(iii) $x_1(n) = -5\left(\frac{1}{2}\right)^n u[-n-1]$

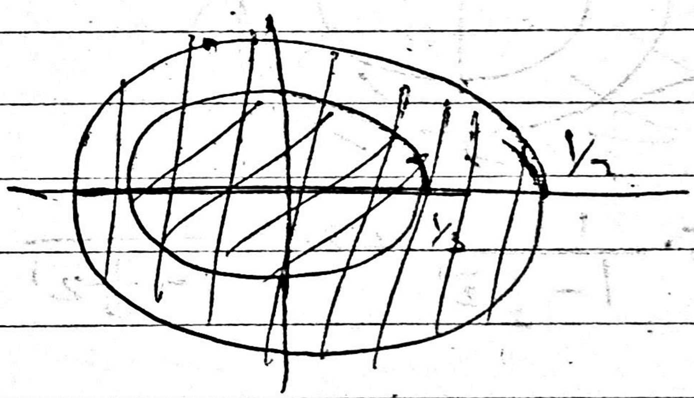
$x_2(n) = -2\left(\frac{1}{3}\right)^n u[-n-1]$

$$X_1(z) = \frac{5}{1-\frac{1}{2}z^{-1}}$$

$$X_2(z) = 2\frac{1}{1-\frac{1}{3}z^{-1}}$$

ROC of $X_1(z)$, $|z| < \frac{1}{2}$

ROC of $X_2(z)$, $|z| < \frac{1}{3}$



ROC of $X(z) \Rightarrow |z| < \frac{1}{3}$

$$X(z) = \frac{5}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}} \; ; |z| < \frac{1}{3}$$

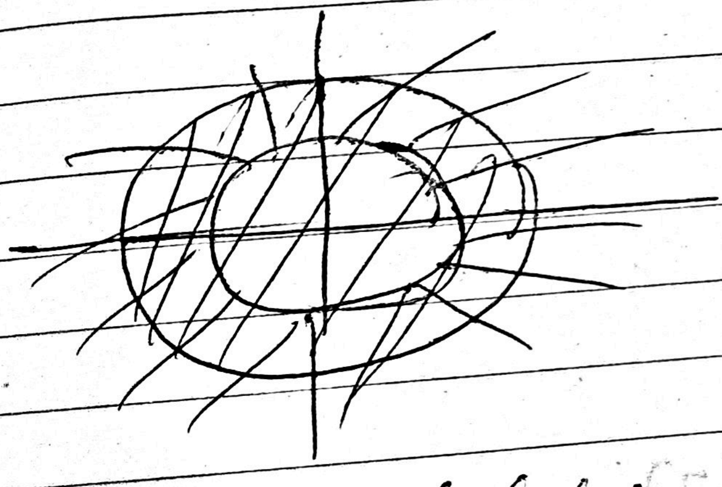
(iv) $x_1[n] = -5\left(\frac{1}{2}\right)^n u(-n-1)$

$x_2[n] = 2\left(\frac{1}{3}\right)^n u[n]$

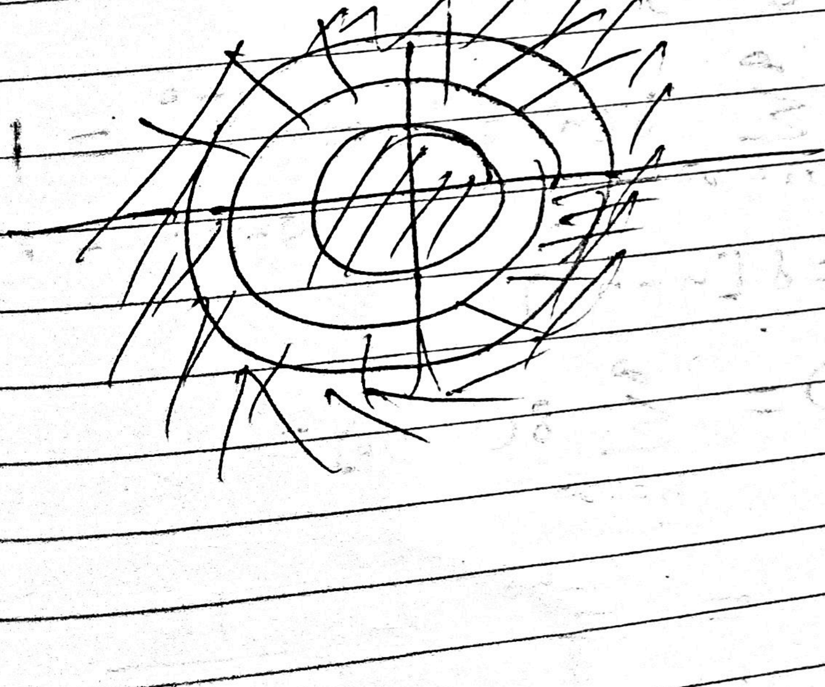
$X_1(z) = \dfrac{5}{1-\frac{1}{2}z^{-1}}$ $\qquad |z| < \frac{1}{2}$

$X_2(z) = \dfrac{2}{1-\frac{1}{3}z^{-1}}$ $\qquad |z| > \frac{1}{2}$

ROC $\Rightarrow$ $\frac{1}{3} < |z| < \frac{1}{2}$



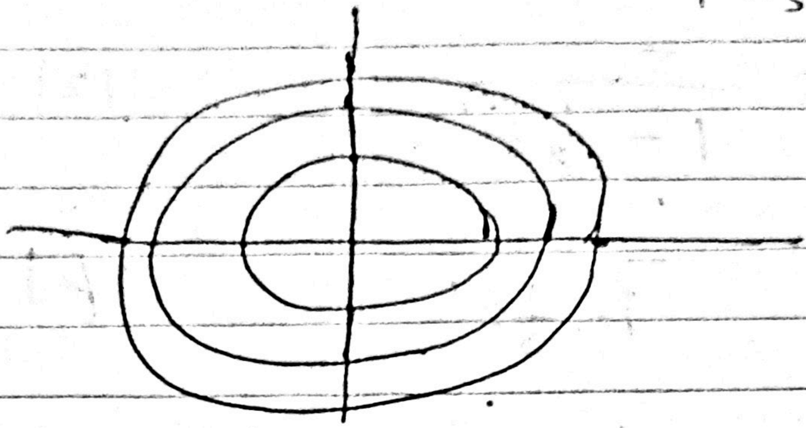
(v)



$|z| < \frac{1}{2}$ $\qquad |z| > \frac{1}{3}$

(vi) $x[n] = 3\left(\frac{1}{2}\right)^n u[-n-1] + 5\left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{5}\right)^n u[n]$

$|z| > \frac{1}{5}$



Q→ $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = z^{-0} = 1$$

entire z-plane.

Q→ $x[n] = \delta[n - n_0]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n - n_0)\, z^{-n}$$

$$= z^{-n_0} = \frac{1}{z^{n_0}}$$

ROC → Entire z-plane except $z = 0$

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Q→ $x[n] = \delta(n+n_0)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n+n_0) z^{-n}$$

$$= z^{n_0}$$

ROC ⇒ Entire z plane except $|z| = \infty$

Q→ $x[n] = u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

$$= \frac{z}{z-1}$$

ROC ⇒ $|z| > 1$

Properties of ROC

① ROC of X(z) is circular stripe centered at origin.

② ROC of X(z) does not contain any pole of X(z).

③ If x[n] is right handed sequence then its ROC lie outside the outermost pole of X(z).

④ If $x[n]$ is left handed sequence then its ROC lies inside the innermost pole of $X(z)$.

⑤ If $x[n]$ is algebric sum of left handed and right handed sequences then ~~its~~ ROC of $X(z)$, area of ~~the~~ overlap of ROCs of individual sequence.

⑥ If $x[n]$ is of finite duration then its ROC is entire z-plane which may or may not include $z=0$ & $z=\infty$.

Inverse Z-Transform :-

$$\boxed{x[n] = \frac{1}{2\pi j}\int_{-\pi}^{\pi} X(z)\, z^{n-1}\, dz}$$

$$\boxed{\int_{-\pi}^{\pi} \leftrightarrow \int_{0}^{2\pi} \leftrightarrow \oint}$$

Q→ Find the Inverse Z-transform of the function $X(z)$ if

(i) $$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

ROC: $|z| > \frac{1}{3}$

$$= \frac{A}{(1-\frac{1}{4}z^{-1})} + \frac{B}{(1-\frac{1}{3}z^{-1})}$$

$$= \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})}$$

Inverse Z - transform

$$x[n] = \left(\frac{1}{4}\right)^n u(n) + 2\left(\frac{1}{3}\right)^n u(n)$$

(ii) Find the Inverse - Z transform of function $X(z)$ if $X(z)$

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} \quad ; |z| < \frac{1}{4}$$

$$= \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})}$$

$$= -\left(\frac{1}{4}\right)^n u(-n-1) - 2\left(\frac{1}{3}\right)^n u(-n-1)$$

(iii) ROC : $\frac{1}{4} < |z| < \frac{1}{3}$

$$x(n) = \left(\frac{1}{4}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(-n-1)$$

ROC: $1 < |z| < 2$

Q→ Find the Inverse z-transform of function $X(z)$ if

$$X(z) = \frac{(1 - z^{-1} + z^{-2})}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})}$$

$$X(z) = \frac{A}{(1 - \frac{1}{2}z^{-1})} + \frac{B}{(1 - 2z^{-1})} + \frac{C}{(1 - z^{-1})}$$

$$= \frac{1}{(1 - \frac{1}{2}z^{-1})} + \frac{2}{(1 - 2z^{-1})} - \frac{2}{(1 - z^{-1})}$$

$$= \left(\frac{1}{2}\right)^n u(n) - 2(2)^n u[-n-1] - 2(1)^n u(n)$$

Q→ Find the inverse z-transform for the function $X(z)$ if

$$X(z) = z^4 + z^2 + z + 5 + 6z^{-1} + 3z^{-3}$$

$$x[n] = \delta[n+4] + \delta[n+2] + \delta[n+1] + 5\delta[n] + 6\delta[n-1] + 3\delta[n-3]$$

$$x[n] = \{1, 0, 1, 1, \underset{\uparrow}{5}, 6, 0, 3\}$$

$$e^x = 1 + \frac{x}{\underline{1}} + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \cdots \infty$$

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Q→ Find the inverse -Z transform of the signal $X(z)$ if

$$X(z) = e^{1/z}$$

Sol^n ⇒

$$= 1 + \frac{\left(\frac{1}{z}\right)}{\underline{1}} + \frac{\left(\frac{1}{z}\right)^2}{\underline{2}} + \frac{\left(\frac{1}{z}\right)^3}{\underline{3}} + \cdots \infty$$

$$= \sum_{n=0}^{\infty} \frac{1}{\underline{n}} \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{\underline{n}} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\underline{n}} u[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Where $\boxed{x[n] = \frac{1}{\underline{n}} u[n]}$

Properties of Z-Transform :-

① Linearity :-

$$Z\{x_1[n] + x_2[n]\} = X_1(z) + X_2(z) \; ; \; R \ni R_1 \cap R_2$$

② Time shifting :-

ⓐ Time delay :-

$$Z\{x(n-k)\} = z^{-k} X(z) + \sum_{m=1}^{k} x[-m] z^{m-k}$$

$$\boxed{\begin{aligned} &Z\{x(n-1)\} = z^{-1} X(z) + x[-1] \\ &Z\{x(n-2)\} = z^{-2} X(z) + x[-1] z^{-1} + x(-2) \end{aligned}}$$

$$Z\{x(n-3)\} = z^{-3} X(z) + x[-1] z^{-2} + x(-2) z^{-1} + x(-3)$$

ⓑ Time Advance :

$$Z\{x[n+k]\} = z^{k} X(z) - \sum_{m=0}^{k-1} x[m] z^{k-m}$$

$$\boxed{\begin{aligned} &Z\{x(n+1)\} = z X(z) - x(0) z \\ &Z\{x(n+2)\} = z^{2} X(z) - x(0) z^{2} - x(1) z \end{aligned}}$$

③ Time Scaling:

$$Z\{\alpha^n x(n)\} = X\left[\frac{z}{\alpha}\right]$$

④ Time Reverral:

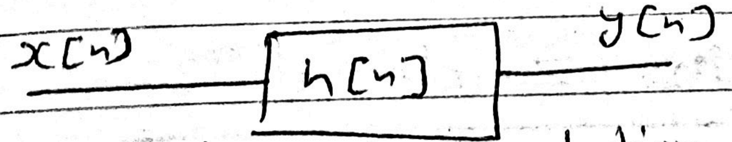
$$Z\{x(-n)\} = X(z^{-1})$$

⑤ Conjugation property:

$$Z\{x^*(n)\} = X^*(z^*)$$

⑥ Convolution Property:

$$Z\left[x_1[n] * x_2[n]\right] = X_1(z)\, X_2(z)$$



Abblication od convolution property to LTI system

$$y(n) = h[n] * x[n]$$

$$Y(z) = H(z)\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$ — Tranffer function

3) Time Scaling:

$$Z\{\alpha^n x(n)\} = X\left[\frac{z}{\alpha}\right]$$

4) Time Reversal:

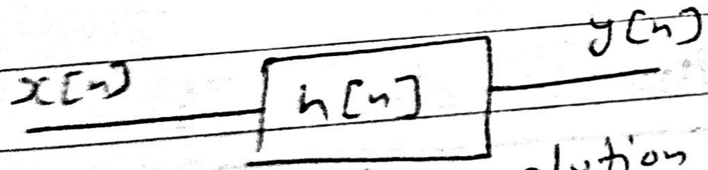
$$Z\{x(-n)\} = X(z^{-1})$$

5) Conjugation property:

$$Z\{x^*(n)\} = X^*(z^*)$$

6) Convolution Property:

$$Z[x_1(n) * x_2[n]] = X_1(z)\, X_2(z)$$



Application of convolution property to LTI system

$$y(n) = h[n] * x[n]$$

$$Y(z) = H(z)\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \text{Transfer function}$$

Q → If $Z\{x(n) * x(n)\} = \left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)^2$; $|z| < \frac{1}{3}$

Then find $x[n] = ($

$Z\{x(n) * x(n)\}$

$X^2(z) = \left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)^2$

$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$

$$\boxed{X(z) = -\left(\frac{1}{3}\right)^n . u(-n-1)}$$

⑦ Differentiation in Z-domain :—

$$Z\{n^k x[n]\} = (-1)^k \frac{d^k}{dz^k} X(z)$$

Q → $x[n] = n\,u[n]$

Sol^n →

$$= (-1)^1 \frac{d}{dz} U[z]$$

$$= (-1) \frac{d}{dz} \frac{z}{z-1}$$

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$$= -\frac{(z-1)1 \cdot -z}{(z-1)^2} = \frac{1}{(z-1)^2}$$

$$= \frac{z^{-2}}{(1-z^{-1})^2}$$

(8) Initial & Final value Theorem:

Initial value:

$$x[0] = \lim_{n \to 0} x[n] = \lim_{z \to \infty} X(z)$$

Final value Theorem:

$$x[\infty] = \lim_{n \to \infty} x[n] = \lim_{z \to 1} [1 - z^{-1}] X(z)$$

$$= \lim_{z \to 1} (z-1) X(z)$$

(9) Multiplication in Time domain:

$$Z\{x_1[n]\, x_2[n]\} = \frac{1}{2\pi j} \oint X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

$(z^{-1}) + (z^{-1})^2 \rightarrow \quad = \frac{1}{1-z^{-1}} = \frac{2}{1-\frac{1}{z}}$

⑩ Z-transform of real part of signal

$$Z[Re\{x[n]\}] = \frac{1}{2}[X(z) + X^*(z^*)]$$

$$Z[Im\{x[n]\}] = \frac{1}{2}[X(z) - X^*(z^*)]$$

Z-Transform of important signals :-

① $Z\{\delta[n]\} = 1$

② $Z\{\delta(n-k)\} = z^{-k}$

③ $Z\{\delta(n+k)\} = z^{k}$

④ $Z\{u[n]\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

Q) Find the z transform of $x(n)$ if

$x[n] = u[-n]$

$$X[z] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=\infty}^{\infty} u(n) z^{-n}$$

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$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{n=\infty}^{0} z^{n} \Rightarrow \text{Replacing } n \text{ by } -n$$

$$= \sum_{n=0}^{\infty} z^{n}$$

$$= 1 + z + z^2 + z^3 + \text{---} \infty$$

$$= \frac{1}{1-z} = -\frac{z^{-1}}{(1-z^{-1})}$$

$$= -\frac{1}{z-1}$$

Q. $x[n] = u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[-n-1] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=\infty}^{1} z^{n}$$

$$= \sum_{n=1}^{\infty} z^{n}$$

$$= z + z^2 + \text{---} \infty$$

$$= \frac{z}{1-z} = -\frac{1}{1-z^{-1}}$$

$x[n] = \cos \omega_0 n \; u[n]$

$$= \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] u[n]$$

$$= \left[\frac{1}{2}\alpha^n + \frac{1}{2}\alpha^{-n}\right] u[n]$$

$$\boxed{X[z] = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}}$$

$x[n] = \sin \omega_0 n \cdot u[n]$

$$\boxed{X[z] = \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}}$$

$x[n] = \alpha^n \cos \omega_0 n \; u[n]$

$$\boxed{X(z) = \frac{1 - \alpha z^{-1}\cos\omega_0}{1 - 2\alpha z^{-1}\cos\omega_0 + \alpha z^{-2}}}$$

$x[n] = \alpha^n \sin \omega_0 n \; u[n]$

$$\boxed{X(z) = \frac{\alpha z^{-1}\sin\omega_0}{1 - 2\alpha z^{-1}\cos\omega_0 + \alpha z^{-2}}}$$

Application of Z-Transform to LTI system

Properties of LTI system:

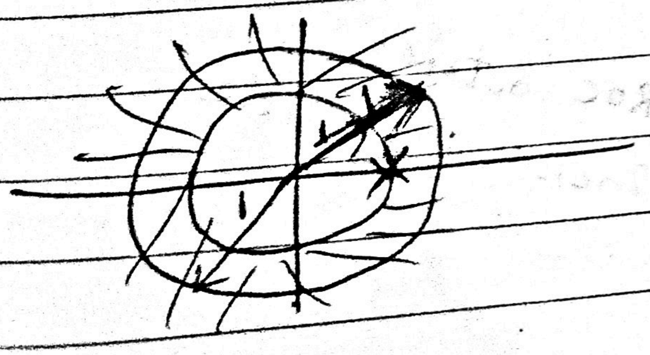
① Causality:

A LTI system is said to be causal if its impulse response is zero for $n<0$. Therefore impulse responce of the system is right handed for the system to be causal. Then poles of Transfer function in z-domain lie ~~out~~ with in outer ~~most~~ pole & ROC lies outside the outer most pole of the transfer function.

② Stability:

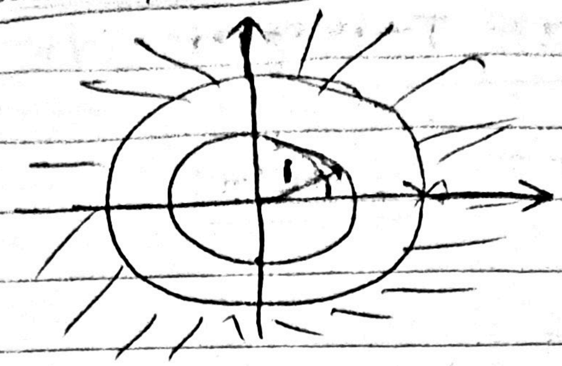
A system is said to be stable if its impulse responce is absolutly summable & ROC of transfer fn include unit circle.

Causal & stable system:

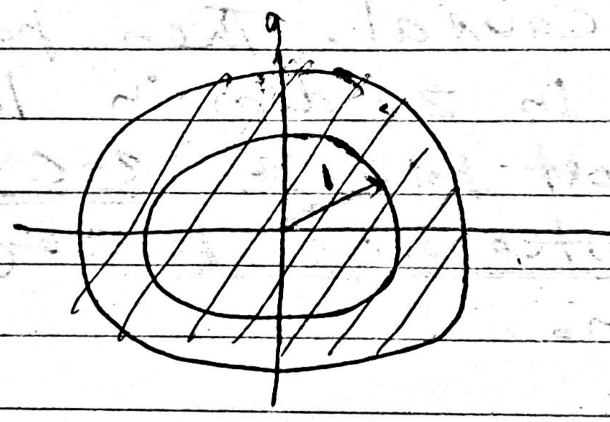


Poles are with in the unit circle

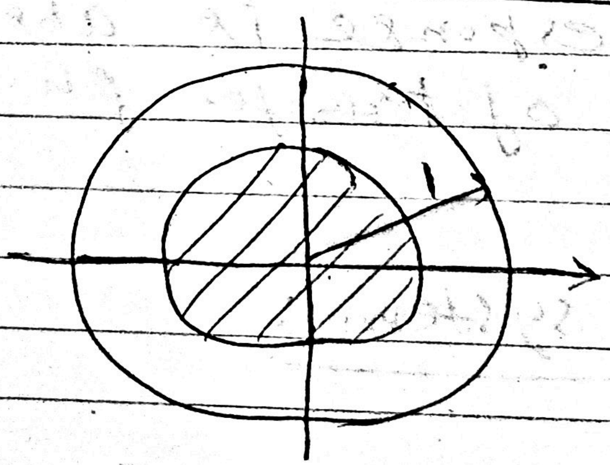
Causal & unstable



Non causal & stable



Non causal & unstable



* Causal → ROC outside
Stable → Include unit circle

Non Causal → Pole outside unit circle

Causal → Pole inside unit circle

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Ex A Causal LTI system is described by the difference Equation

$$2y[n] = a\,y[n-2] - 2x[n] + b\,x[n-1]$$

Find the Condition for stability of system.

Solⁿ Taking z-tr

$$2Y[z] = a\left[z^{-2}Y[z]\right] - 2X[z] + b z^{-1}X[z]$$

$$Y[z]\left[2 - a z^{-2}\right] = \left[b z^{-1} - 2\right]X[z]$$

$$\frac{Y(z)}{X(z)} = \frac{b z^{-1} - 2}{2 - a z^{-2}}$$

$$H(z) = \frac{z^2\left[b z^{-1} - 2\right]}{2\left[z^2 - \frac{a}{2}\right]}$$

Charac. Eqⁿ

$$z^2 - \frac{a}{2} = 0$$

$$z = \pm\sqrt{\frac{a}{2}}$$

$$|z| = \sqrt{\frac{a}{2}}$$

For Causal system to be stable, pole must lie within the unit circle.

$$\sqrt{\frac{a}{2}} < 1$$

$$\boxed{a < 2}$$