



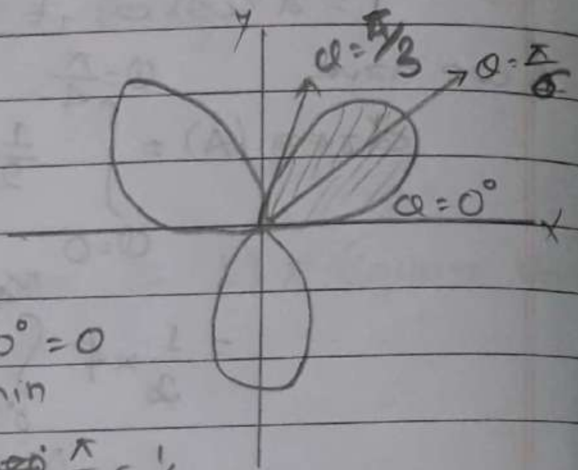
max $\theta = 0°$

min when $\theta = \frac{\pi}{6}$

$\therefore \theta_1 = 0°$

$\theta_2 = \frac{\pi}{6}$

ii) $r = a\sin 3\theta$



$\theta = 0° = 0$

min

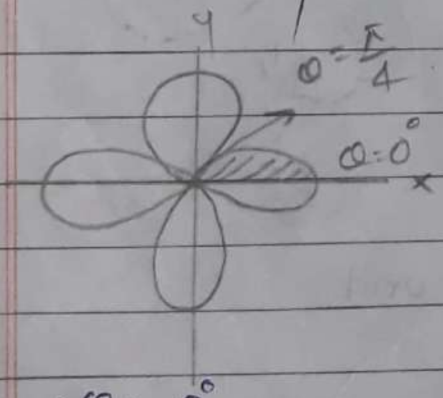
$\theta = \cancel{180}\ \frac{\pi}{3} = \frac{1}{2}$

$\theta_1 = 0°$

$\theta_2 = \frac{\pi}{3}$

* Four leaved Rose:

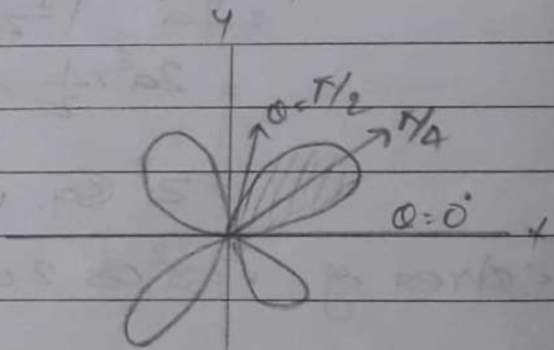
i) $r = a\cos 2\theta$



$\therefore \theta_1 = 0°$

$\therefore \theta_2 = \frac{\pi}{4}$ x8

ii) $r = a\sin 2\theta$



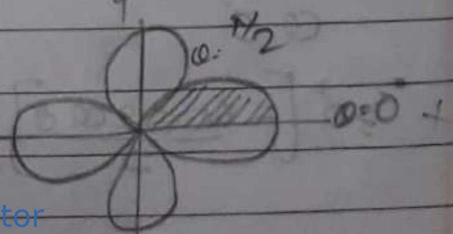
$\theta_1 = 0°$

$\theta_2 = \frac{\pi}{2}$ x4

* Find the area inside one left of the four leaved rose $r = \cos 2\theta$

→ Solution,

Area of one left $= \frac{1}{2} \times 2 \int_0^{\pi/4} (\cos 2\theta)^2 \, d\theta$



$$= \int_{\theta=0^\circ}^{\theta=\pi/4} \cos^2 2\theta \, d\theta = \int_{\theta=0^\circ}^{\theta=\pi/4} \frac{1+\cos 4\theta}{2} d\theta = \frac{1}{2}[\theta]_0^{\pi/4} + \left[\frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] + \left[\frac{\sin 4 \cdot \frac{\pi}{4}}{4} - \frac{\sin 0}{4}\right]$$

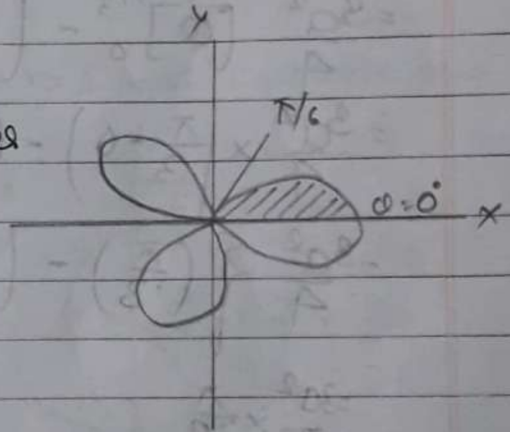
$$= \left(\frac{1}{2} \times \frac{\pi}{4}\right) + \frac{1}{2}\left[\frac{0}{4} - \frac{0}{4}\right]$$

$$= \frac{\pi}{8} \text{ sq. unit}$$

Hence the area of one left on a four leaved rose is $\frac{\pi}{8}$ sq. unit.

* Find the area of three leaved rose $r = a\cos 3\theta$

→ Solution,



Area of three leaved rose $= \frac{1}{2} \times 6 \int_{\theta=0}^{\theta=\frac{\pi}{6}} a^2 \cos^2 3\theta \, d\theta$

$$= 3a^2 \int_0^{\pi/6} \frac{1+\cos 6\theta}{2} d\theta$$

$$= \frac{3a^2}{2}[\theta]_0^{\pi/6} + \left[\frac{\sin 6\theta}{6}\right]_0^{\pi/6}$$

$$= \frac{3a^2}{2}\left[\frac{\pi}{6} - 0\right] + \left[\frac{\sin 6\frac{\pi}{6}}{6} - \frac{\sin 6 \cdot 0}{6}\right]$$

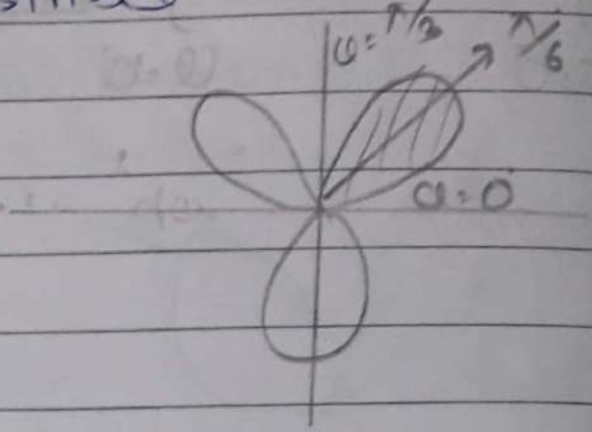
$$= \frac{3a^2}{2} \times \frac{\pi}{6}$$

$$= \frac{\pi a^2}{4} \text{ sq. unit}$$

Hence the area of three leaved rose is $\frac{\pi a^2}{4}$ sq. unit

* Find the area of $r = a\sin 3\theta$

→ Solution, $\theta = \pi/3$



$$Area(A) = \int_{\theta=0}^{\theta=\pi/3} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2} \times 3 \int_{\theta=0} a^2 \sin^2 3\theta \, d\theta$$

$$= \frac{3a^2}{2} \int_{0}^{\pi/3} \left(\frac{1-\cos 6\theta}{2} \right) d\theta$$

$$= \frac{3a^2}{2 \times 2} \int_{0}^{\pi/3} 1 \, d\theta - \int_{0}^{\pi/3} \cos 6\theta \, d\theta$$

$$= \frac{3a^2}{4} [\theta]_0^{\pi/3} - \left[\frac{\sin 6\theta}{6} \right]_0^{\pi/3}$$

$$= \frac{3a^2}{4} \times \left(\frac{\pi}{3} - 0 \right) - \left(\frac{\sin \cancel{6}^{2} \frac{\pi}{\cancel{3}}}{6} - \frac{\sin 6.0}{6} \right)$$

$$= \frac{3a^2}{4} \left\{ \left(\frac{\pi}{3} \right) - \left(\frac{0}{6} - \frac{0}{6} \right) \right.$$

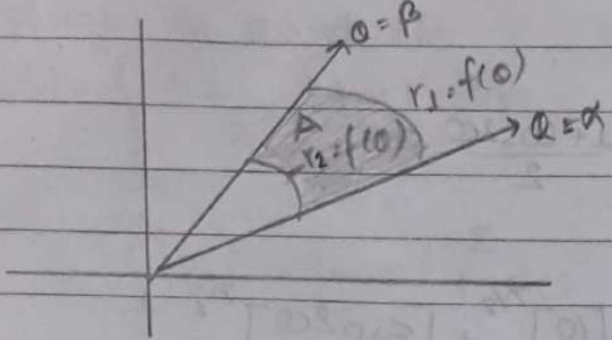
$$= \frac{3a^2}{4} \times \frac{\pi}{3}$$

$$= \frac{\cancel{3}\pi a^2}{\cancel{12}\, 4}$$

$$= \frac{\pi a^2}{4} \text{ sq. unit}$$

Hence the area of $a \sin 3\theta$ is $\frac{\pi a^2}{4}$ sq. unit.

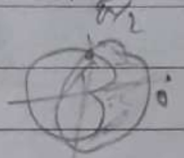
* Area between Polar Curves:



$$\text{Area between Polar Curves} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r_1^2 \, d\theta - \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r_2^2 \, d\theta$$

$$\boxed{\text{Area} = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta}$$

* Find the area inside circle $r=1$ but outside the cardioide $r = 1-\cos\theta$



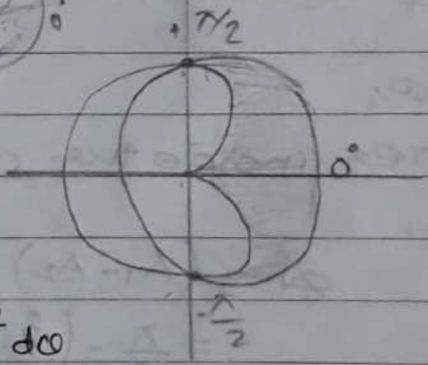
→ Solution,

$r_1 = 1$, $r_2 = 1-\cos\theta$

$\therefore 1 = 1-\cos\theta$

$\cos\theta = 0$

$\therefore \theta = \pm\frac{\pi}{2}$



$$\text{Area of circle}(A_1) = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2} r_1^2 \, d\theta$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \left[\theta\right]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2} \text{ sq. units}$$

$$\text{Area of Cardioide}(A_2) = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2}(1-\cos\theta)^2 \, d\theta$$

$$= \int_{0}^{\pi/2} \frac{1}{\cancel{2}} \times \cancel{2} \, (1-\cos\theta)^2 \, d\theta$$

$$= \int_{\theta=0^\circ}^{\theta=\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_{\theta=0^\circ}^{\theta=\pi/2} 1\, d\theta - 2\int_{\theta=0^\circ}^{\theta=\pi/2} \cos\theta + \int_{\theta=0^\circ}^{\theta=\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= [\theta]_0^{\pi/2} - 2[\sin\theta]_0^{\pi/2} + \frac{1}{2}[\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right) - 2\left[\sin\frac{\pi}{2} - \sin 0^\circ\right) + \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \left(\frac{\sin 2\cdot\frac{\pi}{2}}{2} - \frac{\sin 2\cdot 0^\circ}{2}\right)$$

$$= \frac{\pi}{2} - 2\times 1 + \frac{1}{2}\times\frac{\pi}{2} + \frac{1}{2}\left(\frac{0}{2} - \frac{0}{2}\right)$$

$$= \frac{\pi}{2} - 2 + \frac{\pi}{4}$$

$$= \frac{3\pi}{4} - 2$$

Now,

Area inside the circle but outside the cardioide is,

$$A = (A_1 - A_2)$$

$$= \frac{\pi}{2} - \left(\frac{3\pi}{4} - 2\right)$$

$$= \frac{\pi}{2} - \frac{3\pi}{4} + 2 \quad = \frac{2\pi - 3\pi + 2}{4}$$

$$=$$

$$= 2 - \frac{\pi}{4} \text{ sq. units}$$

* Find the area shared by the circle $r = 2$ and cardioide $r = 2(1 - \cos\theta)$

→ solution,

$r = 2$, $r = 2(1 - \cos\theta)$

$2 = 2(1 - \cos\theta)$

or, $1 - \cos\theta - 1$

$\cos\theta = 1 - 1$

$\theta = \pm \frac{\pi}{2}$

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here,

Area of cardiode

$(A_1) = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2} \times 2 (r)^2 \, d\theta$

$= \int_0^{\pi/2} \{2(1-\cos\theta)\}^2 \, d\theta$

$= 4 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$

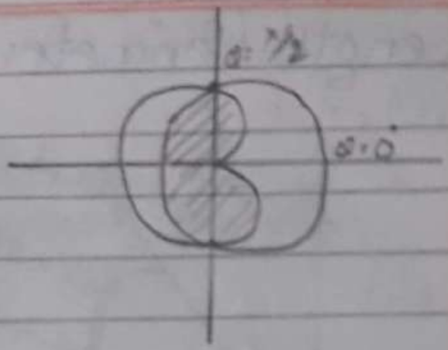
$= 4 \left\{ [\theta]_0^{\pi/2} - 2 [\sin\theta]_0^{\pi/2} \right\} + \frac{1}{2} \left\{ [\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} \right\}$

$= 4 \left\{ \left[\frac{\pi}{2} - 0\right] - 2 \left[\sin\frac{\pi}{2} - \sin 0\right] + \frac{1}{2} \left\{ \left[\frac{\pi}{2} - 0\right] + \frac{\sin 2\frac{\pi}{2}}{2} - \frac{\sin 0}{2} \right\} \right\}$

$= 4 \frac{\pi}{2} - 4 \times 2 (1-0) + \frac{4}{2} \frac{\pi}{2} + 4 \left[\frac{0}{2} - \frac{0}{2}\right]$

$= 2\pi - 8 + \pi + 0$

$= 3\pi - 8$



Again,

Area of Circle $(A_2) = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2} \times 2 \, (2)^2 \, d\theta$

$= 4 \int_0^{\pi/2} 1 \, d\theta$

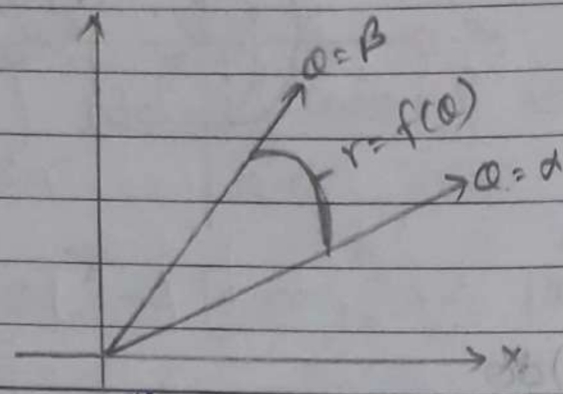
$= 4 [\theta]_0^{\pi/2} = 4 \left[\frac{\pi}{2} - 0\right] = \frac{4\pi}{2} = 2\pi$

Hence Area Covered by circle & Cardioide is

$A = A_1 + A_2 = 3\pi - 8 + 2\pi$

$= (5\pi - 8)$ sq. units

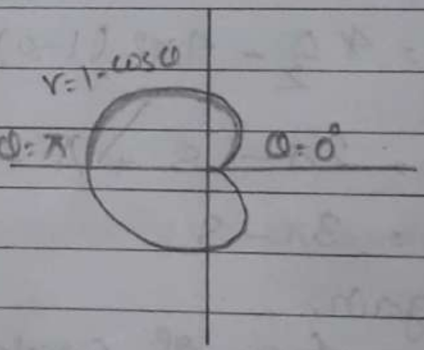
* Length (Perimeter of the Polar Curve):



$$\boxed{L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

* Find the length of Cardiode $r = 1 - \cos\theta$

→ Solution,



$r = 1 - \cos\theta$

$r^2 = 1 - 2\cos\theta + \cos^2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = \frac{d(1-\cos\theta)}{d\theta}$

$= \frac{d1}{d\theta} - \frac{d\cos\theta}{d\theta}$

$= 0 - x - \sin\theta$

$\therefore \left(\frac{dr}{d\theta}\right)^2 = (\sin\theta)^2$

$\therefore \text{length } (L) = \int_{\theta=0°}^{\theta=\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$= \int_{\theta=0}^{\theta=\pi} \sqrt{1 - 2\cos\theta + 1}\, d\theta$ $\quad \because \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \int_0^{\pi} \sqrt{2(1-\cos\theta)}\, d\theta$ $\quad 2\sin^2\theta/_2 = 1 - \cos 2\theta/_2$

$= \int_0^{\pi} \sqrt{2 \cdot 2\sin^2\theta/_2}\, d\theta$ $\quad \therefore 2\sin^2\theta/_2 = 1 - \cos\theta$

$$= \int_0^{\pi} 2 \sin \theta/_2 \, d\theta$$

$$= 2 \left[\frac{-\cos \frac{\theta}{2}}{1/2} \right]_0^{\pi} = 2.2 \left[-\cos \pi/_2 + \cos {}^{0}/_{2} \right]$$

$$= 4 \left[-0 + 1 \right]$$

$$= 4 \times 1 = 4 \times 2 \text{ unit} = 8 \text{ unit}$$

Hence the perimeter of Cardioide is 8 unit.

* Find the length of spiral $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$

→ Solution,

$$r = \theta^2$$

$$r^2 = \theta^4$$

$$\frac{dr}{d\theta} = \frac{d\theta^2}{d\theta} = 2\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = (2\theta)^2 = 4\theta^2$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \theta^4 + 4\theta^2$$

$$= \theta^2(\theta^2 + 4)$$

Now,

$$\text{length } (L) = \int_{\theta_2}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^2(\theta^2 + 4)} \, d\theta$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^2 + 4} \, \theta d\theta$$

let $y = \theta^2 + 4$

diff. $y = \theta^2 + 4$ w.r.to $d\theta$

$$\frac{dy}{d\theta} = 2\theta \qquad \therefore dy = 2\theta d\theta \quad \therefore \frac{dy}{2} = \theta d\theta$$

$y = \theta^2 + 4$

when $\theta = 0$, $y = 4$, when $\theta = \sqrt{5}$

$y = (\sqrt{5})^2 + 4$
$= 9$

Now,

$$L = \int_4^9 \sqrt{y} \cdot \frac{dy}{2}$$

$$= \frac{1}{2} \int_4^9 y^{1/2}\, dy$$

$$= \frac{1}{2} \left[\frac{y^{1/2+1}}{\frac{1}{2}+1} \right]_4^9$$

$$= \frac{1}{2} \left[\frac{y^{3/2}}{3/2} \right]_4^9$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} \left[9^{3/2} - 4^{3/2} \right]$$

$$= \frac{1}{3} \left[3^{2 \cdot 3/2} - 2^{2 \cdot 3/2} \right]$$

$$= \frac{1}{3} (27 - 8)$$

$$= \frac{19}{3}$$

$= 6.33$ unit

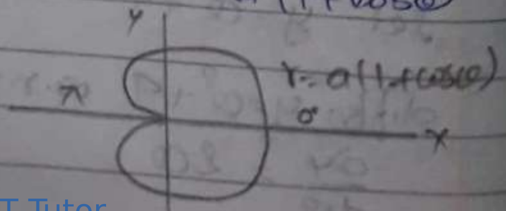
Hence the length of spiral is 6.33 unit.

* Find the length of cardioide $r = a(1+\cos\theta)$

→ For Cardiode

$\theta_1 = 0$

$\theta_2 = \pi$



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$\to r^2 = a(1+\cos\theta)$

$r^2 = a^2(1+2\cos\theta+\cos^2\theta)$

$\frac{dr}{d\theta} = \frac{a\,d(1+\cos\theta)}{dr} = -a\sin\theta$

$\therefore \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$

Now,

$$\text{Length}(L) = \int_{\theta=0}^{\theta=\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sqrt{a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta}\; d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} a\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\; d\theta$$

$$= a\int_{\theta=0}^{\theta=\pi} \sqrt{2(1+\cos\theta)}$$

$\cos^2\theta = \frac{1+\cos2\theta}{2}$

$2\cos^2\theta/2 = 1+\cos2\theta/2$

$\therefore 2\cos^2\theta/2 = 1+\cos\theta$

$$= a\int_{\theta=0}^{\pi} \sqrt{2.2\cos^2\theta/2} = 2\cos\theta/2$$

$$= 2a\left[\frac{+\sin\frac{\theta}{2}}{1/2}\right]_0^{\pi}$$

$$= 2a\times\frac{2}{1}\left[+\sin\frac{\pi}{2} - \sin\frac{0}{2}\right]$$

$= 4a(1-0)$

$= 4a$

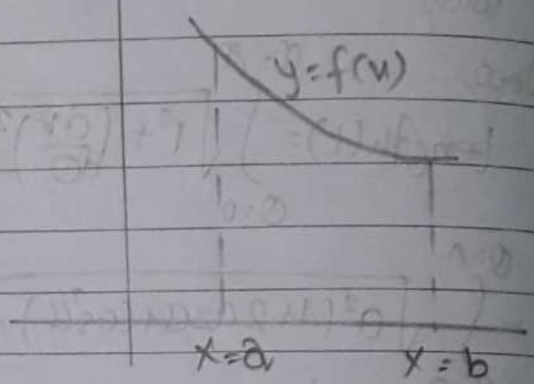
$\therefore$ total length of Cardioide $= 4a\times2$

$= 8a$ units

* Length for a function $y=f(x)$, $a \le x \le b$.

If a function is continuous and differentiable on closed interval $[a,b]$, then,

$$L=\int_{x=a}^{x=b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



For function, $x=f(y)$
$c \le y \le d$

$$L=\int_{x=c}^{x=d} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Note:

If $f(x)$ is not differentiable on closed interval $[a,b]$ then use $f(y)$ and vice-versa.

* Find the length of curve $y=x^{3/2}$, $0 \le x \le 1$.

→ Solution,

$$\frac{dy}{dx}=\frac{d\,x^{3/2}}{dx}=\frac{3}{2}x^{3/2-1}=\frac{3}{2}x^{1/2}$$

∵ It is differentiable at $x \in [0,1]$

Now,

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{3}{2}x^{1/2}\right) = 1+\frac{9}{4}x = \frac{4+9x}{4}$$

Now,

$$\text{length}(L)=\int_{x=0}^{x=1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^1 \sqrt{\frac{4+9x}{4}}\,dx$$

$$=\frac{1}{2}\int_0^1 \sqrt{4+9x}\,dx$$

$$= \frac{1}{2}\int_0^1 (4+9u)^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{(4+9u)^{1/2+1}}{\frac{3}{2}\times 9}\right]_0^1$$

$$= \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{3\times 9}\left[(4+9x)^{3/2}\right]_0^1$$

$$= \frac{1}{27}\,(4+9.1)^{3/2} - (4+9.0)^{3/2}$$

$$= \frac{1}{27}\,(13)^{3/2} - (4)^{3/2}$$

$$= \frac{1}{27}\times 46.87 - 2^{2\cdot\frac{3}{2}}$$

$$= \frac{38.87}{27}$$

$= 1.439$ unit

$\therefore$ length of curve is 1.439 unit.

* Find the length of curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x=0$ to $x=2$

→ Solution,

$$\cancel{\frac{dy}{dx} = \left(\frac{1}{2}\right)^{\frac{2}{3}} \times \frac{2}{3}(x)^{\frac{2}{3}-1} = \frac{1}{3\sqrt{x}}}$$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)^{2/3}\times\frac{2}{3}\,(x)^{\frac{2}{3}-1} = 0.62\times\frac{2}{3\sqrt{x}} = \frac{0.41}{\sqrt{x}}$$

when $x=0$ $= \frac{0.41}{0} = \infty$

so it is not differentiable at $x \in [0,2]$

Again,

$\frac{dx}{dy}$? so, $y = \left(\frac{x}{2}\right)^{2/3}$

$$y^{3/2} = \frac{x}{2} \quad \therefore x = 2y^{3/2}$$

$x = 2y^{3/2}$

when $x = 0$

$0 = 2y^{3/2}$

$\therefore y = 0$

when $x = 2$

$2 = 2y^{3/2}$

$y = 1$

Now,

$\frac{dx}{dy} = 2\frac{dy^{3/2}}{dy} = 2 \times \frac{3}{2} y^{3/2 - 1}$

$= 3y^{1/2}$

$\therefore 1 + \left(\frac{dy}{dy}\right)^2 = 1 + (3y^{1/2})^2 = 1 + 9y$

$\therefore \text{length} (L) = \int_{y=0}^{y=1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_0^1 \sqrt{1 + 9y}\, dy$

$= \int_0^1 (1 + 9y)^{1/2}\, dy$

$= \left[\frac{(1+9y)^{1/2 + 1}}{(1/2 + 1).9} \right]_0^1$

$= \frac{2}{3.9} (1 + 9.1)^{3/2} - (1 + 9.0)^{3/2}$

$= \frac{2}{3.9} (10)^{3/2} - 1$

$= \frac{61.24}{3 \times 9}$

$= \frac{61.24}{27}$

$= 2.26$ unit

Hence the length curve is 2.26 unit.

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* Find the length of curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad 0 \le x \le 1.$

→ Solution,

$$\frac{dy}{dx} = \frac{d\left(\frac{4\sqrt{2}}{3} x^{3/2} - 1\right)}{dx}$$

$$= \frac{4\sqrt{2}}{3} \times \frac{3}{2} x^{\frac{3}{2}-1} - 0$$

$$= 2\sqrt{2}\, x^{1/2}$$

$$= 2\sqrt{2x}$$

Now,

$$1 + \left(\frac{dy}{dx}\right)^2 = (2\sqrt{2x})^2 = 1 + 8x$$

Now,

$$\text{length } (l) = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1+8x}\quad dx$$

$$= \int_0^1 (1+8x)^{1/2}\, dx$$

$$= \left[\frac{(1+8x)^{1/2+1}}{(1/2+1)\cdot 8}\right]_0^1$$

$$= (1+8x)^{3/2} \times \frac{2}{3\times 8}$$

$$= \frac{2}{3} (1+8\cdot 1)^{3/2} - (1+8\times 0)^{3/2}$$

$$= \frac{2}{3\times 8} (3)^{2\cdot 3/2} - 1^{3/2}$$

$$= \frac{2\times 26}{24}$$

$$= 2.16 \text{ unit}$$

Hence the length of Curve is 2.16 unit

* Find the length of curve $y = \frac{1}{3}(x^2+2)^{3/2}$ $0 \le x \le 3$.

→ Solution,

$$\frac{dy}{dx} = \frac{1}{3} \times \frac{3}{2}(x^2+2)^{3/2-1} = \frac{1}{2}(x^2+2)^{1/2}$$

Now,

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{1}{2}(x^2+2)^{1/2}\right)^2$$

$$= 1+\frac{1}{4}(x^2+2)$$

$$= \frac{4+(x^2+2)}{4}$$

$$= \frac{6+x^2}{4}$$

$$\frac{dy}{dx} = \frac{1}{3}\frac{d(x^2+2)^{3/2}}{d(x^2+2)} \times \frac{d(x^2+2)}{dx}$$

$$= \frac{1}{3} \times \frac{3}{2}(x^2+2)^{3/2-1} \times 2x$$

$$= (x^2+2)^{1/2} \cdot 2x$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+4x^2(x^2+2)$$

→ Solution,

$$y = \frac{1}{3}(x^2+2)^{3/2}$$

Now,

$$\frac{dy}{dx} = \frac{1}{3}\frac{d(x^2+2)^{3/2}}{d(x^2+2)} \times \frac{d(x^2+2)}{dx}$$

$$= \frac{1}{3} \times \frac{3}{2}(x^2+2)^{1/2} \times 2x$$

$$= x(x^2+2)^{1/2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left\{x(x^2+2)^{1/2}\right\}^2 = 1+x^2(x^2+2)$$

$$= 1+x^4+2x^2$$

$= 1 + 2x^2 + x^4$

$= (1+x^2)^2$

Now,

$$\text{length}(L) = \int_0^3 \sqrt{(1+x^2)^2}\, dx$$

$$= \int_0^3 (1+x^2)\, dx$$

$$= [x]_0^3 + \left[\frac{x^3}{3}\right]_0^3$$

$$= (3-0) + \left(\frac{3^3}{3} - \frac{0}{3}\right)$$

$$= 3 + \frac{\cancel{27}\,9}{\cancel{3}}$$

$$= 12 \text{ unit.}$$

Hence the length of curve is 12 unit.

* Find the length of curve $x = \left(\frac{y^4}{4}\right) + \frac{1}{8y^2} \quad 1 \le y \le 2$

→ Solution,

$$\frac{dx}{dy} = \frac{d\left(\frac{y^4}{4}\right) + \frac{1}{8y^2}}{dy}$$

$$= \frac{1}{4} \times 4y^3 + \frac{1}{8} \times (-2) y^{-3}$$

$$= y^3 - \frac{1}{4y^3}$$

Now,

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(y^3 - \frac{1}{4y^3}\right)^2$$

$$= \left\{1 + y^3 - \cancel{2}.y^3.\frac{1}{\cancel{4}_2 y^3} + \left(\frac{1}{4y^3}\right)^2\right\}$$

$$= 1 - \frac{1}{2} + y^3 + \left(\frac{1}{4y^3}\right)^2$$

$$= (y)^3 + 2.y^3.\frac{1}{4y^3} + \left(\frac{1}{4y^3}\right)^2$$

$$= \left(y^3 + \frac{1}{4y^3}\right)^2$$

Now,

$$\text{Length} = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\, dy$$

$$= \int_1^2 \left(y^3 + \frac{1}{4y^3}\right) dy$$

$$= \left[\frac{y^4}{4}\right]_1^2 + \frac{1}{4}\left[\frac{y^{-3+1}}{-3+1}\right]_1^2$$

$$= \left(\frac{2^4 - 1}{4}\right) + \frac{1}{4}\left[\frac{2^{-2} - 1^{-2}}{-2}\right]$$

$$= \left(\frac{16-1}{4}\right) + \frac{1}{-8}\left[\frac{1}{2^2} - \frac{1}{1^2}\right]$$

$$= \frac{15}{4} - \frac{1}{8}\left(\frac{1}{4} - 1\right)$$

$$= 3.75 - 0.125\,(-0.75)$$

$$= 3.75 + 0.093$$

$$= 3.84375 \text{ units}$$

Hence the length of curve is 3.84 units.

* Find the length of curve $x = \frac{y^3}{3} + \frac{1}{4y}$, $1 \le y \le 3$

→ Solution,

$$\frac{dx}{dy} = \frac{d\left(\frac{y^3}{3}\right) + \frac{1}{4y}}{dx}$$

$$= \frac{1}{3} \times 3y^2 + \frac{1}{-4} y^{-2}$$

$$= y^2 - \frac{1}{4y^2}$$

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Now,

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^2-\frac{1}{4y^2}\right)^2$$

$$= 1+y^4-\cancel{2}\cdot \cancel{y^2}\cdot\frac{1}{_2 4\cancel{y^2}}+\left(\frac{1}{4y^2}\right)^2$$

$$= 1-\frac{1}{2}+(y^2)^2+\left(\frac{1}{4y^2}\right)^2$$

$$= \frac{1}{2}+(y^2)^2+\left(\frac{1}{4y^2}\right)^2$$

$$= (y^2)^2+2\cdot y^2\cdot\frac{1}{4y^2}+\left(\frac{1}{4y^2}\right)^2$$

$$= \left(y^2+\frac{1}{4y^2}\right)^2$$

Now,

$$\text{length } (L) = \int_1^3 \sqrt{1+\frac{dx}{dy}}\, dy$$

$$= \int_1^3 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\, dy$$

$$= \int_1^3 \left(y^2+\frac{1}{4y^2}\right) dy \qquad +\frac{1}{4y^2} = \frac{1}{4}y^{-2} = \frac{-2}{y^3}$$

$$= \left[\frac{y^3}{3}\right]_1^3 + \frac{-2}{4}\left[\frac{y^{-2+1}}{-2+1}\right]_1^3$$

$$= \left(\frac{3\times3\times3}{3}-\frac{1}{3}\right)+\frac{-2}{4}\left[\frac{y^{-1}}{-1}\right]_1^3$$

$$= \frac{26}{3}+\frac{2}{-4}\left[\frac{1}{3}-\frac{1}{1}\right)$$

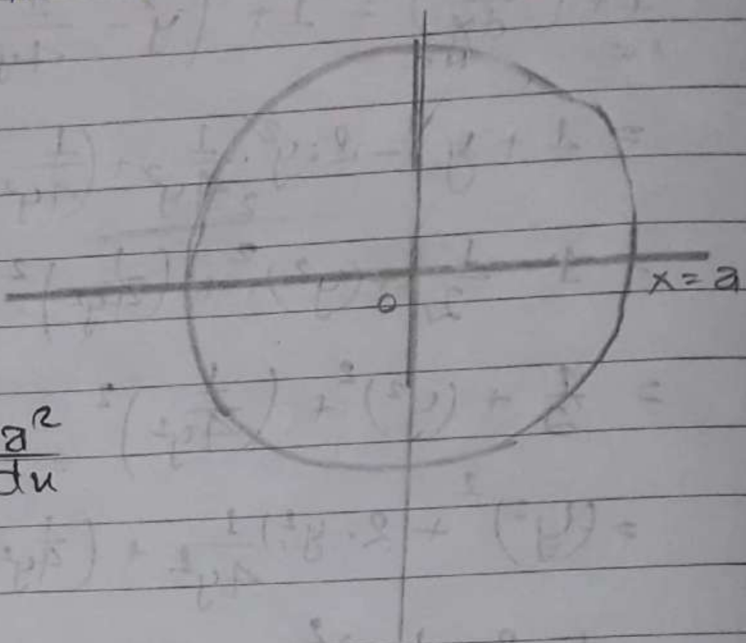
$$= 8.66 - 0.25\times(-0.666)$$

$$= 8.60 + 0.1666$$

$$= 8.766 \text{ unit}$$

$\therefore$ The length of curve is 8.766 unit.

* Find the length of the circle $x^2+y^2=a^2$



→ solution

$$x^2+y^2=a^2$$

differentiating both sides w. r. to x

we get

$$\frac{dx^2}{dx}+\frac{dy^2}{dy}\times\frac{dy}{dx}=\frac{da^2}{dx}$$

$$2x+2y\frac{dy}{dx}=0$$

$$\therefore 2y\frac{dy}{dx}=-2x$$

$$\therefore \frac{dy}{dx}=-\left(\frac{x}{y}\right)$$

$$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{y^2}$$

Now,

$$\text{length } (L)=\int_{x=0}^{x=a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_{x=0}^{x=a}\sqrt{1+\frac{x^2}{y^2}}\,dx$$

$$\boxed{\begin{aligned}\therefore x^2+y^2&=a^2\\ \therefore y^2&=a^2-x^2\end{aligned}}$$

$$=\int_{x=0}^{x=a}\sqrt{\frac{x^2+y^2}{y^2}}\,dx$$

$$=\int_{x=0}^{x=a}\sqrt{\frac{a^2}{a^2-x^2}}\,dx$$

$$=\int_{x=0}^{x=a}\frac{a\,dx}{\sqrt{a^2-x^2}}$$

For $\sqrt{a^2-x^2}$,

Put $x=a\sin\theta$

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diff' both sides,

$$\frac{dx}{d\theta} = \frac{d\,a\sin\theta}{d\theta}$$

or, $dx = a\cos\,d\theta$

when $x = 0$

$0 = a\sin\theta$

$\therefore \theta = \sin^{-1}(0) = 0$

when $x = a$

$a = a\sin\theta$

$\therefore \sin\theta = 1$

$\therefore \theta = \sin^{-1}(1) = \frac{\pi}{2}$

Now,

$$\text{Length}(l) = \int_0^{\pi/2} \frac{a\,dx}{\sqrt{a^2 - x^2}}$$

$$= \int_0^{\pi/2} \frac{a \cdot a\cos\theta\,d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{a^2\cos\theta\,d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \frac{a^2}{a} \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sqrt{\cos^2\theta}}$$

$$= a \int_0^{\pi/2} \frac{\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= a\left[\theta\right]_0^{\pi/2}$$

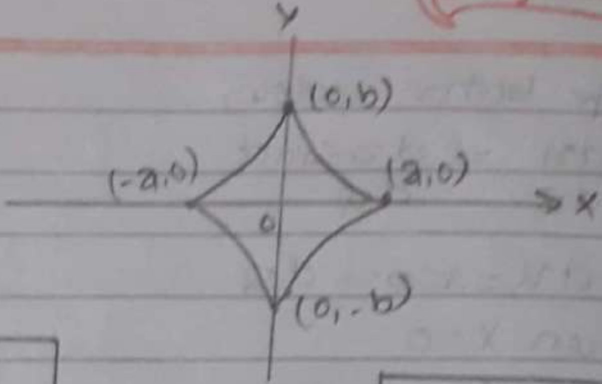
$$= a\left[\pi/2 - 0\right]$$

$$= \frac{\pi a}{2}$$

$\therefore$ Total Area of circle $= \frac{\pi a}{2} \times 4 = 2\pi a$ unit

Hypocycloid

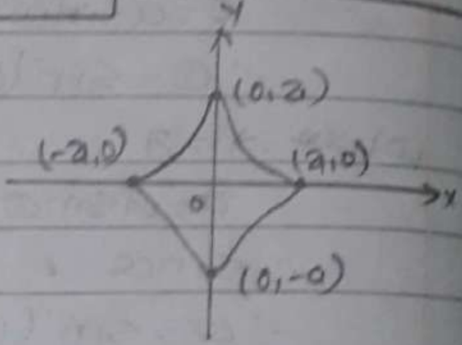
$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$



Astroid $(a=b)$

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$$\therefore (x)^{2/3} + (y)^{2/3} = a^{2/3}$$



* Find the perimeter (length) of asteroid $x^{2/3}+y^{2/3}=a^{2/3}$

→ Solution:

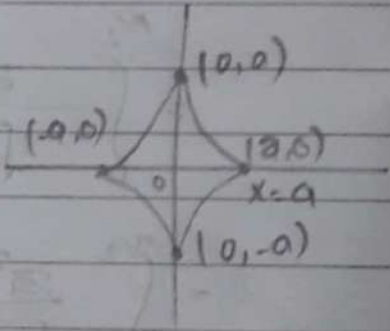
differentiating both sides we get.

$$\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1}\frac{dy}{dx} = 0$$

$$\frac{2}{3\sqrt[3]{x}} + \frac{2}{3\sqrt[3]{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cancel{2}}{\cancel{3}\sqrt[3]{x}} \times \frac{\cancel{3}\sqrt[3]{y}}{\cancel{2}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \left(\frac{\sqrt[3]{y}}{\sqrt[3]{x}}\right)^2 = \frac{y^{2/3}}{x^{2/3}}$$



Now,

$$\text{Length}(L) = \int_0^a \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx$$

$$= \int_0^a \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\,dx \qquad [x^{2/3}+y^{2/3}=a^{2/3}]$$

$$= \int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx$$

$$= \int_0^a \sqrt{\left(\frac{a}{x}\right)^{2/3}}\, dx$$

$$= \int_0^a \left(\frac{a}{x}\right)^{2/3 \times 1/2} dx$$

$$= a^{1/3} \int_0^a \left(\frac{1}{x}\right)^{1/3} dx$$

$$= a^{1/3} \int_0^a x^{-1/3}\, dx$$

$$= a^{1/3} \left[\frac{x^{-1/3+1}}{-1/3+1}\right]_0^a$$

$$= a^{1/3} \cdot \times \frac{3}{2} \left[x^{2/3}\right]_0^a$$

$$= \frac{3a^{1/3}}{2} \left[a^{2/3} - 0^{2/3}\right]$$

$$= \frac{3a^{1/3}}{2} \cdot a^{2/3}$$

$$= \frac{3}{2} \times a^{1/3+2/3}$$

$$= \frac{3a}{2}$$

Total $\therefore$ Area of the asteroid is $= 4 \times \frac{3a}{2} = 6a$ unit.

Length of Parametric Curve:

If $x = f(t)$ and $y = g(t)$ and $a \le t \le b$

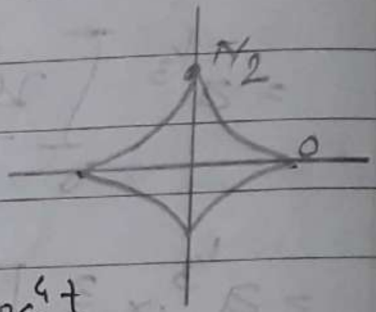
then,

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

* Find the length of asteroid $x = \cos^3 t$ and $y = \sin^3 t$

$0 \le t \le \pi/2$

→ Solution,



$$\frac{dx}{dt} = \frac{d\cos^3 t}{d\cos t} \times \frac{d\cos t}{dt}$$

$$= 3\cos^2 t \times (-\sin t)$$

$$\therefore \frac{dx}{dt} = -3\sin t \cdot \cos^2 t \qquad \left(\frac{dx}{dt}\right)^2 = 9\sin^2 t \cdot \cos^4 t$$

$$\frac{dy}{dt} = \frac{d\sin^3 t}{d\sin t} \times \frac{d\sin t}{dt}$$

$$= 3\sin^2 t \times \cos t$$

$$\therefore \frac{dy}{dt} = 3\sin^2 t \cdot \cos t \qquad \left(\frac{dy}{dt}\right)^2 = 9\sin^4 t \cdot \cos^2 t$$

Now,

$$L = \int_{t=0}^{t=\pi/2} \sqrt{9\sin^2 t \cdot \cos^4 t + 9\sin^4 t \cdot \cos^2 t}\, dt$$

$$= \int_{t=0}^{t=\pi/2} \sqrt{9\sin^2 t \cos^2 t\,(\cos^2 t + \sin^2 t)}\, dt$$

$$= 3 \int_{t=0}^{t=\pi/2} \sin t \cdot \cos t \, dt$$

$$= \frac{3}{2} \int_{t=0}^{t=2\pi} 2 \cdot \sin t \cdot \cos t \, dt$$

$$= \frac{3}{2} \int_{t=0}^{t=2\pi} \sin 2t \, dt$$

$$= \frac{3}{2} \left[\frac{-\cos 2t}{2} \right]_0^{2\pi}$$

$$= \frac{3}{2} \left[\frac{-\cos 2.2\pi}{2} + \frac{\cos 2.0}{2} \right]$$

$$= \frac{3}{4} \left(- \right.$$

$$= \frac{3}{4}$$

∴ Total length of asteroid $= 4\times$

Now,

$$L = \int_{t=0}^{t=\pi/2} \sqrt{9\sin^2 t \cdot \cos^2 t\,(\cos^2 t + \sin^2 t)}\,dt$$

$$= 3\int_0^{\pi/2} \sin t \cdot \cos t \cdot dt$$

$$= \frac{3}{2}\int_0^{\pi/2} 2 \cdot \sin t \cdot \cos t \; dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t \; dt$$

$$= \frac{3}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{3}{2}\left[-\frac{\cos 2.\pi/2}{2} + \frac{\cos 2.0}{2}\right]$$

$$= \frac{3}{4}\,[+1+1]$$

$$= \frac{3}{4}\times 2$$

$$= \frac{3}{2}$$

∴ Total length of asteroid $= 4 \times \frac{3}{2}$

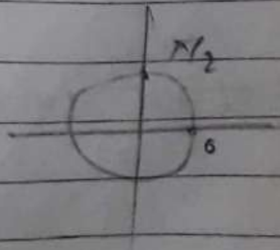
$= 6$ unit.

* Find the length of circle.

$x = a\cos t$, $y = a\sin t$, $0 \le x \le 2\pi$

→ Solution,

here,



we have,

$x = a\cos t$ $\qquad$ $y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ $\qquad$ $\frac{dy}{dt} = a\cos t$

Now,

$$\text{length} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} a\, dt$$

$$= a\,[t]_0^{2\pi}$$

$$= a \times [2\pi - 0]$$

$$= 2\pi a \text{ unit.}$$

Hence length of circle is $2\pi a$ unit.

Mean Value Theorem

1. Rolle's Theorem
2. Langrange's Mean Value Theorum

1> Rolle's Theorem :-

It a function $f(x)$ is,

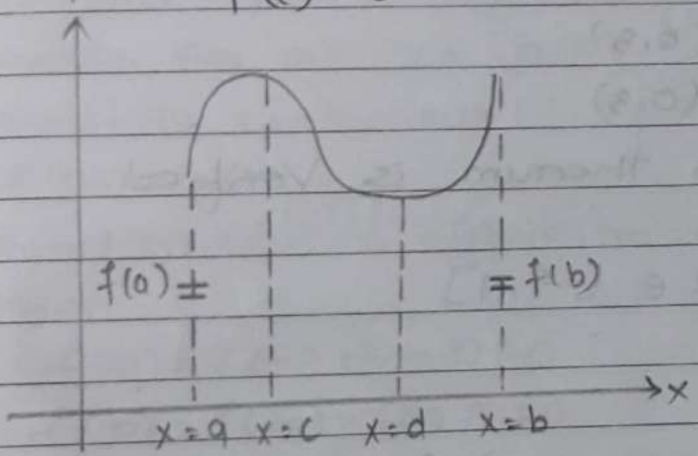
a> continuous in closed interval $[a,b]$

b> differentiable (derivable) in open interval (a,b)

c> $f(a) = f(b)$,

then there exists at least a point $c \in (a,b)$ such that,

$f'(c) = 0$



* verify Rolle's Theorum,

a> $f(x) = x(x-3)^2$, $x \in [0,3]$

→ solution,

$f(x) = x(x^2 - 3.2.x + 9)$

$= x(x^2 - 6x + 9)$

$\therefore f(x) = x^3 - 6x^2 + 9x$

function $f(x)$ is a polynomial function which exists for all $x \in [0,3]$

so it is continuous.

$f'(x) = 3x^2 - 12x + 9$

function $f(x)$ is differentiable for all $x \in (0,3)$

then,

$f(a) = f(b)$

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$f(a) = f(0) = 0^3 - 6\times 0^2 + 0 = 0$
$f(b) = f(3) = 3^3 - 6\cdot 3^2 + 9\cdot 3 = 27 - 54 + 27 = 0$
$\because f(a) = f(b)$
then
there exists at least a point $c \in (0,3)$, such that
$f'(c) = 0$
or, $3c^2 - 12c + 9 = 0$
or, $c^2 - 4c + 3 = 0$
or, $c^2 - 3c - c + 3 = 0$
or, $c(c-3) - 1(c-3) = 0$
or, $(c-3)(c-1) = 0$
Either,
$c = +3, +1$

$\therefore c = 3 \notin (0,3)$
$\therefore c = 1 \in (0,3)$
Hence Rolle's therum is Verified.

b) $f(x) = \sqrt{16 - x^2}$, $x \in [-4, 4]$
→ solution,
$f(x) = \sqrt{16 - x^2}$
function $f(x)$ is polynomial function, so it is continuous, which exists for all $[-4,4]$

$$f'(x) = \frac{d(16-x^2)^{1/2}}{d(16-x^2)} \times \frac{d(16-x^2)}{dx}$$
$$= \frac{1}{2}(16-x^2)^{1/2 - 1} \times -2x$$
$$= \frac{-x}{\sqrt{16-x^2}}$$

function $f(x)$ is differentiable at open interval $x \in (-4, 4)$

Now,
$f(a) = f(-4) = \sqrt{16 - (-4)^2} = 0$
$f(b) = f(4) = \sqrt{16 - 4^2} = 0$
$\therefore f(a) = f(b)$

then there exists at least a point $c \in (-4,4)$,
such that,

$$f'(c) = 0$$

or, $\frac{-c}{\sqrt{16-c^2}} = 0$

or, $c = 0 \in (-4,4)$

Hence Rolle's theorum is verified.

c> verify Rolle's theorum,

$f(x) = \sin x \quad , x \in [0,\pi]$

→ function $f(x)$ is trignometry function which exists for all $x \in [0,\pi]$
so it is continuous.

$f'(x) = \cos x$

function $f(x)$ is differentrable for all $x \in (0,\pi)$
then,

$f(a) = f(0) = \sin 0 = 0$

$f(b) = f(\pi) = \sin \pi = 0$

then there exists at least a point $c \in (0,\pi)$,
Such that,

$$f'(c) = 0$$

$$\cos c = 0$$

$$\therefore c = \cos^{-1}(0)$$

$$\therefore c = \pi/2 \in (0,\pi)$$

Hence Rolle's theorum is Verified.

d> $f(x) = \cos 2x \quad , x \in [\pi,-\pi]$

→ solution

function $f(x)$ is trigonometry function which exists for all $x \in [\pi,-\pi]$
so it is continuous.

$f'(x) = -2\sin 2x$

~~So~~ function $f(x)$ is differentrable for $x \in (-\pi, \pi)$

Now,

$f(a) = f(-\pi) = \cos 2(-\pi) = 1$

$f(b) = f(\pi) = \cos 2\pi = 1$

$\therefore f(a) = f(b)$

then there exists at least a ~~constant~~ point $c \in (-\pi, \pi)$ such that,

$f'(c) = 0$

$-2\sin 2c = 0$

$\sin 2c = 0$

$2c = \sin^{-1}(0)$

$\therefore c = 0/2 = 0 \in (-\pi, \pi)$

Hence Rolle's theorum Verified.

e) $f(x) = e^x(\sin x - \cos x)$ $\quad x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

→ Solution,

function $f(x)$ is trigronometry and exponential function which exists for all $x \in [\pi/4, \frac{5\pi}{4}]$

So, it is continous. $\boxed{\frac{d}{dx} u.v = u\frac{dv}{dx} + v\frac{du}{dx}}$

$f'(x) = e^x \frac{d(\sin x - \cos x)}{dx} + (\sin x - \cos x)\frac{de^x}{dx}$

$= e^x(\cos x + \sin x) + (\sin x - \cos x)e^x$

$= e^x(\cos x + \sin x + \sin x - \cancel{\cos x})$

$= e^x\, 2\sin x$

function $f(x)$ is differentiable for all $x \in (\pi/4, \frac{5\pi}{4})$

then $f(a) = f(b)$

Now.

$f(a) = f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\pi/4} \times (0.7 - 0.7)$
$= 0$

$f(b) = f\left(\frac{5\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right)$

$= e^{\pi/4}(-0.70 + 0.70)$

$= 0$

$\therefore f(a) = f(b)$,

then,

there exists at least a ~~constant~~ point $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

such that,

$f'(c) = 0$

or, $e^c 2\sin c = 0$

or, $\sin c = \frac{0}{2e^c}$

or, $c = \sin^{-1} 0$

or, $c = 0, \pi$

$c = 0 \notin \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Hence Rolle's theorum is verified.

Lagrange Mean Value Theorem (MVT)

If a function $f(x)$ is

a) continuous in closed interval $[a,b]$

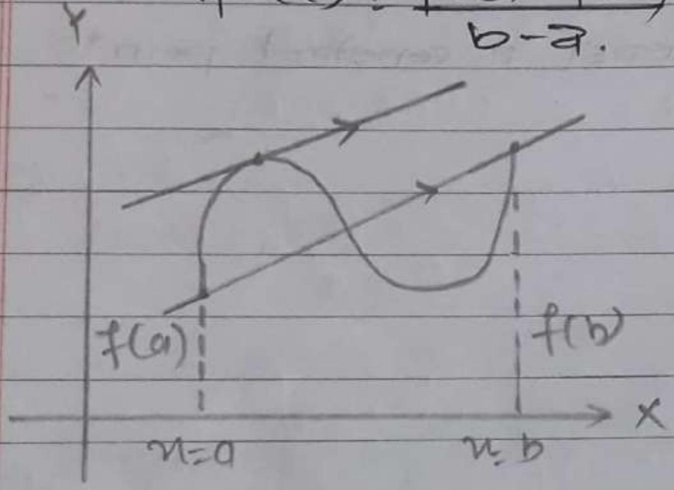
b) differentiable in open interval (a,b)

then,

there exists at least a point $c \in (a,b)$

such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



* verify Mean Value Theorum for the following functions.

a) $f(x) = x(x-1)(x-2) \quad x \in [0, +\frac{1}{2}]$

→ Solution,

$$f(x) = x^2 - x(x-2)$$
$$= x^2(x-2) - x(x-2)$$
$$= x^3 - 2x^2 - x^2 + 2x$$
$$= x^3 - 3x^2 + 2x$$

since the function $f(x)$ is polynomial function which exists for all $x \in [0, \frac{1}{2}]$

$f'(x) = 3x^2 - 6x + 2$

function $f(x)$ is differentiable for all $x \in (0, \frac{1}{2})$

then

there exists at least a point $c \in (0, \frac{1}{2})$

such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$f(a) = f(0) = 0^3 - 3 0^2 + 2\times 0 = 0$

$f(b) = f(\frac{1}{2}) = (\frac{1}{2})^3 - 3(\frac{1}{2})^2 + \frac{1}{2}\times 2$

$= \frac{1}{8} - \frac{3}{4} + 1$

$= \frac{1}{8} - \frac{3}{4} + 1 \quad = \frac{1-6+8}{8} = \frac{3}{8}$

Now,

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0}$

$3c^2 - 6c + 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}^1}{1}$

$12c^2 - 24c + 8 = 3$

$12c^2 - 24c + 5 = 0 \quad \text{---(i)}$

Comparing eq^n (i) with $ax^2 + bx + c$

where $a = 12 \quad b = -24 \quad c = 5$

Now,

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{+24 \pm \sqrt{(-24)^2 - 4\times 12 \times 5}}{2\times 12}$

$= \frac{24 \pm 18.33}{24}$

$= \frac{42.33}{24}, \frac{5.67}{24}$

$= 1.76, 0.236$

$\therefore c = 1.76 \notin (0, \frac{1}{2})$

$\therefore c = 0.236 \in (0, \frac{1}{2})$

Hence Mean value Theorum is verified.

b.) $f(x) = \sqrt{x^2-4}$, $x \in [2, 4]$

→ Solution,

$$f(x) = (x^2-4)^{1/2}$$

function $f(x)$ is polynomial function which exists for all $x \in [2,4]$

so it is continuous

$$f'(x) = \frac{d(x^2-4)^{1/2}}{d(x^2-4)} \times \frac{d(x^2-4)}{dx}$$

$$= \frac{1}{2}(x^2-4)^{1/2-1} \times 2x$$

$$= \frac{1}{\cancel{2}\sqrt{x^2-4}} \times \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2-4}}$$

function $f(x)$ is differentiable at open interval for all $x \in (2,4)$

then,

there exists at least a point $c \in (2,4)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(a) = f(2) = \sqrt{2^2-4} = 0$$

$$f(b) = f(4) = \sqrt{16-4} = \sqrt{12} = 2\sqrt{3}$$

Now,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, $\frac{c}{\sqrt{c^2-4}} = \frac{2\sqrt{3}-0}{4-2}$

squaring on both sides

$$\frac{c^2}{c^2-4} = \frac{\cancel{2}\cancel{2}3}{\cancel{2}}$$

$$c^2 = 3c^2 - 12$$

or, $2c^2 - 12 = 0$

or, $c^2 - 6 = 0$

$\therefore c^2 = 6$

$\therefore c = \sqrt{6}$

$\therefore c = \pm 2.44$

$c = 2.44 \in (2,4)$

Hence, Mean Value Theorum verified.

c) $f(x) = Ax^2 + Bx + C, \quad x \in [a,b]$

→ Solution,

function f(x) is polynomial function which exists for all $x \in [a,b]$

so it is continuous.

$f'(x) = 2Ax + B$

function f(x) is differentiable for all $x \in (a,b)$

then,

there exists at least a point $c \in (a,b)$

such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$f(a) = f(a) = Aa^2 + Ba + C$

$f(b) = Ab^2 + Bb + C$

$$2Ac + B = \frac{Ab^2 + Bb + C - (Aa^2 + Ba + C)}{b-a}$$

$2Abc + \cancel{Bb} - 2Aac - \cancel{aB} = Ab^2 + \cancel{Bb} + \cancel{C} - Aa^2 - \cancel{Ba} - \cancel{C}$

$2Ac(b-a) = A(b^2 - a^2)$

$$2c = \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}}$$

$$c = \frac{(a+b)}{2} \in (a,b)$$

Hence Mean Value Theorum is verified.

d) $f(x) = e^x, \; x \in [0,1]$

→ Solution,

$f(x) = e^x$ is exponential function which exists for all $x \in [0,1]$

so it is continuous.

$f'(x) = e^x$

function $f(x)$ is differentiable for all $x \in (0,1)$

then,

there exists at least one point $c \in (0,1)$

such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f(a) = f(0) = e^0 = 1$$

$$f(b) = f(1) = e^1 = 2.71$$

then,

$$e^c = \frac{2.71 - 1}{1-0}$$

$$e^c = 1.71$$

$$c = \ln(1.71)$$

$$c = 0.53 \in (0,1)$$

Hence Mean Value Theorem is verified.

e.) $f(x) = \log x, \ x \in [1,e]$

→ Solution,

function $f(x)$ is logarithm function which exists for all $x \in [1,e]$. So it is continuous.

$$f'(x) = \frac{d \log x}{dx} = \frac{1}{x}$$

which is differentiable for all $x \in (1,e)$

then,

there exists at least a point $c \in (1,e)$

such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$\therefore f(a) = f(1) = \log(1) = 0$

$\therefore f(b) = f(e) = \log(e) = \log(2.71) = 0.996$

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$\therefore f'(c) = \frac{f(b) - f(a)}{b-a}$

or, $\frac{1}{c} = \frac{0.996 - 0}{e-1}$

or, $0.996c = e-1$

or, $0.996c = 2.71 - 1$

or, $c = \frac{1.71}{0.996} = 1.71 \in (1, e)$

$\therefore c = 1.71 \in (1, e)$

Hence Mean Value Therum is verified.

f) $f(x) = x^3 + x^2 - 6x, \; x \in [-1, 4]$

→ Solution,

function $f(x)$ is polynomial function which exists for all $x \in [-1, 4]$,

So it is continuous.

$f'(x) = 3x^2 + 2x - 6$

function $f(x)$ is differentiable for all $x \in (-1, 4)$

then,

there exists at least a point $c \in (-1, 4)$, such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$f(a) = f(-1) = (-1)^3 + (-1)^2 - 6 \times (-1) = +6$

$f(b) = f(4) = 4^3 + 4^2 - 6 \times 4 = 64 + 16 - 24 = 56$

Now,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 + 2c - 6 = \frac{56 - 6}{4+1}$$

or, $3c^2 + 2c - 6 = \frac{50}{5}$

or, $3c^2 + 2c - 16 = 0$

or $3c^2 + 8c - 6c - 16 = 0$

or $c(3c+8) - 2(3c+8) = 0$

or $(3c+8)(c-2) = 0$

Either

$c = +2 \in (-1, 4)$

$c = -\frac{8}{3}$

Hence MVT verified.

g) $f(x) = (x-1)(x-2)(x-3)$, $x \in [1, 4]$

→ Solution,

function $f(x) = (x-1)(x-2)(x-3)$

$= x(x-2) - 1(x-2)\ (x-3)$

$= (x^2 - 2x - x + 2)(x-3)$

$= (x^2 - 3x + 2)(x-3)$

$= x(x^2 - 3x + 2) - 3(x^2 - 3x + 2)$

$= x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$

$= x^3 - 6x^2 + 11x - 6$

function $f(x)$ is polynomial function which exists for all $x \in [1, 4]$, so

It is continuous.

$f'(x) = 3x^2 - 12x + 11$

so, it is differentiable at open interval for all $x \in (1, 4)$

then,

there exists at least a point $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or, $f(b) = f(4) = 4^3 - 6 \times 4^2 + 11 \times 4 - 6 = 6$

$f(a) = f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$

Now,

$$3c^2 - 12c + 11 = \frac{6-0}{4-1}$$

$3c^2 - 12c + 11 = 2$

or $3c^2 - 12c + 9 = 0$

or $c^2 - 4c + 3 = 0$

or $c^2 - 3c - c + 3 = 0$

or, $c(c-3) - 1(c-3) = 0$

or, $(c-1)(c-3) = 0$

$\therefore c = 1, 3$

$c = 3 \in (1, 4)$

Hence Mean Value Theorum is verified.

h.) $f(x) = x + \frac{1}{x}, x \in [1/2, 2]$

→ Solution,

function f(x) is polynomial function which exists for all $x \in [1/2, 2]$, so it is continuous

$f'(x) = 1 + (-1)x^{-2}$

$= 1 - \frac{1}{x^2}$

it is differentiable at open interval $x \in (1/2, 2)$ then,

there exists at least a point $c \in (1/2, 2)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$\therefore f(a) = f(1/2) = \frac{1}{2} + \frac{1}{1/2} = 0.5 + 2 = 2.5$

$\therefore f(b) = f(2) = 2 + \frac{1}{2} = 2 + 0.5 = 2.5$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $1 - \frac{1}{c^2} = \frac{2.5 - 2.5}{2 - \frac{1}{2}}$

or, $c^2 - 1 = 0$

or, $c^2 = 1$

$\therefore c = \sqrt{1} = \pm 1$

$c = +1 \in (\frac{1}{2}, 2)$

Hence Mean Value Theorum is Verified.

i.> $f(x) = x^{2/3}, x \in [0, 1]$

→ solution,

function $f(x)$ is polynomial function which exists for all $x \in [0, 1]$,

so it is continuous.

$f'(x) = \frac{2}{3} x^{2/3 - 1}$

$= \frac{2}{3x^{1/3}}$

$= \frac{2}{3\sqrt[3]{x}}$

It is differentiable for all $x \in (0, 1)$,

then,

there exists at least a point $c \in (0, 1)$.

Such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \because f(a) = f(0) = 0$$
$$f(b) = f(1) = 1$$

or, $\frac{2}{3\sqrt[3]{c}} = \frac{1 - 0}{1 - 0}$

Cubeing on both sides

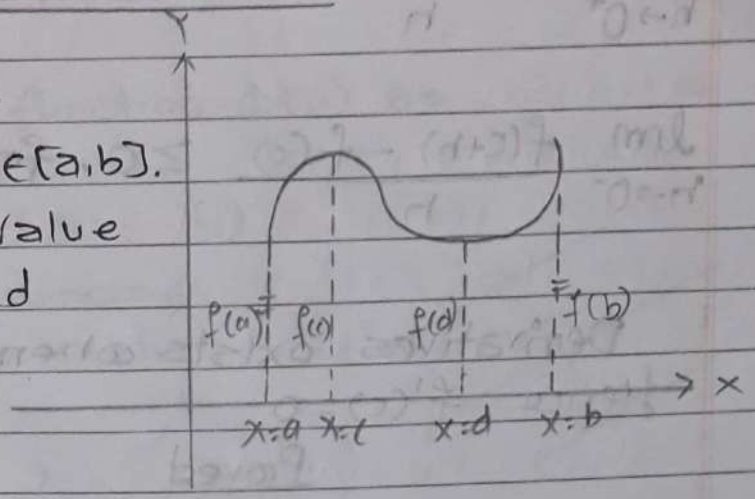
$8 = 27c$

$\therefore c = \frac{8}{27} = 0.29 \in (0, 1)$

Hence MVT is Verified.

* Proof for Rolle's Theorem :

→ Function $f(x)$ is continuous in $x \in [a,b]$. It has greater value say $f(c) = M$ and least value say $f(d) = m$



a.) If $M = m$,
then function is constant and $f'(x) = 0$

b.) If $M \neq m$, then M is different from either $f(a)$ or $f(b)$. Since it is differentiable.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

$$\text{or, } f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If $f(c)$ is greater than $f(c+h)$ for +ve and −ve value of h.

If h is positive :

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

If h is negative :

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

taking limit $h \to 0$ for Right hand Derivatives and LHD.

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0, \text{ for RHD}$$

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0, \text{ for LHD}$$

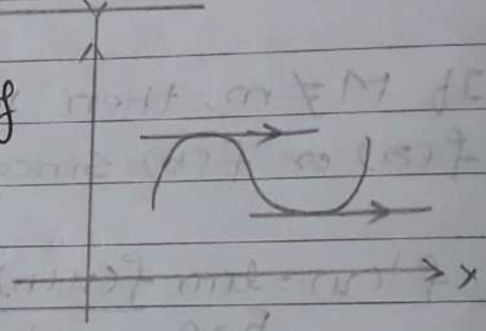
Derivatives exists when RHD = LHD = 0

Hence $f'(c) = 0$

Proved

Geometrical Interpretation :- (Rolle's Theorum)

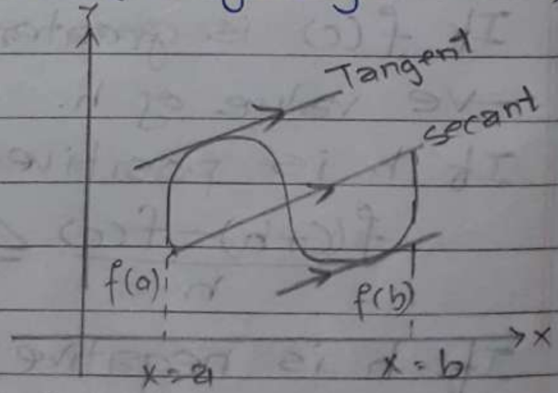
→ In a continuous curve of function $f(x)$, tangent can be drawn from each point of the curve.



There exists at least a point where tangent can be drawn which is parallel to x-axis.

Geometrical Interpretation :- (Langrange MVT)

→ In the continuous curve of function $f(x)$, tangent can be drawn from each point of the curve.



There exists at least a point where tangent can be drawn which is parallel to the secand joining the ends points of the function.

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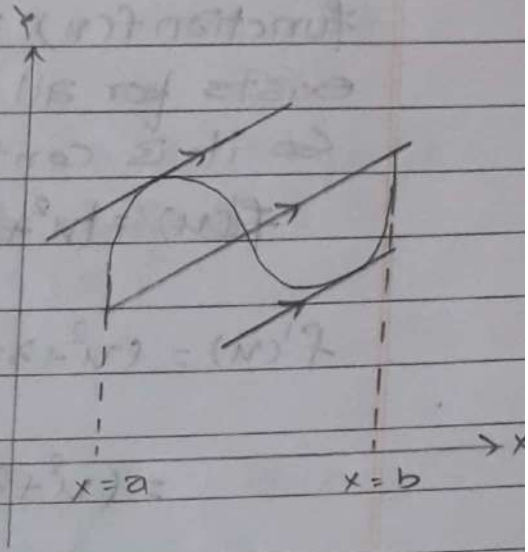
Proof (Lagrange Mean Value Theorem)

Defining a new function $\phi(x)$ as

$$\phi(x) = f(x) + Ax \text{ --- (i)}$$



where A is a constant to be determined such that,

$$\phi(a) = \phi(b)$$

or, $f(a) + Aa = f(b) + Ab$

or, $A(a-b) = f(b) - f(a)$

or, $A = -\dfrac{f(b)-f(a)}{b-a}$

Putting the value of A in eq^n (1),

$$\phi(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$$

and

$$\phi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

Since the function $\phi(x)$ is continuous in $x \in [a,b]$, differentiable in $x \in (a,b)$ and $\phi(a) = \phi(b)$. Then by Rolle's Theorum there exists at least a point $c \in (a,b)$, such that,

$$\phi'(c) = 0$$

or, $f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$

or $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Hence proved

* verify Rolle's Theorum:

$f(x) = x(x+3)e^{-x/2}$, $x \in [-3, 0]$

→ solution,

function f(x) is polynomial function which exists for all $x \in [-3,0]$.
So it is continuous

$$f(x) = (x^2+3x)e^{-x/2}$$

$$f'(x) = (x^2+3x)\frac{d e^{-x/2}}{dx} + e^{-x/2}\frac{d(x^2+3x)}{dx}$$

$$= (x^2+3x)\left(-\tfrac{1}{2}\right)e^{-x/2} + e^{-x/2}(2x+3)$$

$$= e^{-x/2}\left(\frac{(x^2+3x)(-1)}{2} + (2x+3)\right)$$

$$= e^{-x/2}\left(\frac{x^2+3x-4x-6}{2}\right)$$

$$= \frac{e^{-x/2}}{2}(x^2-x-6)$$

which is differentiable for all $x \in (-3,0)$

then,

$$f(a) = f(-3) = (-3)^2 + 3(-3)\,e^{+\frac{3}{2}}$$

$$= 9 - 9\,e^{-3/2}$$

$$= 0.$$

$f(b) = f(0) = 0$

$f(a) = f(b) = 0$

then there exists at least a point $c \in (-3,0)$

Such that

$$f'(c) = 0$$

$$\frac{e^{-c/2}}{2}(c^2-c-6) = 0$$

$$c^2-c-6=0$$

or, $c^2 - 3c + 2c - 6 = 0$

or $c(c-3) + 2(c-3)$

or $(c-3)(c+2) = 0$

either,

$$c = +3 \notin (-3, 0)$$

$$c = -2 \in (-3, 0)$$

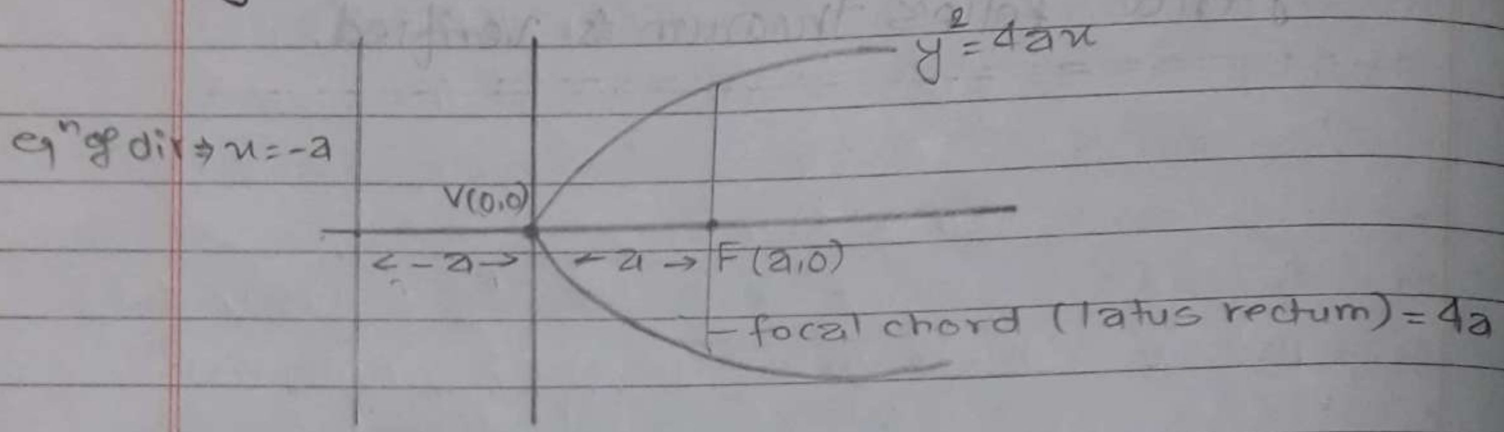
Hence Rolle's theorem is verified.

Conic Section

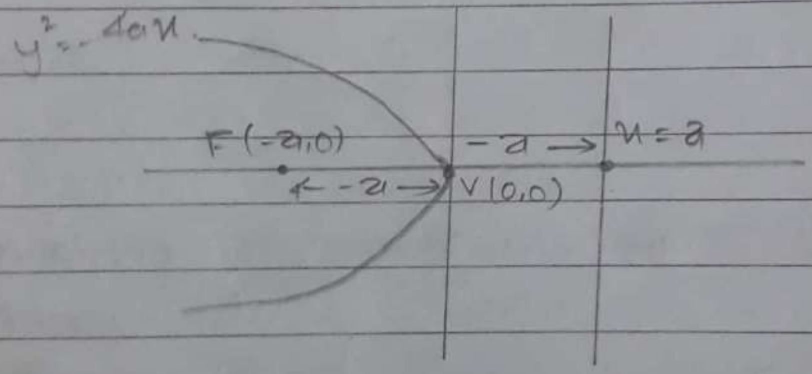
1.) Parabola $(e = 1)$
2.) Ellipse $(e < 1)$
3.) Hyperbola $(e > 1)$

"Parabola"

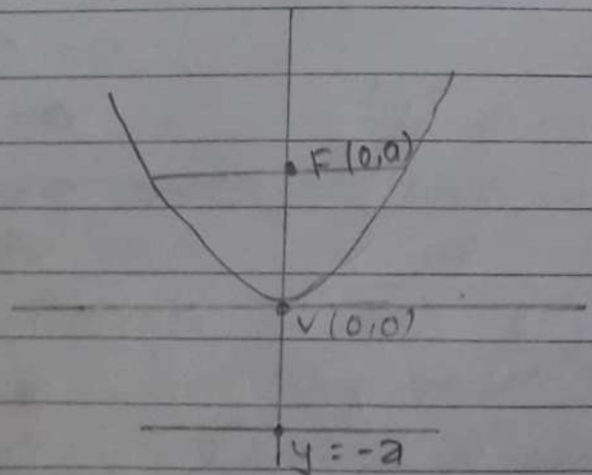
1.) $y^2 = 4ax$



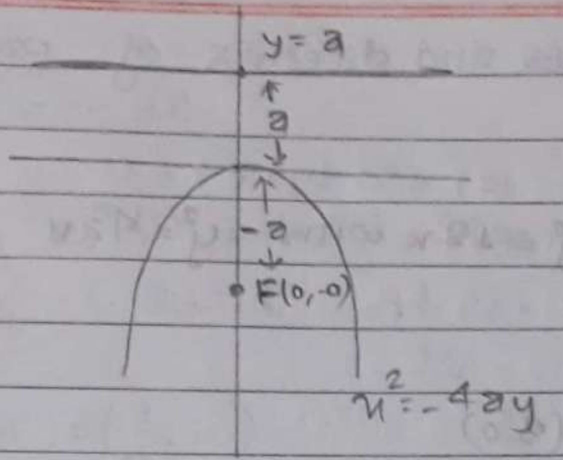
2.) $y^2 = -4ax$



3. $x^2 = 4ay$



4) $x^2 = -4ay$



The important results of parabola are tabulated as.

	Equation of Parabola	Vertex	Focus	Equation of directrix	length of latus rect.
1.	$y^2 = 4ax$ $y^2 = -4ax$	(0,0)	(a,0) (-a,0)	$x = -a$ $x = a$	4a
2.	$x^2 = 4ay$ $x^2 = -4ay$	(0,0)	(0,a) (0,-a)	$y = -a$ $y = +a$	4a
3.	$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	$x = h-a$	4a
4.	$(x-h)^2 = 4a(y-k)$	(h,k)	(h,k+a)	$y = k-a$	4a

* Find the focus and directrix of parabola $y^2 = 10x$.

→ Solution,

$y^2 = 10x$

Comparing $y^2 = 10x$ with $y^2 = 4ax$

$\therefore 4a = 10$

$\therefore a = \frac{5}{2}$

$\therefore$ Focus $= (a,0) = (5/2, 0)$

eq^n of directrix $= x + a = 0$

or, $x + \frac{5}{2} = 0$ $\boxed{\therefore 2x + 5 = 0}$

2. Find the focus and directrix of parabola.

a.) $y^2 = 12x$

→ solution

Comparing $y^2 = 12x$ with $y^2 = 4ax$

$y^2 = 4.3x$

$\therefore a = 3$

$\therefore$ Focus (F) = (3, 0)

equation of directrix is $x = -a$

$\therefore x = -3$

$x + 3 = 0$ //.

b.) $x^2 = 6y$

Comparing $x^2 = 6y$ with eqn $x^2 = 4ay$

$\therefore 4a = 6$

$\therefore a = \frac{6}{4} = \frac{3}{2}$

$\therefore$ Focus (F) = $(0, \frac{3}{2})$ directrix = $y = -3/2$

$y + 3/2 = 0$ //

c.) $x^2 = -8y$

→ Solution,

Comparing $x^2 = -8y$ with $x^2 = -4ay$

$\therefore 4a = 8$

$\therefore a = 2$

$\therefore$ Focus (F) = (0, -2)

directrix = $y = 2$

$\therefore y - 2 = 0$

d.) $y = -8x^2$

→ Solution,

$y = -8x^2$

$x^2 = -\frac{1}{8}y$ ---(i)

Comparing (i) with $x^2 = -4ay$

$\therefore 4a = \frac{1}{8}$

$\therefore a = \frac{1}{32}$ Focus (F) = $(0, -1/32)$

directrix (y) = 1/32

e) $x = -3y^2$

or, $y^2 = -\frac{1}{3}x$ ---(i)

Comparing eqⁿ (i) with $y^2 = -4ax$

$4a = \frac{1}{3}$ $\therefore a = 12^{-1} = \frac{1}{12}$

Focus (F) $= (-\frac{1}{12}, 0)$

directrix $(x) = \frac{1}{3\times4} = \frac{1}{12}$ #.

f) $x = 2y^2$

→ $y^2 = \frac{1}{2}x$ --(i)

Comparing eqⁿ (1) with $y^2 = 4ax$

$\therefore 4a = \frac{1}{2}$

$\therefore a = \frac{1}{8}$ so Focus $= (\frac{1}{8}, 0)$

directrix $(x) = -\frac{1}{8}$

$8x + 1 = 0$

g) $(y+3)^2 = 2(x+2)$

→ Solution,

Comparing $(y+3)^2 = 2(x+2)$ with $(y-k)^2 = 4a(x-h)$

where,

$h = -2, k = -3$ $4a = 2$ $\therefore a = \frac{1}{2}$

we have eqⁿ of directrix $(x) = h - a$

or, $x = -2 - \frac{1}{2}$

$= \frac{-4-1}{2}$

$= -\frac{5}{2}$ #

Vertex $= (h, k) = (-2, -3)$

Focus (F) $= (h+a, k)$

$= (-2 + \frac{1}{2}, -3)$

$= (-\frac{3}{2}, -3)$ #.

h.> $x^2 + 4x + 4y + 16 = 0$

→ Solution,

$x^2 + 2.x.2 + (2)^2 - (2)^2 + 4(y+4) = 0$

or $(x+2)^2 - 4 + 4(y+4) = 0$

or, $(x+2)^2 - 4 + 4y + 16 = 0$

or $(x+2)^2 + 4y + 12 = 0$

or $(x+2)^2 + 4(y+3) = 0$ --(i)

Comparing eq^n (i) with $(x-h)^2 = 4a(y-k)$

$h = -2$

$k = -3$

$4a = -4$

$\therefore a = -1$

Focus $(F) = (h, k+a)$

$= (-2, -3-1)$

$= (-2, -4)$

eq^n of directrix $(y) = k - a$

$y = -3+1 = -2$

* Find the equation of Parabola:

a.> vertex (0,0), Focus (-4,0)

→ Solution,

Since y-coordinates of vertex and focus are equal, so it is parall to x-axis

here,

$V(0,0)$, $F(-4,0)$ is of $y^2 = 4ax$

$\therefore a = -4$

Now

eq^n of Parabola $= y^2 = 4ax$

$y^2 = 4 \times (-4)x$

$\boxed{y^2 = -16x}$ #.

b) vertex (0,0), eq^n of directrix (y) = 2

→ Solution,

eq^n of directrix (y) = 2

so, $y = +a$ ∴ $a = 2$

$x^2 = -4ay$

$x^2 = -4 \times 2y$

$\boxed{x^2 + 8y = 0}$ is req eq^n of parabola

c) vertex (−2,0), eq^n of directrix (x) = 2

→ Solution,

vertex (h,k) = (−2,0) here x=2 in direct.

eq^n of directrix (x) = 2 so eq^n of P is $(y-k)^2$

we have,

$(y-k)^2 = 4a(x-h)$

or, $(y-0)^2 = -4 \times a(x+2)$

For a

we have,

eq^n of directrix (x) = h − a

$2 = -2 - a$

$a = 4$

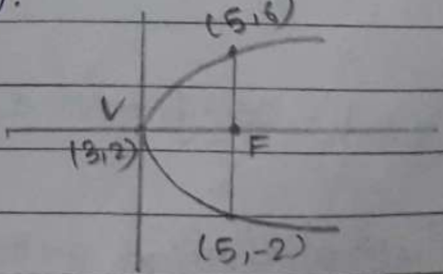
∴ The required eq^n of parabola is $y^2 = -4 \times 4 (x+2)$

$y^2 = -(16x + 32)$

∴ $y^2 + 16x + 32 = 0$

d) vertex (3,2), and end of latus rectum (5,6) and (5,−2).

→ Solution:



Mid point of latus rectum is Focus. so

Mid-point = $\frac{x_1 + x_2}{2}, \frac{y_2 + y_1}{2}$

$= \frac{5+5}{2}, \frac{6-2}{2}$

$= 5, 2$

Focus (5,2) vertex (3,2)

Since y-coordinate of the vertex and focus are equal. so the axis is parall to x-axis,

$x = 5a$ $(x-h)^2 = 4a(y-k)$

here $(h,k) = (3,2)$

$y = 5a$ $(y-k)^2 = 4a(x-h)$

and $(h+a, k) = (5,2)$

$\therefore h+a = 5$

$a = 5-3 = 2$

Now,

eqn of parabola is

$(y-2)^2 = 4a(x-3)$

$y^2 - 4y + 4 = 4\times2(x-3)$

$y^2 - 4y + 4 = 8x - 24$

$\boxed{y^2 - 4y - 8x + 28 = 0}$

which is req. eqn of parabola.

Assignment: 3

Attempt all questions from 2065-2072 'read only'

Year 2065

Group 'A' = 2×10 = 20

1. Verify Rolle's theorem for the function $f(x) = \frac{x^3}{3} - 3x$ on iterval [-3,3].

→ Solution:

Given function (f(x) is polynomial function which exist for all $x \in [-3,3]$

so it is continuous.

$f'(x) = \frac{1}{3} \times 3x^2 - 3$

$\therefore f'(x) = x^2 - 3$

so it is differentiable at open interval $x \in (-3,3)$

then,

$f(a) = f(-3) = -\frac{3^3}{3} + 3\times3 = -9+9 = 0$

$f(b) = f(3) = \frac{3^3}{3} - 9 = +9-9 = 0$

$\therefore f(a) = f(b) = 0$

then,

there exists at least a point $c \in (-3,3)$, such that

$f'(c) = 0$

$c^2 - 3 = 0$

$\therefore c^2 = 3$

$\therefore c = \sqrt{3}$

$\therefore c = 1.73 \in (-3,3)$

Hence Rolle's theorum is verified.

2. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$

→ Solution,

$9x^2 - 16y^2 = 144$

or, $\frac{9x^2}{144} - \frac{16y^2}{144} = 1$

or $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Comparing $\frac{x^2}{(4)^2} - \frac{y^2}{(3)^2} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore a = 4 \ , \ b = 3$

Now,

we have,

eccentricity of hyperbola $= \sqrt{1 + \frac{b^2}{a^2}}$

$\therefore e = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{16+9}{16}}$

$e = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\therefore$ eccentricity of hyperbola is $\frac{5}{4}$.

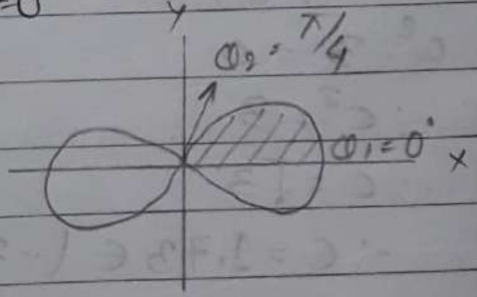
3. Find the area enclosed by the curve $r^2 = 4\cos 2\theta$

→ Solution,

$r^2 = 4\cos 2\theta$

we have,

Area of curve $= \int_{\theta_1 = 0}^{\theta_2 = \pi/4} \frac{1}{2} r^2 \, d\theta$



$= \int_{0°}^{\pi/4} \frac{1}{\cancel{2}} \cancel{4}\cos 2\theta$

$= 2\int_{0}^{\pi/4} \cos 2\theta$

$= 2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$= 2 \times \left[\frac{\sin 2 \cdot \frac{\pi}{\cancel{4}2}}{2} - \frac{\sin 2 \cdot 0°}{2}\right]$

$= 2 \times \left[\frac{1}{2} - \frac{0}{2}\right]$

$= 1$ sq. unit

Page

$\therefore$ The total area enclosed by curve $= 1 \times 4 = 4$ sq. unit.

Year 2066

Group A : $2 \times 10 = 20$

1. Find the length of curve $y = x^{3/2}$ from $x = 0$ to $x = 4$.

→ Solution,

$$y = x^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{3/2 - 1}$$

$$= \frac{3}{2} x^{1/2}$$

$$= \frac{3\sqrt{x}}{2}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{3\sqrt{x}}{2}\right)^2$$

$$= 1 + \frac{9x}{4}$$

$$\text{Length } (L) = \int_{x=0}^{x=4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x=0}^{x=4} \sqrt{1 + \frac{9x}{4}}\, dx$$

$$= \int_{x=0}^{x=4} \left(1 + \frac{9x}{4}\right)^{1/2} dx$$

$$= \left[\frac{\left(1 + \frac{9x}{4}\right)^{1/2 + 1}}{\left(\frac{1}{2} + 1\right) \cdot \frac{9}{4}}\right]_{x=0}^{x=4}$$

$$= \frac{\left(1 + \frac{9x}{4}\right)^{3/2}}{\frac{3}{2} \times}$$

Ellipse

S.No.	Equation of an ellipse	centre	Vertex	Focus	Eccentricities	Length of L.R.
1.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $a>b>0$	$(0,0)$	$(\pm a,0)$	$(\pm ae,0)$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$\frac{2b^2}{a}$
2.	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $b>a>0$	$(0,0)$	$(0,\pm b)$	$(0,\pm be)$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$\frac{2a^2}{b}$
3.	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $a>b>1$	(h,k)	$(h\pm a,k)$	$(h\pm ae,k)$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$\frac{2b^2}{a}$
4.	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}$ $b>a>1$	(h,k)	$(h,k\pm b)$	$(h,k\pm be)$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$\frac{2a^2}{b}$

Hyperbola

S.No.	Equation of Hyperbola	Centre	Vertex	Focus	Trans. axis	Conj. axis	Eccentricities
1	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$(0,0)$	$(\pm a,0)$	$(\pm ae,0)$	$2a$	$2b$	$e=\sqrt{1+\frac{b^2}{a^2}}$
2.	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$	$(0,0)$	$(0,\pm b)$	$(0,\pm be)$	$2b$	$2a$	$e=\sqrt{1+\frac{a^2}{b^2}}$
3.	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	(h,k)	$(h\pm a,k)$	$(h\pm ae,k)$	$2a$	$2b$	$e=\sqrt{1+\frac{b^2}{a^2}}$

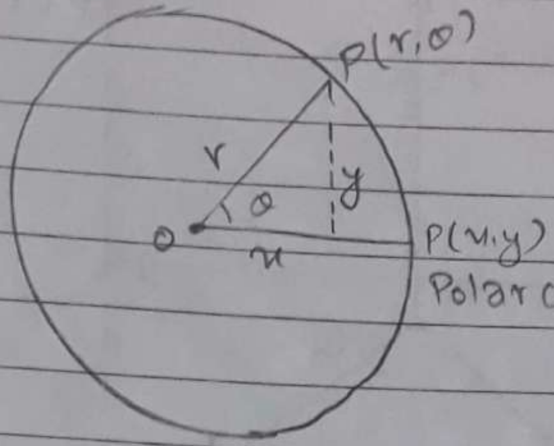
Polar Equation

$\cos\theta = \frac{x}{r}$

$\boxed{\therefore x = r\cos\theta}$

$\sin\theta = \frac{y}{r}$

$\boxed{\therefore y = r\sin\theta}$



Polar co-ordinates.

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\boxed{\therefore r^2 = x^2 + y^2}$$

$$\frac{y}{x} = \cancel{ta}\ \frac{r\sin\theta}{r\cos\theta}$$

$$\therefore \tan\theta = \frac{y}{x}$$

$$\boxed{\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

cartesian equation = (x,y)

find cartesian eqⁿ from polar eqⁿ

$r\sin(\theta + \pi/6) = 2$

or, $r\{\sin\theta \cdot \cos\pi/6 + \sin\pi/6 \cdot \cos\theta\} = 2$

or, $r\left(\sin\theta \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\cos\theta\right) = 2$

$\boxed{\sin(\overset{C+D}{\cancel{A+B}}) = \sin C \cdot \cos D + \cos C \cdot \sin D}$

or, $r\sin\theta\sqrt{3} + r\cos\theta = 4$

$\therefore x + y\sqrt{3} = 4$ #

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Polar Equation of Conic Section

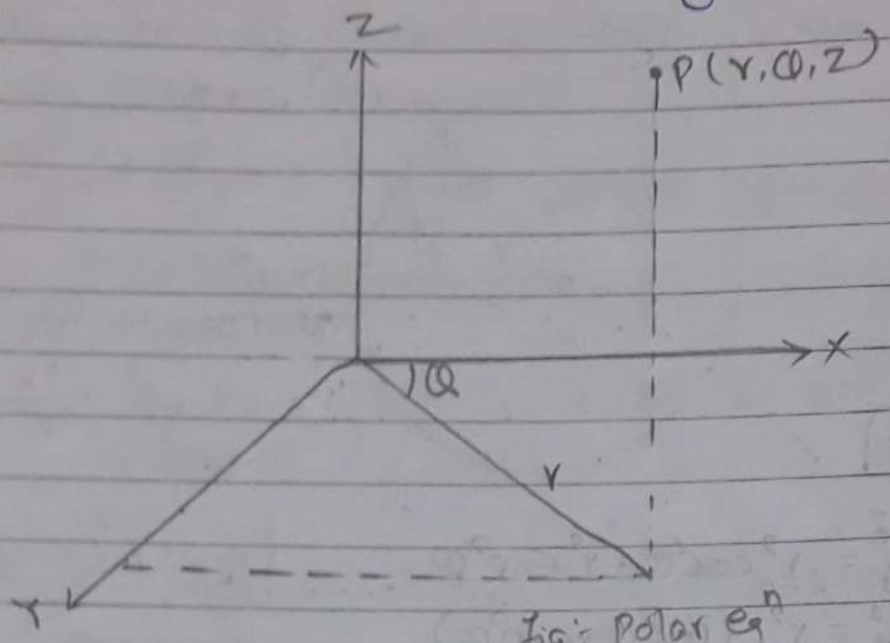


fig: Polar eqⁿ

1.> $r = \dfrac{ke}{1 + e\cos\theta}$

and $x = k$ is the eqⁿ of directrix.

2.> $r = \dfrac{ke}{1 - e\cos\theta}$

and $x = -k$ is the eqⁿ of directrix.

3.> $r = \dfrac{ke}{1 + e\sin\theta}$

and $Y = k$ is the equation of directrix.

4.> $r = \dfrac{ke}{1 - e\sin\theta}$

and $Y = -k$ is the eqⁿ of directrix.

* find Polar eqⁿ of $e = 1$ $x = 2$

$e = 1 \rightarrow$ Parabola
$e > 1 \rightarrow$ hyperbola
$e < 1 \rightarrow$ ellipse

$e = 1$, so parabola

$x = k = 2$

$\therefore$ Polar eqⁿ $= r = \dfrac{ke}{1 + e\cos\theta} = \dfrac{2 \times 1}{1 + 1\cos\theta} = \dfrac{2}{1 + \cos\theta}$ #

"Cylinderical coordinates"

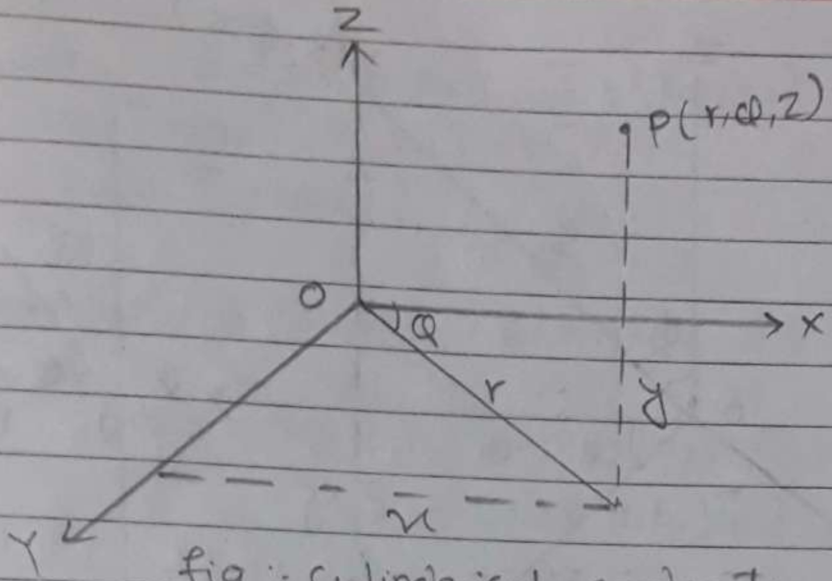


fig: Cylinderical cordinates

→ Cylinderical co-ordinates represent a point 'P' in Space where

a) r and θ are polar co-ordinates for the vertical projection of P on x,y plane

b) Z is the rectangular vertical coordinate.

here,

1. $x = r\cos\theta$
2. $y = r\sin\theta$
3. $Z = Z$

$\tan\theta = \frac{y}{x}$

4. $\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$
5. $x^2 + y^2 = r^2$

* Convert into cylinderical system eqⁿ.

a) $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$

→ $x^2 + y^2 + z^2 - 2z\frac{1}{2} + \frac{1}{4} = \frac{1}{4}$

or, $x^2 + y^2 + z^2 - z = 0$

or, $r^2 + z^2 = \cancel{z}$ #.

$$\boxed{\begin{aligned} \cos\theta &= ar \\ r\cos\theta &= ar^2 \\ x &= ar^2 \\ x &= a(x^2+y^2) \\ \therefore x &= ax^2 + ay^2 \end{aligned}}$$

* convert rect. cordinates (1,0,0) in cylinderical cordinates

→ rect. coordinates $= (x, y, z) = (1, 0, 0)$

$r^2 = x^2 + y^2 =$

$r = \sqrt{x^2 + y^2} = 1$

$\tan\theta = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$\therefore (r, \theta, z) = (1, 0, 0)$

Spherical Co-ordinates

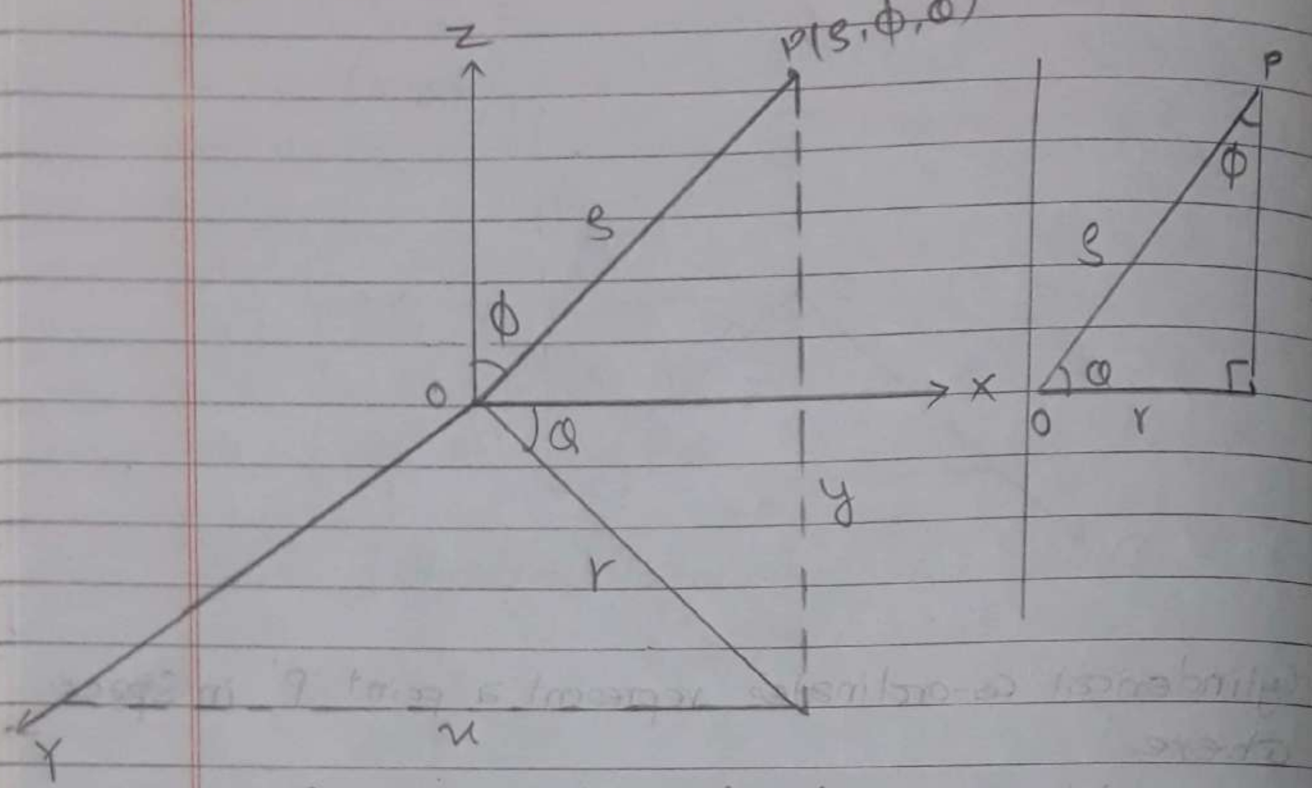


fig: spherical co-ordinates

→ Spherical coordinates represent a point P on the space by ordered triples (ρ, ϕ, θ)
where,
a) ρ is a distance from P to origin (O).
b) ϕ is the angle that $\vec{OP}$ makes with +ve Z axis.
c) θ is the angle from cylinderical coordinates.

$\cos\phi = \frac{z}{\rho}$ $\sin\phi = \frac{r}{\rho}$

$\therefore z = \rho\cos\phi$ $\therefore r = \rho\sin\phi$

1. $x = r\cos\theta = \rho\sin\phi . \cos\theta$	* Find Sp. Co eq^n
2. $y = r\sin\theta = \rho\sin\phi . \sin\theta$	a) $x^2+y^2+(z-1)^2 = 1$
3. $z = \rho\cos\phi$	or, $x^2+y^2+z^2-2z+1 = 1$
4. $\rho^2 = x^2+y^2+z^2$	or, $\rho^2 - 2\rho\cos\phi = 0$
5. $\tan\theta = \left(\frac{y}{x}\right)$	or $\rho(\rho - 2\cos\phi) = 0$
	$\therefore \rho = 0$
	$\rho = 2\cos\phi$

Vector And Vector Valued Function

Dot Product:- $\vec{a} = (a_1\ \cancel{b_2}\ a_2)$, $\vec{b} = (b_1, b_2)$

$\vec{a} \cdot \vec{b} = (a_1\vec{i} + a_2\vec{j}) \cdot (b_1\vec{i} + b_2\vec{j})$

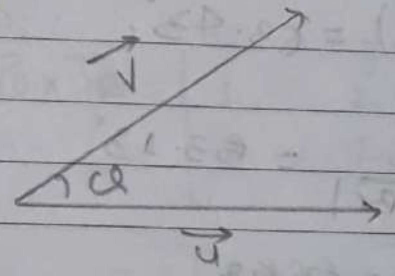
$= (a_1 b_1 \vec{i} + a_2 b_2 \vec{j})$

Cross Product:- $\vec{a}\,(a_1, a_2)$ $\vec{b}\,(b_1, b_2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(a_1 b_2 - b_1 a_2)$$

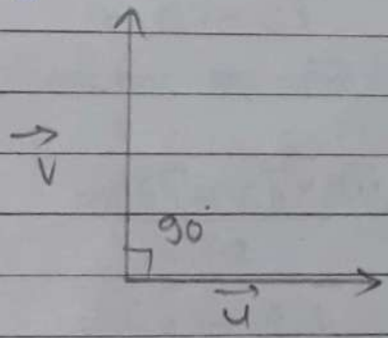
$$= (a_1 b_2 - a_2 b_1)\vec{k}$$

* Angle betn two Vectors:-



$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| . |\vec{v}|}$$

Orthogonal Vectors:



$\vec{u} \cdot \vec{v} = 0$

Unit Vector $= \dfrac{\vec{v}}{|\vec{v}|}$

Q. Find angle betn $\vec{u} = 4\vec{i} - 2\vec{j} - \vec{k}$ and $\vec{v} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

→ $|\vec{u}| = \sqrt{16+4+1} = \sqrt{21}$

$|\vec{v}| = \sqrt{16+4+16} = \sqrt{36} = 6$

Now

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{(4\vec{i} - 2\vec{j} - \vec{k}) \cdot (4\vec{i} - 2\vec{j} + 4\vec{k})}{6 \times \sqrt{21}} = \frac{16+4-4}{6\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) = 54.41^\circ$$

* Find the measure of triangle whose vertices are $A(-1,0)$, $B(2,1)$ and $(1,-2)$.

$\angle A = ?$

→ $\cos\angle A = \dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\cdot|\vec{AC}|}$

$\vec{AB} = \vec{OB} - \vec{OA} = (2,1) - (-1,0) = (3,1)$

$|\vec{AB}| = \sqrt{9+1} = \sqrt{10}$

$\vec{AC} = \vec{OC} - \vec{OA} = (1,-2) - (-1,0) = (2,-2)$

$|\vec{AC}| = \sqrt{4+4} = \sqrt{8}$

$\therefore \cos\angle A = \dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AC}|\cdot|\vec{AB}|} = \dfrac{(3,1)\cdot(2,-2)}{\sqrt{10}\times\sqrt{8}} = \dfrac{6-2}{\sqrt{80}}$

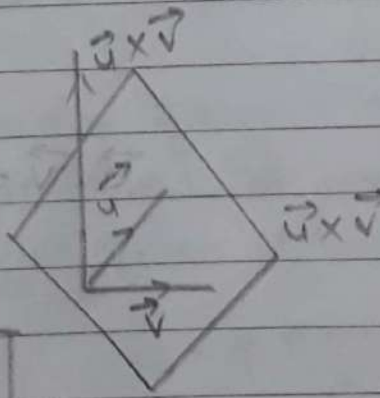
$\therefore \angle A = \cos^{-1}\left(\dfrac{4}{\sqrt{80}}\right) = 63.43^\circ$.

Similarly

find $\cos\angle B = \dfrac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\cdot|\vec{BC}|} = 53.13^\circ$

$\cos\angle C = \dfrac{\vec{CA}\cdot\vec{CB}}{|\vec{CA}|\cdot|\vec{CB}|} = 63.43^\circ$

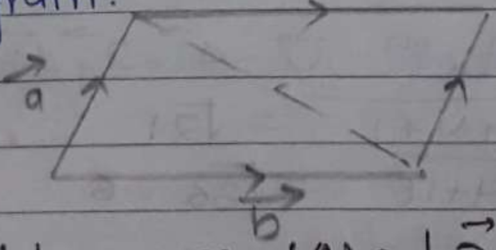
* Cross Vector:



$|\vec{u}\times\vec{v}| = |\vec{u}|.|\vec{v}|\sin\theta$

$$\boxed{\therefore \sin\theta = \frac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}}$$

Area of parallelogram:



Area of parallelogram $(A) = |\vec{a}\times\vec{b}|$

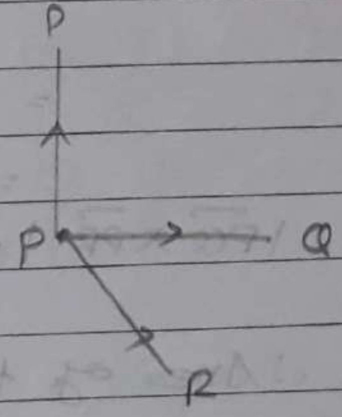
Area of triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$

unit vector perpendicular to $\vec{u}$ and $\vec{v}$ is

$$\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$$

* Find vector perpendicular to plane of P(1,-1,0), Q(2,1,-1) and R(-1,1,2).

→ Solution,



let O be the position Vector,

$\vec{PQ} = \vec{OQ} - \vec{OP} = (2,1,-1) - (1,-1,0)$

$= (1,2,-1)$

$\vec{PR} = \vec{OR} - \vec{OP} = (-1,1,2) - (1,-1,0)$

$= (-2,2,2)$

Now,

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$|\vec{PQ} \times \vec{PR}| = \sqrt{36+36}$

$= \sqrt{72}$

$= \vec{i}(4+2) - \vec{j}(2-2) + \vec{k}(2+4)$

$= 6\vec{i} + 0\vec{j} + 6\vec{k}$

$= 6\vec{i} + 6\vec{k}$

Unit vector of $\vec{PQ} \times \vec{PR} = \dfrac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|}$

$= \dfrac{6\vec{i} + 6\vec{k}}{\sqrt{72}}$

$= \dfrac{6\vec{i}}{6\sqrt{2}} + \dfrac{6\vec{k}}{6\sqrt{2}}$

$= \dfrac{\vec{i}}{\sqrt{2}} + \dfrac{\vec{k}}{\sqrt{2}}$ #.

* Find area of triangle with vertice P(4,2,0), Q(1,3,0) and R(1,1,3)

→ Solution,

let O be the position vector.

$\vec{PQ} = \vec{OQ} - \vec{OP} = (1,3,0) - (4,2,0) = (-3,1,0)$

$\vec{PR} = \vec{OR} - \vec{OP} = (1,1,3) - (4,2,0) = (-3,-1,3)$

Now,

$|\vec{A}\times\vec{B}| = |\vec{PQ}\times\vec{PR}| = ?$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 0 \\ -3 & -1 & 3 \end{vmatrix}$$

$= i(3+0) - j(0-9) + k(3+3)$

$= 3\vec{i} + 9\vec{j} + 6\vec{k}$

$|\vec{PQ}\times\vec{PR}| = \sqrt{9+81+36}$

$= \sqrt{126}$

$\therefore$ Area of triangle $= \dfrac{\sqrt{126}}{2}$ Sq. unit.

* Find area of Parallelogram:

a.) $A(1,0)$ $B(0,1)$, $C(-1,0)$, $D(0,-1)$

→ Solution,

$\vec{AB} = (\vec{OB}-\vec{OA}) = (0,1)-(1,0) = (-1,1)$

$\vec{AC} = (\vec{OC}-\vec{OA}) = (-1,0)-(1,0) = (-2,0)$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -2 & 0 & 0 \end{vmatrix}$$

$= i(0-0) + j(0-0) + \vec{k}(0+2)$

$= 2\vec{k}$

Now,

$|\vec{AB}\times\vec{AC}| = \sqrt{2^2} = 2$

$\therefore$ Area of ||gm $= |\vec{AB}\times\vec{AC}|$

$= 2$ sq. unit.

OR

$\vec{AC}\times\vec{AD}$ w.

Triple Scale Product (Box Product):

$|(u \times v) \cdot w| =$ Volume of box.

$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

* find the Vol. of box detemined by

a) $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{v} = -2\vec{i} + 3\vec{k}$

$\vec{w} = 7\vec{j} - 4\vec{k}$

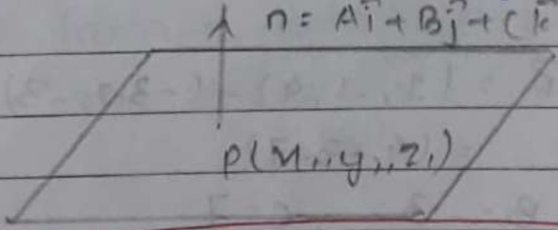
→ Vol. of box $= (u \times v) \cdot w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$

$= \left| 1(0-21) - 2(8-0) - 1(-14-0) \right|$

$= |-23|$

$= 23$ unit Cubed.

Equation of Plane passing through point $P(x_1, y_1, z_1)$ and normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$.



$$\boxed{\therefore Eq^n \text{ of plane} = A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$$

* find the eq^n of plane passing through A(0,0,1) B(2,0,0) C(0,3,0).

→ solution,

$\vec{AB} = \vec{OB} - \vec{OA} = (2,0,0) - (0,0,1) = (2,0,-1)$

$\vec{AC} = \vec{OC} - \vec{OA} = (0,3,0) - (0,0,1) = (0,3,-1)$

Now,

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= i(0+3) - j(-2-0) + \vec{k}(6-0)$$

$$= 3\vec{i} + 2\vec{j} + 6\vec{k} \quad \text{---(i)} \quad \therefore A = 3,\ B = 2,\ C = 6$$

Point $(P) = (0, 0, 1) = (x_1, y_1, z_1)$

$\therefore$ eq^n of Plane $= A(x-x_1) + B(y-y_1) + C(z-z_1)$

$$= 3(x-0) + 2(y-0) + 6(z-1)$$

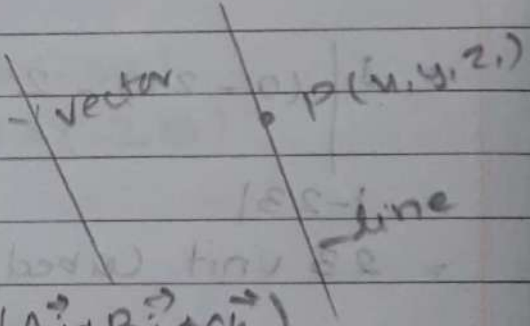
$$= 3x + 2y + 6z = 6 \ \#$$

Parametric eq^n of a line.

→ Parametric eq^n of a line passing through Point $P(x_1, y_1, z_1)$ and Parallel to vector $v = (A\vec{i} + B\vec{j} + C\vec{k})$

$\therefore$ Parametric eq^n are,

$$x = x_1 + At$$
$$y = y_1 + Bt$$
$$z = z_1 + Ct$$



$v = (A\vec{i} + B\vec{j} + C\vec{k})$

* Find the parametric eq^n passing through $A(-3, 2, -3)$ $B(2, -1, 4)$.

→ solution:

$$\vec{AB} = \vec{OB} - \vec{OA} = (1, -1, 4) - (-3, 2, -3)$$
$$= (4, -3, 7)$$

$\therefore A = 4, \quad B = -3 \quad C = 7$

let line $(x_1, y_1, z_1) = (-3, 2, -3)$

$\therefore$ Parametric eq^n is

$x = x_1 + At = -3 + 4t$
$y = y_1 + Bt = 2 - 3t$
$z = z_1 + Ct = -3 + 7t$

or, when $(x_1, y_1, z_1) = (2, -1, 4)$

$x = 1 + 4t$
$y = -1 - 3t$
$z = 4 + 7t$

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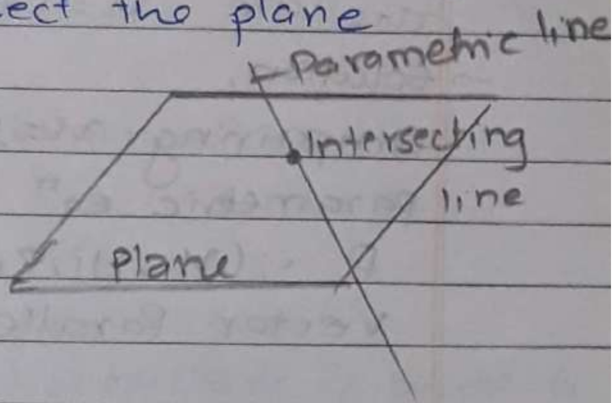
* Find the ~~intersecting~~ point where the line $x = \frac{3}{8} + 2t$, $y = -2t$ and $z = 1 + t$ intersect the plane $3x + 2y + 6z = 6$.



→ Solution,

Substituting the value of x, y, z in eq^n of plane,

$$3x + 2y + 6z = 6$$

or, $3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$

or, $8 + 6t - 4t + 6 + 6t = 6$

or, $8t + 8 = 0$

or, $t = -1$

Now,

$x = \frac{8}{3} + 2t = \frac{8}{3} + 2\times(-1) = \frac{2}{3}$

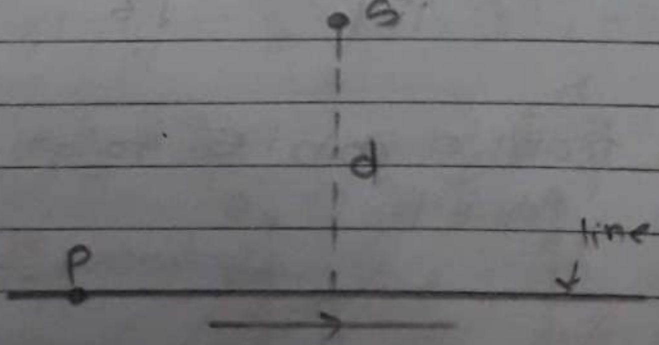
$y = -2t = -2\times -1 = 2$

$z = 1 + t = 1 - 1 = 0$

∴ Required point is $\left(\frac{2}{3}, 2, 0\right)$ #.

From 2nd Term

* Distance from a point S to a line through P parallel to vector $\vec{v}$.



$$\boxed{d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}}$$

$\vec{v} = A\vec{i} + B\vec{j} + C\vec{k}$

* find the distance from point $S(1,1,5)$ to the line $x=1+t,\ y=3-t,\ z=2t$.

→ solution,

Comparing $x=1+t,\ y=3-t,\ z=2t$ with parametric eq^n $x=x_1+At,\ y=y_1+Bt,\ z=z_1+Ct$

$P :(x_1,y_1,z_1)=(1,3,0)$

vector Parallel $(v) = (A,B,C) = (1,-1,2)$

$= \vec{i}-\vec{j}+2\vec{k}$

Given point $S=(1,1,5)$ $\dfrac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$

Now,

distance $(d) = \dfrac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$

$\vec{PS}=\vec{OS}-\vec{OP} = (1,1,5)-(1,3,0) = (0,-2,5)$

$\therefore d = |\vec{PS}\times\vec{v}|/|\vec{v}|$

$$\vec{PS}\times\vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$

$= i(-4+5)-j(0-5)+\vec{k}(0+2)$

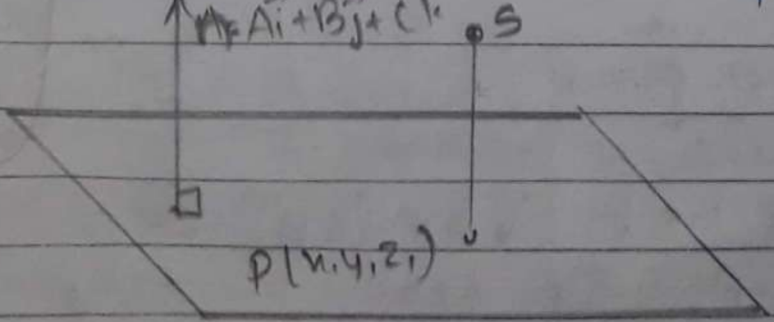
$= \vec{i}+5\vec{j}+2\vec{k}$

$|\vec{PS}\times\vec{v}| = \sqrt{1+25+4} = \sqrt{30}$

$|\vec{v}| = \sqrt{1+1+4} = \sqrt{6}$

$\therefore$ distance $(d) = \dfrac{|\vec{PS}\times\vec{v}|}{|\vec{v}|} = \dfrac{\sqrt{30}}{\sqrt{6}} = \sqrt{\dfrac{30}{6}} = \sqrt{5}$ unit #.

Distance from a point S to a plane.



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modulus

$$\boxed{d = \left| \vec{PS} \cdot \frac{n}{|n|} \right|}$$

where, $n = A\vec{i} + B\vec{j} + C\vec{k}$

P = Point lies on the plane (unknow)

to find P

Put $x=0$, $z=0$ and $y = w$.

* Find the distance from S(1, 1, 3) to Plane $3x+2y+6z=6$

→ Solution,

Given S(1, 1, 3)

normal vector $(\vec{n}) = 3\vec{i} + 2\vec{j} + 6\vec{k}$

for point P, $|\vec{n}| = \sqrt{9+4+36} = \sqrt{49} = 7$

when $x=0$, $z=0$

$3\times 0 + 2y + 6\times 0 = 6$

$\therefore 2y = 6 \quad \therefore y = 3$

$\therefore$ Point P(0, 3, 0)

Now,

$\vec{PS} = \vec{OS} - \vec{OP} = (1, 1, 3) - (0, 3, 0) = \vec{i} - 2\vec{j} + 3\vec{k}$

Then, $\dfrac{\vec{n}}{|n|} = \dfrac{3\vec{i}}{7} + \dfrac{2\vec{j}}{7} + \dfrac{6\vec{k}}{7}$

Now,

$$\text{distance } (d) = \left| \vec{PS} \cdot \frac{\vec{n}}{|n|} \right|$$

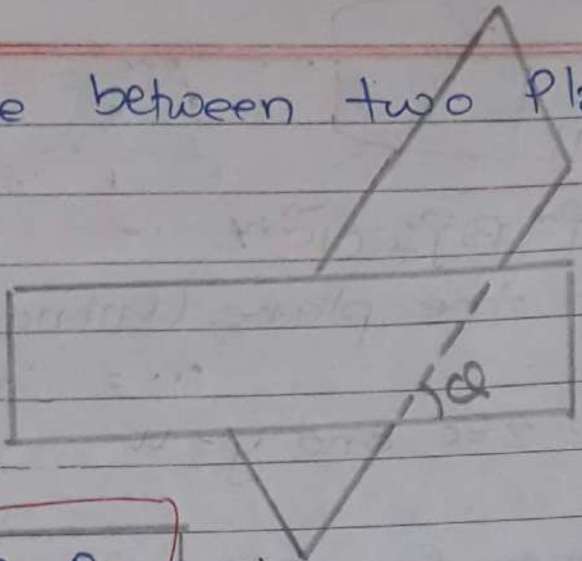
$$= \left| \left(\frac{3}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}\right) \cdot \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) \right|$$

$$= \frac{3}{7} + \frac{4}{7} + \frac{9}{7} \qquad = \frac{3}{7} + \frac{4}{7} + \frac{18}{7}$$

$$= \frac{17}{7} \text{ unit} \qquad = \frac{25}{7}$$

$\frac{17}{7}$ u

* Angle between two Planes:



$$\boxed{\cos Q = \frac{n_1 \cdot n_2}{|n_1||n_2|}}$$

where,
Q = angle betn two planes
n_1 and n_2 are normal for two plane

* Find the angle between two planes
a.> $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.
→ Solution,
normal for two vectors are,

$$n_1 = 3\vec{i} - 6\vec{j} - 2\vec{k}$$

$$|n_1| = \sqrt{9+36+4} = \sqrt{49} = 7$$

$$n_2 = 2\vec{i} + \vec{j} - 2\vec{k}$$

$$|n_2| = \sqrt{4+1+4} = \sqrt{9} = 3$$

Now,

$$\cos Q = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{(3\vec{i} - 6\vec{j} - 2\vec{k}) \cdot (2\vec{i} + \vec{j} - 2\vec{k})}{7 \times 3}$$

$$= \frac{6}{21} - \frac{6}{21} + \frac{4}{21}$$

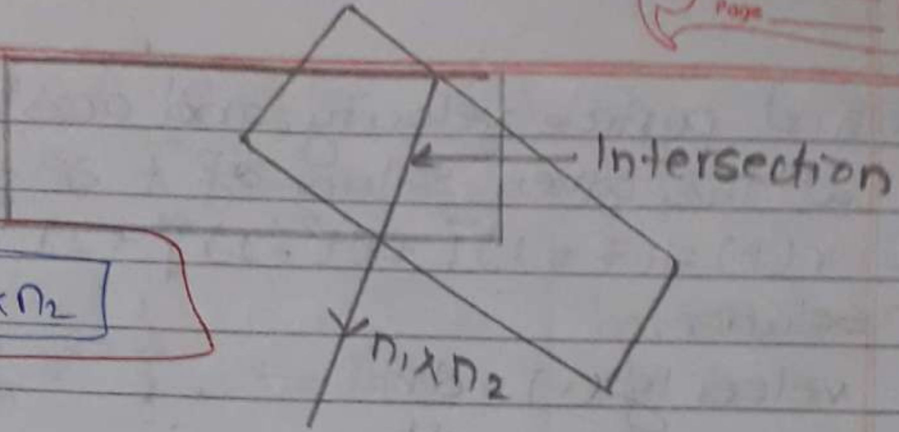
$$\therefore \cos Q = \frac{4}{21}$$

$$\therefore Q = \cos^{-1}\left(\frac{4}{21}\right) = 79.01^\circ$$

∴ Angle betn two plane is 79.01° *

Vector Parallel to the line of intersection of two plane.

Formula $= n_1 \times n_2$



$\boxed{\text{Formula} = n_1 \times n_2}$

* Find the vector parallel to line of intersection of plane, $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

→ Solution,

normal eqⁿ are, $n_1 = 3\vec{i} - 6\vec{j} - 2\vec{k}$

$n_2 = 2\vec{i} + \vec{j} - 2\vec{k}$

Now,

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

Req. Vect. $= 14\vec{i} + 2\vec{j} + 15\vec{k}$ #.

Vector Valued Function:

A vector is defined by $r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is known as Vector Valued function in t.

where,

t = parameters

$f(t), g(t), h(t)$ are components of $r(t)$.

→ if $r(t)$ is a position vector, particle moving along a smooth curve in space, then,

(a) Velocity $(v) = \frac{dr}{dt}$

(b) Speed $(s) = |v|$

(c) Acceleration $(a) = \frac{dv}{dt}$

* Find particle velocity and acceleration vectors at the given value of t.

(a) $r(t) = (t+1)\vec{i} + (t^2-1)\vec{j} + 2t\vec{k}$, $t = 1$.

→ Solution,

velocity (v) $= \frac{dr}{dt}$

$= \frac{d\{(t+1)\vec{i} + (t^2-1)\vec{j} + 2t\vec{k}\}}{dt}$

$= 1\vec{i} + 2t\vec{j} + 2\vec{k}$

when $t = 1$

velocity (v) $= \vec{i} + 2\vec{j} + 2\vec{k}$ #.

speed (s) $= |v| = \sqrt{1+4+4} = \sqrt{9} = 3$ #

acceleration (a) $= \frac{dv}{dt}$

$= \frac{d(1\vec{i} + 2t\vec{j} + 2\vec{k})}{dt}$

$= 2\vec{j}$ #.

(b) $r(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j} + 4t\vec{k}$, $t = \pi/2$.

→ Solution,

Velocity (v) $= \frac{dr}{dt} = -2\sin t\,\vec{i} + 3\cos t\,\vec{j} + 4\vec{k}$

when $t = \pi/2$

$= -2\sin(\pi/2)\vec{i} + 3\cos(\pi/2)\vec{j} + 4\vec{k}$

$= -2 \times 1\vec{i} + 3 \times 0\vec{j} + 4\vec{k}$

$\therefore$ velocity (v) $= -2\vec{i} + 4\vec{k}$

acceleration (a) $= \frac{dv}{dt} = \frac{d}{dt}(-2\sin t)\vec{i} + 3\cos t\,\vec{j} + 4\vec{k})$

$= -2\cos t\,\vec{i} - 3\sin t\,\vec{j}$

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where, $t = \pi/2$

$a = -2\cos\pi/2\,\vec{i} - 3\sin\pi/2\,\vec{j}$
$= -3\vec{j}$

In Calculator do ln 3 first then, ex ln 3 or ln 3 then 2x ln3 after e^{Ans}

ⓒ $r(t) = e^t\vec{i} + \frac{2}{9}e^{2t}\vec{j}$, $t = \ln 3$
→ Solⁿ.

velocity $(v) = \frac{dr}{dt} = e^t\vec{i} + \frac{4}{9}e^{2t}\vec{j}$

when $t = \ln 3$

$v = e^{\ln 3}\vec{i} + \frac{4}{9}e^{2\ln 3}\vec{j}$

$= 3\vec{i} + \frac{9\times 4}{9}\vec{j}$

$= 3\vec{i} + 4\vec{j}$ #.

acceleration $(a) = \frac{dv}{dt} = e^t\vec{i} + \frac{8}{9}e^{2t}\vec{j}$

= when $t = \ln 3$

$a = e^{\ln 3}\vec{i} + \frac{8}{9}e^{2\ln 3}\vec{j}$

$= 3\vec{i} + \frac{8\times\cancel{9}}{\cancel{9}}\vec{j}$

$\therefore a = 3\vec{i} + 8\vec{j}$ #

$|v| = \sqrt{9+16}$
$= 5$ #

* Evaluate the integral of vector value function.

i) $\int_0^1 \{3t^2\vec{i} + 2\vec{j} + (t-3)\vec{k}\}\,dt$

$= \left[\cancel{3}\frac{t^3}{\cancel{3}}\right]_0^1 + 2\left[t\right]_0^1 + \left[\frac{t^2}{2} - 3t\right]_0^1$

$= (1^3 - 0)\vec{i} + 2(1-0)\vec{j} + \left(\frac{1}{2} - 3 - 0 + 0\right)\vec{k}$

$= \vec{i} + 2\vec{j} - \frac{5}{2}\vec{k}$ W.

iii) $\int_1^2 \left\{(6-6t)\vec{i} + 3\sqrt{t}\,\vec{j} + \frac{4}{t^2}\vec{k}\right\}$

$= \left[6t - \frac{6t^2}{2}\right]_1^2 \vec{i} + 3\left[\frac{t^{3/2}}{3/2}\right]_1^2 + 4\int_1^2 t^{-2}\,\vec{k}$

$= \left(12 - 6\times\frac{4}{2} - \left(6 + \frac{6}{2}\right)\right)\vec{i} + \frac{3}{\frac{3}{2}}\left(2^{3/2} - 1\right)\vec{j} + 4\left[\frac{t^{-2+1}}{-2+1}\right]_1^2$

$= -3\vec{i} + 3.65\vec{j} - 4\left[\frac{1}{2} - 1\right]\vec{k}$

$= -3\vec{i} + 3.65\vec{j} + 2\vec{k}$ W.

4.) $\int_1^4 \left[\frac{1}{t}\vec{i} + \frac{1}{5-t}\vec{j} + \frac{1}{2t}\vec{k}\right] dt$

$\rightarrow [\ln t]_1^4 + \int_1^4 \frac{1}{-(5-t)}\vec{j} + \int_1^4 \frac{1}{2}\ln t\,\vec{k}$

$= (\ln 4 - \ln 1)\vec{i} - [\ln(t-5)]_1^4 \vec{j} + \frac{1}{2}[\ln t]_1^4 \vec{k}$

$= \ln\left(\frac{4}{1}\right)\vec{i} - (\ln(-1) - \ln(-4))\vec{j} + \frac{1}{2}(\ln 4 - \ln 1)\vec{k}$

$= \ln 4\,\vec{i} + (\ln(-4) - \ln(-1))\vec{j} + \frac{1}{2}\left(\ln\left(\frac{4}{1}\right)\right)\vec{k}$

$= \ln 4\,\vec{i} + \ln\left(\frac{-4}{-1}\right)\vec{j} + \frac{1}{2}\ln 4\,\vec{k}$

$= \ln 4\,\vec{i} + \ln 4\,\vec{j} + \frac{1}{2}\ln 4\,\vec{k}$ W.

Length of a smootch curve (Arc length formula)

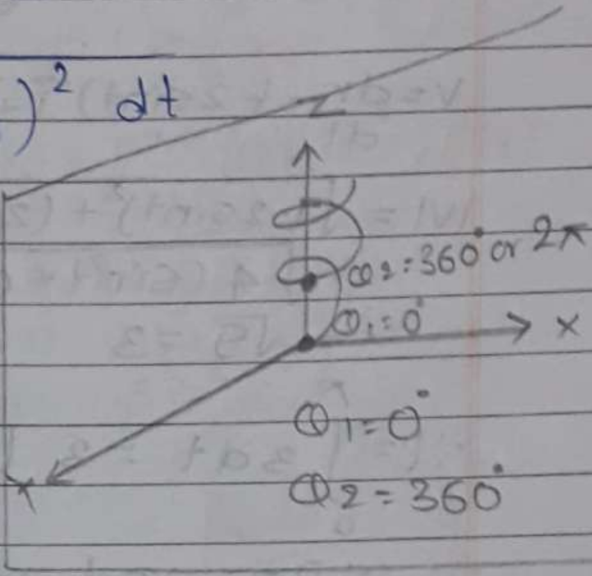
Length of smooth curve is
$r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, $a \le t \le b$

$$\therefore L = \int_{t=a}^{t=b} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

OR

$$\boxed{L = \int_{t=a}^{t=b} |v|\, dt}$$

where $v = \frac{dr}{dt}$



* find the length of 1 turn of helix.
$r(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

→ Solution,

$$V = \frac{dr}{dt} = \frac{d}{dt}\{(\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}\}$$

$$= -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$|\vec{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\therefore \text{length of helix} = \int_{t=0}^{t=2\pi} \sqrt{2}\, dt$$

$$= \sqrt{2}\,[t]_0^{2\pi}$$

$$= \sqrt{2}\,[2\pi - 0]$$

$$= 2\pi\sqrt{2} \text{ unit}$$ //

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* Find the length of indicated portion of curve:

ⓐ $r(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + \sqrt{5}t\,\vec{k}$, $0 \le t \le \pi$

→ Solution,

we have, $\text{Length}(L) = \int_0^{\pi} |v|\,dt$

$$v = \frac{dr}{dt} = (-2\sin t)\vec{i} + (2\cos t)\vec{j} + \sqrt{5}\,\vec{k}$$

$$|v| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2}$$
$$= \sqrt{4(\sin^2 t + \cos^2 t) + 5}$$
$$= \sqrt{9} = 3$$

$$\therefore L = \int_0^{\pi} 3\,dt = 3\,[t]_0^{\pi} = 3\pi \; \#$$

∴ Required length is 3π unit.

ⓑ $r(t) = (6\sin 2t)\vec{i} + (6\cos 2t)\vec{j} + 5t\,\vec{k}$, $0 \le t \le \pi$

→ Solution,

we have, $L = \int_0^{\pi} |v|\,dt$

$$v = \frac{dr}{dt} = (6\times 2\cos 2t)\vec{i} + (-6.2\sin 2t)\vec{j} + 5\vec{k}$$
$$= (12\cos 2t)\vec{i} + (-12\sin 2t)\vec{j} + 5\vec{k}$$

$$|v| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25}$$
$$= \sqrt{144+25} = \sqrt{169} = 13$$

$$\therefore L = \int_0^{\pi} 13\,dt$$
$$= 13\,[t]_0^{\pi}$$
$$= 13\pi \text{ unit}$$

∴ Required length of curve is 13π unit #

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ⓒ $r(t) = t\vec{i} + \frac{2}{3}t^{3/2}\vec{k}$, $0 \le t \le 8$.

→ solution,

$$\text{length } (L) = \int_0^8 |v|\, dt$$

$$v = \frac{dr}{dt} = \vec{i} + \frac{2}{3} \times \frac{3}{2} t^{3/2 - 1}\,\vec{k}$$

$$= \vec{i} + t^{1/2}\,\vec{k}$$

$$|v| = \sqrt{(1)^2 + (t^{1/2})^2} = (1+t)^{1/2}$$

$$\therefore \text{Length} = \int_0^8 (1+t)^{1/2}\, dt =$$

$$= \left[\frac{[1+t]^{1/2+1}}{(1/2+1) \times 1}\right]_0^8$$

$$= \frac{2}{3}\left[\left[1+t\right]^{3/2}\right]_0^8$$

$$= \frac{2}{3}\left\{(9)^{3/2} - (1)^{3/2}\right\}$$

$$= \frac{2}{3}\left\{3^{2 \times 3/2} - 1\right\}$$

$$= \frac{2}{3}(27 - 1)$$

$$= \frac{2 \times 26}{3}$$

$$= 17.33 \text{ unit}$$

$\therefore$ Required length is 17.33 unit.

$$\vec{i} + \frac{2}{3} \times \frac{3}{2} t^{3/2 - 1}$$

$$\vec{i} + \sqrt{t}$$

$$|v| = \sqrt{(1)^2 + (\sqrt{t})^2} = \sqrt{1+t}$$

$$|v| = \sqrt{1 + t^{1/2 \times 2}}$$

$$= (1+t)^{1/2}$$

$$= \left[\frac{(1+t)^{3/2}}{3/2 \times 1}\right]_0^8$$

$$= \frac{2}{3}\left[9^{3/2} - 1^{3/2}\right]$$

$$= \frac{2}{3}\left[3^{2 \times 3/2} - 1\right]$$

$$= \frac{2}{3}[26]$$

$$=$$

$$v = \frac{dr}{dt} = \frac{d\cos^3 t}{d\cos t} \times \frac{d\cos t}{dt}$$

$$= 3\cos^2 t \times (-\sin t)$$

d) $r(t) = (\cos^3 t)\,\vec{j} + (\sin^3 t)\,\vec{k}$, $0 \le t \le \pi/2$

→ solution,

$$\text{Length} = \int_0^{\pi/2} |v|\, dt$$

$$v = \frac{dr}{dt} = \frac{d\cos^3 t}{d\cos t} \times \frac{d\cos t}{dt}$$

$$= 3\cos^2 t\,(-\sin t)$$

$$v = \frac{dr}{dt} = \frac{d(\cos^3 t)}{dt}\vec{i} + \frac{d(\sin^3 t)}{dt}\vec{j}$$

$$= \left(\frac{d\cos^3 t}{d\cos t} \times \frac{d\cos t}{dt}\right)\vec{i} + \left(\frac{d\sin^3 t}{d\sin t} \times \frac{d\sin t}{dt}\right)\vec{j}$$

$$= 3\cos^2 t \cdot (-\sin t)\vec{i} + 3\sin^2 t \cdot \cos t\,\vec{j}$$

$$|v| = \sqrt{9\cos^4 t \cdot \sin^2 t + 9\sin^4 t \cdot \cos^2 t}$$

$$= \sqrt{9\cos^2 t \cdot \sin^2 t\,(\cos^2 t + \sin^2 t)}$$

$$= 3\sin t \cdot \cos t$$

$$= \frac{3}{2}\, 2\sin t \cdot \cos t$$

$$= \frac{3}{2}\sin 2t$$

$$L = \int_0^{\pi/2} \frac{3}{2}\sin 2t$$

$$= \frac{3}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{3}{4}\left[-\cos . 2.\pi/2 + \cos 2.0\right]$$

$$= \frac{3}{4}(-\cancel{\times}-1+1)$$

$$= \frac{3}{4} \times 2$$

$$= \frac{3}{2} \text{ unit}$$

$\therefore$ Required length is $3/2$ unit.

5. $r(t) = t\vec{i} + \frac{\sqrt{6}}{2}t^2\vec{j} + t^3\vec{k}$, $-1 \le t \le 1$

$\rightarrow$ Solution,

$$\text{length } (L) = \int_{-1}^{1} |v|\, dt$$

$$v = \frac{dr}{dt} = \vec{i} + \frac{\sqrt{6}}{2} \times 2t\vec{j} + 3t^2\vec{k}$$

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$|V| = \sqrt{1+6t^2+9t^4} = \sqrt{(1)^2+2.1.3t^2+(3t^2)^2}$

$= \sqrt{(1+3t^2)^2} = (1+3t^2)$

$L = \int_{-1}^{1} (1+3t^2)\,dt$

$= [t]_{-1}^{1} + 3\left[\frac{t^3}{3}\right]_{-1}^{1} = [1-(-1)] + \frac{3}{3}[1^3-(-1)^3]$

$= (1+1) + (1+1) = 2+2 = 4$ unit.

∴ Required length of cure is 4 unit.

$$\boxed{\text{Unit Tangent Vector } (T) = \frac{V}{|V|}}$$

→ It determines the direction of the motion of curve.

$T = \frac{V}{|V|}$, where $V = \frac{dr}{dt}$.

$$\boxed{\text{Curvature}: k = \frac{1}{|V|}\left|\frac{dT}{dt}\right|}$$

The rate at which the unit tangent vector (T) turns per unit length along the curve is known as curvature. i.e.

$$k = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

where T = unit tangent $= \frac{V}{|V|}$

* Show that circle of radius a is $\frac{1}{a}$.

→ Solution,

always for circle

$r(t) = a(\cos t)\vec{i} + a(\sin t)\vec{j}$

we have,

$r(t) = a(\cos t)\vec{i} + a(\sin t)\vec{j}$

$V = \frac{dr}{dt} = (-a\sin t)\vec{i} + a\cos t\,\vec{j}$

$|V| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

Again,

Unit Vector $(T) = \frac{V}{|V|} = \frac{(-\cancel{a}\sin t)}{\cancel{a}}\vec{i} + \frac{\cancel{a}\cos t}{\cancel{a}}\vec{j}$

$= -\sin t\,\vec{i} + \cos t\,\vec{j}$

Again,

Curvature $(k) = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$

$\frac{dT}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$

$\left|\frac{dT}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\therefore k = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{1}{a} \times 1 = 1/a$

$\therefore$ The radius of curvature circle is $1/a$.

* Find the curvature of helix:
$r(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k}$, $a, b \geq 0$, $a^2 + b^2 \neq 0$.

→ Solution,

$v = \frac{dr}{dt} = (-a\sin t)\vec{i} + (a\cos t)\vec{j} + b\vec{k}$

$|v| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$
$= \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2 + b^2}$

Unit tangent Vector $(T) = \frac{v}{|v|} = \frac{(-a\sin t)\vec{i} + (a\cos t)\vec{j} + b\vec{k}}{\sqrt{a^2+b^2}}$

$\frac{dT}{dt} = \frac{(-a\cos t)\vec{i} - (a\sin t)\vec{j} + 0}{\sqrt{a^2+b^2}}$

$\therefore$ Curvature $(k) = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$

$\left|\frac{dT}{dt}\right| = \sqrt{\frac{a^2\cos^2 t + a^2\sin^2 t}{(\sqrt{a^2+b^2})^2}} = \frac{\sqrt{a^2}}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$

$\therefore k = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$

$= \frac{1}{\sqrt{a^2+b^2}} \times \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$

$\therefore$ Required Curvature of Helix is $\frac{a}{a^2+b^2}$ #.

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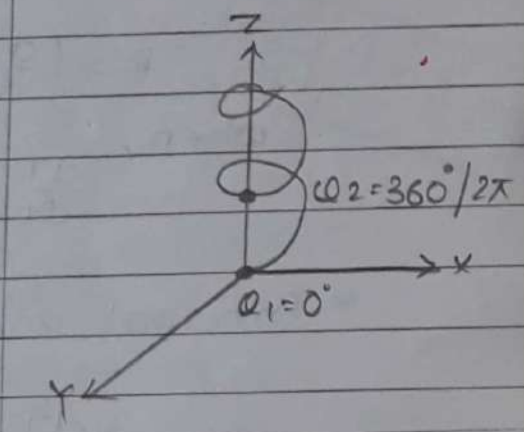
* Length of a smooth curve (Arc length formula):

Length of smooth curve is,
$r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}, \quad a \le t \le b$

$$\therefore L = \int_{t=a}^{t=b} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$



OR

$$\boxed{\text{Length} = \int_{t=a}^{t=b} |V|\, dt}$$

where, $V = \frac{dr}{dt}$

* Unit tangent Vector :- (T)

It determines the direction of the motion of the curve. It is denoted by T and given by,

$$\boxed{T = \frac{V}{|V|}}, \text{ where, } v = \frac{dr}{dt}$$

* Curvature :- (k)

The rate at which the unit tangent vector (T) turns per unit length along the curve is known as curvature.

$$\boxed{k = \frac{1}{|V|}\left|\frac{dT}{dt}\right|}$$

$k = \frac{|v \times a|}{|V|^3}$

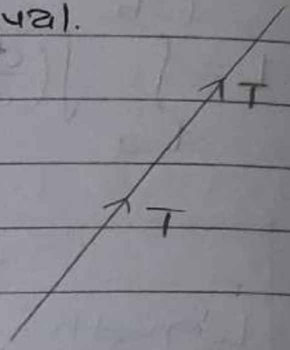
where,

$T = \text{unit Vector} = \frac{V}{|V|}$

for circle $r(t) = a(\cos t)\vec{i} + a(\sin t)\vec{j}$

* Curvature of a straight line :-

The curvature of straight line is zero because the unit tangent vector (T) always points in same direction, so it's components are equal.



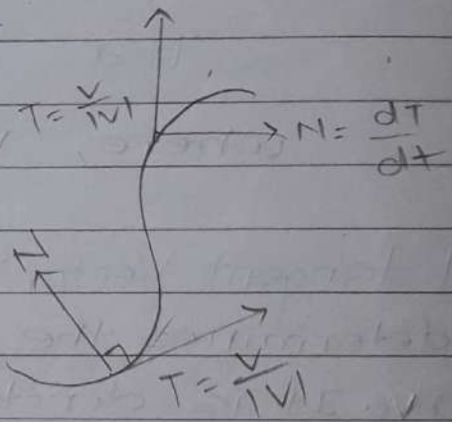
or, $k = \frac{|v \times a|}{|v|^3}$

$$k = \left|\frac{dT}{ds}\right| = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

when,
$$\frac{dT}{dt} = 0 \quad \therefore k = 0$$

* Principal Unit Normal (N) :-

$$\boxed{N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}}$$



Defn :-

Let r(t) be a differentiable vector valued function and T(t) be unit T.V. The P.U.N $= \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

* Find principle Unit Normal Vector and Curvature.

q.> $r(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j}$

$v = \frac{dr}{dt} = (-2\sin 2t)\vec{i} + (2\cos 2t)\vec{j}$

$|v| = \sqrt{4\sin^2 2t + 4\cos^2 2t}$
$= \sqrt{4}$
$= 2$

Now,

$$T = \frac{V}{|V|} = \frac{(-2\sin 2t)\vec{i}}{2} + \frac{(2\cos 2t)\vec{j}}{2}$$

$$T = (-\sin 2t)\vec{i} + (\cos 2t)\vec{j}$$

$$\frac{dT}{dt} = (-2\cos 2t)\vec{i} + (-2\sin 2t)\vec{j}$$

$$=$$

$$\left|\frac{dT}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 t}$$

$$= \sqrt{4} = 2$$

Now,

Principle Unit Normal Vector $(N) = \dfrac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} =$

$$= \frac{(-2\cos 2t)\vec{i}}{2} + \frac{(-2\sin 2t)\vec{j}}{2}$$

$$\therefore N = (-\cos 2t)\vec{i} - (\sin 2t)\vec{j}$$

Curvature $(k) = \frac{1}{|V|}\left|\frac{dT}{dt}\right| = \frac{1}{2} \times 2 = 1$ unit #

b) $r(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$

→ $N = (-\sin t)\vec{i} - (\sin t)\vec{j}$

$k = \frac{3}{25}$ #.

c) $r(t) = (2t+3)\vec{i} + (5-t^2)\vec{j}$

→ Solution,

$V = \frac{dr}{dt} = 2\vec{i} - 2t\vec{j}$, $|V| = \sqrt{4+4t^2} = 2\sqrt{(1+t^2)}$

Now, $T = \frac{V}{|V|} = \frac{2\vec{i} - 2t\vec{j}}{2\sqrt{1+t^2}}$

$$= \frac{\cancel{2}(\vec{i} - t\vec{j})}{\cancel{2}(1+t^2)^{+1/2}}$$

$$\frac{dT}{dt} = \left(\frac{d(1+t^2)^{-1/2}}{d(1+t^2)} \times \frac{d(1+t^2)}{dt}\right)\vec{i} - \frac{d(1+t^2)^{-1/2}}{dt} \cdot t$$

Rough:-

1st part

$$\frac{d(1+t^2)^{-1/2}}{d(1+t^2)} \times \frac{d(1+t^2)}{dt} = -\frac{1}{2}(1+t^2)^{-3/2} \cdot 2t$$

$$= -\frac{t}{(1+t^2)^{3/2}}$$

2nd Part,

$$\frac{d\left(\frac{t}{(1+t^2)^{-1/2}}\right)}{dt} \qquad \boxed{\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$$= \frac{(1+t^2)^{1/2}\frac{dt}{dt} - t\frac{d(1+t^2)^{1/2}}{dt}}{(1+t^2)^{2/2}}$$

$$= (1+t^2)^{1/2} - t\,\frac{1}{2}(1+t^2)^{-1/2} \times 2t \,/\, v^2$$

$$= (1+t^2)^{1/2} - t^2(1+t^2)^{-1/2} \,/\, v^2$$

$$= (1+t^2)^{1/2} - \frac{t^2}{(1+t^2)^{1/2}} \Big/ (\sqrt{1+t^2})^2$$

$$= \frac{\frac{(1+t^2) - t^2}{(1+t^2)^{1/2}}}{1+t^2}$$

$$= \frac{1+t^2-t^2}{(\sqrt{1+t})^{2/2}\cdot(1+t^2)} = \frac{1}{(1+t^2)^{3/2}}$$

$$\therefore \frac{dT}{dt} = \frac{-t}{(1+t^2)^{3/2}}\vec{i} - \frac{1}{(1+t^2)^{3/2}}\vec{j}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\left\{-\frac{t}{(1+t^2)^{3/2}}\right\}^2 + \left\{\frac{-1}{(1+t^2)^{3/2}}\right\}^2}$$

$$= \sqrt{t^2(1+t^2)^{-3} + (1+t^2)^{-3}}$$

$$= \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}}$$

$$= \sqrt{\frac{(t^2+1)}{(1+t^2)^3}}$$

$$= \sqrt{(t^2+1)^{1-3}}$$

$$= \sqrt{(t^2+1)^{-2}}$$

$$= \frac{1}{t^2+1} \qquad \therefore \left|\frac{dT}{dt}\right| = \frac{1}{1+t^2}$$

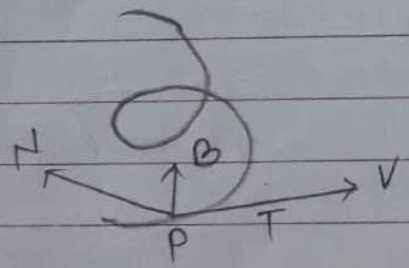
Now,

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = -\frac{t}{\sqrt{1+t^2}}\vec{i} - \frac{1}{\sqrt{1+t^2}}\vec{j}$$

$$\therefore \text{Curvature } (k) = \frac{1}{2}(1+t^2)^{3/2} \ \#$$

Unit Binomial Vector And Torsion

Unit Binomial Vector:-



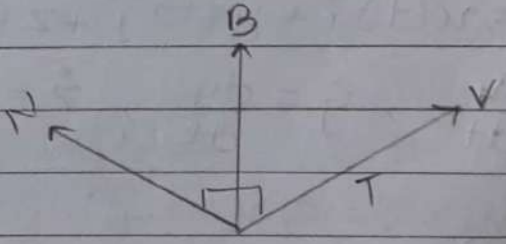


fig: TNB frame

$$\boxed{B = T \times N}$$

Unit Binomnal Vector :- (B)

* TNB are mutually orthogonal to each other and it defines a moving right handed vector frame that play a significant role in calculation paths of particles moving through space. It is called as frenet frame TNB frame.

$B = T \times N$

T = Unit tangent Vector $\rightarrow T = \frac{v}{|v|}$

N = Principle Unit normal $\rightarrow N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

Torsion (τ) :-

Torsion measures how the curve twist. It is negative scalar product of change of binomal with respect to Arc length and principle Unit normal Vector.

$$\boxed{\tau = -\frac{dB}{ds} \cdot N} \quad \text{OR}$$

$$\boxed{\therefore \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}}$$

$\dot{x}$ = 1st derivative = $\frac{dx}{dt}$

$\ddot{x}$ = 2nd derivative = $\frac{d\dot{x}}{dt}$

$\dddot{x}$ = 3rd derivative = $\frac{d\ddot{x}}{dt}$

$r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, $\dot{z} = \frac{dz}{dt}$

Similary $\ddot{x} = \frac{d\dot{x}}{dt}$, $\dddot{x} = \frac{d\ddot{x}}{dt}$

Alternative,

$$\boxed{\text{Curvature } (k) = \frac{|v \times a|}{|v|^3}}$$

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Q. Find the curvature and Torsion for:

a.) $r(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k}$, $a,b \geq 0$, $a^2+b^2 \neq 0$.

→ Solution,

$$v = \frac{dr}{dt} = (-a\sin t)\vec{i} + (a\cos t)\vec{j} + b\vec{k}$$

$$a = \frac{dv}{dt} = (-a\cos t)\vec{i} + (-a\sin t)\vec{j}$$

$$|v| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}$$
$$= \sqrt{a^2+b^2}$$

Now,

$$v \times a = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= i\begin{pmatrix} a\cos t & b \\ -a\sin t & 0 \end{pmatrix} - \vec{j}\begin{pmatrix} -a\sin t & b \\ -a\cos t & 0 \end{pmatrix} + \vec{k}\begin{pmatrix} -a\sin t & a\cos t \\ -a\cos t & -a\sin t \end{pmatrix}$$

$$= (ba\sin t)\vec{i} - (+ab\cos t)\vec{j} + (a^2\sin^2 t + a^2\cos^2 t)\vec{k}$$
$$= (ab\sin t)\vec{i} - (ab\cos t)\vec{j} + a^2\vec{k}$$

Again,

$$|v \times a| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2}$$
$$= \sqrt{a^2b^2(\sin^2 t + \cos^2 t) + a^4}$$
$$= \sqrt{a^2(a^2+b^2)}$$
$$= a\sqrt{a^2+b^2}$$

$$\therefore \text{Curvature} = \frac{|v \times a|}{|v|^3} = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = \frac{a}{(\sqrt{a^2+b^2})^{3-1}} = \frac{a}{a^2+b^2} \#.$$

here,

$x = a\cos t$

$y = a\sin t$

$z = bt$

we have,

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= b \begin{vmatrix} -a\cos t & -a\sin t \\ a\sin t & -a\cos t \end{vmatrix} - 0 + 0$$

$$= b\,(a^2\cos^2 t + a^2\sin^2 t)$$

$$= a^2 b$$

$$\therefore \text{Torsion } (\tau) = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$

$$= \frac{a^2 b}{(a\sqrt{a^2+b^2})^2}$$

$$= \frac{ab}{a(\sqrt{a^2+b^2})^2} \#.$$

$$= \frac{b}{a^2+b^2} \#.$$

b.> $r(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + at\,\vec{k}$

→ Solution,

$$v = \frac{dr}{dt} = (-a\sin t)\vec{i} + (a\,\cancel{\sin t}\cos)\vec{j} + a\vec{k}$$

$$a = \frac{dv}{dt} = (-a\cos t)\vec{i} + -(a\,\cancel{\cos}\sin)\vec{j}$$

$$|v| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2}$$

$$= \sqrt{2a^2}$$

$$v \times a = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$= \vec{i}(a^2\sin t) - \vec{j}(a^2\cos t) + \vec{k}(a^2\sin^2 t + a^2\cos^2 t)$

$= (a^2\sin t)\vec{i} - (a^2\cos t)\vec{j} + a^2\vec{k}$

$|v \times a| = \sqrt{a^4\sin^2 t + a^4\cos^2 t + a^4}$

$= \sqrt{2a^4}$

$= a^2\sqrt{2}$

Curvature $(k) = \dfrac{|v \times a|}{|v|^3} = \dfrac{\cancel{a^2}\cancel{\sqrt{2}}}{2a^3\cancel{\sqrt{2}}} = \dfrac{a^2\sqrt{2}}{a^3 2.\sqrt{2}}$

$= \dfrac{a^2\sqrt{2}}{(a\sqrt{2})^3}$ $\qquad = 1/2a$

$= \dfrac{\cancel{a^2}\cancel{\sqrt{2}}}{a^3 2\cancel{\sqrt{2}}}$

$= \dfrac{1}{2a}$ #.

Now,

$x = a\cos t$, $y = a\sin t$ $\quad z = at$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} -a\sin t & a\cos t & a \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= a\begin{vmatrix} -a\cos t & -a\sin t \\ a\sin t & -a\cos t \end{vmatrix} - 0 + 0$$

$= a(a^2\cos^2 t + a^2\sin^2 t)$

$= a^3$

Now,

$\tau = \dfrac{a^3}{(a^2\sqrt{2})^2} = \dfrac{\cancel{a^3}}{a^4 2} = \dfrac{1}{2a}$ #.

c) $r(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$

→ Solution,

$v = \frac{dr}{dt} = (3\cos t)\vec{i} + (-3\sin t)\vec{j} + 4\vec{k}$

$a = \frac{dv}{dt} = (-3\sin t)\vec{i} + (-3\cos t)\vec{j} + 0$

$|v| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (4)^2}$

$= \sqrt{9(\cos^2 t + \sin^2 t) + 16}$

$= \sqrt{25}$

$= 5$

$$|v \times a| = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} \end{array}$$

$= i(12\cos t) - j(12\sin t) + k(-9\cos^2 t - 9\sin^2 t)$

$= (12\cos t)\vec{i} - (12\sin t)\vec{j} - 9\vec{k}(\cos^2 t + \sin^2 t)$

$= (12\cos t)\vec{i} - (12\sin t)\vec{j} - 9\vec{k}$

$|v \times a| = \sqrt{144\cos^2 t + 144\sin^2 t + 81}$

$= \sqrt{144(\cos^2 t + \sin^2 t) + 81} = \sqrt{225} = 15$

Now,

$\text{curvature}(k) = \frac{|v \times a|}{|v|^3} = \frac{15}{(5)^3} = \frac{15}{125} = 0.12$

$x = 3\sin t \quad z = 4t$

$y = 3\cos t$

$$\therefore \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}$$

$= 4\begin{vmatrix} -3\sin t & -3\cos t \\ -3\cos t & 3\sin t \end{vmatrix} - 0 + 0$

$= 4(-9\sin^2 t - 9\cos^2 t)$

$= 4 \times (-9)(\sin^2 t + \cos^2 t) = -36$

$$\therefore \text{Torsion } (\tau) = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2} = \frac{-36}{(15)^2} = \frac{-36}{225} = -0.16 \#$$

Also find unit binormal vector.

$B = T \times N$

$r(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$

→ Solution,

$v = \frac{dr}{dt} = (3\cos t)\vec{i} + (-3\sin t)\vec{j} + 4\vec{k}$

$|v| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$

Now,

Unit tangent vector $(T) = \frac{v}{|v|} = \frac{(3\cos t)\vec{i} + (-3\sin t)\vec{j} + 4\vec{k}}{5}$

Again,

$\frac{dT}{dt} = \frac{(-3\sin t)\vec{i} + (-3\cos t)\vec{j}}{5}$

$\left|\frac{dT}{dt}\right| = \sqrt{\frac{9\sin^2 t + 9\cos^2 t}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\therefore$ Unit binormal vector $= T \times N$

$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = \frac{(-3\sin t)\vec{i} + (-3\cos t)\vec{j}}{\cancel{5}} \times \frac{\cancel{5}}{3}$

$= (-\sin t)\vec{i} + (-\cancel{3}\cos t)\vec{j}$

$$\therefore B = T \times N = \begin{vmatrix} i & j & k \\ \frac{3\cos t}{5} & \frac{-3\sin t}{5} & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$= (4\cos t)\vec{i} - (4\sin t)\vec{j} + \vec{k}(-3\cos^2 t - 3\sin^2 t)$

$= (4\cos t)\vec{i} - (4\sin t)\vec{j} - \frac{3}{5}\vec{k}(\cos^2 t + \sin^2 t)$

$\therefore$ Unit binormal vector $= (4\cos t)\vec{i} - (4\sin t)\vec{j} - 3\vec{k}$ #.

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d) $r(t) = (e^t \cos t)\vec{i} + (e^t \sin t)\vec{j} + 2\vec{k}$

→ Solution,

$$\frac{d(e^t \cdot \cos t)}{dt} = e^t \cdot \frac{d \cos t}{dt} + \cos t \cdot \frac{de^t}{dt}$$
$$= e^t(-\sin t) + \cos t \cdot e^t$$
$$= e^t(\cos t - \sin t)$$

$$\frac{d(e^t \sin t)}{dt} = e^t \cdot \frac{d \sin t}{dt} + \sin t \frac{de^t}{dt}$$
$$= e^t \cdot \cos t + \sin t(e^t)$$
$$= e^t(\sin t + \cos t)$$

$$v = \frac{dr}{dt} = e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j} + 0$$
$$= e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j}$$

$$\frac{d\, e^t(\cos t - \sin t)}{dt} = e^t(-\sin t - \cos t) + (\cos t - \sin t)e^t$$
$$= e^t(-\sin t - \cancel{\cos t} + \cancel{\cos t} - \sin t)$$
$$= e^t(-2\sin t)$$

$$\frac{d\, e^t(\sin t + \cos t)}{dt} = e^t(\cos t - \sin t) + (\sin t + \cos t).e^t$$
$$= e^t(\cos t - \cancel{\sin t} + \cancel{\sin t} + \cos t)$$
$$= e^t(2\cos t)$$

$$a = \frac{dv}{dt} = e^t(-2\sin t)\vec{i} + e^t(2\cos t)\vec{j}$$

$$|v| = \sqrt{(e^t)^2(\cos t - \sin t)^2 + (e^t)^2(\sin t + \cos t)^2}$$
$$= e^t\sqrt{\cos^2 t - \cancel{2\cos t \cdot \sin t} + \sin^2 t + \sin^2 t + \cancel{2\sin t \cos t} + \cos^2 t}$$
$$= e^t\sqrt{2\cos^2 t + 2\sin^2 t}$$
$$= e^t\sqrt{2(\cos^2 t + \sin^2 t)}$$
$$= e^t\sqrt{2}$$

$$v \times a = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ e^t(-2\sin t) & e^t(2\cos t) & 0 \end{vmatrix}$$

$\vec{k}$)

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$= (e^t)^2 (2\cos^2 t - 2\sin t \cdot \cos t) - (e^t)^2 (-2\sin^2 t - 2\sin t \cdot \cos t)$
$= (e^t)^2 \{ 2\cos^2 t - \cancel{2\sin t \cdot \cos t} + 2\sin^2 t + \cancel{2\sin t \cdot \cos t})$
$= (e^t)^2 \, 2 (\sin^2 t + \cos^2 t)$
$= 2(e^t)^2.$

Now,

$$\text{curvature } (k) = \frac{|v \times a|}{|v|^3}$$

$$= \frac{2e^{2t^2}}{e^{3t^3}(\sqrt{2})^3}$$

$$= \frac{2}{e^t \, 2\sqrt{2}}$$

$$= \frac{1}{e^t\sqrt{2}} \ \#$$

$x = e^t \cos t$, $y = e^t \sin t$ $\quad z = 2$

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ e^t(-2\sin t) & e^t(2\cos t) & 0 \\ e^t(-2\cos t) - (2\sin t) & e^t(2\cos t) - (2\sin t) & 0 \end{vmatrix}$$

$= 0$

$\because$ Now, Torsion $(T) = \dfrac{0}{(2(e^t)^2)^2}$

$= 0 \ \#$

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Formulae from Vector Valued function.

1.) Velocity (v) $= \frac{dr}{dt}$

2.) Speed (s) $= |V|$

3.) acceleration (a) $= \frac{dv}{dt}$

4.) length (L) $= \int_{t=a}^{t=b} |V| \, dt$

5.) Unit Tangent Vector (T) $= \frac{V}{|V|}$

6.) curvature (k) $= \frac{1}{|V|} \left| \frac{dT}{dt} \right|$

or, curvature (k) $= \frac{|V \times a|}{|V|^3}$

7.) Principle Unit Normal (N) $= \frac{\frac{dT}{dt}}{\left| \frac{dT}{dt} \right|}$

8.) Unit Binomal Vector (B) $= T \times N$

9.) Torsion (τ) $= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|V \times a|^2}$

$\dot{x}$ = first der.
$\ddot{x}$ = 2nd der.
$\dddot{x}$ = 3rd der.

10. Radius of curvature (R) $= \frac{1}{k}$

Vector Formula:

i) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|.|\vec{b}|}$ ii) $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

iii) Unit Vect $= \dfrac{\vec{v}}{|\vec{v}|}$ iv) Unit Vect $= \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

v) Area of parallelogram $= |\vec{a}\times\vec{b}|$

vi) Equation of plane $= A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

vii) Parametric eq^n ; $x = x_1 + At$ $y = y_1 + Bt$ $z = z_1 + ct$

viii) Distance from s to line $(d) = \dfrac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$

ix) Distance from s to a plane $(d) = \left|\vec{PS}\cdot\dfrac{n}{|n|}\right|$

x) Angle between two plane,

$$\cos\theta = \frac{n_1 . n_2}{|n_1|.|n_2|}$$

xi) Vector parallel to line $= n_1 \times n_2$

xii) Polar equation

$$x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2+y^2$$

$$\theta = \tan^{-1}(y/x)$$

xiii) $r = \dfrac{ke}{1+e\cos\theta}$, $x = k$ xiv) $r = \dfrac{ke}{1-e\sin\theta}$, $y = -k$

xv) Cylinderical co-ordinates $= (r, \theta, z)$

xvi) Spherical coordinates $= (\rho, \phi, \theta)$, $\rho^2 = x^2+y^2+z^2$, $z = \rho\cos\phi$

$$\tan\theta = \left(\frac{y}{x}\right) \qquad x = \rho\sin\phi.\cos\theta \qquad y = \rho\sin\phi.\sin\theta$$

Infinite Series:

Expression in the form of

$a_1 + a_2 + a_3 + \dots\dots + a_n + \dots\dots$ is called infinite series, where,

a_n is n^{th} term of infinite series.

Sequence form;

$S_1 = a_1$

$S_2 = a_2 + a_1$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots\dots + a_n$

$$\boxed{\therefore S_n = \sum_{n=1}^{n} a_n}$$

is a sequence of partial sum.

where,

S_n is a n^{th} partial sum.

Convergent Series:-

If the sequence of the partial sum of series converses to the certain value or limit, then this series is known as Convergent series.

$$a_1 + a_2 + a_3 + \dots\dots + a_n + \dots\dots = \sum_{n=1}^{\infty} a_n = L$$

or, $= \sum_{n=1}^{\infty} a_n = 0$, it is called divergent series.

Test:

i) Geometric series Test

ii) Ratio test

iii) Root Test

iv) limit Comparison test

v) Integral test.

* Geometric Series test :-

$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$

Common ratio $(r) = \frac{t_2}{t_1} = \frac{ar}{a} = r$

when,

$|r| < 1$

then the series converges to $S_\infty = \frac{a}{1-r}$

$|r| > 1$ or infinite,

then the series is diverges.

Q. Test whether the given series are convergent or divergent.

1.> $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

→ Solution,

series $= 1 - \frac{1}{4} + \frac{1}{4^2} - \ldots\ldots + \infty$

Common ratio $(r) = \frac{t_2}{t_1} = -\frac{\frac{1}{4}}{1} = -\frac{1}{4}$

$|r| = \frac{1}{4} < 1$, so series is converged.

Sum of series $(S_\infty) = \frac{a}{1+\frac{1}{4}} = \frac{4}{5} = 0.8$

2.> $\sum_{n=1}^{\infty} \frac{7}{4^n}$

→ Series $= \frac{7}{4} + \frac{7}{4^2} + \frac{7}{4^3} + \ldots + \infty$

∴ common diff$(r) = \frac{t_2}{t_1} = \frac{7}{4^2} \times \frac{4}{7} = \frac{1}{4}$

$|r| = \frac{1}{4} < 1$, So it is convergent

sum$(S_\infty) = \frac{a}{1-r}$

$= \frac{1}{1-\frac{1}{4}}$

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$= \frac{7}{\cancel{4}} \times \frac{\cancel{4}}{3} = \frac{7}{3}$ #.

3) $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

→ Series $= +5 - \frac{5}{4} + \frac{5}{4^2} + \cdots + \infty$

$r = \frac{t_2}{t_1} = -\frac{5}{4} \times \frac{1}{5} = -\frac{1}{4}$

$|r| = \frac{1}{4} < 1$, so it is convergent

Sum of series $(S_\infty) = \frac{a}{1-r} = \frac{5}{1+\frac{1}{4}} = \frac{5}{1} \times \frac{4}{5} = 4$ #.

4) $\sum_{n=1}^{\infty} 9^{-n+2} . 4^{n+1}$

→ Series $= 9.4^2 + 9.4^3 + 9^{-1}.4^4 + \cdots + \infty$

$r = \frac{t_2}{t_1} = \frac{64}{144} = 0.44$

$\therefore |r| = 0.44 < 1$.

So, it is convergent, Sum $= \frac{a}{1-r} = \frac{144}{1-0.44} = \frac{144}{0.56} = 259.2$

5) $\sum_{n=6}^{\infty} (-1)^n \left(\frac{2^{n+3}}{3^n}\right)$

→ Series $= \frac{2^9}{3^6} - \frac{2^{10}}{3^7} + \frac{2^{11}}{3^8} \cdots \infty$

$|r| = \left|\frac{t_2}{t_1}\right| = \left|-\frac{2^{10}}{3^7} \times \frac{3^6}{2^9}\right| = \frac{2^{10-9}}{3^{7-6}} = \frac{2}{3} = 0.66$

$= \frac{746496}{1119744}$

$|r| < 1$, so Converged.

$S_\infty = \frac{a}{1-r} = \frac{0.702}{1+0.66} = 0.42$ #.

6. $\sum_{n=1}^{\infty} \frac{2^{n+1}+9^{n+2}}{5^n}$

→ series $= \frac{2^2+9^3}{5} + \frac{5^3+9^4}{5^2} + \frac{5^4+9^5}{5^3} + \ldots\ldots$

$a = \frac{2^2+9^3}{5} = 146.6$

$r = \frac{t_2}{t_1} = \frac{267.44}{146.6} = 1.82$

$|r| > 1$, so it is divergent.

7.> $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

$= \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$

$= \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) + \left(1 - \frac{1}{5} + \frac{1}{5^2} - \ldots\right)$

Common difference $(r_1) = \frac{1}{2} \times 1 = \frac{1}{2}$

Common diff. $(r_2) = -\frac{1}{5} \times 1 = -\frac{1}{5}$

$|r_1| =< 1$, so it is converged.

$|r_2| < 1$, so it is also converged

Sum $(S_{\infty 1}) = \frac{a_1}{1-r_1} = \frac{1}{1-\frac{1}{2}} = \frac{1}{1} \times \frac{2}{1} = 2$

Sum $(S_{\infty 2}) = \frac{a_1}{1-r_2} = \frac{1}{1+\frac{1}{5}} = \frac{1}{1.2} = 0.83$

$\therefore$ Sum $(S_\infty) = 2 + 0.83 = 2.83$ Ans.

8.> $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$.

Series $= 1 - \frac{64}{1} + \frac{4096}{5} + \cdots \infty$

$r = \frac{-64}{1} = -64$ $\quad |r| = 64 > 1$, so it is divergent.

9. $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

→ solution,

$$\sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} \right) - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) - \left(1 + \frac{1}{6} + \frac{1}{6^2} + \cdots \right)$$

Common diff $(r_1) = \frac{1}{2}$ $\quad \because |r| < 1$, so,

Common diff $(r_2) = \frac{1}{6}$ $\quad |r| < 1$, they are convergent

Sum $(S_{\infty_1}) = \frac{a_1}{1 - r_1} = \frac{1}{1 - 1/2} = 2$

Sum $(S_{\infty_2}) = \frac{a_1}{1 - r_2} = \frac{1}{1 - 1/6} = \cancel{12}\ \frac{1}{1} \times \frac{6}{5} = 1.2$

$\therefore$ Sum $(S_\infty) = 2 - 1.2 = 0.8$ #.

* Ratio Test :-

Let $\sum a_n$ be series with positive terms such that,

$$\boxed{\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L}$$

then,

a) If $L < 1$, then convergent series

b) If $L > 1$ or ∞, then divergent series.

c) If $L = 1$, then test is inconclusive

Note:- When the question is in factorial form, then Use ratio test. $n!$ ....

* Investigate the convergence of following series.

a) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

→ Solution,

Given that, $a_n = \frac{2^n+5}{3^n}$

$a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

Now,

$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \Big/ \frac{2^n+5}{3^n}$

or, $L = \lim_{n\to\infty} \frac{2^n.2+5}{\cancel{3^n}.3} \times \frac{\cancel{3^n}}{2^n+5}$

or, $L = \lim_{n\to\infty} \frac{\cancel{2^n}(2+\frac{5}{2^n})}{3.\cancel{2^n}(1+\frac{5}{2^n})}$

take n common now as you can take n common always

or, $L = \frac{(2+\frac{5}{\infty})}{3(1+\frac{5}{\infty})}$

$[\infty+1 = 0]$

$[n^\infty = 0]$

$2^n = 2^\infty = \infty$

$\frac{1}{\infty} = 0$

or, $L = \frac{2}{3}$

$\therefore L < 1$, so it is Convergent.

b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

→ Given, $a_n = \frac{2^n}{n!}$ $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$n! = n(n-1)(n-2)!$

$\infty + 1 = \infty \qquad \frac{1}{\infty} = 0$

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Now,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n} = \frac{\cancel{a^n}\cdot a}{(n+1)\,\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{a^n}}$$

$$\text{or, } L = \lim_{n\to\infty} \frac{a}{n+1}$$

$$\text{or } L = \frac{a}{\infty} = 0 < 1,$$

So it is convergent.

3. $\sum_{n=1}^{\infty} \frac{4^n n!\, n!}{(2n)!}$

$\rightarrow a_n = \frac{4^n n!\, n!}{(2n)!}, \quad a_{n+1} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!}$

Now,

$$L = \lim_{n\to\infty} \frac{4^n\cdot 4\,(n+1)!\,(n+1)!}{(2n+2)!} \times \frac{(2n)!}{4^n\, n!\, n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{4^n}\cdot 4\,(n+1)\,\cancel{n!}\,(n+1)\,\cancel{(n)!}}{(2n+2)(2n+1)\,\cancel{(2n)!}} \times \frac{\cancel{(2n)!}}{\cancel{4^n}\,\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{4\ \cancel{n}(1+1/n)\,\cancel{n}(1+1/n)}{\cancel{n}(2+2/n)\,\cancel{n}(2+1/n)}$$

$$= \frac{4\,(1+0)(1+0)}{(2+0)(2+0)}$$

$$= \frac{4}{4}$$

$$= 1$$

$\therefore L = 1$, so it is inconclusive.

4. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

$\rightarrow$ Solution $a_n = \frac{n^2}{3^n}, \quad a_{n+1} = \frac{(n+1)^2}{3^{(n+1)}}$

$$L = \lim_{n\to\infty} \frac{(n+1)^2}{3^{(n+1)}} \times \frac{3^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{\cancel{3^n}.3} \times \frac{\cancel{3^n}}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}\right)$$

$$= \lim_{n\to 0} \frac{1}{3}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$= \frac{1}{3}(1+0+0) = L = 1/3 < 1, \text{ so it is convergent.}$$

5.) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

→ $a_n = \frac{(2n)!}{n!\,n!}$. $a_{n+1} = \frac{(2n+1)!}{(n+1)!\,(n+1)!}$

Now,

$$L = \lim_{n\to\infty} \frac{(2n+1)!}{(n+1)!\,(n+1)!} \times \frac{n!\,n!}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)\,\cancel{(2n)!}\,(2n+1)}{(n+1)\,\cancel{n!}\,(n+1)\,\cancel{n!}} \times \frac{\cancel{n!\,n!}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+2/n)\,\cancel{n}(2+1/n)}{\cancel{n}(1+1/n)\,\cancel{n}(1+1/n)}$$

$$= (2+0)\,(2+0)$$

$$= 4$$

$\therefore L = 4 > 1$, so it is divergent.

6. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\left(\text{Formula: } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right)$

$\rightarrow a_n = \frac{(n)!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Now,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$$

$\left(1+\frac{x}{n}\right)^x = e^x$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\, n!}{(n+1)^n \cdot \cancel{(n+1)^1}} \times \frac{n^n}{\cancel{n!}}$$

$\left(1+\frac{x}{n}\right)^x = e^x$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left[\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right]^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{(1+1/n)^n}\right) \quad \left[\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^x = e^x\right.$$

$$= \frac{1}{e}$$

$\frac{1}{(1+1/n)^n}$

7) $\sum_{n=1}^{\infty} \frac{n\, 2^n (n+1)!}{3^n\, n!}$

$\rightarrow a_n = \frac{n\, 2^n (n+1)!}{3^n\, n!}$, $a_{n+1} = \frac{(n+1)\, 2^{n+1} (n+1+1)!}{3^{n+1} (n+1)!}$

Now

$$L = \lim_{n\to\infty} \frac{(n+1)\, \cancel{2^n} \cdot 2\, (n+2)!}{\cancel{3^n} \cdot 3\, (n+1)!} \times \frac{\cancel{3^n}\, n!}{n\, \cancel{2^n} \cdot (n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\, 2\, (n+2)\, \cancel{(n+1)!}}{3(n+1)\, \cancel{(n)!}} \times \frac{\cancel{n!}}{n\, \cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{\cancel{n}(1+1/n) \cdot \cancel{n}(1+2/n)}{\cancel{n}(1+1/n) \cdot \cancel{n}}$$

$$= \frac{2}{3} \frac{(1+0)(1+0)}{1+0}$$

$L = \frac{2}{3} < 1$, so it is Convergent.

$3! = 3 \times 2 \times 1 = 6$

8.) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$

→ Solution

$a_n = \frac{(n+3)!}{3!\, n!\, 3^n}$, $a_{n+1} = \frac{(n+1+3)!}{3!\,(n+1)!\, 3^{n+1}}$

$L = \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+4)!}{\cancel{3!}\,(n+1)!\,\cancel{3^n}.3} \times \frac{\cancel{3!}\, n!\, \cancel{3^n}}{(n+3)!}$

$= \lim_{n\to\infty} \frac{(n+4)\,\cancel{(n+3)!}}{(n+1)\,\cancel{(n)!}\; 3} \times \frac{\cancel{n!}}{\cancel{(n+3)!}}$

$= \lim_{n\to\infty} \frac{\cancel{n}(1+4/n)}{\cancel{n}(1+1/n)\,3}$

$= \frac{1+0}{(1+0)3}$

$\therefore L = \frac{1}{3} < 1$, so it is convergent.

Root Test:-

Let $\sum a_n$ be a series, such that,

$\lim_{n\to\infty} \sqrt[n]{a_n} = L$

$\sqrt[n]{a_n} = (a_n)^{1/n}$

$n^{1/n} = 1$

then,

a.) if $L < 1$, the series is convergent.

b.) if $L > 1$, or infinite, then series is divergent.

c.) if $L = 1$, then inconclusive.

Note:

if $\lim_{n\to\infty} \sqrt[n]{n} = n^{1/n} = 1$

$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

$\lim_{n\to\infty} \frac{d(a)^x}{dx} = a^x \ln a$

Test the convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

→ Solution, $a_n = \frac{n^2}{2^n}$

$L = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n}$

$= \lim_{n\to\infty} \frac{((n)^{1/n})^2}{2^{\cancel{n} \times 1/\cancel{n}}}$

$\therefore L = \frac{1}{2} < 1$, so it is convergent.

2. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

→ Solution, $a_n = \frac{2^n}{n^2}$

$\therefore L = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{2^{\cancel{n} \times 1/\cancel{n}}}{(n^{1/n})^2} = \frac{2}{1}$

$\therefore L = 2 > 1$, so it is divergent.

3. $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

→ Solution,

$a_n = \left(\frac{1}{1+n}\right)^n$

$L = \lim_{n\to\infty} \left(\frac{1}{1+n}\right)^{\cancel{n} \times 1/\cancel{n}} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0$

$\therefore L = 0 < 1$, so it is convergent.

4. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

→ Solution;

$$L = \lim_{n\to\infty} \left(\frac{n^{10}}{10^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{(n^{1/n})^{10}}{10^{n\times 1/n}}$$

$$= \frac{1^{10}}{10}$$

$\therefore L = \frac{1}{10} = 0.1 < 1$, so it is convergent.

5. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

→ $a_n = \left(\frac{n}{3n+1}\right)^n$

$$L = \lim_{n\to\infty} \left(\frac{n}{3n+1}\right)^{n\times 1/n} = \lim_{n\to\infty} \frac{n}{n(3+1/n)} = \frac{1}{3+0} = \frac{1}{3}$$

$\therefore L = 1/3 < 1$. so it is convergent.

6. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

→ Solution, $a_n = \left(1+1/n\right)^{-n^2}$

$$L = \lim_{n\to\infty} \left(1+1/n\right)^{-n^2\times 1/n} \qquad \left[\therefore \left(1+\frac{x}{n}\right)^n = e^x\right]$$

$$= \lim_{n\to\infty} \left(1+1/n\right)^{-n}$$

$\therefore L = e^{-1} = 1/e = 0.36 < 1$, so it is convergent.

7. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

or, $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

or, $L = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2\times 1/n}$

or, ~~$L = \lim_{n\to 0} \left(\frac{n}{n} + \frac{n}{1}\right)^n$~~

~~$= \lim_{n\to 0} (1+$~~

or, $L = \lim_{n\to\infty} \left(\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right)^n$

or, $L = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$

or, $L = \frac{1}{e} = 0.36 < 1$, So it is convergent.

8. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$ $\left[\frac{d(a^x)}{dx} = a^x \ln a\right]$

→ Solution, $a_n = \frac{n^n}{2^{n^2}}$

$\therefore L = \lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{\cancel{n} \times 1/\cancel{n}} = \lim_{n\to\infty} \left(\frac{n}{2^n}\right)$

$= \lim_{n\to\infty} \frac{1}{2^n \ln 2}$ [$\because$ L' Hospital Rule]

$= \frac{1}{\infty}$

$\therefore L = 0 < 1$, So, it is convergent.

* Integral Test (P-Series Test) :-

$\int_{x=a}^{x=b} f(x) = dx =$ finite Value ⇒ convergent series.

$\int_{x=a}^{x=b} f(x) = dx =$ infinite value ⇒ Divergent series.

P-series Test (P-Test) :-

$\left(\sum_{n=1}^{\infty} \frac{1}{n^P} = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \dots\dots\right)$

$$\int_{x=1}^{x=\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty}$$

$$= \lim_{a\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^{a}$$

$$= \frac{1}{1-p}\lim_{a\to\infty}\left[x^{-p+1}\right]_1^{a}$$

when,

$p > 1$ let $p = 2$ $\quad x^{-2+1} = x^{-1} = \frac{1}{x}$

$p > 1$

$$\lim_{a\to\infty} = \frac{1}{1-p}\left[\frac{1}{x^{p-1}}\right]_1^{a}$$

$$= \frac{1}{1-p}\lim_{a\to\infty}\left[\frac{1}{a^{p-1}} - \frac{1}{1^{p-1}}\right]$$

$$= \frac{1}{1-p}(0-1)$$

$$= \frac{1}{p-1} = (\text{finite})$$

Hence the Series is convergent.

Eg when $p > 1$.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$ convergent.

$p < 1$ $\quad \infty$

when $p < 1$.

$$= \frac{1}{1-p}\lim_{a\to\infty}\left[x^{1-p}\right]_1^{a}$$ let $p = -1$ $\quad x^{1+1} = x^2$

$$= \frac{1}{1-p}\lim_{a\to\infty}\left[a^{1-p} - 1^{1-p}\right]$$

$$\boxed{\begin{array}{l}\infty + 1 = \infty \\ \infty - 1 = \infty \\ \infty \times 1 = \infty \\ \infty / 1 = \infty\end{array}}$$

$$= \frac{1}{1-p}\times\infty$$

$$= \infty \quad \text{divergent.}$$

$\frac{1}{\infty} = 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots\cdots$$

when $p = 1$ $\qquad p = 1$

$$= \frac{1}{1-P} \lim_{a \to \infty} \left[x^{1-P} \right]_1^a$$

$$= \frac{1}{1-P} \lim_{a \to \infty} \left[a^{1-P} - 1^{1-P} \right]$$

$$= \frac{1}{0} \left[a^{1-1} - 1^{1-1} \right]$$

$$= \infty \times 1$$

$$= \infty \quad \text{(divergent)}$$

e.g. $\sum_{n=1}^{\infty} 1/n = 1/1 + 1/2 + 1/3 + \cdots\cdots$

$\therefore$ Harmonic Series = divergent.

* P-Test by Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{n^P} = \frac{1}{1^P} + \frac{1}{2^P} + \frac{1}{3^P} + \cdots\cdots$$

when $P > 1$, then series is convergent.

when $P \le 1$, Then series is divergent

when $P = 1$, Then series is inconclusive.

$$\sum_{n=1}^{\infty} \frac{1}{n^{P=1}} = 1 + 1/2 + 1/3 + 1/4 + \cdots\cdots$$

Harmonic Series are always divergent.

$$\sum_{x=1}^{\infty} \frac{1}{x}\,dx = \left[\log x\right]_1^{\infty} = \log\infty - \log 1$$
$$= \infty$$

Harmonic Series is Divergent.

$\frac{1}{\infty} = 0$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

→ Ratio Test, $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) \times \frac{n}{1} = \frac{\cancel{n}}{\cancel{n}(1+1/n)} = 1$$

∴ $L = 1$, so test is inconclusive.

example:- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots\ldots$

$1 + \frac{1}{2} = 1.5$ $\frac{2}{4} = \frac{1}{2} = \frac{1}{3} + \frac{1}{4} = 0.58$ $\frac{4}{8} = \frac{1}{2}$

$0.58 > \frac{2}{4} = \frac{1}{2}$ $= 0.634 > \frac{4}{8} > \frac{1}{2}$

* Limit Comparison Test:

let $\sum a_n$ and $\sum b_n$ be series then

a) when $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then series both converges or diverges.

b) when $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges then $\sum a_n$ also diverges.

to find b_n compare power and put highest power n

Questions:

a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

→ Solution,

$$a_n = \frac{2n+1}{n^2+2n+1}, \quad b_n = \frac{2n}{n^2} = \frac{2}{n}$$

$b_n = \frac{2}{n}$, $P = 1$, So by p-test b_n is divergent.

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \times \frac{n}{2} = \lim_{n\to\infty} \frac{\cancel{n^2}(2+\frac{1}{n})}{\cancel{n^2}(1+\frac{2}{n}+\frac{1}{n^2})2}$

$$= \frac{2 + \frac{1}{\infty}}{\left(1 + \frac{2}{\infty} + \frac{1}{\infty^2}\right) \times 2}$$

$$= \frac{2}{2} = 1$$

$\therefore a_n$ is also divergent series.

2.) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

→ Solution, $a_n = \frac{\sqrt{n}}{n^2+1}$, $b_n = \frac{\sqrt{n}}{n^2}$

Now,

$$b_n = \frac{n^{1/2}}{n^2} = \frac{1}{n^{2-1/2}} = \frac{1}{n^{3/2}}$$

Since $P > 1$, so it is convergent by p-test.

We have,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \times \frac{n^2}{\sqrt{n}} = \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{1+0} = 1$$

$\therefore L = 1 > 0$, so a_n also a convergent series.

4.) $\sum_{n=1}^{\infty} \frac{n^4+5}{n^5}$

→ Solution, $a_n = \frac{n^4+5}{n^5}$, $b_n = \frac{n^4}{n^5} = \frac{1}{n}$

$\therefore b_n = \frac{1}{n}$, $p = 1$, so it is divergent by p-test

Now,

$$\frac{a_n}{b_n} = \frac{n^4+5}{\cancel{n^5}} \times \frac{\cancel{n^5}}{n^4} = \lim_{n\to\infty} \frac{\cancel{n^4}\left(1+\frac{5}{n^4}\right)}{\cancel{n^4}} = 1 + 0 = 1$$

$\therefore L = 1 > 0$, so it is also divergent series.

3) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)$

→ $a_n = \sqrt{n^2+1} - n$

$= \frac{\sqrt{n^2+1} - n}{1} \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$

$= \frac{(\sqrt{n^2+1})^2 - (n)^2}{\sqrt{n^2+1}+n}$

$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

$\therefore a_n = \frac{1}{\sqrt{n^2+1}+n}$, $b_n = \frac{1}{n}$

$\therefore P = 1$, so it is divergent from P-Test.

we know,

$\frac{a_n}{b_n} = \frac{1}{\sqrt{n^2+1}+n} \times \frac{n}{1}$

$\lim_{n\to\infty} = \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+1/n^2}+1\right)}$

$\lim_{n\to\infty} = \frac{1}{\sqrt{1+0}+1}$

$\therefore L = \frac{1}{2} = 0.5 > 0$

So it is also divergent series.

5) $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

→ solution, $a_n = \frac{2n-1}{n(n+1)(n+2)}$

$= \frac{2n-1}{n^2(n+2)+n(n+2)}$ $= \frac{2n-1}{n^3+2n^2+n^2+2n}$

$= \frac{2n-1}{n^3+3n^2+2n}$

$\therefore b_n = \frac{2n}{n^3} = \frac{2}{n^2}$

$\because P > 1$, so it is convergent series by P-test.

Now,

$\frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n-1)}{n^3+3n^2+2n} \times \frac{n^2}{2}$

$= \lim_{n\to\infty} \frac{n^3(2 - 1/n^3)}{n^3(1+3/n+2/n^2)2} = \lim_{n\to\infty} \frac{2 - 1/n^3}{2(1+3/n+2/n^2)} = \frac{2}{2} = 1$

$l > 0$. / $\therefore l = 1$, so a_n is also convergent series

6. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+1}$.

→ $a_n = \frac{1}{2\sqrt{n}+1}$, $b_n = \frac{1}{2\sqrt{n}} = \frac{1}{2n^{1/2}}$

$\because P < 1$, so it is divergent by P-test.

Now,

$\frac{a_n}{b_n} = \frac{1}{2\sqrt{n}+1} \times \frac{2\sqrt{n}}{1} = \frac{\sqrt{n}(2)}{\sqrt{n}(2+1/\sqrt{n})} = \frac{2}{2+0} = 1$

$\therefore L > 0$, so a_n also a divergent series.

7. $\sum_{n=1}^{\infty} \tan(1/n)$

→ Solution, $a_n = \tan(1/n)$, $b_n = 1/n$

$b_n = 1/n = P = 1$, so it is divergent series.

Now,

$\frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{\sec^2(1/n)\cancel{(-1/n^2)}}{\cancel{(-1/n^2)}}$

[L-Hospital Rule]

$= \sec^2(0) = \frac{1}{\cos^2 0} = 1 > 0$

So it is also divergent series.

8) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

→ Solution, $a_n = \frac{1}{2^{n-1}}$, $b_n = \frac{1}{2^n}$

For b_n, using root test

$b_n = \frac{1}{2^n} = \left(\frac{1}{2^n}\right)^{1/n} = \left(\frac{1}{2}\right)^{n \times 1/n} = \frac{1}{2} = 0.5$

$\therefore L = 0.5 < 1$, so it is convergent series.

Now,

$\frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^{n-1}} \times \frac{2^n}{1} = \frac{2^n}{2^n \cdot 1} = 1$

So a_n is also convergent series.

n-term test for divergence:-

The necessary condition for the convergence of infinite series Σa_n is.

$\lim_{n\to\infty} a_n = 0$

but this is not sufficient.

Exception:-

$\sum_{n=1}^{\infty} \frac{1}{n}$ = divergent Series

$\lim_{n\to\infty} \frac{1}{n} = 0$ Convergent

* Questions :- $\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

a) $\sum_{n=1}^{\infty} n^2 = \lim_{n\to\infty} n^2 = \infty$, so it is divergent.

b) $\sum_{n=1}^{\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1)} = \lim_{n\to\infty} 1+\frac{1}{n} = 1$, Divergent.

c) $\sum_{n=1}^{\infty} \frac{-n}{2n+5} = \lim_{n\to\infty} \frac{-\cancel{n}}{\cancel{n}(2+5/n)} = -1/2$, Divergent.

d) $\sum_{n=1}^{\infty} \left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{1/2} = e^{1\cdot 1/2} = 1.64 \neq 0$
Divergent.

Alternating Series :-

when, $n = 1, 2, 3 \ldots$,

1. $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$

$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \ldots$

2. $a_1 > a_2 > a_3 > \ldots$ decreasing Orders.

3. Alternating series test :- [Leibniz Test]

* Questions :-

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

i.e. $a_1 > a_2 > a_3 > \ldots$

and also in $(+,-,+,-)$ alternative Sign

* Leibniz Test:-

$\lim_{n\to\infty} \frac{1}{n} = 0$, Convergent series.

* Absolute Convergent:-

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely of the corresponding series of $\sum |a_n|$ converges.

* $1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Leibniz Test:-

$\lim_{n\to\infty} \frac{1}{n^2} = 0$, convergent.

Absolute Convergent:-

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n^2}\right| = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$

$\sum_{n=1}^{\infty} |a_n| = \frac{1}{n^2}$ (if Leibniz & absol. come con & div/alter then series is Cond. Converged)

$\therefore$ by P test, $P>1$, so convergent.

So the series is absolutely convergent.

* Conditional Convergent:-

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

absolute value:

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$

$\therefore \sum|a_n| = \sum\left|\frac{1}{n}\right|$, by P-test, $P=1$, so series is divergent

Leibniz Test:-

$\lim_{n\to\infty} \frac{1}{n} = 0$, so convergent

$p = 1$

∴ Series is Conditional Convergent.

* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$

$\Rightarrow 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots + \frac{1}{n^3} + \cdots$

Leibniz Test:

$\lim_{n\to\infty} \frac{1}{n^3} = 0$, so Series is Convergent by This test.

Absolute Value:

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^3} \right| = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$

$\Sigma |a_n| = \Sigma \left| \frac{1}{n^3} \right|$,

Since $P > 1$, So Convergent.

So Series is Convergent

* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

$2^u du = 2^u \ln 2$

$\Rightarrow \frac{n}{2^n} = \frac{1}{2} - \frac{2}{2^2} + \frac{3}{2^3} - \frac{4}{2^4} + \cdots + \frac{n}{2^n} + \cdots$

Leibniz Test:

$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2}$ {L-Hospital}

$= 0$

So Convergent

Absolute value

$$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{n}{2^n}\right| = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$$

$$\therefore \left|\sum_{n=1}^{\infty}\right| \left|\frac{n}{2^n}\right| =$$

$$* \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n}$$

$\rightarrow$ series $= -\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots + \frac{1}{2^n} - \cdots$

Leibniz test,

$$\lim_{n\to\infty} \frac{1}{2^n} = \frac{0}{2n \ln 2} = 0, \text{ Converged.}$$

Absolute value

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{2^n}\right| = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

$$\sum_{n=1}^{\infty} \left|\frac{1}{2^n}\right| =$$

Summary

i) Geometric Series:-

If $|r| \geq 1$, then series diverges, otherwise it converges.

ii) For non-negative terms:-

Use integral test, ratio test, root test, and limit comparison test.

iii) Alternating series:-

Leibniz's test.

iv) n term test for divergence:-

Unless $a_n \to 0$

$\lim_{n\to\infty} a_n = 0$, convergent, otherwise divergent.

v) Series with absolute convergent:-

$\sum |a_n|$ = Convergent (Absolute Convergent)

Power Series:-

A series about $x=0$ in the form of,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots + a_n x^n + \dots\dots$$

is called power series.

At $x=a$

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots$$

where, a = Counter and $a_0, a_1, a_2, a_3, \dots a_n$ = Coefficient/constant.

* Test the convergence for the given power series.

a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$\rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = (-1)^{n-1+1} \frac{x^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1} x^n}$

$$= (-1)^n \frac{x^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1} x^n}$$

$$= \cancel{(-1)^n} \frac{\cancel{x^n} \cdot x}{(n+1)} \times \frac{n}{\cancel{(-1)^n} (-1)^{-1} \cancel{x^n}}$$

$$= \left| \frac{n}{n+1} \cdot x \right| \qquad = \frac{n}{n(1+1/n)} = \frac{1}{1+0} = 1$$

$$= |x|$$

$\therefore |x| < 1$, then converges absolutely.

$\therefore |x| > 1$, then divergent.

b:) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

→ Solution:

$a_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$, $a_{n+1} = (-1)^{n-1+1} \frac{x^{2(n+1)-1}}{(2(n+1)-1}$

Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{\cancel{(-1)^n} x^{2n+1}}{(2n+1)} \times \frac{(2n-1)}{\cancel{(-1)^n} \cdot (-1)^{-1} \cdot x^{2n-1}}$$

$$= \frac{\cancel{x^{2n}} \cdot x (2n-1)}{(2n+1) (-1)^{-1} \cancel{x^{2n}} \cdot x^{-1}}$$

$$= \frac{x^2 (2n-1)}{(2n+1)}$$

$$= |x^2|$$

$|x^2| < 1$, Converges absolutely

$|x^2| > 1$, then Divergent.

c:) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

→ $a_{n+1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \frac{x^{n+1}}{(n+1)?}$

Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \cancel{x} + \cancel{x} + \cancel{\frac{x^2}{2!}} + \cancel{\frac{x^3}{3!}} + \ldots$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} = \frac{\cancel{x^n}.x}{(n+1)\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{x^n}} = \frac{x}{n+1}$$

for all value of x, it is zero. i.e. <1, so converges absolutely.

d.) $\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + x^3.3! + \ldots$

$\rightarrow a_{n+1} = (n+1)!\;x^{n+1}$, $a_n = n!\,x^n$

Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\cancel{n!}\,\cancel{x^n}.x}{\cancel{n!}\,\cancel{x^n}} = (n+1)\,|x|$$

Divergent for all values of x. exception case.

Taylor Series And Maclaurin Series

Let the function 'f' with all derivatives of all orders throughout some interval containing 'a' as an interior point then the Taylor Series generated by f at $x=a$ is

Taylor polynom of order n

of order n is

$$\sum_{k=0}^{\infty} \frac{f^{k}(a)\,(x-a)^{k}}{k!}$$

Taylor series at $x=a$.

$$\sum_{k=0}^{\infty} \frac{f^{k}(a)\,(x-a)^{k}}{k!}$$

$$= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots + f^{n}(a)\frac{(x-a)^n}{n!} + \cdots$$

Maclaurin series: or

(Taylor Series at $x=0$): with no center (a).

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)\cdot x^{k}}{k!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{n}(0)\frac{x^n}{n!} + \cdots$$

Q. Find the Taylor series and polynomial for the following at $x=0$.

a) $f(x) = e^{x}$ at $x=0$

→ solution,

given function $f(x) = e^{x}$ $f(0) = e^{0} = 1$

$f'(x) = e^{x}$ $f'(0) = e^{0} = 1$

$f''(x) = e^{x}$ $f''(0) = e^{0} = 1$

$f'''(x) = e^{x}$ $f'''(0) = e^{0} = 1$

$\vdots$ $\vdots$

$f^{n}(x) = e^{x}$ $f^{n}(0) = e^{0} = 1$

Taylor series at $x=0$ (Maclaurin series) is,
(upto ∞)

$$\sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!} = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$+ \cdots\cdots + f^n(0)\frac{x^n}{n!} + \cdots\cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^n}{n!} + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Taylor Polynomial, (only upto n)

$$\sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^n}{n!}$$

~~Ex~~ Q.) $f(x) = e^{-x}$ at $x=0$

→ Given function,

$f(x) = e^{-x}$ $\quad\quad$ $f(0) = 1$

$f'(x) = -e^{-x}$ $\quad\quad$ $f'(0) = -1$

$f''(x) = e^{-x}$ $\quad\quad$ $f''(0) = 1$

$f'''(x) = -e^{-x}$ $\quad\quad$ $f'''(0) = -1$

$\vdots$ $\quad\quad$ $\vdots$

$f^n(x) = (-1)^n e^{-x}$ $\quad\quad$ $f^n(0) = (-1)^n$

Mach. ~~Taylor~~ series $= \sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!} = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!}$

$$+ f'''(0)\frac{x^3}{3!} + \cdots\cdots + \frac{x^n}{n!} + \cdots\cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots + (-1)^n\frac{x^n}{n!} + \cdots\cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots + \frac{(-x)^n}{n!} + \dots\dots$$

$$= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$

Taylor polynomial:

$$\sum_{k=0}^{n} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots + \frac{(-x)^n}{n!} \#.$$

c) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

→ Solution,

As we know,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\dots\dots$$

Now,

$$e^x + e^{-x} = \left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \dots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \dots\right)$$

$$\therefore e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots + \frac{x^{2n}}{(2n)!} + \dots\dots$$

$$\therefore \text{Taylor series} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\therefore \text{Taylor polynomial} = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$$

d) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

→ Solution,

As we know,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)$$

or, $$e^x - e^{-x} = \left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cdots\right) - \cancel{1} + x - \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} - \cdots$$

or, $$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots$$

or, $$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{(2n+1)}}{(2n+1)!} + \cdots$$

$$\therefore \text{Taylor series} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\therefore \text{Taylor polynomial} = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

e) $f(x) = e^{4x}$ from derivative also you can but alt. method:

→ Solution,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$f(x) = e^x$ $\qquad$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad$ $f'(0) = e^0 = 1$

$\vdots$ $\qquad$ $\vdots$

$f^n(x) = e^x$ $\qquad$ $f^n(0) = e^0 = 1$

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Similarly

for e^{4x}.

Taylor series $= 1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \cdots\cdots$

$= \sum_{k=0}^{\infty} \frac{(4x)^k}{k!}$ #.

Taylor polynomial $= \sum_{k=0}^{n} \frac{(4x)^k}{k!}$

for $e^{-7x} = f(x)$

→ Solution

~~As~~ we know,

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots$

$f(x) = e^{-x}$ $\quad f(0) = e^{-0} = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -e^{-0} = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = e^{0} = 1$

$\vdots$ $\quad \vdots$

$f^n(x) = (-1)^n e^{-x}$ $\quad f^n(0) = (-1)^n$

$\therefore$ Taylor Series $= \sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!}$

$= f(0) + f'(0)\, x + f''(0) \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \cdots$

$= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$

Similarly,
for e^{-7x}

Taylor series $= \sum_{k=0}^{\infty} \frac{(-7x)^k}{k!}$ #.

$= 1 - 7x + \frac{(7x)^2}{2!} - \frac{(7x)^3}{3!} + \cdots + \frac{(-7x)^n}{n!} + \cdots$

Taylor polynomial $= \sum_{k=0}^{n} \frac{(-7x)^k}{k!}$ #.

* Find the Taylor series and polynomial for the following function.

$f(x) = \cos x$. even derivative one side } odd one side.

(sin x doesn't contribute in ser

→ solution,

$f(x) = \cos x$ $f(0) = 1$ $f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$ $f'''(x) = +\sin x$ $f'''(0) = 0$

$f^{2n}(x) = (-1)^n \cos x$ $f^{2n}(0) = (-1)^n$ $f^{2n+1}(x) = (-1)^{n+1} \sin x$

$f^{2n+1}(0) = 0$

Now,

Taylor series at $x = 0$ (maclaurin series)

$$\sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^n(0)\frac{x^n}{n!} + \cdots$$

$$= 1 + 0 \times x + (-1) \times \frac{x^2}{2!} + 0 \times \frac{x^3}{3!} + \cdots + f^{2n}(0)\frac{x^{2n}}{2n!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \cdots\cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$\therefore$ Taylor polynomial $= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ #.

$$= 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \text{ #.}$$

b) $f(x) = \sin x$ at $x = 0$

→ Solution,

$f(x) = \sin x$ $\quad f(0) = 0$ $\quad f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$ $\quad f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{2n}(x) = (-1)^n \sin x$ $\quad f^{2n+1}(x) = (-1)^n \cos x$

$f^{2n}(0) = 0$ $\quad f^{2n+1}(0) = (-1)^n$

Now,

Taylor series $= \sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!} = f(0) + f'(0)\, x + f''(0) \frac{x^2}{2!} + \cdots\cdots + f^n(0) \frac{x^n}{n!} + \cdots$

$$= 0 + x + 0 - 1 \times \frac{x^3}{3!} + 0 - \cdots\cdots + f^{2n+1}(0) \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= x - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \text{ #.}$$

Taylor polynomial $= x - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} \text{ #.}$$

c) $f(x) = \sin 4x$ at $x = 0$

→ Solution,

for $\sin x$

$f(x) = \sin x$ $\quad f(0) = 0$ $\quad f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\cancel{\cos x}\sin x$ $\quad f''(0) = 0$ $\quad f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{2n}(x) = (-1)^n \sin x$ $\quad f^{2n}(0) = 0$ $\quad f^{2n+1}(x) = (-1)^n \cos x$

$f^{2n+1}(0) = (-1)^n$

∴ Taylor Series for $\sin x$

$$= \sum_{k=0}^{\infty} \frac{f^k(0)\, x^k}{k!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots + f^n(0)\frac{x^n}{n!} + \dots$$

$$= 0 + 1 \times x + 0 \times \frac{x^2}{2!} + (-1) \times \frac{x^3}{3!} + \dots + f^{2n+1}(0)\frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$= x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \#.$$

Similarly for $\sin 4x$

$$\text{Taylor Series} = \sum_{k=0}^{\infty} (-1)^k \frac{(4x)^{2k+1}}{(2k+1)!} \quad \#.$$

$$= 4x - \frac{(4x)^3}{3!} + \dots + (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!} + \dots$$

Taylor polynomial for $\sin x$ is

$$= x - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \#$$

Similarly for $\sin 4x$.

Taylor polynomial $= \sum_{k=0}^{n} (-1)^k \frac{(4x)^{2k+1}}{(2k+1)!}$

$$= 4x - \frac{(4x)^3}{3!} + \cdots\cdots + (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!} \#$$

d) $f(x) = x \sin x$

→ Solution,

for $f(x) = \sin x$ $\quad f(0) = 0 \quad f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0 \quad f'''(x) = -\cos x \quad f'''(0) = -1$

$\vdots$

$f^{2n}(x) = (-1)^n \sin x \qquad f^{2n+1}(x) = (-1)^n \cos x$

$f^{2n}(0) = 0$

$f^{2n+1}(0) = (-1)^n$

Now,

Taylor series for $\sin x = \sum_{k=1}^{\infty} \frac{f^k(0)\, x^k}{k!}$

$$= f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^n(0)\frac{x^n}{n!}$$

$$= 0 + 1 \times x + 0 \times \frac{x^2}{2!} + (-1) \times \frac{x^3}{3!} + \cdots + f^{2n+1}(0)\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= x - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \#$$

Similarly
Taylor series for $x \sin x = x\left(x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+1} \cdot x}{(2n+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+2}}{(2n+1)!} \#.$$

Taylor ~~series~~ polynomial of $\sin x$

$$= \left(x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Similarly
Taylor polynomial of $x \sin x$

$$= x\left(x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+2}}{(2k+1)!} \#.$$

e.) $f(x) = \frac{1}{x}$ at $x = 1$ (Taylor series)
→ Solution,

$f(x) = 1/x = x^{-1}$	$f(1) = 0$
$f'(x) = -x^{-2}$	$f'(1) = -1$
$f''(x) = 2x^{-3}$	$f''(1) = 2$
$f'''(x) = -6x^{-4}$	$f'''(1) = -6$
⋮	⋮
f	⋮

Taylor series at $x=1$ is $a=1$

$$\sum_{k=0}^{\infty} f^{k}(a) \frac{(x-a)^k}{k!} = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots$$

$$= f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + \cdots$$

$$= 1 - (x-1) + 2\frac{(x-1)^2}{2!} - 6\frac{(x-1)^3}{3!} + \cdots$$

Taylor series $= \sum_{k=0}^{\infty} \frac{f^k(1)(x-1)^k}{k!}$ #.

Taylor polynomial $= \sum_{k=0}^{n} \frac{f^k(1)(x-1)^k}{k!}$ #

* Taylor Polynomial of order n

let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x=a$ is the Polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n.$$

Multiple Integral :-

$\int dx$ = Single integral

$\iint dx\,dy$ = Double integral

$\iiint dx\,dy\,dz$ = Triple integral

a.) $\int_0^3 \int_0^2 (4-y^2)\,dy\,dz$

$= \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dz$

$= \int_0^3 \left[8 - \frac{8}{3}\right] dz$

$= \int_0^3 \left[\frac{24-8}{3}\right] dz$

$= \frac{16}{3} [z]_0^3 = \frac{16}{\cancel{3}} \times \cancel{3} = 16$ #.

b.) $\int_0^3 \int_0^2 (x^2y - 2xy)\,dy\,dx$

$= \int_0^3 \left[\frac{x^2y^2}{2} - \frac{\cancel{2}xy^2}{\cancel{2}}\right]_0^2 dx$

$= \int_0^3 \left[\frac{4x^2}{2} - \frac{4}{2}\,2x\right] dx$

$= \frac{4}{2} \int_0^3 (x^2 - 2x)\,dx$

$= \frac{4}{2} \left[\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^3$

$= \frac{4}{2} \left[\frac{27}{3} - 9\right]$

$= \frac{4}{2} \times 0 = 0$ #.

c) $\int_{-1}^{0}\int_{-1}^{1}(x+y+1)\,dx\,dy$

$= \int_{-1}^{0}\left[\frac{x^2}{2}+xy+x\right]_{-1}^{1} dy$

$= \int_{-1}^{0}\left[\frac{(1)^2}{2}-\frac{(-1)^2}{2}+1\times y-(-1)\times y+1-(-1)\right] dy$

$= \int_{-1}^{0}\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}+2y+2\right) dy$

$= 2\times\frac{0}{2}-2\times\frac{(-1)^2}{2}+2\times 0-2\times(-1) = -1+2 = 1$ #.

d) $\int_{\pi}^{2\pi}\int_{0}^{\pi}(\sin x+\cos y)\,dx\,dy$

$= \int_{\pi}^{2\pi}\left[-\cos x+x\cos y\right]_{0}^{\pi} dy$

$= \int_{\pi}^{2\pi}\left[x\cos y-\cos x\right]_{0}^{\pi} dy$

$= \int_{\pi}^{2\pi}\left[\pi\cos y-\cos\pi+\cos 0^\circ\right] dy$

$= \int_{0}^{2\pi}\left[\pi\cos y+1+1\right] dy$

$= \left[\pi\sin y+2y\right]_{\pi}^{2\pi}$

$= \pi(\sin 2\pi-\pi)+2(2\pi-\pi)$

$= \pi\times 0+2(\pi)$

$= 2\pi$ #.

$= \cancel{\pi\sin 2\pi+2}$

e) $\int_{1}^{2}\int_{y}^{y^2}dx\,dy$

$= \int_{1}^{2}\left[y^2-y\right] dy$

$= \left[\frac{y^3}{3}-\frac{y^2}{2}\right]_{1}^{2}$

$= \left(\frac{2\times2\times2}{3}-\frac{1}{3}-\frac{2\times2}{2}-\frac{1}{2}\right)$

$= \frac{7}{3}-\frac{3}{2}$

$= \frac{14-9}{6}$

$= 5/6$ #.

* Triple integral.

Evaluate

a) $\int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dz$

$= \int_0^1 \int_x^1 [z]^{y-x} dy\, dx = \int_0^1 \int_x^1 [y-x]\, dy\, dx$

$= \int_0^1 \left[\frac{y^2}{2} - xy\right]_x^1 dx = \int_0^1 \left[\frac{(1)^2}{2} - \frac{x^2}{2} - (x - x^2)\right] dx$

$= \int_0^1 \left[\frac{1}{2} + \frac{x^2}{2} - x\right] dx = \left[\frac{1}{2}x + \frac{x^3}{2\times 3} - \frac{x^2}{2}\right]_0^1$

$= \cancel{\frac{1}{2}} + \frac{1}{6} - \cancel{\frac{1}{2}} = \frac{1}{6}$ #

b) $\int_0^1 \int_0^{1-z} \int_0^2 dx\, dy\, dz$

$= \int_0^1 \int_0^{1-z} [x]_0^2\, dy\, dz = \int_0^1 \int_0^{1-z} (2)\, dy\, dz$

$= 2\int_0^1 [y]_0^{1-z} dz = 2\int_0^1 [1-z]\, dz$

$= 2\left[z - \frac{z^2}{2}\right]_0^1$

$= 2\left[1 - \frac{1}{2}\right]$

$= \cancel{2} \times \frac{1}{\cancel{2}}$

$= 1$ #

c.) $\int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy = 1$ d.) $\int_0^2 \int_0^1 \int_0^{1-y} dz\, dy\, dx = 1 \Rightarrow$ Same

e.) $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\, dy\, dx$

$= \int_0^1 \int_0^{3-3x} [z]^{3-3x-y} dy\, dx$

$= \int_0^1 \int_0^{3-3x} [3-3x-y]\, dy\, dx$

$= \int_0^1 \left[3y - 3xy - \frac{y^2}{2}\right]_0^{3-3x} dx$

$= \int_0^1 \left[\frac{6y - 6xy - y^2}{2}\right]_0^{3-3x} dx$

$= \frac{1}{2} \int_0^1 6(3-3x) - 6x(3-3x) - (3-3x)^2\, dx$

$= \frac{1}{2} \int_0^1 (18 - \cancel{18x} - 18x + 18x^2 - 9 + \cancel{18x} - 9x^2)\, dx$

$= \frac{1}{2} \int_0^1 (9x^2 - 18x + 9)\, dx$

$= \frac{1}{2} \left[\frac{9x^3}{3} - \frac{18x^2}{2} + 9x\right]_0^1$

$= \frac{1}{2} \left[3x^3 - 9x^2 + 9x\right]_0^1$

$= \frac{1}{2} [3 - \cancel{9} + \cancel{9}]$

$= 3/2$ Ans

f.) $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dx\, dy$

→ Solution:-

$$= \int_0^{\sqrt{2}} \int_0^{3y} [z]_{x^2+3y^2}^{8-x^2-y^2} dx\, dy$$

$$= \int_0^{\sqrt{2}} \int_0^{3y} [8-x^2-y^2-x^2-3y^2]\, dx\, dy$$

$$= \int_0^{\sqrt{2}} \int_0^{3y} [8-2x^2-4y^2]\, dx\, dy$$

$$= \int_0^{\sqrt{2}} \left[8x - 2\frac{x^3}{3} - 4xy^2\right]_0^{3y} dy$$

$$= \int_0^{\sqrt{2}} \left[8\times 3y - 2\times\frac{(3y)^3}{3} - 4.3y.y^2\right] dy$$

$$= \int_0^{\sqrt{2}} [24y - 18y^3 - 12y^3]\, dy$$

$$= \left[24\frac{y^2}{2} - \frac{18y^4}{4} - \frac{12y^4}{4}\right]_0^{\sqrt{2}}$$

$$= \left[12y^2 - \frac{30y^4}{4}\right]_0^{\sqrt{2}}$$

$$= 12(\sqrt{2})^2 - \frac{30(\sqrt{2})^4}{4}$$

$$= 12\times 2 - 30\times\frac{\cancel{4}}{\cancel{4}}$$

$$= 24 - 30$$

$$= -6 \#.$$

g.) $\int_1^e \int_1^e \int_1^e \frac{1}{xyz}\,dx\,dy\,dz$ $\boxed{\int \frac{1}{x}dx = \log x}$

$= \int_1^e \int_1^e [\log x]_1^e \frac{1}{yz}\,dy\,dz$ $\boxed{\begin{array}{l}\log e = 1 \\ \log 1 = 0\end{array}}$, $e = 2.71$, $\ln(2.71) = 1$

$= \int_1^e \int_1^e [\log e - \log 1] \frac{1}{yz}\,dy\,dz$

$= \int_1^e \int_1^e [1-0] \frac{1}{yz}\,dy\,dz$

$\int \frac{1}{y}dy = \log y$

$= \int_1^e [\log y]_1^e \frac{1}{z}\,dz$

$= \int_1^e [\log e - \log 1] \frac{1}{z}\,dz$

$= \int_1^e [1-0] \frac{1}{z}\,dz$

$= [\log z]_1^e$

$= [\log e - \log 1]$

$= [1-0]$

$= 1$ #.

h.) $\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2\sin^2\theta + z^2)\,dz\; r\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^1 \left[z r^2 \sin^2\theta + \frac{z^3}{3}\right]_{-1/2}^{1/2} r\,dr\,d\theta$

$= \int_0^{2\pi} \int_0^1 \left[\frac{1}{2} r^2\sin^2\theta - \left(-\frac{1}{2}\right) r^2 \sin^2\theta + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(-\frac{1}{2}\right)^3}{3}\right] r\,dr\,d\theta$

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$$= \int_0^{2\pi}\int_0^{1}\left[\frac{r^2\sin^2\theta}{2}+\frac{r^2\sin^2\theta}{2}+\frac{1}{24}+\frac{1}{24}\right]r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left[r^2\sin^2\theta+\frac{1}{12}\right]r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left[r^3\sin^2\theta+\frac{r}{12}\right]dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4\sin^2\theta}{4}+\frac{r^2}{12\times 2}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\frac{\sin^2\theta}{4}+\frac{1}{24}\right]d\theta$$

$$= \int_0^{2\pi}\left[\frac{1-\cos 2\theta}{2\times 4}+\frac{1}{24}\right]d\theta$$

$$= \left[\frac{[\theta]_0^{2\pi}}{8}-\frac{[\sin 2\theta]_0^{2\pi}}{2\times 8}+\frac{1}{24}[\theta]_0^{2\pi}\right]$$

$$= \frac{2\pi}{8}-\frac{1}{16}\times(0-0)+\frac{1}{24}\times 2\pi$$

$$= \frac{\pi}{4}+\frac{\pi}{12}$$

$$= \frac{3\pi+\pi}{12}$$

$$= \frac{4\pi}{12}$$

$$= \frac{\pi}{3} \#.$$

4) $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{\theta/2\pi} [z]_0^{3+24r^2}\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{\theta/2\pi} [3+24r^2]\times r\; dr\,d\theta$

$= \int_0^{2\pi}\int_0^{\theta/2\pi} [3r+24r^3]\, dr\,d\theta$

$= \int_0^{2\pi} \cancel{\int} \left[\frac{3r^2}{2}+\frac{24r^4}{4}\right]_0^{\theta/2\pi} d\theta$

$= \int_0^{2\pi} \left[\frac{3\frac{\theta^2}{4\pi^2}}{2} + \frac{24\frac{\theta^4}{16\pi^4}}{4}\right] d\theta$

$= \int_0^{2\pi} \left[\frac{3\theta^2}{4\pi^2}\times\frac{1}{2} + \frac{\overset{6\,3}{\cancel{24}}\theta^4}{\underset{8}{\cancel{16}}\pi^4}\times\frac{1}{\cancel{4}}\right] d\theta$

$= \int_0^{2\pi} \left[\frac{3\theta^2}{8\pi^2} + \frac{3\theta^4}{8\pi^4}\right] d\theta$

$= \frac{3}{8\pi^2}\left[\frac{\theta^3}{3}\right]_0^{2\pi} + \frac{3}{8\pi^4}\left[\frac{\theta^5}{5}\right]_0^{2\pi}$

$= \frac{\cancel{3}}{8\pi^2}\times\frac{1}{\cancel{3}}(2\pi)^3 + \frac{3}{8\pi^4}\times\frac{1}{5}(2\pi)^5$

$= \frac{\cancel{8\pi^3}}{\cancel{8\pi^2}} + \frac{32\times3\times\pi^5}{40\,\pi^4}$

$= \pi + \frac{96}{40}\pi$

$= \pi + \frac{12\pi}{5} = \frac{17\pi}{5}$ #.

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5.) $\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1+\cos\theta} 4r \, dr \, d\theta \, dz.$

$= \int_{-1}^{1}\int_{0}^{2\pi} \left[\frac{4r^2}{2}\right]_0^{1+\cos\theta} d\theta \, dz$

$\therefore \cos^2\theta = \frac{1+\cos2\theta}{2}$

$= \int_{-1}^{1}\int_{0}^{2\pi} \left[2r^2\right]_0^{1+\cos\theta} d\theta \, dz$

$= \int_{-1}^{1}\int_{0}^{2\pi} 2\left[1+\cos\theta\right]^2 d\theta \, dz$

$= \int_{-1}^{1}\int_{0}^{2\pi} 2\left(1+2\cos\theta+\cos^2\theta\right) d\theta \, dz$

$= \int_{-1}^{1}\int_{0}^{2\pi} 2.\left(1+2\cos\theta+\frac{1+\cos2\theta}{2}\right) d\theta \, dz$

$= 2\int_{-1}^{1} \left[[\theta]_0^{2\pi} + 2[\sin\theta]_0^{2\pi} + \frac{1}{2}\left\{[\theta]_0^{2\pi} + \left[\frac{\sin2\theta}{2}\right]_0^{2\pi}\right\}\right] dz$

$= 2\int_{-1}^{1} (2\pi-0) + 2[\sin2\pi - \sin0^\circ] + \frac{1}{2}[2\pi-0] + \frac{1}{4}[\sin2.2\pi - \sin2.0^\circ] \, dz$

$= 2\int_{-1}^{1} \left(2\pi + \frac{2\pi}{2} + \frac{1}{4}\times 0\right) dz$

$= 2\int_{-1}^{1} 3\pi$

$= 6\pi\,[z]_{-1}^{1}$

$= 6\pi\,[1+1]$

$= 12\pi$ #.

Integration by parts

$= \int u.v\,dx \quad = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

$= u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

= first part x integration of second - Integration { derivative of ~~sec~~ first x integration of second }

Rule for integration by parts:-

ILATE Rule

I → Inverse → $\sin^{-1}x$ ...

L → Logarithm → log

A → Algebraic → x, y, z ...

T → Trignometric → $\sin x, \cos x$ ...

E → exponential → e^x

e.g.

$\int x.e^x\,dx$, $= x\int e^x dx - \int\left\{\frac{dx}{dx}\int e^x dx\right\}dx$

→ here,

$u = x$, $v = e^x$ $= xe^x - \int e^x dx$

$= xe^x - e^x + C$

$\Rightarrow x\int e^x dx - \int\left\{\frac{dx}{dx}\int e^x dx\right\}dx$

$\Rightarrow x.e^x - \int e^x dx$

$\Rightarrow x.e^x - e^x + C$ #.

e.g. $\int y.e^y\,dy$

$= y\int e^y dy - \int\left\{\frac{dy}{dy}\int e^y dy\right\}dx$

$= ye^x - e^y + C$ #

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* Evaluate :-

$\therefore e^{\ln y} = y$

$\therefore e^{\ln 8} = 8$

$$* \int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx\, dy$$

$$= \int_1^{\ln 8} \int_0^{\ln y} e^{x} . e^{y}\, dx\, dy$$

$$= \int_1^{\ln 8} \left[e^{y} . [e^{x}]_0^{\ln y} \right] dy$$

$$= \int_1^{\ln 8} \left[e^{y} . [e^{\ln y} - e^{0}] \right] dy$$

$$= \int_1^{\ln 8} \left[e^{y} . (y-1) \right] dy$$

$$= \int_1^{\ln 8} \left[e^{y} . y - e^{y} \right] dy$$

$$= \int e^{y} . y\, dy$$

$$= y \int e^{y} dy - \int \left\{ \frac{dy}{dy} \int e^{y} dy \right\} dy$$

$$= y . e^{y} - e^{y} + c$$

$$= ye^{y} - e^{y} + c$$

$$= \left[\underbrace{ye^{y} - e^{y}}_{e^{y}(y-1)} \right]_1^{\ln 8} - \left[e^{y} \right]_1^{\ln 8}$$

$$= \left[ye^{y} - 2e^{y} \right]_1^{\ln 8}$$

$e^{\ln 8}(\ln 8 - 1) - \cancel{e^{\ln 8}} + e^{y} = 8 \ln 8$

$$= \left[\ln 8 \; e^{\ln 8} - 1 \times e^{1} - 2e^{\ln 8} + 2e^{1} \right]$$

$$= 8 \ln 8 - e - 2 \times 8 + 2e$$

$$= 8 \ln 8 - 16 + e \quad \underline{\text{Answer}}$$

Fubini's Theorem :-

1) First Form :-

If $f(x,y)$ is continuous on region R : $a \le x \le b$ and $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\,dx\,dy$$

$$= \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\,dy\,dx$$

This form gives the volume.

* evaluate :-

$$f(x,y) = 1-6x^2y \quad , -1 \le x \le 1, \quad 0 \le y \le 2$$

→ Solution,

At first :- At second

$$\int_{y=0}^{2} \int_{x=1}^{1} (1-6x^2y)\,dx\,dy = \int_{x=-1}^{1} \int_{y=0}^{2} (1-6x^2y)\,dy\,dx$$

$$= \int_0^2 \int_{-1}^{1} (1-6x^2y)\,dx\,dy$$

$$= \int_0^2 \left[x - \frac{\cancel{6}^{2} x^3 y}{\cancel{3}}\right]_{-1}^{1} dy$$

$$= \int_0^2 \left[1-(-1) - \{2y - 2(-1)^3 y\}\right] dy$$

$$= \int_0^2 (2-2y-2y)\,dy$$

$= \left[2y - \frac{4y^2}{2}\right]_0^2$

$= 2\times2 - 2(2)^2 - 0$

$= 4 - 8$

$= -4$

$\iiint_R dx\,dy\,dz = \text{Volume}$

$\iint_R dx\,dy = \text{Area}$

At second,

$\int_{-1}^{1}\int_{0}^{2} (1 - 6x^2y)\,dy\,dx$

$= \int_{-1}^{1} \left[y - \frac{\cancel{6}^{3}x^2y^2}{\cancel{2}}\right]_0^2 dx$

$= \int_{-1}^{1} \left[2 - 3x^2(2)^2\right] dx$

$= \int_{-1}^{1} \left[2 - 12x^2\right] dx$

$= \left[2x - \frac{\cancel{12}^{4}x^3}{\cancel{3}}\right]_{-1}^{1}$

$= \left[2x - 4x^3\right]_{-1}^{1}$

$= 2\times1 - 2\times(-1) - \{4\times(1)^3 - 4\times(-1)^3\}$

$= 2 + 2 - (4 + 4)$

$= 4 - 8$

$= -4$

$\therefore$ Both $\int dy\,dx$ and $\int dx\,dy$ have same value.

*** Fubini's Theorem (stronger Form)**

Function $f(x,y)$ is continuous on region R, then,

a.) if R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, then,

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

b.) if R is defined by $a \le y \le b$, $g_1(y) \le x \le g_2(y)$

$$\iint_R f(x,y)\,dA = \int_{y=a}^{y=b} \int_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx\,dy$$

Find the area between $y^2 = 4x$ and $x^2 = 2y$

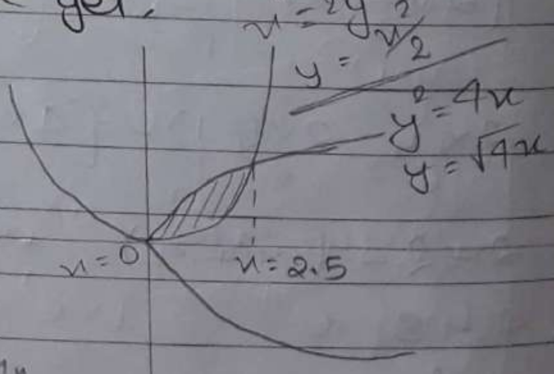
→ Solution

$y^2 = 4x$ ----(i)

$x^2 = 2y$

$y = \frac{x^2}{2}$ ---(ii)

solving eqn ① and ⑪, we get,



$\left(\frac{x^2}{2}\right)^2 = 4x$

$\frac{x^4}{4} = 4x$

or $x^4 = 16x$

or $x(x^3 - 16) = 0$

Either $x = 0$

$x^3 - 16 = 0$

$\therefore x = \sqrt[3]{16} = 2.5$

$$\int_{x=0}^{x=2.5} \int_{y=x^2/2}^{y^2=4x} dy\,dx$$

from Single integration:-

$$= \text{Area} = \int_{x=0}^{x=2.5} \sqrt{4x}\,dx - \int_{x=0}^{x=2.5} \frac{x^2}{2}\,dx$$

OR

Double integration:

$$\text{Area} = \int_{x=0}^{x=2.5} \int_{y=x^2/2}^{y=\sqrt{4x}} dy\,dx$$

$$= \int_{x=0}^{x=2.5} [y]_{x^2/2}^{\sqrt{4x}}\,dx$$

$$= \int_{x=0}^{x=2.5} \left(\sqrt{4x} - \frac{x^2}{2}\right) dx$$

$$= \int_{x=0}^{x=2.5} \left(\sqrt{4}\,x^{1/2} - \frac{x^2}{2}\right) dx$$

$$\cancel{= \sqrt{4}\left[\frac{x^{3/2}}{3/2} - \frac{\sqrt{4}x^3}{2\times 3}\right]_0^{2.5}}$$

$$\cancel{= 2 \times \frac{2}{3}\left[(2.5)^{3/2} - \frac{\sqrt{4}}{6}(2.5)^3\right]}$$

$$\cancel{= 1.33\,(3.95 - 2.60 \times \sqrt{4})}$$

$$\cancel{= 1.33 \times}$$

$=$

$$= \sqrt{4}\left(\frac{x^{3/2}}{3/2}\right)_0^{2.5} - \left[\frac{x^3}{6}\right]_0^{2.5}$$

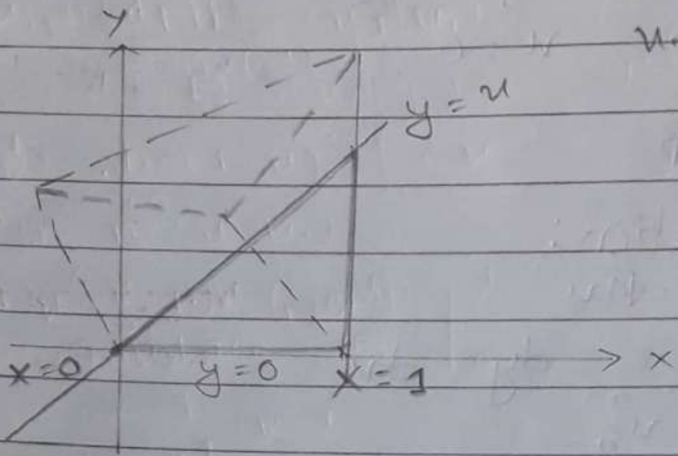
$$= 2 \times \frac{2}{3} \times (3.95) - \frac{(2.5)^3}{6}$$

$$= \frac{4}{3} \times 3.95 - 2.60$$

$$= 5.2 - 2.60$$

$$= 2.66$$

Find the Volume of Prism whose base is the triangle in the xy-plane bounded by the X-axis and the line $y=x$ and $x=1$ and whose lies in the plain $z=f(x,y)=3-x-y$



$$\int_{x=0}^{x=1}\int_{y=x}^{y=x}(3-x-y)\,dy\,dx$$

$$= \text{Volume} = \int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x,y)\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x}(3-x-y)\,dy\,dx$$

$$= \int_{x=0}^{x=1}\left[3y-xy-\frac{y^2}{2}\right]_0^x dx$$

$$= \int_{x=0}^{x=1}\left[3x-x^2-\frac{x^2}{2}\right]dx$$

$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}-\frac{x^3}{2\times3}\right]_0^1$$

$$= \frac{3}{2}-\frac{1}{3}-\frac{1}{6}$$

$$= \frac{9-2-1}{6}$$

$$= \frac{6}{6}$$

$= 1$ Cubic units #.

∴ Required Volume of prism is 1 cm^3.

* Find the Volume of Solid under the surface $z = f(x,y) = x^2 + y^2$ over the triangular region whose vertices are $(0,0)$, $(1,0)$ and $(0,1)$

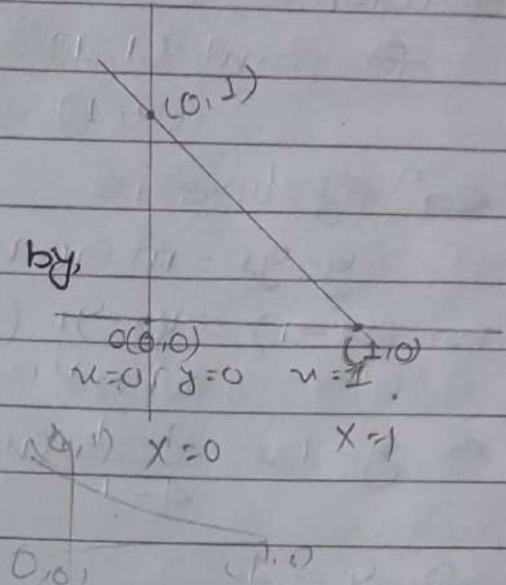
→ Solution:-

Point $(x_1, y_1) = (1, 0)$

Point $(x_2, y_2) = (0, 1)$

equation of a line is given by,

$y - y_1 = m(x - x_1)$



or, $(y - 0) = \frac{y_2 - y_1}{x_2 - x_1}(x - 1)$

or, $(y - 0) = \frac{(1 - 0)}{(0 - 1)}(x - 1)$

or $y = -x + 1$

$\therefore y = (1 - x)$

Now,

$$\text{Volume} = \int_{x=0}^{x=1} \int_{y=0}^{y=(1-x)} x^2 + y^2 \, dy \, dx$$

$$= \int_0^1 \left[y x^2 + \frac{y^3}{3} \right]_0^{1-x} dx$$

or use $\frac{(1-x)^{3+4}}{3 \times 4 \times (-1)}$

$$= \int_0^1 \left[(1-x) x^2 + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{1}{3} (1)^3 - 3(1)^2 \cdot x + 3(1) \cdot x^2 - x^3 \right] dx$$

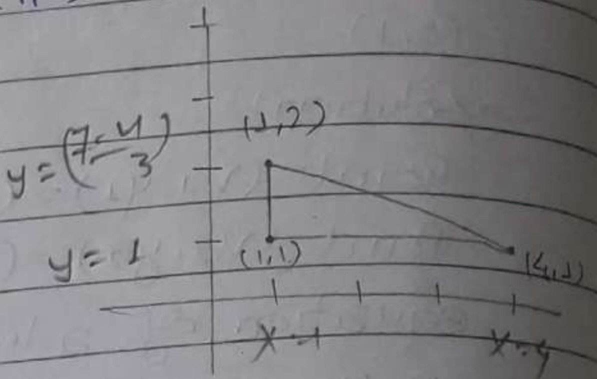
$$= \frac{1}{3} \int_0^1 (3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3) \, dx$$

$$= \frac{1}{3} \left[\frac{\cancel{3} x^3}{\cancel{3}} - \frac{3x^4}{4} + x - \frac{3x^2}{2} + \frac{3x^3}{\cancel{3}} - \frac{x^4}{4} \right)_0^1$$

$$= \frac{1}{3} \times \left[1 - \frac{3}{4} + \frac{1}{\cancel{4}} - \frac{3}{2} + 1 - \frac{1}{4} \right) = \frac{1}{3} \times \left(\frac{4 - 3 + 4 - 6 + 4 - 1}{4} \right)$$

$$= \frac{1}{3} \times \frac{2}{4_2} = \frac{1}{3} \times \frac{1}{2} = 1/6 \ \#.$$

* Find volume, $z = f(x,y) = xy$
vertices $(1,1)$, $(4,1)$ and $(1,2)$



→ Solution,

for point $(1,1) = (x_1, y_1)$
$(4,1) = (x_2, y_2)$

eq^n of line is

$$y - y_1 = m(x - x_1)$$

or $(y-1) = \dfrac{y_2 - y_1}{x_2 - x_1}(x-1)$

or $y - 1 = \dfrac{1-1}{4-1}(x-1)$

or $y = 1$

$\therefore$ lower limit $y = 1$

for point $(1,2) = (x_1, y_1)$ and $(4,1) = (x_2, y_2)$

eq^n of line is

$$(y-2) = \frac{1-2}{4-1}(x-1)$$

or $y - 2 = -\dfrac{1}{3}(x-1)$

or $3y - 6 = -x + 1$

or $3y = 7 - x$

$\therefore y = \left(\dfrac{7-x}{3}\right)$ $\quad \therefore$ Upper limit of $y = \dfrac{(7-x)}{3}$

$$\text{Volume} = \int_{x=1}^{x=4} \int_{y=1}^{y=\left(\frac{7-x}{3}\right)} xy \, dy \, dx$$

$$= \int_1^4 \left[x \frac{y^2}{2} \right]_1^{\frac{7-x}{3}} dx$$

$\frac{y - y_1}{} = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

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$$= \int_1^4 \left[\frac{x \cdot \left(\frac{7-x}{3}\right)^2}{2} - \frac{x}{2}\right] dx$$

$$= \int_1^4 \left[\frac{x(49 - 14x + x^2)}{9 \times 2} - \frac{x}{2}\right] dx$$

$$= \int_1^4 \left[\frac{1}{18} 49x - 14x^2 + x^3 - \frac{x}{2}\right] dx$$

$$= \frac{1}{18} \int_1^4 \left[49x - 14x^2 + x^3 - 9x\right] dx$$

$$= \frac{1}{18} \int_1^4 \left[40x - 14x^2 + x^3\right] dx$$

$$= \frac{1}{18} \left[\frac{40x^2}{2} - \frac{14x^3}{3} + \frac{x^4}{4}\right]_1^4$$

$$= \frac{1}{18} \left[20x^2 - \frac{14x^3}{3} + \frac{x^4}{4}\right]_1^4$$

$$= \frac{1}{18} \left[20 \times (4)^2 - \frac{14 \times (4)^3}{3} + \frac{(4)^4}{4} - 20 \times 1 + \frac{14}{3} \times 1 - \frac{1}{4}\right]$$

$$= \frac{1}{18} \left[320 - 298.66 + 64 - 20 + 4.66 - 0.25\right]$$

$$= \frac{69.75}{18}$$

$$= 3.87$$

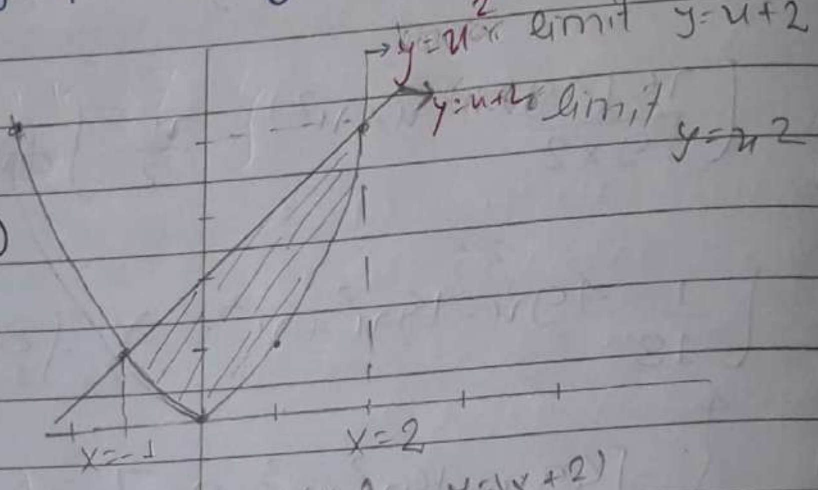
$\therefore$ Required Volume is 3.87 cubic unit

* Find the area of parallelogram $y=x^2$ and $y=x+2$.

→ Solution,

$y = x^2$ --- (i)

$y = x+2$ --- (ii)



Solving eqⁿ ① & ②

$x^2 = x+2$

or, $x^2 - x - 2 = 0$

or $x^2 - 2x + x - 2 = 0$

or $x(x-2) + 1(x-2) = 0$

or $(x-2)(x+1) = 0$

Either,

$x_1 = -1$, $y_1 = 1$

$x_2 = 2$, $y_2 = 4$

$$\int_{x=-1}^{x=2} \int_{y=x^2}^{y=(x+2)} dy\, dx$$

eqⁿ of parabola is $y = x^2$

when $y = 0$, $x = 0$

$y = 1$, $x = \pm 1$

$y = 4$, $x = \pm 2$

$A = A_1 + A_2$

$\int_{-1} (y+2)\, dx$

$$= \text{Area} = \int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} dy\, dx$$

$$= \int_{x=-1}^{x=2} [\,y\,]_{x^2}^{x+2}\, dx$$

$$= \int_{-1}^{2} [x+2-x^2]\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{12+24-16-3+12-2}{6}$$

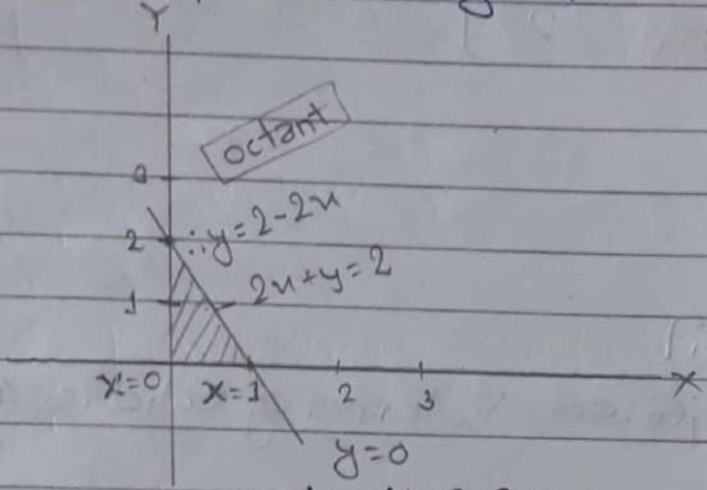
$$= \frac{27}{6}$$

$= 4.5$ ##.

octant means first qudrant

* Find the Volume of Solid in the first octant bounded by the co-ordinate plans, the paraboloid, $z = x^2+y^2+1$, and the plane $2x+y=2$

→



In eq^n $2x+y=2$

when,

$x=0, y=2$

$x=1, y=0$

$x=2, y=-2$

$$\therefore \text{Volume } (v) = \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} (x^2+y^2+1)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[x^2y + \frac{y^3}{3} + y \right]_0^{2-2x} dx$$

$$= \int_{x=0}^{x=1} \left[x^2(2-2x) + \frac{(2-2x)^3}{3} + (2-2x) \right] dx$$

$$= \int_{x=0}^{x=1} \left[2x^2 - 2x^3 + \frac{(2)^3 - 3.(2)^2.2x + 3.2(2x)^2 - (2x)^3}{3} + (2-2x) \right] dx$$

$$= \frac{1}{3} \int_{x=0}^{x=1} \left[6x^2 - 6x^3 + 8 - 24x + 24x^2 - 8x^3 + 6 - 6x \right] dx$$

$$= \frac{1}{3} \int_{x=0}^{x=1} \left[14 - 30x + 30x^2 - 14x^3 \right] dx$$

$$= \frac{1}{3}\left\{ [14x]_0^1 - 30\left[\frac{x^2}{2}\right]_0^1 + 30\left[\frac{x^3}{3}\right]_0^1 - 14\left[\frac{x^4}{4}\right]_0^1 \right\}$$

$$= \frac{1}{3}\left\{ 14 - 15 + 10 - \frac{7}{2} \right\}$$

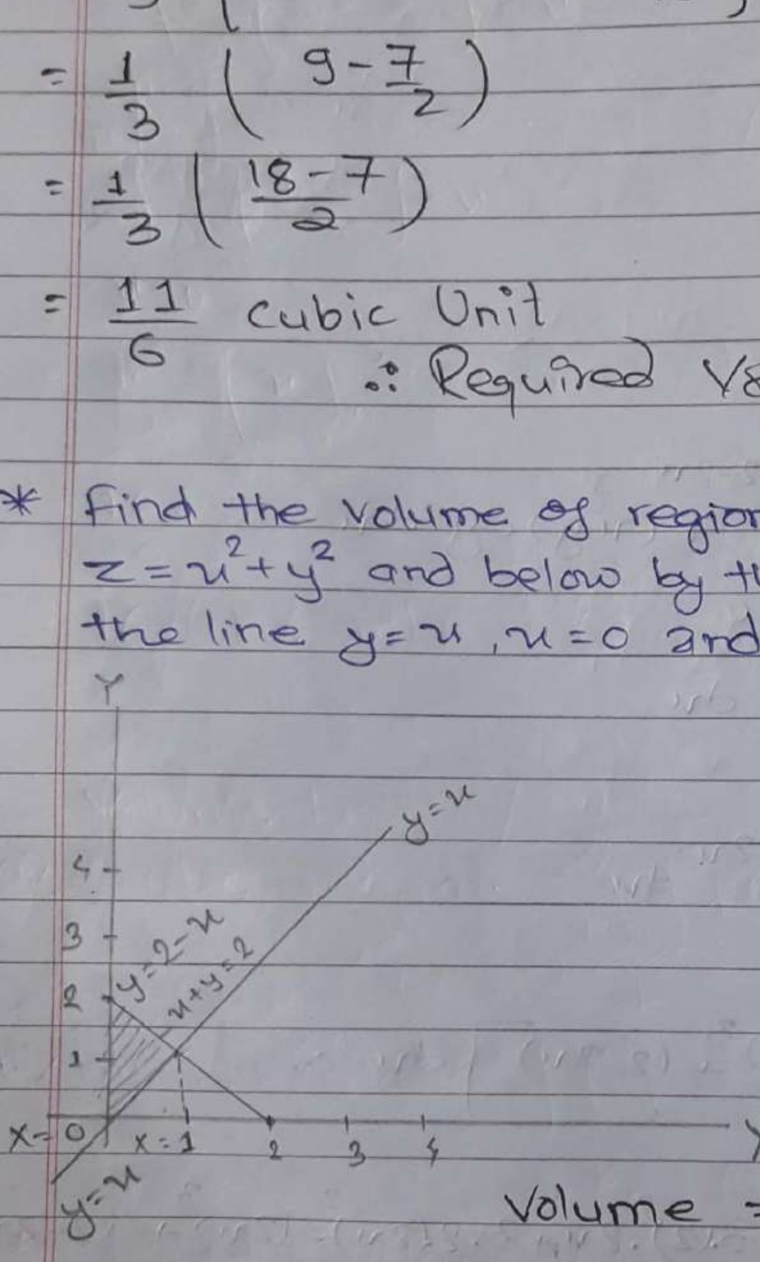
$$= \frac{1}{3}\left(9 - \frac{7}{2} \right)$$

$$= \frac{1}{3}\left(\frac{18-7}{2} \right)$$

$$= \frac{11}{6} \text{ cubic Unit}$$

$\therefore$ Required Volume of Solid is $\frac{11}{6}$ Cubic Unit

* Find the volume of region bounded by paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the line $y = x$, $x = 0$ and $x + y = 2$ in the xy plane



for $x + y = 2$

when $x = 0$, $y = 2$

when $x = 1$, $y = 1$

when $x = 2$, $y = 0$

$$\text{Volume} = \int_{x=0}^{x=1} \int_{y=x}^{y=2-x} (x^2 + y^2)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{y^3}{3} \right]_{y=x}^{y=2-x} dx$$

$$= \int_{x=0}^{x=1} \left[x^2(2-x) - x^2(x) + \frac{(2-x)^3}{3} - \frac{(x)^3}{3} \right] dx$$

$$= \int_{x=0}^{x=1} \left[2x^2 - x^3 - x^3 + \frac{(2)^3 - 3.(2)^2.x + 3(2).(x)^2 - (x)^3}{3} - \frac{x^3}{3}\right] dx$$

$$= \frac{1}{3}\int_{x=0}^{x=1} \left[6x^2 - 3x^3 - 3x^3 + 8 - 12x + 6x^2 - x^3 - x^3\right] dx$$

$$= \frac{1}{3}\int_{x=0}^{x=1} \left[8 - 12x + 12x^2 - 8x^3\right] dx$$

$$= \frac{1}{3}\left\{8[x]_0^1 - 12\left[\frac{x^2}{2}\right]_0^1 - 8\left[\frac{x^4}{4}\right]_0^1\right\} + 12\left[\frac{x^3}{3}\right]_0^1 \}$$

$$= \frac{1}{3}\left\{8\times 1 - \overset{6}{12\times\frac{1}{2}} - \overset{2}{8\times\frac{1}{4}}\right\} + \overset{4}{12\times\frac{1}{3}} \}$$

$$= \frac{1}{3}\{8 - 6 - 2 + 4\}$$

$$= \frac{1}{3}(12 - 8)$$

$= \frac{4}{3}$ cubic unit. ∴ Required Volum of Region is $\frac{4}{3}$ cubic unit.

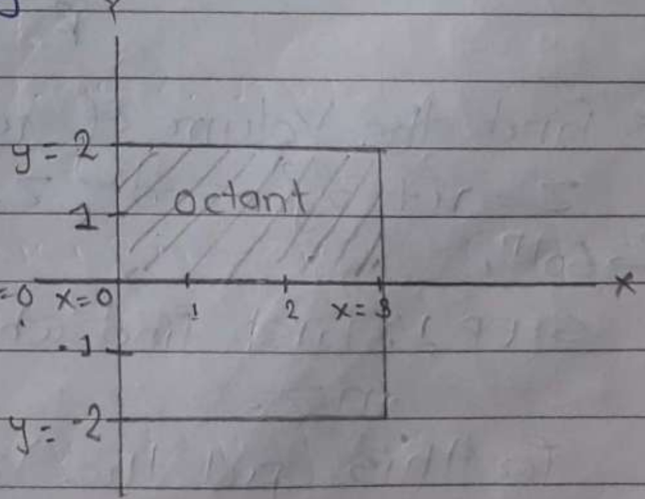
* Find the Volume of Solid in the first octant bounded by coordinate planes, the planes, $x = 3$ and the parabolic cylinder, $z = 4 - y^2$.

eqⁿ $z = 4 - y^2$

when $z = 0$

$y^2 = 4$

$x = 3$ ∴ $y = \pm 2$



$$\rightarrow V = \int_{x=0}^{3}\int_{y=0}^{y=2} (4 - y^2)\, dy\, dx$$

$$= \int_{x=0}^{x=3} \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_{x=0}^{x=3} \left[8 - \frac{8}{3}\right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{16}{3}\right] dx$$

$$= \frac{16}{3} \left[x\right]_0^3$$

$$= \frac{16}{3} \times 3$$

= 16 Cubic unit

Hence the required Volume of Solid is 16 Cubic unit.

Volume by:

a) Double integral. $= \iint_R f(x,y)\, dy\, dx$

b) Triple integral $= \iiint_R dz\, dy\, dx$

* Find the Volume of region enclosed by Surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

→ Soln,

STEP 1:- First find which is upper and lower limit in z.

for this, put the Value of x and y as 0.

then,

which is greater that is Upper & which value is lower that is lower limit.

i.e.

$$z = 0^2 + 3 0^2 = 0 \rightarrow \text{lower limit} = x^2 + 3y^2$$

$$z = 8 - 0^2 - 0^2 = 8 \rightarrow \text{Upper limit} = 8 - x^2 - y^2$$

STEP 2: To find limit of y,

Solve the equation as,

$$x^2 + 3y^2 = 8 - x^2 - y^2$$

or, $2x^2 + 4y^2 = 8$

or, $x^2 + 2y^2 = 4$ which is eq^n of ellipse

i.e. $\dfrac{x^2}{(2)^2} + \dfrac{y^2}{(\sqrt{2})^2} = 1$

or, $2y^2 = 4 - x^2$

or, $y^2 = \dfrac{4 - x^2}{2}$

$\therefore a = 2, \quad b = \sqrt{2}$

$\therefore y = \pm\sqrt{\dfrac{4 - x^2}{2}}$

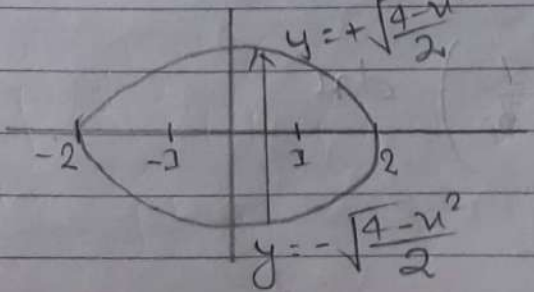
For y, Upper limit $= +\sqrt{\dfrac{4 - x^2}{2}}$

lower limit $= -\sqrt{\dfrac{4 - x^2}{2}}$

STEP 3: To find limit of x.

by solving,

eq^n is ellipse, i.e. $\dfrac{x^2}{(2)^2} + \dfrac{y^2}{(\sqrt{2})^2} = 1$



in this eq^n

$a = 2$

$\therefore$ Upper limit $= +2$

$\therefore$ lower limit $= -2$

Now,

$$\text{Volume} = \int_{x=-2}^{x=+2} \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=+\sqrt{\frac{4-x^2}{2}}} \int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-x^2-y^2-x^2-3y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[8y - 2x^2y - 4\frac{y^3}{3} \right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[(8-2x^2)\times 2\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\times 2 \left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

Common

$$= \int_{-2}^{2} \left[2(4-x^2)^1 \frac{(4-x^2)^{1/2}}{\sqrt{2}} - \frac{4}{3}\times 2 \left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$2^{3/2} = 2\sqrt{2}$

$$= \int_{-2}^{+2} \left[\frac{4}{\sqrt{2}}(4-x^2)^{3/2} - \frac{4\times \cancel{2}}{3\times \cancel{2}\sqrt{2}} (4-x^2)^{3/2} \right] dx$$

take common

$$= \int_{-2}^{2} \frac{4}{\sqrt{2}} (4-x^2)^{3/2} \left(1 - \frac{1}{3}\right) dx$$

$= \frac{3-1}{3} = \frac{2}{3}$

> if $\int_{y=-a}^{y=+a} dy$
> then,
> $= [y]_{-a}^{a}$
> $= 2a$
> if same lower & upper limit with +ve & neg. sig we can write 2 x that limit

$$= \frac{8}{3\sqrt{2}} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

$$\int_{-a}^{+a} f(x)\, dx = 2\int_{0}^{+a} f(x)\, dx$$

$$= \frac{4\times\sqrt{2}\times\cancel{\sqrt{2}}}{3\cancel{\sqrt{2}}} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

$$= \frac{4\sqrt{2}}{3}\times 2 \int_{0}^{2} (4-x^2)^{3/2}\, dx$$

when we go limit from 0 to 2 we should $\times 2$.

$$= \frac{8\sqrt{2}}{3} \int_{0}^{2} (4-x^2)^{3/2}\, dx$$

which is in the form of (a^2-x^2). So (2^2-x^2) $\therefore a=2$

put $x = a\sin\theta$

$x = 2\sin\theta$

differentiating both sides with respect to θ.

$$\frac{dx}{d\theta} = 2\frac{d\sin\theta}{d\theta}$$

$\therefore dx = 2\cos\theta\, d\theta$

when $x=2$

$2 = 2\sin\theta$

$\therefore \theta = \sin^{-1}(1) = \pi/2$

when $x=0$

$0 = 2\sin\theta$

$\therefore \theta = \sin^{-1}(0) = 0^\circ$

$$\therefore V = \frac{8\sqrt{2}}{3} \int_{0}^{\pi/2} (4-4\sin^2\theta)^{3/2}\, 2.\cos\theta\, d\theta$$

$$= \frac{8\sqrt{2}}{3} \int_{0}^{\pi/2} 4^{3/2}(1-\sin^2\theta)^{3/2}.\, 2\cos\theta\, d\theta$$

$$= \frac{8\sqrt{2}}{3} \int_{0}^{\pi/2} 8\, {\cos^2\theta}^{3/2} \cdot 2\cos\theta\, d\theta$$

$$= \frac{8\sqrt{2}}{3} \times 16 \int_0^{\pi/2} \cos^4\theta \, d\theta \qquad \because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\therefore \cos^4\theta = (\cos^2\theta)^2$$

$$= \frac{8 \times 16\sqrt{2}}{3} \int_0^{\pi/2} (\cos^2\theta)^2 d\theta$$

$$= \frac{8 \times 16\sqrt{2}}{3} \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{8 \times 16\sqrt{2}}{3} \int_0^{\pi/2} \left\{\frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)\right\} d\theta$$

$$= \frac{8 \times 16\sqrt{2}}{3} \int_0^{\pi/2} \left\{\frac{1}{4}\left(\overset{\times 2}{1} + \overset{\times 2}{2\cos 2\theta} + \frac{1+\cos 4\theta}{2}\right)\right\} d\theta$$

L.C.m.

$$= \frac{8 \times 16\sqrt{2}}{3} \int_0^{\pi/2} \frac{1}{8}\left(2 + 4\cos 2\theta + 1 + \cos 4\theta\right) d\theta$$

$$= \frac{\cancel{8} \times 16\sqrt{2}}{3} \times \frac{1}{\cancel{8}} \int_0^{\pi/2} (3 + 4\cos 2\theta + \cos 4\theta) d\theta$$

$$= \frac{16\sqrt{2}}{3}\left[3[\theta]_0^{\pi/2} + 4\cdot\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} + \left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}\right]$$

$$= \frac{16\sqrt{2}}{3}\left[\frac{3\pi}{2} + 4\left[\frac{\sin 2\cdot\pi/2}{2} - \frac{\sin 2\cdot 0^\circ}{2}\right] + \left[\frac{\sin \overset{2}{\cancel{4}}\cdot\pi/\cancel{2} - \sin 0}{4}\right]\right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2} + 4\left(\frac{0}{2} - \frac{0}{2}\right) + \left(\frac{0-0}{4}\right)\right)$$

$$= \frac{\overset{8}{\cancel{16}}\sqrt{2}}{\cancel{3}} \times \frac{\cancel{3}\pi}{\cancel{2}}$$ ~~cubic~~ unit

$$= 8\pi\sqrt{2}$$ Answer

$\therefore$ Required Volume is $8\pi\sqrt{2}$ Cubic unit

* find the Volume bounded above by paraboloid $z = 5 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

→ Solution,

for Upper and lower limit in Z, put $x=0$, $y=0$

$z = 5 - x^2 - y^2 = 5 - 0 - 0 = 5 \Rightarrow$ Upper limit

$z = 4x^2 + 4y^2 = 0 + 0 = 0 \Rightarrow$ lower limit.

for Upper limit and lower limit in y, solve 2 eqn

$4x^2 + 4y^2 = 5 - x^2 - y^2$

or, $5x^2 + 5y^2 = 5$

or, $x^2 + y^2 = 1$ which is eqn of circle

or, $y^2 = 1 - x^2$ $\quad x^2 + y^2 = a^2$

$\therefore y = \pm\sqrt{1 - x^2}$ $\quad \therefore a = 1$

$\therefore$ Upper limit in $y = +\sqrt{1 - x^2}$

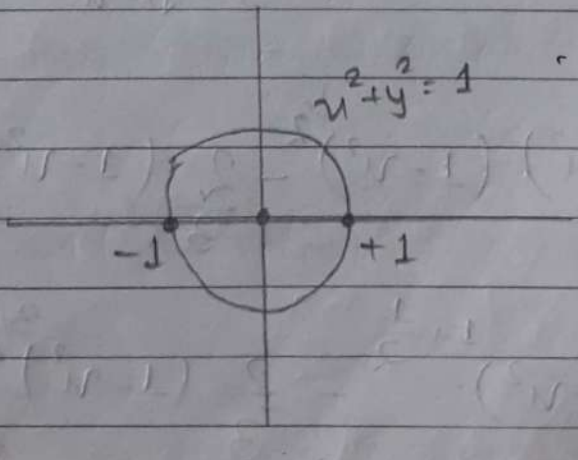
$\therefore$ lower limit in $y = -\sqrt{1 - x^2}$

for Upper and lower limit for x.

eqn is $x^2 + y^2 = 1$

$\therefore$ eqn of circle

so,



$\therefore$ Upper limit $= +1$

lower limit $= -1$

$$\therefore \text{Volume} = \int_{x=-1}^{x=+1} \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} \int_{z=4x^2+4y^2}^{z=5-x^2-y^2} dz\, dy\, dx.$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[z \right]_{4x^2+4y^2}^{5-x^2-y^2} dy\, dx \qquad = 5-x^2-y^2-4x^2-4y^2$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[5-5x^2-5y^2 \right] dy\, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 5\left[(1-x^2-y^2) \right] dy\, dx$$

$$= 5\int_{-1}^{1} \left[y - x^2 y - \frac{y^3}{3} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \qquad \int_{-a}^{+a} f(x) = 2a$$

$$= 5\int_{-1}^{1} \left[(1-x^2)y - \frac{y^3}{3} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 5\int_{-1}^{1} \left[2(1-x^2)(1-x^2)^{1/2} - \frac{2}{3}(1-x^2)^{3/2} \right] dx$$

$$= 5\int_{-1}^{1} \left[2 \cdot (1-x^2)^{1+\frac{1}{2}} - \frac{2}{3}(1-x^2)^{3/2} \right] dx$$

$$= \frac{5\times2}{3}\int_{-1}^{+1}\left[2(1-x^2)^{3/2}\right]dx$$

we should multi. by 2 when limit is from 0 to 1

$$= \frac{20}{3}\int_{-1}^{1}(1-x^2)^{3/2}dx = \frac{20\times2}{3}\int_{0}^{1}(1-x^2)^{3/2}$$

Which is in the form of (a^2-x^2), i.e. (1^2-x^2)

$\therefore a = 1$

Put $x = a\sin\theta$

$\therefore x = \sin\theta$

differentiating w.r. to θ

$$\frac{dx}{d\theta} = \frac{d(\sin\theta)}{d\theta}$$

$\therefore dx = \cos\theta\, d\theta$

when,

$x = 1$

$1 = \sin\theta$

$\therefore \theta = \sin^{-1}(1) = \pi/2$

when, $x = 0$

$0 = \sin\theta$

$\therefore \theta = \sin^{-1}(0) = 0$

$$\therefore \text{Volume} = \int_0^{\pi/2} \frac{20\times2}{3}(1-\sin^2\theta)^{3/2}\cdot\cos\theta\, d\theta$$

$$= \frac{20\times2}{3}\int_0^{\pi/2}\cos^{2\cdot\frac{3}{2}}\theta\cdot\cos\theta\, d\theta$$

$$= \frac{20\times2}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{20\times2}{3}\int_0^{\pi/2}(\cos^2\theta)^2 d\theta$$

$$= \frac{20\times 2}{3}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$$

$$= \frac{20\times 2}{3}\int_0^{\pi/2} \frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right) d\theta$$

$$= \frac{20\times 2}{3}\int_0^{\pi/2} \frac{1}{4}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{20\times 2}{3}\int_0^{\pi/2} \frac{1}{4}\left(\frac{2+4\cos 2\theta+1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{20\times 2}{3}\times\frac{1}{8}\int_0^{\pi/2} \left(3+4\cos 2\theta+\cos 4\theta\right) d\theta$$

$$= \frac{20\times 2}{3\times 8}\left[3\left[\theta\right]_0^{\pi/2}+4\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}+\cancel{\cos}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}\right]$$

$$= \frac{20\times 2}{3\times 8}\left[3.\frac{\pi}{2}+4\left[\frac{\sin 2.\pi/2-\sin 0}{2}\right]+\left[\frac{\sin 4\,\pi/2-\sin 0}{4}\right]\right]$$

$$= \frac{20\times 2}{3\times 8}\times\left[\frac{3\pi}{2}+0+0\right]$$

$$= \frac{\overset{10\times 2}{\cancel{20}}}{\cancel{3}\times 8}\times\frac{\cancel{3}\pi}{\cancel{2}}$$

$$= \frac{\overset{5}{\cancel{10}}\pi\times 2}{\underset{4}{\cancel{8}}}$$

$$\therefore V = \frac{2\times 5\pi}{4} \text{ Cubic Unit} = \frac{5\pi}{2}$$

Hence Required Volume is $\frac{5\pi}{\cancel{4}2}$ Cubic unit.

* find the Volume bounded above by paraboloid $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

→ solution,

for Upper and lower limit of z, Put $x = 0, y = 0$

$z = 8 - x^2 - y^2 = 8 - 0 - 0 = 8$ → Upper limit

$z = x^2 + y^2 = 0 + 0 = 0$ → lower limit.

for Upper and lower limit of y. solving eqn $8 - x^2 - y^2$ and $x^2 + y^2$

or, $8 - x^2 - y^2 = x^2 + y^2$

or, $2x^2 + 2y^2 = 8$

or, $x^2 + y^2 = 4$ eqn of circle

or, $y^2 = 4 - x^2$

$\therefore y = \pm\sqrt{4 - x^2}$

$\therefore$ lower limit of $y = -\sqrt{4 - x^2}$

Upper limit of $y = +\sqrt{4 - x^2}$

for Upper & lower limit of z,

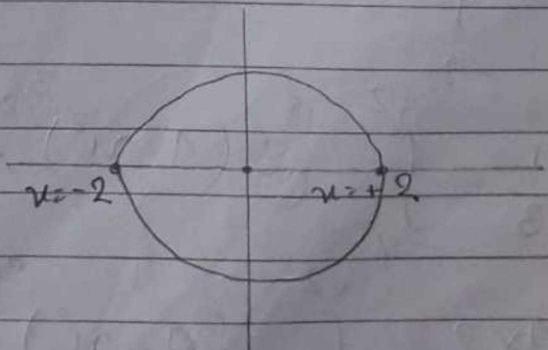
$x^2 + y^2 = (2)^2$ is the eqn of circle

so, $x^2 + y^2 = a^2$

$\therefore a = 2$

Upper limit $= 2$

lower limit $= -2$



$$\therefore \text{Volume}(V) = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=x^2+y^2}^{z=8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{+2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_{x^2+y^2}^{8-x^2-y^2} dy\, dx$$

$$= \int_{-2}^{+2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[8-2x^2-2y^2\right] dy\, dx$$

$$= \int_{-2}^{2} \left[8y - 2x^2y - \frac{2y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{+2} \left[(8-2x^2)y - \frac{2}{3}y^3\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

if $\int_{-a}^{+a} f(x) = 2\times a$ upp = low = but

do = 2a

$$= \int_{-2}^{+2} \left[2(4-x^2)y - \frac{2}{3}y^3\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{+2} \left[2\times 2(4-x^2)(\sqrt{4-x^2}) - \frac{2\times 2}{3}(\sqrt{4-x^2})^3\right] dx$$

$$= 4\int_{-2}^{+2} \left[(4-x^2)^1(4-x^2)^{1/2} - \frac{1}{3}(4-x^2)^{3/2}\right] dx$$

$$= \frac{4}{3}\int_{-2}^{+2} \left[3(4-x^2)^{3/2} - (4-x^2)^{3/2}\right] dx$$

$$= \frac{4\times 2}{3}\int_{-2}^{+2} (4-x^2)^{3/2} dx$$

$= \frac{8}{3} \int_{-2}^{+2} (4-x^2)^{3/2} \, dx$ if we do $\int_{-2}^{+2}$ to $\int_{0}^{2}$ we have to x2.

$= \frac{8 \times 2}{3} \int_{0}^{+2} (4-x^2)^{3/2} \, dx$

which is in the form of (a^2-x^2) so, $a=2$

put ~~x~~ $x = a\sin\theta = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$\therefore dx = 2\cos\theta \, d\theta$

when $x=0$

$0 = 2\sin\theta$

$\sin\theta = 0$

$\therefore \theta = \sin^{-1}(0) = 0^\circ$

when $x=2$

$2 = 2\sin\theta$

$\sin\theta = 1$

$\therefore \theta = \sin^{-1}(1) = \pi/2$

then,

$\text{Volume (v)} = \frac{16}{3} \int_{0}^{\pi/2} (4-4\sin^2\theta)^{3/2} \cdot 2\cos\theta \, d\theta$

$= \frac{16}{3} \int_{0}^{\pi/2} (4)^{3/2} \cdot (1-\sin^2\theta)^{3/2} \cdot 2\cos\theta \, d\theta$

$= \frac{16}{3} \int_{0}^{\pi/2} 2^{2 \cdot 3/2} (\cos^2\theta)^{3/2} \cdot 2\cos\theta \, d\theta$

$= \frac{16}{3} \times 8 \times 2 \int_{0}^{\pi/2} \cos^4\theta \, d\theta$

$= \frac{16 \times 16}{3} \int_{0}^{\pi/2} (\cos^2\theta)^2 \, d\theta$

$= \frac{16 \times 16}{3} \int_{0}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{16\times16}{3}\int_0^{\pi/2} \frac{1}{4}\left(1+2\cancel{4}\cos 2\theta + \cancel{4}\cos^2 2\theta\right) d\theta$$

$$= \frac{16\times16}{3}\int_0^{\pi/2} \frac{1}{4}\left(1+\cancel{4}\cos 2\theta + \cancel{4}\left(\frac{1+\cos 4\theta}{2}\right)\right) d\theta$$

$$= \frac{16\times16}{3}\int_0^{\pi/2} \frac{1}{4}\left(\frac{2+2\cos 2\theta + 1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{\overset{2}{\cancel{16}}\times 16}{3}\times\frac{1}{\cancel{8}}\int_0^{\pi/2}\left(3+2\cos 2\theta+\cos 4\theta\right) d\theta$$

$$= \frac{32}{3}\left[3\left[\theta\right]_0^{\pi/2} + 2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} + \left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}\right]$$

$$= \frac{32}{3}\left[3\times\frac{\pi}{2} + 2\times 0 + 0\right]$$

$$= \frac{\overset{16}{\cancel{32}}}{\cancel{3}}\times\frac{\cancel{3}\pi}{\cancel{2}}$$

$= 16\pi$ Cubic unit

Hence the required Volume is 16π Cubic unit.

~~NB~~ volume nikaldo jhukene place.

⇒ $\int_{-a}^{+a}$ xa vane ⇒ $2a$

⇒ $(4-x^2)$ = yeslai put garne $x = a\sin\theta$

⇒ $\int_0^{a}$ ⇒ limit 0 to a baraune mult by $\times 2$.

⇒ ani $(4-4\sin^2\theta)^{3/2}$ thau ma Common lida $4^{3/2} = 8$ hunchha $(1-\sin^2\theta)^{3/2}$

"Formulae In 2 dimension"

Mass and First Moment formula

Mass :-

$$M = \iint_R \delta(x,y)\,dA$$

$$\left[\because \delta(x,y) \text{ denotes density at } (x,y)\right]$$

First Moments :-

$$M_x = \iint_R y\,\delta(x,y)\,dA$$

$$M_y = \iint_R x\,\delta(x,y)\,dA$$

Center of mass (centroid) $= \bar{x} = \dfrac{M_y}{M}$

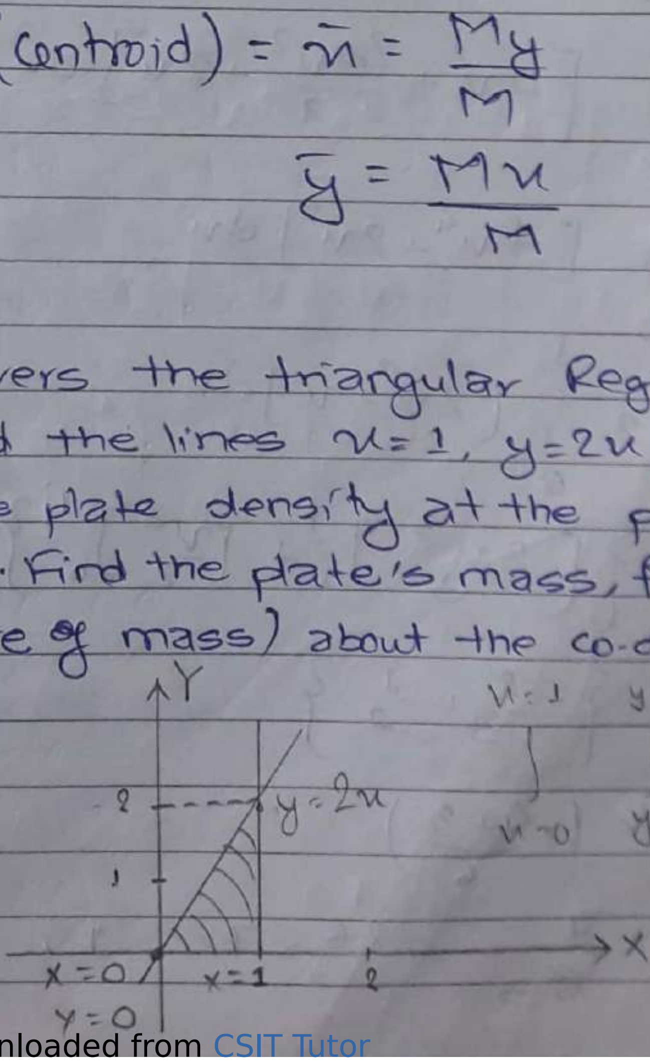
$$\bar{y} = \frac{M_x}{M}$$

* A thin plate covers the triangular Region bounded by the x-axis and the lines $x=1$, $y=2x$ in the first quadrant. The plate density at the point is $\delta(x,y) = 6x+6y+6$. Find the plate's mass, first moments and centroid (Centre of mass) about the co-ordinate axes).

→ from eq^n $y=2x$

when $x=0, y=0$

$x=1, y=2$



$$\int_{x=0}^{x=1}\int_{y=0}^{y=2x} \delta(x,y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} \delta(x,y)\, dy\, dx$$

$\therefore$ we know mass $= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} (6x+6y+6)\, dy\, dx.$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} 6(x+y+1)\, dy\, dx$$

$$= 6 \int_0^1 \left[xy + \frac{y^2}{2} + y \right]_{y=0}^{y=2x} dx$$

$$= 6 \int_0^1 \left[x(2x) + \frac{(2x)^2}{2} + 2x \right] dx$$

$$= 6 \int_0^1 \left[2x^2 + \frac{\cancel{2}4x^2}{2} + 2x \right] dx$$

$$= 6 \int_0^1 \left[4x^2 + 2x \right] dx$$

$$= 6 \times 2 \int_0^1 \left[2x^2 + x \right] dx$$

$$= 12 \left\{ \left[\frac{2x^3}{3} \right]_0^1 + \left[\frac{x^2}{2} \right]_0^1 \right\}$$

$$= 12 \left\{ \frac{2}{3} + \frac{1}{2} \right\}$$

$$= 12 \times \left(\frac{4+3}{6} \right)$$

$$= \cancel{12}^{6} \times \frac{7}{\cancel{6}_3} = \frac{2\cancel{6}}{\cancel{3}} \times 7 = 14$$

$\therefore$ mass (m) = 14

first Moments,

$$M_x = \iint_R y\, \delta(x,y)\, dA$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} y\,(6x+6y+6)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} (6xy + 6y^2 + 6y)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[\left[\frac{\overset{3}{\cancel{6}} x y^2}{\cancel{2}} \right]_0^{2x} + \left[\frac{\overset{2}{\cancel{6}} y^3}{\cancel{3}} \right]_0^{2x} + \left[\frac{\overset{3}{\cancel{6}} y^2}{\cancel{2}} \right]_0^{2x} \right] dx$$

$$= \int_{x=0}^{x=1} \left[3x.(2x)^2 + 2\,(2x)^3 + 3\,(2x)^2 \right] dx$$

$$= \int_{x=0}^{x=1} \left[12x^3 + 16x^3 + 12x^2 \right] dx$$

$$= \int_{x=0}^{x=1} \left[28x^3 + 12x^2 \right] dx$$

$$= 28 \left[\frac{x^4}{4} \right]_0^1 + 12 \left[\frac{x^3}{3} \right]_0^1$$

$$= \overset{7}{\cancel{28}} \times \frac{1}{\cancel{4}} + \overset{4}{\cancel{12}} \times \frac{1}{\cancel{3}}$$

$$= 7 + 4 = 11 \qquad \therefore M_x = 11$$

$$M_y = \iint_R x\, \delta(x,y)\, dA$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} x\,(6x+6y+6)\, dy\, dx$$

$$= \int_0^1 \int_0^{2x} (6x^2+6xy+6x)\, dy\, dx$$

$$= \int_0^1 \left[\left[6x^2 y + \cancel{6}^{3} x \frac{y^2}{\cancel{2}} + 6xy \right]_0^{2x} \right] dx$$

$$= \int_0^1 \left[6x^2.(2x) + \cancel{6}^{3} x\,(2x)^2 + 6x.(2x) \right] dx$$

$$= \int_0^1 \left[12x^3 + 12x^3 + 12x^2 \right] dx$$

$$= \int_0^1 12 \left[2x^3 + x^2 \right] dx$$

$$= 12 \left\{ \left[\cancel{2} \frac{x^4}{\cancel{4}_2} \right]_0^1 + \left[\frac{x^3}{3} \right]_0^1 \right\}$$

$$= 12. \left(\frac{1}{2} + \frac{1}{3} \right)$$

$$= 12 \times \left(\frac{3+2}{6} \right)$$

$$= \cancel{12}^{2} \times \frac{5}{\cancel{6}} = \underline{10}$$

$$\therefore M_y = \underline{10}$$

$\therefore$ Centre of mass (Centroid) $= \bar{x} = \frac{M_y}{M}$

$$\bar{x} = \frac{10}{14} = \frac{5}{7}$$

$$\bar{y} = \frac{M_x}{M} = \frac{11}{14}$$

$\therefore$ Centroid $= \left(\frac{5}{7}, \frac{11}{14}\right)$

Second Moment (Moment of Inertia):-

1. About X-axis:-

$$I_x = \iint_R y^2 \delta(x,y)\, dA$$

2. About Y-axis:-

$$I_y = \iint_R x^2 \delta(x,y)\, dA$$

3. About Origin:-

$$I_0 = I_x + I_y$$

Radii of Gyration:-

1. About X-axis:-

$$R_x = \sqrt{\frac{I_x}{M}}$$

2. About Y-axis:-

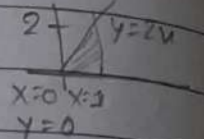
$$R_y = \sqrt{\frac{I_y}{M}}$$

3. About Origin:-

$$R_0 = \sqrt{\frac{I_0}{M}}$$

* Find the moment of inertia [Second moment] and radii of Gyration from eqn $f(x,y) = 6x+6y+6$ about co-ordinate axes and origin. $x=1$, $y=2x$ in 1st q.

→ Solution,



$$\delta(x,y) = 6x+6y+6$$

Moment of Inertia (Second Moment)

About X-axis,

$$I_x = \iint_R y^2\,\delta(x,y)\,dA$$

$$= \int_0^1 \int_0^{2x} y^2(6x+6y+6)\,dy\,dx$$

$$= \int_0^1 \int_0^{2x} (6xy^2+6y^3+6y^2)\,dy\,dx$$

$$= \int_0^1 \left[\left[2\cancel{6}x\frac{y^3}{\cancel{3}}\right]_0^{2x} + 6\left[\frac{y^4}{4}\right]_0^{2x} + 2\cancel{6}\left[\frac{y^3}{\cancel{3}}\right]_0^{2x} \right] dx$$

$$= \int_0^1 \left[2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_0^{2x} dx$$

$$= \int_0^1 \left[2x(2x)^3 + \frac{3}{2}(2x)^4 + 2(2x)^3 \right] dx$$

$$= \int_0^1 \left[16x^4 + 3\times\frac{16}{2}x^4 + 16x^3 \right] dx$$

$$= \int_0^1 16\left[x^4 + \frac{3x^4}{2} + x^3 \right] dx$$

$$= \int_0^1 16\left[\frac{2x^4+3x^4+2x^3}{2} \right] dx$$

$$= \frac{16}{2}\int_0^1 (5x^4+2x^3)\,dx$$

$$= \frac{16}{2}\left\{5\left[\frac{x^5}{5}\right]_0^1 + 2\left[\frac{x^4}{4}\right]_0^1\right\}$$

$$= \frac{16}{2}\left\{\frac{\cancel{5}}{\cancel{5}} + \frac{2}{\cancel{4}2}\right\}$$

$$= \frac{16}{2}\left(\frac{3}{2}\right)$$

$$= 8\times\frac{3}{2} = 4\times 3 = 12$$

About :-

For Y-axis:-

$$I_y = \iint_R \delta(x,y)\,dA\,x^2$$

or, $$I_y = \int_0^1\int_0^{2x} x^2(6x+6y+6)\,dy\,dx$$

$$= \int_0^1\int_0^{2x} (6x^3+6x^2y+6x^2)\,dy\,dx$$

$$= \int_0^1 \left[6x^3y + \frac{6x^2y^2}{2} + 6x^2y\right]_0^{2x} dx$$

$$= \int_0^1 \left[6x^3\cdot(2x) + 3x^2(2x)^2 + 6x^2(2x)\right] dx$$

$$= \int_0^1 (12x^4 + 12x^4 + 12x^3)\,dx$$

$$= \int_0^1 (12x^3 + 24x^4)\,dx$$

$= 12\int_0^1 (2x^4 + x^3)\,dx$

$= 12\int_0^1 (2x^4 + x^3)\,dx \quad = 12\left[2\frac{x^5}{5} + \frac{x^4}{4}\right]_0^1$

$= 12\left(\frac{2}{5} + \frac{1}{4}\right)$

$= {}^{3}\cancel{12} \times \frac{13}{\cancel{20}\,5}$

$= \frac{39}{5}$

About origin,

$I_0 = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}$

∴ Radii of Gyration,

About x-axis $(R_x) = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{12}{14}} = \sqrt{\frac{6}{7}}$

$\therefore R_x = \underline{0.925}$

About Y-axis $(R_y) = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{39}{5} \times \frac{1}{14}} = \sqrt{\frac{39}{70}}$

$\therefore R_y = 0.746$

About Origin $= (R_0) = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{99}{5} \times \frac{1}{14}} = \sqrt{\frac{99}{76}}$

$\therefore R_0 = 1.189$

* Find the centroid of the region in the first qudrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

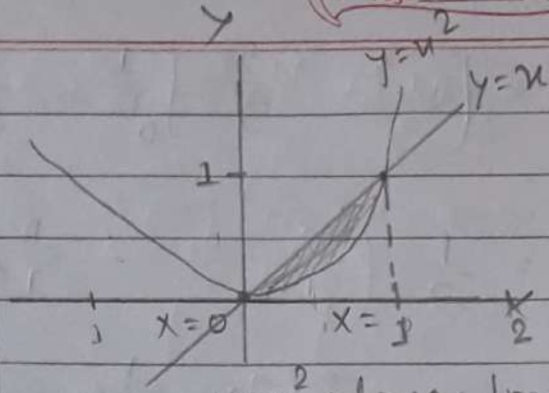
→ $y = x \quad y = x^2$

$x^2 = x$

or, $x^2 - x = 0 \quad x(x-1) \quad \therefore x = 0,\ x = 1$

$y = 0 \quad y = 1$

when $x=0 \quad y=0$
$x=\pm 1 \quad y=1$
$x=2 \quad y=4$



$y=x^2 \rightarrow$ lower limit because parabola is down from line $y=x$
$y=x \rightarrow$ Upper limit

$\Rightarrow$ First Mass is given by

$$\text{Mass} = \iint_R \delta(x,y)\,dA$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} 1\,dy\,dx$$

$$= \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 [x-x^2]\,dx = \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$\therefore \text{Mass } (M) = \frac{1}{6}$$

Now,
first Moment,

$$M_x = \iint_R y\,\delta(x,y)\,dA$$

$$= \int_0^1\int_{x^2}^{x} y\,dy\,dx = \int_0^1\int_{x^2}^{x} y\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 \left[(x)^2 - (x^2)^2\right] dx$$

$$= \frac{1}{2}\int^1 (x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{2}\left(\frac{2}{15}\right) = \frac{1}{15}$$

$$\therefore M_x = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x \, dy \, dx$$

$$= \int_0^1 \left[xy\right]_{x^2}^{x} dx = \int_0^1 \left[x.(x) - x.(x^2)\right] dx$$

$$= \int_0^1 \left[x^2 - x^3\right] dx = \int_0^1 x^2 dx - \int_0^1 x^3 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore M_y = \frac{1}{12}$$

Now,

we have, Centroid is given by

$$\bar{x} = \frac{M_y}{M} = \frac{1}{12} \times \frac{6}{1} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{15} \times \frac{6}{1} = \frac{6}{15} = \frac{2}{5}$$

$$\therefore \text{Centroid} = \left(\frac{1}{2}, \frac{2}{5}\right) \#.$$

Ans. for moment of inertia,

$I_x = \frac{1}{28}$, $I_y = \frac{1}{20}$

$I_0 = \frac{3}{35}$

x-axis. $R_x = \sqrt{\frac{I_x}{M}} = 0.$

y-axis: $R_y = \sqrt{\frac{I_y}{M}} = 0.5$

Orgin - $R_0 = \sqrt{\frac{I_0}{M}} = 0.50$

"Formula in three dimension"

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1. Masses And Moments in three dimension

i) Mass $(M) = \iiint_R \delta \, dv$ $[\because \delta = \text{density at } x, y, z]$

2.) First Moments in Co-ordinate planes.

i) $M_{xy} = \iiint_R z \, \delta \, dv$ – In about xy-plane

$[\because \delta \, dv = dx \cdot dy \cdot dz \, \delta]$

ii) $M_{xz} = \iiint_R y \, \delta \, dv$ – About xz-plane

iii) $M_{yz} = \iiint_R x \, \delta \, dv$ – About yz-plane

3.) Centre of mass (Centroid):

i) $\bar{x} = \frac{M_{yz}}{M}$

ii) $\bar{y} = \frac{M_{xz}}{M}$

iii) $\bar{z} = \frac{M_{xy}}{M}$

4. Motion of Inertia [Second Moment]:

i) $I_x = \iiint_R (y^2 + z^2) \, \delta \, dv$

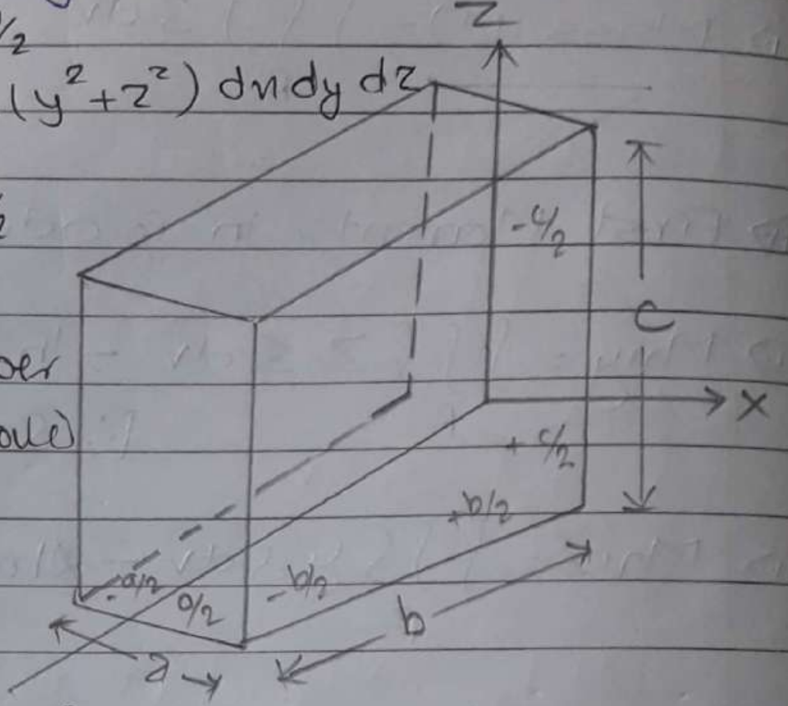
ii) $I_y = \iiint_R (x^2 + z^2) \, \delta \, dv$

iii) $I_z = \iiint (x^2 + y^2) \, \delta \, dv$

* Find I_x, I_y, I_z for rectangular solid of constant density δ as in fig. (Put $M = abc\delta$)

→ Solution,

$$I_x = \int_{z=-c/2}^{z=+c/2} \int_{y=-b/2}^{y=b/2} \int_{x=-a/2}^{x=a/2} \delta(y^2+z^2)\,dx\,dy\,dz$$



when we go from lower limit 0 to $+c/2$ we should Multiply by 2.
Similarly in 0 to $b/2$ and in 0 to $a/2$ then,

$$I_x = 2\times2\times2 \int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2+z^2)\,dx\,dy\,dz\,\delta$$

$$= 8\delta \int_0^{c/2}\int_0^{b/2} \left[xy^2 + xz^2\right]_0^{a/2} dy\,dz$$

$$= 8\delta \int_0^{c/2}\int_0^{b/2} \left[\frac{a}{2}y^2 + \frac{a}{2}z^2\right] dy\,dz$$

$$= 8\delta \cdot \frac{a}{2} \int_0^{c/2}\int_0^{b/2} \left[y^2+z^2\right] dy\,dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + yz^2\right]_0^{b/2} dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{b^3}{8\times3} + \frac{b}{2}z^2\right] dz$$

$$= 4a\delta\frac{b}{2}\int_0^{c/2}\left[\frac{b^2}{12}+z^2\right]dz$$

$$= 2ab\delta\left[\frac{b^2}{12}z+\frac{z^3}{3}\right]_0^{c/2}$$

$$= 2ab\delta\left[\frac{cb^2}{2\cdot 12}+\frac{c^3}{8\times 3}\right]$$

$$= 2ab\delta\left[\frac{cb^2}{24}+\frac{c^3}{24}\right]$$

$$= \frac{2abc\delta}{24}\left[b^2+c^2\right]$$

$$= \frac{abc\delta}{12}(b^2+c^2)$$

$$\therefore I_x = \frac{M}{12}(b^2+c^2)$$

Now,

$$I_y = \int_0^{c/2}\int_0^{b/2}\int_0^{a/2} 8\delta(x^2+z^2)\,dx\,dy\,dz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2}\left[\frac{x^3}{3}+xz^2\right]_0^{a/2}dy\,dz$$

$$= 8\delta\int_0^{c/2}\int_0^{b/2}\left[\frac{a^3}{2^3\times 3}+\frac{a}{2}z^2\right]dy\,dz$$

$$= 8\delta\cdot\frac{a}{2}\int_0^{c/2}\int_0^{b/2}\left[\frac{a^2}{12}+z^2\right]dy\,dz$$

$$= 4a\delta\int_0^{c/2}\left[\frac{a^2y}{12}+z^2y\right]_0^{b/2}dz$$

$$= 4a\delta \int_0^{c/2} \frac{b}{2}\left(\frac{a^2}{12} + z^2\right) dz$$

$$= 4a\delta \frac{b}{2} \left[\frac{a^2 z}{12} + \frac{z^3}{3}\right]_0^{c/2}$$

$$= 2ab\delta \left[\frac{a^2 c}{12\times 2} + \frac{c^3}{24}\right]$$

$$= 2abc\frac{\delta}{24}\left(a^2 + c^2\right)$$

$$\therefore I_y = \frac{M}{12}\left(a^2 + c^2\right)$$

Similarly,

$$I_z = 8\delta \int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (x^2+y^2)\, dx\, dy\, dz$$

$$= \int_0^{c/2}\int_0^{b/2} 8\delta \left[\frac{x^3}{3} + xy^2\right]_0^{a/2} dy\, dz$$

$$= \int_0^{c/2}\int_0^{b/2} 8\delta \left[\frac{a^3}{24} + \frac{a}{2}y^2\right] dy\, dz$$

$$= 8\delta \frac{a}{2} \int_0^{c/2}\int_0^{b/2} \left[\frac{a^2}{12} + y^2\right] dy\, dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{a^2 y}{12} + \frac{y^3}{3}\right]_0^{b/2} dz$$

$$= 4a\delta \int_0^{c/2} \left[\frac{a^2 b}{24} + \frac{b^3}{24}\right] dz$$

$$= \frac{4 a \delta b}{24 c}\left[a^2 z + b^2 z\right]_0^{c/2}$$

$$= \frac{ab\delta}{6}\left[a^2 \frac{c}{2} + b^2 \frac{c}{2}\right]$$

$$= \frac{abc\delta}{6\times 2}\left(a^2+b^2\right)$$

$$\therefore I_z = \frac{M}{12}\left(a^2+b^2\right) \#$$

$$\therefore I_y = \frac{M}{12}\left(a^2+c^2\right) \#$$

$$\therefore I_x = \frac{M}{12}\left(b^2+c^2\right) \#$$

Double Integral in Polar form:-

Note :-

$$dx\,dy = dy\,dx = r\,dr\,d\theta$$

$$x^2+y^2 = r^2$$

$\boxed{x^2+y^2=1}$ (eqⁿ of circle)

$$* \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dy\,dx.$$

$$= 2\times 2\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx$$

$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx$$

$x^2+y^2=1^2$

In Polar, when

always $\to y=0$

$\to r=0$

$y=\sqrt{1-x^2}$ in

$$y = \sqrt{1-x^2}$$

$$x^2+y^2 = 1$$

$$r = 1$$

when $y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$

$\therefore r=1$

$x = 0$ $\qquad$ $x = 1$ $\qquad$ x is for φ → up & lower limit

$\varphi = 0$ $\qquad$ $\varphi = \pi/2$ $\qquad$ y is for r → up & low limit

Now,

$$\Rightarrow 4 \int_{\varphi_1 = 0}^{\varphi_2 = \pi/2} \int_{r=0}^{r=1} r\,dr\,d\varphi \qquad [\because dy\,dx = dx\,dy = r\,dr\,d\varphi]$$

$$= 4 \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^1 d\varphi = 4 \int_0^{\pi/2} \left[\frac{1}{2}\right] d\varphi$$

$$= 4 \times \frac{1}{2} \left[\varphi\right]_0^{\pi/2}$$

$$= 2 \cdot \frac{\pi}{2}$$ $\qquad$ ∴ Area of circle is π Sq. unit.

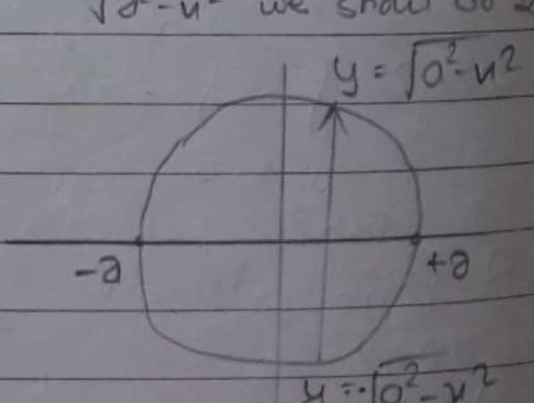
$$= \pi \text{ Ans.}$$

Double integral in Polar form:

Q. $$\int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} dy\,dx.$$

when we go from 0 to a & 0 to $\sqrt{a^2-x^2}$ we should do 2×2.

$$= \int_{-a}^{+a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$



$$= 2 \times \int_0^a \times 2 \int_0^{\sqrt{a^2-x^2}} dy\,dx$$

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$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$$

to change in polar,

when $y = 0$ | $y = \sqrt{a^2 - x^2}$
$r = 0$ | then,
$x \neq 0$ | $y^2 = a^2 - x^2$ $[\because x^2+y^2=r^2]$
$\theta = 0$ | $\therefore x^2 + y^2 = a^2$
$\therefore r^2 = a^2 \quad \therefore r = a$

when,

$x = 0 \qquad x = a$

$\theta_1 = 0° \qquad \theta_2 = \pi/2$

Now,

$$= 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\, dr\, d\theta \qquad [dy\,dx = dx\,dy = r\,dr\,d\theta]$$

$$= 4\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^a d\theta = 4\int_0^{\pi/2} \frac{a^2}{2} d\theta = 2a^2 \int_0^{\pi/2} d\theta$$

$$= 2a^2 \left[\theta\right]_0^{\pi/2} = 2a^2 \times \frac{\pi}{2} = \pi a^2 \#$$

* $$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$$

→ Solution,

when, | when,
$y = 0$ | $y = \sqrt{1-x^2}$
$r = 0$ | then, Sq.
| $y^2 = 1 - x^2$
| $x^2 + y^2 = 1$
| $\therefore r^2 = 1 \quad \therefore r = 1$

when, when,

$x=0$ $x=1$

$\theta_1=0°$ $\theta_2=\pi/2$

Now,
$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{r=1}(x^2+y^2)\,r\,dr\,d\theta$$

$[\because x^2+y^2=r^2]$

$$=\int_0^{\pi/2}\int_0^1 r^2.r.dr\,d\theta$$

$$=\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 d\theta=\int_0^{\pi/2}\frac{1}{4}\,d\theta=\frac{1}{4}\left[\theta\right]_0^{\pi/2}$$

$$=\frac{1}{4}\times\frac{\pi}{2}=\frac{\pi}{8}\ \#$$

Q. $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\frac{1}{(1+x^2+y^2)^2}\,dy\,dx$$

$$\Rightarrow 2\int_0^1 2\int_0^{\sqrt{1-x^2}}\frac{1}{(1+r^2)^2}\,dy\,dx$$

$$=4\int_0^1\int_0^{\sqrt{1-x^2}}\frac{1}{(1+r^2)^2}\,dy\,dx$$

when,

$y=0$ $y=\sqrt{1-x^2}$

$r=0$ $r=1$

Rough

$y=\int\frac{1}{(1+x^2)^2}\,x\,dx$

let

$y=1+x^2$

$\frac{dy}{dx}=2x$

$\therefore \frac{dy}{2}=x\,dx$

then,

$y=\int\frac{1}{y^2}\frac{dy}{2}$

$=\frac{1}{2}\frac{y^{-2+1}}{-2+1}$

$=-\frac{1}{2y}$

$=-\frac{1}{2(1+x^2)}$

when,

$x = 0 \qquad x = 1$

$\theta_1 = 0 \qquad \theta_2 = \pi/2$

$\therefore \int \frac{1}{(1+x^2)^2} x\,dx$

$= -\frac{1}{2(1+x^2)}$

$= 4\int_0^{\pi/2}\int_0^1 \frac{1}{(1+r^2)^2} \cancel{dy\,dx}\; r\,dr\,d\theta$

$r\,dr =$

$= 4\int_0^{\pi/2} \left[-\frac{1}{2(1+r^2)}\right]_0^1 d\theta = 4\int_0^{\pi/2} \left[-\frac{1}{2\times 2} + \frac{1}{2}\right] d\theta$

$= 4\int_0^{\pi/2} \left[\frac{1}{2} - \frac{1}{4}\right] d\theta = 4\int_0^{\pi/2} \left(\frac{1}{4}\right) d\theta = \frac{4}{4}\,[\theta]_0^{\pi/2}$

$= \pi/2$ #.

Q. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\,dx$

when, $y = 0$, $r = 0$

when, $y = \sqrt{1-x^2}$, $x^2 + y^2 = 1$, $\therefore r^2 = 1^2 \therefore r = 1$

when $x = 0$, $\theta = 0$; $x = 1$, $\theta = \pi/2$

$= 2\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx$

$= 2\int_0^{\pi/2}\int_0^1 r\,dr\,d\theta$

$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^1 d\theta = 2\int_0^{\pi/2} \frac{1}{2} d\theta = 2 \times \frac{1}{2}\,[\theta]_0^{\pi/2}$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$ #.

Substitutions in Multiple Integrals

Definition of Jacobian:-

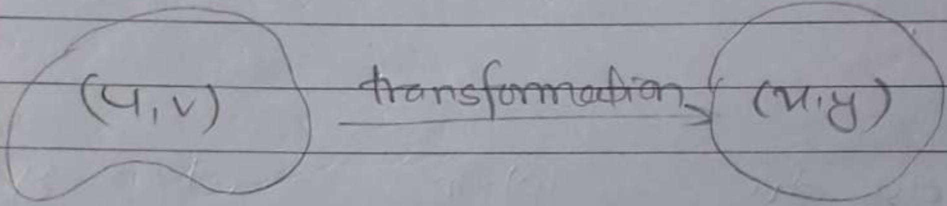
- The Jacobian determinant or Jacobian of the co-ordinate transformation $x = g(u,v)$, $y = h(u,v)$ is,

$$J(u,v) = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{\delta x}{\delta u}\frac{\delta y}{\delta v} - \frac{\delta y}{\delta u}\frac{\delta x}{\delta v}$$

The Jacobian is denoted by,

$$J(u,v) = \frac{\delta(x,y)}{\delta(u,v)}$$

$x = g(u,v)$
$y = h(u,v)$



$$\iint_R f(x,y)\,\underset{dx\,dy}{dA} \xrightarrow{\text{"transformation"}} \iint_R H(u,v)\,|J(u,v)|\,\underset{du\,dv}{dA}$$

* Jacobian determinant in 3-D.

$x = g(u,v,w)$
$y = h(u,v,w)$
$z = k(u,v,w)$

$$J(u,v,w) = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} & \frac{\delta z}{\delta w} \end{vmatrix}$$

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$$\iiint_R f(x,y,z)\,dx\,dy\,dz \xrightarrow{\text{transformation}} \iiint_R H(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

* Find the Jacobian of following :- determinant

a.) $x = uv$, $y = \frac{u}{v}$

→ solution,

$\frac{\delta x}{du} = \frac{\delta uv}{\delta u} = v$

$\frac{\delta x}{dv} = \frac{\delta uv}{\delta v} = u$

$\frac{\delta y}{du} = \frac{\delta u/v}{\delta u} = \frac{1}{v}$

$\frac{\delta y}{dv} = \frac{\delta u/v}{\delta v} = u\frac{\delta v^{-1}}{\delta v}$

$= -u.\frac{1}{v^2} = -\frac{u}{v^2}$

Partial derivative garda j ko respect ma xa tyo matra garne ani constant line mult form a ma raye, +, - form ma raye 0.

e.g. $\frac{\delta uv}{\delta u} = v \times 1 = v$

e.g. $\frac{\delta u+v}{\delta u} = \frac{\delta u+0}{\delta u} = 1$

$\therefore J(u,v) = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$

$= \cancel{v} \times (-\frac{u}{v^2}) - \frac{u}{v}$ $\quad J(u,v) = \frac{\delta(x,y)}{\delta(u,v)}$

$= -\frac{2u}{v}$ #.

b.) $x = \frac{u+v}{3}$, $y = \frac{v-2u}{3}$

→ Solution,

$\frac{\delta x}{\delta u} = \frac{\delta \frac{u+v}{3}}{\delta u} = \frac{1}{3}\frac{\delta u+0}{\delta u} = 1/3$

$$\frac{\delta x}{\delta v} = \frac{\delta\left(\frac{u+v}{3}\right)}{\delta v} = \frac{1}{3}\frac{\delta v}{\delta v} = \frac{1}{3}$$

$$\frac{\delta y}{\delta u} = \frac{\delta\left(\frac{v-2u}{3}\right)}{\delta u} = \frac{1}{3}\frac{\delta(2u)}{\delta u} = -\frac{2}{3}$$

$$\frac{\delta y}{\delta v} = \frac{\delta\left(\frac{v-2u}{3}\right)}{\delta v} = \frac{1}{3}\frac{dv}{dv} = \frac{1}{3}$$

Now,

$$J(u,v) = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = 1/3 \ \#$$

c.) $x = u\cos v \qquad y = u\sin v$

→ Solution,

$$\frac{\delta x}{\delta u} = \frac{\delta u\cos v}{\delta u} = \cos v$$

$$\frac{\delta x}{\delta v} = \frac{\delta u\cos v}{\delta v} = u(-\sin v) = -u\sin v$$

$$\frac{\delta y}{\delta u} = \frac{\delta u\sin v}{\delta u} = \sin v$$

$$\frac{\delta y}{\delta v} = \frac{\delta u\sin v}{\delta v} = u\cos v$$

$$\therefore J(u,v) = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v$$

$$= u(\cos^2 v + \sin^2 v)$$

$$= u \ \#$$

d.) $x = u\sin v \qquad , y = u\cos v$

→ Solution,

$$\frac{\delta x}{\delta u} = \frac{\delta u\sin v}{\delta u} = \sin v$$

$$\frac{\delta x}{\delta v} = \frac{\delta\, 4\sin v}{\delta v} = 4\cos v$$

$$\frac{\delta y}{\delta u} = \frac{\delta\, u\cos v}{\delta u} = \cos v$$

$$\frac{\delta y}{\delta v} = \frac{\delta\, 4\cos v}{\delta v} = -4\sin v$$

$$\therefore J(u,v) = \begin{vmatrix} \sin v & 4\cos v \\ \cos v & -4\sin v \end{vmatrix} = -4\sin^2 v - 4\cos^2 v$$

$$= -4(\sin^2 v + \cos^2 v)$$

$$= -4 \ \#$$

* Evaluate the double integral $\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2}\, dx\, dy$ by applying transformation, $u = \frac{2x-y}{2}$, $v = y/2$ and integrating over an appropriate region in the uv-plane.

→ Solution:-

$u = \frac{2x-y}{2}$ $\qquad$ $v = y/2$

or, $2u = 2x - y$ $\qquad$ or, $y = 2v$

or, $2x = 2u + y$ $\quad [\because y = 2v]$

$\therefore x = \frac{2u+2v}{2}$

or, $x = \frac{2(u+v)}{2}$

$\therefore x = (u+v)$ $\quad v = y/2$

$y = 2v$

$x = u + v \qquad y = 2v$

Now,

$$\frac{\delta x}{\delta u} = \frac{\delta(u+v)}{\delta u} = 1$$

$$\frac{\delta x}{\delta v} = \frac{\delta(u+v)}{\delta v} = 1$$

$$\frac{\delta y}{\delta u} = \frac{\delta 2v}{\delta u} = 0$$

$$\frac{\delta y}{\delta v} = \frac{\delta 2v}{\delta v} = 2$$

$$\therefore J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = |2-0| = 2 \ \#.$$

Now,

we have to change limit from table.

$(x = u+v) \qquad y = 2v$

xy eqⁿ	Corresp. uv eqⁿ	Simplified uv
$x = \frac{y}{2} + 1$	or, $u+v = \frac{2v}{2} + 1$	$\therefore u = v - v + 1 = 1$
$x = \frac{y}{2}$	or, $u+v = \frac{2v}{2}$	$\therefore u = v - v = 0$
$y = 4$	$2v = 4$	$\therefore v = \frac{4}{2} = 2$
$y = 0$	$2v = 0$	$\therefore v = 0$

$$= \int_{y=0}^{y=4} \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy$$

$$= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \, |J(u,v)| \, du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \times 2 \, du \, dv$$

$$= 2 \int_{v=0}^{v=2} \left[\frac{u^2}{2} \right]_0^1 dv$$

$$= \frac{2}{2} \int_0^2 1 \, dv$$

$$= [v]_0^2$$

$$= 2$$

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Substitution in Multiple Integrals.

Def^n of Jacobian:-

The Jacobian determinant or Jacobian of the co-ordinate transformation $x = g(u,v)$, $y = h(u,v)$ is,

$$J(u,v) = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{\delta x}{\delta u}\frac{\delta y}{\delta v} - \frac{\delta y}{\delta u}\frac{\delta x}{\delta v}$$

* Evaluate:-

$$\int_{z=0}^{z=3}\int_{y=0}^{y=4}\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

by applying the transformation,

$$u = \frac{(2x-y)}{2}, \quad v = y/2, \quad w = \frac{z}{3}$$

→ Solution:-

$$u = \frac{2x-y}{2} \qquad y = 2v \qquad z = 3w$$

or, $2u = 2x - y$

or, $2x = 2u + y$

or, $2x = 2u + 2v$

or, $x = u + v$

Now,

$$x = u+v \qquad y = 2v \qquad z = 3w$$

For $J(u,v)$

$\frac{\delta x}{\delta u} = \frac{\delta(u+v)}{\delta u} = 1$ $\quad \frac{\delta y}{\delta u} = \frac{\delta 2v}{\delta u} = 0$ $\quad \frac{\delta z}{\delta w} = \frac{\delta 3w}{\delta w} = 3$

$\frac{\delta x}{\delta v} = \frac{\delta(u+v)}{\delta v} = 1$ $\quad \frac{\delta y}{\delta v} = \frac{\delta 2v}{\delta v} = 2$ $\quad \frac{\delta z}{\delta w} =$

For $J(u,v)$

$\frac{\delta x}{\delta u} = \frac{\delta(u+v)}{\delta u} = 1$ $\frac{\delta y}{\delta u} = \frac{\delta 2v}{\delta u} = 0$

$\frac{\delta x}{\delta v} = \frac{\delta(u+v)}{\delta v} = 1$ $\frac{\delta y}{\delta v} = \frac{\delta 2v}{\delta v} = 2$

$\frac{\delta x}{\delta w} = \frac{\delta(u+v)}{\delta w} = 0$ $\frac{\delta y}{\delta w} = \frac{\delta 2v}{\delta w} = 0$

$\frac{\delta z}{\delta u} = \frac{\delta 3w}{\delta u} = 0$ $\frac{\delta z}{\delta v} = \frac{\delta 3w}{\delta v} = 0$ $\frac{\delta z}{\delta w} = \frac{\delta 3w}{\delta w} = 3$

Now,

$$J(u,v) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \quad \begin{aligned} x &= u+v \\ y &= 2v \\ z &= 3w \end{aligned}$$

$$= 1(6-0) = 6$$

Now,

xyz eqⁿ	uvw eqⁿ	Simplified uvw
$x = \frac{y}{2}+1$	$u+v = \frac{2v}{2}+1$	$\therefore u = 1$
$x = \frac{y}{2}$	$u+v = \frac{2v}{2}$	$\therefore u = 0$
$y = 0$	$2v = 0$	$\therefore v = 0$
$y = 4$	$2v = 4$	$\therefore v = 2$
$z = 3$	$3w = 3$	$\therefore w = 1$
$z = 0$	$3w = 0$	$\therefore w = 0$

Then,

$$\int_{w=0}^{w=1}\int_{v=0}^{v=2}\int_{u=0}^{u=1} (u+w)\,|J(u,v)|\,du\,dv\,dw$$

$$= \int_0^1\int_0^2\int_0^1 (u+w)\times 6\,du\,dv\,dw$$

$$= 6\int_0^1\int_0^2 \left[\frac{u^2}{2}+uw\right]_0^1 dv\,dw$$

$$= 6\int_0^1\int_0^2 \left[\frac{1}{2}+w\right] dv\,dw$$

$$= 6\int_0^1 \left[\frac{1}{2}v+vw\right]_0^2 dw$$

$$= 6\int_0^1 \left[1+2w\right] dw$$

$$= 6\left\{\left[w+\frac{2w^2}{2}\right]_0^1\right\}$$

$$= 6\{1+1\}$$

$$= 6\times 2$$

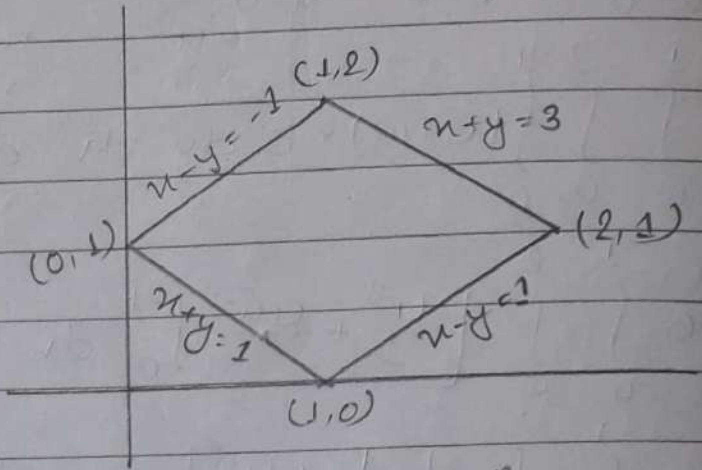
$$= 12$$

$\therefore$ Required answer is 12.

* Evaluate :- $$\iint_R (x-y)^4 e^{x+y}\,dx\,dy$$

applying transformation, $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$

[R is square with vertices (1,0) (2,1) (1,2) and (0,1)]



$\rightarrow$ To find eqⁿ (1,2) & (2,1)

$(x_1 y_1 \quad x_2 y_2)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or, $(y-2) = \frac{1-2}{2-1}(x-1)$

or, $y - 2 = -1(x-1)$

or, $y - 2 = -x + 1$

$\therefore \boxed{x + y = 3}$ like other same eqⁿ given in qus.

Table :-

xy eqⁿ	uv eqⁿ	simplified uv eqⁿ
$x+y=1$	$\frac{u+v}{2} + \frac{u-v}{2} = \frac{u+\cancel{v}+u-\cancel{v}}{2} = 1$	$\frac{2u}{2} = 1$ $\therefore u = 1$
$x+y=3$	$\frac{u+v}{2} + \frac{u-v}{2} = 3$	$\therefore u = 3$
$x-y=1$	$\frac{\cancel{u}+v-\cancel{u}+v}{2} = 1$	$\therefore v = 1$
$x-y=-1$	$\frac{\cancel{u}+v-\cancel{u}+v}{2} = -1$	$\therefore v = -1$

$x - y = \frac{u+v}{2} - \frac{u-v}{2} =$

or, $x - y = \frac{\cancel{u}+v-\cancel{u}+v}{2}$

or, $v = x - y$

$x + y = \frac{u+v}{2} + \frac{u-v}{2}$

or, $x + y = \frac{2u}{2}$

$\therefore u = x + y$

$$\frac{\delta x}{\delta u} = \frac{\delta\left(\frac{u+v}{2}\right)}{\delta u} = \frac{1}{2} \qquad \frac{\delta x}{\delta v} = \frac{\delta\left(\frac{u+v}{2}\right)}{\delta v} = \frac{1}{2}$$

$$\frac{\delta y}{\delta u} = \frac{\delta\left(\frac{u-v}{2}\right)}{\delta u} = \frac{1}{2} \qquad \frac{\delta y}{\delta v} = \frac{\delta\left(\frac{u-v}{2}\right)}{\delta v} = -\frac{1}{2}$$

then,

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{-4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

Now,

$$\iint_R (x-y)^4 e^{x+y}\, dx\, dy \qquad [\because x-y=v]\ [\because x+y=u]$$

$$= \int_{v=-1}^{v=1} \int_{u=1}^{u=3} (v)^4 . e^u \left| J(u,v) \right| du\, dv$$

$$= \int_{-1}^{1} v^4 \left[e^u\right]_1^3 dv \times -\frac{1}{2}$$

$$= -\frac{1}{2} \int_{-1}^{1} \left[e^3 - e^1\right] v^4\, dv$$

$$= -\frac{1}{2}\left(e^3 - e^1\right) \times 2 \left[\frac{v^5}{5}\right]_0^1$$

$$= -\frac{1}{5}\left(e^3 - e^1\right)$$ #

Jacobian in Polar co-ordinate :-

$$J(r,\theta) = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{vmatrix}$$

* Find Jacobian of :-

a.) $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\delta r\cos\theta}{\delta r} & \frac{\delta r\cos\theta}{\delta \theta} \\ \frac{\delta r\sin\theta}{\delta r} & \frac{\delta r\sin\theta}{\delta \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r \;\#.$$

$$J(r,\theta,z) = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} & \frac{\delta x}{\delta z} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} & \frac{\delta y}{\delta z} \\ \frac{\delta z}{\delta r} & \frac{\delta z}{\delta \theta} & \frac{\delta z}{\delta z} \end{vmatrix}$$

* find Jacobian of;

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

→ Solution,

$\frac{\delta x}{\delta r} = \frac{\delta r\cos\theta}{\delta r} = \cos\theta \qquad \frac{\delta y}{\delta r} = \frac{\delta r\sin\theta}{\delta r} = \sin\theta$

$\frac{\delta x}{\delta \theta} = \frac{\delta r\cos\theta}{\delta \theta} = -r\sin\theta \qquad \frac{\delta y}{\delta \theta} = \frac{\delta r\sin\theta}{\delta \theta} = r\cos\theta$

$\frac{\delta x}{\delta z} = \frac{\delta r\cos\theta}{\delta z} = 0 \qquad \frac{\delta y}{\delta z} = \frac{\delta r\sin\theta}{\delta z} = 0$

$\frac{\delta z}{\delta r} = \frac{\delta z}{\delta r} = 0 \qquad \frac{\delta z}{\delta \theta} = \frac{\delta z}{\delta \theta} = 0 \qquad \frac{\delta z}{\delta z} = 1$

$$\therefore J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\,|r\cos^2\theta + r\sin^2\theta|$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r \; \#.$$

Partial Derivatives :-

$\frac{\delta f}{\delta x} = f_x$	$\frac{\delta f}{\delta y} = f_y$	$\frac{\delta^2 f}{\delta x^2} = f_{xx}$	$\frac{\delta^2 f}{\delta y^2} = f_{yy}$
$\frac{\delta^2 f}{\delta x \delta y} = f_{yx}$	$\frac{\delta^2 f}{\delta y \delta x} = f_{xy}$		

* Find value of $\frac{\delta f}{\delta x} = f_x$ and $\frac{\delta f}{\delta y} = f_y$ at $(4,-5)$ if,

$$f(x,y) = x^2 + 3xy + y - 1.$$

→ solution,

$$f(x,y) = x^2 + 3xy + y - 1$$

$$\therefore \frac{\delta f}{\delta x} = f_x = 2x + 3y$$

at $(4,-5)$

$$f(x) = 2\times 4 + 3\times(-5)$$
$$= 8 - 15$$
$$= -7 \;\#$$

Similarly,

$$\frac{\delta f}{\delta y} = f_y = 3x + 1$$

at $(4,-5)$

$$= 3\times 4 + 1$$
$$= 13 \;\#$$

* Find $\frac{\delta f}{\delta y}$ if $f(x,y) = x\cos xy$. $\frac{\delta f}{\delta x} = ?$

→ solution,

$$\frac{\delta f}{\delta y} = f_y = \frac{\delta x \cos xy}{\delta y}$$

$$= x\frac{\delta \cos xy}{\delta xy} \times \frac{\delta xy}{\delta y}$$

$$= x^2(-\sin xy)$$

$$= -x^2 \sin xy$$

Product rule $= \frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient rule $= \frac{d}{dx}\left(\frac{u}{v}\right) = v\frac{du}{dx} - u\frac{dv}{dx} / v^2$

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$$\frac{\delta f}{\delta x} = f_y = \frac{\delta \overset{u}{x}.\cos \overset{v}{xy}}{\delta x}$$

$$= x\frac{\delta \cos xy}{\delta x} + \cos xy \frac{\delta x}{\delta x}$$

$$= x\frac{\delta \cos xy}{\delta xy} \times \frac{\delta xy}{\delta x} + \cos xy \times 1$$

$$= xy(-\sin xy) + \cos xy$$

$$= -xy\sin xy + \cos xy.$$

* Find f_x and f_y if $f(x,y) = \frac{2y}{y+\cos x}$ ⇒ quotient.

$$f_x = \frac{\delta\left(\frac{2y}{y+\cos x}\right)}{\delta x}$$

$$= \frac{(y+\cos x)\frac{\delta 2y}{\delta x} - 2y\frac{\delta(y+\cos x)}{\delta x}}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)\times 0 - 2y(0+(-\sin x))}{(y+\cos x)^2}$$

$$\therefore f_x = \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\delta\left(\frac{2y}{y+\cos x}\right)}{\delta y} = \frac{(y+\cos x)\frac{\delta 2y}{\delta y} - 2y\frac{\delta(y+\cos x)}{\delta y}}{(y+\cos x)^2}$$

$$= (y+\cos x)\times 2 - 2y(1+0)/v^2$$

$$= 2(\cancel{y}+\cos x - \cancel{y})/v^2$$

$$\therefore f_y = \frac{2\cos x}{(y+\cos x)^2} \quad \#.$$

* find $\frac{\delta f}{\delta x} = f_x$ and $\frac{\delta f}{\delta y} = f_y$.

a) $f(x,y) = \sqrt{x^2+y^2}$

$$\rightarrow f(x) = \frac{\delta(x^2+y^2)^{1/2}}{\delta x} = \frac{1}{2}(x^2+y^2)^{1/2-1} \times 2x$$

$$= \frac{\delta(x^2+y^2)^{1/2}}{\delta(x^2+y^2)} \times \frac{\delta(x^2+y^2)}{\delta x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore f(x) = \frac{x}{\sqrt{x^2+y^2}} \quad \#.$$

$$f_y = \frac{\delta(x^2+y^2)^{1/2}}{\delta y}$$

$$= \frac{\delta(x^2+y^2)^{1/2}}{\delta(x^2+y^2)} \times \frac{\delta(x^2+y^2)}{\delta y}$$

$$= \frac{1}{\cancel{2}\sqrt{(x^2+y^2)}} \times \cancel{2}y$$

$$\therefore f_y = \frac{y}{\sqrt{x^2+y^2}} \quad \#.$$

b) $f(x,y) = e^{x+y+1}$

$$\therefore f_x = \frac{\delta e^{x+y+1}}{\delta(x+y+1)} \times \frac{\delta(x+y+1)}{\delta x}$$

$= e^{x+y+1} \times 1$

$fx = e^{x+y+1}$ #

$$fy = \frac{\delta(e^{x+y+1})}{\delta(x+y+1)} \times \frac{\delta(x+y+1)}{\delta y}$$

$\therefore fy = e^{x+y+1}$ #

c.) $f(x,y) = \ln(x+y)$

→ Solution,

$$fx = \frac{\delta \ln(x+y)}{\delta(x+y)} \times \frac{\delta(x+y)}{\delta x} \qquad \frac{d \log x}{dx} = \frac{1}{x}$$

$$= \frac{1}{(x+y)} \times 1$$

$$\therefore f(x) = \frac{1}{(x+y)} \#$$

$$fy = \frac{\delta \ln(x+y)}{\delta(x+y)} \times \frac{\delta(x+y)}{\delta y}$$

$$\therefore fy = \frac{1}{(x+y)} \#$$

Implicit Differentiation :

*. Find $\frac{\delta z}{\delta x}$ if the eqn $yz - \ln z = x+y$.

→ Solution,

$$\frac{\delta(yz - \ln z)}{\delta x} = \frac{\delta(x+y)}{\delta x}$$

$$\text{or, } \frac{\delta yz}{\delta x} - \frac{\delta \ln z}{\delta x} = 1$$

or, $y \frac{\delta z}{\delta x} - \frac{\delta(\ln z)}{\delta z} \times \frac{\delta z}{\delta x} = 1$

or, $\frac{\delta z}{\delta x}\left(y - \frac{1}{z}\right) = 1$

$\therefore \frac{\delta z}{\delta x} = \left(\frac{z}{yz-1}\right)$ #.

* If $f(x,y) = x\cos y + ye^x$

→ find, $\frac{\delta^2 f}{\delta x^2} = f_{xx}$, f_{yy}, f_{xy}, f_{yx}

→ Solution

for f_{xx}

we have to do first f_x then do f_x again with first for answer.

i.e,

$f_x = \frac{\delta(x\cos y + ye^x)}{\delta x}$

$= \frac{\delta x\cos y}{\delta x} + \frac{ye^x}{\delta x}$

$\therefore f_x = \cos y + ye^x$ ///.

$\therefore f_{xx} = \frac{\delta(\cos y + ye^x)}{\delta x}$

$= \frac{\delta\cos y}{\delta x} + \frac{(ye^x)\delta}{\delta x}$

$= 0 + ye^x$

$\therefore f_{xx} = ye^x$ ///.

$f_{yy} = ?$

$$f_y = \frac{\delta(x\cos y + ye^x)}{\delta y}$$

$$= \frac{\delta x\cos y}{\delta y} + \frac{\delta(ye^x)}{\delta y}$$

$$= -x\sin y + e^x$$

$$f_{yy} = \frac{\delta(-x\sin y + e^x)}{\delta y}$$

$$= -x\cos y$$

$$f_{xy} = \frac{\delta^2(f,y)}{\delta y\,\delta x} = f_{xy}$$ — first do f_x second f_y

$\therefore f_x = \cos y + ye^x$

$$\therefore f_{xy} = \frac{\delta(\cos y + ye^x)}{\delta y}$$

$$= -\sin y + e^x$$

$f_y = -x\sin y + e^x$

$$f_{yx} = \frac{\delta(-x\sin y + e^x)}{\delta x}$$

$$= -\sin y + e^x$$

$$\boxed{\therefore f_{xy} = f_{yx} = -\sin y + e^x}$$

This is called Euler's theorum / Mixed Derivative Theorum.

* Euler's Theorum (Mixed Derivative Theorum):-

$$f_{xy} = f_{yx}$$

$$\frac{\delta^2 f}{\delta y \delta x} = \frac{\delta^2 f}{\delta x \delta y}$$

* Find f_{yxyz} if, $f(x,y,z) = 1 - 2xy^2z + x^2y$

(order of differentiation marked: y → 1st, x → 2nd, y → 3rd, z → 4th)

→ Solution,

first do from first y, then x . . . z.

$$f_y = \frac{\delta(1 - 2xy^2z + x^2y)}{\delta y}$$

$$= -2 \times 2xyz + x^2$$

$$f_{yx} = \frac{\delta(-4xyz + x^2)}{\delta x}$$

$$= -4yz + 2x$$

& $$f_{yxy} = \frac{\delta}{\delta y}(-4yz + 2x)$$

$$= -4z + 0$$

$$= -4z$$

$$f_{yxyz} = \frac{\delta(-4z)}{\delta z}$$

$$= -4$$

$$\boxed{\therefore f_{yxyz} = -4}$$

* Verify Euler's Theorum for the following:-

a.) $f(x,y) = x^2 + 5xy + \sin x + 7e^x$

→ Solution,

To verify euler's theorum,

$$f_{xy} = f_{yx}$$

$$\therefore f_x = \frac{\delta(x^2 + 5xy + \sin x + 7e^x)}{\delta x}$$

$$= 2x + 5y + \cos x + 7e^x$$

$$\therefore f_{xy} = \frac{\delta(2x + 5y + \cos x + 7e^x)}{\delta y}$$

$$= 0 + 5 + 0 + 0$$

$$= 5$$

$$f_y = \frac{\delta(x^2 + 5xy + \sin x + 7e^x)}{\delta y}$$

$$= 0 + 5x + 0 + 0$$

$$= 5x$$

$$\therefore f_{yx} = \frac{\delta\, 5x}{\delta x}$$

$$= 5$$

$\boxed{\therefore f_{xy} = f_{yx} = 5}$ Hence euler's Theorum is Verified.

b.) $f(x,y) = y + x^2y + 4y^3 - \ln(y^2+1)$

→ Solution,

$$f_x = \frac{\delta(y + x^2y + 4y^3 - \ln(y^2+1))}{\delta x}$$

$$= 0 + 2xy + 0 - 0 = 2xy$$

$$\therefore f_{xy} = \frac{\delta(2xy)}{\delta y} = 2x$$

Similarly,

$$f_y = \frac{\delta(y + x^2y + 4y^3 - \ln(y^2+1))}{\delta y}$$

$$= \frac{\delta y}{\delta y} + \frac{\delta(x^2y)}{\delta y} + \frac{\delta\, 4y^3}{\delta y} - \frac{\delta(\ln(y^2+1))}{\delta(y^2+1)} \times \frac{\delta(y^2+1)}{\delta y}$$

$$= 1 + x^2 + 4\times 3y^2 - \frac{1}{(y^2+1)} \times 2y$$

$$\therefore f_{yx} = \frac{\delta}{\delta x}\left(1 + x^2 + 12y^2 - \frac{1}{(y^2+1)} \times 2y\right)$$

$$= 0 + 2x + 0 - 0$$

$$= 2x$$

$$\boxed{\therefore f_{xy} = f_{yx} = 2x}$$

Hence euler's therum is verified.

c) $f(x,y) = x \ln xy$

→ Solution,

$$f_x = \frac{\delta(x \ln xy)}{\delta x} \qquad \frac{d(u.v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x\frac{\delta \ln xy}{\delta xy} \times \frac{\delta xy}{\delta x} + \ln xy \frac{\delta x}{\delta x}$$

$$= \frac{\cancel{x}}{x\cancel{y}} \times \cancel{y} + \ln xy$$

$$= 1 + \ln xy$$

$$\therefore f_{xy} = \frac{\delta(1+\ln xy)}{\delta y}$$

$$= 0 + \frac{\delta(\ln xy)}{\delta xy} \times \frac{\delta xy}{\delta y}$$

$$= \frac{1}{\cancel{x}y} \times \cancel{x} \qquad \therefore f_{xy} = \frac{1}{y}$$

Similarly,

$$f_y = \frac{\delta(x \ln xy)}{\delta y}$$

$$= x \frac{\delta(\ln xy)}{\delta(xy)} \times \frac{\delta xy}{\delta y} + \ln xy \frac{\delta x}{\delta y}$$

$$= x \frac{1}{\cancel{x}y} \times \cancel{x} + 0$$

$$= \frac{x}{y}$$

$$\therefore f_{yx} = \frac{\delta\left(\frac{x}{y}\right)}{\delta x}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{y\frac{\delta x}{\delta x} - x\frac{\delta y}{\delta x}}{y^2}$$

$$= \frac{\cancel{y} \times 1 - x \times 0}{y^{\cancel{2}}}$$

$$= \frac{\cancel{y}}{y^{\cancel{2}}} = \frac{1}{y}$$

$$\boxed{\therefore f_{xy} = f_{yx} = \frac{1}{y}}$$

Hence euler's theorem is verified.

d) $f(x,y) = x\sin y + y\sin x + xy$

→ solution,

$$f_x = \frac{\delta(x\sin y + y\sin x + xy)}{\delta x}$$

$$= \sin y + y\cos x + y$$

$$\therefore f_{xy} = \frac{\delta(\sin y + y\cos x + y)}{\delta y}$$

$$\therefore f_{xy} = \cos y + \cos x + 1$$

$$f_y = \frac{\delta(x\sin y + y\sin x + xy)}{\delta y}$$

$$= x\cos y + \sin x + x$$

$$\therefore f_{yx} = \frac{\delta(x\cos y + \sin x + x)}{\delta x}$$

$$= \cos y + \cos x + 1$$

$$\boxed{\therefore f_{xy} = f_{yx} = \cos y + \cos x + 1}$$

Hence euler's theorum is verified.

eg) $f(x,y) = e^x + x\ln y + y\ln x$

→ solution,

$$f_x = \frac{\delta(e^x + x\ln y + y\ln x)}{\delta x}$$

$$= e^x + \ln y + y \times \frac{1}{x}$$

$$f_{xy} = \frac{\delta(e^x + \ln y + \frac{y}{x})}{\delta y}$$

$$= 0 + \frac{1}{y} + \frac{1}{x}$$

$$\therefore f_{xy} = \frac{1}{x} + \frac{1}{y}$$

$$f_y = \frac{\delta(e^x + x\ln y + y\ln x)}{\delta y} = 0 + \frac{x}{y} + \ln x$$

$$\delta_{yx} = \frac{\delta(\frac{x}{y} + \ln x)}{\delta x}$$

$$= \frac{1}{y} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$$

$$\boxed{\therefore f_{xy} = f_{yx} = \frac{1}{x} + \frac{1}{y}}$$

Hence Euler's theorum is verified.

* Find the value of $\frac{\delta z}{\delta x}$ at point $(1,1,1)$ if eqⁿ $xy+z^3x-2yz=0$.

→ solution:-

z is function,

so we cannot put z as constant.

$xy+z^3x-2yz=0$ (Product rule)

differentiating on both side, we get

$$\frac{\delta(xy)}{\delta x}+\frac{\delta(z^3x)}{\delta x}-2\frac{\delta(yz)}{\delta x}=0$$

or, $y+z^3\frac{\delta x}{\delta x}+x\frac{\delta z^3}{\delta z}\times\frac{\delta z}{\delta x}-2\left(y\frac{\delta z}{\delta x}+z\frac{\delta y}{\delta x}\right)$

or, $y+z^3+3xz^2\frac{\delta z}{\delta x}-2y\frac{\delta z}{\delta x}+z\times0=0$

or, $y+z^3+\frac{\delta z}{\delta x}(3xz^2-2y)=0$

or, $\frac{\delta z}{\delta x}(3xz^2-2y)=-y-z^3$

or, $\frac{\delta z}{\delta x}=\frac{-y-z^3}{(3xz^2-2y)}$ At $(1,1,1)$

$\therefore \frac{\delta z}{\delta x}=\frac{-1-1}{3\times1\times1-2\times1}=\frac{-2}{3-2}=-2$ #.

* Find the value of $\frac{\delta x}{\delta z}$ at $(1,-1,-3)$ if eqⁿ $xz+y\ln x-x^2+4=0$

→ Solution,

$xz+y\ln x-x^2+4=0$

differentiating both side, we get,

$$\cancel{\frac{\delta(xz)}{\delta z}} \; \cancel{\frac{\delta(y\ln y)}{\delta z}} \quad \frac{\delta(xz)}{\delta z} + \frac{\delta(y\ln x)}{\delta z} - \frac{\delta x^2}{\delta z} + \frac{\delta 4}{\delta z} = 0$$

(Product above $\frac{\delta(xz)}{\delta z}$ and above $\frac{\delta(y\ln x)}{\delta z}$)

or, $x\frac{\delta z}{\delta z} + z\frac{\delta x}{\delta z} + y\frac{\delta \ln x}{\delta x} \times \frac{\delta x}{\delta z} - \frac{\delta x^2}{\delta x} \times \frac{\delta x}{\delta z} + 0 = 0$

or, $x + z\frac{\delta x}{\delta z} + \frac{y}{x}\frac{\delta x}{\delta z} - 2x\frac{\delta x}{\delta z} = 0$

or, $x + \frac{\delta x}{\delta z}\left(z + \frac{y}{x} - 2x\right) = 0$

at point $(1, -1, -3)$

$\frac{\delta x}{\delta z}\left(-3 + -\frac{1}{1} - 2\times 1\right) = -1$

or, $\frac{\delta x}{\delta z} = \frac{-1}{(-3-1-2)} = \frac{\cancel{-}1}{\cancel{-}6} = \frac{1}{6}$

Hence the required value of $\frac{\delta x}{\delta z}$ is $1/6$.

One Dimension Wave Equation:-

$$\frac{\delta^2}{\delta t^2} = c^2\frac{\delta^2 w}{\delta x^2}$$

$$\boxed{\text{i.e. } w_{tt} = c^2 w_{xx}}$$

where,

w = wave height

x = distance variable

t = time variable

c = Velocity with which waves are propagated.

* Show that the following functions are solution of the following wave eq^n:

a.) $w = \sin(x+ct)$

→ Solution,

To be the solution of wave eq^n,

$$w_{tt} = c^2 w_{xx}.$$

Taking L.H.S

$$w_t = \frac{\delta(\sin(x+ct))}{\delta t} = \frac{\delta(\sin(x+ct))}{\delta(x+ct)} \times \frac{\delta(x+ct)}{\delta t}$$

$$= \cos(x+ct) \times c$$

$$= c\cos(x+ct)$$

$$w_{tt} = \frac{\delta}{\delta t} c(\cos(x+ct)) = \frac{c\,\delta(\cos(x+ct))}{\delta(x+ct)} \times \frac{\delta(x+ct)}{\delta t}$$

$$= c(-\sin(x+ct)) \times c$$

$$\therefore w_{tt} = -c^2\sin(x+ct) \ \#.$$

Taking R.H.S.

$$= c^2 w_{xx}$$

$$w_x = \frac{\delta(\sin(x+ct))}{\delta x} = \frac{\delta(\sin(x+ct))}{\delta(x+ct)} \times \frac{\delta(x+ct)}{\delta x}$$

$$= \cos(x+ct) \times 1 = \cos(x+ct)$$

$$w_{xx} = \frac{\delta(\cos(x+ct))}{\delta x} = \frac{\delta(\cos(x+ct))}{\delta(x+ct)} \times \frac{\delta(x+ct)}{\delta x}$$

$$= -\sin(x+ct) =$$

$$\therefore c^2 w_{xx} = -c^2\sin(x+ct) \ \#.$$

$$\boxed{\therefore L.H.S = R.H.S = -c^2\sin(x+ct)}$$

Hence the given function is the sol^n of wave eq^n.

b) $w = \cos(2x + 2ct)$

→ Solution,

To be sol^n of wave eq^n, it should satisfy,

$$w_{tt} = c^2 w_{xx}$$

Taking L.H.S $= w_{tt}$

$$w_t = \frac{\delta(\cos(2x+2ct))}{\delta(2x+2ct)} \times \frac{\delta(2x+2ct)}{\delta t}$$

$$= -\sin(2x+2ct) \times 2c$$

$$= -2c\sin(2x+2ct)$$

$$w_{tt} = \frac{\delta(-2c(\sin(2x+2ct)))}{\delta t}$$

$$= -2c\,\frac{\delta(\sin(2x+2ct))}{\delta(2x+2ct)} \times \frac{\delta(2x+2ct)}{\delta t}$$

$$= -2c\cos(2x+2ct) \times 2c$$

$$= -4c^2\cos(2x+2ct) \ \#.$$

Taking R.H.S $= w_{xx}\,c^2$

$$w_x = \frac{\delta(\cos(2x+2ct))}{\delta x} = \frac{\delta(\cos(2x+2ct))}{\delta(2x+2ct)} \times \frac{\delta(2x+2ct)}{\delta x}$$

$$= -2\sin(2x+2ct) \times 1$$

$$w_{xx} = \frac{\delta(-2\sin(2x+2ct))}{\delta x}$$

$$= \frac{\delta(-2\sin(2x+2ct))}{\delta(2x+2ct)} \times \frac{\delta(2x+2ct)}{\delta x}$$

$$= -2\cos(2x+2ct) \times 2$$

$$= -4\cos(2x+2ct)$$

$$\therefore c^2 w_{xx} = -4c^2\cos(2x+2ct)$$

$$\boxed{\therefore \text{L.H.S} = \text{R.H.S} = -4c^2\cos(2x+2ct)}$$

Hence the given function is the sol^n of wave eq^n.

Verify Euler's theorum for

a) $f(x,y) = y + \left(\frac{x}{y}\right)$

→ Solution,

to verify $f_{xy} = f_{yx}$, so

$f_x = \frac{\delta}{\delta x}\left(y + \frac{x}{y}\right)$

$= 0 + \frac{1}{y} \times 1 = \frac{1}{y}$

$\therefore f_{xy} = \frac{\delta}{\delta y}\left(\frac{1}{y}\right)$

$= \frac{\delta(y^{-1})}{\delta y} = -\frac{1}{y^2}$

$f_y = \frac{\delta\left(y + \frac{x}{y}\right)}{\delta y}$

$= 1 + x(-y^{-2})$

$= 1 + \left(-\frac{x}{y^2}\right)$

$\therefore f_{yx} = \frac{\delta}{\delta x}\left(1 - \frac{x}{y^2}\right)$

$= 0 - \frac{1}{y^2} \times 1$

$= -\frac{1}{y^2}$

$\boxed{\therefore f_{xy} = f_{yx} = -\frac{1}{y^2}}$

Hence, Euler's theorum is verified.

b) $f(x,y) = x \sin y + e^y$

→ Solution,

$f_x = \frac{\delta}{\delta x}(x \sin y + e^y) = \sin y$

$\therefore f_{xy} = \frac{\delta(\sin y)}{\delta y} = \cos y$

$f_y = \frac{\delta}{\delta y}(x \sin y + e^y)$

$= x \cos y + e^y$

$\therefore f_{yx} = \frac{\delta(x \cos y + e^y)}{\delta x}$

$= 1 \times \cos y = \cos y$

$\boxed{\therefore f_{xy} = f_{yx} = \cos y}$ Hence Euler's theorum is verified.

$\Rightarrow f(x,y) = \tan^{-1}(y/x)$ [Note: $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$]

→ Solution,

$$f_x = \frac{\delta(\tan^{-1}(\frac{y}{x}))}{\delta(\frac{y}{x})} \times \frac{\delta(y/x)}{\delta x}$$

$$= \frac{1}{1+(\frac{y}{x})^2} \times -\frac{y}{x^2} = \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{-y}{x^2}$$

$$\cancel{f_{xy}} = \frac{1}{1} \times \frac{x^2}{x^2+y^2} \times \frac{(-y)}{\cancel{x^2}}$$

$$\therefore f_x = -\frac{y}{x^2+y^2}$$

$$f_{xy} = \frac{\delta}{\delta y}\left(\frac{-y}{x^2+y^2}\right) \quad \text{[quotient rule]}$$

$$= \frac{(x^2+y^2)\frac{\delta(-y)}{\delta y} - (-y)\frac{\delta(x^2+y^2)}{\delta y}}{(x^2+y^2)^2}$$

$$= (x^2+y^2) \times (-1) + y\,(0+2y) / (x^2+y^2)^2$$

$$= -x^2 - y^2 + 2y^2 / (x^2+y^2)^2$$

$$= -x^2 + y^2 / (x^2+y^2)^2$$

$$f_y = \frac{\delta(\tan^{-1}(\frac{y}{x}))}{\delta(y/x)} \times \frac{\delta(y/x)}{\delta y}$$

$$= \frac{1}{1+(y/x)^2} \times \frac{1}{x}$$

$$= \frac{\cancel{x^2}}{x^2+y^2} \times \frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}$$

$$\therefore f_{yx} = \frac{\delta\left(\frac{x}{x^2+y^2}\right)}{\delta x}$$

$$= (x^2+y^2)\frac{\delta x}{\delta x} - x\frac{\delta(x^2+y^2)}{\delta x} \Big/ (x^2+y^2)^2$$

$$= \frac{(x^2+y^2)\times 1 - x(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)}$$

$$\boxed{\therefore f_{xy} = f_{yx} = \frac{-x^2+y^2}{(x^2+y^2)}}$$

$\therefore$ Hence Euler's theorem is verified.

d) $f(x,y) = y + x^2y + 4y^3 - \ln(y^2+1)$

→ Solution,

$$f_x = \frac{\delta(y+x^2y+4y^3-\ln(y^2+1))}{\delta x}$$

$$= 0 + 2xy + 0 - 0$$

$$= 2xy$$

$$f_{xy} = \frac{\delta(2xy)}{\delta y} = 2x$$

$$f_y = \frac{\delta(y+x^2y+4y^3-\ln(y^2+1))}{\delta y} \qquad \frac{\delta(\ln(y^2+1))}{\delta(y^2+1)} \times \frac{\delta(y^2+1)}{\delta y}$$

$$= 1 + x^2 + 12y^2 - \frac{1\ 2y}{(y^2+1)} \qquad = \frac{1}{(y^2+1)} \times 2y$$

$$f_{yx} = \frac{\delta\left(1+x^2+12y^2-\frac{1}{y^2+1}\right)}{\delta x} \quad = \cancel{1 \times 0 + 2y}$$

$$= 0 + 2x + 0 - 0 = 2x$$

$\boxed{\therefore f_{xy} = f_{yx} = 2x}$ Hence Euler's theorem is verified.

partial deriv. → we take constant which is not w.r.to..
Derivatives → we do chain rule or .... No ~~constant~~ is taken

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chain Rule :- (2-D)

$$\frac{dw}{dt} = \frac{\delta f}{\delta x} \times \frac{dx}{dt} + \frac{\delta f}{\delta y} \times \frac{dy}{dt}$$

where $w = f(x,y)$ = function.

$$\frac{dw}{dt} = \frac{\delta f}{\delta x} \times \frac{dx}{dt} + \frac{\delta f}{\delta y} \times \frac{dy}{dt} + \frac{\delta f}{\delta z} \times \frac{dz}{dt}$$

where $w = f(x,y,z)$ = function.

* Find $\frac{dw}{dt}$

1.> $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$ at ~~$\pi = \pi$~~

→ Solution,

$$\frac{dw}{dt} = \frac{\delta w}{\delta x} \times \frac{dx}{dt} + \frac{\delta w}{\delta y} \times \frac{dy}{dt}$$

$$= \frac{\delta(x^2+y^2)}{\delta x} \times \frac{d(\cos t)}{dt} + \frac{\delta(x^2+y^2)}{\delta y} \times \frac{d(\sin t)}{dt}$$

$$= 2x \times (-\sin t) + 2y \cos t$$

Put value of x & y

$$= 2\cos t \cdot (-\sin t) + 2\sin t \cdot \cos t$$

$$= \cancel{-2\sin t \cdot \cos t} + \cancel{2\sin t \cdot \cos t}$$

$$= 0$$ //.

2.> $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$ ~~at~~

→ Solution,

$$\frac{dw}{dt} = \frac{\delta w}{\delta x} \times \frac{dx}{dt} + \frac{\delta w}{\delta y} \times \frac{dy}{dt}$$

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$$= \frac{\delta(x^2+y^2)}{\delta x} \times \frac{d}{dt}(\cos t + \sin t) + \frac{\delta(x^2+y^2)}{\delta y} \times \frac{d}{dt}(\cos t - \sin t)$$

$$= 2x \times \left((-\sin t) + \cos t\right) + 2y \times \left((-\sin t) - \cos t\right)$$

$$= 2(\cos t + \sin t) \times (\cos t - \sin t) + 2(\cos t - \sin t)(-\cos t - \sin t)$$

$$= 2(\cos^2 t - \sin^2 t) \cancel{+ 2(\cos t)(-\cos t - \sin t)} \rightarrow \text{common}$$

$$+ 2(\cos t - \sin t) - (\cos t + \sin t)$$

$$= 2\cancel{(\cos^2 t - \sin^2 t)} - 2\cancel{(\cos^2 t - \sin^2 t)}$$

$$= 0 \ \#$$

3.) $w = x^2y - y^2$, $x = \sin t$, $y = e^t$ at $t = 0$

→ solution,

$$\frac{dw}{dt} = \frac{\delta w}{\delta x} \times \frac{dx}{dt} + \frac{\delta w}{\delta y} \times \frac{dy}{dt}$$

$$= \frac{\delta(x^2y - y^2)}{\delta x} \times \frac{d}{dt}(\sin t) + \frac{\delta(x^2y - y^2)}{\delta y} \times \frac{d}{dt}(e^t)$$

$$= 2xy \times \cos t + (x^2 - 2y)e^t$$

$$= 2\sin t . e^t \times \cos t + (\sin t)^2 - 2(e^t).e^t$$

at $t = 0$

$$= 2\sin 0^\circ . e^0 \times \cos 0 + (\sin 0)^2 - 2(e^0).e^0$$

$$= 2 \times 0 + 0 - 2 \times 1 \times 1$$

$$= -2 \ \#$$

4.) $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$, at $t = 0$

→ solution

$$\frac{dw}{dt} = \frac{\delta(w)}{\delta x} \times \frac{dx}{dt} + \frac{\delta w}{\delta y} \times \frac{dy}{\delta t} + \frac{\delta w}{\delta z} \times \frac{dz}{dt}$$

$$= \frac{\delta(xy+z)}{\delta x} \times \frac{d(\cos t)}{dt} + \frac{\delta(xy+z)}{\delta y} \times \frac{d(\sin t)}{dt}$$

$$+ \frac{\delta(xy+z)}{\delta z} \times \frac{d(t)}{dt}$$

$= y \times -\cancel{\cos}\sin t + x \times \cos t + 1 \times 1$

$= \sin t \times (-\sin t) + \cos t \times \cos t + 1$

$= -\sin^2 t + \cos^2 t + 1$

at $t = 0$

$= (-\sin 0^\circ)^2 + (\cos 0^\circ)^2 + 1$

$= 0 + 1 + 1$

$= 2$ #.

Chain Rule : (3-D) all partial

For two independent variables (r, s) and three intermediate (x, y, z) variables

$w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, $z = k(r, s)$

$$\boxed{\frac{\delta w}{\delta r} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta r} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta r} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta r}}$$

$$\boxed{\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta s} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta s} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta s}}$$

* Find $\frac{\delta w}{\delta r}$ and $\frac{\delta w}{\delta s}$ if,

Q.) $w = x + 2y + z^2$, $x = r/s$, $y = r^2 + \ln s$, $z = 2r$

→ Solution:

$$\frac{\delta w}{\delta r} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta r} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta r} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta r}$$

$$= \frac{\delta(x+2y+z^2)}{\delta x} \times \frac{\delta(r/s)}{\delta r} + \frac{\delta(x+2y+z^2)}{\delta y} \times \frac{\delta(r^2+\ln s)}{\delta r}$$

$$+ \frac{\delta(x+2y+z^2)}{\delta z} \times \frac{\delta(2r)}{\delta r}$$

$\frac{\delta x}{\delta y} = 0$ $\quad \frac{\delta 2y}{\delta y} = 2$ $\quad \frac{\delta z^2}{\delta z} \times \frac{\delta z}{\delta y}$ $\quad = 2z \frac{\delta z}{\delta y}$

$$= 1 * \frac{1}{s} + 2 \times 2r + 2z \times 2$$

$$= \frac{1}{s} + 4r + 4z \qquad [z = 2r]$$

$$= \frac{1}{s} + 4r + 4 \times 2r$$

$$\boxed{\frac{\delta w}{\delta r} = \frac{1}{s} + 12r}$$

$$\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta s} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta s} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta s}$$

$$= \frac{\delta(x+2y+z^2)}{\delta x} \times \frac{\delta(r/s)}{\delta s} + \frac{\delta(x+2y+z^2)}{\delta y} \times \frac{\delta(r^2+\ln s)}{\delta s}$$

$$+ \frac{\delta(x+2y+z^2)}{\delta z} \times \frac{\delta(2r)}{\delta s}$$

$$= 1 \times \left(-\frac{r}{s^2}\right) + 2 \times \frac{1}{s} + 2z \times 0$$

$$= -\frac{r}{s^2} + \frac{2}{s}$$

$$\boxed{\therefore \frac{\delta w}{\delta s} = -\frac{r}{s^2} + \frac{2}{s}}$$

* Define partial differential eq^n of Second order with suitable e.g.

→ Partial differential equation

if a dependent variable is a function of two or more than two independent variable then an equation involving with partial coefficient it is known as partial differential eq^n. This is the relation of dependent variable, independent variable and partial differential coefficient.

$\frac{d^2z}{\delta x^2} + \frac{d^2}{dxdy} + \frac{2d^2}{dy^2} = 0$ is Second order partial equation

$$\boxed{\frac{d^2z}{dx^2} + \frac{d^2}{dxdy} + \frac{2d^2}{dy^2} = 0}$$

* find $\frac{\delta w}{\delta r}$ and $\frac{\delta w}{\delta s}$ if :-

a) $w = x^2 + y^2$, $x = r - s$ $y = r + s$

→ Solution

$$\frac{\delta w}{\delta r} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta r} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta r}$$

$$= \frac{\delta(x^2+y^2)}{\delta x} \times \frac{\delta(r-s)}{\delta r} + \frac{\delta(x^2+y^2)}{\delta y} \times \frac{\delta(r+s)}{\delta r}$$

$$= 2x \times 1 + 2y \times 1$$

$$= 2(r-s) + 2(r+s)$$

$$= 2r - \cancel{2s} + 2r + \cancel{2s}$$

$$= 4r$$

$$\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta s} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta s}$$

$$= \frac{\delta(x^2+y^2)}{\delta} \times \frac{\delta(r-s)}{\delta s} + \frac{\delta(x^2+y^2)}{\delta y} \times \frac{\delta(r+s)}{\delta s}$$

$= 2x \times (-1) + 2y \times 1$

$= -2(r-s) + 2(r+s)$

$= -2r + 2s + 2r + 2s$

$= 4s$

$\therefore \frac{\delta w}{\delta r} = 4r$

$\therefore \frac{\delta w}{\delta s} = 4s$ #.

b.) $w = (x+y+z)^2$, $x = r-s$, $y = \cos(r+s)$, $z = \sin(r+s)$

→ Solution, when $r = 1$ $s = -1$

$$\frac{dw}{dr} = \frac{\delta(w)}{\delta x} \times \frac{\delta x}{\delta r} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta r} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta r}$$

$$= \frac{\delta(x+y+z)^2}{\delta(x+y+z)} \times \frac{\delta(x+y+z)}{\delta x} \times \frac{\delta(r-s)}{\delta r} + \frac{\delta(x+y+z)^2}{\delta(x+y+z)} \times \frac{\delta(x+y+z)}{\delta y}$$

$$\times \frac{\delta \cos(r+s)}{\delta(r+s)} \times \frac{\delta(r+s)}{\delta r} + \frac{\delta(x^2+y^2+z)^2}{\delta(x+y+z)} \times \frac{\delta(x+y+z)}{\delta z}$$

$$\times \frac{\delta \sin(r+s)}{\delta(r+s)} \times \frac{\delta(r+s)}{\delta r}$$

$= 2(x+y+z) \times 1 \times 1 + 2(x+y+z) \times 1 \times (-\sin(r+s)) \times 1 +$
$2(x+y+z) \times 1 \times \cos(r+s) \times 1$

$= 2(x+y+z)\,[1 + (-\sin(r+s)) + \cos(r+s)]$

Putting the value of x, y, z

$= 2(r-s+\cos(r+s)+\sin(r+s))\,[1+(-\sin(1-1)+\cos(1-1)]$

$= 2(1+1+\cos(1-1)+\sin(1-1))\,[1+(-\sin(0)+\cos 0]$

$= 2(2+1+0)[1+0+1]$

$= 2 \times 3\,[2] = 6 \times 2 = 12$ #.

$$\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x} \times \frac{\delta x}{\delta s} + \frac{\delta w}{\delta y} \times \frac{\delta y}{\delta s} + \frac{\delta w}{\delta z} \times \frac{\delta z}{\delta s}$$

$$= \frac{\delta (x+y+z)^2}{\delta (x+y+z)} \times \frac{\delta (x+y+z)}{\delta x} \times \frac{\delta (r-s)}{\delta s}$$

$$+ \frac{\delta (x+y+z)^2}{\delta (x+y+z)} \times \frac{\delta (x+y+z)}{\delta y} \times \frac{\delta (\cos(r+s))}{\delta (r+s)} \times \frac{\delta (r+s)}{\delta s}$$

$$+ \frac{\delta (x+y+z)^2}{\delta (x+y+z)} \times \frac{\delta (x+y+z)}{\delta z} \times \frac{\delta (\sin(r+s))}{\delta (r+s)} \times \frac{\delta (r+s)}{\delta s}$$

$$= 2(x+y+z) \times 1 \times (-1) + 2(x+y+z) \times 1 \times -\sin(r+s) \times 1$$
$$+ 2(x+y+z) \times 1 \times \cos(r+s) \times 1$$

$$= 2(x+y+z) \left[-1 + (-\sin(r+s) + \cos(r+s) \right]$$

when $r = 1 \quad s = -1$

$$= 2(r-s+\cos(r+s)+\sin(r+s)\left[-1+0+1\right]$$
$$= 2(1+1+1+0) \times (-1+1)$$
$$= 2 \times 3 \times 0$$
$$= 0 \#.$$

$\therefore \frac{\delta w}{\delta s} = 0 \qquad \frac{\delta w}{\delta r} = 12 \#$

Implicit Differentiation:-

$f(x,y)$ is differentiable and eq^n is in the form $f(x,y) = 0$ and $f_y \neq 0$, then,

$$\boxed{\frac{dy}{dx} = -\frac{f(x)}{f(y)}}$$

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* Find $\frac{dy}{dx}$ if, a) $y^2 - x^2 - \sin xy = 0$

→ Solution,

to find $\frac{dy}{dx}$ we have to find f_x and f_y

$$\therefore f_x = \frac{\delta (y^2 - x^2 - \sin xy)}{\delta x}$$

$$= 0 - 2x - \frac{\delta \sin xy}{\delta xy} \times \frac{\delta xy}{\delta x}$$

$$= -2x - y \cos xy$$

$$\therefore f_y = \frac{\delta (y^2 - x^2 - \sin xy)}{\delta y}$$

$$= 2y - 0 - \cos xy \times y$$

$$= 2y - \cancel{y} \cos xy$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-2x - y\cos xy)}{2y - \cancel{y}\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy} \#$$

b) $y^3 + y^2 - 5y - x^2 + 4 = 0$

→ Solution,

$$f_x = \frac{\delta (y^3 + y^2 - 5y - x^2 + 4)}{\delta x} = -2x$$

$$f_y = \frac{\delta (y^3 + y^2 - 5y - x^2 + 4)}{\delta y} = 3y^2 + 2y - 5$$

$$\therefore \frac{dy}{dx} = -\frac{f(x)}{f(y)} = \frac{2x}{(3y^2 + 2y - 5)} \#$$

c) $xy + y^2 - 3x - 3 = 0$ at $(-1, 1)$

→ Solution,

$f_x = y + 0 - 3 - 0$

$= y - 3$

$f_y = x + 2y$

$\therefore \frac{dy}{dx} = -\frac{y+3}{x+2y}$

at $-1, 1$

$\frac{dy}{dx} = \frac{-1+3}{-1+2\times 1} = \frac{2}{1} = 2$ #

d) $xe^y + \sin xy + y - \ln 2 = 0$ at $(0, \ln 2)$

→ Solution

$f_x = \frac{\delta}{\delta x}(xe^y + \sin xy + y - \ln 2)$

$= e^y + \cos xy \times y$

$= e^y + y\cos xy$

$f_y = \frac{\delta}{\delta y}(xe^y + \sin xy + y - \ln 2)$

$= xe^y + \cos xy \times x + 1$

$= xe^y + x\cos xy + 1$

$\therefore \frac{dy}{dx} = -\frac{(e^y + y\cos xy)}{xe^y + x\cos xy + 1}$ at $(0, \ln 2)$

$= -\frac{(e^{\ln 2} + \ln 2 \cos(0 \times \ln 2))}{0 \times e^y + 0 \times \cos(0 \times \ln 2) + 1}$

$= -(2 + \ln 2)$

$= -2.69$ #.

Directional Derivative And Gradient Vector

Gradient Vector ($\bar{\nabla} f$):

$\bar{\nabla} f$ at point $P(x,y)$

$$\bar{\nabla} f = \frac{\delta f}{\delta x}\vec{i} + \frac{\delta f}{\delta y}\vec{j}$$
$$= f_x \vec{i} + f_y \vec{j}$$

$$\boxed{\therefore \bar{\nabla} f \text{ (Gradient Vector)} = f_x \vec{i} + f_y \vec{j}}$$

$\bar{\nabla} f$ at point $P(x,y,z)$ is,

$$\boxed{\bar{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}}$$

* Find Gradient Vector ($\bar{\nabla} f$).

a) $f(x,y) = \ln(x^2+y^2)$ at $(1,1)$

→ Solution,

$$f(x) = \frac{\delta(\ln(x^2+y^2))}{\delta(x^2+y^2)} \times \frac{\delta(x^2+y^2)}{\delta x}$$
$$= \frac{1}{(x^2+y^2)} \times 2x \quad \text{at } (1,1) = \frac{2}{(1+1)} = \frac{2}{2} = 1.$$

$$f(y) = \frac{\delta(\ln(x^2+y^2))}{\delta(x^2+y^2)} \times \frac{\delta(x^2+y^2)}{\delta y}$$
$$= \frac{1}{(x^2+y^2)} \times 2y \quad \text{at } (1,1) = \frac{2\times 1}{(1+1)} = \frac{2}{2} = 1$$

$$\therefore \bar{\nabla} f = f(x)\vec{i} + f_y \vec{j}$$
$$= \vec{i} + \vec{j}$$

b) $f(x,y,z) = x^3 - xy^2 - z$ at $(1,1,0)$

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→ Solution,

$$f(x) = \frac{\delta(x^3 - xy^2 - z)}{\delta x} = 3x^2 - y^2 \quad \text{at } (1,1,0)$$

$$f(x) = 3-1$$
$$= 2$$

$$f(y) = \frac{\delta(x^3 - xy^2 - z)}{\delta y} = -2xy \quad \text{at } (1,1,0)$$

$$f(y) = -2\times1\times1$$
$$= -2$$

$$f(z) = \frac{\delta(x^3 - xy^2 - z)}{\delta z} = -1$$

$$\therefore \bar{\nabla} f = 2\vec{i} - 2\vec{j} - \vec{k}$$

* Directional Derivatives:-

If $f(x,y)$ is differentiable at point $p(x,y)$ then it is denoted by $D_u f$ is is given by

$$\boxed{D_u f = \bar{\nabla} f \cdot u} \quad \text{at that point } p(x,y).$$

where, $u = \frac{v}{|v|}$ = Unit vector

Directional derivative is a dot product of gradient vector and unit vector.

$$D_u f = \bar{\nabla} f \cdot u = |\bar{\nabla} f|\,|u| \cos\theta$$
$$= |\bar{\nabla} f| \cos\theta$$

Properties:-

i) $D_u f$ increase when $\cos\theta = 1$ i.e. $\theta = 0^\circ$

ii) ~~$D_u f$ decrease~~ when $\cos\theta = -1$, i.e. $\theta = 180^\circ$

* find the directional derivatives of $f(x,y) = x^2 \sin 2y$

(a) at point $(1, \pi/2)$ in the direction of $v = 3\vec{i} - 4\vec{j}$.

→ Solution,

given vector $(\vec{v}) = 3\vec{i} - 4\vec{j}$

Unit vector $(u) = \dfrac{v}{|v|} = \dfrac{3\vec{i} - 4\vec{j}}{\sqrt{9+16}} = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}$

given function $= f(x,y) = x^2 \sin 2y$

$f(x) = \dfrac{\delta(x^2 \sin 2y)}{\delta x} = 2x \dfrac{\sin 2y}{\delta 2y} \times \dfrac{\delta 2y}{\delta x}$

at point $(1, \pi/2)$

$f(x) = 2 \times 1 \sin 2 \times \pi/2 = 0$

$f(y) = \dfrac{\delta(x^2 \cdot \sin 2y)}{\delta y} = x^2 \cancel{2}(-\cos 2y) = -x^2 y \cos 2y$

at point $(1, \pi/2)$

$\dfrac{\delta(\sin 2y)}{\delta 2y} \times \dfrac{\delta y}{2y}$ $\quad f(y) = -1 \times 2\cos 2 \cdot \pi/2 = -2$

$-\cos(2y) \times 2$

∴ Gradient vector $(\nabla f) = 0\vec{i} - \vec{j} = -2\vec{j}$

Unit vector $(u) = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}$

∴ directional derivatives $= \nabla f \cdot u$

$= -2\vec{j} \cdot \left(\dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j}\right)$

$= 0 + \dfrac{8}{5}$

$= \dfrac{8}{5}$ # $+ \dfrac{8}{5}$

(b) $f(x,y) = x \cdot e^y + \cos(xy)$ at $(2,0)$ in direction of $v = 3\vec{i} - 4\vec{j}$

→ Solution,

given vector $(\vec{v}) = 3\vec{i} - 4\vec{j}$

∴ Unit Vector (u) $= \frac{\vec{V}}{|V|} = \frac{3\vec{i} - 4\vec{j}}{\sqrt{9+16}} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$

given function $= f(x,y) = x \cdot e^y + \cos(xy)$

$f(x) = \frac{\delta(x \cdot e^y + \cos(xy))}{\delta x}$

$= e^y + \frac{\delta \cos(xy)}{\delta(xy)} \times \frac{\delta(xy)}{\delta x}$

$= e^y + (-\sin xy) \times y$

at (2,0)

$= e^0 + (-\sin 0 \times 2) \times 0$

$f(x) = 1$

$f(y) = \frac{\delta(x \cdot e^y + \cos xy)}{\delta y}$

$= xe^y + (-\sin xy) \times x$

at (2,0)

$f(y) = 2 \times e^0 + (-\sin 0 \times 2) \times 0$

$= 2$

Now,

Gradient Vector $(\vec{\nabla} f) = \vec{i} + 2\vec{j}$

∴ directional derivatives $(D_u f) = \vec{\nabla} f \cdot u$

$= (\vec{i} + 2\vec{j}) \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$

$= \frac{3}{5} - \frac{8}{5}$

$= \frac{-5}{5} = -1$ #.

* $f(x,y,z) = x^3 - xy^2 - z$ at (1,1,0) in the direction of $v = 2\vec{i} - 3\vec{j} + 6\vec{k}$

→ Solution,

Unit Vector $(u) = \frac{v}{|v|} = \frac{2\vec{i} - 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}} = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$

then,

$f(x) = \frac{\delta}{\delta x}(x^3 - xy^2 - z) = 3x^2 - y^2$ at $(1,1,0)$

$\therefore f(x) = 3\times1 - 1 = 2$ #.

$f(y) = \frac{\delta(x^3 - xy^2 - z)}{\delta y} = -2xy$ at $(1,1,0)$

$\therefore f(y) = -2\times1\times1 = -2$

$f_z = \frac{\delta(x^3 - xy^2 - z)}{\delta z} = -1$

$\therefore$ gradient vector $(\bar{\nabla}f) = 2\vec{i} - 2\vec{j} - \vec{k}$

$\therefore$ directional derivatives $(D_u f) = \bar{\nabla}f \cdot u$

$= (2\vec{i} - 2\vec{j} - \vec{k}) \cdot \left(\frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}\right)$

$= \frac{4}{7} + \cancel{\frac{6}{7}} - \cancel{\frac{6}{7}}$

$= \frac{4}{7}$ #

* Equation of Tangent at points (x_1, y_1, z_1):

when,

$\vec{\nabla}f = A\vec{i} + B\vec{j} + C\vec{k}$ at (x_1, y_1, z_1)

$\boxed{\text{eq}^n \text{ of tangent} = A(x - x_1) + B(y - y_1) + C(z - z_1)}$

* Equation of normal line at (x_1, y_1, z_1)
  eq^n are,

$$x = x_1 + At$$
$$y = y_1 + Bt$$
$$z = z_1 + Ct$$

* Find the eq^n of tangent and normal line of the surface. $f(x,y,z) = x^2+y^2+z-9 = 0$ at point $P(1,2,4)$.

$$\vec{\nabla f} = A\vec{i} + B\vec{j} + C\vec{k} \quad \text{at } (1,2,4)$$

→ Solution,

given function $= f(x,y,z) = x^2+y^2+z-9=0$

$$f_x = \frac{\delta(x^2+y^2+z-9)}{\delta x} = 2x \quad \text{at } (1,2,4)$$
$$f(x) = 2\times 1 = 2$$

$$f_y = \frac{\delta(x^2+y^2+z-9)}{\delta y} = 2y \quad \text{at } (1,2,4)$$
$$f_y = 2\times 2 = 4$$

$$f_z = \frac{\delta(x^2+y^2+z-9)}{\delta z} = 1$$

$\therefore$ gradient Vector $(\bar{\nabla} f) = 2\vec{i} + 4\vec{j} + \vec{k}$ ---- (i)

Comparing eq^n (i) with $A\vec{i} + B\vec{j} + C\vec{k}$

$\therefore A = 2, \; B = 4 \quad C = 1$

Now, Point $(x_1, y_1, z_1) = (1,2,4)$

$\therefore$ Required eq^n of tangent is,

$= A(x-x_1) + B(y-y_1) + C(z-z_1)$

$= 2(x-1) + 4(y-2) + 1(z-4)$

$= 2x - 2 + 4y - 8 + z - 4$

$\therefore eq^n$ of tangent $= 2x + 4y + z - 14$

$\boxed{2x+4y+z=14}$ eqⁿ of tangent.

eqⁿ of normal are,

$x = x_1 + At = 1+2t$

$\therefore x = 1+2t$ #

$y = y_1 + Bt$

$\therefore y = 2+4t$ #

$z = z_1 + Ct$

$\therefore z = 4+t$ #.

* Find the eqⁿ of tangent to surface $z = 1 - \frac{1}{10}(x^2+4y^2)$ at $(1, 1, 1/2)$.

→ Solution:- For eqⁿ of T to S $= A(x-x_1) + B(y-y_1) - (z-z_1)$

given function $= z = f(x,y) = 1 - \frac{1}{10}(x^2+4y^2)$

$f_x = \frac{\delta}{\delta x}\left((1 - \frac{1}{10}(x^2+4y^2)\right)$

$= 0 - \frac{1}{10} \times 2x$ at $(1,1)$

$f(x) = -\frac{1}{5}$

$f(y) = \frac{\delta}{\delta y}\left(1 - \frac{1}{10}(x^2+4y^2)\right)$

$= 0 - \frac{1}{10}(0+8y)$

at $(1,1)$

$f(y) = -\frac{1}{10} \times 8 \times 1 = -\frac{4}{5}$

Plane tangent to surface $z = f(x,y)$

$$\boxed{A(x-x_1) + B(y-y_1) - (z-z_1) = 0}$$

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$$f_z = \frac{\delta}{\delta z}\left(1 - \frac{1}{10}(x^2+4y^2)\right)$$
$$= 0$$

Now,

Vector gradient $(\vec{\nabla} f) = -\frac{1}{5}\vec{i} - \frac{4}{5}\vec{j}$

Comparing $\vec{\nabla} f$ with $A\vec{i} + B\vec{j} + C\vec{k}$

$\therefore A = -\frac{1}{5}$, $B = -\frac{4}{5}$, $C = 0$

$\therefore$ eq^n of tangent $= A(x-x_1) + B(y-y_1) - (z-z_1) = 0$

or, $-\frac{1}{5}(x-1) - \frac{4}{5}(y-1) - (z-\frac{1}{2}) = 0$

or, $-\frac{1}{5}x + \frac{1}{5} - \frac{4}{5}y + \frac{4}{5} - z + \frac{1}{2} = 0$

or, $-\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} = 0$ #.

$\frac{1}{5}x + \frac{4}{5}y + z = \frac{3}{2}$

Req. eq^n

Extreme Values And Saddle Points.

1. First derivative test :-

If a function $f(x,y)$ is continuous and differentiable.

$$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \text{maximum or minimum.}$$

2. Second derivative test :-

If a function $f(x,y)$ is continuous and it's first and second derivative is differentiable at point (a,b) then;

a) If $f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$
Then function is minimum.

b) If $f_{xx} < 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$, then the Function is maximum.

c) If $f_{xx} \cdot f_{yy} - f^2_{xy} < 0$, then there is saddle point.

d) If $f_{xx} \cdot f_{yy} - f^2_{xy} = 0$, then function is inconclusive.

* Find the extreme value of the following function :-

a) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

→ Solution:

The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to,

$$f_x = \frac{\delta}{\delta x}(xy - x^2 - y^2 - 2x - 2y + 4) \quad \text{or,}$$

for stationary point

$f_x = y - 2x - 2$

$f_y = x - 2y - 2$

$f_x = f_y = 0$

$$\therefore f_x = y - 2x - 2 = 0 \text{ ---(i)}$$

$$f_y = \frac{\delta}{\delta y}(xy - x^2 - y^2 - 2x - 2y + 4)$$

$$f_y = x - 2y - 2 = 0 \text{ ---(ii)}$$

$\therefore f_x = y - 2x - 2 = 0$

or, $y = 2x + 2$

Putting $y = 2x + 2$ in eq^n (ii)

$x - 2(2x + 2) - 2) = 0$

or, $x - 4x - 4 - 2 = 0$

or, $-3x = 6$

$\therefore x = -2$

$\therefore y = 2 \times (-2) + 2 = -2$

$\therefore (x, y) = (-2, -2)$

Now,

$$f_{xx} = \frac{\delta}{\delta x}(y - 2x - 2) = -2$$

$$f_{yy} = \frac{\delta}{\delta y}(x - 2y - 2) = -2$$

$$f_{xy} = \frac{\delta}{\delta y}(y - 2x - 2) = 1$$

$\therefore f_{xx} = -2 < 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-2) \times (-2) - (1)^2 = 4 - 1$

$= 3 > 0$

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Thus the function is maximum at $(-2,-2)$. The value of f at $f(-2,-2)$ is

$$f(-2,-2) = (-2)\times(-2) - (-2)^2 - (-2)^2 - 2\times(-2) - 2\times(-2) + 4$$
$$= 4 - 4 - 4 + 4 + 4 + 4$$
$$= 8 \#$$

b.) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

→ Solution:

The function is defined and differentiable for all x and y and its domain has no boundary point. The function therefore has extereme values only at the point where f_x and f_y are simultaneously 0. This leads

$$f_x = 2y - 10x + 4 = 0 \quad --(i)$$
$$f_y = 2x - 4y + 4 = 0 \quad --(ii)$$
$$2y = 10x - 4$$
$$\therefore y = 5x - 2 \quad --(iii)$$

put $y = 5x - 2$ in eq^n (ii)

$$2x - 4(5x-2) + 4 = 0$$
or, $2x - 20x + 8 + 4 = 0$
or, $-18x = -12$
$$\therefore x = \frac{12}{18} = \frac{2}{3}$$

Put $x = 2/3$ in eq^n (iii)

$$y = 5\times\frac{2}{3} - 2 = \frac{10}{3} - 2 = \frac{4}{3}$$

$$\therefore (x,y) = (2/3, 4/3)$$

Now,

$$f_{xx} = -10$$
$$f_{yy} = -4$$
$$f_{xy} = 2$$

Here,

$$f_{xx} = -10 < 0$$

and,

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = -10 \times -4 - (2)^2$$
$$= 40 - 4$$
$$= 36 > 0$$

So the function f has local maximum at $(2/3, 4/3)$ $(\cancel{-10, -4})$ and it's value is,

$$f(2/3, 4/3) = 2 \times \frac{2}{3} \times \frac{4}{3} - 5 \times \left(\frac{2}{3}\right)^2 - 2 \times \left(\frac{4}{3}\right)^2 * 4 \times \frac{2}{3} + 4 \times \frac{4}{3} - 4$$

$$= \frac{16}{9} - \frac{20}{9} - \frac{32}{9} + \frac{8}{3} \times \frac{3}{3} + \frac{16}{3} + \frac{3}{3} - \frac{4}{1} \times \frac{9}{9}$$

$$= \frac{16 - 20 - 32 + 24 + 48 - 36}{9}$$

$$= \frac{88 - 88}{9} = 0$$

$\therefore$ The value of f at $f(2/3, 4/3)$ is 0 #.

b) $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$

→ Solution,

The function is defined and differentiable for all x and y. and its domain has no boundary point. The function therefore has the extreme value only at the point where f_x and f_y are simultaneously zero (0). This leads,

$f_x = 5y - 14x + 3 = 0$ --- (i)
$f_y = 5x \cancel{+1/3} - 6 = 0$ --- (ii) $\therefore x = \frac{6}{5}$

$5y - 14 \times \frac{6}{5} + 3 = 0$

or $5y = \frac{84}{5} - 3$

or $5y = \frac{69}{5}$

$\therefore y = \frac{69}{25}$

$f_{xx} = \frac{\delta(5y-14x+3)}{\delta x} = -14$

$f_{yy} = \frac{\delta}{\delta y}(5x-6) = 0$

$f_{xy} = \frac{\delta(5y-14x+3)}{\delta y} = 5$

$f_{xx} = -14 < 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = -14 \times 0 - 5^2 = 0 - 25 = -25 < 0$

Since $f_{xx} \cdot f_{yy} - f_{xy}^2 = -25 < 0$, so the function f_{xy} has saddle point at $\left(\frac{6}{5}, \frac{69}{25}\right)$

c) $f(x,y) = x^2 + xy + 3x + 2y + 5$

→ Solution,

Given function is defined and differentiable for all x and y and its domain has no boundary point. The function therefore has extreme value only at point where f_x and f_y are simultaneously 0. This leads.

$f_x = \frac{\delta(x^2 + xy + 3x + 2y + 5)}{\delta x}$

$f_x = 2x + y + 3 = 0$

$f_y = \frac{\delta(x^2 + xy + 3x + 2y + 5)}{\delta y}$

$f_y = x + 2 = 0 \qquad \therefore x = -2$

then,

$2x + y + 3 = 0$

or, $2 \times (-2) + y + 3 = 0$

or, $-4 + y + 3 = 0$

$\therefore y = 1 \qquad \therefore (x, y) = (-2, 1)$

Now,

$f_{xx} = \frac{\delta(2x + y + 3)}{\delta x} = 2$

$f_{yy} = \frac{\delta(x + 2)}{\delta y} = 0$

$f_{xy} = \frac{\delta(2x + y + 3)}{\delta y} = 1$

Now,

$\therefore f_{xx} = 2 > 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \times 0 - 1 = -1$

$\therefore -1 < 0$

$\therefore$ The function has saddle point at $(-2, 1)$.

d.) $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$

→ Solution,

given function is defined and differentiable for all x and y and its domain has no boundary point. The function therefore has extreme value only at point where f_x and f_y are simultaneously 0.

then,

$$f_x = \frac{\delta(x^2+xy+y^2+3x-3y+4)}{\delta x}$$

$\therefore f_x = 2x+y+3 = 0$

$f_y = x+2y-3$

$\therefore f_y = x+2y-3 = 0$

$\therefore x = 3-2y$

Now,

$2x+y+3 = 0$

or, $2(3-2y)+y+3 = 0$

or, $6-4y+y+3 = 0$

or, $-3y+9 = 0$

or, $3y = 9$

$\therefore y = 3$

put $y=3$ in $x = 3-2y$

$\therefore x = 3-2\times 3 = 3-6 = -3$

$\therefore (x,y) = (-3,3)$

we know,

$$f_{xx} = \frac{\delta(2x+y+3)}{\delta x} = 2$$

$$f_{xy} = \frac{\delta(2x+y+3)}{\delta y} = 1$$

$$f_{yy} = \frac{\delta(x+2y-3)}{\delta}$$

$\therefore f_{yy} = 2$

Now,

$f_{xx} = 2 > 0$

$f_{xx}\cdot f_{yy} - f_{xy}^2 = 2\times 2 - 1^2 = 3 > 0$

So the function f has local minimum at $(-3,3)$ and its value at $f(2,2)$ is

$f(2,2) = (2)^2 + 2\times 2 + (2)^2 + \cancel{3\times 2 - 3\times 2} + 4$

$f(-3,3) = 4+4+4+4 = 16$ #

$= 4 - 13$ ~~L~~

e.) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

→ Solution,

given function is defined and differentiable for all x and y and its domain has no boundary point. The function therefore has extreme value only at point where f_x and f_y are simultaneously zero. This leads,

$$f_x = \frac{\delta(x^3+y^3+3x^2-3y^2-8)}{\delta x}$$

$$= 3x^2 + 6x = 0$$

$$f_y = \frac{\delta(x^3+y^3+3x^2-3y^2-8)}{\delta y}$$

$$= 3y^2 - 6y = 0$$

$f_x = 3x^2 + 6x = 0$ --- (1)

$f_y = 3y^2 - 6y = 0$ --- (11)

$3x^2 + 6x = 0$

or, $3x(x+2) = 0$

Either $x = 0, -2$

$3y^2 - 6y = 0$

or, $3y(y-2) = 0$

Either $y = 0, y = 2$

$\therefore x = (0, -2)$ $(y = 0, 2)$

∴ The points are $(0,0), (0,2)$ $(-2,0), (-2,2)$

Now,

$f_{xx} = 6x + 6$ $f_{xy} = 0$

$f_{yy} = 6y - 6$

(a) at point $(0,0)$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 6 \times (-6) - 0 = -36 < 0$

$f_{xx} = 6 > 0$

then the function has saddle point at (0, 0).

(b) at point (0,2)

$f_{xx} = 6x+6 = 6\times0+6 = 6$ $\quad f_{xy}=0$

$f_{yy} = 6y-6 = 6\times2-6 = 12-6 = 6$

$f_{xx} = 6 > 0$

$f_{xx}\cdot f_{yy} - f_{xy}^2 = 6\times6-0 = 36 > 0$

Hence the function have local minimum at (0,2) and its value is

$f(0,2) = 0^3 + 2^3 + 3\times0^2 - 3\times(2)^2 - 8$

$= \cancel{8} - 12 \cancel{-8}$

$= -12$ #

(c) at point (−2,0)

$f_{xx} = 6\times(-2)+6 = -12+6 = -6$

$f_{yy} = 6\times0-6 = -6$

$f_{xy} = 0$

$f_{xx} = -6 < 0$

$f_{xx}\cdot f_{yy} - f_{xy}^2 = -6\times-6-0 = +36 > 0$

Hence the function have local maximum at (−6,−6) and its value is,

$f(-2,0) = (-2)^3 + (0)^3 + 3(-2)^2 - 3\times0^2 - 8$

$= -8+12-8$

$= -4$ #.

(d) at point (2,2)

$f_{xx} = 6\times(2)+6 = \cancel{18}\ -6$, $\quad f_{yy} = 6\times(2)-6 = \cancel{18}\ -12-6$ $\quad f_{xy}=0$

$f_{xx}<0$ $\cancel{f_{xx} = 18 > 0}$, $\quad f_{xx}\cdot f_{yy} - f_{xy} \overset{<}{=} 18\times6-0 = 108 > 0$

Hence the function have local minimum at point ~~(2,2)~~ (18,6) and its value at f(2,2) is,

$f(2,2) = 2^3 + 2^3 + 3\cdot2^2 - 3\cdot2^2 - 8 = 8+8-8 = 8$ #.

Saddle point at (−2,2)

"Absolute Maximum And Minimum"

STEPS:

1. List the interior points of the region R, where function may have local maxima and minima and evaluate function at this point. Those points are critical /stationary point.

2. List the boundary point of the Region R, where the function have local maxima and local minima and evaluate function at this point.

3. Look for maximum and minimum values of function.

* Find the absolute maximum and minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular region in the first quadrant bounded by line $x=0$, $y=0$ and $y=9-x$.

→ Solution,

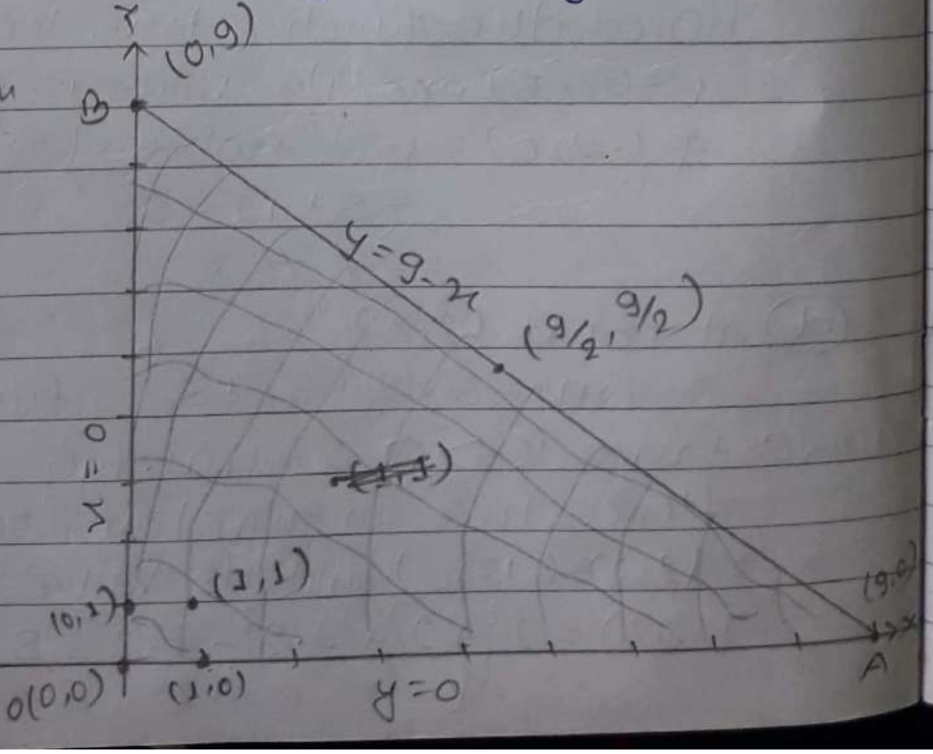
solving eq^n $y=9-x$

when $x=0$

$y=9$

when $y=0$

$x=9$



→ Solution :-

given function,

$f(x,y) = 2+2x+2y-x^2-y^2$

$f_x = \frac{\delta(2+2x+2y-x^2-y^2)}{\delta x} = 2-2x = 0$

$\therefore 2x = 2 \quad \therefore x = 1$

$f_y = \frac{\delta(2+2x+2y-x^2-y^2)}{\delta y} = 2-2y = 0$

$\therefore 2-2y = 0 \quad \therefore y = 1$

$\therefore f(1,1) = 2+2\times1+2\times1-1-1$ → In given function

$= 2+2+2-2$

$= 4$

Boundary points

a) line segment OA , $y=0$

$y=0$

$f(x,0) = 2+2x+2\times0-x^2-0 = 2x-x^2+2$

$\therefore f(x) = 2-2x = 0$

$\therefore 2x = 2 \quad$ so $x=1$, $y=0$

Critical point is $(1,0)$

$f(1,0) = 2+2\times1+2\times0-1^2-0^2$

$= 2+2-1 = 3$

At point A(9,0)

$f(9,0) = 2+2\times9+2\times0-9^2-0^2 = 2+18-81 = -61$

At point O (0,0)

$f(0,0) = 2+2\times0+2\times0-0^2-0^2 = 2$

b) line segment OB , $x=0$

$x=0$

$f(0,y) = 2 + 2\times0 + 2y - 0^2 - y^2 = 2y - y^2 + 2$

$f_y = 2 - 2y = 0$

$2 = 2y \quad \therefore y = 1$

critical point $(0,1)$

$\therefore f(0,1) = 2 + 2\times0 + 2\times1 - 0^2 - 1^2 = 2 + 2 - 1 = 3$

at point B $(0,9)$

$f(0,9) = 2 + 2\times0 + 2\times9 - 0^2 - 9^2 = 2 + 18 - 81 = -61$

c) line segment AB, $y = 9 - x$

$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= 2 + 2x + 18 - 2x - x^2 - (81 - 2.9.x + x^2)$

$= 20 + \cancel{2x} - \cancel{2x} - x^2 - 81 + 18x - x^2$

$= -61 - 2x^2 + 18x$

$\therefore f(x) = -4x + 18$

$f(x) = 0$

$\therefore 4x = 18 \quad \therefore x = \frac{18}{4} = \frac{9}{2}$

Put $x = 9/2$ in eq^n $y = 9 - x$

critical point $= (9/2, 9/2)$ $\therefore y = 9 - \frac{9}{2} = \frac{9}{2}$

$\therefore f(9/2, 9/2) = 2 + 2\times\frac{9}{2} + 2\times\frac{9}{2} - \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2$

$= 2 + \frac{18}{2} + \frac{18}{2} - \frac{81}{4} - \frac{81}{4} = \frac{8+36+36-81-81}{4}$

$= -\frac{41}{2} = -20.5$

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$\therefore f(1,1) = 4$, $f(1,0) = 3$, $f(0,1) = 3$, $f(9/2, 9/2) = -20.5$

at A $f(9,0) = -61$

at B $f(0,9) = -61$

at O $f(0,0) = 2$

$\therefore$ The absolute maximum is 4 at (1, 1) and absolute minimum is -61 at (0,9) and (9,0).

* Find absolute max. and min. values of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on closed triangular plate bounded by line $x=0$, $y=2$, $y=2x$ in first quadrant.

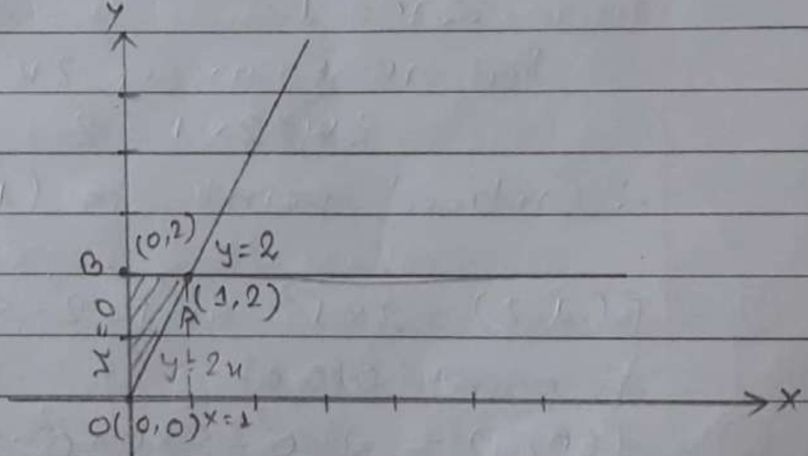
→ eq^n $y = 2x$

when

$x=0$ $y=0$

$x=1$ $y=2$

$x=2$ $y=4$



→ Solution,

given function,

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

here,

$f(x) = \frac{\delta(2x^2 - 4x + y^2 - 4y + 1)}{\delta x} = 4x - 4$

$\therefore f_x = 0$

$4x - 4 = 0$

$\therefore x = 1$

$f_y = 2y - 4 = 0$

$2y = 4$

$\therefore y = 2$

$\therefore f(1,2) = 2 \times 1^2 - 4 \times 1 + 2^2 - 4 \times 2 + 1$

$= 2 - \cancel{4} + \cancel{4} - 8 + 1$

$= -5$ —(1)

boundary points

a) line segment OA , $y = 2x$

$y = 2x$

$f(x, 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1$

$= 2x^2 - 4x + 4x^2 - 8x + 1$

$= 6x^2 - 12x + 1$

$f(x) = \frac{\delta(6x^2 - 12x + 1)}{\delta x}$

$= 12x - 12$

$f_x = 0$

$12x - 12 = 0$

$\therefore x = 1$

Put $x = 1$ in $y = 2x$

$\therefore y = 2 \times 1 = 2$

$\therefore$ Critical point is $(1, 2)$

② – $f(1,2) = 2 \times 1^2 - 4 \times 1 + 2^2 - 4 \times 2 + 1 = 2 - \cancel{4} + \cancel{4} - 8 + 1 = -5$

at point $O(0,0)$

③ – $f(0,0) = 2 \times 0^2 - 4 \times 0 + 0^2 - 4 \times 0 + 1 = 1$

at point $A(1,2)$

④ – $f(1,2) = -5$

b) line segment OB , $x = 0$

$x = 0$

$f(0, y) = 2 \times 0^2 - 4 \times 0 + y^2 - 4y \cancel{+1/2} + 1 = y^2 - 4y + 1$

$f_y = 2y - 4 = 0$

$2y = 4$

$\therefore y = 2$

⑤ – $f(0,2) = 2 \times 0^2 - 4 \times 0 + 2^2 - 4 \times 2 + 1 = 4 - 8 + 1 = -3$

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⑥- at point B(0,2)
$f(0,2) = -3$

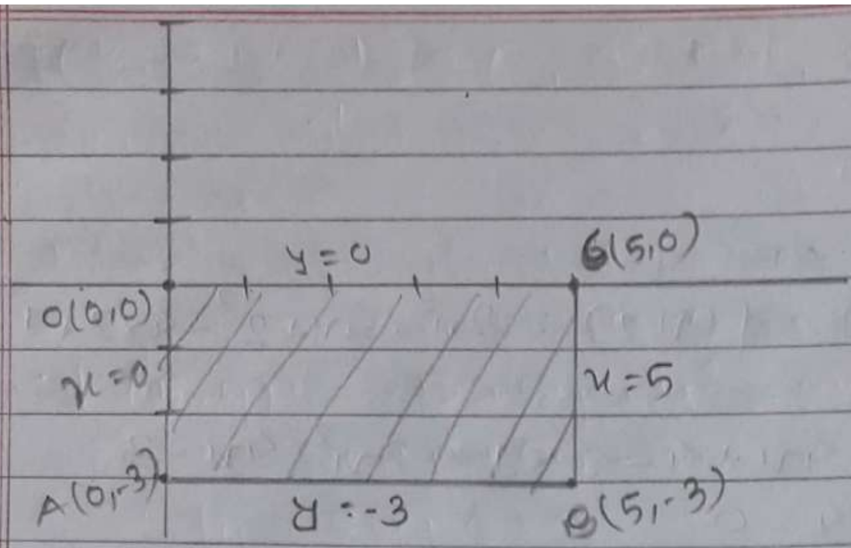
c.7 line segment AB $y=2$
~~f(0,2)~~ $f(x,y) = f(x,2) = 2x^2 - 4x + 2^2 - 4\times 2 + 1$

$f(x,2) = 2x^2 - 4x + 4 - 8 + 1 = 2x^2 - 4x - 3$
$f_x = 4x - 4 = 0$
$\therefore x = 1$

~~put~~
⑦- $\therefore f(1,2) = -5$

Hence,
$f(1,2) = -5$
$f(0,0) = 1$
$f(0,2) = -3$
Hence absolute max. is 1 at (0,0) and absolute minimum is -5 at (1,2).

Q. Find absolute maximum and minimum for $f(x,y) = x^2 + xy + y^2 - 6x + 2$ on rectangular plate
$0 \le x \le 5$, $-3 \le y \le 0$. , $x=0$ $x=5$ $y=-3$ $y=0$
→ Solution,
points are $x = 0, 5$
$y = -3, 0$
$\therefore$ Rectangular points are $(0,-3), (0,0), (5,-3), (5,0)$



→ Solution,

given function,

$$f(x,y) = x^2 + xy + y^2 - 6x + 2$$

here,

$$f_x = 2x + y - 6$$
$$f_y = x + 2y$$

Since, $f_x = f_y = 0$

$$2x + y = 6 \quad , \quad x = -2y$$

Now,

$$2(-2y) + y = 6$$

or, $-4y + y = 6$

or, $-3y = 6$

$$\therefore y = -6/3 = -2$$

Put y in $x = -2y$ $\quad \therefore x = -2 \times -2 = 4$

$\therefore$ Critical point $= (+4, -2)$

$$f(+4,-2) = (-4)^2 + (+4) \times (-2) + (-2)^2 - 6 \times (+4) + 2$$
$$= 16 - 8 + 4 - 24 + 2 = \cancel{54} = -10$$

Boundary points,

~~a> line segment OA~~

at point O(0,0)

$f(0,0) = 0^2+0\times0+0^2-6\times0+2 = 2$

at point A (0,-3)

$f(0,-3) = 0^2+0\times-3+(-3)^2-6\times0+2 = 9+2 = 11$

at point B(5,-3)

$f(5,-3) = 5^2+5\times(-3)+(-3)^2-6\times5+2$

$= 25-15+9-30+2 = -9$

at point c (5,0)

$f(5,0) = 5^2+5\times0+(0)^2-6\times5+2$

$= 25-30+2 = -3$

a> at line segment OA

$x=0$

$f(0,y) = 0^2+0\times y+y^2-6\times0+2 = y^2 = y^2$

$\therefore f_y = 2y = 0$

$\therefore y=0$ $\quad\therefore$ critical point $=(0,0)$

$f(0,0) = 0^2+0\times0+0^2-6\times0+2 = 2$

b> at line segment AB, $y=-3$

$f(\cancel{x},-3) = x^2+xy+y^2-6x+2$

$= x^2+x\times(-3)+(-3)^2-6x+2$

$= x^2-3x+9-6x+2$

$= x^2-9x+11$

$\therefore f(x) = 2x-9 = 0$

$\therefore x = \frac{9}{2}$

$f(9/2,-3) = \left(\frac{9}{2}\right)^2+\frac{9}{2}\times(-3)+(-3)^2-6\times\frac{9}{2}+2$

$f(9/2,-3) = \frac{81}{4} - \frac{27}{2} + 9 - \frac{54}{2} + 2 = \frac{-37}{4}$

c) at line segment BC,

$x = 5$

$\therefore f(5,y) = 25 + 5y + y^2 - 6\times5 + 2$

$= y^2 + 5y - 3$

$\therefore f_y = 2y + 5 = 0$

$\therefore 2y = -5$

$y = -5/2$

$\therefore$ Critical point $= (5, -5/2)$

$\therefore f(5,-5/2) = (5)^2 + 5\times(-5/2) + (-5/2)^2 - 6\times5 + 2$

$= 25 - \frac{25}{2} + \frac{25}{4} - 30 + 2$

$= -\frac{37}{4}$

$\therefore f(5,-5/2) = -37/5$

d) at line segment OC, $y = 0$

$f(x,0) = x^2 + x\times0 + 0^2 - 6x + 2 = x^2 - 6x + 2$

$\therefore f_x = 2x - 6 = 0$

$\therefore x = \frac{6}{2} = 3$

$\therefore f(3,0) = 3^2 * 3\times0 + 0^2 - 6\times3 + 2 = 9 - 18 + 2 = -7$

$\therefore$ Hence absolute maximum is 11 at $(0,-3)$, and absolute minimum is -10 at $(4,-2)$.

Linearization :-

$$L(x,y) = f(x_1,y_1) + f_x(x_1,y_1)(x-x_1) + f_y(x_1,y_1)(y-y_1)$$

$$\boxed{\therefore f(x,y) \approx L(x,y)}$$

* Find the linearization of :

a) $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at point $(3,2)$.

→ Solution :-

given function $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$

and point $(x_1,y_1) = (3,2)$

Now,

$f_x = \delta(x^2 - xy + \frac{1}{2}y^2 + 3) = 2x - y$

at $(3,2)$

$f_x = 2\times3 - 2 = 6 - 2 = 4$

$f_y = \frac{\delta(x^2 - xy + \frac{1}{2}y^2 + 3)}{\delta y} = -x + \frac{1}{2}\times 2y$

$f_y = -x + y$ at $(3,2)$

$f_y = -3 + 2 = -1$

$f(3,2) = f(x_1,y_1)$

$= 3^2 - 3\times2 + \frac{1}{2}\times 2^2 + 3$

$= 9 - 6 + 2 + 3$

$= 9 - 1 = 8$

$\cancel{= 3^2 - 3\times2 + \frac{1}{2}\times 2^2 + 3}$

$\cancel{= 9 - 6 + \frac{4}{2} + 3}$

$\cancel{= \frac{18 - 12 + 4 + 6}{2}}$

$\cancel{= 18 - 12 + 2 + 6}$

$L(3,2) = f(x_1,y_1) + f_x(x_1,y_1)(x-x_1) + f_y(x_1,y_1)(y-y_1)$

$= 8 + 4(x-3) + (-1)(y-2)$

$= 8 + 4x - 12 - y + 2$

$$\boxed{\therefore L(3,2) = 4x - y - 2 = 0}$$

Method of Lagrange Multiplier :

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and to find the local maximum and local minimum values of the function subject to constrains $g(x,y,z)=0$, we need to find the value of x,y,z and λ that simultaneously satisfy the equation,

$$\bar{\nabla} f = \lambda \bar{\nabla} g \quad \text{---(i)}, \text{and}$$

$$g(x,y,z) = 0 \text{ -- (ii)}$$

where,

λ = langrange Multiplier

$\bar{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

$\bar{\nabla} g = g_x \vec{i} + g_y \vec{j} + g_z \vec{k}$

* Find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

→ Solution,

given function, $f(x,y) = xy$

subject to constrain $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Then, Vector gradient $\vec{\nabla} f$ of $f(x,y)$

$f_x = y \quad f_y = x$

$\therefore \bar{\nabla} f = y\vec{i} + x\vec{j}$

Then, Vector gradient $\bar{\nabla} f$ of $g(x,y)$

$$g_x = \frac{\delta\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)}{\delta x}$$

$$= \frac{2x}{8} = \frac{x}{4}$$

$$g_y = \frac{\delta\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)}{\delta y}$$

$$= \frac{2y}{2} = y$$

$$\therefore \bar{\nabla} g = g_x \vec{i} + g_y \vec{j}$$

$$\bar{\nabla} g = \frac{x}{4} \vec{i} + y \vec{j}$$

Now,

$$\bar{\nabla} f = \lambda \bar{\nabla} g$$

$$y \vec{i} + x \vec{j} = \lambda\left(\frac{x}{4} \vec{i} + y \vec{j}\right)$$

Equating both sides we get,

$$y = \frac{\lambda x}{4} \quad \text{and} \quad x = \lambda y$$

$$\therefore y = \frac{\lambda . \lambda y}{4}$$

or, $4y - \lambda^2 y = 0$

or, $y(4 - \lambda^2) = 0$

$$\therefore y = 0$$
$$\lambda = \pm 2$$

then,

$$g(x, y) = 0$$

$\frac{x^2}{8}+\frac{y^2}{2}-1=0$

when $y=0$,

$\frac{x^2}{8}+\frac{0}{2}=1 \quad x^2=8 \quad \therefore x=\pm\sqrt{8}$

$\therefore x=\pm 2\sqrt{2}$ why? $\quad f(2\sqrt{2},0)=$ funct $f(x,y)=0$

$f(-2\sqrt{2},0)=f(x,y)=0$

$\therefore y=0, x=\pm 2\sqrt{2}, \lambda=\pm 2$

$x=\lambda y$ by equating $\lambda=\pm 2 \to$ by solving.

Put $x=\pm 2y$ in $g(x,y)$

Now

$\frac{x^2}{8}+\frac{y^2}{2}=1$

or, $\frac{(\pm 2y)^2}{8}+\frac{y^2}{2}=1$

or, $\frac{4y^2}{8_2}+\frac{y^2}{2}=1$

or, $2y^2=2$

$\therefore y=\pm 1, \quad x=\pm 2y$

when $y=+1$ In eq^n $f(x,y)=xy$

$x=\pm 2\times(+1)=\pm 2$

$\therefore (+2,+1)\ (-2,+1)$

$f(2,1)=2\times 1=2$

$f(-2,1)=-2\times 1=-2$

when $y=-1$

$x=\pm 2\times(-1)=\pm 2$

$\therefore (2,-1)\ (-2,-1)$

$f(2,-1)=-1\times 2=-2$

$f(-2,-1)=-1\times -2=2$

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$\therefore$ The greatest value is 2 at $(-2,-2)$ and $(2,1)$ and the smallest value is -2 at $(-2,1)$ and $(2,2)$.

* Find the maximum and minimum values of function $f_{xy} = 3x + 4y$ on circle $x^2 + y^2 = 1$

→ Solution,

we have to find the value of x, y and λ which satisfy the condition,

$$\bar{\nabla}f = \lambda \bar{\nabla}g \text{ and } g(x,y) = 0$$

Now,

$f_x = 3$, $f_y = 4$

$\therefore \bar{\lambda}f = 3\vec{i} + 4\vec{j}$

$g_x = 2x$, $g_y = 2y$

$\therefore \bar{\nabla}g = 2x\vec{i} + 2y\vec{j}$

$\therefore 3\vec{i} + 4\vec{j} = (2x\vec{i} + 2y\vec{j})\lambda$

$3\vec{i} + 4\vec{j} = 2x\lambda\vec{i} + 2y\lambda\vec{j}$

Equating on both sides we get,

$2x\lambda = 3$, $4 = 2y\lambda$

$\therefore x = \dfrac{3}{2\lambda}$ $\quad\therefore y = \dfrac{4}{2\lambda}$

Substituting thes values in $g(x,y) = 0$, we get

$$x^2 + y^2 = 1$$

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 1$$

or, $\dfrac{9}{4\lambda^2} + \dfrac{16}{4\lambda^2} = 1$

or, $9 + 16 = 4\lambda^2$

$25 = 4\lambda^2$

$\therefore \lambda^2 = \frac{25}{4}$

$\therefore \lambda = \sqrt{\frac{25}{4}} = \pm \frac{5}{2}$

Since $x = \frac{3}{2\lambda}$ and $y = \frac{4}{2\lambda}$, x and y have ~~sign~~ at $\lambda = 5/2$

For $x = \frac{3}{2\lambda}$ $\therefore x = \frac{3}{2\times 5/2} = \frac{3}{5}$

at $\lambda = -5/2$

$x = \frac{3}{2\times -5/2} = -\frac{3}{5}$

$\therefore x = \left(\frac{3}{5}, -\frac{3}{5}\right)$

For $y = \frac{4}{2\lambda}$, $y = \frac{4}{2\times \frac{5}{2}} = \frac{4}{5}$ at $\lambda = 5/2$

at $\lambda = -5/2$

$x(1,2)$
$y(4,5)$
then
point are
$(1,4)\,(1,5)\,(2,4)\,(2,5)$

$y = \frac{4}{2\times -5/2} = -4/5$

$\therefore y = \left(\frac{4}{5}, -\frac{4}{5}\right)$

as same

then points are $\left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{3}{5}, -\frac{4}{5}\right), \left(-\frac{3}{5}, \frac{4}{5}\right)$

$\left(-\frac{3}{5}, -\frac{4}{5}\right)$

at $f_{xy} = 3x + 4y$

$\therefore f(3/5, 4/5) = 3 \times \frac{3}{5} + 4 \times \frac{4}{5} = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$

$\therefore f(3/5, -4/5) = 3 \times \frac{3}{5} + 4 \times \left(-\frac{4}{5}\right) = \frac{9}{5} - \frac{16}{5} = -\frac{7}{5}$

$\therefore f\left(-\frac{3}{5}, \frac{4}{5}\right) = 3 \times \left(-\frac{3}{5}\right) + 4 \times \frac{4}{5} = -\frac{9}{5} + \frac{16}{5} = \frac{7}{5}$

$\therefore f\left(-\frac{3}{5}, -\frac{4}{5}\right) = 3 \times \left(-\frac{3}{5}\right) + 4 \times \left(-\frac{4}{5}\right) = -\frac{9}{5} - \frac{16}{5} = -\frac{25}{5} = -5$

$\therefore$ The maximum value of function is 5 at $f(3/5, 4/5)$

The minimum value of function is -5 at $f(-3/5, -4/5)$.

Partial Differentiable Equation (PDE):-

Review of Ordinary differentiable equation

$$\frac{dy}{dx} + xy = x^2 \text{ ------ ODE}$$

$$\frac{\delta y}{\delta x} + \frac{\delta^2 y}{\delta x^2} = 1 \text{ ------ PDE}$$

i) Separate Variable

ii) Homogeneous Equation

iii) Linear Equation

* Separate Variable :-

a.) $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$

→ Solution,

$$\sqrt{1-x^2}\,dy = -\sqrt{1-y^2}\,dx$$

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

or, Integrating both sides, we get,

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}} \qquad \left(\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\right)$$

or, $\sin^{-1}y = -\sin^{-1}x + c$

$\boxed{\therefore \sin^{-1}x + \sin^{-1}y = c}$ is the required equation.

b.) $\frac{dy}{dx} + y = 1$

→ Solution,

$$\frac{dy}{dx} = 1 - y$$

or, $\frac{dy}{(1-y)} = dx$

Integrating on both sides, we get,

$$\int \frac{dy}{(1-y)} = \int dx \qquad \left(\frac{dx}{1-x} = -\log(1-x)\right)$$

or, $-\log(1-y) = x + c$

or, $\boxed{x + \log(1-y) = c}$ is required equation.

c.) $dy = e^{x-y}dx + x \cdot e^{-y}dx$

take Common e^{-y}

or, $dy = e^{x} \cdot e^{-y}dx + x \cdot e^{-y}dx$

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or, $dy = e^{-y}(e^x dx + x dx)$

or, $\frac{dy}{e^{-y}} = e^x dx + x dx$

Integrating both sides,

$\int \frac{dy}{e^{-y}} = \int e^x dx + \int x dx$

or, $\int e^y dy = e^x + \frac{x^2}{2}$

or, $e^y = e^x + \frac{x^2}{2} + c$

$\boxed{\therefore e^x + \frac{x^2}{2} - e^y = c}$ is the required equation.

d.) $(1+x^2)dy = (1+y^2)dx$

→ Solution,

$\frac{dy}{(1+y^2)} = \frac{dx}{(1+x^2)}$ $\left[\because \int \frac{dy}{(1+y^2)} = \tan^{-1} y\right]$

Integrating both sides,

$\int \frac{dy}{(1+y^2)} = \int \frac{dx}{(1+x^2)}$

or, $\tan^{-1} y = \tan^{-1} x + c$

$\boxed{\therefore \tan^{-1} y - \tan^{-1} x = c}$ is the required equation.

* Homogeneous eq^n

* Homogeneous Equation:

- equation in the form of $\phi\left(\frac{y}{x}\right)$

put $v = \frac{y}{x}$

$\therefore y = vx \rightarrow$ Product rule

differentiating both side w.r. to x

$$\frac{dy}{dx} = \frac{d(v.x)}{dx}$$

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$$

$$\boxed{\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}} \qquad \boxed{\frac{y}{x} = v}$$

Q.) $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x^2}$

change $\frac{y}{x}$ to v. and Put $\frac{dy}{dx} = v + x\frac{dv}{dx}$

or, $\cancel{v} + x\frac{dv}{dx} - \cancel{v} = -(v)^2$

or, $\frac{dv}{v^2} = -\frac{dx}{x}$

or, $v^{-2}\,dv = -x^{-1}\,dx$

Integrating both side

or, $\int v^{-2}\,dv = -\int \frac{1}{x}\,dx$

or, $\frac{v^{-2+1}}{-2+1} = -\ln x$

or, $-\frac{1}{v} = -\ln x + c$

$[\ln a + \ln b = \ln(ab)]$

$[\ln a - \ln b = \ln(a/b)]$

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or, $-\dfrac{1}{y/x} = -\ln x + \ln c$

or, $-\dfrac{1}{1} \times \dfrac{x}{y} = \ln\left(\dfrac{c}{x}\right)$

or, $-\dfrac{x}{y} = \ln\left(\dfrac{c}{x}\right)$

or $e^{-x/y} = \dfrac{c}{x}$

$\boxed{\therefore x e^{-x/y} = c}$ is required eq^n.

b.) $\dfrac{dy}{dx} = \dfrac{y-2x}{x}$

$y = vx$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

→ Solution,

$\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{2x}{x}$

or, $v + x\dfrac{dv}{dx} = v - 2$

or, $x\,dv = -2\,dx$

or, $dv = -2\,\dfrac{dx}{x}$

Integrating on both sides,

$\int dv = -2\int \dfrac{dx}{x}$

$\boxed{\int \dfrac{1}{x}\,dx = \ln x}$

or, $v = -2\ln x$

or, $\dfrac{y}{x} = -2\ln x + c$

OR

or, $\dfrac{y}{x} = -2\ln x + \ln c$

or, $\dfrac{y}{2x} = \ln(c/x)$

or, $\dfrac{c}{x} = e^{y/2x}$

$c = x e^{y/2x}$

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* Linear Equation:-

$$\frac{dy}{dx} + py = Q$$

where P and Q are function of x (not y)

I.F = integrating factor

i.e.

$$I.F = e^{\int P dx}$$

Multiplying the linear eq^n with I.F

$$\frac{d}{dx}(y.IF) = Q.IF$$

or, $\int d(y \times IF) = \int Q dx \times I.F$

$$y \times IF = \int Q dx \times IF + c$$

Q.) $\tan x \frac{dy}{dx} + y = \sec x$

→ changing this eq^n to $\frac{dy}{dx} + Py = Q$

dividing both sides by $\tan x$, we get

$$\frac{dy}{dx} + \frac{1}{\tan x} y = \frac{\sec x}{\tan x}$$

or $\frac{dy}{dx} + \cot x . y = \frac{1}{\cos x} \times \frac{\cos x}{\sin x}$ — — (i)

Comparing eq^n ① with $\frac{dy}{dx} + Py = Q$

$\therefore P = \cot x$

$Q = \frac{1}{\sin x}$

Now,

Integrating Factor (I.F) $= e^{\int p\,dx}$

$= e^{\int \cot x\,dx}$ $\left[\because \int \cot x\,dx = \ln \sin x\right]$

$= e^{\ln \sin x}$

$= \sin x$

multiplying both sides by I.F, we get,

$\sin x \left(\frac{dy}{dx} + \cot x \cdot y\right) = \frac{1}{\cancel{\sin x}} \times \cancel{\sin x}$

always $\frac{d(y \times I.F)}{dx}$

or, $d(y \times \sin x) = dx$

Integrating both side, we get

$\int d(y \cdot \sin x) = \int dx$

or, $\boxed{y \cdot \sin x = x + C}$ which is required eq^n.

b.) $\frac{dy}{dx} + y = e^x$

→ Comparing $\frac{dy}{dx} + y = e^x$ with $\frac{dy}{dx} + Py = Q$

$\therefore P = 1 \qquad Q = e^x$

Now,

$= \int P\,dx = \int dx = x$

Integrating Factor $= e^{\int P\,dx} = e^x$

multiplying both sides by e^x

$$e^x\left(\frac{dy}{dx}+y\right)=e^x.e^x$$

or, $d(y \times e^x) = e^{2x}\,dx$

Integrating, we get

or, $\int d(y.e^x) = \int e^{2x}\,dx$

or, $\boxed{y.e^x = \frac{1}{2}e^{2x}+C}$ is the required equation.

* Linear differentiable equation of Second Order

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+py=Q$$

$\frac{d}{dx} = D =$ Differential operation

$\boxed{D^2y+D+Dy=Q} = \boxed{m^2+m+p=Q=0}$

STEPS for solution :-

1.) Auxillary Equation

$\frac{d}{dx} = D = m$

$y=1$ $\boxed{m = \text{two values roots}}$

(2) Solution :-

(a) m = roots ~~and~~ are real and different.

$\boxed{\therefore y = C_1 e^{m_1 x} + C_2 e^{m_2 x}}$

C_1 and C_2 are Constant.

b) When roots are real and equal.

$$\boxed{y = (C_1 + C_2 x) e^{mx}}$$ $\boxed{m = m_1 = m_2}$

c) When roots are imaginary and distinct.

$m = a \pm ib$

$$\boxed{y = e^{ax} (C_1 \cos bx + C_2 \sin bx)}$$

* Solve :-

a) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

→ Solution,

Axilluary equation is,

$m^2 + 5m + 6 = 0$

or $m^2 + 2m + 3m + 6 = 0$

or $m(m+2) + 3(m+2) = 0$

Either $m_1 = -2$ and $m_2 = -3$

∴ Required eq^n is,

roots m_1 and m_2 are real and different.

so,

$$\boxed{y = C_1 e^{-2x} + C_2 e^{-3x}}$$

b) $\frac{d^2y}{dx^2} + \omega^2 y = 0$ $\omega = a + ib$ = Imaginary

Axilliulary eq^n is,

a, $m^2 + \omega^2 = 0$

∴ $m = \pm \omega i$ $0 + \omega i$

$m = a \pm ib$

$\therefore a = 0$ Roots are imaginary & distinct.

$b = \omega$

$\therefore y = e^0 (C_1 \cos \omega x + C_2 \sin \omega x)$ #.

3.> $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 12y = 0$???

→ Solution,

given axilliary eq^n is

$m^2 + 4m + 12 = 0$ — ①

Comparing eq^n ① with

$ax^2 + bx + c = 0$

$\therefore a = 1$

$b = 4$

$c = 12$

$m^2 + 6m - 2m + 12 = 0$

$m(m+6) - 2(m+6) = 0$

$(m-2)(m\pm6)$

impossible !!!!

so

Use quadratic eq^n

Now,

$\cancel{x = \pm b \frac{\sqrt{b^2-4ac}}{2a}}$

$\cancel{= \pm 4 \frac{\sqrt{16-4\times1\times12}}{2\times1}}$

$\cancel{= \pm 4 \sqrt{\frac{-32}{2}}}$

$\cancel{= \pm 2\sqrt{-32}}$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\cancel{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

$-b \pm \frac{\sqrt{b^2-4ac}}{2a}$

$= -4 \pm \frac{\sqrt{16-4\times1\times12}}{2\times1}$

$= -4 \pm \sqrt{\frac{-32}{2}}$

d.) $(D+3)^2 y = 0$ $\quad D = \frac{dy}{dx} \quad \frac{d^2y}{dx^2} = D = m$

→ Auxilliary Equation is

$(m+3)^2 = 0$

roots are, real and equal, so

$(m+3)(m+3) = 0$

Either, $m = -3, -3$

Required eqⁿ is,

$\therefore$ $\boxed{y = (C_1 + C_2 x) e^{-3x}}$ #.

Partial Differential Equation :-

→ The equation in the form of $f(x, y, z, p, q) = 0$ is called partial differential equation of first order.

(x, y, z) are independent variable

$p = \frac{\delta z}{\delta x}$, $q = \frac{\delta z}{\delta y}$

second order :-

$P(x, y, z, p, q, r, s, t \ldots\ldots) = 0$

$r = \frac{\delta^2 z}{\delta x^2}$, $s = \frac{\delta^2 z}{\delta x \delta y}$, $t = \frac{\delta^2 z}{\delta y^2}$, ...etc.

Origin of first order PDE

→ eliminating constants

→ eliminating orbitrary function.

1.) Form PDE by eliminating constants a and b from:

$$(x-a)^2+(y-b)^2+z^2=1 \quad \text{—— (A)}$$

2.) Form PDE by eliminating orbitrary function f from the eqn,

$$z=xy+f(x^2+y^2).$$

1st step w.r. to x

$$\Rightarrow \frac{\delta(x-a)^2}{\delta(x-a)}\times\frac{\delta(x-a)}{\delta x}+\frac{\delta(y-b)^2}{\delta x}+\frac{\delta z^2}{\delta z}\times\frac{\delta z}{\delta x}=0$$

$$\left(\because \frac{\delta z}{\delta x}=p\right)$$

or, $2(x-a)\times 1+0+2z\times p=0$

or, $2(x-a)=-2zp$

$$\left(\because \frac{\delta z}{\delta y}=q\right)$$

or, $x-a=-zp$ ---- (i)

Partial derivative w.r. to y

$$\frac{\delta(x-a)^2}{\delta(x-a)}\times\frac{\delta(x-a)}{\delta y}+\frac{\delta(y-b)^2}{\delta(y-b)}\times\frac{\delta(y-b)}{\delta y}+\frac{\delta z^2}{\delta z}\times\frac{\delta z}{\delta y}=0$$

or, $2(x-a)\times 0+2(y-b)\times 1+2z\times q=0$

$$y-b=-zq \text{ --- (ii)}$$

from eqn (i) and (ii) we get, in eqn — (A)

$(x-a)^2+(y-b)^2+z^2$

$$(-zp)^2+(-zq)^2+z^2=1$$

$$\boxed{z^2(p^2+q^2+1)=1}$$

P.D.E

2nd GRP

$z = xy + f(x^2+y^2)$

partial derivative w.r. to x.

$$\frac{\delta z}{\delta x} = y + \frac{\delta f(x^2+y^2)}{\delta(x^2+y^2)} \times \frac{\delta(x^2+y^2)}{\delta x}$$

$$p = y + f'(x^2+y^2) \times 2x \text{ ---- (i)}$$

partial derivatives w.r. to y.

$$\frac{\delta z}{\delta y} = x + \frac{\delta f(x^2+y^2)}{\delta(x^2+y^2)} \times \frac{d(x^2+y^2)}{\delta y}$$

$$q = x + f'(x^2+y^2) \times 2y \text{ ---- (ii)}$$

from eq^n (i) and (ii)

from eq^n (i)

$$f'(x^2+y^2) = \frac{p-y}{2x} \text{ — (iii)}$$

from (ii)

$$f'(x^2+y^2) = \frac{q-x}{2y} \text{ — (iv)}$$

equating eq^n (iii) & (iv)

$$\frac{p-y}{2x} = \frac{q-x}{2y}$$

$$\frac{p-y}{2x} = \frac{q-y}{2y}$$

$$\frac{p-y}{x} - \frac{q+x}{y} = 0$$

or, $\boxed{py - y^2 = qx - x^2}$

$py - y^2 - qx + x^2 = 0$

PDE ~~#~~ $py - y^2 = qx - x^2$

$py - y^2$

Q. Form PDE by eliminating ϕ from

$$lx+my+nz=\phi(x^2+y^2+z^2)$$

→ Solution,

differentiating both sides w.r. to x

$$\frac{\delta(lx)}{\delta x}+\frac{\delta(my)}{\delta x}+\frac{\delta(nz)}{\delta x}=\frac{\delta\phi(x^2+y^2+z^2)}{\delta x}$$

or $$l+0+np=\frac{\delta\phi(x^2+y^2+z^2)}{\delta(x^2+y^2+z^2)}\times\frac{\delta(x^2+y^2+z^2)}{\delta x}$$

$$\frac{\delta z^2}{\delta z}\times\frac{\delta z}{\delta x}$$

or $$l+np=\phi'(x^2+y^2+z^2)\times(2x+2zp)\quad\text{---(i)}$$

$$2z\times p$$

diff. w.r. to y on both sides,

$$\frac{\delta(lx)}{\delta y}+\frac{\delta(my)}{\delta y}+\frac{\delta nz}{\delta y}=\frac{\delta\phi(x^2+y^2+z^2)}{\delta(x^2+y^2+z^2)}\times\frac{\delta(x^2+y^2+z^2)}{\delta y}$$

or $$0+m+nq=\phi'(x^2+y^2+z^2)\times(2y+2zq)$$

$$\therefore \phi'(x^2+y^2+z^2)=\frac{m+nq}{(2y+2zp)}\quad\text{----(ii)}$$

$$\therefore \phi'(x^2+y^2+z^2)=\frac{l+np}{(2x+2zp)}\quad\text{--- from eq}^n\ (i)$$

equating eq^n (i) and (ii), we get,

$$\frac{m+nq}{\cancel{2}(y+zp)}=\frac{l+np}{\cancel{2}(x+zq)}$$

Q. $mx + nqx = ly + npq$

$\therefore$ P.D.E

$(x + zq)(m + nq) = (l + np)(y + zp)$

Linear P.D.E of first order:

$Pp + Qq = R$

Langrange Auxilliary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \left[p = \frac{\delta z}{\delta x},\ q = \frac{\delta z}{\delta y} \right]$$

Solution is $\phi(u, v) = 0$ [P, Q, are the function of x, y, z]

where,

$u(x, y, z) = C_1$

$v(x, y, z) = C_2$

Solve : $y^2 pz + x^2 qz = xy^2$

→ Solution,

Comparing with $Pp + Qq = R$

$\therefore P = y^2 z \quad Q = x^2 z \quad R = xy^2$

Now,

Langrange Auxilliary eq^n,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

taking only two variable

$$\frac{dx}{y^2z} = \frac{dy}{x^2z}$$

Integrating both sides,

$$\int x^2 dx = \int y^2 dy$$

or, $\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$

or, $\frac{c_1}{\cancel{3}} = \frac{x^3 - y^3}{\cancel{3}}$ $\therefore c_1 = x^3 - y^3$

Now,

again taking two variables,
taking first and 3rd

$$\frac{dx}{y^2z} = \frac{dz}{xy^2}$$

~~taking~~ Integrating both sides,

$$\int x\,dx = \int z\,dz$$

or, $\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$

or, $\frac{c_2}{\cancel{2}} = \frac{x^2 - z^2}{\cancel{2}}$ $\therefore c_2 = x^2 - z^2$

Solution is,

$$\phi(c_1, c_2) = 0$$

$$\phi(x^3 - y^3, x^2 - z^2) = 0 \ \#$$

* $p + q = x$

→ Solution,

Comparing question with $Pp + Qq = R$

$\therefore P = 1 \quad Q = 1 \quad R = x$

Now,

Langrang Auxilliary eq^n is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x}$$

taking first and second,

Integrating

$$\int dx = \int dy$$

or $x - y = C_1$

taking first and 3rd

$$dx = \frac{dz}{x}$$

or, $x\,dx = dz$

Integrating both side,

$$\int x\,dx = \int dz$$

or $\frac{x^2}{2} = z + C_2$

or, $\frac{x^2}{2} - \frac{2z}{2} = \frac{C_2}{2}$

or, $C_2 = x^2 - 2z$

Solution is $\phi(c_1, c_2) = 0$

$$\phi(x-y, x^2-2z) = 0 \#$$

b.) $(y-z)\frac{\delta z}{\delta x} + (x-y)\frac{\delta z}{\delta y} = z-x$

→ Solution,

$(y-z)p + (x-y)q = z-x$ $\left[\because \frac{\delta z}{\delta x} = p, \frac{\delta z}{\delta y} = q\right)$

Comparing with $Pp + Qq = R$

$\therefore P = (y-z)$ $Q = (x-y)$, $R = (z-x)$

langrange Auxilliary eq^n is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $$\frac{dx}{(y-z)} = \frac{dy}{(x-y)} = \frac{dz}{(z-x)}$$

Using multiplier 1, 1, 1, we get,

$$\frac{dx+dy+dz}{\cancel{y}-\cancel{z}+\cancel{x}-\cancel{y}+\cancel{z}-\cancel{x}} = k$$

or, $dx+dy+dz = k$

Integrating,

$c_1 + x+y+z = 0$

$x+y+z = c_1$

Using multiplier x, $\cancel{y}$ z and y

$$\frac{xdx + zdy + ydz}{\cancel{xy} - xz + zx - zy + yz - \cancel{xy}} = k$$

or, $\int xdx + \int zdy + \int ydz = 0$

or, $\frac{x^2}{2} + zy + yz = C_2$

$\therefore$ Solution is $\phi(C_1, C_2) = 0$

$$\Rightarrow \phi\left(x+y+z, \frac{x^2}{2} + zy + yz\right) = 0 \ \#.$$

(c) $(mz - ny)p + (nx - lz)q = ly - mx$

→ Solution,

Comparing question with $Pp + Qq = R$

$\therefore P = (mz - ny)$, $Q = (nx - lz)$ $R = ly - mx$

then,

Auxilliary eq^n is

$$\frac{dx}{(mz - ny)} = \frac{dy}{(nx - lz)} = \frac{dz}{(ly - mx)}$$

Using multiplier x, y, z, we get,

$$\frac{xdx + ydy + zdz}{mzx - nyx + nyx - lyz + lyz - mxz} = k$$

or, Integrating both side

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_1}{2}$$

$\therefore C_1 = x^2 + y^2 + z^2$

Using multiplier l, m, n

$$\frac{l\,dx + m\,dy + n\,dz}{mzl - \cancel{my}l + mnx - \cancel{ml}z + \cancel{nly} - nmx} = k$$

or, $\int l\,dx + \int m\,dy + \int n\,dz$

$\therefore C_2 = \cancel{l+m+n}\ lx + my + nz$

$\therefore$ Solution is $\phi(x^2+y^2+z^2, \cancel{l}x + my + nz)$

d.) $x^2 \frac{\delta z}{\delta x} + y^2 \frac{\delta z}{\delta y} = (x+y)z$

→ Solution,

$$x^2 p + y^2 q = (x+y)z \quad \text{--- (i)}$$

Comparing eq^n (i) with $Pp + Qq = R$

$\therefore P = x^2 \quad Q = y^2 \quad R = (x+y)z$

then langrange ~~ext~~ auxilliary eq^n is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(xz+yz)}$

taking 1st and 2nd

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating, $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

or, $\int x^{-2}\,dx = \int y^{-2}\,dy$

or, $\frac{x^{-2+1}}{-2+1} = \frac{y^{-2+1}}{-2+1}$

or, $\cancel{-}\frac{1}{x} = \cancel{-}\frac{1}{y} \mp C_1$

or, $\frac{1}{x} - \frac{1}{y} = C_1$

or, $\frac{y-x}{xy} = C_1$

taking 1st and 3rd variable.

$$\int \frac{dx}{x^2} = \int \frac{dz}{xz+yz}$$

or, $\int x^{-2}\, dx = \int \frac{dz}{xz} + \int \frac{dz}{yz}$

or, $\frac{x^{-2+1}}{-2+1} = \frac{\ln z}{x} + \frac{\ln z}{y}$

or, $-\frac{1}{x} = \frac{\ln z}{x} + \frac{\ln z}{y} + C_2$

Taking 1st and 2nd

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$= x^{-2} \int dx = \int y^{-2} dy$

$= +\frac{1}{x} - \frac{1}{y} = C_1$

$\therefore C_1 = \frac{1}{x} - \frac{1}{y}$

~~Taking~~ Again,

$$\frac{dx - dy}{(x^2-y^2)} = \frac{dz}{z(x+y)}$$

$$\frac{d(x-y)}{(x-y)\cancel{(x+y)}} = \frac{dz}{z\cancel{(x+y)}}$$

$\int d = \int z^{-1} dz$

$C_2 = \log(1-2) = \log z + \log x$

$(1-2) = c.x$

$c\left(1-\frac{2}{x}\right)2$

(e) $y^2z\frac{\delta z}{\delta x}+x^2z\frac{dz}{\delta y}=xy^2$

→ Comparing $y^2zp+x^2zq=xy^2$ with $Pp+Qq=R$

$\therefore P=y^2z \;,\; Q=x^2z \;,\; R=xy^2$

Now,

langrange Auxillary eq^n is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

or, $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$

Using ~~multiplier -x, y, -z.~~

or, $\cancel{\frac{-xdx+ydy+zdz}{-xy^2z-yx^2z}}$ taking 1st and 2nd

$$\int zx^2dx=\int y^2dy\,z$$

or, $z\frac{x^3}{3}=z\frac{y^3}{3}+\frac{c_1}{3}$

$\therefore C_1=z(x^3-y^3)$

taking 1st and 3rd $\frac{dx}{y^2z}=\frac{dz}{xy^2}$

or, $\int xy^2dx=\int y^2zdz$

or, $y^2\frac{x^2}{2}=y^2\frac{z^2}{2}+\frac{c_2}{2}$

or, $C_2=y^2(x^2-z^2)$

$\therefore$ Solution is $\phi(z(x^3-y^3), y^2(x^2-z^2))$ #

f.) $xp - yq = y^2 - x^2$

→ Solution,

Comparing $xp - yq = y^2 - x^2$ with $Pp + Qq = R$

$\therefore P = x \quad Q = y, \quad R = y^2 - x^2$

Langrang Auxilliary eqⁿ is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{y^2 - x^2}$

Using multiplier $x, -y, 1$

or $\frac{x\,dx + (-y)dy + dz}{x^2 - y^2 + y^2 - x^2} = k$

or, $x\,dx - y\,dy + dz = 0$

Integrating we get,

or, $\int x\,dx - \int y\,dy + \int dz = 0$

or, $\frac{x^2}{2} - \frac{y^2}{2} + z = \frac{c_1}{2}$

$\therefore c_1 = (x^2 - y^2 + 2z)$

taking 1st and 2nd

$\int \frac{dx}{x} = \int \frac{dy}{y}$

or, $\ln x = \ln y + \ln c_2$

or, $c_2 = \ln x - \ln y = \ln\left(\frac{x}{y}\right)$

taking 1st and 3rd

$\int \frac{dx}{x} = \int \frac{dz}{y^2 - x^2}$

or, $\ln x =$

8) $x^2p + q = z^2$

→ Solution,

Comparing eqⁿ with $Pp + Qq = R$

$\because P = x^2 \quad Q = 1 \quad R = z^2$

Now,

Lagrange eqⁿ of Auxilliary is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{x^2} = \frac{dy}{1} = \frac{dz}{z^2}$$

Integrating 1st and 2nd

or $$\int \frac{dx}{x^2} = \int dy$$

or $$\frac{x^{-2+1}}{-2+1} = y + C_1$$

$$\therefore C_1 = -\frac{1}{x} - y$$

Integrating 1st and 3rd

$$\int \frac{dx}{x^2} = \int \frac{dz}{z^2}$$

or, $$\frac{x^{-2+1}}{-2+1} = \frac{z^{-2+1}}{-2+1} + C_2$$

or $$-\frac{1}{x} = -\frac{1}{z} + C_2$$

$$\therefore C_2 = \frac{1}{z} - \frac{1}{x}$$

$\therefore$ Solution is $\phi(C_1, C_2)$

$$\phi = \left(-\frac{1}{x} - y, \frac{1}{z} - \frac{1}{x}\right) \#$$

PDE of Second Order :-

PDE of the form :-

$$\boxed{F(D, D')z = f(x, y)}$$ (Partial diff. Eqⁿ)

is PDE of second order.

where,

$$D = \frac{\delta}{\delta x} \quad \text{and} \quad D' = \frac{\delta}{\delta y}$$

$r = D^2 = \frac{\delta^2 z}{\delta x^2}$ $\quad s = DD' = \frac{\delta^2 z}{\delta x . \delta y}$ $\quad t = D'^2 = \frac{\delta^2 z}{\delta y^2}$

$\frac{\delta^2 z}{\delta x^2}$ $\quad \frac{\delta^2 z}{\delta y^2}$

* Complementary Function :- (CF)

Solution of $F(D, D')z = 0$ is Called complementary function.

→ Solution :-

1) When roots are real and distinct, diff. then

$$CF = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \dots\dots$$

2) When roots are real and same (equal)

$$C.F = \phi_1(y + m_1 x) + x\,\phi_2(y + m_2 x) + x^2\phi_3(y + m_3 x)$$

* Particular Integral (P.I)

$$z = \frac{f(x, y)}{F(D, D')}$$ is

Called particular integral.

Complete Solution :-

$$\boxed{Z = C.f + P.I}$$

Note:- For P.I

Binomial Expansion

$(1+D)^{-1} = 1-D+D^2-D^3+\ldots\ldots$

$(1-D)^{-1} = 1+D+D^2+D^3+\ldots\ldots$

$(1+D)^{-2} = 1-2D+3D^2-4D^3+\ldots\ldots$

$(1-D)^{-2} = 1+2D+3D^2+4D^3+\ldots\ldots$

Particular Integral:

1> $f(x,y) = x^m \cdot y^m$

$$Z = F(D,D')^{-1} x^m \cdot y^m$$

Use binomial expansion.

2> $f(x,y) = \cos(ax+by)$ or $f(x,y) = \sin(ax+by)$

$$Z = \frac{\cos(ax+by)}{F(+D^2, +DD', +D'^2)} = \frac{\cos(ax+by)}{F(-a^2, -ab, -b^2)}$$

3> $f(x,y) = \dfrac{e^{ax+by}}{F(D,D')} = \dfrac{e^{ax+by}}{F(a,b)}$

$\{$provided that $F(a,b) \neq 0\}$

if $F(a,b) = 0$, differentiate denominator with respect to D and multiply by x.

Q. Solve :-

$$(D^2 - D'^2) z = x - y$$

→ Solution :-

for Complementary function,

$$F(D, D') = 0$$

$D = m$, $D' = 1$

Auxilliary equation is put always $D = m$ and $D' = 1$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\therefore m_1 = 1, \quad m_2 = -1$$

$$\therefore C.F = \phi_1 (y + x) + \phi_2 (y - x)$$

Now,

Particular Integral (P.I),

$$z = \frac{f(x,y)}{F(D, D')} \text{ is}$$

$$z = \frac{(x-y)}{(D^2 - D'^2)}$$

doing binomial expansion
take D^2 common

$$z = \frac{x - y}{D^2\left(1 - \frac{D'^2}{D^2}\right)}$$

$$\frac{\delta(\cos(lx+my))}{\delta(lx+my)} \times \frac{\delta(lx+my)}{\delta y} = -\sin(lx+my) \times m$$

$$= \frac{-A[l^2\cos(lx+my) - D'\cos(lx+my)]}{(l^4+m^2)}$$

P. $= -\left(\frac{A(l^2\cos(lx+my)}{l^4+m^2} + \frac{\sin(lx+my) \times m}{l^4+m^2}\right)$

$$P.I = -\frac{\cancel{A}\ 1}{(l^4+m^2)}\left(\cancel{A}(l^2\cos(lx+my) - \sin(lx+my).m\right)$$

$Z = P.I + C.I \qquad x^m y^n = f(xy)$

$$(1-D)^{-1} = 1+D+D^2+D^3+\cdots$$
$$(1+D)^{-1} = 1-D+D^2-D^3+\cdots$$
$$(1-D)^{-2} = 1+2D+3D^2+4D^3+\cdots$$
$$(1+D)^{-2} = 1-2D+3D^2-4D^3+\cdots$$

$f(xy) = \cos(ax+by) / \sin(ax+by)$

$$\frac{F(\cos(ax+by)}{F(D^2, DD', D'^2)} \qquad \frac{\cos(ax+by)}{(-a^2, -ab, -b^2)}$$

$D'^2 \times 2$

classmate
Date
Page

* Important long question :-

* Solve $(D^2 - D'^2) z = x - y$

→ Solution :-

for complementry function
$F(D, D') = 0$

where $D = m$ and $D' = 1$

$\therefore m^2 - 1 = 0$ (Auxilliary eqn)

$\therefore m^2 = 1$

$\therefore m = \pm 1$ $m_1 = +1$ $m_2 = -1$

$\therefore$ Complementry function (C.F) =
$\phi_1 (y + x) + \phi_2 (y - x)$

Now,

Particular integral (P.I) =
$$Z = \frac{F(x, y)}{(D, D')}$$

$$Z = \frac{(x - y)}{(D^2 - D'^2)}$$

the $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \ldots$

let D

$$= \frac{(x - y)}{D^2 \left(1 - \frac{D'^2}{D^2}\right)} = \frac{1}{D^2} \left(1 - \frac{D'^2}{D^2}\right)^{-1} (x - y)$$

$$= \frac{1}{D^2} \left(1 + \frac{D'^2}{D^2} + \frac{D'^4}{D^4} + \ldots\ldots\right) (x - y)$$

$$= \frac{1}{D^2} \left[(x - y) + (x - y) \frac{D'^2}{D^2}\right]$$

$$= \frac{1}{D^2} \left[(x - y) + 0\right]$$

$= D^{-2}(x-y)$

Integrating w.r. to x because $D = \frac{\delta}{\delta x}, D' = \frac{\delta}{\delta y}$

$= \cancel{\frac{\delta}{\delta x}}(D^{-2}(x-y))$

$= D^{-1}\left(\frac{x^2}{2} - xy\right)$

$= \cancel{\frac{\delta}{\delta x}}\left(D^{-1}\left(\frac{x^2}{2} - xy\right)\right)$

$= \frac{x^3}{6} - \frac{x^2 y}{2}$

$\therefore$ Particular integral is $\frac{x^3}{6} - \frac{x^2 y}{2}$

$\therefore$ Comple Sol^n $(z) = \phi_1(y+x) + \phi_2(y-x) + \frac{x^3}{6} - \frac{x^2}{2}y$ #.

* Solve:-

$(D^2 - D')Z = A\cos(lx + my)$

→ Solution,

for Complementary function,

$F(D, D') = 0$

$D = m \quad , D' = 1$

$\therefore$ Auxilliary eq^n is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \sqrt{1}$

$m = \pm 1$

$\therefore m_1 = 1, \quad m_2 = -1$

Since roots are real and distinct.

so,

Complementary function (C.F) $= \phi_1(y+x) + \phi_2(y-x)$

Now,

for particular integral

$$Z = \frac{f(x,y)}{F(D,D')}$$

$f(x,y)$ is cos, so $\frac{\cos(lx+my)}{F(D^2, DD', D'^2)} = \frac{\cos(lx+my)}{F(-l^2, -lm, -m^2)}$

$\therefore D^2 = -l^2, \; D'^2 = -m^2$

$$Z = \frac{A\cos(lx+my)}{(D^2 - D')}$$

we can't put $D' = -m$

so make D' to D'^2

to put $-m^2$

$$\text{or, } Z = \frac{A\cos(lx+my)}{(-l^2 - D')}$$

$$\text{or, } Z = \frac{A\cos(lx+my)}{-(l^2 + D')} \times \frac{(l^2 - D')}{(l^2 - D')}$$

$$\text{or, } Z = \frac{A\cos(lx+my) \times (l^2 - D')}{-(l^4 - D'^2)}$$

$D'^2 = -m^2$

$$\text{or, } Z = -\frac{A\cos(lx+my) \times (l^2 - D')}{(l^4 - \cancel{D'^2}(-m^2))}$$

$$\text{or, } Z = -\frac{(A\cos(lx+my)\, l^2 - D' A\cos(lx+my))}{(l^4 + m^2)}$$

$$\text{or, } Z = -\frac{A l^2 \cos(lx+my) + \frac{\delta}{\delta y}(D' A\cos(lx+my)}{(l^4 + m^2)}$$

or, $Z = -\frac{Al^2}{(l^4+m^2)} \cos(lx+my) + \frac{AS\cos(lx+my)}{\delta(lx+my)} \times \frac{\delta(lx+my)}{\delta y}$

or, $Z = -\frac{Al^2}{(l^4+m^2)} \cos(lx+my) + (-)\sin(lx+my) \times m \times A$

$\therefore Z = -\frac{Al^2}{(l^4+m^2)} \cos(lx+my) - \frac{Am\sin(lx+my)}{(l^4+m^2)}$

$\therefore$ Complete solution,

$Z = C.F + P.I$

$= \phi_1(y+x) + \phi_2(y-x) - \frac{Al^2}{(l^4+m^2)} \cos(lx+my) - m\sin(lx+my)$

$\#$

* Find particular integral of eq^n $(D^2-D')Z = 2y-x^2$

$D = \frac{\delta}{\delta x}$, $D' = \frac{\delta}{\delta y}$ $\quad x^2y - \frac{x^4}{4} + \frac{x^3}{3}$

→ Solution,

Particular integral is given as

$Z = \frac{F(x,y)}{F(D,D')} = \frac{f(2y-x^2)}{(D^2-D')}$

Using binomial expansion,

$Z = \frac{1}{D^2\left(1-\frac{D'}{D^2}\right)}(2y-x^2)$.

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let $\left(\frac{D'}{D^2}\right)$ be D $(1-D)^{-1} = 1+D+D^2+\cdots$

$$z = \frac{1}{D^2}\left(1-\frac{D'}{D^2}\right)^{-1}(2y-x^2)$$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D^2}+\frac{D'^2}{D^4}+\cdots\right)(2y-x^2) \qquad D'^2 \to 0$$

$$= \frac{1}{D^2}\left[(2y-x^2)+\frac{1}{D^2}\left(D'(2y-x^2)\right)\right]$$

$$= \frac{1}{D^2}\left[(2y-x^2)+\frac{1}{D^2}\frac{\delta(2y-x^2)}{\delta y}\right]$$

$$= \frac{1}{D^2}\left[(2y-x^2)+\frac{1}{D^2}\times 2\right]$$

$$= \frac{1}{D^2}(2y-x^2)+\frac{1}{D^4}2$$

$$= \frac{\delta(2y-x^2)}{\delta x}D+\frac{1}{D^3}2x$$

$$= \frac{\delta}{\delta x}\left(2xy-\frac{x^3}{3}\right)+\frac{1}{D^2}\frac{2x^2}{2}$$

$$= \frac{2x^2}{2}y-\frac{x^4}{12}+\frac{2x^3}{2\times 3}\frac{1}{D}$$

$$= x^2y-\cancel{\frac{x^4}{12}}+\cancel{\frac{x^4}{3\times 4}}$$

$$= x^2y \;\#$$

$(D^2-D')z = 2y-x^2$

$$\frac{\delta^2 z}{\delta x^2}-\frac{\delta^2 z}{\delta y^2}$$

i.e. $(D^2-D'^2)z = 2y-x^2$

$$\text{P.I} = \frac{1}{D^2-D'^2}\,2y-x^2$$

$\frac{1}{D^2}\left((2y-x^2)\times\frac{1}{D}2y\right)$

$\frac{1}{D^2}(2y-x^2)\times 2y/2$

$\frac{1}{D^2}(2xy)$ $= \frac{1}{D}\,2xy$ $= x^2y$

$\frac{1}{D^2}(2y-x^2)+\frac{1}{D^2}2x$

$\frac{1}{D}\left(2xy-\frac{x^3}{3}\right)+\frac{1}{D}2x$

$x^2y-\frac{x^4}{12}$

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* Solve:

1.) $\frac{\delta^2 z}{\delta x^2} - \frac{2\delta^2 z}{\delta x \delta y} + \frac{\delta^2 z}{\delta y^2} = \sin(2x+3y)$

$\Rightarrow (D^2 - 2DD' + D'^2)z = \sin(2x+3y)$

for Complementary function

$D = m \quad D' = 1$

The auxilliary eqn is,

$m^2 - 2m + 1 = 0$

or, $m^2 - m - m + 1 = 0 \quad$ or, $m(m-1) - 1(m-1) = 0$

Either $m_1 = 1 \quad m_2 = +1$

The roots are equal and real.

$\therefore C.F = \phi_1(y+x) + \phi_2(y+x).x$

for particular Integration.

$$Z = \frac{f(x,y)}{F(D,D')} = \frac{\sin(2x+3y)}{D^2 - 2DD' + D'^2}$$

$$= \frac{\sin(2x+3y)}{-2^2 + 2\times2\times3 - 3^2} = -1 \sin(2x+3y)$$

$-4+12-9 \quad -13+12$

$\therefore Z = -\sin(2x+3y)$

$\therefore$ Complete soln is $Z = \phi_1(y+x) + x\phi_2(y+x) - \sin(2x+3y)$

2.) $r + 3s + 2t = x + y$

$$\left(\frac{\delta^2 z}{\delta x^2} + 3\frac{\delta^2 z}{\delta x \delta y} + 2\frac{\delta^2 z}{\delta y^2}\right) = x + y$$

or $D^2 + 3DD' + 2D'^2 = x + y$

for particular integral.

$$Z = \frac{f(x,y)}{F(D,D')} = \frac{x+y}{(D^2 + 3DD' + 2D'^2)}$$

Neglect this

$$= \frac{1}{D^2}\left(1 + \frac{3D'}{D}\right)^{-1}(x+y)$$

$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D} + \frac{3^2 D'^2}{D^2} + \dots\right)(x+y)$$

$$= \frac{1}{D^2}\left(x+y - (x+y)\frac{3D'}{D} + 0\right)$$

$\frac{1}{D^2} \rightarrow$ Integ. w.r.to x 2 time

$D' =$ deriv. w.r.to y

$$= \frac{1}{D^2}\left((x+y) - \frac{3}{D}(0+1)\right)$$

$$= \frac{1}{D^2}\left[(x+y) - 3x\right]$$

$$= \frac{1}{D^2}(y-2x)$$

$$= xy - \frac{2x^2}{2} D^{-1}$$

$$= \frac{x^2y}{2} - \frac{x^3}{3}$$

for Complimentary function,

$D' = 1 \qquad D = m$

$\therefore$ auxillary eqn is, $m^2+3m+2=0$

or, $m^2(m+2)+1(m+2)=0$

$m_1 = -2 \qquad m_2 = -1$

$\therefore$ Roots are real and distinct

$\therefore C.F = \phi_1(y-2x) + \phi_2(y-x)$

$$\therefore z = \frac{x^2y}{2} - \frac{x^3}{3} + \phi_1(y-2x) + \phi_2(y-x) \#$$

* $\frac{\delta^2 z}{\delta x^2} - 2\frac{\delta^2 z}{\delta x \delta y} + \frac{\delta^2 z}{\delta y^2} = e^{x+2y}$

$\Rightarrow (D^2 - 2DD' + D'^2)z = e^{x+2y}$.

$\rightarrow$ for particular integral.

$$\therefore \frac{e^{ax+by}}{F(D,D')} = \frac{e^{ax+by}}{(a,b)}$$

$$P.I = \frac{e^{ax+by}}{(D^2-2DD'+D'^2)}$$

$$= \frac{\cancel{e^{ax+by}}}{\cancel{\partial}} = \frac{e^{x+2y}}{1^2 - 2\times1\times2+2^2} = \frac{e^{x+2y}}{1}$$

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for complementry function:

$D = m \qquad D' = 1$

Auxilliary eqⁿ is, $m^2 - 2m + 1 = 0$

or, $m^2 - m - m + 1 = 0$

Either $m_1 = 1 \quad m_2 = 1$

$\therefore C.F = \phi_1(y+x) + \phi_2\, x(y+x)$

$\therefore$ Complete solⁿ,

$$Z = \phi_1(y+x) + \phi_2\, x(y+x) + e^{x+2y} \#$$

4) $(r - a^2 t = x^2)$

$= \frac{\delta^2 z}{\delta x^2} - a^2 \frac{\delta^2 z}{}$

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5) $\left[D^2 + (a+b)DD' + abD'^2\right]z = xy$

$$P.I = \frac{f(x,y)}{F(D,D')} \qquad D^2\left(1 + (a+b)\frac{D'}{D}\right)$$

$$= \frac{xy}{D^2\left(1 + (a+b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right)}$$

$$= \frac{1}{D^2}\left(1 + (a+b)\frac{D'}{D}\right)^{-1} xy$$

$$= \frac{1}{D^2}\left(xy - (a+b)\frac{D'}{D}(xy) + (a+b)\,xy\,\frac{D'^2}{D^2}\right)$$

$$= \frac{1}{D^2}\left(xy - \frac{(a+b)}{D}x + 0\right]$$

$$= \frac{1}{D^2}\left(xy - \frac{(a+b)x^2}{2}\right)$$

$$= \frac{x^3y}{3\times2} - \frac{(a+b)x^4}{2\times3\times4}$$

$$P.I = \frac{x^3y}{6} - \frac{x^4}{24}(a+b)$$

$m^2 + (a+b)m + ab = 0$

$m^2 + am + bm + ab = 0$

$m(m+a) + b(m+a) = 0$

$(m+a)(m+b) = 0$

$\therefore m = -a$

$m = -b$

$\therefore C.F = \phi_1(y-ax) + \phi_2(y-bx)$

D'Alembert's Solution of Wave Equation

→ The ~~one dimensional~~ wave equation is,

$$\frac{\delta^2 u}{\delta t^2} = c^2 \frac{d^2 u}{\delta x^2}$$

where $c^2 = \frac{T}{S}$

Then $u(x,t)$ is a function of v and w, $v = x + ct$

$\frac{\delta v}{\delta t} = c$ and $\frac{\delta v}{\delta x} = 1$

The solution can be obtained by introducing the two independent variable v and w defined by

$v = x + ct$ and $w = x - ct$

$w = x - ct$

$\frac{\delta w}{\delta t} = -c$ $\quad \frac{\delta w}{\delta x} = 1$

Differentiating v w.r.t. x

$$\frac{\delta v}{\delta x} = \frac{\delta u}{\delta v} \cdot \frac{\delta v}{\delta x} + \frac{\delta u}{\delta w} \cdot \frac{dw}{\delta x}$$

$$= \frac{\delta u}{\delta v} + \frac{\delta u}{\delta w}$$

Again diff. w.r. to x

$\frac{\delta^2}{\delta v}\Big($

D'Alembert's solution of wave equation

The wave eqⁿ is $\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2}$ --- (i) $\quad u_{tt} = c^2 u_{xx}$

let $v = x + ct$ and $w = x - ct$

Then $u(x,t)$ is a function of v and w

$$v = x + ct$$

$$\frac{\delta v}{\delta t} = c \quad \text{and} \quad \frac{\delta v}{\delta x} = 1$$

$$w = x - ct$$

$$\frac{\delta w}{\delta t} = -c \qquad \frac{\delta w}{\delta x} = 1$$

by chain rule,

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta v} \cdot \frac{\delta v}{\delta x} + \frac{\delta u}{\delta w} \cdot \frac{\delta w}{\delta x}$$

$$= \frac{\delta u}{\delta v} \times 1 + \frac{\delta u}{\delta w} \times 1$$

$$= \frac{\delta u}{\delta v} + \frac{\delta u}{\delta w}$$

Assuming that all the partial derivatives are continuous.

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta v}\right) + \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta w}\right)$$

or $$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta v}\left(\frac{\delta u}{\delta v} + \frac{\delta u}{\delta w}\right) \cdot \frac{\delta v}{\delta x} + \frac{\delta}{\delta w}\left(\frac{\delta u}{\delta v} + \frac{\delta u}{\delta w}\right) \cdot \frac{\delta w}{\delta x}$$

or $$\frac{\delta^2 u}{\delta x^2} = \left(\frac{\delta^2 u}{\delta v^2} + \frac{\delta^2 u}{\delta v \cdot \delta w}\right) 1 + \left(\frac{\delta^2 u}{\delta w \cdot \delta v} + \frac{\delta^2 u}{\delta w^2}\right) \cdot 1$$

or $$\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 u}{\delta v^2} + 2\frac{\delta^2 u}{\delta v \cdot \delta w} + \frac{\delta^2 u}{\delta w^2}$$

$$\therefore u_{xx} = u_{vv} + 2u_{vw} + u_{ww} \text{ ---- } (2)$$

Again for $u_{tt}\left(\frac{\delta^2 u}{\delta t^2}\right)$

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta v}\cdot\frac{\delta v}{\delta t} + \frac{\delta u}{\delta w}\cdot\frac{\delta w}{\delta t}$$

$$= \frac{\delta u}{\delta v}\cdot c + \frac{\delta u}{\delta w}(-c)$$

$$= c\left[\frac{\delta u}{\delta v} - \frac{\delta u}{\delta w}\right]$$

$$\frac{\delta^2 u}{\delta t^2} = c\frac{\delta}{\delta t}\left[\frac{\delta u}{\delta v} - \frac{\delta u}{\delta w}\right]$$

$$= c\frac{\delta}{\delta v}\left(\frac{\delta u}{\delta v} - \frac{\delta u}{\delta w}\right)\frac{\delta v}{\delta t} + \frac{c\delta}{\delta w}\left(\frac{\delta u}{\delta v} - \frac{\delta u}{\delta w}\right)\cdot\frac{\delta w}{\delta t}$$

$$= c\left(\frac{\delta^2 u}{\delta v^2} - \frac{\delta^2 u}{\delta v\cdot\delta w}\right)\cancel{\delta v}\, c + c\left[\frac{\delta^2 u}{\delta u\cdot\delta v} - \frac{\delta^2 u}{\delta w^2}\right]\cdot(-c)$$

$$= c^2\left[u_{vv} - u_{vw} - u_{vw} + u_{ww}\right]$$

$$u_{tt} = c^2\left[u_{vv} - 2u_{vw} + u_{ww}\right] \text{ --- (3)}$$

putting eq^n (2) and eq^n (3) in eq^n (1)

$$u_{tt} = c^2 u_{xx}$$

$$\cancel{c^2}\left[u_{vv} - 2u_{v.w} + u_{ww}\right] = \cancel{c^2}\left[u_{vv} + 2u_{v} + u_{ww}\right]$$

or $\cancel{u_{vv}} - 2u_{v.w} + \cancel{u_{ww}} - \cancel{u_{vv}} - 2u_{uv} - \cancel{u_{ww}} = 0$

or, $-4u_{v.w} = 0$

or, $u_{v.w} = 0$

or $\frac{\delta^2 u}{\delta v.\delta w} = 0$ Integrating we get,

$$u = \int f(w)\,dw + \phi(v)$$

or $u = \psi(w) + \phi(v)$

where,

$$\psi(w) = \int f(w)\,dw$$

and $\psi(w)$ and $\phi(v)$ are arbitrary function.

$$u = \phi(x+ct) + \psi(x-ct)$$

$$\boxed{\therefore u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

is the required ~~equation~~ D'Alembert's soln of wave equation.

D' Alembert's Soln of wave eqn when initial condition are given.

→ we have,

wave eqn is $u_{tt} = c^2 u_{xx}$ --- (1)

initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

$u(x,t) = \phi(x+ct) + \psi(x-ct)$ --- (2)

soln of wave equation.

we impose given integration constants in order to remove arbitary function ϕ and ψ.

Diff. eqn (2) w.r. to t

we get,

$$\frac{\delta u}{\delta t} = \frac{\delta \phi(x+ct)}{\delta(x+ct)} \times \frac{\delta(x+ct)}{\delta t} + \frac{\delta \psi(x-ct)}{\delta(x-ct)} \times \frac{\delta(x-ct)}{\delta t}$$

or, $u_t = \phi'(x+ct) \cdot c + \psi'(x-ct) \times (-c)$

$u_t = c\phi'(x+ct) - c\psi'(x-ct)$ ---- (3)

Using initial condition in eqn (2), we get,

$$u(x,0) = \phi(x+0) + \psi(x-0)$$
$$u(x,0) = \phi(x) + \psi(x)$$

$\phi(x) + \psi(x) = f(x)$ ---- (4)

Using initial condition:

$$u_t(x,0) = g(x)$$

from eq^n 3

$$u_t(x,0) = c\phi'(x+0) - c\psi'(x-0)$$

$$\therefore g(x) = c\phi'(x) - c\psi'(x) \quad \text{---- (5)}$$

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

Integrating both sides,

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{s=a}^{s=x} g(s)\,ds \quad \text{--- (6)}$$

Adding eq^n 4 and 6

$$2\phi(x) = f(x) + \frac{1}{c}\int_{s=a}^{s=x} g(s)\,ds$$

or, $$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{a}^{x} g(s)\,ds$$

or, $$\phi(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{a}^{x+ct} g(s)\,ds \quad \text{— (7)}$$

Subtracting eq^n 4 and 6

$$2\psi(x) = f(x) - \frac{1}{2c}\int_{s=a}^{s=x} g(s)\,ds$$

or $$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{s=a}^{s=x} g(s)\,ds$$

or $$\psi(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_{s=a}^{s=x-ct} g(s)\,ds \quad \text{— (8)}$$

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Putting $\phi(x+ct)$ and $\psi(x-ct)$ from eqⁿ (7) and (8) into eqⁿ (2)

we get

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{s=a}^{s=x+ct} g(s)\,ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{s=a}^{s=x-ct} g(s)\,ds \quad \left[\because -\int_a^b f(x)\,dx = \int_b^a f(x)\,dx\right]$$

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_a^{x+ct} g(s)\,ds + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{s=x-ct}^{s=a} g(s)\,ds \quad \left[\because \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx\right]$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{s=x-ct}^{s=x+ct} g(s)\,ds \quad \text{--- (9)}$$

when initial Velocity is zero.

$$u_t(x,0) = g(x) = 0$$

eqⁿ (9) becomes,

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \#.$$

which is the required initial condition of D'Almebert's solⁿ of wave equation.

Solution of heat eq^n by Fourier Series :-

$$PDE = \frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2},$$

→ The heat equation is,

$$u_t = c^2 u_{xx} \quad \text{----(i)}$$

Boundary Condition:

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(l,t) &= 0 \end{aligned}\right\} t > 0$$

Initial Condition:

$$u(x,0) = f(x)$$

Let $u(x, ct) = XT$ ---- (2)

where,

$X(x)$ = function of x only.

$T(t)$ = function of t only.

Diff. w.r.to x and t, we get from eq^n (2)

$$\frac{\delta u}{\delta t} = XT', \quad \frac{\delta u}{\delta x} = X'T \text{ and } \frac{\delta^2 u}{\delta x^2} = XT''$$

Using these values in eq^n (1)

$$XT' = c^2 X''T$$

Separating Variables,

$$\frac{X''}{X} = \frac{T'}{c^2 T} = k \text{ (say)}$$

From above

$\frac{X''}{X} = k$	$\frac{T'}{c^2 T} = k$
	$T' = kc^2 T$
$X'' - kX = 0$	$T - kc^2 T = 0$
$\left(\frac{d^2}{dx^2} - k\right) X = 0$ --- (3)	$\left(\frac{d}{dt} - kc^2\right) T = 0$ --- (4)

which are ODE

Solving eq^n of ODE

Case 1:

where $k = \lambda^2 > 0$

from eq^n (3)

Auxilrary equation is,

$m^2 - \lambda^2 = 0$

$m = \pm\lambda$

$\therefore y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$$\boxed{\therefore X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}}$$

from eq^n (4)

$m = c^2\lambda^2$

$m_1 = c^2\lambda^2$

$$\boxed{\therefore T(t) = C_3 e^{c^2\lambda^2 t}}$$

Case 2:

when $k = 0$

from eq^n (3)

$m^2 = 0$

$\therefore X(x) = C_4 e^0 + C_5 e^0$

$$\boxed{\therefore X(x) = x C_4 + C_5}$$

from eq^n (4)

$m \cancel{+\lambda^2 c^2} = 0$

$\therefore m = \cancel{-\lambda^2 c^2}\ 0$

$\therefore T_t = C_6 e^0$

$$\boxed{\therefore T_t = C_6}$$

Case 3:

when $k = -\lambda^2 < 0$

from eq^n (3)

$m^2 + \lambda^2 = 0$

$m = \sqrt{-\lambda^2}$

$m = \pm\lambda i$

$= a + ib$

$y = Ce^{ax}(C_1 \cos bx + C_2 \sin bx)$

$$\boxed{\therefore X(x) = C_7 \cos\lambda x + C_8 \sin\lambda x}$$

from eq^n (4)

$m + \lambda^2 c^2 = 0$

$m = -\lambda^2 c^2$

$$\boxed{\therefore T(t) = C_9 e^{-\lambda^2 c^2}}$$

among those three cases, Solution of Case 3 is Consistent. It is because temperature is a predict function of x and t which must contain trignometry function.

The feasible sol^n is,

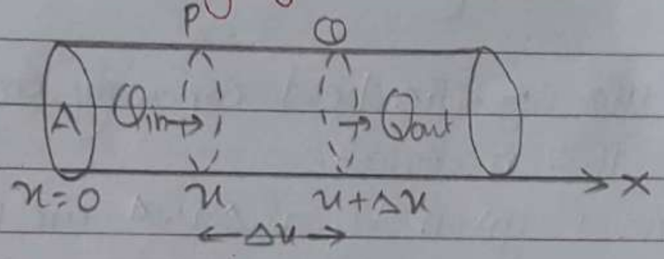
$$u(x,t) = C_9 e^{-\lambda^2 c^2 t} + C_7 \cos \lambda x + C_8 \sin \lambda x$$

$$\therefore U(x,t) = e^{-\lambda^2 c^2 t}(A \cos \lambda x + B \sin \lambda x)$$

where, $A = C_7 C_9$

$B = C_8 C_9$

Mathematical Modeling of one dimensional Heat flow



Let us consider flow of heat by Conduction in an uniform bar with sides insulated so that the loss of heat from sides by conduction is neglegible.

Physical Assumption

i) The bar is made up of heat conducting material.

ii) The bar is uniform and thin so that the temp at all points of cross section is constant.

Let one end of the bar is placed at origin which is at higher temp. and u denotes temperature function. Then u at any point of the bar depends upon distance (x) and time (t). i.e. $u = u(x,t)$

The amount of heat flowing inside the section of bar depends on

a) Cross sectional area of bar. i.e. A

b) Thermal Conductivity of bar i.e. K

c) Rate of temp w.r. to position i.e. $\frac{\delta u}{\delta x}$ (templ. gradient)

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Let P and Q be two nearly points on bar at x and $x+\Delta x$ from origin.
The amount of heat flowing into section PQ from P is given by
$$Q_{in} = -kA\left(\frac{\delta u}{\delta x}\right)_x \quad \text{--- (i)}$$
the amount of heat flowing out from Q is given by
$$Q_{out} = -kA\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} \quad \text{--- (ii)}$$
[(-) indicate that x increases and u decreases]

from eq^n (i) and eq^n (ii)
$$Q_{in} - Q_{out} = -kA\left(\frac{\delta u}{\delta x}\right)_x + kA\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x}$$
heat gained in section PQ is given by,
$$\therefore Q_{in} - Q_{out} = kA\left[\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x\right] \quad \text{--- (iii)}$$

Let s be the specific heat capacity and ρ be the density of the material.
The volume is given by $x.\Delta x.A$ and mass $= \rho.\Delta x.A$
Then,
amount of heat gained by section PQ is,
$= m \times s \times$ rate of change of temp per unit time
$$= \rho.\Delta x.A \times \frac{\delta u}{\delta t}.s \quad \text{--- (iv)}$$

Since eq^n (iii) and (iv) are equal, so,
$$\frac{\delta u}{\delta t} \rho\, \Delta x\, s\, \cancel{A} = k\cancel{A}\left[\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x\right]$$
$$\therefore \frac{\delta u}{\delta t} = \frac{k}{\rho s}\,\frac{\left[\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x\right]}{\Delta x}$$

taking limit $\Delta x \to 0$ on both sides.
$$\lim_{\Delta x \to 0} \frac{\delta u}{\delta t} = \lim_{\Delta x \to 0} \frac{k}{\rho s}\,\frac{\left[\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x\right]}{\Delta x}$$

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or, $\frac{\delta u}{\delta t} = \frac{k}{ss} \cdot \frac{\delta^2 u}{\delta x^2}$

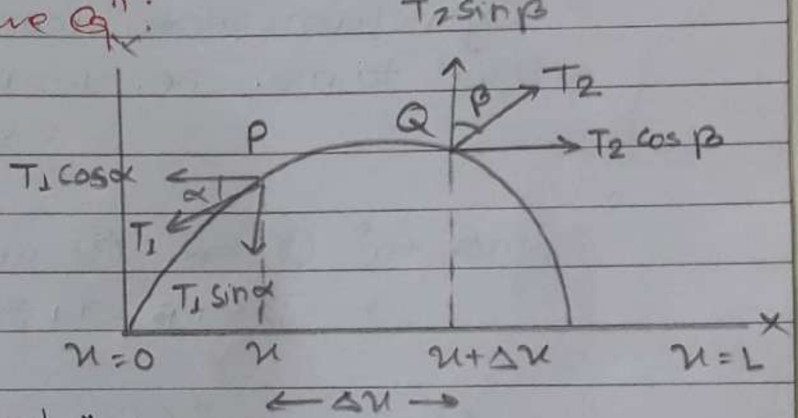
or, $\boxed{\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2}}$ where $c^2 = \frac{k}{ss}$ is diffusivity of material.

$\boxed{\therefore u_t = c^2 u_{xx}}$ is required P.D.E of One dimensional heat flow eq^n.

Mathematical modeling of wave eq^n:

Physical assumption:

i) The motion takes place on the vertical plane only and each particle of string execute transvers vibration only.



ii) The string is perfectly elastic and it transmit tension only but not bending and shearing force.

iii) The motion of string is subject to only a constant tension (T) and no other external forces.

Consider an elastic string of length 'L' (when it is stretched). Fix two ends of string at x=0 and x=L along x-axis.

let u(x,t) denotes deflection of string at any position 'x' & time 't'. Consider a portion PQ of the string. Since there is no resistance to bending the tension is traditional to the string at each point.

Let T_1 and T_2 be the tension at P and Q respectively.

Since the points of string moved vertically, there is no motion in the horizontal direction, so that the tension produced at horizontal component must be constant. i.e.

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{Constant} \quad \text{--- (1)}$$

The vertical component T_1 and T_2 are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ at P and Q respectively.

Resultant force acting on the portion PQ of the string is given by

$$T_2 \sin\beta - T_1 \sin\alpha \quad \text{---} \; (2)$$

Let ρ be the mass of undeflected string per unit length.

i.e. $\rho = \dfrac{m}{\Delta x}$, $m = \rho . \Delta x$

By newton second law of motion, The resultant force is equal to ma. i.e. $F = ma$, $a = \dfrac{\delta^2 u}{\delta t^2}$

$$= \rho . \Delta x . \frac{\delta^2 u}{\delta t^2} \quad \text{---} \; (3)$$

Since eq^n (2) and (3) are equal so,

$$T_2 \sin\beta - T_1 \sin\alpha = \rho . \Delta x . \frac{\delta^2 u}{\delta t^2} \; - (4)$$

Dividing eq^n (4) by eq^n 1

$$\frac{\cancel{T_2} \sin\beta}{\cancel{T_2} \cos\beta} - \frac{\cancel{T_1} \sin\alpha}{\cancel{T_1} \cos\alpha} = \frac{\rho \Delta x}{T} . \frac{\delta^2 u}{\delta t^2}$$

$$\therefore \frac{\delta^2 u}{\delta t^2} = \frac{T}{\rho \Delta x} (\tan\beta - \tan\alpha)$$

Since $\tan\beta$ and $\tan\alpha$ are slopes of string at $x + \Delta x$ and x.

$$\tan\alpha = \left(\frac{\delta u}{\delta x}\right)_x \qquad \tan\beta = \left(\frac{\delta u}{\delta x}\right)_{x+\Delta x}$$

then eq^n (4) becomes,

$$\frac{\delta^2 u}{\delta t^2} = \frac{T}{\rho \Delta x} \left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x$$

taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x \to 0} \frac{\delta^2 u}{\delta t^2} = \lim_{\Delta x \to 0} \frac{T}{\rho} \frac{\left(\frac{\delta u}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta u}{\delta x}\right)_x}{\Delta x}$$

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or, $\frac{\delta^2 u}{\delta t^2} = \frac{T}{s} \frac{\delta^2 u}{\delta x^2}$ $\boxed{c^2 = \frac{T}{s}}$

$\boxed{\therefore u_{tt} = c^2 u_{xx}}$ is the p.d.e of wave eq^n