UAS EXERCISE

VECTORS

1. Cross and Dot Products

Given vectors $\bar{a} = [1, 2, 3]$
$\bar{b} = [-4, 1, 0]$
$\bar{c} = [3, -2, 5]$

Calculate $|\bar{a} \times \bar{b}|\bar{c} + (2\bar{a} \cdot 3\bar{c})\bar{b} - 5\bar{a}$

- $\bar{a} \times \bar{b}$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -4 & 1 & 0 \end{vmatrix} \rightarrow \text{reduksi baris } 1 \rightarrow \text{always!}$$

$$= i \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -4 & 1 \end{vmatrix}$$

$$= i\,(2(0) - 3(1)) - j\,(1(0) - 3(-4)) + k\,(1(1) - 2(-4))$$

$$= -3i + 12j + 9k \rightarrow [-3, 12, 9]$$

- $|\bar{a} \times \bar{b}|$

$$= \sqrt{(-3)^2 + 12^2 + 9^2}$$
$$= \sqrt{9 + 144 + 81}$$
$$= \sqrt{234}$$
$$= 3\sqrt{26}$$

- $|\bar{a} \times \bar{b}|\bar{c}$

$$= 3\sqrt{26} \cdot [3, -2, 5]$$
$$= [9\sqrt{26}, -6\sqrt{26}, 15\sqrt{26}]$$

A

- $2\bar{a} \cdot 3\bar{c}$

$$= 2[1, 2, 3] \cdot 3[3, -2, 5]$$
$$= [2, 4, 6] \cdot [9, -6, 15]$$
$$= 2(9) + 4(-6) + 6(15)$$
$$= 18 - 24 + 90$$
$$= 84$$

- $(2\bar{a} \cdot 3\bar{c})\bar{b}$

$$= 84 \cdot [-4, 1, 0]$$
$$= [-336, 84, 0]$$

B

- $5\bar{a}$

$$= 5 \cdot [1, 2, 3]$$
$$= [5, 10, 15]$$

C

SOLUTIONS:

$$A = [9\sqrt{26}, -6\sqrt{26}, 15\sqrt{26}]$$
$$B = [-336, 84, 0] \quad +$$
$$[-290.1, 53.4, 76.5]$$
$$C = [5, 10, 15] \quad -$$
$$[-295.1, 43.4, 61.5] \,//$$

② Line & Plane Equation

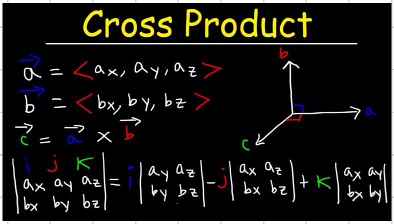
Find plane equation passing through $\overset{P_1}{(1,2,3)}$, $\overset{P_2}{(4,5,7)}$, and $\overset{P_3}{(-1,-2,-3)}$

- Vector normal

↳ define 2 vectors

$$\boxed{\begin{aligned} V_1 &= P_2 - P_1 = (3, 3, 4) \\ V_2 &= P_3 - P_1 = (-2, -4, -6) \end{aligned}}$$

↳ $n = V_1 \times V_2$



$$= \begin{vmatrix} i & j & k \\ 3 & 3 & 4 \\ -2 & -4 & -6 \end{vmatrix}$$

$$= i\begin{vmatrix} 3 & 4 \\ -4 & -6 \end{vmatrix} - j\begin{vmatrix} 3 & 4 \\ -2 & -6 \end{vmatrix} + k\begin{vmatrix} 3 & 3 \\ -2 & -4 \end{vmatrix}$$

$$= i\,(3(-6)-4(-4)) - j\,(3(-6)-4(-2)) + k\,(3(-4)-3(-2))$$

$$= -2i + 10j - 6k$$

$$= [\underset{a}{-2}, \underset{b}{10}, \underset{c}{-6}]$$

- vector form → $P = P_0 + t_1\overline{V_1} + t_2\overline{V_2}$

$$(x, y, z) = (1, 2, 3) + t_1(3, 3, 4) + t_2(-2, -4, -6) \,/\!/$$

- Parametric Form

$$x = 1 + 3t_1 - 2t_2$$
$$y = 2 + 3t_1 - 4t_2$$
$$z = 3 + 4t_1 - 6t_2 \,/\!/$$

- Symmetric form

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
$$-2(x-1) + 10(y-2) - 6(z-3) = 0$$
$$-2x + 2 + 10y - 20 - 6z + 18 = 0$$
$$-2x + 10y - 6z = 0 \,/\!/$$

eigen

Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}$

① Find characteristic equation

$|A - \lambda I| = 0$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 4 & 5-\lambda \end{bmatrix} = 0$$ → ekspansi baris 1

$$(1-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 4 & 5-\lambda \end{vmatrix} - 0 + 0 = 0$$

$$(1-\lambda)\left((3-\lambda)(5-\lambda) - 0(4)\right) = 0$$

$$(1-\lambda)(15 - 8\lambda + \lambda^2) = 0$$

$$15 - 8\lambda + \lambda^2 - 15\lambda + 8\lambda^2 - \lambda^3 = 0$$

$$-\lambda^3 + 9\lambda^2 - 23\lambda + 15 = 0 \mathbin{/\!/}$$

② Find eigenvalues

METODE HORNER

1	-1	9	-23	15
	0	-1	8	-15
+	-1	8	-15	0

0 → sisa bagi

$$(\lambda - 1)(-\lambda^2 + 8\lambda - 15) = 0$$

$$(\lambda - 1) \cdot -(\lambda - 5)(\lambda - 3) = 0$$

$$\lambda_1 = 1 \mathbin{/\!/} \cup \lambda_2 = 5 \mathbin{/\!/} \cup \lambda_3 = 3 \mathbin{/\!/}$$

3. Find eigenvectors → $(A-\lambda I)x = 0$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 4 & 5-\lambda \end{bmatrix}$$

•) for $\lambda = 1$

$$\begin{array}{c} \\ \text{i} \\ \text{ii} \end{array}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i) $2X_1 + 2X_2 = 0$

$\boxed{X_1 = -X_2}$

ii) $4X_2 + 4X_3 = 0$

$\boxed{X_3 = -X_2}$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -X_2 \\ X_2 \\ -X_2 \end{bmatrix} = X_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

eigenvectors $= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ //

•) for $\lambda = 5$

$$\begin{array}{c} \text{i} \\ \text{ii} \\ \text{iii} \end{array}\begin{bmatrix} -4 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i) $-4X_1 = 0$

$\boxed{X_1 = 0}$

iii) $4X_2 = 0$

$\boxed{X_2 = 0}$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ X_3 \end{bmatrix} = X_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

eigenvector : $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ //

•) for $\lambda = 3$

$$\begin{array}{c} \text{i} \\ \text{ii} \\ \text{iii} \end{array}\begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i) $-2X_1 = 0$

$\boxed{X_1 = 0}$

iii) $4X_2 + 2X_3 = 0$

$\boxed{X_2 = -\frac{1}{2} X_3}$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{X_3}{2} \\ X_3 \end{bmatrix} = X_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

eigenvector : $\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ //

4. Diagonalize matrix A → $P^{-1}AP$

matrix $P = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$

$$[P \mid I] \xrightarrow{OBE} [I \mid P^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-b_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[b_3+b_1]{b_2-b_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & 1 & 1 & 0 \end{array}\right] \xrightarrow{-2b_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & -2 & 0 \end{array}\right] \xrightarrow{b_2-b_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & -2 & 0 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -2 & 0 \end{bmatrix}}_{P^{-1}}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 5 & 10 & 5 \\ -6 & -6 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ → diagonalize // ✓

5. Find A^{10} → $A^k = PD^kP^{-1}$

$$A^{10} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{10} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$

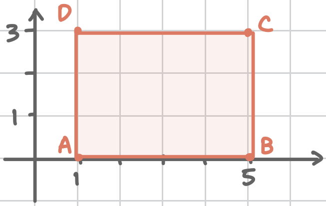
$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 5^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$

=

Linear Transformation

1. Given rectangle ABCD : A(1,0), B(5,0), C(5,3), D(1,3)

It's a) reflected about $y=x$,
then b) rotated 90° counterclockwise,
then c) dilated with factor $k=2$
find the final image!



CHEAT!

Translation by (a,b)

$(x,y) \longrightarrow (x+a, y+b)$

Rotation

90° CW / 270° CCW : $(x,y) \rightarrow (y,-x)$

180° CW / 180° CCW : $(x,y) \rightarrow (-x,-y)$

270° CW / 90° CCW : $(x,y) \rightarrow (-y,x)$

Reflection

x-axis : $(x,y) \rightarrow (x,-y)$

y-axis : $(x,y) \rightarrow (-x,y)$

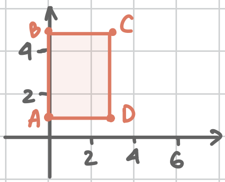
Dilation by k

$(x,y) \rightarrow (kx,ky)$

a) Reflected $y=x$

$(x,y) \rightarrow (y,x)$

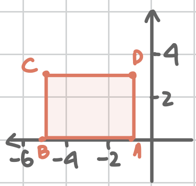
A(1,0)		A'(0,1)
B(5,0)	$\Rightarrow$	B'(0,5)
C(5,3)		C'(3,5)
D(1,3)		D'(3,1)



b) Rotation 90° counterclockwise

$(x,y) \longrightarrow (-y,x)$

A'(0,1)		A''(-1,0)
B'(0,5)	$\Rightarrow$	B''(-5,0)
C'(3,5)		C''(-5,3)
D'(3,1)		D''(-1,3)



c) Dilated by $k=2$

$(x,y) \rightarrow (kx,ky)$

A''(-1,0)		A'''(-2,0)
B''(-5,0)	$\overset{2}{\Rightarrow}$	B'''(-10,0)
C''(-5,3)		C'''(-10,6)
D''(-1,3)		D'''(-2,6)